Introduction to Linear Algebra

Linear algebra plays an important role in machine learning and general mathematics.

- Linear system of equations
- Matrix algebra
- Eigenvalues and eigenvectors
- Vector spaces

Linear Algebra - Chapter 1

Systems of Linear Equations

OUR GOAL

Elementary Operations

Gaussian Elimination

Homogeneous Equations

1.1. Solutions and Elementary Operations

•
$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

is called a linear equation

- If $a_1s_1+a_2s_2+...+a_ns_n=b$
- \rightarrow (s₁,s₂,...,s_n) is called solution of the equation
- A system may have:
 - no solution 0 unique solution 1 an infinite family of solutions ∞

Inconsistent	Consistent		
No solutions	Unique solution	Infinitely many solutions	

Example 1

$$\begin{cases} x + 2y = 1 \\ x + 2y = 3 \end{cases}$$

no solution

$$\begin{cases} x + y - z = 1 \\ x + y + z = 3 \end{cases}$$

(0,2,1), (2,0,1) (t,2-t,1)

Inconsistent

Consistent

(infinitely many solutions)

(t,2-t,1): general solution, given in parametric form, t is parameter (t is arbitrary)

Algebraic Method

$$3x_{1} + 2x_{2} - x_{3} + x_{4} = -1$$

$$2x_{1} - x_{3} + 2x_{4} = 0$$

$$3x_{1} + x_{2} + 2x_{3} + 5x_{4} = 2$$

$$3x_{1} + x_{2} + 2x_{3} + 5x_{4} = 2$$

$$3x_{1} + x_{2} + 2x_{3} + 5x_{4} = 2$$

$$3x_{1} + x_{2} + 2x_{3} + 5x_{4} = 2$$

$$3x_{1} + x_{2} + 2x_{3} + 5x_{4} = 2$$

$$\begin{vmatrix} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{vmatrix}$$

augmented matrix

coefficient matrix

constant matrix

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Example 2

Consider the system

$$\begin{cases} x - y = 1 \\ x + 2y = -2 \end{cases}$$

augmented matrix
$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$$

Example 2

$$\begin{cases} x - y = 1 \\ x + 2y = -2 \end{cases} \Leftrightarrow \begin{cases} x - y = 1 \\ 0x + 3y = -3 \end{cases} \Leftrightarrow \begin{cases} x - y = 1 \\ 0x + y = -1 \end{cases} \Leftrightarrow \begin{cases} x + 0y = 0 \\ 0x + y = -1 \end{cases}$$
$$\langle \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} \rangle \Rightarrow \langle \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \end{vmatrix} \rangle \Rightarrow \langle \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix} \rangle \Rightarrow \langle \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} \rangle$$

 \Rightarrow Solution (0,-1)

Elementary Operations (phép biến đổi sơ cấp)

Interchange two equations (type I)

$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow{interchange} \begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases}$$

Multiply one equation by a nonzero number (type II)

$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow{multiplied \ by - 2} \begin{cases} 2x + 3y = 5 \\ -2x + 4y = 2 \end{cases}$$

 Add a multiple of one equation to a different equation (type III)

$$\begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases} \xrightarrow{add\ a\ multiple} \begin{cases} x - 2y = -1 \\ 0x + 7y = 7 \end{cases}$$

Elementary operations

Interchange two rows

$$\begin{bmatrix} 0 & -2 & 3 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

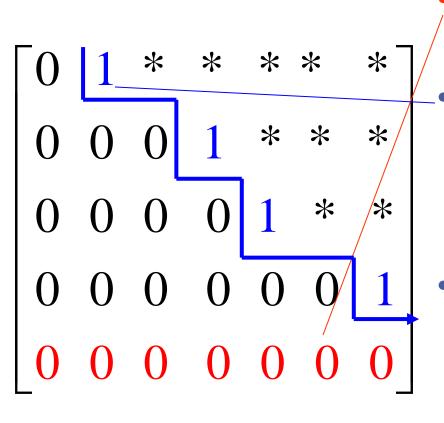
Multiply one row by a nonzero number

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 2 & 0 & 1 \end{bmatrix}$$

Add a multiple of one row to different row

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{-2r_1+r_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 0 & 2 & -3 \end{bmatrix}$$

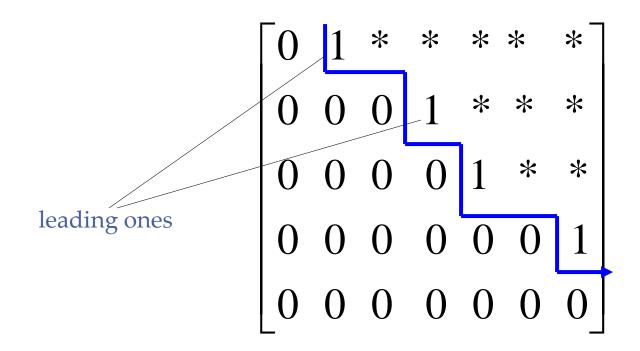
A row-echelon matrix has 3 properties



- All the zero rows are at the bottom
- The first nonzero entry from the left in each nonzero row is a 1, called the leading 1 for that row
- right of all leading 1's in the rows above it

Row-echelon matrix

The row-echelon matrix has the "staircase" form



(for any choice in *-position)

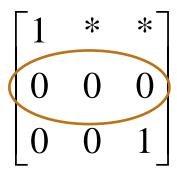
Which matrices are in row-echelon form?

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * \\ 0 & 2 & * \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

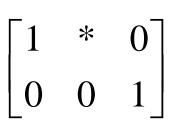


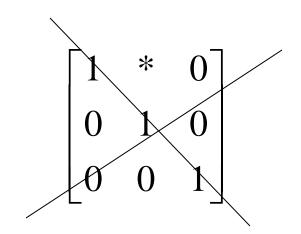
A reduced row-echelon matrix

- It is a row-echelon matrix
- Each leading 1 is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

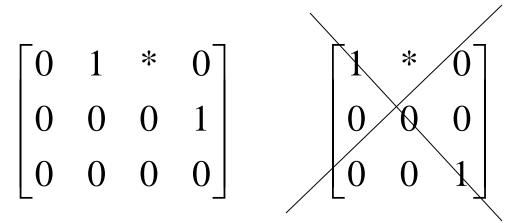
Which is a reduced row- echelon matrix?





$\lceil 1 \rceil$	0	*	0
0	1	*	0
0	0	0	1

$$\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



How to carry a matrix to (reduced) row-echelon form?

Gaussian Algorithm

Theorem. Every matrix can be brought to (reduced) rowechelon form by a series of elementary row operations

- Step 1. If all row are zeros, stop
- Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a) and move the row containing a to the top position
- Step 3. Multiply that row by 1/a to creat the leading 1
- Step 4. By subtracting multiples of that row from the rows below it, make each entry below the leading 1 zero
- Step 5. Repeat step 1-4 on the matrix consisting of the remaining rows

Gaussian Algorithm

Example

Carry the matrix

$$\begin{bmatrix} 2 & 6 & -2 & 2 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{bmatrix}$$

- to row-echelon matrix
- to reduced rowechelon matrix

$$\begin{bmatrix} 2 & 6 & -2 & 2 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 3 & -1 & 1 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{bmatrix} \xrightarrow{\frac{2r_1+r_2}{-3r_1+r_3}} \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 3 & 9 & 6 \\ 0 & 2 & 6 & -3 \end{bmatrix}$$

row-echelon matrix

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2r_3+r_2} \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3r_2+r_1} \begin{bmatrix} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced row-echelon matrix

Gauss-Jordan Elimination (for solving a system of linear equations)

- Step 1.Using elementary row operations, augmented matrix -> reduced row-echelon matrix
- Step 2. If a row [0 0 0...0 1] occurs, the system is **inconsistent** redu
- Step 3. Otherwise, assign the nonleading variables as parameters, solve for the leading variables in terms of parameters

$$\begin{bmatrix} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{cases} x + 0y - 10z = 0 \\ 0x + 1y + 3z = 0 \\ 0x + 0y + 0z = 0 \end{cases}$$

reduced row echelon matrix

$$\begin{bmatrix} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} 1x + 0y - 10z = 0 \\ 0x + 1y + 3z = 0 \\ 0x + 0y + 0z = 0 \end{cases}$$

z=t (parameter)

z is nonleading variable

Solve the following system of equations

$$x-2y = 3$$

$$2x-3y+2z = 5$$

$$-3x+7y+2z = 10$$

Solution.

Carry the augmented matrix to reduced row-echelon form

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & 2 & 5 \\ -3 & 7 & 2 & 10 \end{bmatrix} \xrightarrow{\stackrel{-2r_1+r_2}{3r_1+r_3}} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 19 \end{bmatrix} \xrightarrow{\stackrel{-r_2+r_3}{-r_2+r_3}} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

inconsistent

Example 3: Solve the following system of equations.

$$x_1 - 2x_2 - x_3 + 3x_4 = 1$$
$$2x_1 - 4x_2 + x_3 = 5$$
$$x_1 - 2x_2 + 2x_3 - 3x_4 = 4$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

 x_2, x_4 are <u>nonleading</u> <u>variables</u>, so we set x_2 =t and x_4 =s (parameters) and then compute x_1, x_3

$$x_1 = 2 + 2t - s$$

$$x_2 = t$$

$$x_3 = 1 + 2s$$

$$x_4 = s$$

reduced row echelon matrix

Theorem 2

Suppose a system of m equations in **n variables** has a solution. If the **rank** of the augment matrix is **r** then the set of solutions involves exactly **n-r** parameters

$$\begin{bmatrix}
1 & -2 & -1 & 3 & | 1 \\
2 & -4 & 1 & 0 & | 5 \\
1 & -2 & 2 & -3 & | 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & -1 & 3 & | 1 \\
0 & 0 & 3 & -6 & | 3 \\
0 & 0 & 3 & -6 & | 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & -1 & 3 & | 1 \\
0 & 0 & 1 & -2 & | 1 \\
0 & 0 & 0 & 0 & | 0
\end{bmatrix}$$
rankA=2

4(number of variables)- 2(rankA) = 2 (two parameters : x_2 =t, x_4 =s)

1.3. Homogeneous Equations (phương trình thuần nhất)

- The system is called homogeneous if the constant matrix has all the entry are zeros
- Note that every homogeneous system has at least one solution (0,0,...,0), called trivial solution
- If a homogeneous system of linear equations has nontrivial solution then it has infinite family of solutions.

Example 1: Show that the following homogeneous system has nontrivial solutions.

$$x_1 - x_2 + 2x_3 + x_4 = 0$$
$$2x_1 + 2x_2 - x_4 = 0$$
$$3x_1 + x_2 + 2x_3 + x_4 = 0$$

Solution

The reduction of the augmented matrix to reduced row – echelon form is outined below.

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 4 & -4 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The leading variables are x_1, x_2 and x_4 , so x_3 is assigned as a parameter – say $x_3 = t$. Then the general solution is

$$x_1 = -t$$

$$x_2 = t$$

$$x_3 = t$$

$$x_4 = 0$$

Hence, taking t = 1, we get a nontrivial solution: $x_1 = -1, x_2 = 1, x_3 = 1, x_4 = 0$.

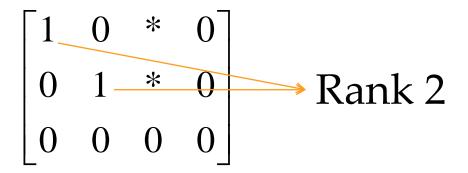
Theorem 1

If a homogeneous system of linear equations has more variables than equations, then it has nontrivial solution (in fact, infinitely many)

Note that the converse of theorem 1 is not true

The rank of a matrix

 The rank of the matrix A, rankA = the number of leading ones in the reduced row-echelon form of A.



Summary

Elementary Operations

Gaussian Elimination

Homogeneous Equations

THANKS