Exercise Book MAE 101 (có hướng dẫn)

Discrete Mathematics (FPT University)

Name:..

Class:..

Mathematics for Engineering

Exercise Book

Trần Trọng Huỳnh - 2018

CALCULUS

Chapter 1: Function and Limit

1. Find the domain of each function:

a. $f\left(x\right)=\dfrac{2}{(x+2)\sqrt{x+1}}$

b. $f\left(x\right)=\dfrac{2x-1}{\sqrt{x\left|x-4\right|}}$

c. $f\left(x\right)=\ln\left(x+1\right)-\dfrac{x}{\sqrt{x-1}}$

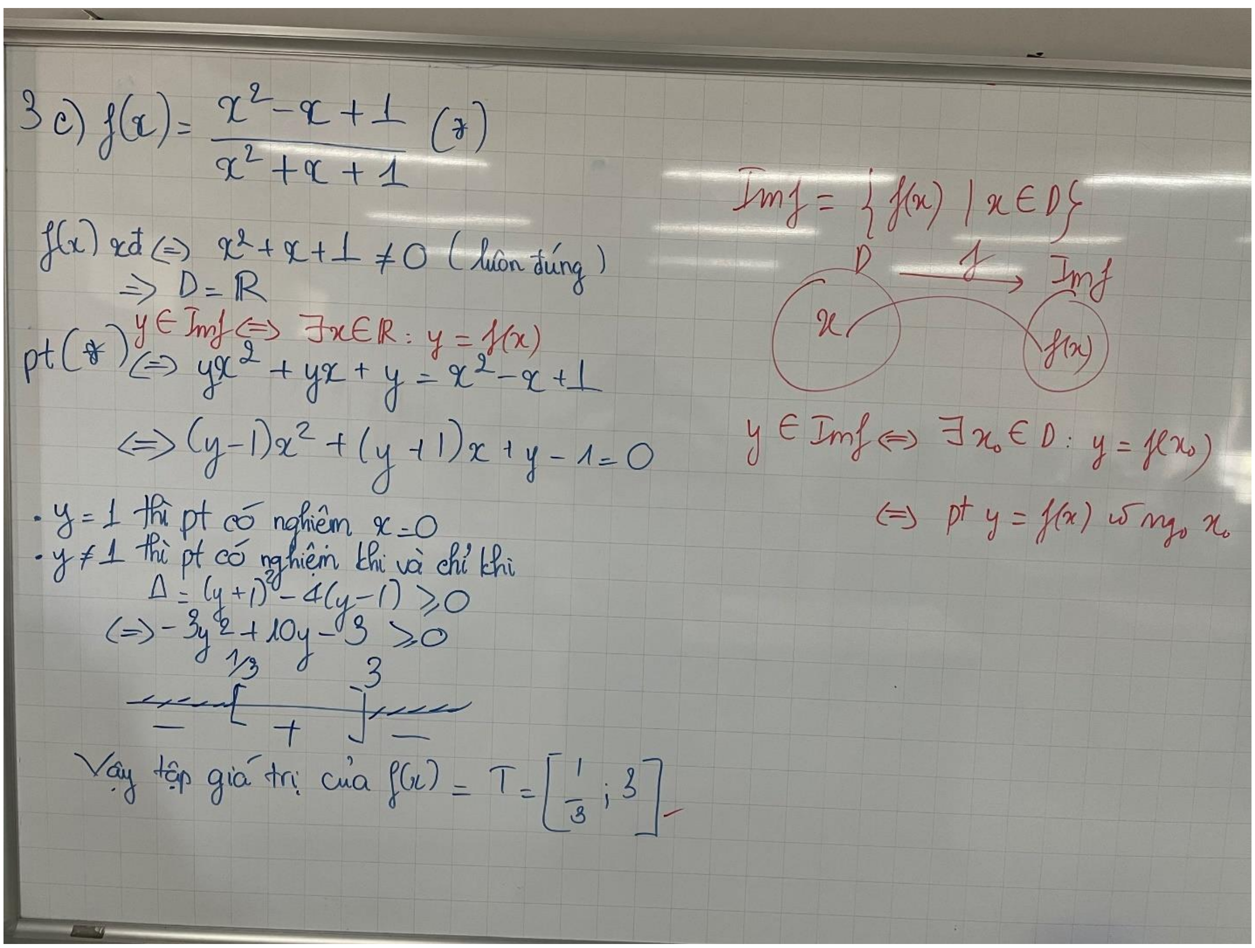
d. $f(x)=\dfrac{\sqrt{x^2-4x+3}}{\lg(x-2)}$

2. Find the range of each function:

a. $f\left(x\right)=\dfrac{3x+5}{2x-1}$

b. $f\left(x\right)=x^2-2x$

c. $f\left(x\right)=\dfrac{x^2-x+1}{x^2+x+1}$



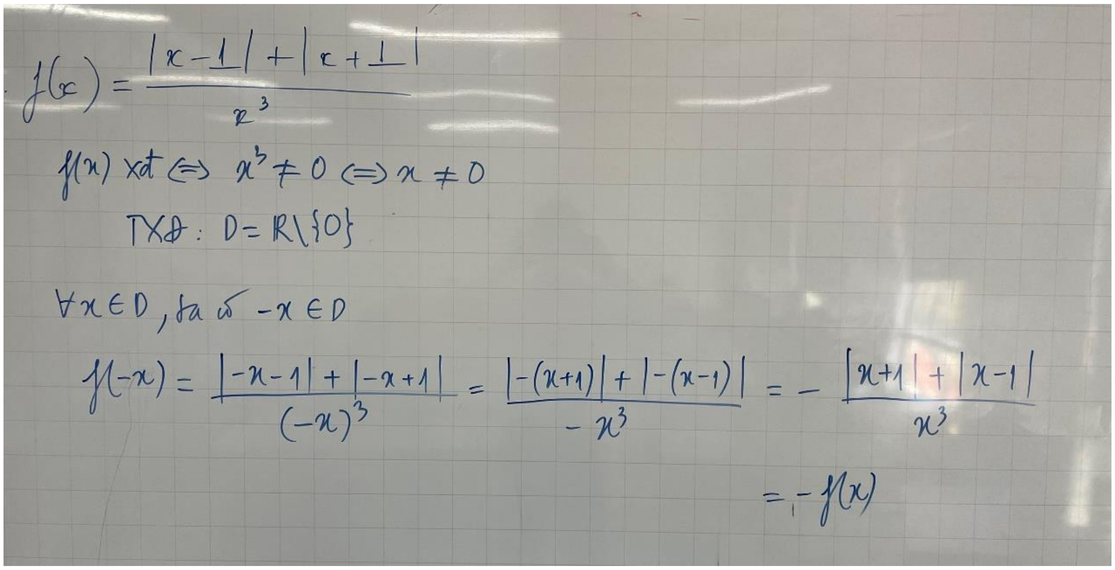
3. Determine whether is even, odd, or neither

$a) f(x) = \dfrac{|x-1|+|x+1|}{x^3}$

$c) f(x) = \ln\left(x+\sqrt{1+x^2}\right)$

$b) f(x) = \dfrac{1}{2}\left(a^x + a^{-x}\right)$

$d) f(x) = \lg\dfrac{1+x}{1-x}$



$f(x) = \dfrac{|x-1|+|x+1|}{x^3}$

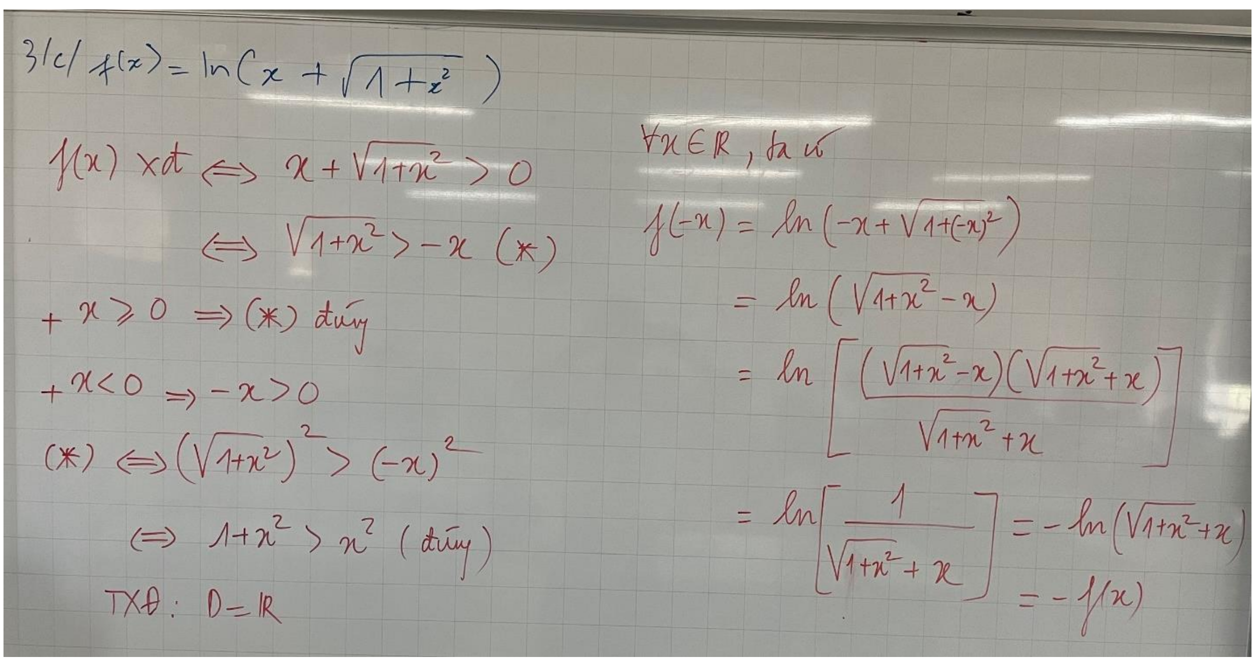
$f(x)$ xđ $\Leftrightarrow x^3 \neq 0 \Leftrightarrow x \neq 0$

TXĐ: $D = \mathbb{R}\setminus\{0\}$

$\forall x \in D$, ta có $-x \in D$

$f(-x) = \dfrac{|-x-1|+|-x+1|}{(-x)^3} = \dfrac{|-(x+1)|+|-(x-1)|}{-x^3} = -\dfrac{|x+1|+|x-1|}{x^3}$

$= -f(x)$



3/c/ $f(x) = \ln\left(x+\sqrt{1+x^2}\right)$

$f(x)$ xđ $\Leftrightarrow x+\sqrt{1+x^2} > 0$

$\Leftrightarrow \sqrt{1+x^2} > -x \quad (*)$

$+\ x \geq 0 \Rightarrow (*)$ đúng

$+\ x < 0 \Rightarrow -x > 0$

$(*) \Leftrightarrow \left(\sqrt{1+x^2}\right)^2 > (-x)^2$

$\Leftrightarrow 1+x^2 > x^2$ (đúng)

TXĐ: $D = \mathbb{R}$

$\forall x \in \mathbb{R}$, ta có

$f(-x) = \ln\left(-x+\sqrt{1+(-x)^2}\right)$

$= \ln\left(\sqrt{1+x^2}-x\right)$

$= \ln\left[\dfrac{\left(\sqrt{1+x^2}-x\right)\left(\sqrt{1+x^2}+x\right)}{\sqrt{1+x^2}+x}\right]$

$= \ln\left[\dfrac{1}{\sqrt{1+x^2}+x}\right] = -\ln\left(\sqrt{1+x^2}+x\right)$

$= -f(x)$

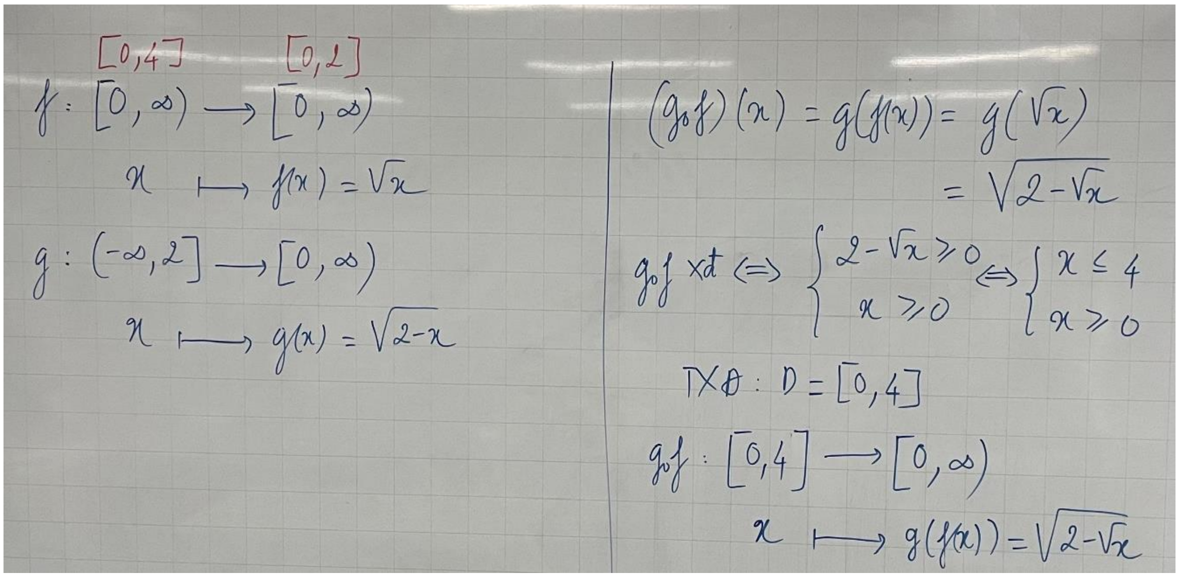
4. Explain how the following graphs are obtained from the graph of $f(x)$

a. $f(x-4)$ b. $f(x)+3$ c. $f(x-2)-3$ d. $f(x+5)-4$

5. Suppose that the graph of $f(x)=\sqrt{x}$ is given. Describe how the graph of the function $y=\sqrt{x-1}+2=f(x-1)+2$ can be obtained from the graph of f.

6. Let $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$. Find each function

a. $f_o g$ b. $g_0 f$ c. $g_o g$ d. $f_o f$



7. Let $f(x)=\dfrac{x^2+x+1}{x}=x+1+\dfrac{1}{x}$. Find

a. $f\left(x+\dfrac{1}{x}\right)$ b. $f(2x-1)$

8. Use the table to evaluate each expression

a. f(g(1)) b. g(f(1)) c. f(f(1)) d. g(g(1))

e. $(g_o f)(3)=g(f(3))$ f. $(g_o f)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

9. Evaluate the following limits

a. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x-3}$ b. $\lim\limits_{x\to 1}\dfrac{x^6-1}{x^{10}-1}$ c. $\lim\limits_{x\to 0}\dfrac{\tan 3x+2x}{\tan 5x-\sin x}$ d. $\lim\limits_{h\to 0}\dfrac{(3+h)^2-9}{h}$

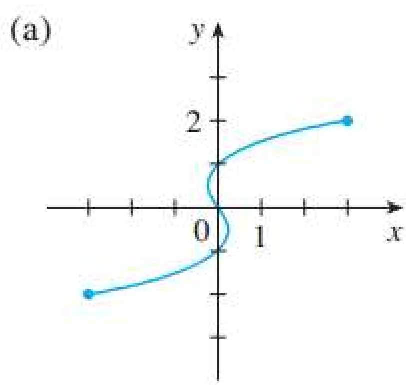
$$\lim_{x\to1}\frac{x^6-1}{x^{10}-1}=\lim_{x\to1}\frac{\left(x^3-1\right)\left(x^3+1\right)}{\left(x^5-1\right)\left(x^5+1\right)}=\lim_{x\to1}\frac{(x-1)(x^2+x+1)(x^3+1)}{(x-1)(x^4+x^3+x^2+x+1)\left(x^5+1\right)}$$

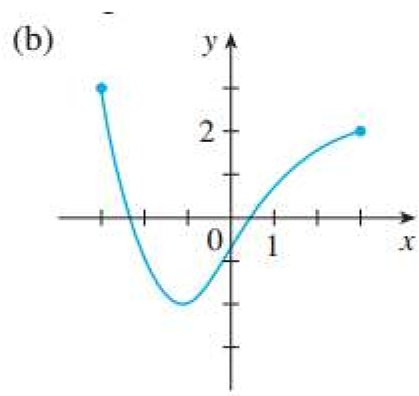
$$=\lim_{x\to1}\frac{(x^2+x+1)(x^3+1)}{(x^4+x^3+x^2+x+1)\left(x^5+1\right)}=\frac{(1+1+1)(1+1)}{(1+1+1+1+1)(1+1)}=\frac{3}{5}$$

e. $\lim_{t\to0}\frac{\sqrt{t^2+9}-3}{t^2}$ f. $\lim_{x\to\infty}\frac{x^2+x-12}{x^3-3}$ g. $\lim_{x\to0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ h. $\lim_{x\to1}\frac{x^2-1}{|x-1|}$

a-h b-g c-f d-e

10. Determine whether each curve is the graph of a function of *x*. If it is, state the domain and range of the function.





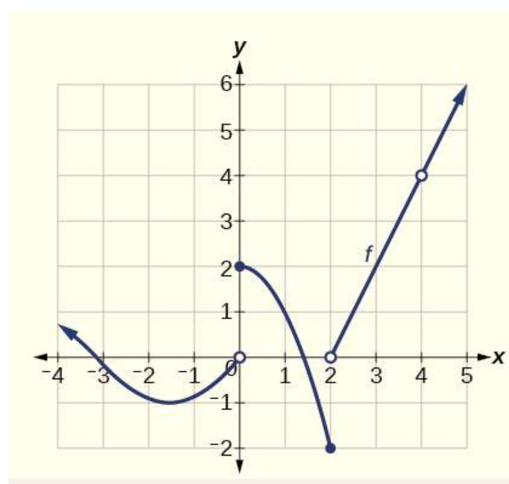
11. The graph of f is given.

a. Find each limit, or explain why it does not exist.

i. $\lim_{x\to0^+}f(x)$, $\lim_{x\to0^-}f(x)$ and $\lim_{x\to0}f(x)$

ii. $\lim_{x\to1}f(x)$ and $\lim_{x\to4}f(x)$

b. At what numbers is discontinuous?

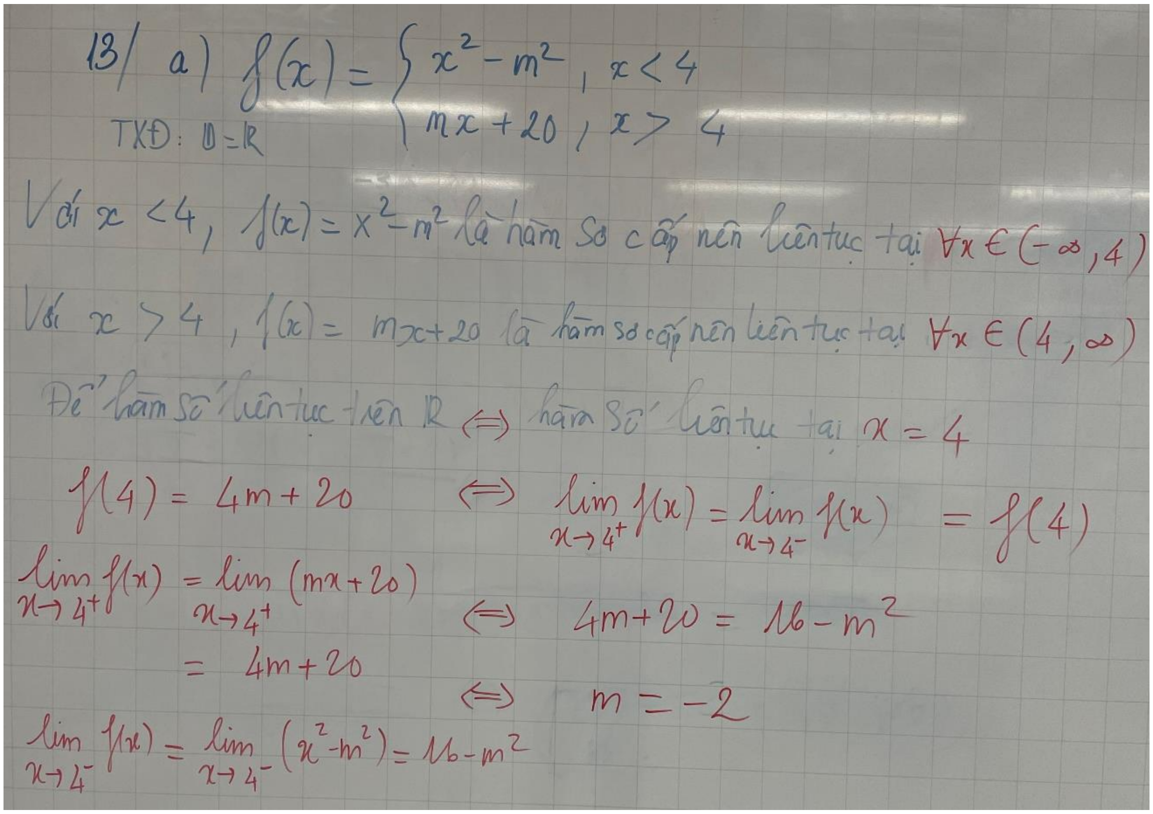


12. Determine where the function $f(x)$ is continuous

a. $f(x)=\frac{2x^2+x-1}{x-2}$ b. $f(x)=\frac{x-9}{\sqrt{4x^2+4x+1}}$ c. $f(x)=\ln(2x+5)$

13. Find the constant *m* that makes *f* continuous on its domain

a. $f\left(x\right)=\begin{cases}x^2-m^2, & x<4\\ mx+20, & x\geq 4\end{cases}$ b. $f\left(x\right)=\begin{cases}mx^2+2x, & x<2\\ x^3-mx, & x\geq 2\end{cases}$



c. $f\left(x\right)=\begin{cases}\dfrac{e^{2x}-1}{x}, & x\neq 0\\ m, & x=0\end{cases}$ d. $f\left(x\right)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ m+1, & x=1\end{cases}$

c).

$$f(x) = \begin{cases} \dfrac{e^{2x}-1}{x} & ; x \neq 0 \\ m & ; x = 0 \end{cases} \qquad TXĐ: D = R.$$

- Với $x \neq 0$ thì $f(x) = \dfrac{e^{2x}-1}{x}$.

Đây là hàm phân thức hữu tỉ có TXĐ $= (-\infty, 0) \cup (0, +\infty)$

Nó liên tục trên mỗi khoảng $(-\infty, 0)$ và $(0; +\infty)$

- $x = 0$ thì $f(0) = m$

- $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x} = \lim\limits_{x\to 0} 2.\left(\dfrac{e^{2x}-1}{2x}\right) = 2.\lim\limits_{x\to 0}\left(\dfrac{e^{2x}-1}{2x}\right) = 2.1 = 2$

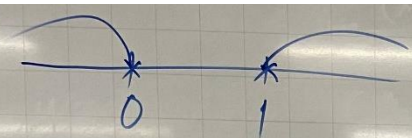
Để h/s liên tục trên TXĐ: $D = R$ thì:

$$f(0) = \lim_{x\to 0} f(x) = 2$$

$$\Leftrightarrow m = 2$$

Vậy $m = 2$ thì h/s liên tục trên TXĐ của nó

14. Find the numbers at which the function $f(x) = \begin{cases} x+2, & x < 0 \\ 2x^2, & 1 \geq x \geq 0 \\ 2-x, & x > 1 \end{cases}$ is discontinuous.



TXĐ : $D = \mathbb{R}$

* $x < 0$: $f(x) = x + 2$ là hssc nên $f(x)$ ltục trên $(-\infty, 0)$

* $x > 1$: $f(x) = 2 - x$ ______ $(1, \infty)$

* $0 < x < 1$: $f(x) = 2x^2$ ______ $(0, 1)$

* $x = 0$, $f(0) = 0$

* $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (2x^2) = 0$

$\neq$ $\Rightarrow \nexists \lim\limits_{x \to 0} f(x) \Rightarrow f(x)$ gián đoạn tại $x = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 2) = 2$

* $x = 1$, $f(1) = 2.1^2 = 2$

* $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1$

$\neq$ $\Rightarrow \nexists \lim\limits_{x \to 1} f(x) \Rightarrow f(x)$ gián đoạn tại $x = 1$

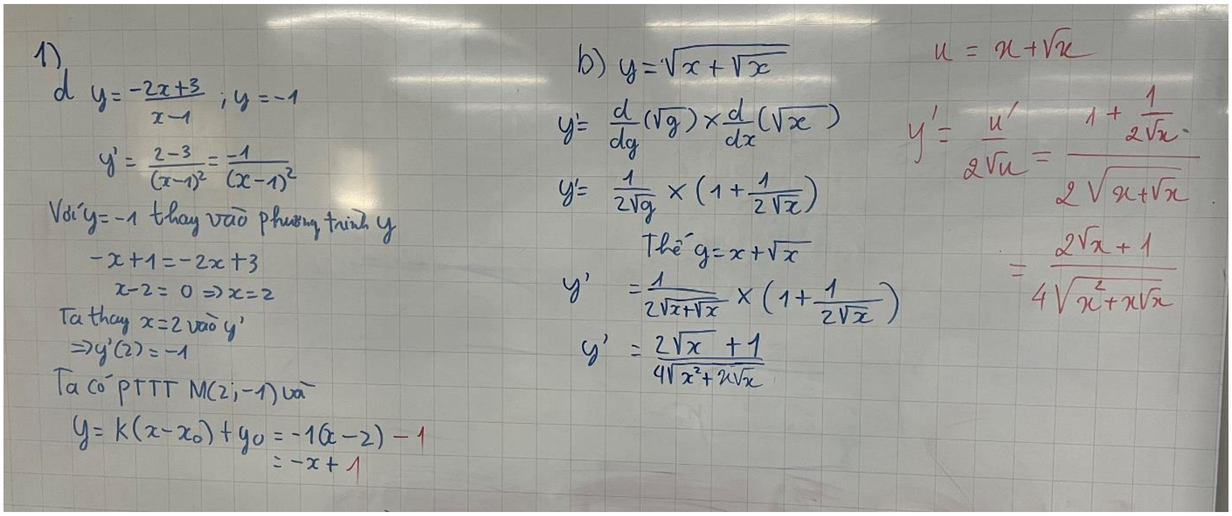
$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (2x^2) = 2$

Chapter 2: Derivatives

2. Find an equation of the tangent line to the curve at the given point:

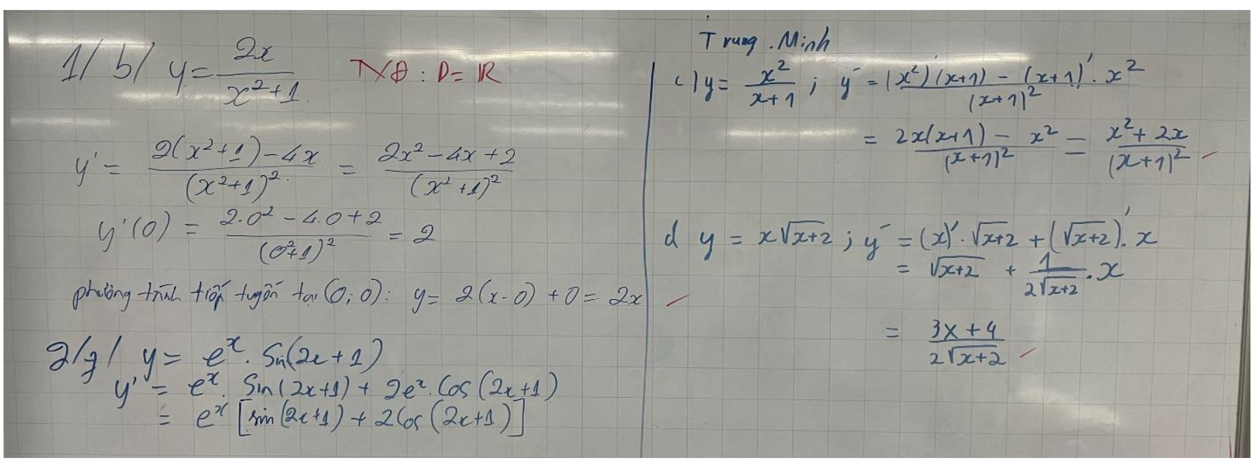
a. $y=\dfrac{x-1}{x-2}, \quad (3,2)$

b. $y=\dfrac{2x}{x^2+1}, \quad (0,0)$



c. $y=3-2x+x^2, \quad x=1$

d. $y=\dfrac{3-2x}{x-1}, \quad y=-1$



3. Find y'

a. $y=x^2-x\sqrt{x}+\dfrac{1}{x}+2$

b. $y=\sqrt{x+\sqrt{x}}$

c. $y=\dfrac{x^2}{x+1}$

d. $y=x\sqrt{x+2}$

e. $y=\ln\left(x^2+1\right)-\dfrac{1}{x}$

f. $y=e^x\sin\left(2x+1\right)$

4. Find y''

a. $y = xe^{3x-1}$ b. $y = \sqrt[3]{2x+1}$ c. $y = e^{-x}\cos x$

5. Find dy/dt for:

a. $y = x^3 + x + 2, dx/dt = 2$ and $x = 1$ b. $y = \ln x, dx/dt = 1$ and $x = e^2$

3c) $y = e^{-x} \cos x$

$y' = (e^{-x}\cos x)' = (-e^{-x} . \cos x) + (-e^{-x} \sin x)$

$= -e^{-x}(\sin x + \cos x)$

$y'' = (-e^{-x}(\sin x + \cos x))'$

$= (e^{-x}.(\sin x + \cos x)) - e^{-x}(\cos x - \sin x))$

$= e^{-x}(\sin x + \cos x - \cos x + \sin x)$

$= e^{-x}(2 \sin x) = 2e^{-x} \sin x$

4a) $y = x^3 + x + 2$ $dx/dt = 2$ and $x = 1$

Có $\frac{dy}{dx} = (x^3 + x + 2)' = 3x^2 + 1$

$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} . \frac{dx}{dt} = (3x^2 + 1) . \frac{dx}{dt}$

$= (3x^2 + 1) . 2 = 6x^2 + 2$

Thay $x = 1$ ta được:

$\frac{dy}{dx} = 6 . 1^2 + 2 = 8$

c. $y = \tan\sqrt{t}$ and $t = \frac{\pi^2}{16}$ d. $\begin{cases} y = \sin\varphi \\ t = \cos\varphi \end{cases}$ and $\varphi = \frac{\pi}{3}$

$$\frac{dy}{d\varphi} = \frac{dy}{dt} . \frac{dt}{d\varphi} \Rightarrow \frac{dy}{dt} = \frac{\frac{dy}{d\varphi}}{\frac{dt}{d\varphi}} = \frac{\cos\varphi}{-\sin\varphi} = -\cot\varphi$$

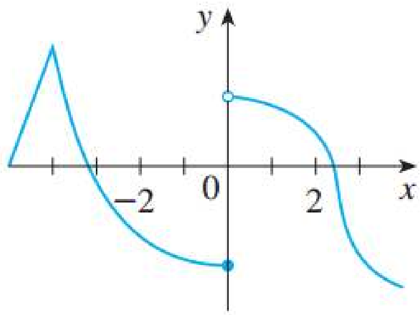
6. Find dy for: $dy = y'(x)dx$

a. $y = \frac{1}{x^2+1}$ b. $y = \sqrt{x+1}, x = 3$ c. $y = \ln(x^2+1)$, $x = 1$ and $dx = 0.1$

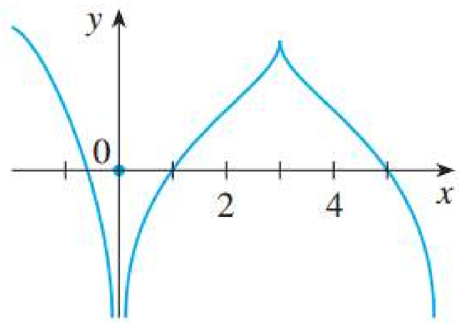
$$dy = \left[\ln(x^2+1)\right]' dx = \frac{2x}{x^2+1} dx$$

7. The graph of is given. State the numbers at which is not differentiable

a.



b.



8. A table of values for f, f', g and g' is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

a. If $h(x) = f(g(x))$, find $h'(1)$

b. If $H(x) = g_o f(x)$, find $H'(1)$

$h'(x) = \left[f(g(x))\right]' = f'(g(x)).g'(x)$

$\Rightarrow h'(1) = f'(g(1)).g'(1) = f'(2).g'(1) = 5.6 = 30$

c. If $F(x) = f_o f(x)$, find $F'(2)$

d. If $G(x) = g_o g(x)$, find $G'(3)$

9. If $h(x) = \sqrt{4+3f(x)}$, where $f(1) = 7, f'(1) = 4$, find $h'(1)$.

$h'(x) = \left(\sqrt{4+3f(x)}\right)' = \dfrac{3f'(x)}{2\sqrt{4+3f(x)}}$

10. For the circle $x^2 + y^2 = 25$.

a. Find dy/dx

b. Find an equation of the tangent to the circle at the point (3, 3).

11. Let $(L): x^3 + y^3 = 6xy$

a. Find dy/dx $\quad y' = \dfrac{2y - x^2}{y^2 - 2x} \Rightarrow y'(3) = -1$

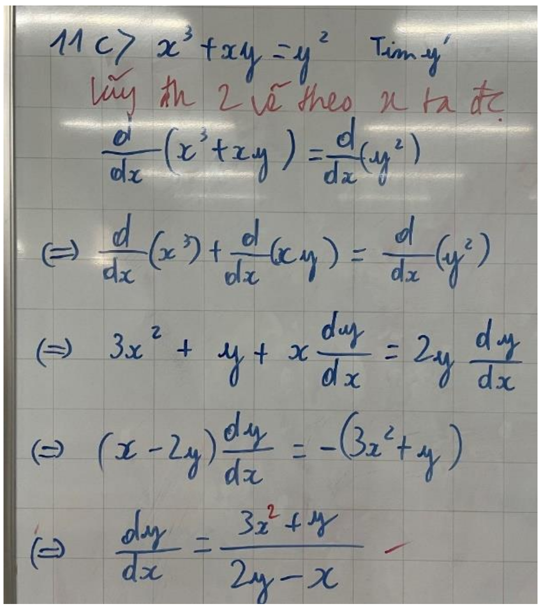
b. Find an equation of tangent to the curve (L) at the point (3, 3)

$y = y'(3)(x-3) + y(3)$

$\Rightarrow y = -1(x-3) + 3 = -x + 6$

12. Find y' by implicit differentiation

a. $x^4 + y^4 = 16x + y$ b. $\sqrt{x} + \sqrt{y} = 4$ c. $x^3 + xy = y^2$



13. Find f' in terms of g'

a. $f(x) = g(\sin 2x)$ b. $f(x) = g\left(e^{1-3x}\right)$

14. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2 ?

15. If $x^2 + y^2 = 25$ and $dy/dt = 6$, find dx/dt when y = 4 and x > 0.

13/ Có: $\frac{dl}{dt} = 6$ cm/s

$S = l^2 = 16 \text{ cm}^2 \Rightarrow l = 4$ cm.

$\frac{dS}{dt} = 2l\frac{dl}{dt}$

$= 2.4.6 = \underline{48} \text{ (cm}^2\text{/s)}$

14/ $x^2 + y^2 = 25$

$\frac{dx}{dt} 2x + \frac{dy}{dt} 2y = 0.$

$\Rightarrow \frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt}$

Khi $y = 4$ thì $x = 3$ $(x > 0)$, ta có

$\frac{dx}{dt} = -\frac{4}{3} \cdot 6 = -8$

16. If $z^2 = x^2 + y^2 \quad (z > 0)$, $dx/dt = 2, dy/dt = 3$, find dz/dt when $x = 5, y = 12$

lấy đạo hàm 2 vế theo t ta được

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dt}{dt}$$

khi $x = 5, y = 12$ thì $z = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = 13 \ (z > 0)$

vậy $\frac{dz}{dt} = \frac{x}{z}.\frac{dx}{dt} + \frac{y}{z}.\frac{dt}{dt} = \frac{5}{13}.2 + \frac{12}{13}.3 = \frac{46}{13}$

17. Find the linearization $L(x)$ of the function at a.

a. $f(x) = \frac{1}{\sqrt{2+x}}, \qquad a = 2$ b. $f(x) = \sqrt[3]{5-x}, \qquad a = -3$

$$f'(x) = \left[(2+x)^{-\frac{1}{2}} \right]' = -\frac{1}{2}(2+x)^{-3/2} = -\frac{1}{2\sqrt{(2+x)^3}}$$

$$L(x) = f'(2)(x-2) + f(2) = -\frac{1}{2.8}(x-2) + \frac{1}{2} = -\frac{1}{16}x + \frac{5}{8}$$

18. The equation of motion is $s(t) = 3\sin t - 4\cos t + 1$ for a particle, where s is in meters and t is in seconds. Find the acceleration (in m/s^2) after 3 seconds.

$$v(t) = s'(t) = 3\cos t + 4\sin t$$

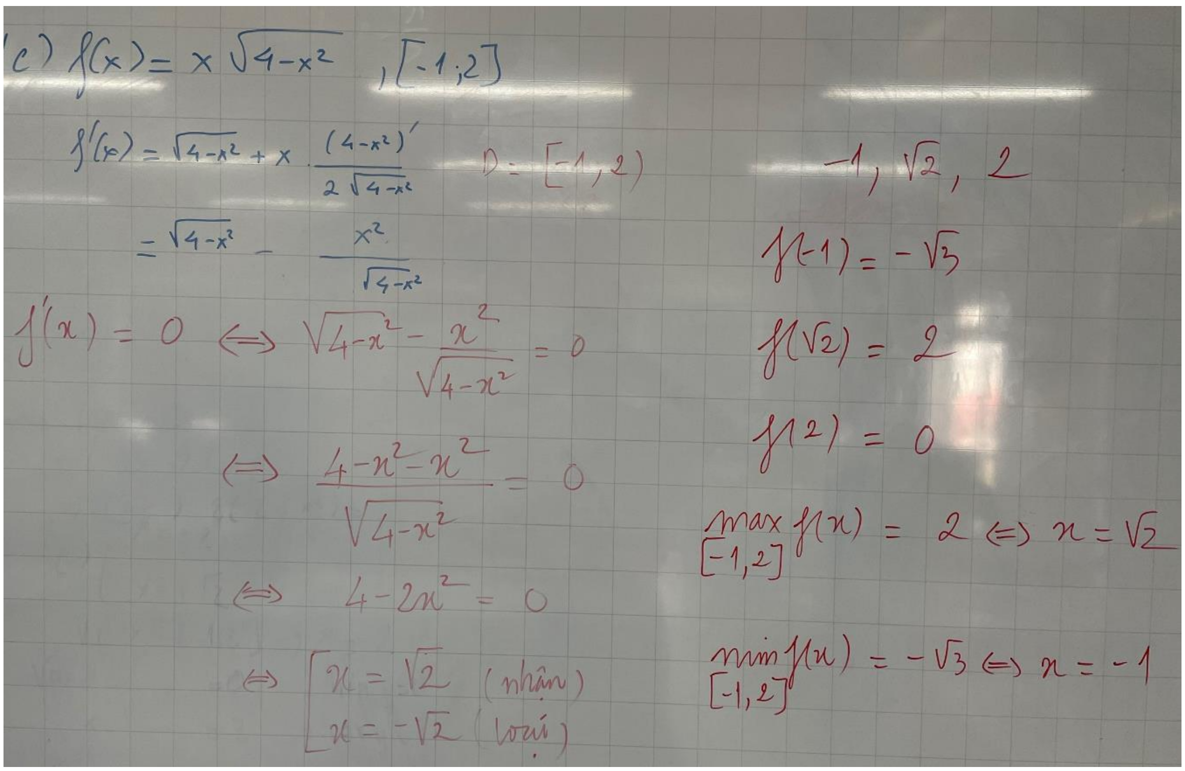
$$a(t) = v'(t) = 4\cos t - 3\sin t$$

Chapter 3: Applications of Differentiation

1. Find the absolute maximum and absolute minimum values of the function on the given interval

a. $f(x) = 3x^2 - 12x + 5, [0;3]$ b. $f(x) = x^3 - 3x + 5, \quad [0;3]$

c. $f(x) = x\sqrt{4 - x^2}, \quad [-1;2]$ d. $f(x) = x - \ln x, \quad \left[\frac{1}{2};2\right]$

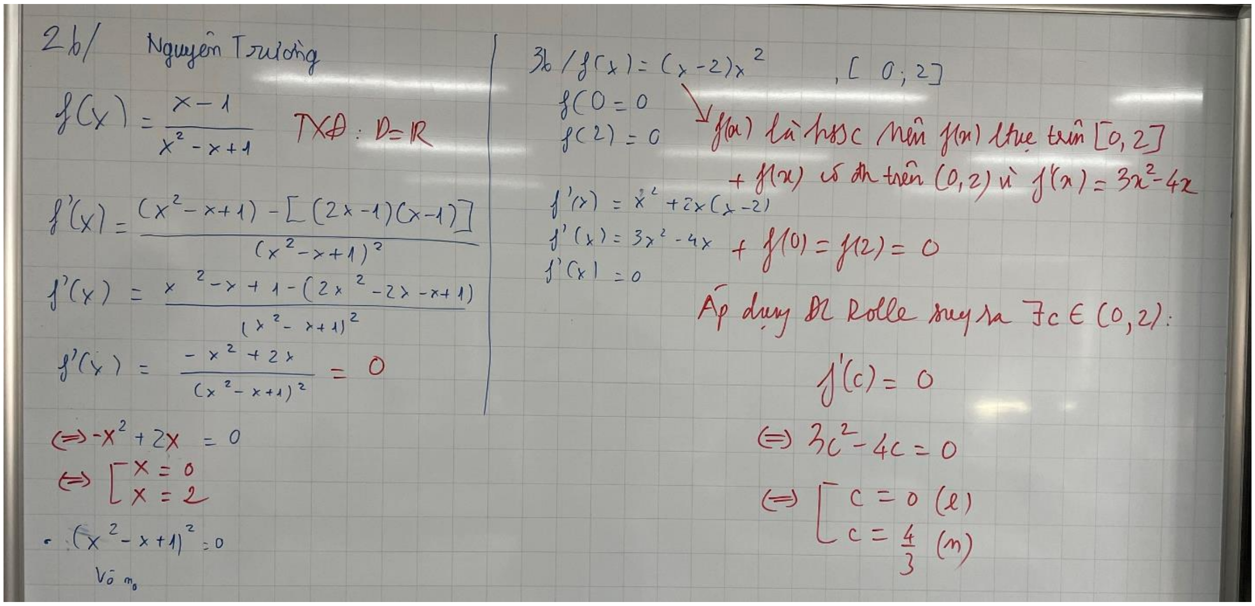


2. Find the critical numbers of the function

a. $f(x) = 5x^2 + 4x$ b. $f(x) = \dfrac{x - 1}{x^2 - x + 1}$ c. $f(x) = x\ln x$

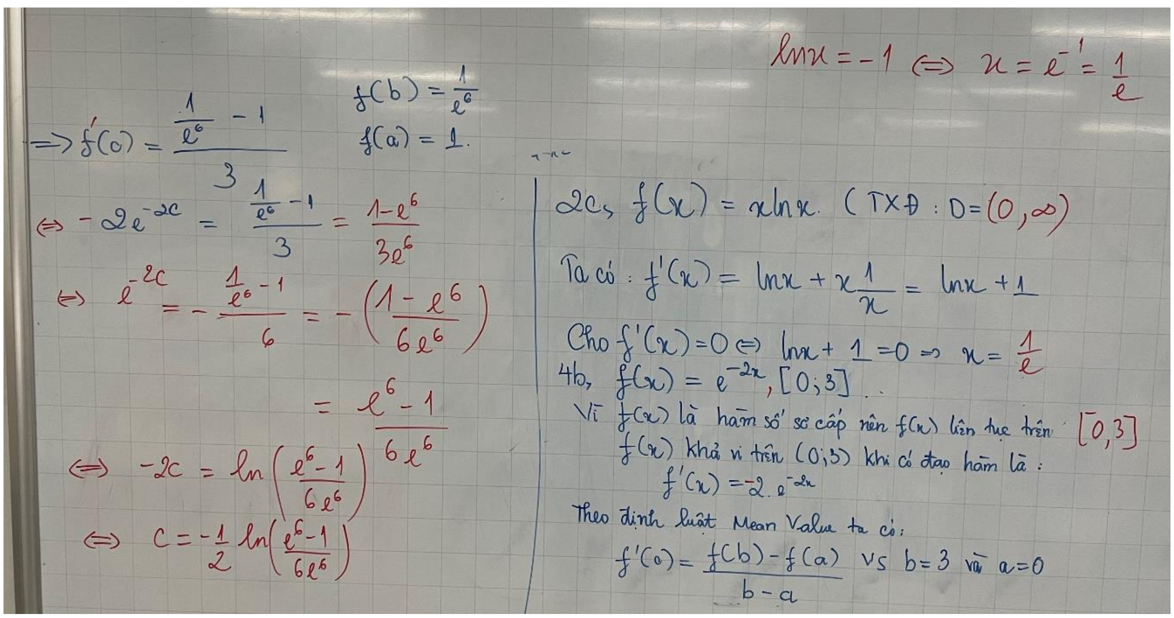
3. Find all numbers that satisfy the conclusion of the Rolle's Theorem

a. $f(x) = x\sqrt{x + 2}, \quad [-2;0]$ b. $f(x) = (x - 2)x^2, \quad [0;2]$

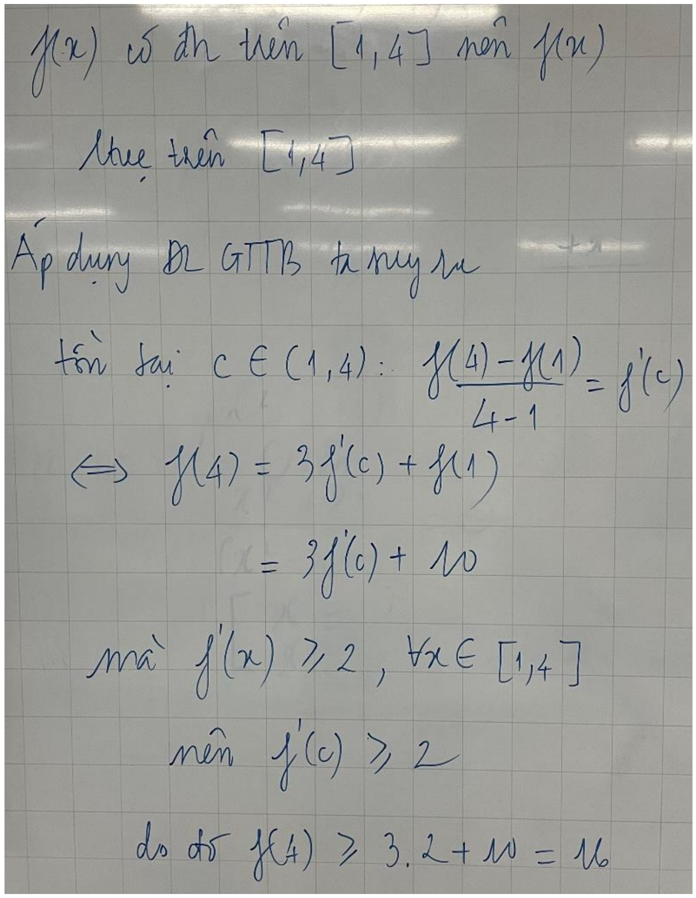


4. Find all numbers that satisfy the conclusion of the Mean Value Theorem

a. $f\left(x\right)=3x^2+2x+5, \left[-1;1\right]$ b. $f\left(x\right)=e^{-2x}, \left[0;3\right]$



5. If $f\left(1\right)=10$ and $f'\left(x\right)\geq 2, \forall x\in\left[1;4\right]$, how small can $f\left(4\right)$ possibly be?



$f(x)$ có đh trên $[1,4]$ nên $f(x)$

liên tục trên $[1,4]$

Áp dụng ĐL GTTB ta suy ra

tồn tại $c \in (1,4)$: $\dfrac{f(4)-f(1)}{4-1} = f'(c)$

$\Leftrightarrow f(4) = 3f'(c) + f(1)$

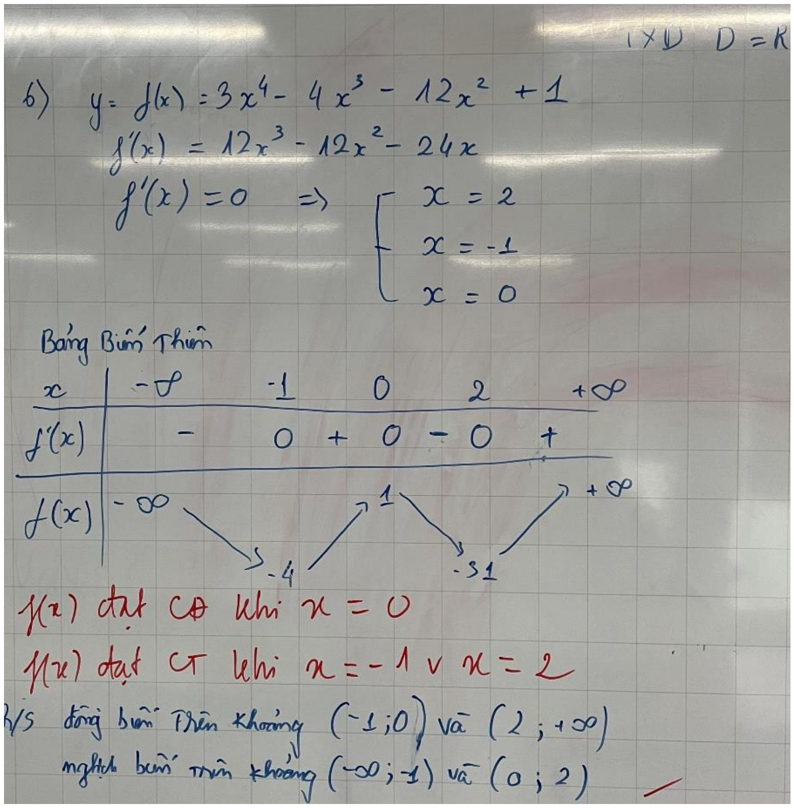
$= 3f'(c) + 10$

mà $f'(x) \geq 2, \forall x \in [1,4]$

nên $f'(c) \geq 2$

do đó $f(4) \geq 3.2 + 10 = 16$

6. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is increasing and where it is decreasing.



7. Find the inflection points for the function

a. $f(x) = x^4 - 4x + 1$ b. $f(x) = x^6$ c. $f(x) = xe^x$

8. Find $f(x)$ for $f'(x) = \sqrt{2x+1}$ and $f(0) = 1$.

$$\int \sqrt{2x+1}dx = \int (2x+1)^{1/2}\,dx = \frac{1}{2}\frac{(2x+1)^{1+\frac{1}{2}}}{1+\frac{1}{2}} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

9. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1;4)$

$$M\left(\frac{y^2}{2}, y\right) \in (P)$$

$$AM = \sqrt{\left(\frac{y^2}{2}-1\right)^2 + (y-4)^2} = \sqrt{\frac{y^4}{4} - 8y + 17}$$

$$AM \to \min \Leftrightarrow AM^2 \to \min \Leftrightarrow \frac{y^4}{4} - 8y + 17 \to \min$$

$$f(y) = \frac{y^4}{4} - 8y + 17$$

$$f'(y) = y^3 - 8 = 0 \Leftrightarrow y = 2$$

$$f''(y) = 3y^2 \geq 0 \Rightarrow f''(2) = 12 > 0$$

$$\Rightarrow f(y) \to \min \Leftrightarrow y = 2$$

$$\Rightarrow M(2,2)$$

10. Find two numbers whose difference is 100 and whose product is a minimum.

11. Find two positive numbers whose product is 100 and whose sum is a minimum.

12. Use Newton's method with the specified initial approximation x_1 to find x_3

a. $x^3 + 2x - 4 = 0,\ x_1 = 1$

b. $x^5 + 2 = 0,\ x_1 = -1$

c. $\ln\left(x^2+1\right) - 2x - 1 = 0,\ \ x_1 = 1$

d. $\ln\left(4 - x^2\right) = x,\ x_1 = 1$

$$x^3 + 2x - 4 = 0,\ x_1 = 1$$

$$f(x) = x^3 + 2x - 4 \Rightarrow f'(x) = 3x^2 + 2 \neq 0,\ \forall x \in \mathbb{R}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\ 3} + 2x_n - 4}{3x_n^{\ 2} + 2} = \frac{2x_n^{\ 3} + 4}{3x_n^{\ 2} + 2}\ (n \geq 1)$$

$$x_2 = \frac{2x_1^{\ 3} + 4}{3x_1^{\ 2} + 2} = \frac{2.1 + 4}{3.1 + 2} = \frac{6}{5} = 1.2$$

$$x_3 = \frac{2x_2^{\ 3} + 4}{3x_2^{\ 2} + 2} = \frac{2.\left(\frac{6}{5}\right)^3 + 4}{3.\left(\frac{6}{5}\right)^2 + 2} = \frac{466}{395} \approx 1.1797$$

14. Find the most general anti-derivative of the function.

a. $f\left(x\right) = 6x^2 - 2x + 3$

b. $f\left(x\right) = \sqrt[6]{x} + \frac{1}{x^2}$

$$\int f(x)dx = \int\left(\sqrt[6]{x} + \frac{1}{x^2}\right)dx = \frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1} - \frac{1}{x} + C = \frac{6}{7}x^{\frac{7}{6}} - \frac{1}{x} + C$$

c. $f\left(x\right) = \dfrac{x^2 + x + 2}{x}$ d. $f\left(x\right) = 2x\left(x^2 + 1\right)$

15. Find the anti-derivative of that satisfies the given condition

a. $f\left(x\right) = 5x^4 - 2x^5, F\left(0\right) = 4$ b. $f\left(x\right) = 4 - \dfrac{2x}{x^2 + 1}, F\left(0\right) = 1$

$$F(x) = \int\left(4 - \frac{2x}{x^2 + 1}\right)dx = \int 4dx - \int\frac{2x}{x^2 + 1}dx = 4x - \int\frac{(x^2 + 1)'}{x^2 + 1}dx = 4x - \ln(x^2 + 1) + C$$

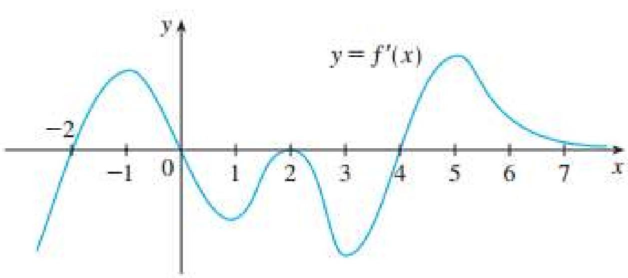
16. A particle is moving with the given data. Find the position of the particle

a. $v\left(t\right) = \sin t - \cos t,\ s\left(0\right) = 0$

b. $v\left(t\right) = 10\sin t + 3\cos t,\ s\left(\pi\right) = 0$

c. $v\left(t\right) = 10 + 3t - 3t^2,\ s\left(2\right) = 10$

17. The figure shows the graph of the derivative f' of a function f



a. On what intervals is f increasing or decreasing?

b. For what values of x does f have a local maximum or minimum?

Chapter 4 - 6: Integration

1. Estimate the area under the graph of $y = f\left(x\right)$ using 6 rectangles and left endpoints

a. $f\left(x\right) = \dfrac{1}{x} + x \;,\; x \in \left[1,4\right]$ b. $f\left(x\right) = x^2 - 2 \;,\; x \in \left[-1,2\right]$

$$S = \int_a^b \left|f(x)\right| dx$$

$$S = \int_{-1}^{2} \left|x^2 - 2\right| dx$$

Chia $[-1,2]$ thành 6 đoạn con, mỗi đoạn có độ dài $\Delta x = \dfrac{2-(-1)}{6} = \dfrac{1}{2} = 0,5$ bởi các điểm

$x_0 = -1,\ x_1 = -0.5,\ x_2 = 0,\ x_3 = 0.5,\ x_4 = 1,\ x_5 = 1.5,\ x_6 = 2$

[-1 , -0.5] ; [-0.5 , 0] ; [0 , 0.5] ; [0.5 , 1] ; [1 , 1.5] ; [1.5 , 2]

1b) Trên mỗi đoạn con $[x_{i-1}, x_i]\ (i = 1,2,3,4,5,6)$ chọn $x_i^* = x_{i-1}$

$$S = \int_{-1}^{2} \left|x^2 - 2\right| dx \approx \sum_{i=1}^{6} \left|\left(x_i^*\right)^2 - 2\right| \Delta x = \sum_{i=1}^{6} \left|x_{i-1}^2 - 2\right| \Delta x = \Delta x \left(\left|x_0^2 - 2\right| + \left|x_1^2 - 2\right| + \ldots + \left|x_5^2 - 2\right|\right)$$

3b) Trên mỗi đoạn con $[x_{i-1}, x_i]\ (i = 1,2,3,4,5,6)$ chọn $x_i^* = x_i$

$$S = \int_{-1}^{2} \left|x^2 - 2\right| dx \approx \sum_{i=1}^{6} \left|\left(x_i^*\right)^2 - 2\right| \Delta x = \sum_{i=1}^{6} \left|x_i^2 - 2\right| \Delta x = \Delta x \left(\left|x_1^2 - 2\right| + \left|x_2^2 - 2\right| + \ldots + \left|x_6^2 - 2\right|\right)$$

c. A table of values for f is given

x	1	2	3	4	5	6	7
$f(x)$	5	6	3	2	7	1	2

3. Repeat part (1) using right endpoints

4. For the function $f\left(x\right) = x^3, x \in \left[-2,2\right]$. Estimate the area under the graph of using four approximating rectangles and taking the sample points to be

a. Right endpoints

b. Left endpoints

c. Midpoints

$$S = \int_{-2}^{2} \left| x^3 \right| dx$$

Chia $[-2,2]$ thành 4 đoạn con, mỗi đoạn có độ dài $\Delta x = \dfrac{2-(-2)}{4} = 1$ bởi các điểm

$x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2$

[-2,-1] ; [-1,0] ; [0,1] ; [1,2]

a) Trên mỗi đoạn con $[x_{i-1}, x_i]$ $(i = 1,2,3,4)$ chọn $x_i^* = x_i$

$$S = \int_{-2}^{2} \left| x^3 \right| dx \approx \sum_{i=1}^{4} \left| \left(x_i^* \right)^3 \right| \Delta x = \sum_{i=1}^{4} \left| x_i^3 \right| \Delta x = \left(\left| x_1^3 \right| + \left| x_2^3 \right| + \left| x_3^3 \right| + \left| x_4^3 \right| \right) \Delta x = \ldots$$

b) Trên mỗi đoạn con $[x_{i-1}, x_i]$ $(i = 1,2,3,4)$ chọn $x_i^* = x_{i-1}$

$$S = \int_{-2}^{2} \left| x^3 \right| dx \approx \sum_{i=1}^{4} \left| \left(x_i^* \right)^3 \right| \Delta x = \sum_{i=1}^{4} \left| x_{i-1}^3 \right| \Delta x = \left(\left| x_0^3 \right| + \left| x_1^3 \right| + \left| x_2^3 \right| + \left| x_3^3 \right| \right) \Delta x = \ldots$$

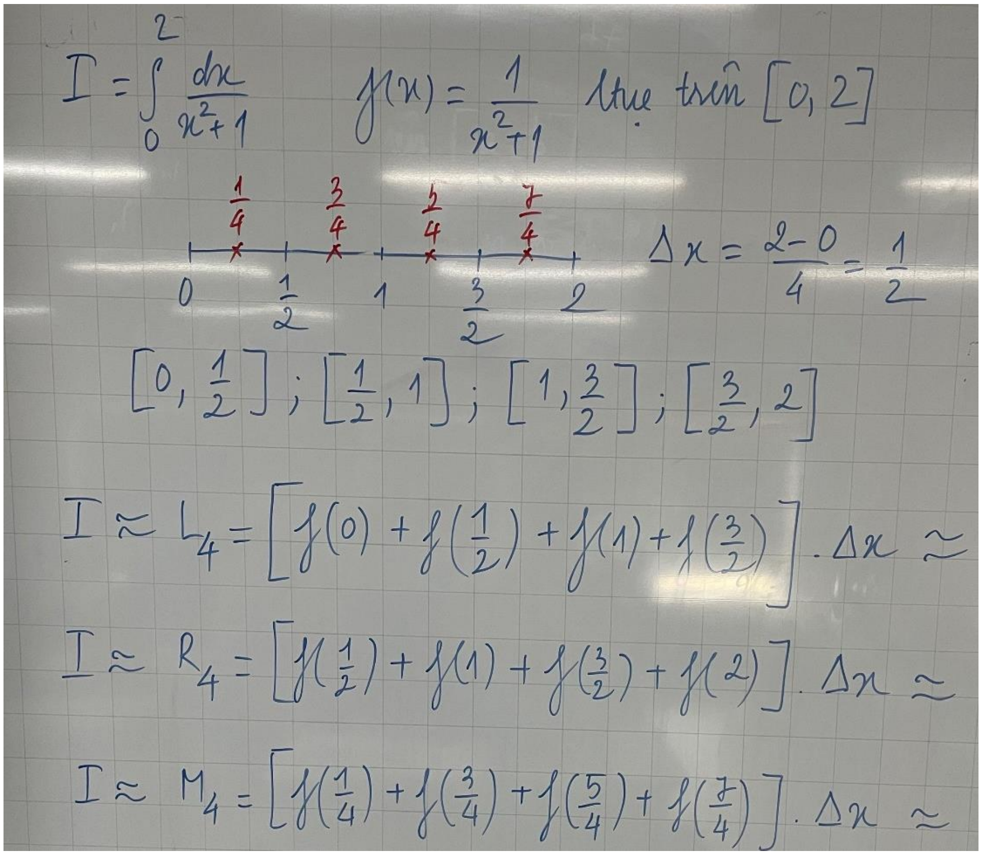
c) Trên mỗi đoạn con $[x_{i-1}, x_i]$ $(i = 1,2,3,4)$ chọn $x_i^* = \overline{x_i} = \dfrac{x_{i-1} + x_i}{2}$

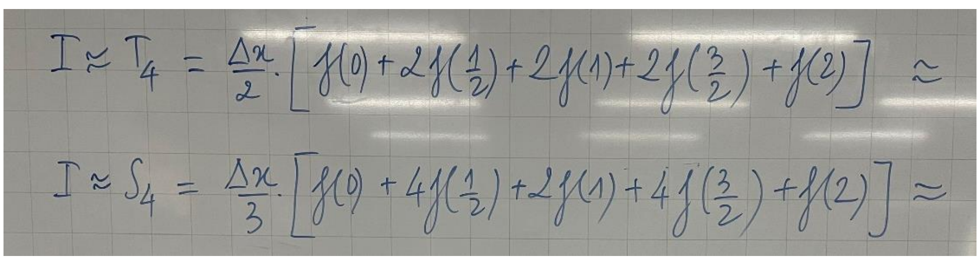
$$S = \int_{-2}^{2} \left| x^3 \right| dx \approx \sum_{i=1}^{4} \left| \left(x_i^* \right)^3 \right| \Delta x = \sum_{i=1}^{4} \left| \overline{x_i} \right|^3 \Delta x = \left(\left| \overline{x_1} \right|^3 + \left| \overline{x_2} \right|^3 + \left| \overline{x_3} \right|^3 + \left| \overline{x_4} \right|^3 \right) \Delta x = \ldots$$

5. Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of *n*.

a. $\displaystyle\int_0^3 \sqrt{x}\,dx,\qquad n = 4$ b. $\displaystyle\int_1^3 \frac{\sin x}{x}\,dx\qquad , n = 6$

6. Let $I = \displaystyle\int_0^2 \frac{dx}{x^2+1}$. Find the approximations L_4, R_4, M_4, T_4 and S_4 for I.





7. Find the derivative of the function $g(x) = \int_0^x \sqrt{t^2+1}\,dt$

8. Find g'

a. $g(x) = \int_1^{x^4} \frac{1}{\cos t}\,dt$

b. $g(x) = \int_1^{\sqrt{x}} \frac{\sin u}{u}\,du$

c. $g(x) = \int_{2x}^{x^2+x+2} \frac{e^t}{t}\,dt$

d. $g(x) = \int_{\sin x}^{\cos x} \left(1+v^2\right)^{10}\,dv$

$$g'(x)=\left(\int_{\sin x}^{\cos x}\left(1+v^2\right)^{10}dv\right)'=(\cos x)'.\left(1+\cos^2 x\right)^{10}-(\sin x)'.\left(1+\sin^2 x\right)^{10}$$

$$=-\sin x\left(1+\cos^2 x\right)^{10}-\cos x\left(1+\sin^2 x\right)^{10}$$

9. Find the average value of the function on the given interval

a. $f(x)=x^2, \quad [-1,1]$ b. $f(x)=\frac{1}{x}, \quad [1,5]$

c. $f(x)=x\sqrt{x}, \; [1,4]$ d. $f(x)=x\ln x, \left[1,e^2\right]$

$$f_{ave}=\frac{1}{e^2-1}\int_1^{e^2}x\ln x dx$$

Tính $I=\int_1^{e^2}x\ln x dx$

$$u=\ln x\Rightarrow du=\frac{1}{x}dx$$
$$dv=xdx\Rightarrow v=\frac{x^2}{2} \quad \ldots\ldots$$

10. A particle moves along a line so that its velocity at time t is v(t) = t^2 – t – 6 (m/s)

a. Find the displacement of the particle during the time period $1 \le t \le 4$

b. Find the distance traveled during this time period

11. Suppose the acceleration function and initial velocity are a(t)= t + 3 (m/s^2), v(0)=5 (m/s). Find the velocity at time t and the distance traveled when $0 \le t \le 5$.

$$v(t)=\int a(t)dt=\int(t+3)dt=\frac{t^2}{2}+3t+C$$
$$v(0)=5\Rightarrow C=5$$
$$\Rightarrow v(t)=\frac{t^2}{2}+3t+5$$
$$d=\int_0^5\left|v(t)\right|dt=\int_0^5\left|\frac{t^2}{2}+3t+5\right|dt=\int_0^5\left(\frac{t^2}{2}+3t+5\right)dt$$ vì $v(t)=\frac{t^2}{2}+3t+5>0,\forall t\in[0,5]$

12. A particle moves along a line with velocity function $v(t)=t^2-t$, where is measured in meters per second. Find the displacement and the distance traveled by the particle during the time interval $t\in[0,2]$.

$$s=\int_0^2 v(t)dt=\int_0^2 (t^2-t)dt=...$$

$$d=\int_0^2 |v(t)|dt=\int_0^2 |t^2-t|dt$$

$$v(t)=t^2-t=t(t-1)\Rightarrow\begin{cases} v(t)\geq 0,\forall t\in[1,2] \\ v(t)\leq 0,\forall t\in[0,1] \end{cases}$$

$$d=\int_0^2 |v(t)|dt=\int_0^2 |t^2-t|dt=\int_0^1 |t^2-t|dt+\int_1^2 |t^2-t|dt=\int_0^1 (t-t^2)dt+\int_1^2 (t^2-t)dt=....$$

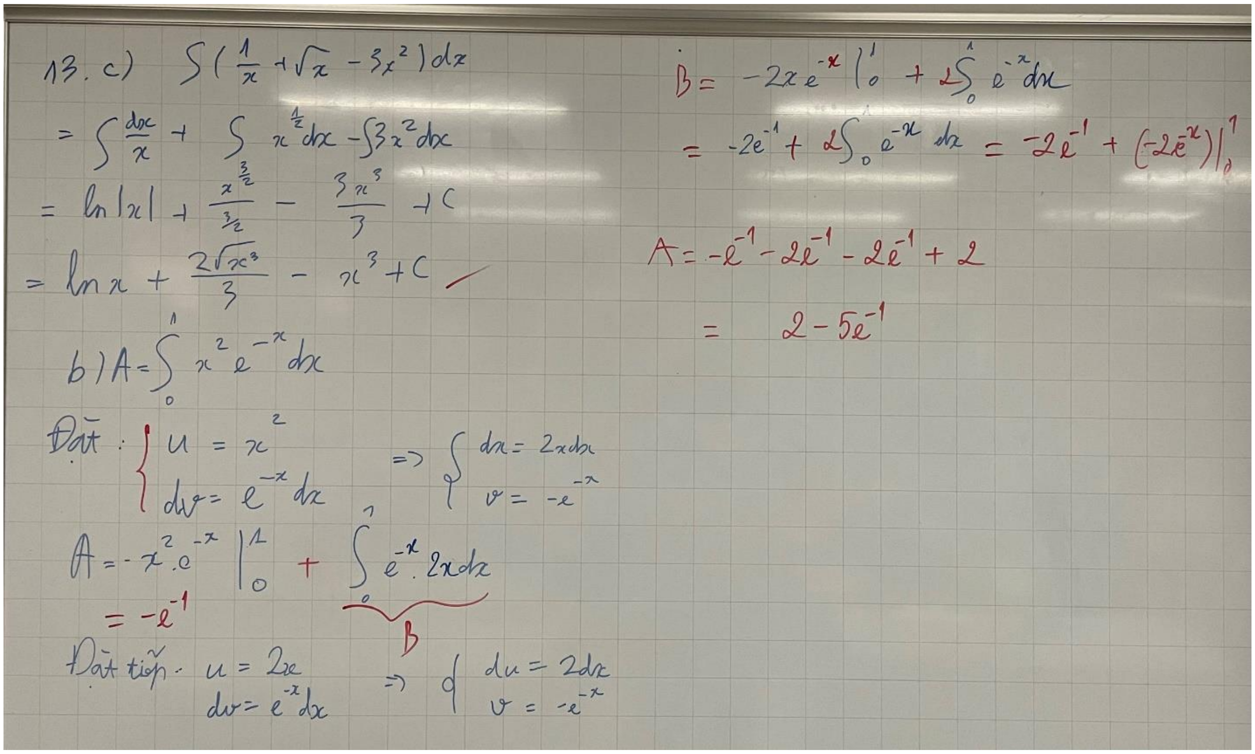
13. Evaluate the integral

a. $\int_0^2 x^2.\sqrt{x^3+1}\,dx$ b. $\int xe^{x^2}dx$ c. $\int\left(\frac{1}{x}+\sqrt{x}-3x^2\right)dx$

d. $\int_0^1 y\left(1+y^2\right)^5 dy$ e. $\int\frac{\ln x}{x}dx$ f. $\int\frac{t}{t^2+1}dt$

14. Evaluate the integral

a. $\int xe^x dx$ b. $\int_0^1 x^2e^{-x}dx$ c. $\int x\sin x dx$



d. $\int \ln x dx$ e. $\int_1^e x \ln x dx$ f. $\int e^{\sqrt{x}} dx$

$I = \int e^{\sqrt{x}} dx$

$t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow dx = 2t dt$

$I = 2\int t e^t dt$

15. Suppose $f(x)$ is differentiable, f(1) = 4 and $\int_0^1 f(x) dx = 5$. Find $\int_0^1 x f'(x) dx$

16. Suppose $f(x)$ is differentiable, $f(1) = 3$, $f(3) = 1$ and $\int_1^3 x f'(x) dx = 13$. What is the average value of f on the interval [1,3]?

$$f_{ave} = \frac{1}{3-1}\int_1^3 f(x) dx$$

$$\int_1^3 xf'(x)dx = xf(x)\Big|_1^3 - \int_1^3 f(x)dx$$

$$\Rightarrow \int_1^3 f(x)dx = xf(x)\Big|_1^3 - \int_1^3 xf'(x)dx = 3f(3) - f(1) - 13 = -13$$

$$\Rightarrow f_{ave} = \frac{1}{3-1}\int_1^3 f(x)dx = -\frac{13}{2}$$

17. Let $f(x) = \begin{cases} -x-1, & -3 \le x \le 0 \\ -\sqrt{1-x^2}, & 0 < x \le 1 \end{cases}$. Evaluate $\int_{-3}^{1} f(x)dx$

$$\int_{-3}^{1} f(x)dx = \int_{-3}^{0} f(x)dx + \int_{0}^{1} f(x)dx = \int_{-3}^{0} (-x-1)dx + \int_0^1 \left(-\sqrt{1-x^2}\right)dx$$

$$I = \int_0^1 \left(-\sqrt{1-x^2}\right)dx$$

$$x = \cos t \Rightarrow dx = -\sin t dt$$

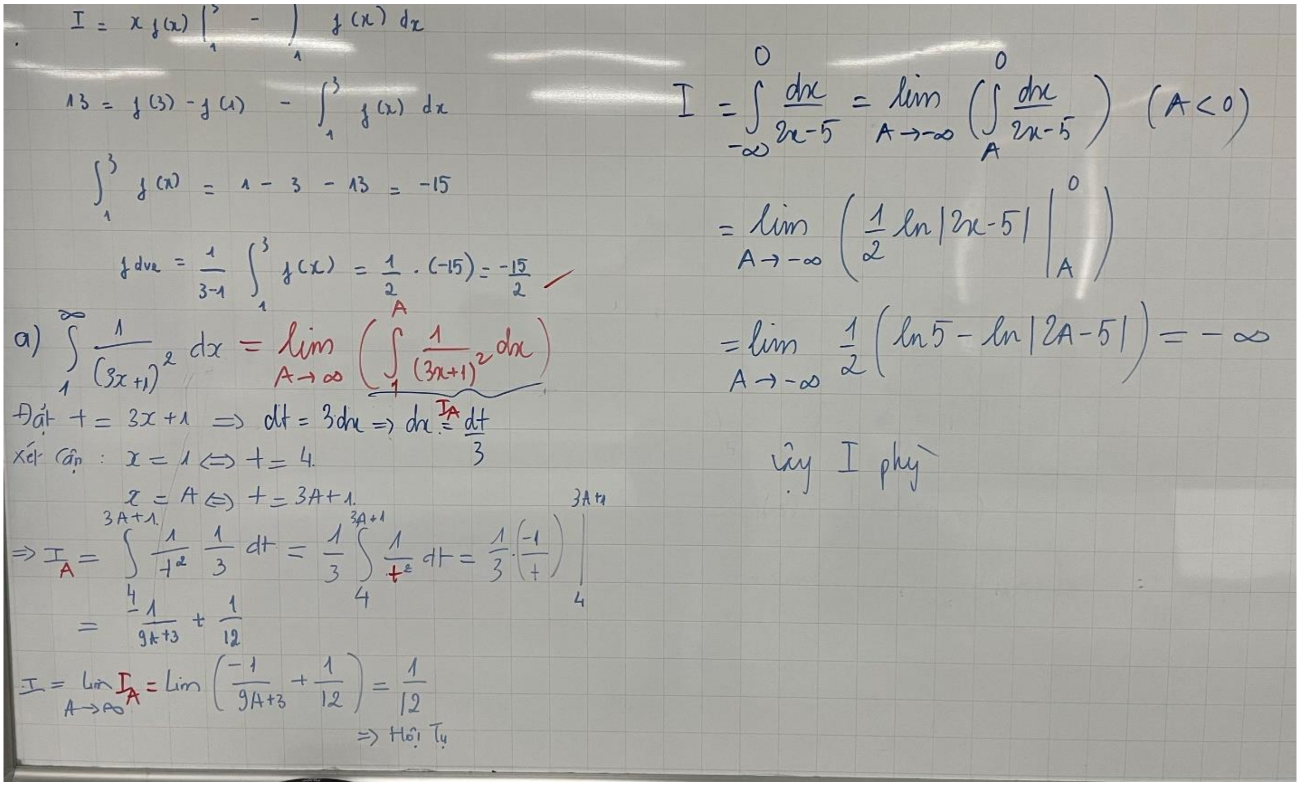
$$I = \int_0^{\pi/2} \sqrt{1-\cos^2 t}\sin t dt = \int_0^{\pi/2} \sin t.\sin t dt = \int_0^{\pi/2} \sin^2 t dt = \int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)dt = \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \ldots$$

18. Find $g'(0)$ for

a. $g(x) = \int_x^{x^2} e^{2x+1}dx$ b. $\int_{2x-1}^{x^3} x\sqrt{x+1}dx$

19. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a. $\int_1^\infty \frac{dx}{(3x+1)^2}$ b. $\int_{-\infty}^0 \frac{dx}{2x-5}$ c. $\int_{-\infty}^{-1} \frac{dx}{\sqrt{2-x}}$



d. $I = \int_0^\infty \frac{xdx}{\left(x^2+2\right)^2} = \lim_{A\to\infty}\left(\int_0^A \frac{xdx}{\left(x^2+2\right)^2}\right)$ $(A > 0)$

tính $I_A = \int_0^A \frac{xdx}{\left(x^2+2\right)^2}$

đặt $t = x^2 + 2 \Rightarrow dt = 2xdx$

$x = 0 \Rightarrow t = 2$

$x = A \Rightarrow t = A^2 + 2$

$$I_A = \frac{1}{2}\int_2^{A^2+2}\frac{dt}{t^2} = \frac{1}{2}\left(-\frac{1}{t}\right)\Bigg|_2^{A^2+2} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{A^2+2}\right) = \frac{1}{4} - \frac{1}{2(A^2+2)}$$

$$I = \lim_{A\to\infty} I_A = \lim_{A\to\infty}\left(\frac{1}{4} - \frac{1}{2(A^2+2)}\right) = \frac{1}{4} - 0 = \frac{1}{4}$$

e. $\int_{4}^{\infty} e^{-\frac{y}{2}} dy$ f. $\int_{-\infty}^{-1} e^{-2t} dt$ g. $\int_{-\infty}^{\infty} xe^{-x^2} dx$

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{0} xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx$$

$$I_1 = \int_{-\infty}^{0} xe^{-x^2} dx = \lim_{B \to -\infty} \left(\int_{B}^{0} xe^{-x^2} dx \right) = ...$$

$$I_2 = \int_{0}^{\infty} xe^{-x^2} dx = \lim_{A \to \infty} \left(\int_{0}^{A} xe^{-x^2} dx \right) = ...$$

$$I_1 = \int_{-\infty}^{0} xe^{-x^2} dx = \lim_{B \to -\infty} \left(\int_{B}^{0} xe^{-x^2} dx \right)$$

$$I_B = \int_{B}^{0} xe^{-x^2} dx$$

$$t = x^2 \Rightarrow dt = 2xdx$$

$$I_B = \int_{B^2}^{0} \frac{1}{2} e^{-t} dt = \left(-\frac{1}{2} e^{-t} \right) \Bigg|_{B^2}^{0} = \frac{1}{2}\left(e^{-B^2} - 1 \right) = \frac{1}{2}\left(\frac{1}{e^{B^2}} - 1 \right)$$

$$I_1 = \lim_{B \to -\infty} \frac{1}{2}\left(\frac{1}{e^{B^2}} - 1 \right) = \frac{1}{2}(0-1) = -\frac{1}{2}$$

i. $\int_{0}^{1} \frac{dx}{4x-1}$ j. $\int_{3}^{4} \frac{dx}{\sqrt{x-3}}$ k. $\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}} dx$ l. $\int_{0}^{1} \frac{dx}{\sqrt{x}}$

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}} dx = \int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}} dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x^2}} dx$$

$$\int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}} dx = \lim_{t \to 0^-} \left(\int_{-1}^{t} \frac{1}{\sqrt[3]{x^2}} dx \right) = \lim_{t \to 0^-} \left(\int_{-1}^{t} x^{-2/3} dx \right) = \lim_{t \to 0^-} \left(3x^{1/3} \Big|_{-1}^{t} \right) = \lim_{t \to 0^-} \left(3t^{1/3} + 3 \right) = 0 + 3 = 3$$

$$\int_{0}^{1} \frac{1}{\sqrt[3]{x^2}} dx = \lim_{t \to 0^+} \left(\int_{t}^{1} \frac{1}{\sqrt[3]{x^2}} dx \right) = \lim_{t \to 0^+} \left(\int_{t}^{1} x^{-2/3} dx \right) = \lim_{t \to 0^+} \left(3x^{1/3} \Big|_{t}^{1} \right) = \lim_{t \to 0^+} \left(3 - 3t^{1/3} \right) = 3 - 0 = 3$$

20. Determine whether the integral is convergent or divergent

a. $\int_{1}^{\infty} \frac{\cos^2 x dx}{1+x^2}$

$$0 \le f(x) = \frac{\cos^2 x}{1+x^2} \le \frac{1}{1+x^2} \le \frac{1}{x^2} \quad , \forall x \in [1, \infty)$$

Mà $\int_1^{\infty} \frac{1}{x^2} dx$ hội tụ nên $\int_1^{\infty} \frac{\cos^2 x dx}{1+x^2}$ hội tụ

b. $\int_1^{\infty} \frac{2+e^{-x}}{x} dx$

$$f(x) = \frac{2+e^{-x}}{x} > \frac{1}{x} > 0 \text{ , } \forall x \in [1, \infty)$$

Mà $\int_1^{\infty} \frac{1}{x} dx$ phân kỳ nên $\int_1^{\infty} \frac{2+e^{-x}}{x} dx$ phân kỳ

c. $\int_1^{\infty} \frac{dx}{x+e^{2x}}$

$$0 < h(x) = \frac{1}{x+e^{2x}} < \frac{1}{e^{2x}} = e^{-2x} \quad , \forall x \in [1, \infty)$$

$$\int_1^{\infty} e^{-2x} dx = \lim_{A \to \infty} \left(\int_1^{A} e^{-2x} dx \right) = \lim_{A \to \infty} \left(\frac{-1}{2} e^{-2x} \Big|_1^A \right) = \lim_{A \to \infty} \left(\frac{1}{2e^2} - \frac{1}{2e^{2A}} \right) = \frac{1}{2e^2}$$

Mà $\int_1^{\infty} e^{-2x} dx$ hội tụ nên $\int_1^{\infty} \frac{dx}{x+e^{2x}}$ hội tụ

d. $\int_1^{\infty} \frac{xdx}{\sqrt{1+x^6}}$

$$f(x) = \frac{x}{\sqrt{1+x^6}} > 0 \text{ , } \forall x \in [1, \infty)$$

$$g(x) = \frac{1}{x^2} > 0 \text{ , } \forall x \in [1, \infty)$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^3}{\sqrt{1+x^6}} = \lim_{x \to \infty} \frac{1}{\sqrt{\frac{1}{x^6}+1}} = \frac{1}{\sqrt{1+0}} = 1$$

Mà $\int_1^{\infty} \frac{1}{x^2} dx$ hội tụ nên $\int_1^{\infty} \frac{xdx}{\sqrt{1+x^6}}$ hội tụ

$$e.\ I = \int_0^{\frac{\pi}{2}} \frac{\cos x dx}{\sqrt{\sin x}} = \lim_{t \to 0^+} \left(\int_t^{\frac{\pi}{2}} \frac{\cos x dx}{\sqrt{\sin x}} \right) \ (t > 0)$$

$$I(t) = \int_t^{\frac{\pi}{2}} \frac{\cos x dx}{\sqrt{\sin x}}$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$I(t) = \int_{\sin t}^{1} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{\sin t}^{1} = 2 - 2\sqrt{\sin t}$$

$$I = \lim_{t \to 0^+} I(t) = \lim_{t \to 0^+} \left(2 - 2\sqrt{\sin t} \right) = 2$$

f. $\displaystyle\int_0^1 \frac{2dx}{\sqrt{x^3}}$

Chapter 8: Series

1. Determine whether the sequence converges or diverges. If it converges, find the limit

a. $a_n = \dfrac{3+2n^2}{n+n^2}$ b. $a_n = \dfrac{\sqrt{n}}{\sqrt{2n+1}+3}$ c. $a_n = \dfrac{n}{\sqrt{n}+1}$ d. $a_n = \left(1+\dfrac{2}{n}\right)^n$

$$\left\{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, ...\right\}$$

$$a_1 = \sqrt{2} = 2\cos\frac{\pi}{4} = 2\cos\frac{\pi}{2^2}$$

$$a_2 = \sqrt{2+\sqrt{2}} = \sqrt{2+2\cos\frac{\pi}{2^2}} = 2\cos\frac{\pi}{8} = 2\cos\frac{\pi}{2^3}$$

e. ....

$$a_n = \sqrt{2+....\sqrt{2+\sqrt{2}}} = 2\cos\frac{\pi}{2^{n+1}}$$

$$\Rightarrow \lim_{n\to+\infty} a_n = 2$$

$$\begin{cases} a_1 = \sqrt{2} \\ a_{n+1} = \sqrt{a_n+2}\ ,\ n \ge 1 \end{cases}$$

f. $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, ...\right\}$

g. $\{0.12,\ 0.1212,\ \ 0.121212,\ ...\}$

2. Find the limit of the sequence $\{a_n\}$

a. $a_1 = \sqrt{5},\ a_{n+1} = \sqrt{5+a_n}$ b. $a_1 = 2,\ a_{n+1} = \dfrac{1}{3-a_n}$ c. $a_1 = 1,\ a_{n+1} = \dfrac{1}{1+a_n}$

3. Determine whether the sequence is increasing, decreasing or not monotonic

a. $u_n = \dfrac{1}{2n^2-n+1}$ b. $u_n = \dfrac{\sqrt{n+5}}{n+1}$ c. $\begin{cases} u_1 = 1 \\ u_{n+1} = \dfrac{u_n}{3-u_n} \end{cases}$

4. Find the formula for the nth term of the sequence

a. $\{1,3,5,7,...\}$ b. $\begin{cases} u_1 = 1 \\ u_n = 2u_{n-1}+1 \end{cases}$ c. $\begin{cases} u_1 = u_2 = 1 \\ u_{n+2} = u_{n+1}+u_n \end{cases}$

5. Suppose that $f(1)=1, f(2)=-2$ and $f(n+2)=-2f(n+1)+3f(n)$.

a. Find $f(5)$

b. Determine the formula for $f(n)$

6. Determine whether the series is convergent or divergent. If it is convergent, find its sum

a. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ b. $\sum_{n=2}^{\infty}\frac{n^2+n-1}{n(n-1)}$ c. $\sum_{n=2}^{\infty}\frac{1}{3.2^{n-1}}$

d. $\sum_{n=1}^{\infty}\sin n$ e. $\sum_{n=1}^{\infty}\frac{1+3^n}{2^n}$ f. $\sum_{n=1}^{\infty}\left(0,8^n+0,3^{n-1}\right)$

7. Determine whether the series is convergent or divergent

a. $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ b. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ c. $\sum_{n=1}^{\infty}\left(\frac{1}{n^6}+\frac{4}{n\sqrt{n}}\right)$ d. $\sum_{n=1}^{\infty}ne^{-n}$

e. $\sum_{n=1}^{\infty}\frac{1}{2n+3}$ f. $\sum_{n=1}^{\infty}\frac{4+3^n}{2^n}$ g. $\sum_{n=1}^{\infty}\frac{n!}{n^2 2^n}$ h. $\sum_{n=1}^{\infty}\frac{\cos n}{n^2+1}$

8. Determine whether the series is convergent or divergent

a. $\sum_{n=1}^{\infty}\frac{n}{2^n}$ b. $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{\sqrt{n}}$ c. $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n^2+n+1}$ d. $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$

e. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+3}$ f. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\ln n}{n}$ g. $\sum_{n=1}^{\infty}\frac{\cos \pi n}{\sqrt{n+1}}$ h. $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+2n+3}\right)^n$

9. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

a. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n^3}$ b. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ c. $\sum_{n=1}^{\infty}\frac{(-10)^n}{n!}$ d. $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$

e. $\sum_{n=1}^{\infty}\frac{\sin 4n}{n^2}$ f. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\sqrt{n+1}}$ g. $\sum_{n=2}^{\infty}\frac{\cos \pi n}{\ln n}$ h. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

10. Find the radius of convergence and interval of convergence of the series

a. $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ b. $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$ c. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ d. $\sum_{n=0}^{\infty}\frac{(-1)^n (x-1)^n}{2n+1}$

e. $\sum_{n=0}^{\infty} n!(2x-1)^n$ f. $\sum_{n=0}^{\infty}\frac{x^n}{n^2 3^n}$ g. $\sum_{n=0}^{\infty}\sqrt{n+1}x^n$ h. $\sum_{n=0}^{\infty}\frac{(-2)^n (x-3)^n}{\sqrt[4]{n}}$

11. Find the first n terms in the Maclaurin series for the given function

a. $f(x)=x\sin x,\ n=4$ b. $f(x)=x\cos 2x,\ n=3$

c. $f(x)=\ln(1+x^2),\ n=4$ d. $f(x)=e^x\sin x,\ n=3$

12. Approximate f by a Taylor polynomial with degree at the number a

a. $f(x)=\sqrt{x+1},\ n=1,\ a=0$ b. $f(x)=\frac{1}{x},\ n=3,\ a=1$

c. $f(x)=e^{x^2},\ n=3,\ a=0$ d. $f(x)=\cos x,\ n=4,\ a=\frac{\pi}{3}$

LINEAR ALGEBRA

Chapter 1: Systems of Linear Equations

1. Write the augmented matrix for each of the following systems of linear equations and then solve them.

a. $\begin{cases} -x+y+2z=1 \\ 2x+3y+z=-2 \\ 5x+4y+2z=4 \end{cases}$ b. $\begin{cases} 2x+3y+z=10 \\ 2x-3y-3z=22 \\ 4x-2y+3z=-2 \end{cases}$

c. $\begin{cases} x+y+z=0 \\ 2x-y+2z=0 \\ x+z=0 \end{cases}$ d. $\begin{cases} x_1+2x_2-x_3+x_4=0 \\ 2x_1+3x_2-2x_3+3x_4=0 \\ x_1+x_2-3x_3+x_4=0 \end{cases}$

2. Compute the rank of each of the following matrices.

a. $A=\begin{pmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$ b. $B=\begin{pmatrix} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{pmatrix}$

$$B=\begin{pmatrix} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{pmatrix} \xrightarrow{d_1\to d_1+d_2} \begin{pmatrix} 1 & -1 & 4 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{pmatrix} \xrightarrow[d_3\to d_3+5d_1]{d_2\to d_2-3d_1} \begin{pmatrix} 1 & -1 & 4 \\ 0 & -1 & -11 \\ 0 & 2 & 22 \end{pmatrix} \xrightarrow{d_3\to d_3+2d_2} \begin{pmatrix} 1 & -1 & 4 \\ 0 & -1 & -11 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow r(B)=2$

c. $C=\begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 5 & 8 \end{pmatrix}$ d. $D=\begin{pmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{pmatrix}$

3. Find all values of k for which the system has nontrivial solutions and determine all solutions in each case.

a. $\begin{cases} x-y+2z=0 \\ -x+y-z=0 \\ x+ky+z=0 \end{cases}$ b. $\begin{cases} x-2y+z=0 \\ x+ky-3z=0 \\ x-6y+5z=0 \end{cases}$

$$A=\begin{bmatrix}1 & -1 & 2\\ -1 & 1 & -1\\ 1 & k & 1\end{bmatrix}$$

Hệ pt (1) có nghiệm không tầm thường $\Leftrightarrow r(A)<3$

$$\Leftrightarrow \det A = 0$$

$$\Leftrightarrow \begin{vmatrix}1 & -1 & 2\\ -1 & 1 & -1\\ 1 & k & 1\end{vmatrix}=0$$

$$\Leftrightarrow 1+1-2k-2+k-1=0$$

$$\Leftrightarrow k=-1$$

* Có thể sử dụng pp Gauss để giải

$$A=\begin{bmatrix}1 & -1 & 2\\ -1 & 1 & -1\\ 1 & k & 1\end{bmatrix}\xrightarrow[d_3\to d_3-d_1]{d_2\to d_2+d_1}\begin{bmatrix}1 & -1 & 2\\ 0 & 0 & 1\\ 0 & k+1 & -1\end{bmatrix}\xrightarrow{d_2\leftrightarrow d_3}\begin{bmatrix}1 & -1 & 2\\ 0 & k+1 & -1\\ 0 & 0 & 1\end{bmatrix}=A'$$

* $k+1\neq 0\Leftrightarrow k\neq -1\Rightarrow r(A)=3\Rightarrow$ hpt (1) có nghiệm tầm thường

* $k+1=0\Leftrightarrow k=-1$. Khi đó,

$$A'=\begin{bmatrix}1 & -1 & 2\\ 0 & 0 & -1\\ 0 & 0 & 1\end{bmatrix}\xrightarrow{d_3\to d_3+d_2}\begin{bmatrix}1 & -1 & 2\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}$$

Suy ra $r(A)=r(A')=2<3$ hay hpt (1) có nghiệm ko tầm thường

Vậy $k=-1$

$$\text{hpt (1)} \Leftrightarrow \begin{cases}x-y+2z=0\\ -z=0\end{cases}\Leftrightarrow\begin{cases}x=y\\ z=0\end{cases}\Leftrightarrow\begin{cases}x=a\\ y=a \quad (a\in\mathbb{R})\\ z=0\end{cases}$$

Nghiệm của hpt (1) là $(a,a,0)$ $(a\in\mathbb{R})$

c. $\begin{cases}x+y+z=0\\ x+y-z=0\\ x+y+kz=0\end{cases}$

d. $\begin{cases}x+y-z=0\\ ky-z=0\\ x+y+kz=0\end{cases}$

4. Determine the values of m such that the system of linear equations has exactly one solution.

a. $\begin{cases} x-y+2z=m \\ -x+y-z=0 \\ -x+my-z=1-m \end{cases}$ b. $\begin{cases} mx+y+z=1 \\ x+my+z=m \\ x+y+mz=m^2 \end{cases}$

$$A=\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ -1 & m & -1 \end{bmatrix}$$

Hệ pt (1) có nghiệm duy nhất $\Leftrightarrow$ hpt (1) là hệ Cramer

$$\Leftrightarrow \det A \neq 0$$
$$\Leftrightarrow -1-1-2m+2+m+1 \neq 0$$
$$\Leftrightarrow m \neq 1$$

* Có thể sử dụng pp Gauss để giải

$$\overline{A}=\left[\begin{array}{ccc|c} 1 & -1 & 2 & m \\ -1 & 1 & -1 & 0 \\ -1 & m & -1 & 1-m \end{array}\right] \xrightarrow[d_3 \to d_3+d_1]{d_2 \to d_2+d_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & m \\ 0 & 0 & 1 & m \\ 0 & m-1 & 1 & 1 \end{array}\right] \xrightarrow{d_2 \leftrightarrow d_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & m \\ 0 & m-1 & 1 & 1 \\ 0 & 0 & 1 & m \end{array}\right]$$

Hệ pt (1) có nghiệm duy nhất $\Leftrightarrow r(A)=r(\overline{A})=3$

$$\Leftrightarrow m-1 \neq 0$$
$$\Leftrightarrow m \neq 1$$

Hpt (1) $\Leftrightarrow \begin{cases} x-y+2z=m \\ (m-1)y+z=1 \\ z=m \end{cases} \Leftrightarrow \begin{cases} x=-m-1 \\ y=-1 \\ z=m \end{cases}$

c. $\begin{cases} x+y-z=1 \\ x+my+2z=m \\ x+2y+z=2 \end{cases}$ d. $\begin{cases} x+my-mz=m \\ 2x+y-z=2 \\ x+y+z=0 \end{cases}$

5. Determine the values of m such that the system of linear equations is inconsistent.

a. $\begin{cases} x-y+2z=m \\ -x+y-z=0 \\ x-y+3z=1-m \end{cases}$ b. $\begin{cases} x-2y+2z=m \\ x+my+z=0 \\ 2x+y+mz=2-m \end{cases}$

$$\overline{A}=\left[\begin{array}{ccc|c}1&-2&2&m\\1&m&1&0\\2&1&m&2-m\end{array}\right]\xrightarrow[d_3\to d_3-2d_1]{d_2\to d_2-d_1}\left[\begin{array}{ccc|c}1&-2&2&m\\0&m+2&-1&-m\\0&5&m-4&2-3m\end{array}\right]\xrightarrow{d_2\leftrightarrow d_3}\left[\begin{array}{ccc|c}1&-2&2&m\\0&5&m-4&2-3m\\0&m+2&-1&-m\end{array}\right]$$

$$\xrightarrow{d_3\to d_3-\frac{m+2}{5}d_2}\left[\begin{array}{ccc|c}1&-2&2&m\\0&5&m-4&-m\\0&0&-\frac{1}{5}(m+1)(m-3)&\frac{1}{5}(3m^2-m-4)\end{array}\right]$$

Hpt (1) vô nghiệm $\Leftrightarrow r(A)\neq r(\overline{A})$

$$\Leftrightarrow\begin{cases}r(A)=2\\ r(\overline{A})=3\end{cases}\Leftrightarrow\begin{cases}(m+1)(m-3)=0\\ 3m^2-m-4\neq 0\end{cases}\Leftrightarrow m=3$$

6. Find a, b and c so that the system $\begin{cases}x+ay+cz=0\\ bx+cy-3z=1\\ ax+2y+bz=5\end{cases}$ has the solution $(3,-1,2)$

$(3,-1,2)$ là nghiệm của hpt khi và chỉ khi $\begin{cases}3-a+2c=0\\ 3b-c-6=1\\ 3a-2+3b=5\end{cases}\Leftrightarrow\begin{cases}a=1\\ b=2\\ c=-1\end{cases}$

7. Consider the matrix $A=\begin{pmatrix}2&-1&3\\-4&2&k\\4&-2&6\end{pmatrix}$

a. If A is the augmented matrix of a system of linear equations, determine the number of equations and the number of variables.

b. If A is the augmented matrix of a system of linear equations, find the value(s) of k such that the system is consistent.

8. Find all values of k so that the system of equations has no solution.

a. $\begin{cases}x+y-z=2\\ -2y+z=3\\ 4y-2z=k\end{cases}$

b. $\begin{cases}x+y-z=1\\ 2x+(k+5)y-2z=4\\ x+(k+3)y+(k-1)z=k+3\end{cases}$

9. Find all values of a and b for which the system of equations $\begin{cases} x+y+3z=2 \\ x+2y+5z=1 \\ 2x+2y+az=b \end{cases}$

is inconsistent.

$$\overline{A}=\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 5 & 1 \\ 2 & 2 & a & b \end{array}\right] \xrightarrow[d_3\to d_3-2d_1]{d_2\to d_2-d_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & a-6 & b-4 \end{array}\right]$$

Hpt vô nghiệm $\Leftrightarrow r(A)\neq r(\overline{A})$

$$\Leftrightarrow \begin{cases} r(A)=2 \\ r(\overline{A})=3 \end{cases} \Leftrightarrow \begin{cases} a-6=0 \\ b-4\neq 0 \end{cases}$$

10. Solve the system of linear equation corresponding to the given augmented matrix

a. $A=\begin{pmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ b. $B=\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

11. Determine the values of m such that the rank of the matrix is 2

a. $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 5 \\ 1 & 2 & m \end{pmatrix}$ b. $\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 5 \\ -3 & 6 & 1 & m \end{pmatrix}$ c. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & 1 & 2 \\ m & 3 & 5 \end{pmatrix}$

$$B=\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 1 & 5 \\ -3 & 6 & 1 & m \end{pmatrix} \xrightarrow[d_3\to d_3+3d_1]{d_2\to d_2-2d_1} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & -1 & -3 \\ 0 & 12 & 4 & m+12 \end{pmatrix} \xrightarrow{d_3\to d_3+4d_2} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & -1 & -3 \\ 0 & 0 & 0 & m \end{pmatrix}$$

$r(B)=2\Leftrightarrow m=0$

12. Solve the system $\begin{cases} x+2y=12 \\ 3x-y=8 \\ -x+5y=16 \end{cases}$

Chapter 2: Matrix Algebra

1. Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}, B=\begin{pmatrix}-1 & 1\\ 3 & 2\end{pmatrix}$ and $C=\begin{pmatrix}1 & 3 & -4\\ -1 & 2 & 1\end{pmatrix}$. Compute the matrix

a. $2A-B^T$ b. AB c. BA d. AC

e. CC^T f. C^TC g. A^3 h. B^2A^T

2. Suppose that A and B are nxn matrices. Simplify the expression

a. $(A+B)^2-(A-B)^2$

$$(A+B)^2-(A-B)^2=(A+B).(A+B)-(A-B).(A-B)$$
$$=A^2+AB+BA+B^2-\left(A^2-AB-BA+B^2\right)$$
$$=A^2+AB+BA+B^2-A^2+AB+BA-B^2$$
$$=2.(AB+BA)$$

b. $A(BC-CD)+A(C-B)D-AB(C-D)$

$$=ABC-ACD+(AC-AB)D-ABC+ABD$$
$$=ABC-ACD+ACD-ABD-ABC+ABD$$
$$=\theta$$

3. Let $A=\begin{pmatrix}3 & 1 & 2\\ 4 & 8 & 0\\ 0 & 1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}0 & 5 & 2 & 1\\ 1 & 8 & 0 & -6\\ 1 & 4 & 3 & 7\end{pmatrix}$.

a. Compute AB

b. Compute $f(A)$ if $f(x)=x^2-3x+2=x^2-3x+2x^0$

$$f(A)=A^2-3A+2I_3$$

$$=\begin{pmatrix}3 & 1 & 2\\ 4 & 8 & 0\\ 0 & 1 & 2\end{pmatrix}^2-3.\begin{pmatrix}3 & 1 & 2\\ 4 & 8 & 0\\ 0 & 1 & 2\end{pmatrix}+2.\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

4. Find the inverse of each of the following matrices.

a. $\begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix}$ b. $\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}$ c. $\begin{pmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{pmatrix}$ d. $\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{pmatrix}$

5. Given $A^{-1} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{pmatrix}$. Find a matrix X such that

a. $AX = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ b. $AX = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ c. $XA = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$

$$\Leftrightarrow X = A^{-1}.\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \ldots$$

6. Find A when

a. $(3A)^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ b. $(I + 2A)^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ c. $(A^{-1} - 2I)^T = -2\begin{pmatrix} 1 & 4 \\ 3 & 11 \end{pmatrix}$

$$(A^{-1} - 2I_2)^T = -2\begin{pmatrix} 1 & 4 \\ 3 & 11 \end{pmatrix} = \begin{pmatrix} -2 & -8 \\ -6 & -22 \end{pmatrix}$$

$$(A^{-1} - 2I_2) = \left[(A^{-1} - 2I_2)^T\right]^T = \begin{pmatrix} -2 & -6 \\ -8 & -22 \end{pmatrix}$$

$$A^{-1} = 2I_2 + \begin{pmatrix} -2 & -6 \\ -8 & -22 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & -6 \\ -8 & -22 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ -8 & -20 \end{pmatrix}$$

$$A = (A^{-1})^{-1} = \begin{pmatrix} 0 & -6 \\ -8 & -20 \end{pmatrix}^{-1} = \begin{pmatrix} 5/12 & -1/8 \\ -1/6 & 0 \end{pmatrix}$$

7. Write the system of linear equations in matrix form and then solve them.

a. $\begin{cases} 2x - y = 4 \\ 3x + 2y = -4 \end{cases}$ b. $\begin{cases} 2x + 3y + z = 10 \\ 2x - 3y - 3z = 22 \\ 4x - 2y + 3z = -2 \end{cases}$ c. $\begin{cases} x + y = a \\ 2x + 3y = 1 - 2a \end{cases} (a \in R)$

8. Find A^{-1} if

a. $A^2 - 6A + 5I = 0$ b. $A^2 + 3A - I = 0$ c. $A^4 = I$

$$A^2 - 6A + 5I = 0$$
$$\Leftrightarrow 5I = 6A - A^2$$
$$\Leftrightarrow I = \frac{6}{5}A - \frac{1}{5}A^2 = A.\left(\frac{6}{5}I - \frac{1}{5}A\right) = \left(\frac{6}{5}I - \frac{1}{5}A\right).A$$
$$\Rightarrow A^{-1} = \left(\frac{6}{5}I - \frac{1}{5}A\right)$$

9. Solve for X

a. $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & -1 \\ 3 & 3 \end{pmatrix}$ b. $ABXC = B^T$ c. $AX^TBC = B$

$$\Leftrightarrow X = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} . \begin{pmatrix} 1 & -1 \\ 3 & 3 \end{pmatrix} = ...$$

$$X = (AB)^{-1}.B^T.C^{-1} = B^{-1}.A^{-1}.B^T.C^{-1}$$

(where A, B and C are nxn invertible matrices)

10. Compute $\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}^{101}$

$$\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}.\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$
$$\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}^2 . \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} = I_2.\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}^{101} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$$

11. Let $T : R^2 \to R^2$ be a linear transformation, and assume that $T(1,2) = (-1,1)$ and $T(0,3) = (-3,3)$

a. Compute $T(11,-5)$ b. Compute $T(1,11)$

c. Find the matrix of T d. Compute $T^{-1}(2,3)$

$$T(x,y)=(ax+by,cx+dy)\Rightarrow A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T(1,2)=(a+2b,c+2d)=(-1,1)\Rightarrow \begin{cases} a+2b=-1 \\ c+2d=1 \end{cases}$$

$$T(0,3)=(3b,3d)=(-3,3)\Rightarrow \begin{cases} 3b=-3 \\ 3d=3 \end{cases} \Leftrightarrow \begin{cases} b=-1 \\ d=1 \end{cases} \Rightarrow \begin{cases} a=1 \\ c=-1 \end{cases}$$

$$\Rightarrow A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

12. Let $T: R^2 \to R^2$ be a linear transformation such that the matrix of T is $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

Find $T^{-1}(3,-2)$

Vì ma trận của T là $A=\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ nên suy ra $A^{-1}=\begin{pmatrix} \frac{3}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}$. Do đó

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=A^{-1}.\begin{bmatrix} x \\ y \end{bmatrix}==\begin{pmatrix} \frac{3}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}.\begin{bmatrix} x \\ y \end{bmatrix}=\begin{pmatrix} \frac{3}{5}x-\frac{2}{5}y \\ \frac{1}{5}x+\frac{1}{5}y \end{pmatrix}$$

Suy ra $T^{-1}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)=\begin{pmatrix} \frac{3}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}=\begin{bmatrix} \frac{13}{5} \\ \frac{1}{5} \end{bmatrix}$

13. The (2;1)-entry of the product $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 5 & 1 \\ 4 & -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 5 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix}$

$$c_{21}=0.4+2.2+5.5+1.0=29$$

Chapter 3: Determinants and Diagonalization

1. Evaluate the determinant

a. $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix}$ b. $\begin{vmatrix} -2 & 0 & 0 \\ 4 & 6 & 0 \\ -3 & 7 & 2 \end{vmatrix}$ c. $\begin{vmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{vmatrix}$ d. $\begin{vmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{vmatrix}$

e. $\begin{vmatrix} x & y & 1 \\ -1 & -2 & 1 \\ 1 & 5 & 1 \end{vmatrix}$ f. $\begin{vmatrix} m & -1 & 0 \\ 1 & 2 & 1 \\ 2 & m & -3 \end{vmatrix}$

2. Find the minors and the cofactors of the matrix

a. $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ b. $B = \begin{pmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{pmatrix}$ c. $C = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & m \end{pmatrix}$

3. Find the adjugate and the inverse of the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$

4. Let $A = \begin{pmatrix} 1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Find $|A| = 1.(-1).2.2 = -4$

a. $|2A^{-1}| = 2^4.|A^{-1}| = 16.\dfrac{1}{|A|} = 16.\dfrac{-1}{4} = -4$ b. $|AA^T| = |A|.|A^T| = |A|^2$

c. $|\text{adj}\, A| = |A|^3 = -64$ d. $|-A^3| = (-1)^4.|A^3| = |A|^3$

e. $|(2A)^{-1}| = \dfrac{1}{|2A|} = \dfrac{1}{2^4.|A|}$

f. $|A^{-1} - 2\text{adj}\, A| = |A^{-1} - 2.|A|.A^{-1}| = |A^{-1} + 8A^{-1}| = |9.A^{-1}| = 9^4.|A^{-1}|$

5. Let A and B be square matrices of order 4 such that $|A| = -5$ and $|B| = 3$. Find

a. $|2AB| = 16|A|.|B|$ b. $|adj(AB)| = |AB|^3 = |A|^3.|B|^3$

c. $\left|5A^{-1}B^{T}\right|$ d. $\left|A^{T}B^{-1}A^{2}\right|$

6. Find all values of m,k for which the matrix is not invertible

a. $A=\begin{pmatrix}1 & 3\\ k & 2\end{pmatrix}$ b. $B=\begin{pmatrix}m & 1 & 3\\ 1 & 3 & 2\\ -1 & 4 & 5\end{pmatrix}$ c. $C=\begin{pmatrix}m & 2 & 0\\ 1 & m & 1\\ 2 & 3 & 1\end{pmatrix}$

7. Find the characteristic polynomial of the matrix

a. $A=\begin{pmatrix}3 & 5\\ 1 & 2\end{pmatrix}$ b. $B=\begin{pmatrix}1 & 3\\ 1 & 3\end{pmatrix}$

$$P_A(\lambda)=\left|\lambda I_2-A\right|=\begin{vmatrix}\lambda-3 & -5\\ -1 & \lambda-2\end{vmatrix}=(\lambda-3).(\lambda-2)-5=\lambda^2-5\lambda+1$$

c. $C=\begin{pmatrix}1 & 0 & 0\\ -2 & 2 & 1\\ 1 & 2 & 1\end{pmatrix}$ d. $D=\begin{pmatrix}1 & 2 & -1\\ 0 & 1 & 2\\ -1 & 1 & 1\end{pmatrix}$

8. Find the eigenvalues and corresponding eigenvectors of the matrix

a. $A=\begin{pmatrix}-3 & 5\\ 10 & 2\end{pmatrix}$ b. $B=\begin{pmatrix}5 & 4\\ 2 & 1\end{pmatrix}$

c. $C=\begin{pmatrix}1 & 0 & 0\\ -2 & 3 & 0\\ 0 & 0 & 4\end{pmatrix}$ d. $D=\begin{pmatrix}-3 & 2 & -1\\ 0 & 1 & 0\\ 4 & 1 & 1\end{pmatrix}$

9. Find the determinant of the matrix $A=\begin{pmatrix}5 & 1 & 2 & 4\\ 1 & 0 & -1 & -3\\ 1 & 1 & 6 & 1\\ 1 & 0 & 0 & -4\end{pmatrix}$

Khai triển định thức theo dòng 4, ta được

$$\left|A\right|=1.A_{41}+0.A_{42}+0.A_{43}+(-4).A_{44}$$
$$=1.(-1)^{4+1}.\begin{vmatrix}1 & 2 & 4\\ 0 & -1 & -3\\ 1 & 6 & 1\end{vmatrix}+(-4).(-1)^{4+4}.\begin{vmatrix}5 & 1 & 2\\ 1 & 0 & -1\\ 1 & 1 & 6\end{vmatrix}$$

$$|A| = \begin{vmatrix} 5 & 1 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{vmatrix} \overset{d_1 \leftrightarrow d_2}{=\!=\!=} - \begin{vmatrix} 1 & 0 & -1 & -3 \\ 5 & 1 & 2 & 4 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{vmatrix} \overset{\substack{d_2 \to d_2 - 5d_1 \\ d_3 \to d_3 - d_1 \\ d_4 \to d_4 - d_1}}{=\!=\!=\!=\!=} - \begin{vmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 7 & 19 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

$$\overset{d_3 \to d_3 - d_2}{=\!=\!=\!=} - \begin{vmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 7 & 19 \\ 0 & 0 & 0 & -15 \\ 0 & 0 & 1 & -1 \end{vmatrix} \overset{d_3 \leftrightarrow d_4}{=\!=\!=} \begin{vmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 7 & 19 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -15 \end{vmatrix} = 1.1.1.(-15) = -15$$

10. Find the (1, 2)-cofactor and (3,1) - cofactor of the matrix $\begin{bmatrix} -1 & 3 & -2 \\ 4 & 5 & -7 \\ 7 & 8 & 1 \end{bmatrix}$

$$A_{12} = (-1)^{1+2}.\begin{vmatrix} 4 & -7 \\ 7 & 1 \end{vmatrix} = -53$$

11. Let $A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & x \end{pmatrix}$. For which values of x is A invertible ?

Chapter 5: The Vector Space $\mathbb{R}^n$

1. Let $x=(-1,-2,-2), u=(0,1,4), v=(-1,1,2)$ and $w=(3,1,2)$ in R^3.

Find scalars *a*, *b* and *c* such that $x = au + bv + cw$

Xét bt $x = au + bv + cw$

$$\Leftrightarrow (-1,-2,-2) = a.(0,1,4) + b.(-1,1,2) + c.(3,1,2)$$

$$\Leftrightarrow \begin{cases} -b+3c=-1 \\ a+b+c=-2 \\ 4a+2b+2c=-2 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=-2 \\ c=-1 \end{cases}$$

$$x = u - 2v - w$$

2. Write *v* as a linear combination of *u* and *w*, if possible, where $u=(1,2), w=(1,-1)$

a. $v=(0,1)$ b. $v=(2,3)$ c. $v=(1,4)$ d. $(-5,1)$

Xét bt $v = au + bw$

$$\Leftrightarrow (2,3) = a.(1,2) + b.(1,-1)$$

$$\Leftrightarrow \begin{cases} a+b=2 \\ 2a-b=3 \end{cases} \Leftrightarrow \begin{cases} a=5/3 \\ b=1/3 \end{cases}$$

$$v = \frac{5}{3}u + \frac{1}{3}w$$

3. Determine whether the set S is linearly independent or linearly dependent in corresponding vector spaces.

a. $S=\{(-1,2),(3,1),(2,1)\}$ b. $S=\{(-1,2,3),(1,3,5)\}$

c. $S=\{(1,-2,2),(2,3,5),(3,1,7)\}$ d. $S=\{(-1,2,1),(2,4,0),(3,1,1)\}$

e. $S=\{(1,-2,2,1),(1,2,3,5),(-1,3,1,7)\}$

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & 2 & 3 & 5 \\ -1 & 3 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 4 & 1 & 4 \\ 0 & 1 & 3 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 4 & 1 & 4 \\ 0 & 0 & 11/4 & 7 \end{bmatrix}$$

4. For which values of *k* is each set linearly independent in corresponding vector spaces.

a. $S=\{(-1,2,1),(k,4,0),(3,1,1)\}$ b. $S=\{(-1,k,1),(1,1,0),(2,-1,1)\}$

c. $S=\{(k,1,1),(1,k,1),(1,1,k)\}$ d. $S=\{(1,2,1,0),(-2,1,1,-1),(-1,3,2,k)\}$

$$A=\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix} \Rightarrow \det A = k^3-3k+2$$

S đltt $\det A = k^3-3k+2\neq 0$

5. Find all values of m such that the set S is a basis of R^3

a. $S=\{(1,2,1),(m,1,0),(-2,1,1)\}$ b. $S=\{(-1,m,1),(1,1,0),(m,-1,-1)\}$

$$A=\begin{bmatrix} 1 & 2 & 1 \\ m & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \Rightarrow \det A = 1+m+2-2m=3-m$$

Vì S có 3 vectơ nên S là một cơ sở của R^3 khi và chỉ khi S đltt trong R^3

$\Leftrightarrow \det A = 3-m\neq 0$

$\Leftrightarrow m\neq 3$

6. Find a basis for and the dimension of the subspace U

a. $U=\{(2s-t,s,s+t)\mid s,t\in R\}$ b. $U=\{(s-t,s,t,s+t)\mid s,t\in R\}$

$U=\{(2s-t,s,s+t)\mid s,t\in R\}$

$\dim U=2$

$s=0,t=1\Rightarrow u_1=(-1,0,1)$

$s=1,t=0\Rightarrow u_2=(2,1,1)$

c. $U=\{(0,t,-t)\mid t\in R\}$ d. $U=\{(x,y,z)\mid x+y+z=0\}$

e. $U=\{(x,y,z)\mid x+y+z=0, x-y=0\}$ f. $U=span\{(1,2,3),(2,3,4),(3,5,7)\}$

U là không gian nghiệm của hệ phương trình thuần nhất $\begin{cases} x+y+z=0 \\ x-y=0 \end{cases}$ (*)

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{d_2\to d_2-d_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

Hpt (*) $\Leftrightarrow \begin{cases} x+y+z=0 \\ -2y-z=0 \end{cases} \Leftrightarrow \begin{cases} x=-\dfrac{z}{2} \\ y=-\dfrac{z}{2} \end{cases} \Leftrightarrow \begin{cases} x=-\dfrac{a}{2} \\ y=-\dfrac{a}{2} \; (a\in\mathbb{R}) \\ z=a \end{cases}$

Nghiệm của hpt (*) là $U=\left\{\left(-\dfrac{a}{2},-\dfrac{a}{2},a\right),a\in\mathbb{R}\right\}$

$\Rightarrow \dim U = 1$

chọn $a=2$ ta có 1 cơ sở của U là $u=(-1,-1,2)$

g. $U=span\{(1,2,4),(-1,3,4),(2,3,1)\}$ h. $U=span\{(1,2,1,1),(2,1,-1,0),(3,3,0,1)\}$

$$A=\begin{array}{c}\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 0 & 1 \end{bmatrix}\end{array}\begin{array}{l} u_1 \\ u_2 \\ u_3 \end{array} \xrightarrow[d_3\to d_3-3d_1]{d_2\to d_2-2d_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -3 & -2 \\ 0 & -3 & -3 & -2 \end{bmatrix}\begin{array}{l} u_1 \\ u_2 \\ u_3 \end{array}$$

$$\xrightarrow{d_3\to d_3-d_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{l} u_1 \\ u_2 \\ u_3 \end{array}$$

$r(A)=2 \Rightarrow \dim U = 2$

Một cơ sở của U là $\{u_1,u_2\}$

7. Find a basis for and the dimension of the solution space of the homogeneous system of linear equations.

a. $\begin{cases} -x+y+z=0 \\ 3x-y=0 \\ 2x-4y-5z=0 \end{cases}$ b. $\begin{cases} x+2y-4z=0 \\ -3x-6y+12z=0 \end{cases}$ c. $\begin{cases} x+y+z+t=0 \\ 2x+3y+z=0 \\ 3x+4y+2z+t=0 \end{cases}$

8. Find all values of m for which x lies in the subspace spanned by S

a. $x=(-3,2,m)$ and $S=\{(-1,-1,1),(2,-3,-4)\}$

$x=(-3,2,m)\in Span\{(-1,-1,1),(2,-3,-4)\}$

$\Leftrightarrow \exists a,b\in\mathbb{R}:(-3,2,m)=a.(-1,-1,1)+b.(2,-3,-4)$

$$\Leftrightarrow\begin{cases}-a+2b=-3\\ -a-3b=2 \quad has\ solutions\\ a-4b=m\end{cases}$$

$$\Leftrightarrow\begin{cases}a=1\\ b=-1\\ m=5\end{cases}$$

b. $x=(4,5,m)$ and $S=\{(1,-1,1),(2,-3,4)\}$

c. $x=(m+1,5,m)$ and $S=\{(1,1,1),(2,3,1),(3,4,2)\}$

d. $x=(3,5,7,m)$ and $S=\{(1,1,1,-1),(1,2,3,1),(2,3,4,0)\}$

9. Find the dimension of the subspace

$U=span\{(-2,\ 0,\ 3),(1,\ 2,\ -1),(-2,\ 8,\ 5),(-1,\ 2,\ 2)\}$

10. Let $A=\begin{pmatrix}1&2&2&-1\\3&6&5&0\\2&2&1&2\end{pmatrix}$. Find $\dim(\operatorname{col}A)$ and $\dim(\operatorname{row}A)$

$colA=Span\{(1,3,2),\ (2,6,2),\ (2,5,1),\ (-1,0,2)\}$

$rowA=Span\{(1,2,2,-1),\ (3,6,5,0),\ (2,2,1,2)\}$

$\dim(colA)=\dim(rowA)=rank(A)$

11. Which of the following are subspaces of R^3 ?

$(i)\quad \{(2+a,b-a,b)\mid a,b\in R\}$

$(ii)\quad \{(a+b,a,b)\mid a,b\in R\}$

$(iii)\quad \{(2a+b,0,ab)\mid a,b\in R\}$

12. Let $u=(1,-3,-2),v=(-1,1,0)$ and $w=(1,2,-3)$. Compute $\|u-v+w\|$

13. Let $u,v\in\mathbb{R}^3$ such that $\|u\|=3,\|v\|=4$ and $u\bullet v=-2$. Find

a. $\|u+v\|$ b. $\|2u+3v\|$

$$\|u+v\|^2 = (u+v)\bullet(u+v) = u\bullet u + u\bullet v + v\bullet u + v\bullet v$$
$$= \|u\|^2 + 2.u\bullet v + \|v\|^2 = 9 - 2.2 + 16 = 21$$
$$\Rightarrow \|u+v\| = \sqrt{21}$$

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