Chapter 3

Determinants and Diagonalization

OBJECTIVES

- Determinants of nxn matrices
- Properties:
- Determinants and inverse of a matrix
  - An nxn matrix has an inverse if and only if $\det(A) \neq 0$
  - $A^{-1} = \mathrm{adj}(A)/\det A$
- Diagonalization
  - Characteristic polynomial
  - Eigenvalues
  - Eigenvectors

Determinant of a square matrix

- Determinant of an nxn matrix A are denoted by det(A) or |A|.
- For 2 x 2 matrices:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Or

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- 3 x 3matrices:

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \end{array}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

det(A) =

$+$a.det$\begin{pmatrix} e & f \\ h & i \end{pmatrix}$ $-$ b.det$\begin{pmatrix} d & f \\ g & i \end{pmatrix}$ $+$ c.det$\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

= aei – afh – (bdi – bgf) + cdh – cge

Example

- Find the determinant of the matrix

$$A = \begin{pmatrix} -3 & -2 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & -3 \end{pmatrix}$$

The determinant of 3x3 matrix (only)

$$\begin{array}{ccc} col1 & col2 & col3 \\ a & b & c \\ d & e & f \\ g & h & i \end{array}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

$Find\ \det(A)$,

$$where\ A = \begin{pmatrix} -3 & -2 & 1 \\ 2 & 0 & -1 \\ 2 & 1 & -3 \end{pmatrix} \begin{matrix} -3 & -2 \\ 2 & 0 \\ 2 & 1 \end{matrix}$$

$$\det A = aei + bfg + cdh - ceg - afh - bdi$$

Definition

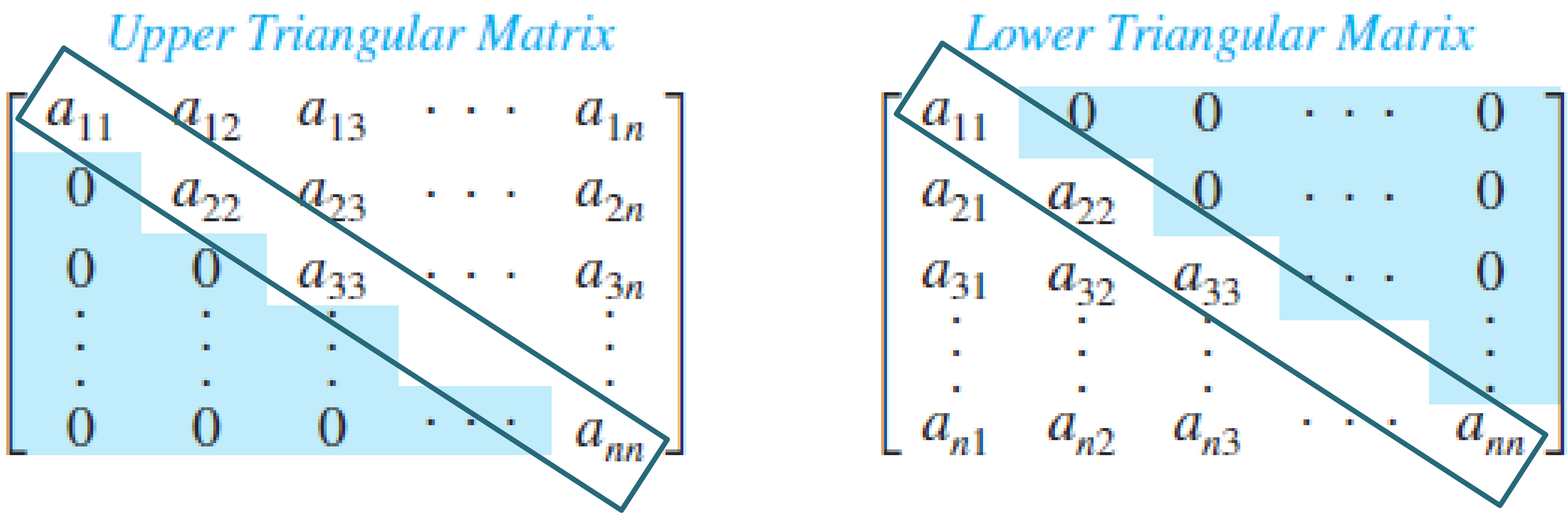
$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \underset{a_{11}}{a}.\underbrace{(+).\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{(1,1)-cofactor} + \underset{a_{12}}{b}.\underbrace{(-).\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{(1,2)-cofactor} + \underset{a_{13}}{c}.\underbrace{(+).\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{(1,3)-cofactor}$$

If A is an mxm matrix then the **determinant** of A is defined by

- **detA**$=a_{i1}c_{i1}(A)+a_{i2}c_{i2}(A)+\ldots+a_{im}c_{im}(A)$
- or **detA**$= a_{1j}c_{1j}(A)+a_{2j}c_{2j}(A)+\ldots+a_{mj}c_{mj}(A)$

$$\begin{vmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ 6 \\ 7 \\ 1 \end{matrix} & \begin{matrix} -1 \\ 4 \\ -1 \\ 8 \end{matrix} & \begin{matrix} 5 \\ 0 \\ 0 \\ 2 \end{matrix} \end{vmatrix} = 1\begin{vmatrix} 6 & 4 & 0 \\ 7 & -1 & 0 \\ 1 & 8 & 2 \end{vmatrix} = -68$$

The determinant of triangular matrices



- Determinant = $a_{11}.a_{22}\ldots a_{nn}$

Examples

- Find the determinants

$$|A| = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{vmatrix} = -1.3.(-2) = 6,$$

$$\begin{vmatrix} 0 & 3 & 2 \\ -1 & 2 & 3 \\ 0 & 0 & -2 \end{vmatrix} = -6$$ // from A, interchange row 1 and row 2

$$\text{and} \begin{vmatrix} 2 & -4 & -6 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{vmatrix} = -12$$ // from A, -2.(row 1)

Examples

- Find the determinants

$$\begin{vmatrix} -1 & 2 & -2 \\ 0 & 5 & 1 \\ 2 & -4 & 5 \end{vmatrix} = -5$$

And

$$\begin{vmatrix} -1 & 2 & -2 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -5$$

The second matrix is obtained from the first matrix by (2*row1 + row3), they have the same determinants.

Properties

Example

If $det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 5$, find $det\begin{bmatrix} a+2x & b+2y & c+2z \\ x+p & y+q & z+r \\ 3p & 3q & 3r \end{bmatrix}$.

Examples

$$\begin{vmatrix} 0 & 2 & -1 & 9 \\ 2 & 2 & -4 & 6 \\ 3 & 2 & -2 & 1 \\ -3 & 4 & 2 & 0 \end{vmatrix} \overset{r_1 \leftrightarrow r_2}{=} - \begin{vmatrix} 2 & 2 & -4 & 6 \\ 0 & 2 & -1 & 9 \\ 3 & 2 & -2 & 1 \\ -3 & 4 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & 2 & -1 & 9 \\ 3 & 2 & -2 & 1 \\ -3 & 4 & 2 & 0 \end{vmatrix} \overset{\substack{-3r_1+r_3 \\ 3r_1+r_4}}{=} -2 \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & 2 & -1 & 9 \\ 0 & -1 & 4 & -8 \\ 0 & 7 & -4 & 9 \end{vmatrix}$$

$$\overset{r_2 \leftrightarrow r_3}{=} -(-2) \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 2 & -1 & 9 \\ 0 & 7 & -4 & 9 \end{vmatrix} \overset{\substack{2r_2+r_3 \\ 7r_2+r_4}}{=} 2 \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 0 & 7 & -7 \\ 0 & 0 & 24 & -47 \end{vmatrix} = 2.7 \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 24 & -47 \end{vmatrix}$$

$$\overset{-24r_3+r_4}{=} 2.7 \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 4 & -8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -23 \end{vmatrix} = 2.7.1.(-1).1.(-23)$$

Do yourself: Find $\begin{vmatrix} 1 & 2 & -1 & 5 \\ 0 & 6 & 4 & 3 \\ 1 & -3 & 4 & 6 \\ 1 & 2 & 4 & 5 \end{vmatrix}$

Diagonal matrices

- An nxn matrix is called diagonal matrix if all its entries off the main diagonal are zeros

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$$

- For example

$$diag(3, -2, 1, 4) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Diagonalization

- Diagonalizing a matrix A is to find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix $P^{-1}AP=\mathrm{diag}(\lambda_1, \lambda_2,\ldots, \lambda_n)$

For example,

- Given a matrix A, $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$
- Find a matrix P, $P = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}$
- Compute $P^{-1}AP$,

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

How to find P?

- Find $c_A(x)=\det(xI-A)$: $\begin{vmatrix} x-2 & 4 \\ 1 & x+1 \end{vmatrix} = (x-2)(x+1)-4 = x^2-x-6$ Đa thức đặc trưng (characteristic polynomial)
- $c_A(x)=0 \Leftrightarrow x=3 \vee x=-2$ Các giá trị riêng (eigenvalues) của A
- $\boxed{x=3}$ solve the system $(3I-A)X=0 \sim \begin{cases} x+4y=0 \\ x+4y=0 \end{cases}$

$\Leftrightarrow \begin{cases} y=t \\ x=-4t \end{cases} \Leftrightarrow X=\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ Nếu t≠0 thì X=(-4t,t) được gọi là véc tơ riêng (eigenvectors) ứng với giá trị riêng x=3

- $\boxed{x=-2}$ solve the system $(-2I-A)X=0 \sim \begin{cases} -4x+4y=0 \\ x-y=0 \end{cases}$

$\Leftrightarrow \begin{cases} y=t \\ x=t \end{cases} \Leftrightarrow X=\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Nếu t≠0 thì X=(t,t) được gọi là véc tơ riêng ứng với x=-2

Choose P= $\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} \boxed{3} & 0 \\ 0 & \boxed{-2} \end{bmatrix}$

Relationship between eigenvalues and eigenvectors

λ: eigenvalue (a number)

X: λ-eigenvector (remember: vector X≠0)

(λI-A)X=0 ⇒ AX=λX

$P=\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, P=\begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix}, P=\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}, P=\begin{bmatrix} -1 & -4 \\ -1 & 1 \end{bmatrix}$,...are allowed. In case P= $\begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}, P^{-1}AP=\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

Example

Find the eigenvalues ang eigenvectors and then *diagonalize* the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

The characteristic polynomial of A is

$$c_A(x) = \begin{vmatrix} x-1 & -1 \\ -2 & x-2 \end{vmatrix} = (x-1)(x-2) - 2 = x(x-3) = 0 \Leftrightarrow x = 0 \vee x = 3$$

$x = 0, 3$ are $\boxed{\text{eigenvalues}}$

$x = 0$: Solve the system $(0I\text{-}A)X=0$

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow X = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$x = 3$: Solve the system $(3I\text{-}A)X=0$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow X = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

When is A diagonalizable?

Theorem

A is diagonalizable iff every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors, that is, iff the general solution of the system $(\lambda I-A)X=0$ has exactly m parameters.

For example,

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \text{ is not diagonalizable.}$$

$$\text{In fact, } c_A(x)=\det(xI-A)=\begin{vmatrix} x & -1 \\ 1 & x-2 \end{vmatrix}=x(x-2)+1=x^2-2x+1=(x-1)^2=0 \Leftrightarrow x=1$$

$$x=\boxed{1\text{ (multiplicity 2):}}\text{ solve the system } (1I-A)X=0 \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$X=t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\text{one parameter}} \Rightarrow \text{not diagonalizable}$$

When is A diagonalizable?

$$A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 2 & 0 & 0\end{bmatrix} \text{ is not diagonalizable.}$$

In fact, $c_A(x)=\det(xI-A)=\begin{vmatrix}x & -1 & -1\\ -1 & x & -1\\ -2 & 0 & x\end{vmatrix}=x^3-3x-2=(x+1)^2(x-2)=0 \Leftrightarrow x=-1 \vee x=2$

$x=\boxed{-1 \text{ (multiplicity 2)}}$: solve the system $(-1I-A)X=0 \sim \left[\begin{array}{ccc|c}-1 & -1 & -1 & 0\\ -1 & -1 & -1 & 0\\ -2 & 0 & -1 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 2 & 1 & 0\end{array}\right]$

$\Rightarrow$ $\boxed{\text{one parameter}}$ $\Rightarrow$ not diagonalizable

When is A diagonalizable?

$$A=\begin{bmatrix}0&1&1\\1&0&1\\2&0&0\end{bmatrix} \text{ is not diagonalizable.}$$

In fact, $c_A(x)=\det(xI-A)=\begin{vmatrix}x&-1&-1\\-1&x&-1\\-2&0&x\end{vmatrix}=x^3-3x-2=(x+1)^2(x-2)=0 \Leftrightarrow x=-1 \vee x=2$

$x=$ $\boxed{-1 \text{ (multiplicity 2)}}$: solve the system $(-1I-A)X=0 \sim \left[\begin{array}{ccc|c}-1&-1&-1&0\\-1&-1&-1&0\\-2&0&-1&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&2&1&0\end{array}\right]$

$\Rightarrow$ $\boxed{\text{one parameter}}$ $\Rightarrow$ not diagonalizable

(i,j)-cofactor

- **(i,j)-cofactor** of a matrix $[a_{ij}]$

is defined by

$$c_{ij} = (-1)^{i+j}\det(A_{ij}),$$

where A_{ij} is the matrix obtained from A by deleting row i^{th} and column j^{th}

For example, given

$$A = \begin{array}{cl} \begin{pmatrix} -2 & 3 & 1 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} & \begin{array}{l} \\ \text{row 2} \\ \\ \end{array} \\ \text{column 3} & \end{array}$$

Then, $c_{23} = (-1)^{2+3}\det\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$

$= -1.(-1) = 1$

Do yourself. Find all cofactors c_{ij} of the matrix A

How to find A^{-1}?

- An nxn matrix A is **invertible** if and only if ***$det(A) \neq 0$***

Furthermore,

$$A^{-1} = \frac{1}{\det(A)} adj(A),$$

where the adjugate matrix adj(A) is defined as

$$\mathrm{adj(A)} = [(\mathrm{i,j}) - \mathrm{cofactors}]^{\mathrm{Transpose}}$$

Adjugate matrix

- The ***adjugate*** matrix of A is the matrix

$$adjA = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

- For example,

$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$. We have $c_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3$, $c_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3$,

$$\boxed{c_{11} = 3, \quad c_{12} = -3, \quad c_{13} = -6}$$

$$c_{21} = 2, \quad c_{22} = 1, \quad c_{23} = -4,$$

$$c_{31} = 2, \quad c_{32} = 1, \quad c_{33} = 5$$

$$\Rightarrow adjA = \begin{bmatrix} 3 & 2 & 2 \\ -3 & 1 & 1 \\ -6 & -4 & 5 \end{bmatrix}$$

Theorem of Adjugate Formula

If A is any square matrix, then

- A(adjA)=(detA)**I**
- In particular, if **detA≠0** then A is **invertible** and

$$A^{-1} = \frac{1}{\det A} adjA$$

- For example,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det A = 2 \text{ and adjA=} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

SUMMARY

- Determinants of nxn matrices
- Properties:
    - $\det(AB) = \det(A)\det(B)$
    - $\det(cA) = c^n\det(A)$
    - $\det(A^T) = \det(A)$
    - $\det(A^{-1}) = 1/\det(A)$
    - Determinants and elementary operators
- Determinants and inverse of a matrix
    - An nxn matrix has an inverse if and only if $\det(A) \neq 0$
    - $A^{-1} = \mathrm{adj}(A)/\det A$
- Diagonalization
    - Characteristic polynomial
    - Eigenvalues
    - Eigenvectors

THANKS