LA-Chapter 2

Matrix algebra

OUR GOAL

- Matrices
- Special matrices
- Operations on matrices:
    - Addition
    - Difference
    - Transposition
    - Scalar multiplication
    - Matrix multiplication
- Inverse of a square matrix
- Matrices and linear systems of equations
- Matrices and linear transformations

Definition

- An mxn matrix is *rectangular array* of numbers

$$\left.\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}}\right\} m \text{ rows}$$

- (m x n): size of the matrix *m by n*
- A = [a_{ij}] // a_{ij} is called (i, j)-entry

Matrices - examples

- A 2x3 matrix // 2 rows, 3 columns
- Read: **two by three** matrix

$$A = \begin{pmatrix} 7 & -3 & 1/2 \\ 3 & -5 & 0 \end{pmatrix}$$

(1,3)-entry
a[1,3] = 1/2
$a_{13} = 1/2$

$$\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix}$$

3 x 3 matrix,
a **square** matrix

$$\begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix}$$

3 x 1 matrix
column matrix

Special matrices

- **Zero** matrix 0_{mxn}

$$0_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Main diagonal** of a matrix

$$\begin{bmatrix} 3 & -1 & 7 \\ 0 & 2 & 3 \\ -2 & 4 & -1 \end{bmatrix}, \begin{bmatrix} -4 & 1 & 0 \\ -2 & 3 & 5 \end{bmatrix}$$

Identity matrices

Identity matrix: square matrix $[a_{ij}]$ where $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular matrices, diagonal matrices

Upper triangular matrix $\begin{bmatrix} 3 & 13 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Lower triangular matrix $\begin{bmatrix} 3 & 0 & 0 \\ 11 & -1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$

Diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Transpose of a matrix

mother	Bob	Alice	Minh	Nam
Eva	0	1	0	0
Susan	1	0	0	0
Lan	0	0	1	1

son/daughter	Eva	Susan	Lan
Bob	0	1	0
Alice	1	0	0
Minh	0	0	1
Nam	0	0	1

Transpose of a matrix

- The **transpose** of an mxn matrix $[a_{ij}]$ is an nxm matrix $[a_{ji}]$
- Notation: A^T // the transpose of A
- Example

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 0 \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ -1 & 0 \end{bmatrix}$$

Symmetric matrices

- Square matrix $[a_{ij}]$ where $a_{ij} = a_{ji}$

$or\ A^T = A$

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & 3 & 7 \\ 5 & 7 & 4 \end{bmatrix}$$

$$A^T = A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & 3 & 7 \\ 5 & 7 & 4 \end{bmatrix}$$

Matrix Operations

- Addition
- Scalar multiplication
- Matrix multiplication

Addition. Difference
Scalar multiplication

day 1

$$\begin{array}{l} \\ \textit{South stand} \\ \textit{North stand} \\ \textit{West stand} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Peanuts} & \textit{Hot Dogs} & \textit{Soda} \end{array} \\ \begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \end{array}$$

day 2

$$\begin{array}{l} \\ \textit{South stand} \\ \textit{North stand} \\ \textit{West stand} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Peanuts} & \textit{Hot Dogs} & \textit{Soda} \end{array} \\ \begin{bmatrix} 110 & 230 & 280 \\ 300 & 155 & 389 \\ 35 & 117 & 201 \end{bmatrix} \end{array}$$

addition

difference

Scalar multiplication

day 1 + day 2?

day 1 – day 2?

2(day 1)?

Properties

Suppose A, B, C are mxn matrices, k is a number:

1. $A + B = B + A$ // commutative law
2. $A + (B + C) = (A + B) + C$ // associative law
3. $k(A + B) = kA + kB$ // distributive law
4. $(A + B)^T = A^T + B^T$

Matrix multiplication - introduction

	peanuts	soda	hot dogs
group A	8	5	12
group B	15	7	13

selling price	store 1	store 2	store 3	store 4
peanuts	2	2.5	2	2.5
soda	2.5	2	2.75	2
hot dogs	3	3	2.5	3

8x2.5 + 5x2 + 12x3 = 66$

	store 1	store 2	store 3	store 4
group A	64.5	66	59.75	66
group B	86.5	87.5	81.75	90.5

Matrix multiplication

- $A_{m \times n} \cdot B_{n \times p} = C_{m \times p}$ //suitable size

- The entry c_{ij} = (row i of A).(column j of B)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1.\ 3 & \cdot 2\ 4 & 1 & 2 \\ -1 & -2 & -1 & 0 \\ -2 & 0 & 2 & -4 \end{pmatrix}$$

Some Properties

1. $A(B + C) = AB + AC$ //distributive law
2. $A(BC) = (AB)C$ //associative law
3. $(\mathbf{A}B)^T = B^T\mathbf{A}^T$

Note:

- In general, $AB \neq BA$ ➔ Not commutative
- $AB = 0 \not\Rightarrow A = 0$ or $B = 0$
- $AB = AC \not\Rightarrow B = C$

The inverse of a matrix

B is called the **inverse** of an nxn matrix A if

$$\mathbf{AB = BA = I_n}$$

- Denoted by $\mathbf{A^{-1}}$

Example

- $A = \begin{pmatrix} 2 & -3 \\ 1 & -4 \end{pmatrix}$
- $A^{-1} = \dfrac{1}{-5} \begin{pmatrix} -4 & 3 \\ -1 & 2 \end{pmatrix}$

$-4.2 - 3.(-1) = -5$

// determinant of A, denoted by det(A)

The Inversion Algorithm

The Inversion algorithm:

$$[\mathbf{A} \mid I_n] \rightarrow \ldots \rightarrow [I_n \mid \mathbf{A}^{-1}]$$

For example,

$$\left(\begin{array}{ccc|ccc} \overbrace{\begin{matrix}1 & -1 & 2\\ 0 & -1 & 3\\ 0 & 2 & -5\end{matrix}}^{\mathbf{A}} & \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\end{array}\right) \xrightarrow{-r_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 2 & -5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-2r_2 + r_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right) \xrightarrow[3r_3 + r_2]{-2r_3 + r_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -4 & -2 \\ 0 & 1 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix} & \underbrace{\begin{matrix}1 & 1 & 1\\ 0 & 5 & 3\\ 0 & 2 & 1\end{matrix}}_{\mathbf{A}^{-1}}\end{array}\right)$$

Exercise

Find (x, y) such that

$$\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

Linear equation and matrix multiplication

$$\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

A X = B

$$\begin{pmatrix} -2x + y \\ 3x - 2y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$-2x + y = -1$

$3x - 2y = 5$

$AX = B$

$\Leftrightarrow X = A^{-1}B$

$\Leftrightarrow X = \begin{pmatrix} -2 & -1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \end{pmatrix} \Rightarrow x = -3, y = -7$

Matrix and linear transformation

- Example of a transformation

T(x, y) = (x, -y)

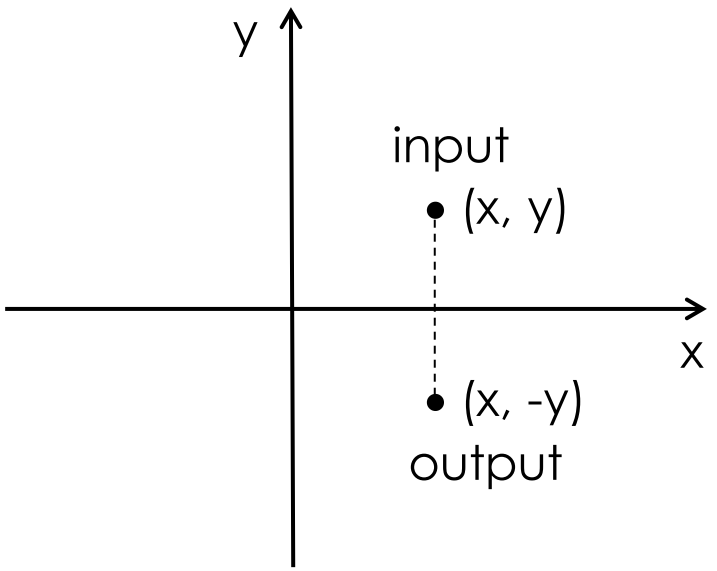
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

input

output

Matrix of
The transformation

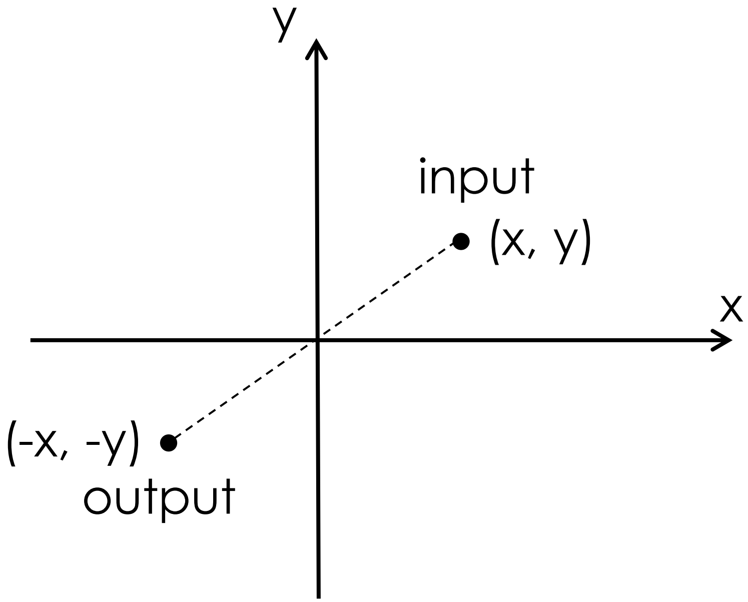


Matrix and linear transformation

- Example of a transformation

$S(x, y) = ?$

Find the matrix of S?



Suppose T is a linear transformation given by the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$$

Find T(1, 2, -3).

$$T(1, 2, -3) = T\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$$

The composition of transformations

Given T(x, y) = (x, y-x)

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y - x \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

And S(x, y) = (x-y, y)

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Find the composite transformation

(T∘S)(x, y) defined by

(T∘S)(x, y) = T(S(x, y))

Matrix of T ∘ S:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

➔ $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix}$

Theorem

If the matrix of T is A, then the matrix of T^{-1} is A^{-1}

Example. Given T(x, y) = (x – y, -x + 2y),
find T^{-1}, the inverse of T.
Solution.

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ -x + 2y \end{pmatrix}$ has the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

➔ T^{-1} has the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

➔ $T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + y \end{pmatrix}$

Note that $(T \circ T^{-1})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

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- Thanks