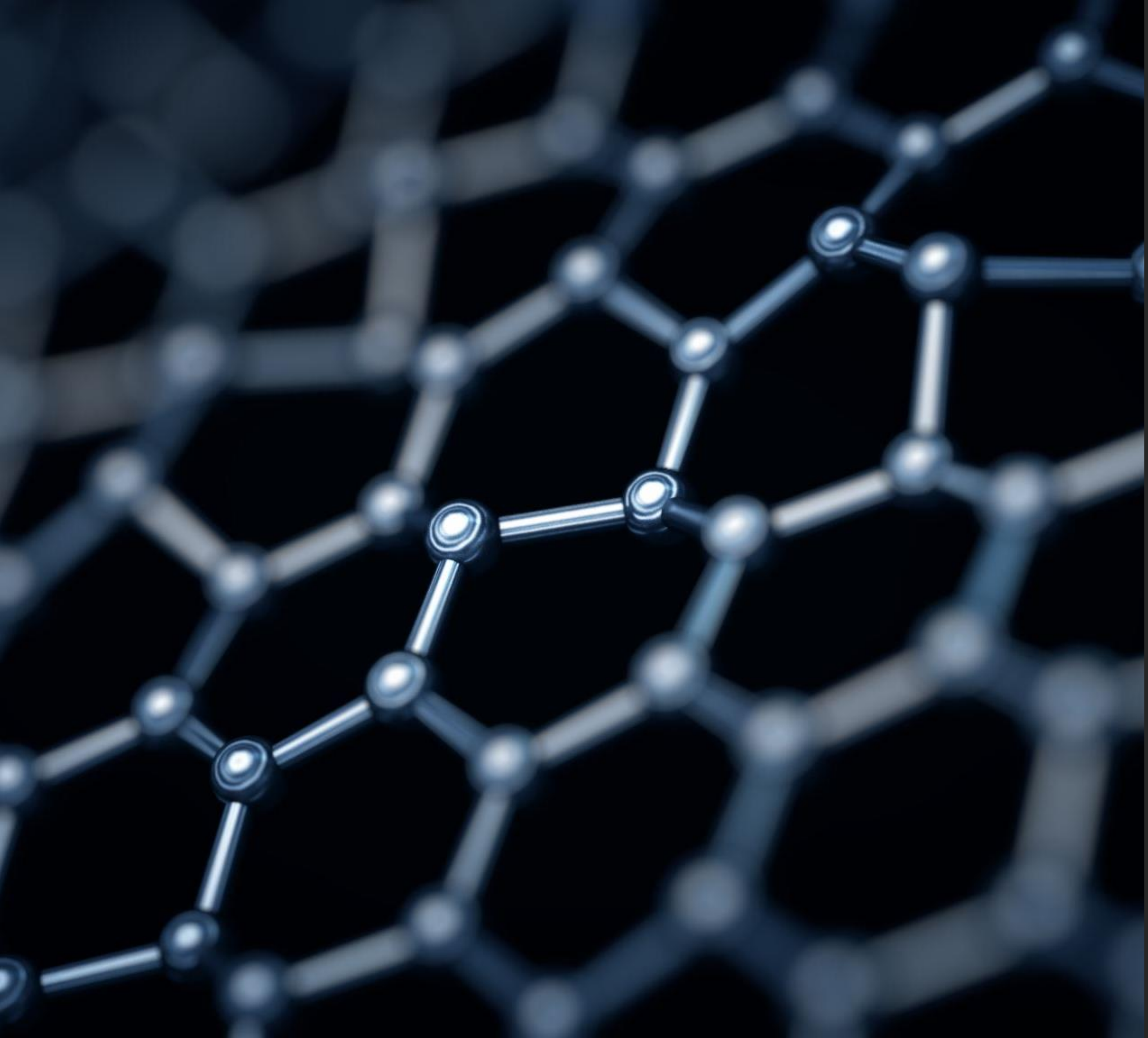


4

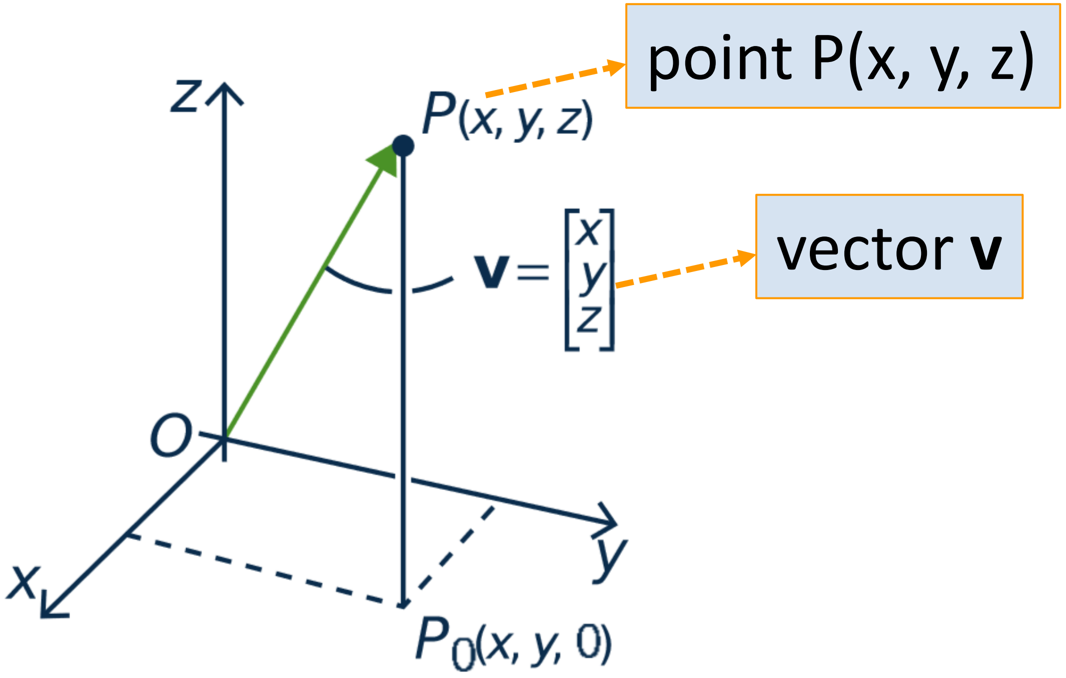
VECTOR GEOMETRY

Chapter Outline

4.1. Vectors and Lines

Vectors in $\mathbb{R}^3$

- The terms ***vector*** and ***point*** are interchangeable.



Vectors in $\mathbb{R}^3$

Length.

If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ıen $\|v\| = \sqrt{x^2 + y^2 + z^2}$

Ex.

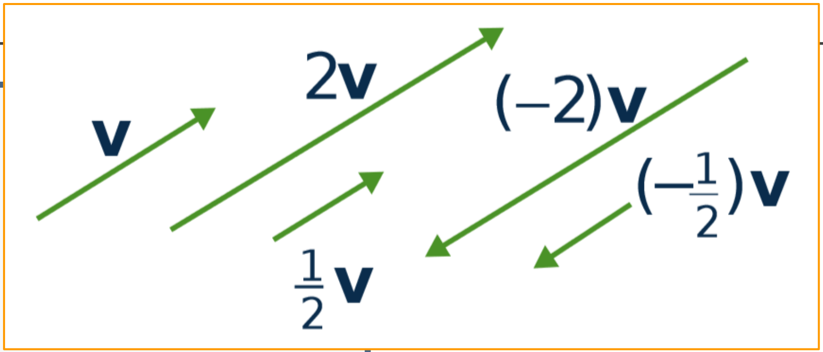
If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ then $\|\mathbf{v}\| = \sqrt{4+1+9} = \sqrt{14}$.

Scalar Multiple Law

If a is a real number and $\mathbf{v} \neq \mathbf{0}$ is a vector then:

1. The length of $a\mathbf{v}$ is $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.

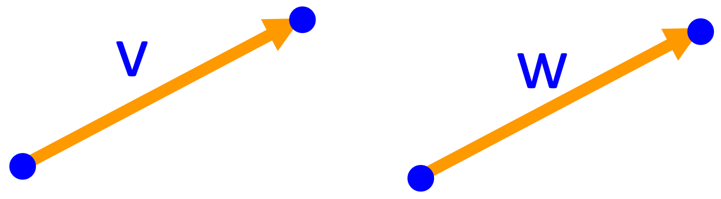
2. If $a\mathbf{v} \neq \mathbf{0}$,[9] the direction of $a\mathbf{v}$ is $\begin{cases} \text{the same as } \mathbf{v} \text{ if } a > 0, \\ \text{opposite to } \mathbf{v} \text{ if } a < 0. \end{cases}$



Equality

Let $v \neq 0$ and $w \neq 0$ be vectors in R^3.

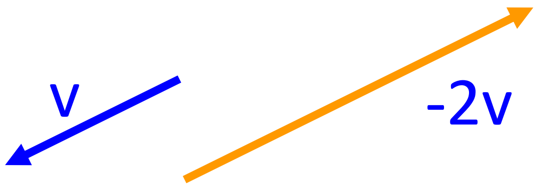
Then **v = w** if and only if v and w have the same direction and the same length.



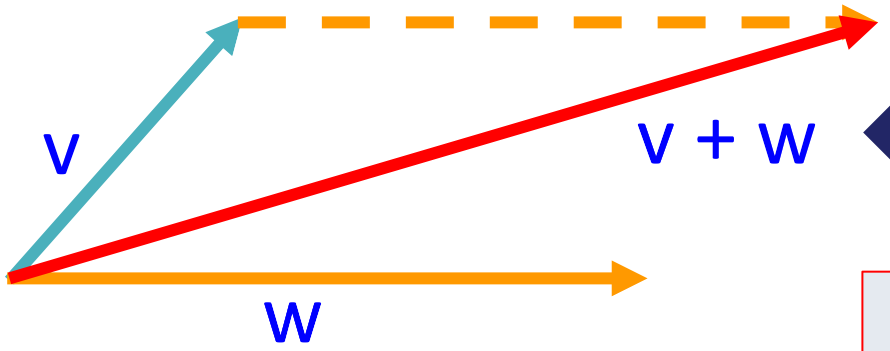
➔ The same geometric vector can be positioned anywhere in space.

Parallel vectors

- Two nonzero vectors are called ***parallel*** if they have the same or opposite direction.
- v and w are parallel $\Leftrightarrow$ v = kw for some scalar k



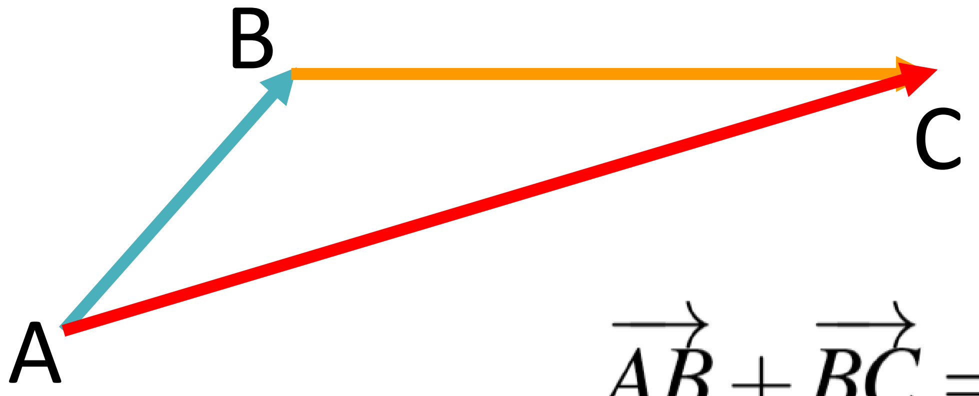
The Parallelogram Law



← first v then w

? $v + w = w + v$

Tip-to-tail rule



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Distance between two points

Theorem 4.1.4

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then:

1. $\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$.

2. The distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Lines in Space

- Given the point P_0 (with vector p_0) and the *direction vector* d $\neq 0$.
- Then *line* parallel to d through the point P_0 is given by:

$$p = p_0 + td,$$

(t is any number)

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}}$$

4.2 Projections and Planes

The Dot Product

Definition 4.4

Given vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* **dot product** $\mathbf{v} \cdot \mathbf{w}$ *is a number defined*

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

The Dot Product

Example 4.2.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot (-1) = -5$.

Properties of dot product

Theorem 4.2.1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v} \cdot \mathbf{w}$ *is a real number.*
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
3. $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$.
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
5. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{v}) = \mathbf{v} \cdot (k\mathbf{w})$ *for all scalars* k.
6. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

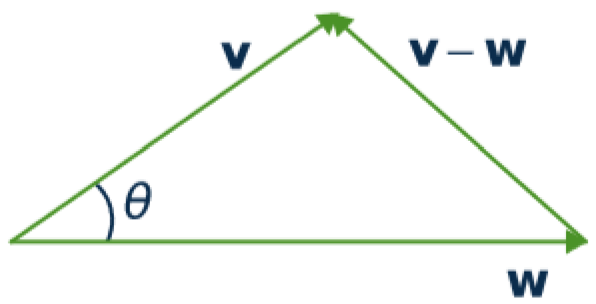
Ex. Find (v + w)•(v – 2w) if $\|v\| = 3$, $\|w\| = 2$, and v•w = -1.

Angles between vectors

Theorem 4.2.2

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be nonzero vectors. If* θ *is the angle between* $\mathbf{v}$ *and* $\mathbf{w}$*, then*

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$



Example.

Compute the angle between $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Orthogonality

- Two vectors v and w are said to be ***orthogonal*** if

$$v \bullet w = 0$$

Example 4.2.4

Show that the points $P(3, -1, 1)$, $Q(4, 1, 4)$, and $R(6, 0, 4)$ are the vertices of a right triangle.

Solution. The vectors along the sides of the triangle are

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \overrightarrow{PR} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \text{ and } \overrightarrow{QR} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Evidently $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 2 - 2 + 0 = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors. This means sides PQ and QR are perpendicular—that is, the angle at Q is a right angle.

Exercise 4.2.3 Find all real numbers x such that:

a. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

b. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors, then $(2\mathbf{u} - 3\mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$ is:

a) -5

b) -1

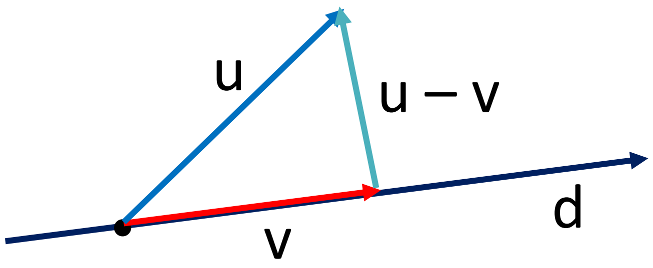
c) 0

d) 1

e) 5

f) not computable with the given data

Projection



1. v ∥ d
2. u − v ⊥ d

Theorem 4.2.4

Let $\mathbf{u}$ *and* $\mathbf{d} \neq \mathbf{0}$ *be vectors.*

1. *The projection of* $\mathbf{u}$ *on* $\mathbf{d}$ *is given by* $\boxed{\operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}.}$
2. *The vector* $\mathbf{u} - \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ *is orthogonal to* $\mathbf{d}$.

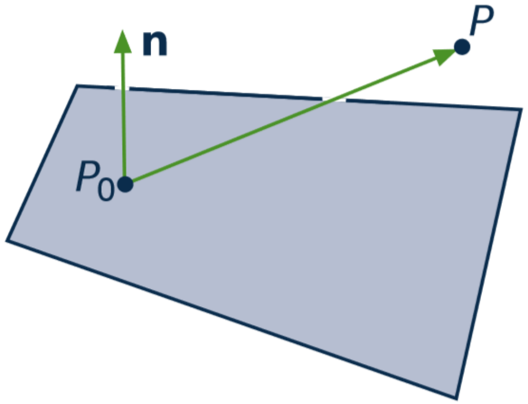
If $\mathbf{u} = (-2, 1, 1)$ and $\mathbf{v} = (1, 0, 1)$, then $\|\text{proj}_{\mathbf{v}}\mathbf{u}\|$ is:

a) $\frac{\sqrt{6}}{6}$ b) 1 c) $\frac{\sqrt{2}}{2}$ d) 0 e) $\frac{1}{2}$

Planes

Definition 4.7

A nonzero vector **n** *is called a* **normal** *for a plane if it is orthogonal to every vector in the plane.*



$$\boxed{\mathbf{n} \cdot \overrightarrow{P_0P} = 0}$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

The Cross Product

Definition 4.8

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* **cross product** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

If $\mathbf{u} = (3, -1, 4)$ and $\mathbf{v} = (-1, 6, -5)$, what is $\mathbf{u} \times \mathbf{v}$?

a) $(17, -10, 11)$

b) $(-19, 11, 17)$

c) $(-3, -6, -20)$

d) $(-19, -11, 17)$

e) $(-17, -10, 11)$

f) $(3, -6, 20)$

Determinant Form of the Cross Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are two vectors, then*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix}$$

where the determinant is expanded along the first column.

If $\mathbf{u} = (3, -1, 4)$ and $\mathbf{v} = (-1, 6, -5)$, what is $\mathbf{u} \times \mathbf{v}$?

a) $(17, -10, 11)$

b) $(-19, 11, 17)$

c) $(-3, -6, -20)$

d) $(-19, -11, 17)$

e) $(-17, -10, 11)$

f) $(3, -6, 20)$

Properties of cross product

Theorem 4.2.5

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

1. $\mathbf{v} \times \mathbf{w}$ *is a vector orthogonal to both* $\mathbf{v}$ *and* $\mathbf{w}$.
2. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are nonzero, then* $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ *if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel.*

An equation for the plane passing through the points $(1, 2, 3)$, $(1, 0, -1)$ and $(4, -2, 0)$ is:

a) $x = 1$

b) $5x + 6y - 3z + 8 = 0$

c) $5x + 6y - 3z = 8$

d) $6x - 5y - 3z + 8 = 0$

e) $2x + 2y - z = 3$

f) $3x - 2y - 3z = 3$

4.3 More on Cross Product

Theorem 4.3.1

If $\mathbf{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *then* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$

Let $\mathbf{u} = (-4, 2, 7)$, $\mathbf{v} = (2, 1, 2)$, $\mathbf{w} = (1, 2, 3)$. Then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ equals:

a) 15 b) -15 c) 16 d) -16 e) 17

Let $\mathbf{u} = (-4, 2, 7)$, $\mathbf{v} = (2, 1, 2)$, $\mathbf{w} = (1, 2, 3)$. Then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ equals:

a) 15 b) -15 c) 16 d) -16 e) 17 __________

Theorem 4.3.3: Lagrange Identity[11]

If $\mathbf{u}$ *and* $\mathbf{v}$ *are any two vectors in* $\mathbb{R}^3$, *then*

$$\|\boldsymbol{u} \times \mathbf{v}\|^2 = \|\boldsymbol{u}\|^2 \|\mathbf{v}\|^2 - (\boldsymbol{u} \cdot \mathbf{v})^2$$

Let $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{2}$ and $\mathbf{u} \cdot \mathbf{v} = 1$. Then $\|\mathbf{u} \times \mathbf{v}\|^2$ is:

a) 0 b) 1 c) 2 d) 3 e) $\sqrt{2}$

Theorem 4.3.4

If $\mathbf{u}$ *and* $\mathbf{v}$ *are two nonzero vectors and* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*, then*

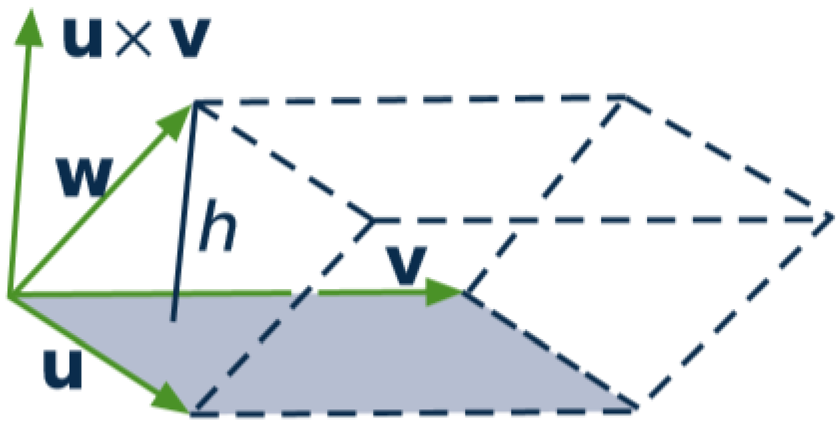
1. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ = *area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.
2. $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The area of a parallelogram determined by the vectors $\mathbf{u} = (1, -1, 0)$ and $\mathbf{v} = (2, -3, 1)$ is:

a) $\sqrt{3}$ b) 3 c) -3 d) $3\sqrt{3}$ e) 27 f) 9

Find the area of the triangle with vertices $A(-1, 5, 0)$, $B(1, 0, 4)$ and $C(1, 4, 0)$.

a) 1 b) 2 c) 3 d) 4 e) 5 f) 6



Find the volume of the parallelepiped determined by the vectors $\mathbf{u} = (1, 1, -1)$, $\mathbf{v} = (2, 0, 1)$ and $\mathbf{w} = (1, -1, 3)$.

a) -2 b) 4 c) 6 d) 8 e) 16 f) 2

The volume of the pyramid with vertices $(0,0,0)$, $(-2,8,14)$, $(-6,7,-3)$ and $(4,0,2)$ is:

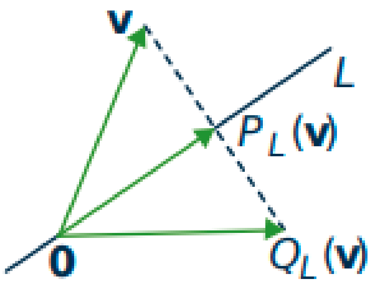
a) 35 b) 45 c) 60 d) 70 e) 75 f) 85

Linear Operators on R^3

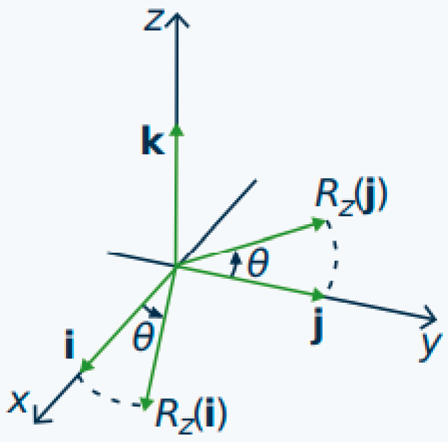
- Reflections and Projections
- Rotations
- Translations

Reflections and Projections

Q_m has matrix $\dfrac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$ and P_m has matrix $\dfrac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$.

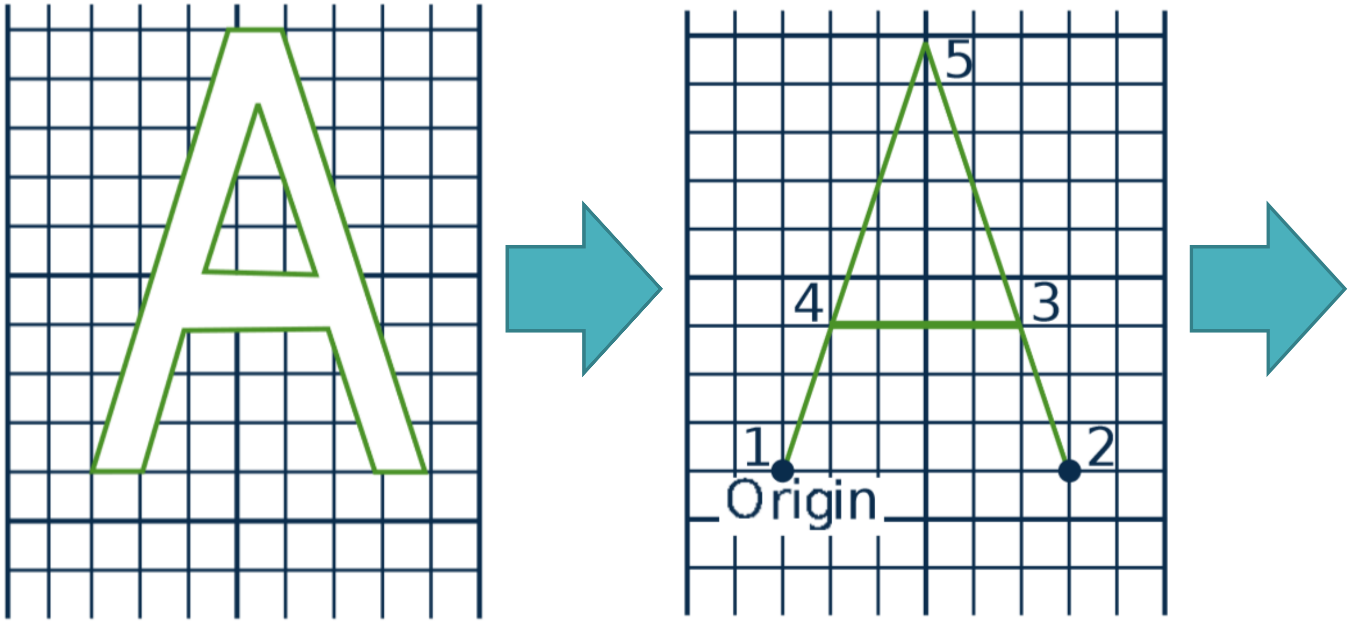


Rotations



$$\begin{bmatrix} R_{z,\theta}(\mathbf{i}) & R_{z,\theta}(\mathbf{j}) & R_{z,\theta}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

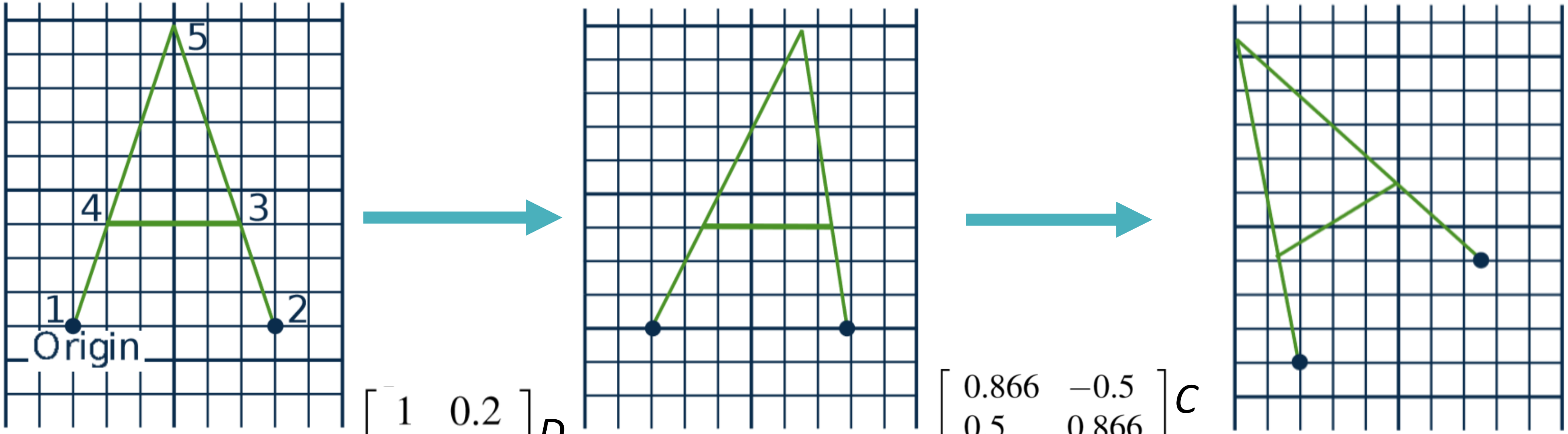
4.5 An application in Computer Graphics



$$\begin{array}{cc} \textit{Vertex} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ D = & \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \end{array}$$

Data matrix D

How to change image?



$$D = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} D} \underbrace{\begin{bmatrix} 0 & 6 & 5.6 & 1.6 & 4.8 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}}_{C} \xrightarrow{\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} C} \begin{bmatrix} 0 & 5.196 & 2.83 & -0.634 & -1.902 \\ 0 & 3 & 5.098 & 3.098 & 9.294 \end{bmatrix}$$

Computer graphics

- Image ➔ matrix D
- Matrices of transformations $A_1, A_2, \ldots, A_n$
- Compute $A_n \ldots A_3 A_2 A_1 D$

n^{th} transformation

3^{rd} transformation

2^{nd} transformation

1^{st} transformation

Data matrix

Matrices of Translations

- Need a clever way to give these matrices (Read yourself in the text book)

Homogeneous coordinate

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous coordinate of v

Then translation by $\mathbf{w} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be achieved by multiplying by a 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+p \\ y+q \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\mathbf{w}}(\mathbf{v}) \\ 1 \end{bmatrix}$$

Example 4.5.1

Rotate the letter A in Figure 4.5.2 through $\frac{\pi}{6}$ about the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Solution.

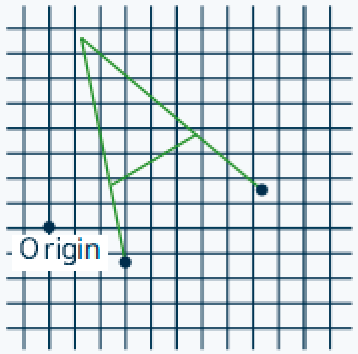


Figure 4.5.6

Using homogenous coordinates for the vertices of the letter results in a data matrix with three rows:

$$K_d = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

If we write $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, the idea is to use a composite of transformations: First translate the letter by $-\mathbf{w}$ so that the point $\mathbf{w}$ moves to the origin, then rotate this translated letter, and then translate it by $\mathbf{w}$ back to its original position. The matrix arithmetic is as follows (remember the order of composition!):

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Summary

- 4.1. Vectors and Lines
- 4.2. Projections and Planes
- 4.3. More on the Cross Product
- 4.4. Linear Operations on $\mathbb{R}^3$
- 4.5. An application to Computer Graphics