4

INTEGRALS

Contents

4.1
Areas and Distances

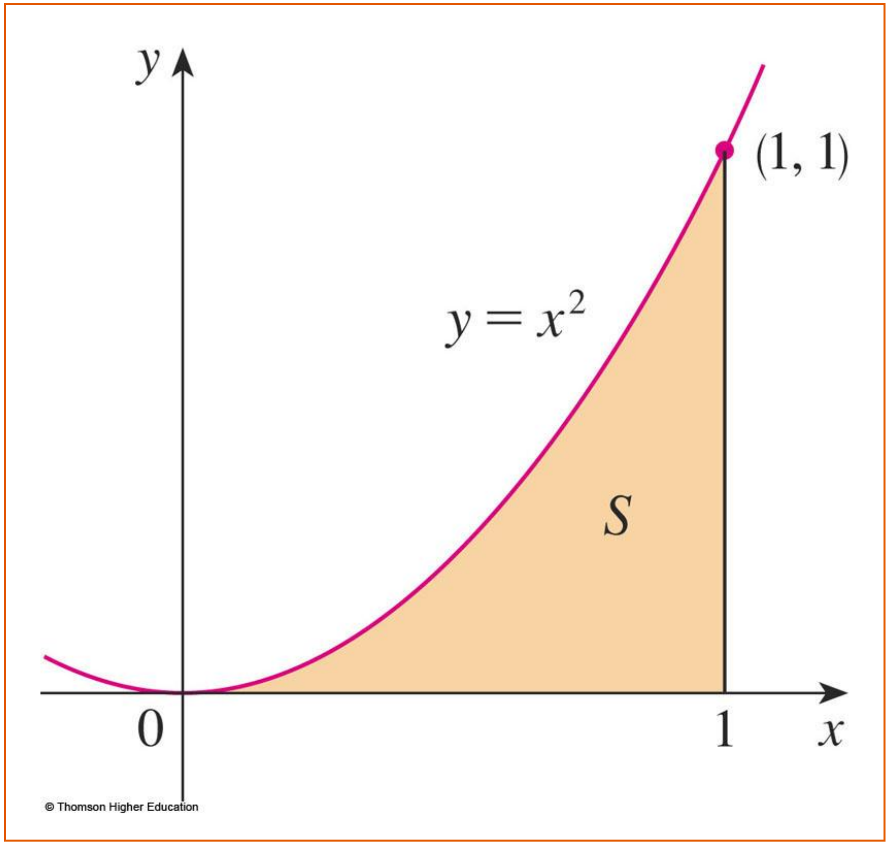
In this section, we will learn that:

We get the same special type of limit in trying to find the area under a curve or a distance traveled.

AREA PROBLEM

We begin by attempting to solve the area problem:

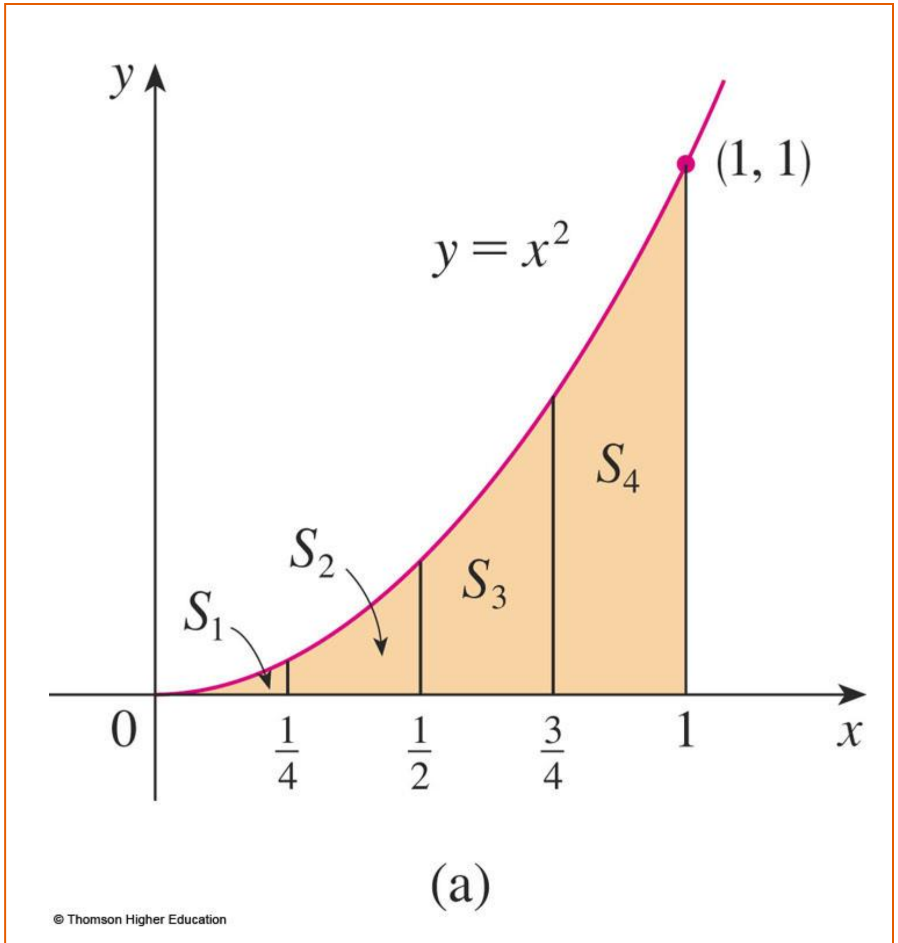
Find the area of the region *S* that lies under the curve $y = f(x)$ from *a* to *b*.



AREA PROBLEM

Example 1

Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = ¼$, $x = ½$, and $x = ¾$.



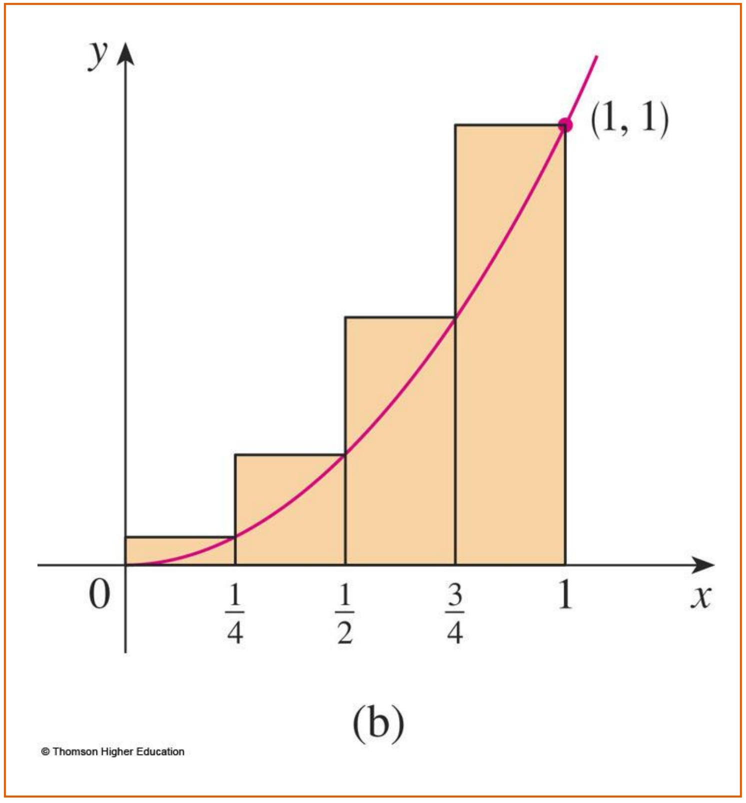
AREA PROBLEM

Example 1

The heights of these rectangles are the values of the function $f(x) = x^2$ at the **right endpoints** of the subintervals

[0, ¼],[¼, ½], [½, ¾],

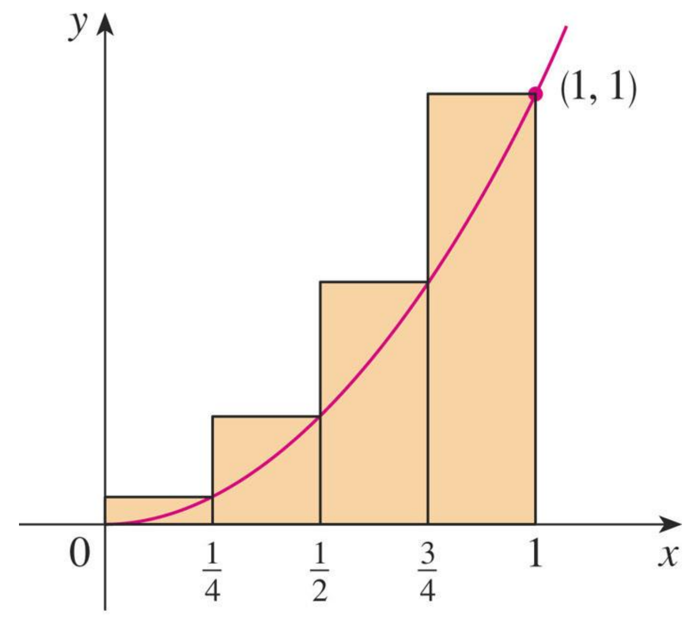
and [¾, 1].



AREA PROBLEM

Example 1

Each rectangle has width ¼ and the heights are $(\frac{1}{4})^2$, $(\frac{1}{2})^2$, $(\frac{3}{4})^2$, and 1^2.



(b)

$$R_4 = \tfrac{1}{4}\cdot\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4}\cdot 1^2$$

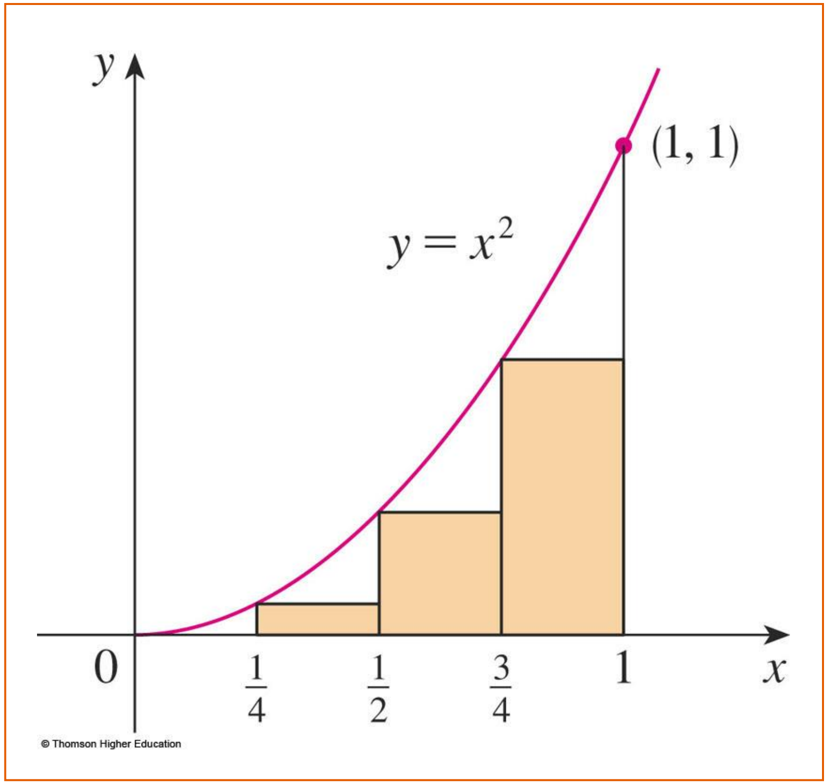
$$= \tfrac{15}{32}$$

$$= 0.46875$$

AREA PROBLEM

Example 1

Here, the heights are the values of *f* at the **left endpoints** of the subintervals.



$$L_4 = \tfrac{1}{4}\cdot 0^2 + \tfrac{1}{4}\cdot\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\cdot\left(\tfrac{3}{4}\right)^2$$

$$= \frac{7}{32}$$

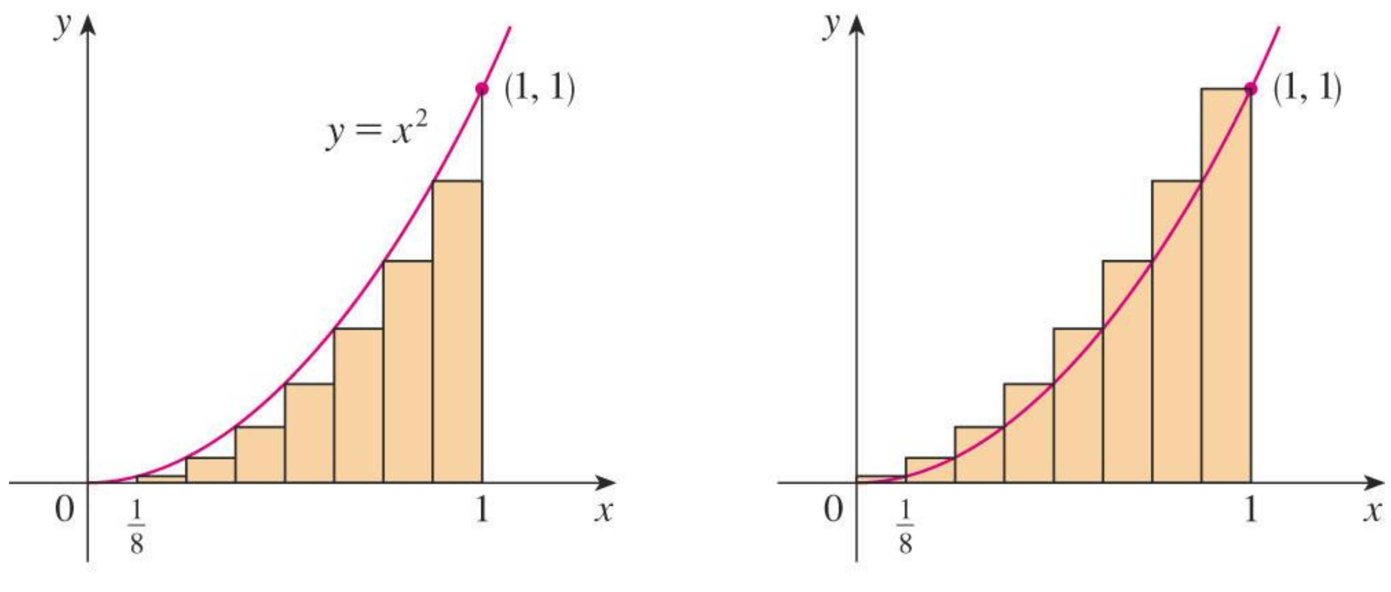
$$= 0.21875$$

AREA PROBLEM

Example 1

The figure shows what happens when we divide the region *S* into eight strips of equal width.

$$0.2734375 < A < 0.3984375$$



(a) Using left endpoints

(b) Using right endpoints

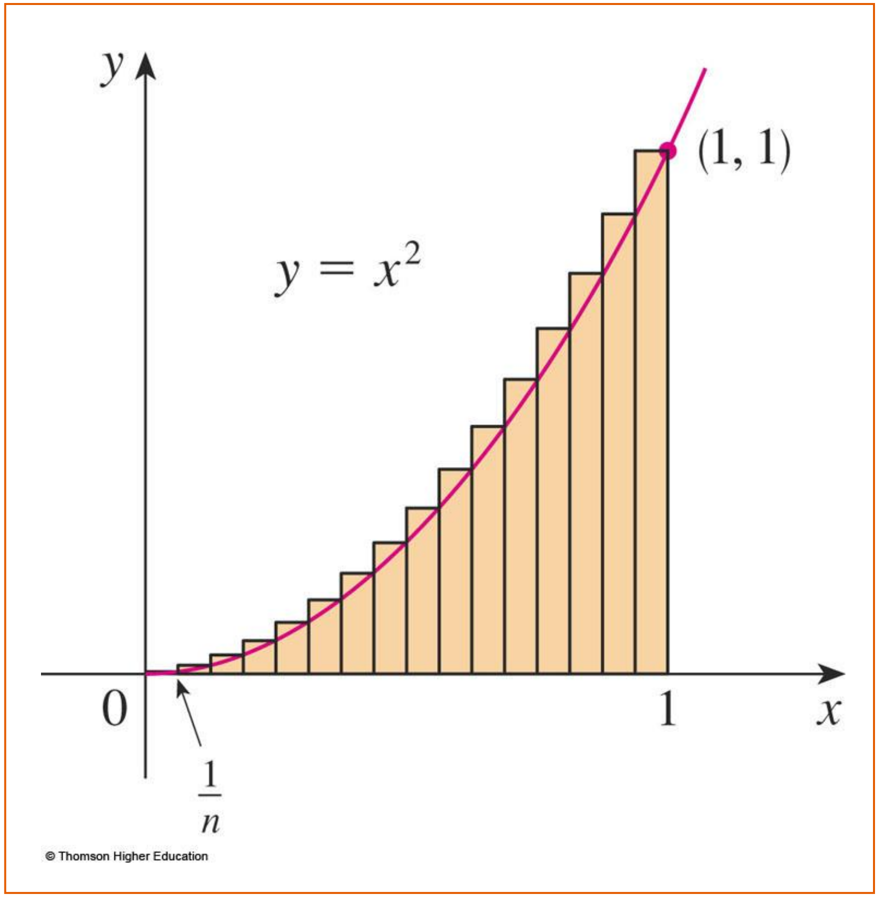
AREA PROBLEM

Example 2

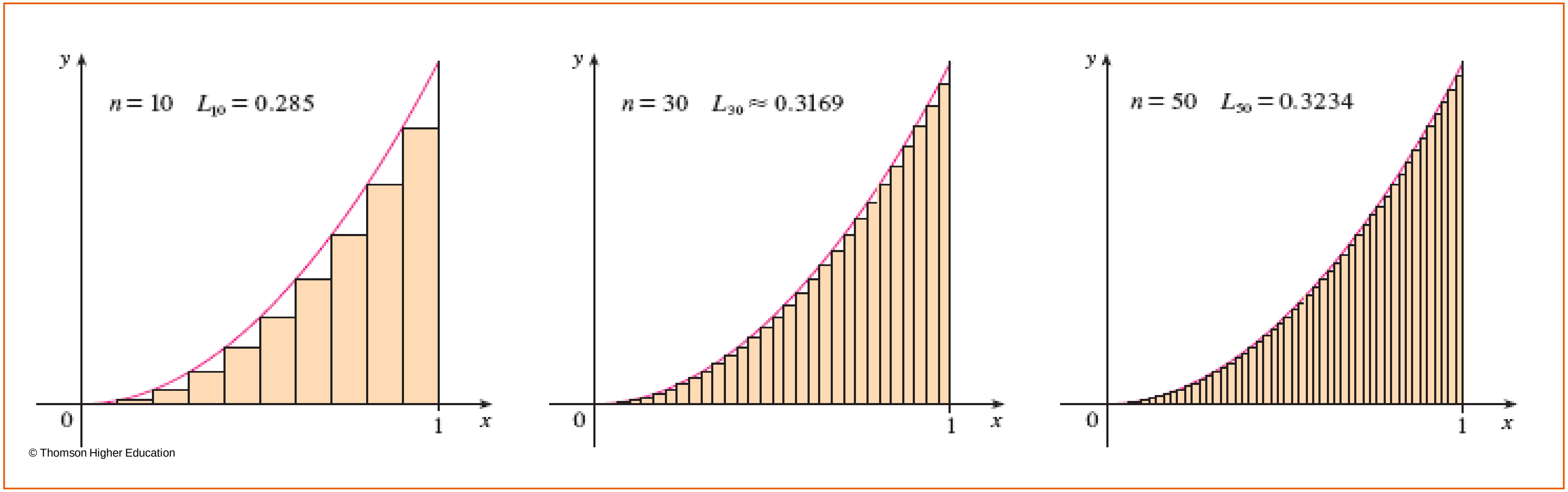
R_n is the sum of the areas of the n rectangles.

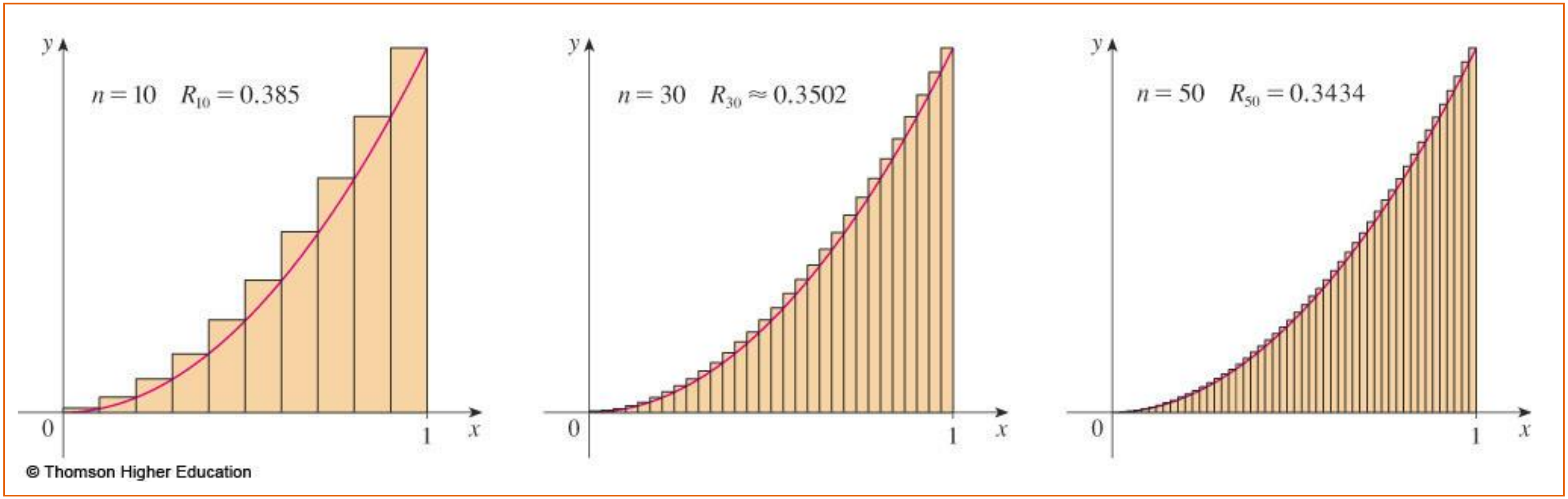
- Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$.

- That is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$.



AREA PROBLEM



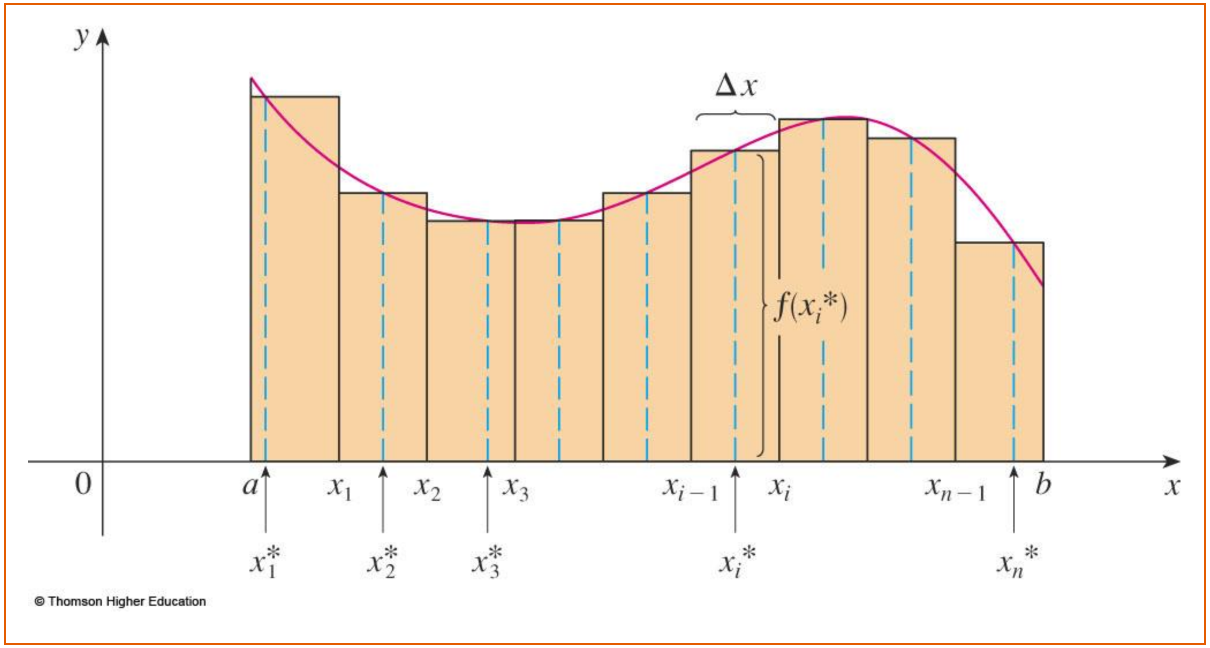


AREA PROBLEM

The figure shows approximating rectangles when the sample points are not chosen to be endpoints.

x_i* in the i th subinterval $[x_{i-1}, x_i]$

(the sample points)



$$A = \lim_{n\to\infty}[\,f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

AREA PROBLEM

Hence,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum.

DEFINITE INTEGRAL

Definition 2

Then, the definite integral of *f* from *a* to *b* is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

provided that this limit exists.

☞ If it does exist, we say *f* is integrable on [*a*, *b*].

☞ The symbol $\int$ was introduced by Leibniz and is called an integral sign.

DEFINITE INTEGRAL **Note 2**

The definite integral $\int_a^b f(x)dx$ is a number.

It does not depend on *x.*

In fact, we could use any letter in place of *x* without changing the value of the integral:

$$\boxed{\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr}$$

MIDPOINT RULE **Example 5**

Use the Midpoint Rule with $n = 5$ to approximate

$$\int_1^2 \frac{1}{x}\,dx$$

- The endpoints of the five subintervals are: 1, 1.2, 1.4, 1.6, 1.8, 2.0
- So, the midpoints are: 1.1, 1.3, 1.5, 1.7, 1.9

MIDPOINT RULE **Example 5**

– The width of the subintervals is:

$\Delta x = (2 - 1)/5 = 1/5$

– So, the Midpoint Rule gives:

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

example

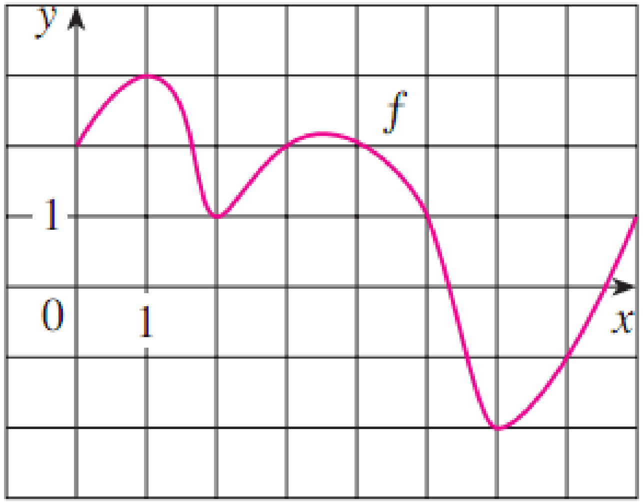
Estimate the area under the graph of $y=x^2 - x$ from $x=0$ to 4 using four subintervals and left endpoints.
8
20
12
None of the others

example

Find the Riemann sum for $f(x)=x^2$,$0 \le x \le 1$, with four equal subintervals, taking the sample points to be left endpoints.	
a	0.21873
b	0.21875
c	0.21874
d	0.33333

Do yourself

The graph of a function f is given. Estimate $\int_0^8 f(x)\,dx$ using four subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



Do yourself

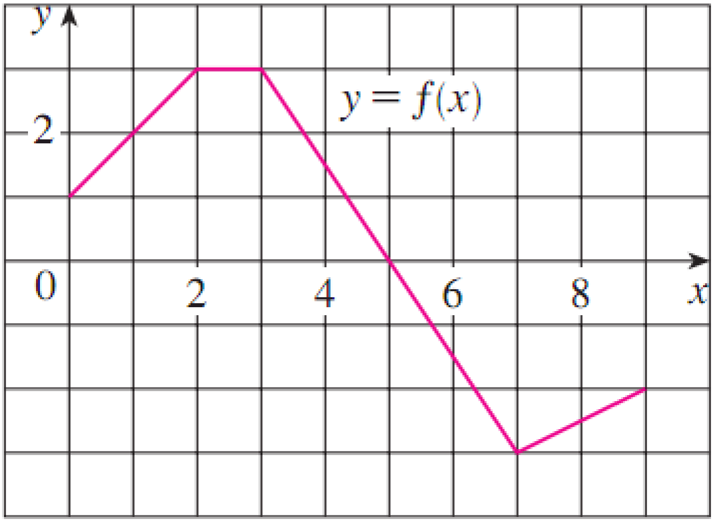
The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 f(x)\,dx$

(b) $\int_0^5 f(x)\,dx$

(c) $\int_5^7 f(x)\,dx$

(d) $\int_0^9 f(x)\,dx$



PROPERTIES OF THE INTEGRAL

We assume *f* and *g* are continuous functions.

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

COMPARISON PROPERTIES OF THE INTEGRAL

These properties, in which we compare sizes of functions and sizes of integrals, are true only if *a* ≤ *b*.

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

4.4

The Fundamental Theorem of Calculus

In this section, we will learn about:

The Fundamental Theorem of Calculus and its significance.

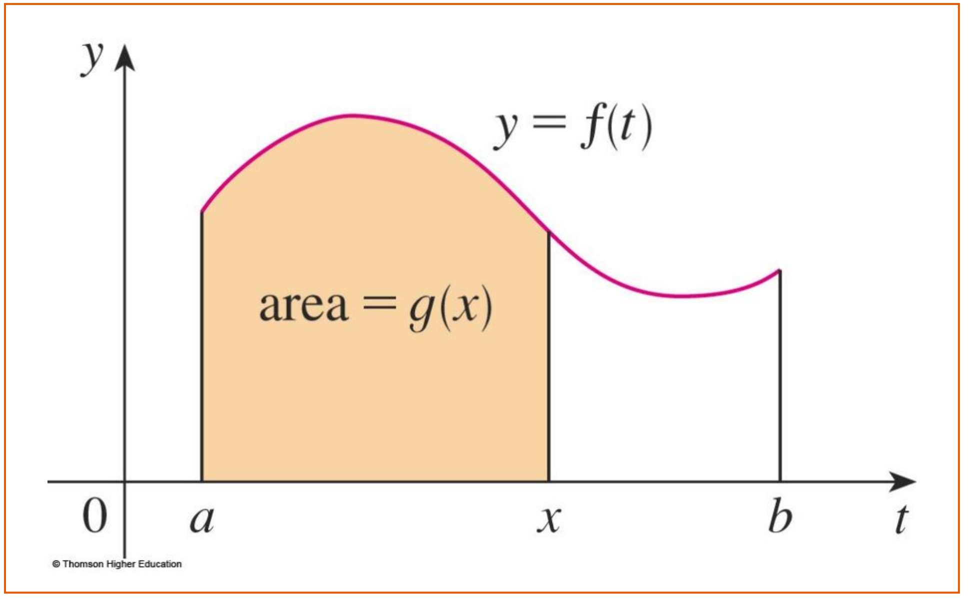
FTC

Equation 1

The first part of the FTC deals with functions defined by an equation of the form:

$$g(x) = \int_a^x f(t)\,dt$$

where *f* is a continuous function on [*a*, *b*] and *x* varies between *a* and *b*.



FTC1

If *f* is continuous on [*a*, *b*], then the function *g* defined by

$$\boxed{g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b}$$

is continuous on [*a*, *b*] and differentiable on (*a*, *b*), and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write the FTC1 as

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

when *f* is continuous.

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f(u(x)) - v'(x)f(v(x))$$

If $\quad g(x) = \int_1^{x^2} (t^2 - t)dt \quad ,1 \le x$

Find g'(x)

$$g(x) = \int_1^{x^2} (t^2 - t)dt$$

$$g'(x) = u'(x)f(u(x)) = 2x(x^4 - x^2), u(x) = x^2$$

example

Find $$\frac{d}{dx}\int_{3}^{1+x^2} \ln t\,dt$$
$2x\ln(1+x^2)$
$2x/(1+x^2)$
$\ln(1+x^2)-\ln 3$
None of the others.

example

Suppose $g(x)=\int_{1}^{x^2}\sin(t-1)dt$ Find g'(x).	
a	$g'(x)=\sin(x-1)$
b	$g'(x)=\sin(x^2-1)$
c	$g'(x)=\cos(x-1)$
d	$g'(x)=2x\cos(x^2-1)$
e	$g'(x)=2x\sin(x^2-1)$

FTC2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

NET CHANGE THEOREM

- If the mass of a rod measured from the left end to a point *x* is *m*(*x*), then the linear density is *ρ(x)* = *m'*(*x*). So, *m(b)-m(a)* is the *mass* of the segment of the rod that lies between *x* = *a* and *x* = *b*.

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

- f the rate of growth of a population is *dn*/*dt*, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the *net change in population* during the time period from t_1 to t_2.

- If an object moves along a straight line with position function *s*(*t*), then its velocity is *v*(*t*) = *s'*(*t*).So,

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net *change of position, or displacement*, of the particle during the time period from t_1 to t_2.

Distance vs displacement

- Displacement= $s(t_2)-s(t_1)$= *change in position*
- Distance

$$\int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

- The acceleration of the object is $a(t) = v'(t)$.
  - So, $$\int_{t_1}^{t_2} a(t)\, dt = v(t_2) - v(t_1)$$

    is the *change in velocity* from time t_1 to time t_2.

A particle moves along a line so that its velocity at time t is:

$$v(t) = t^2 - t - 6 \text{ (in meters per second)}$$

a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

b) Find the distance traveled during this time period.

Do yourself

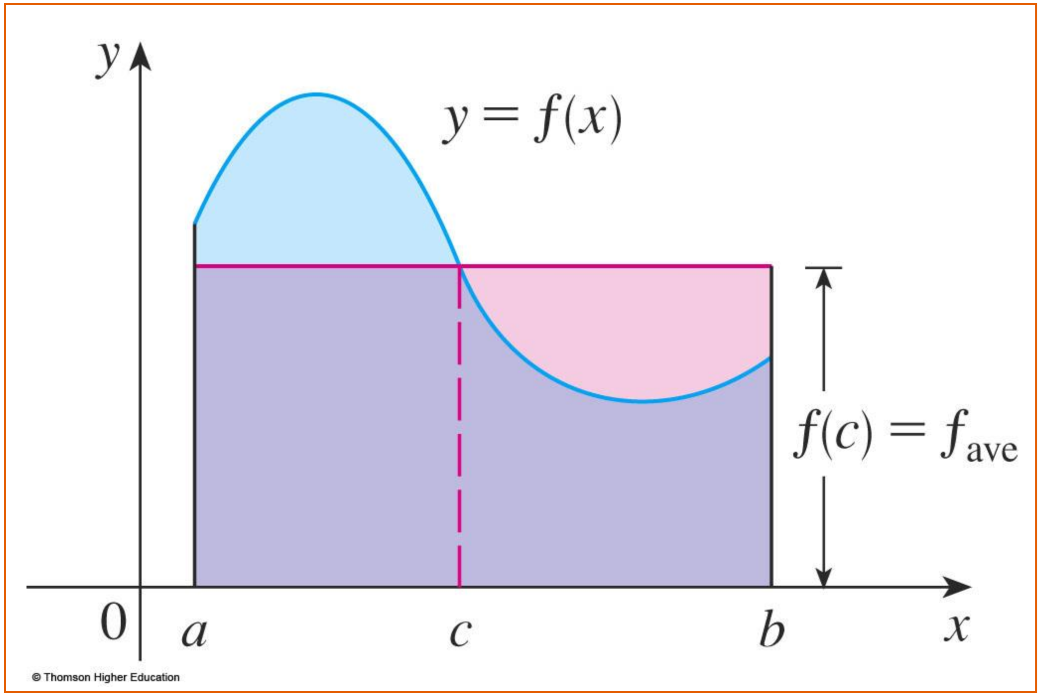
The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

An animal population is increasing at a rate of $200 + 50t$ per year (where t is measured in years). By how much does the animal population increase between the fourth and tenth years?

MEAN VALUE THEOREM

The geometric interpretation of the Mean Value Theorem for Integrals is as follows.

- For ‘positive’ functions *f*, there is a number *c* such that the rectangle with base [*a*, *b*] and height *f*(*c*) has the same area as the region under the graph of *f* from *a* to *b*.



MEAN VALUE THEOREM

If *f* is continuous on [*a*, *b*], then there exists a number *c* in [*a*, *b*] such that

that is, $$\boxed{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

example

Find the average value of the function $y=x^2 -2x$ on the interval [0,3].
0
3/2
-1/2
1
None of the others.

INDEFINITE INTEGRAL

Due to the relation given by the FTC between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an indefinite integral.

Thus, $\boxed{\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)}$

TABLE OF INDEFINITE INTEGRALS **Table 1**

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

4.5
The Substitution Rule

In this section, we will learn:

To substitute a new variable in place of an existing expression in a function, making integration easier.

SUBSTITUTION RULE

Equation 4

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)\,du}$$

INTRODUCTION **Equation 1**

However, our antidifferentiation formulas don’t tell us how to evaluate integrals such as

$$\int 2x\sqrt{1+x^2}\,dx$$

SUBSTITUTION RULE **Example**

Find $\int x^3 \cos(x^4 + 2)\, dx$

- We make the substitution $u = x^4 + 2$.
- This is because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral.

SUB. RULE FOR DEF. INTEGRALS

Example 7

Evaluate $$\int_1^2 \frac{dx}{(3-5x)^2}$$

- Let $u = 3 - 5x$.
- Then, $du = -5 \, dx$, so $dx = -du/5$.
- When $x = 1$, $u = -2$, and when $x = 2$, $u = -7$.

INTEGS. OF SYMM. FUNCTIONS

Theorem 6

Suppose f is continuous on $[-a, a]$.

a. If f is even, $[f(-x) = f(x)]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

b. If f is odd, $[f(-x) = -f(x)]$, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

Thanks