ĐẠI HỌC FPT

5 The Vector Space $\mathbb{R}^n$

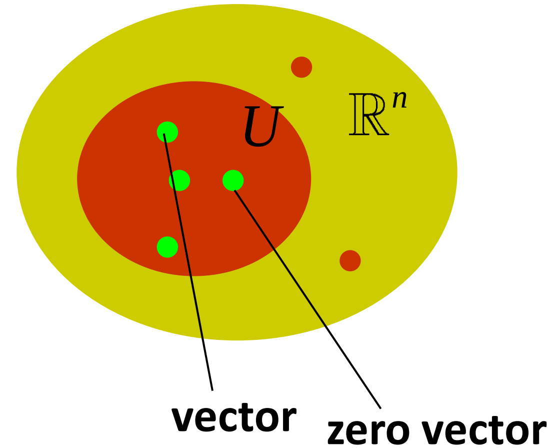
Objectives

- Subspaces and Spanning sets
- Independence and Dimension
- Orthogonality
- Rank of a Matrix

Subspace of R^n

Definition of subspace of R^n.

- Let $\emptyset \neq U$ be a subset of R^n
- U is called a *subspace* of R^n if:



❶ S_1. The zero vector **0** is in U

❷ S_2. If **X**,**Y** are in U then **X**+**Y** is in U

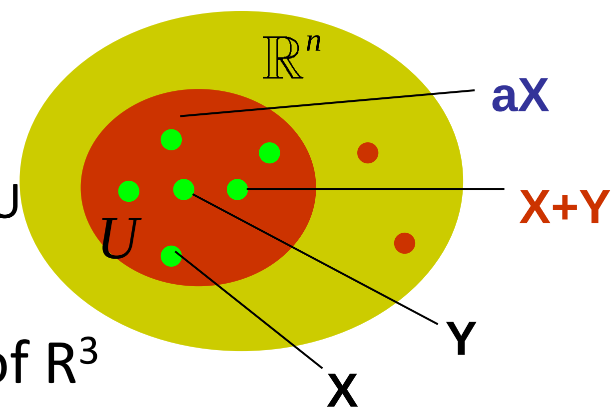
❸ S_3. If **X** is in U then **a**X is in U for all real number **a**.

- *Ex1*. $U=\{(a,a,0) | a \in R\}$ is a ***subspace*** of R^3

❶ the zero vector of R^3, $(0,0,0) \in U$

❷ $(a,a,0), (b,b,0) \in U \Rightarrow (a,a,0)+(b,b,0)=(a+b,a+b,0) \in U$

❸ If $(a,a,0) \in U$ and $k \in R$, then $k(a,a,0)=(ka,ka,0) \in U$



- *Ex2*. $U=\{(a,b,1): a,b \in R\}$ is not a ***subspace*** of R^3

❶ $(0,0,0) \notin U \Rightarrow$ U is not a ***subspace***

- *Ex3*. $U=\{(a,|a|,0) | a \in R\}$ is not a ***subspace*** of R^3

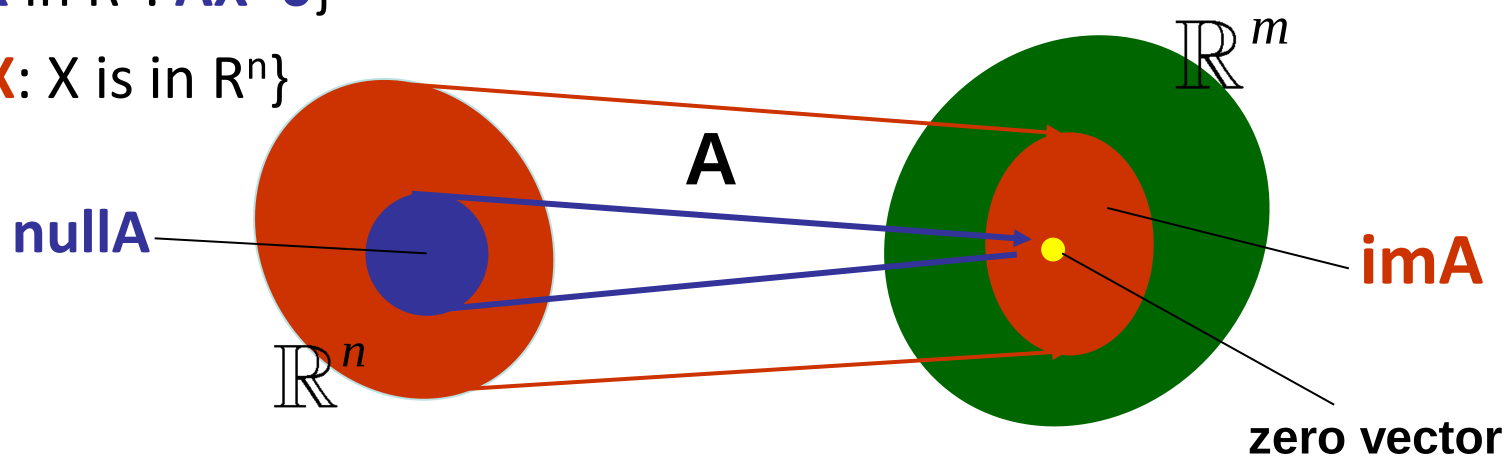
❷ $(-1,|-1|,0), (1,|1|,0) \in U$ but $(0,2,0) \notin U \Rightarrow$ U is not a ***subspace***

Subspaces or not - Do yourself

- $V=\{[0\ a\ \ 0]^T$ in $\mathbb{R}^3$: $a \in Z\}$
- $U=\{[a\ \ 0\ \ a+1]^T$ in $\mathbb{R}^3$: $a \in R\}$
- $W=\{[a\ \ \ b\ \ \ a-b]^T$ in $\mathbb{R}^3$: $a,b \in R\}$
- $Q=\{[a\ \ b\ \ |a+b|]^T$: $a \in \mathbb{R}\}$
- $H=\{[a\ \ \ b\ \ ab]^T$: $a \in \mathbb{R}\}$
- $P=\{(x,y,z)|\ x-2y+z=0$ and $2x-y+3z=0\}$. P is called the *solution space* of the system $x-2y+z=0$ and $2x-y+3z=0$.

Null space and image space of a matrix

- A is an mxn matrix, if X is nx1 matrix then AX is mx1 matrix
- **nullA** = {**X** in R^n: **AX=0**}
- **imA** = {**AX**: X is in R^n}



nullA ={X∈ R^n:AX=0} is a subspace of R^n:

❶ A.0=0⇒0∈nullA

❷ X,Y ∈nullA ⇒AX=0, AY=0 ⇒A(X+Y)=AX+AY=0 ⇒(X+Y) ∈nullA

❸ X ∈nullA, a ∈R ⇒ AX=0 ⇒ A(aX)=a(AX)=0 ⇒ aX∈nullA

imA ={AX:X ∈ R^n}is a subspace of R^m:

❶ 0=A.0⇒0∈nullA

❷ AX,AY ∈imA ⇒AX+AY=A(X+Y)=AZ ⇒AX+AY ∈imA

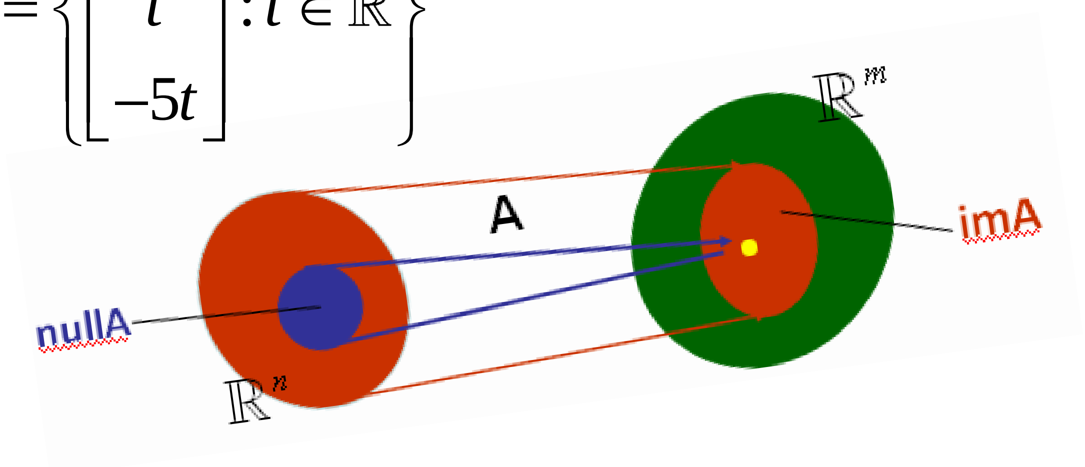
❸ AX ∈imA, a ∈R ⇒ a(AX)=A(aX)=AZ ⇒ a(AX)∈imA

Null space nullA={X:AX=0}

- For example, $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 1\end{bmatrix}_{2\times 3}$

$$nullA=\left\{\begin{bmatrix}x\\ y\\ z\end{bmatrix}: A\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\right\}=\left\{\begin{bmatrix}x\\ y\\ z\end{bmatrix}:\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\right\}$$

$$=\left\{\begin{bmatrix}x\\ y\\ z\end{bmatrix}:\begin{cases}x-y=0\\ 2x+3y+z=0\end{cases}\right\}=\left\{\begin{bmatrix}t\\ t\\ -5t\end{bmatrix}: t\in\mathbb{R}\right\}$$



Eigenspaces (không gian riêng)

- Suppose A is an nxn matrix and λ is an eigenvalue of A
- $E_\lambda(A)=\{X: AX=\lambda X\}$ is an subspace of $\mathbf{R}^n$
- For example,

$$A=\begin{bmatrix}-3 & -1\\ 0 & 2\end{bmatrix} \Rightarrow c_A(x)=\det(xI-A)=\begin{vmatrix}x+3 & 1\\ 0 & x-2\end{vmatrix}=(x+3)(x-2)$$

$$c_A(x)=0 \Leftrightarrow x=-3 \vee x=2$$

$$x=-3:\left[\begin{array}{cc|c}0 & 1 & 0\\ 0 & -5 & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}0 & 1 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow X=\begin{bmatrix}t\\ 0\end{bmatrix}\ (\text{or X}=(t,0))$$

$$x=2:\left[\begin{array}{cc|c}5 & 1 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow X=\begin{bmatrix}t\\ -5t\end{bmatrix}$$

$$E_{-3}=\{X: AX=-3X\}=\{(t,0):t\in\mathbb{R}\}$$

$$E_2=\{X: AX=2X\}=\{(t,-5t):t\in\mathbb{R}\}$$

Các không gian riêng ứng với GTR

Spanning sets

- $Y=k_1X_1+k_2X_2+\ldots+k_nX_n$ is called a *linear combination* of the vectors $X_1,X_2,\ldots,X_n$
- The <u>set</u> of all *linear combinations* of the the vectors $X_1,X_2,\ldots,X_n$ is called the *span* of these vectors, denoted by span$\{X_1,X_2,\ldots,X_n\}$.
- This means, span$\{X_1,X_2,\ldots,X_n\}$ = $\{k_1X_1+k_2X_2+\ldots+k_nX_n : k_i \in R$ is arbitrary$\}$
- span$\{X_1,X_2,\ldots,X_n\}$ is a subspace of R^n.
- For example, span{(1,0,1),(0,1,1)}={a(1,0,1)+b(0,1,1) :a,b$\in$R}.
- And we have (1,2,3)$\in$span{(1,0,1),(0,1,1)} because (2,-3,-1)= 2(1,0,1)+ -3(0,1,1).
- (2,3,2)$\notin$span{(1,0,1),(0,1,1)} because (2,3,2)$\neq$a(1,0,1)+b(0,1,1) for all a,b.

Examples

- If $x=(1,3,-5)$ is expressed as a linear combination of the vectors $v_1 = (1, 1, 1)$; $v_2 =(1,1,-1)$; $v_3 = (1, 0, 2)$; then the coefficient of v_3 is:

A. 2 B. 3 C. -2 ✓ D. 1 E. 0

- x is expressed as a linear combination of v_1, v_2, v_3 means $x=av_1+bv_2+cv_3$ for some a,b,c and c is called the ***coefficient*** of v_3.
- the system is

$$\begin{cases} a + b + c = 1 \\ a + b = 3 \\ a - b + 2c = -5 \end{cases}$$

1	1	1	1
1	1	0	3
1	-1	2	-5

1	1	1	1
0	0	-1	2
0	-2	1	-6

1	1	1	1
0	-2	1	-6
0	0	-1	2

$\Rightarrow a = 1$

$\Rightarrow b = 2$

$\Rightarrow c = -2$

- Which of the vectors below is a *linear combination* of u=(1,1,2); v=(2,3,5)?

A. (0,1,1) ✓ B. (1,1,0) C. (1,1,1) D. (1,0,1) ✓ E. (0,0,1)

- Có thể giải bằng biến đổi sơ cấp trên ma trận chứa các vector cột như sau:

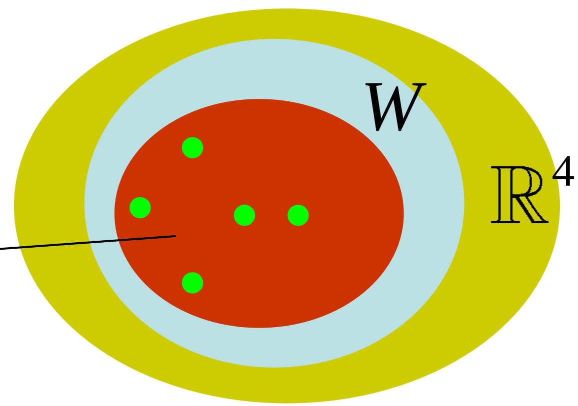
u	v	A	B	C	D	E
➊	2	0	1	1	1	0
1	3	1	1	1	0	0
2	5	1	0	1	1	1

u	v	A	B	C	D	E
➊	-2	0	1	1	1	0
0	➊	1	0	0	-1	0
0	1	1	-2	-1	-1	1

u	v	A	B	C	D	E
➊	-2	0	1	1	1	0
0	➊	1	0	0	-1	0
0	0	0	-2	-1	0	1

Theorem

- $U=\text{span}\{X_1,X_2,\ldots,X_n\}$ is in R^n and U is a subspace of R^n
- If W is a subspace of R^n such that X_i are in W then $U \subseteq W$



$$U = span\{x_1, x_2, x_3, x_4, x_5\}$$

Fpt University
ĐẠI HỌC FPT

Linear Independence

A set of vectors in R^m $\{X_1, X_2, \ldots, X_n\}$ is called **linearly independent**

if $t_1X_1 + t_2X_2 + \ldots + t_nX_n = 0$ $\boxed{\Rightarrow t_1 = t_2 = \ldots = t_n = 0 \text{ only}}$

numbers in R **vectors in R^m**

Ex1. The set $\{[1\ \ -1]^T, [2\ \ 3]^T\} \subset R^2$ is called linearly independent since $t_1[1\ \ -]^T + t_2[2\ \ 3]^T = [0\ \ 0]^T$ follows $t_1 = t_2 = 0$.

Ex2. A set of vectors that containing zero vector never linearly independent.

Ex3. The set $\{(0,1,1), (1,-1,0), (1,0,1)\}$ is *not linearly independent* because the system $t_1(0,1,1) + t_2(1,-1,0) + t_3(1,0,1) = (0,0,0)$ has one *solution $t_1=1$, $t_2=1$, $t_3=1$*

Examples

- Show that {(1,1,0);(0,1,1);(1,0,1)} is linearly independent in R^3

$$t_1(1,1,0)+t_2(0,1,1)+t_3(1,0,1)=(0,0,0)$$

$$\Rightarrow \ldots \Rightarrow t_1=t_2=t_3=0$$

$$t_1(1,1,0)+t_2(0,1,1)+t_3(1,0,1)=(0,0,0)$$

$$\Leftrightarrow \begin{cases} 1t_1+0t_2+1t_3=0 \\ 1t_1+1t_2+0t_3=0 \\ 0t_1+1t_2+1t_3=0 \end{cases} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow t_1=t_2=t_3=0 \Rightarrow \textit{independent}$$

__More ex__. {(1,0,-2), (2,1,0), (0,1,5), (-1,1,0)} is *__not linearly independent__* *(number of* leading 1s = number of vectors)

1	2	0	-1
0	1	1	1
-2	0	5	0

1	2	0	-1
0	1	1	1
0	4	5	2

1	2	0	-1
0	1	1	1
0	0	1	-2

Examples – do yourself

- Determine whether each the following sets is linearly independent or linearly dependent.
- {(-1,2,0)}
- {(0,0,0); (1,2,3); (-3,0,1)}
- {(1,1,-1); (-1,1,1); (1,-1,1)}
- {(-2,3,4,1); (4,-1,5,0); (-2,1,0,3)}
- {(1,1,0); (-2,3,1); (5,0,1); (-1,0,2)}
- {X-Y+Z,3X+Z,X+Y-Z}, where {X,Y,Z} is an independent set of vectors. (see below)

1	3	1
-1	0	1
1	1	-1

1	3	1
0	3	2
0	-2	-2

1	3	1
0	1	1
0	3	2

1	3	1
0	1	1
0	0	-1

1	3	1
0	1	1
0	0	1

$\Rightarrow$independent

Fundamental Theorem

- *Theorem.* Let U be a subspace of R^n is spanned by m vectors, if U contains k linearly independent vectors, then k≤m
- This implies if *k>m*, then the set of k vectors is always linear dependece.
- For example, Let U be the space spanned by {(1,0,1), (0,-1,1)} and S={(1,0,1), (0,-1,1), (2,-1,3)} ⊂U. Then, S is *not linearly independent* (*m=2, k=3*).

Basis and dimension

- ***Definition of basis***: Suppose U is a *subspace* of R^n, a *set* $\{X_1, X_2, \ldots, X_k\}$ is called a *basis* of U if U=span$\{X_1, X_2, \ldots, X_k\}$ and $\{X_1, X_2, \ldots, X_k\}$ is *linear independence*
- *Ex1*. Let U=$\{(a,-a) | a \in R\}$. Then U is a *subspace* of R^2. Consider the set B=$\{(1,-1)\}$. B is *linearly independent* and U=$\{(a,-a): a \in R\}$=$\{a(1,-1): a \in R\}$ =span$\{(1,-1)\}$. So, B is a *basis* of U.
- Note that B'=$\{(-1,1)\}$ is also a *basis* of U.
- But $\{(1,1)\}$ is not a *basis* of U because U can not be spanned by $\{(1,1)\}$
- *Ex2*. Given that V=span$\{(1,1,1), (1,-1,0), (0,2,1)\}$. Then, B=$\{(1,1,1), (1,-1,0), (0,2,1)\}$ is not linearly independent, because $(0,2,1)=(1,1,1) – (1,-1,0) \Rightarrow$ B is not a *basis* of U.
- Consider B'=$\{(1,1,1), (1,-1,0)\}$. B' is *linearly independent* and all vectors in V are spanned by B' because $a(1,1,1)+ b(1,-1,0)+ c(0,2,1) =a(1,1,1)+ b(1,-1,0)+ c(1,1,1) –c(1,-1,0) = (a+c)(1,1,1)+(b-c)(1,-1,0)$. So, B' is a *basis* of V.

Some important theorems

- ***Theorem 1***. The following are ***equivalence*** for an nxn matrix A.

  ❶ A is invertible.

  ❷ the columns of A are linearly independent.

  ❸ the columns of A span R^n.

  ❹ the rows of A are linearly independent.

  ❺ the rows of A span the set of all 1xn rows.

- ***Theorem 2***. (Invariance theorem). If $\{a_1,a_2,..,a_m\}$ and $\{b_1,b_2,...,b_k\}$ are bases of a subspace U of R^n, then m=k. In this case, m=k is called ***dimension*** of U and we write ***dimU=m***.
- *Ex1*. Let $U=\{(a,-a)|a\in R\}$ be a subspace of R^2. Then, B={(1,-1)} is a *basis* of U and B'={(-1,1)} is also a *basis* of U. In this case, dimU=1.
- *Ex2*. {(1,0), (0,1)} is a basis of R^2 and {(1,-2), (2,0)} is also a basis of R^2. But {(1,0), (-1,1), (1,1)} is not a basis of R^2. We have $dimR^2=2$.
- The basis {(1,0), (0,1)} is called ***standard basis*** of R^2.
- *Ex3*. Which of the following is a basis of R^3?

  ❶ {(1,0,1), (0,0,1)}

  ❷ {(2,1,0), (-1,0,1), (1,0,1), (0,-1,1)}

  ❸ {(0,1), (1,0)}

  ❹ None of the others

Some important theorems

- ***Theorem 3***. Let U≠0 be a subspace of R^n. Then:

  ❶ U has a basis and $\dim U \le n$.

  ❷ Any independent set of U can be enlarged (by adding vectors) to a basis of U.

  ❸ If B spans U, then B can be cut down (by deleting vectors) to a basis of U.

  Ex1. Let U=span{(1,1,1), (1,0,1), (1,-2,1)} be a subspace of R^3. This means, B= {(1,1,1), (1,0,1), (1,-2,1)} spans U.

  ❶ $\Rightarrow$U has a basis and $\dim U \le 3$,

  ❸ $\Rightarrow$ B can be cut down to a basis of U: {(1,0,1), (1,1,1)} is a basis of U, ***dimU=2***

  ❷ $\Rightarrow$construct a basis for U: $\varnothing \rightarrow$ {(1,0,1)} $\rightarrow$ {(1,0,1), (1,1,1)}.

1	1	1
1	0	-2
1	1	1

1	1	1
0	-1	-3
0	0	0

1	1	1
0	1	3
0	0	0

- ***Theorem 4.*** Let U be a subspace of R^n and $B=\{X_1,X_2,\ldots,X_m\}\subset U$, where dimU=m. Then B is independent if and only if B spans U.
- ***Theorem 5***. Let $U\subseteq V$ be subspaces of R^n. Then:

  ❶ $\dim U \le \dim V$.

  ❷ If dimU=dimV, then U=V.

Examples

Determine whether U is a subspace of R^3.
U={[0 a b]^T: a,b $\in$ R} ✓
U={[0 1 s]^T:s $\in$ R}
U={[a b a+1] ^{T}:a,b $\in$ R}
U={[a b a^2]^T: a, b $\in$ R}

1	0	2
0	1	m
1	1	1

1	0	2
0	1	m
0	1	-1

1	0	2
0	1	m
0	0	-1-m

Find all m such that the set {(2,m,1),(1,0,1),(0,1,1)} is linearly independent.
m≠-1 ✓
m=-1 only
m=0 only
m≠0
None of the others

A basis for the subspace U={[a b a-b]^T: a,b $\in$R} is…
a. {[1 0 1]^T, [0 1 -1]^T} ✓
b. {[1 1 0]^T}
c. {[1 0 1]^T, [-1 0 -1]^T, [0 1 -1]^T}
d. None of the others.

ĐẠI HỌC FPT

Exercises

The dimension of the subspace U=span{(-2, 0, 3),(1, 2, -1),(-2, 8, 5),(-1, 2, 2)} is…
a. 2 ✓
b. 4
c. 3
d. 1

- không thể là b vì dimU $\leq$ dimR^3=3
- kiểm tra bằng biến đổi sơ cấp

1	-2	-1	-2
2	8	2	0
-1	5	2	3

➊	-2	-1	-2
0	12	4	4
0	3	1	1

➊	-2	-1	-2
0	➊	1/3	1/3
0	0	0	0

} only 2 ➊

Let u and v be vectors in R^3 and w $\in$ span{u,v}. Then …
a. {u,v,w} is linearly dependent. ✓
b. {u,v,w} is linearly independent.
c. {u,v,w} is a basis of R^3
d. the subspace is spanned by {u,v,w} has the dimension 3.

Let {u,v,w,z} be independent. Then …. is also independent.
a. {u,v+w,z} ✓
b. {u,v,v-z-u,z}
c. {u+v,u-w,z, v+z+w}
d. {u,v,w,u-v+w}

1	1	0	0
1	0	0	1
0	-1	0	1
0	0	1	1

➊	1	0	0
0	-1	0	1
0	-1	0	1
0	0	1	1

➊	1	0	0
0	➊	0	-1
0	0	➊	1
0	0	0	0

Exercises

- Let U=span{(1,-1,1), (0,2,1)}. Find all value(s) of m for which (3,-1,m)$\in$U.
- (3,-1,m) $\in$U$\Leftrightarrow$ (3,-1,m)= a(1,-1,1) + b(0,2,1) for some a,b. Solve for a,b$\Rightarrow$m=4

- Find all values of m so that {(2,-1,3); (0,1,2); (-4,0; m)} spans R^3.
- ***Theorem 4.*** Let U be a subspace of R^n and B={X_1,X_2,…,X_m}$\subset$U, where dimU=m. Then B is independent if and only if B spans U.
- So, {(2,-1,3); (0,1,2); (3,1; m)} spans R^3 $\Leftrightarrow$ it is linearly independent $\Leftrightarrow$ m≠10

2	0	-4
-1	1	0
3	2	m

①	-1	0
2	0	-4
3	2	m

①	-1	0
0	2	-4
0	5	m

①	-1	0
0	①	2
0	5	m

①	-1	0
0	①	2
0	0	m-10

- Find a basis for the solution space to the homogeneous system

$$\begin{aligned} x - y + 2z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

①	-1	2	0
2	1	1	0

①	-1	2	0
0	3	-3	0

①	-1	2	0
0	①	-1	0

Solution: z=t, y=t, x=-t
Solution space: U={(-t,t,t)| t$\in$R}
={t(-1,1,1)| t$\in$R}=span{(-1,1,1)}
A basis for U: {(-1,1,1)}
dimU=1

Definitions

- *Dot product:*

  Suppose $X= [x_1 \ \ x_2 \dots x_n]^T$, $Y= [y_1 \ \ y_2 \dots y_n]^T$ are vectors in R^n . The ***dot product*** of two vectors X and Y, denoted as X•Y, is a number defined by

  $X \bullet Y = x_1y_1 + x_2y_2 + \dots + x_ny_n$

- *Length:*

  The length of the vector $X=[x_1 \ x_2 \ \dots \ x_m]^T$ (or $(x_1,x_2,\dots,x_n)$) is

$$\|X\| = \sqrt{X \bullet X} = \sqrt{{x_1}^2 + {x_2}^2 + \dots + {x_n}^2}$$

- *Unit vector:* a vector with length 1
- *Distance:* $d(X,Y)= \|X-Y\|$

Theorem

- Let X,Y, and Z denote vectros in R^n. Then:
  - ❶ $X \bullet Y = Y \bullet X$
  - ❷ $X \bullet (Y+Z) = XY + XZ$
  - ❸ $(aX) \bullet Y = a(X \bullet Y) = X \bullet (aY)$
  - ❹ $X \bullet X = \|X\|^2$
  - ❺ $\|X\| \geq 0$, $\|X\| = 0 \Leftrightarrow X = 0$
  - ❻ $\|aX\| = |a| . \|X\|$
- *Ex.* Suppose that $R^n = \text{span}\{F_1, F_2, \ldots, F_n\}$. If $X \bullet F_i = 0$ for each I, where X is a vector in R^n. Show that X=0 (zero vector).

  $R^n = \text{span}\{F_1, F_2, \ldots, F_n\}$ and $X \in R^n \Rightarrow X = k_1F_1 + k_2F_2 + \ldots + k_nF_n$ for some $k_i \in R^n$.

  We have $\|X\|^2 = X \bullet X = X \bullet (k_1F_1 + k_2F_2 + \ldots + k_nF_n) = k_1X \bullet F_1 + k_2X \bullet F_2 + \ldots + k_nX \bullet F_n = 0 \Rightarrow \|X\| = 0 \Leftrightarrow X = 0$

Theorem

- If X, Y and Z are vectors in R^n, we have:
  - ❶ $d(X,Y) \geq 0$
  - ❷ $d(X,Y) = 0 \Leftrightarrow X=Y$
  - ❸ $d(X,Y)=d(Y,X)$
  - ❹ triangle inequality: $d(X,Y)+d(Y,Z) \geq d(X,Z)$
- ***Corolary.*** $||X+Y|| \leq ||X|| + ||Y||$
- ***Cauchy Inequality:*** $\mathbf{X \bullet Y} \leq ||\mathbf{X}||.||\mathbf{Y}||$

Definitions

- *Orthogonal set*

  A set $\{x_1, x_2, \ldots, x_m\}$ is called **orthogonal set** if x_i is not zero vector and $x_i \bullet x_j = 0$ for all $i \neq j$.

  For example, {(1,-1);(1,1)} is an orthogonal set in R^2

  {(1,1,1);(-1,0,1);(0,1,0)} is not a orthogonal set but {(-1,0,1);(0,1,0)} is a orthogonal set.

- *Orthonormal set*

  A orthogonal set $\{x_i\}$ is called orthonormal set (hệ trực chuẩn) is x_i is unit vector for all i. For example, (1,0,0);(0,1,0)} is orthonormal.

  {(-3,0,4);(4,5,3)} is a orthogonal set, not a orthonormal set.

  However, the set $\left\{\frac{1}{5}(-3,0,4);\frac{1}{5\sqrt{2}}(4,5,3)\right\}$ is orthonormal

Examples

- The standard basis of R^n $\{E_1,E_2,\ldots,E_n\}$ is orthonormal
- If $\{F_1,F_2,\ldots,F_k\}$ is orthogonal then $\{a_1F_1,a_2F_2,\ldots,a_kF_k\}$ is also orthogonal for any nonzero scalar a_i
- Every orthogonal set is a linearly independent set
- If u,v are <u>*unit*</u> <u>*orthogonal*</u> vectors then $(3u-5v)\bullet(4u+2v)=12u\bullet u+6u\bullet v-20v\bullet u-10v\bullet v=12\|u\|^2-10\|v\|^2=12-10=2$

Pythagoras's Theorem

- If $\{F_1,F_2,\ldots,F_k\}$ is orthogonal then

$$\left\|F_1+F_2+\ldots+F_k\right\|^2=\left\|F_1\right\|^2+\left\|F_2\right\|^2+\ldots+\left\|F_k\right\|^2$$

Expansion Theorem

- Let $\{F_1, F_2, \ldots, F_k\}$ be a orthogonal basis of a subspace U and X is in U. Then

$$X = \frac{X \bullet F_1}{\|F_1\|^2} F_1 + \frac{X \bullet F_2}{\|F_2\|^2} F_2 + \ldots + \frac{X \bullet F_n}{\|F_n\|^2} F_k$$

5.4. Rank of a matrix

Rank of a matrix

- If A is carried to row-echelon form then rankA=number of leading 1's
- If a is an **mxn** matrix then **rankA≤min{n,m}**
- rankA=rank(A^T)

rowA and colA subspaces

- rowA=span{rows of matrix A}
- colA=span{columns of A}
- dim(rowA)=dim(colA)=rankA
- For example, find bases of colA and rowA if

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 3 & 2 & 0 & 5 \\ -2 & -3 & 3 & -4 \\ 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

$$A=\begin{bmatrix}1&1&-1&2\\3&2&0&5\\-2&-3&3&-4\\1&1&-1&3\\0&1&-1&2\end{bmatrix}\to\begin{bmatrix}1&1&-1&2\\0&-1&3&-1\\0&-1&1&0\\0&0&0&1\\0&1&-1&2\end{bmatrix}$$

$$\to\begin{bmatrix}1&1&-1&2\\0&-1&3&-1\\0&0&-2&1\\0&0&0&1\\0&0&-3&3\end{bmatrix}\to\begin{bmatrix}1&1&-1&2\\0&-1&3&-1\\0&0&-2&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

A basis of **rowA** is $\{r_1,r_2,r_3,r_4\}$ and **dim(rowA)=4**

A basis of **colA** is $\{c_1,c_2,c_3,c_4\}$ and **dim(colA)=4**

Theorem

❶ An nxn matrix A is invertible if and only if rankA=n

❷ If an mxn matrix B has rank n then the n columns of B is linearly independent

❸ If A is mxn matirx and m>n then the set of m rows of A is not independent

For example, If A is an 3x5 matrix with rank 3 then 5 columns are dependent and 3 rows are independent.

Theorem

If an mx**n** matrix A has **rank r** then:

❶ The equation AX=0 has **n**-**r** basic solutions $X_1, X_2, \ldots, X_{n-r}$

❷ $\{X_1, X_2, \ldots, X_{n-r}\}$ is a **basis** of **nullA={X:AX=0}**

❸ dim**nullA**=**n**-**r**

❹ **imA=colA** and

❺ dim**imA**=dim**colA**=rankA=r

Thanks