Chapter 3: Determinants and Diagonalization

1. Evaluate the determinant

a. $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix}$ b. $\begin{vmatrix} -2 & 0 & 0 \\ 4 & 6 & 0 \\ -3 & 7 & 2 \end{vmatrix}$ c. $\begin{vmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{vmatrix}$ d. $\begin{vmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{vmatrix}$

e. $\begin{vmatrix} x & y & 1 \\ -1 & -2 & 1 \\ 1 & 5 & 1 \end{vmatrix}$ f. $\begin{vmatrix} m & -1 & 0 \\ 1 & 2 & 1 \\ 2 & m & -3 \end{vmatrix}$

2. Find the minors and the cofactors of the matrix

a. $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ b. $B = \begin{pmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{pmatrix}$ c. $C = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & m \end{pmatrix}$

3. Find the adjugate and the inverse of the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$

4. Let $A = \begin{pmatrix} 1 & * & * & * \\ 0 & -1 & * & * \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Find

a. $\left|2A^{-1}\right|$ b. $\left|AA^T\right|$ c. $\left|\operatorname{adj} A\right|$

d. $\left|-A^3\right|$ e. $\left|(2A)^{-1}\right|$ f. $\left|A^{-1} - 2\operatorname{adj} A\right|$

5. Let A and B be square matrices of order 4 such that $|A| = -5$ and $|B| = 3$. Find

a. $|2AB|$ b. $|adj(AB)|$ c. $\left|5A^{-1}B^T\right|$ d. $\left|A^T B^{-1} A^2\right|$

6. Find all values of k for which the matrix is not invertible

a. $A=\begin{pmatrix}1 & 3\\ k & 2\end{pmatrix}$ b. $B=\begin{pmatrix}m & 1 & 3\\ 1 & 3 & 2\\ -1 & 4 & 5\end{pmatrix}$ c. $C=\begin{pmatrix}m & 2 & 0\\ 1 & m & 1\\ 2 & 3 & 1\end{pmatrix}$

7. Find the characteristic polynomial of the matrix

a. $A=\begin{pmatrix}3 & 5\\ 1 & 2\end{pmatrix}$ b. $B=\begin{pmatrix}1 & 3\\ 1 & 3\end{pmatrix}$

c. $C=\begin{pmatrix}1 & 0 & 0\\ -2 & 2 & 1\\ 1 & 2 & 1\end{pmatrix}$ d. $D=\begin{pmatrix}1 & 2 & -1\\ 0 & 1 & 2\\ -1 & 1 & 1\end{pmatrix}$

8. Find the eigenvalues and corresponding eigenvectors of the matrix

a. $A=\begin{pmatrix}-3 & 5\\ 10 & 2\end{pmatrix}$ b. $B=\begin{pmatrix}5 & 4\\ 2 & 1\end{pmatrix}$

c. $C=\begin{pmatrix}1 & 0 & 0\\ -2 & 3 & 0\\ 0 & 0 & 4\end{pmatrix}$ d. $D=\begin{pmatrix}-3 & 2 & -1\\ 0 & 1 & 0\\ 4 & 1 & 1\end{pmatrix}$

9. Find the determinant of the matrix $A=\begin{pmatrix}5 & 1 & 2 & 4\\ 1 & 0 & -1 & -3\\ 1 & 1 & 6 & 1\\ 1 & 0 & 0 & -4\end{pmatrix}$

10. Find the (1, 2)-cofactor and (3,1) - cofactor of the matrix $\begin{bmatrix}-1 & 3 & -2\\ 4 & 5 & -7\\ 7 & 8 & 1\end{bmatrix}$

11. Let $A=\begin{pmatrix}1 & 3 & 1\\ 0 & 1 & 0\\ 2 & -1 & x\end{pmatrix}$. For which values of x is A invertible ?

Chapter 4: Vector Geometry

1. Find the equations of the line through the points $P_0(2, 0, 1)$ and $P_1(4, -1, 1)$.

2. Find the equations of the line through $P0(3, -1, 2)$ parallel to the line with equations:

$$\begin{cases} x = -1 + 2t \\ y = 1 + t \\ z = -3 + 4t \end{cases}$$

3. Determine whether the following lines intersect and, if so, find the point of intersection.

$$\begin{cases} x = 1 - 3t \\ y = 2 + 5t \\ z = 1 + t \end{cases}, \begin{cases} x = -1 + s \\ y = 3 - 4s \\ z = 1 - s \end{cases}$$

4. Compute $||v||$ if v equals:

a. (2,-1,2) b. 2(1,1,-1) c. -3(1,1,2) d. (1,2,3) - (4,1,2)

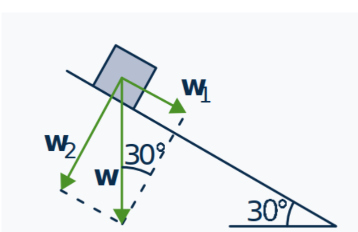
5. Find a unit vector in the direction from (3,-1,4) to (1,3,5).

6. Find $||v - 3w||$ when $||v|| = 2$, $||w|| = 1$, and $v \cdot w = 2$

7. Compute the angle between u = (-1,1,2) and v = (-1,2,1).

8. Show that the points $P(3, -1, 1)$, $Q(4, 1, 4)$, and $R(6, 0, 4)$ are the vertices of a right triangle.

9. Suppose a ten-kilogram block is placed on a flat surface inclined 30° to the horizontal as in the diagram. Neglecting friction, how much force is required to keep the block from sliding down the surface?



10. Find the projection of u = (2,-3,1) on d = (-1,1,3) and express $u = u_1 + u_2$ where u1 is parallel to d and u_2 is orthogonal to d.

11. Find an equation of the plane through $P_0(1, -1, 3)$ with n = (-3,-1,2) as normal.

12. Find an equation of the plane through $P_0(3, -1, 2)$ that is parallel to the plane with equation $2x - 3y - z = 6$.

13. Find the shortest distance from the point P(2, -1, − 3) to the plane with equation $3x - y + 4z = 1$. Also find the point Q on this plane closest to P.

14. Find the equation of the plane through P(1, 3, − 2), Q(1, 1, 5), and R(2, − 2, 3).

15. Find the shortest distance between the nonparallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

16. Compute u · v where:

a. u = (2,-1,3), v = (-1,1,1) b. u = (-2,1,4), v = (-1,5,1)

17. Find all real numbers x such that:

a. (3,-1,2) and (3,-2,x) are orthogonal.

b. (2,-1,1) and (1,x,2) are at an angle of $\pi/3$.

18. Find the three internal angles of the triangle with vertices:

a. A(3, 1, − 2), B(3, 0, − 1), and C(5, 2, − 1)

b. A(3, 1, − 2), B(5, 2, − 1), and C(4, 3, − 3)

19. Find the equations of the line of intersection of the following planes.

a. $2x - 3y + 2z = 5$ and $x + 2y - z = 4$.

b. $3x + y - 2z = 1$ and $x + y + z = 5$.

20. Find the area of the triangle with vertices P(2, 1, 0), Q(3, − 1, 1), and R(1, 0, 1)

21. Find the volume of the parallelepiped determined by the vectors u = (1,2,-1), v = (3,4,5) and w = (-1,2,4).

22. In each case show that that T is either projection on a line, reflection in a line, or rotation through an angle, and find the line or angle

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-y \\ y-x \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -x-y \\ x-y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

e. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$

f. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

23. Determine the effect of the following transformations.

a. Rotation through $\pi/2$, followed by projection on the y axis, followed by reflection in the line y = x.

b. Projection on the line y = x followed by projection on the line y = −x.

c. Projection on the x axis followed by reflection in the line y = x.

24. Find the reflection of the point P in the line y = 1 + 2x in R^2 if:

a. P = P(1, 1)

b. P = P(1, 4)

25. Find the angle between the following pairs of vectors.

a. u = (1,-1,4), v = (5,2,-1)

b. u = (2,1,5), v = (0,3,1)

26. In each case, compute the projection of u on v.

a. $\mathbf{u} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

27. Find the shortest distance between the following pairs of nonparallel lines and find the points on the lines that are closest together.

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix};$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix};$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix};$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Chapter 5: The Vector Space R^n

1. Let $x=(-1,-2,-2), u=(0,1,4), v=(-1,1,2)$ and $w=(3,1,2)$ in R^3. Find scalars a, b and c such that $x=au+bv+cw$

2. Write v as a linear combination of u and w, if possible, where $u=(1,2), w=(1,-1)$

a. $v=(0,1)$ b. $v=(2,3)$ c. $v=(1,4)$ d. $(-5,1)$

3. Determine whether the set S is linearly independent or linearly dependent

a. $S=\{(-1,2),(3,1),(2,1)\}$ b. $S=\{(-1,2,3),(1,3,5)\}$

c. $S=\{(1,-2,2),(2,3,5),(3,1,7)\}$ d. $S=\{(-1,2,1),(2,4,0),(3,1,1)\}$

e. $S=\{(1,-2,2,1),(1,2,3,5),(-1,3,1,7)\}$

4. For which values of k is each set linearly independent?

a. $S=\{(-1,2,1),(k,4,0),(3,1,1)\}$ b. $S=\{(-1,k,1),(1,1,0),(2,-1,1)\}$

c. $S=\{(k,1,1),(1,k,1),(1,1,k)\}$ d. $S=\{(1,2,1,0),(-2,1,1,-1),(-1,3,2,k)\}$

5. Find all values of m such that the set S is a basis of R^3

a. $S=\{(1,2,1),(m,1,0),(-2,1,1)\}$ b. $S=\{(-1,m,1),(1,1,0),(m,-1,-1)\}$

6. Find a basis for and the dimension of the subspace U

a. $U=\{(2s-t,s,s+t) \mid s,t\in R\}$ b. $U=\{(s-t,s,t,s+t) \mid s,t\in R\}$

c. $U=\{(0,t,-t) \mid t\in R\}$ d. $U=\{(x,y,z) \mid x+y+z=0\}$

e. $U=\{(x,y,z) \mid x+y+z=0, x-y=0\}$ f. $U=span\{(1,2,3),(2,3,4),(3,5,7)\}$

g. $U=span\{(1,2,4),(-1,3,4),(2,3,1)\}$ h. $U=span\{(1,2,1,1),(2,1,-1,0),(3,3,0,1)\}$

7. Find a basis for and the dimension of the solution space of the homogeneous system of linear equations.

a. $\begin{cases} -x+y+z=0 \\ 3x-y=0 \\ 2x-4y-5z=0 \end{cases}$ b. $\begin{cases} x+2y-4z=0 \\ -3x-6y+12z=0 \end{cases}$ c. $\begin{cases} x+y+z+t=0 \\ 2x+3y+z=0 \\ 3x+4y+2z+t=0 \end{cases}$

8. Find all values of m for which x lies in the subspace spanned by S

a. $x=(-3,2,m)$ and $S=\{(-1,-1,1),(2,-3,-4)\}$

b. $x=(4,5,m)$ and $S=\{(1,-1,1),(2,-3,4)\}$

c. $x=(m+1,5,m)$ and $S=\{(1,1,1),(2,3,1),(3,4,2)\}$

d. $x=(3,5,7,m)$ and $S=\{(1,1,1,-1),(1,2,3,1),(2,3,4,0)\}$

9. Find the dimension of the subspace
$U=span\{(\text{-}2,\ 0,\ 3),(1,\ 2,\ \text{-}1),(\text{-}2,\ 8,\ 5),(\text{-}1,\ 2,\ 2)\}$

10. Let $A=\begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 2 & 2 & 1 & 2 \end{pmatrix}$. Find $\dim(\operatorname{col} A)$ and $\dim(\operatorname{row} A)$

11. Which of the following are subspaces of R^3?

(i) $\{(2+a,b-a,b) \mid a,b\in R\}$

(ii) $\{(a+b,a,b) \mid a,b\in R\}$

(iii) $\{(2a+b,0,ab) \mid a,b\in R\}$

12. Let $u=(1,-3,-2), v=(-1,1,0)$ and $w=(1,2,-3)$. Compute $\|u-v+w\|$

13. Let $u,v\in\mathbb{R}^3$ such that $\|u\|=3, \|v\|=4$ and $u\bullet v=-2$. Find

a. $\|u+v\|$ b. $\|2u+3v\|$ c. $||2u - v||$