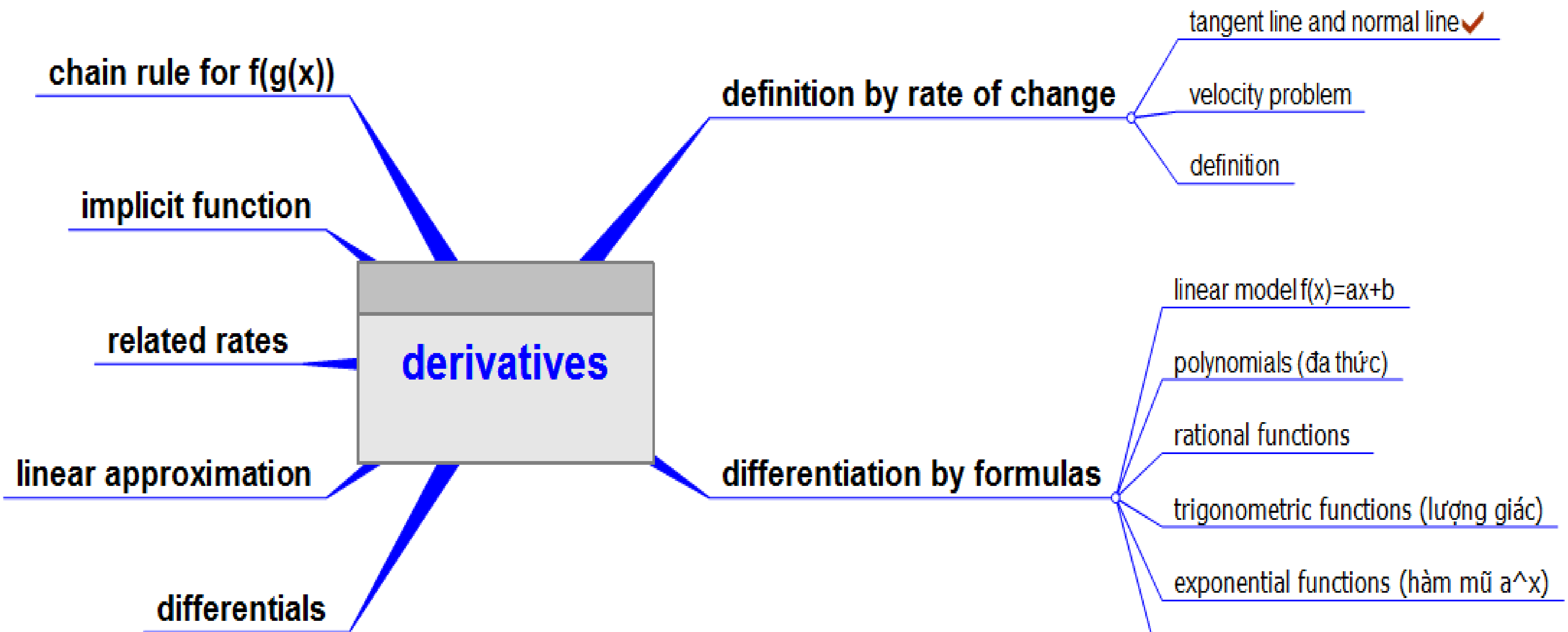
2

DERIVATIVES



derivatives
chain rule for f(g(x))
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linear model f(x)=ax+b
polynomials (đa thức)
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algorithmic functions

2.1

Derivatives and Rates of Change

In this section, we will learn:

How the derivative can be interpreted as a rate of change in any of the sciences or engineering.

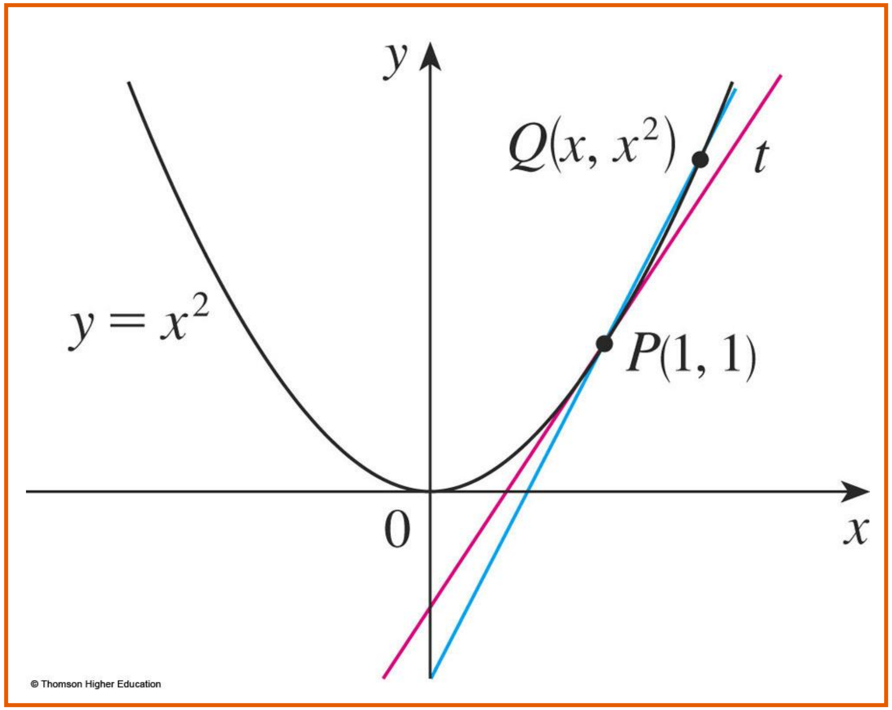
THE TANGENT PROBLEM

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

- We will be able to find an equation of the tangent line as soon as we know its slope m.

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$



THE TANGENT PROBLEM

Example 1

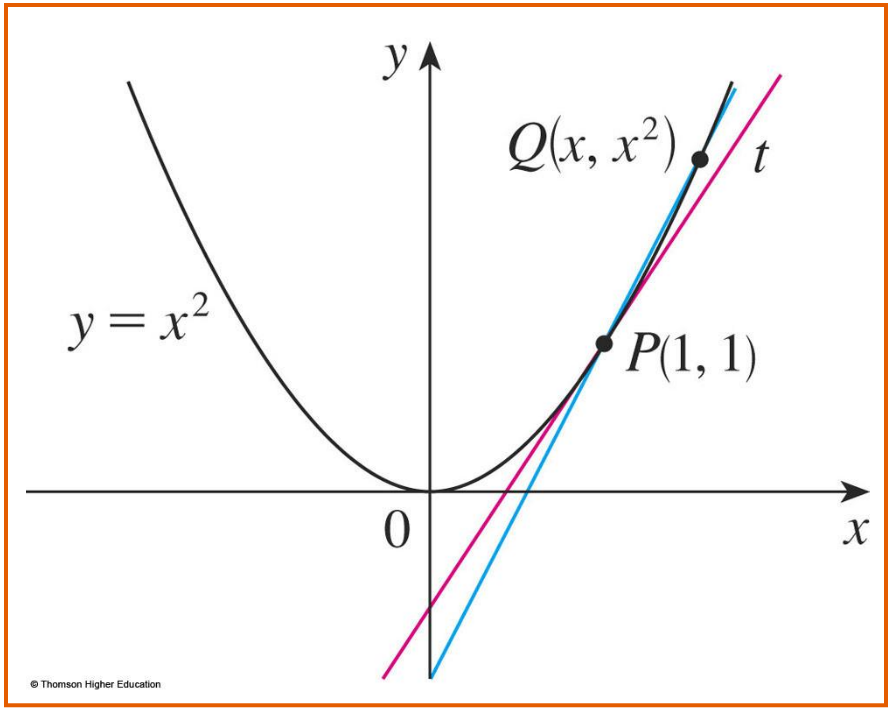
- The slope of the tangent line is said to be the limit of the slopes of the secant lines.

$$\lim_{Q \to P} m_{PQ} = m$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

The equation of the tangent line through (1, 1) as:

$$y = 2x - 1$$



TANGENTS

1. Definition

- The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P **with slope**

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- provided that this limit exists.

THE VELOCITY PROBLEM

Example 3

Investigate the example of a falling ball.

- Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground.
- Find the velocity of the ball after 5 seconds.

THE VELOCITY PROBLEM

Example 3

- If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the following equation.

- $s(t) = 4.9t^2$

THE VELOCITY PROBLEM

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1} = 49.49 \text{ m/s}$$

Thus, the (instantaneous) velocity after 5 s is:

v = 49 m/s

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

3. Definition

- We define the **velocity** (or instantaneous velocity) $v(a)$ **at time *t* = *a*** to be the limit of these average velocities:

$$\boxed{v(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

4. Definition

- The derivative of a function *f* at a number *a*, denoted by *f'*(*a*), is:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

- if this limit exists. Or

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

2.2

The Derivative as a Function

In this section, we will learn about:
The derivative of a function *f*.

DERIVATIVES

1. Equation

- In the preceding section, we considered the derivative of a function *f* at a fixed number *a*:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- If we replace *a* in Equation 1 by a variable *x*, we obtain:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

OTHER NOTATIONS

- Some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols *D* and *d/dx* are called differentiation operators.

The symbol *dy/dx* is called Leibniz notation

OTHER NOTATIONS

- If we want to indicate the value of a derivative *dy/dx* in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \quad or \quad \left.\frac{dy}{dx}\right]_{x=a}$$

- which is a synonym for $f'(a)$.

3. Definition

- A function f is differentiable at a if $f'(a)$ exists.

It is differentiable on an open interval D if it is differentiable at every number in the interval D.

HOW CAN A FUNCTION FAIL TO BE DIFFERENTIABLE?

Theorem

If f is differentiable at a,

then f is continuous at a.

$\Longrightarrow$ This theorem states that, if f is not continuous at a, then f is not differentiable at a.

HIGHER DERIVATIVES

- If f is a differentiable function, then its derivative f' is also a function.
- So, f' may have a derivative of its own, denoted by

  $(f')' = f''$.

This new function f'' is called the second derivative of f.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

HIGHER DERIVATIVES

- The process can be continued.
  - In general, the *n*th derivative of *f* is denoted by $f^{(n)}$ and is obtained from *f* by differentiating *n* times.
  - If $y = f(x)$, we write:

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

2.3

Differentiation Formulas

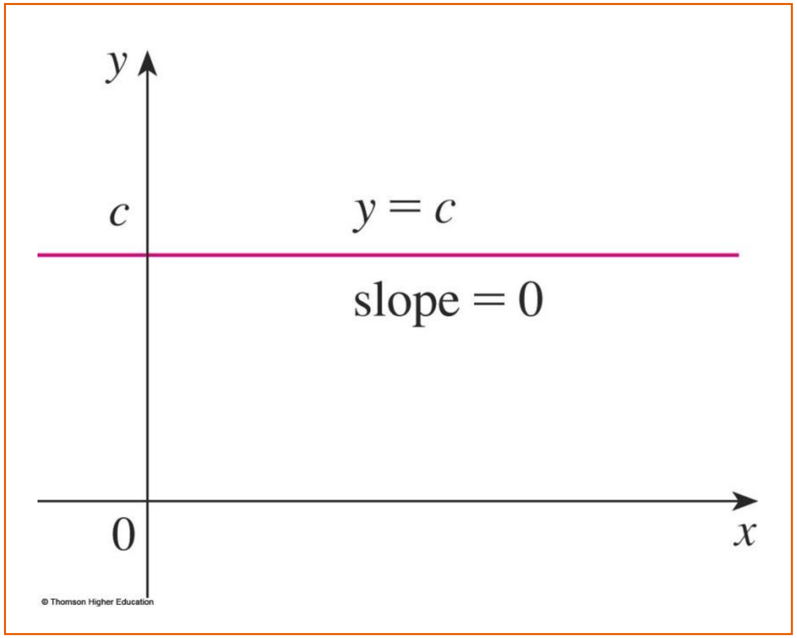
In this section, we will learn:

How to differentiate constant functions, power functions, polynomials, and exponential functions.

CONSTANT FUNCTION—DERIVATIVE

- In Leibniz notation, we write this rule as follows.

$$\boxed{\frac{d}{dx}(c) = 0}$$



(Reference Pages, p.5)

DIFFERENTIATION FORMULAS

•Here's a summary of the differentiation formulas we have learned so far.

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

TANGENT AND NORMAL LINES

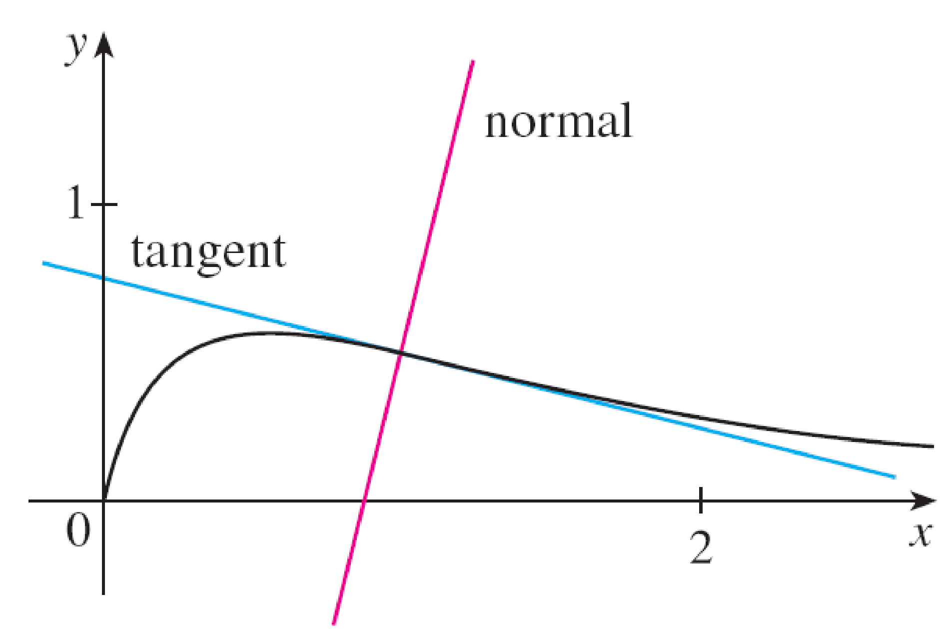
Example 12

•Find equations of the **tangent line** and **normal line** to the curve

$$y = \sqrt{x}/(1+x^2)$$

at the point (1, ½).

$$y = -\tfrac{1}{4}x + \tfrac{3}{4}$$



THE CHAIN RULE

- If g is differentiable at x and f is differentiable at $g(x)$, the composite function $F = f \circ g$ is differentiable at x and F' is given by the product:

$$\boxed{\bullet\ F'(x) = f'(g(x)) \cdot g'(x)}$$

  – In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Let $f(x)=g(\sin 3x)$. Find f' in terms of g'.	
A	$3\cos 3x\, g'(x)$
B	$3\cos 3x\, g'(\sin 3x)$
C	$\cos 3x\, g'(\sin 3x)$

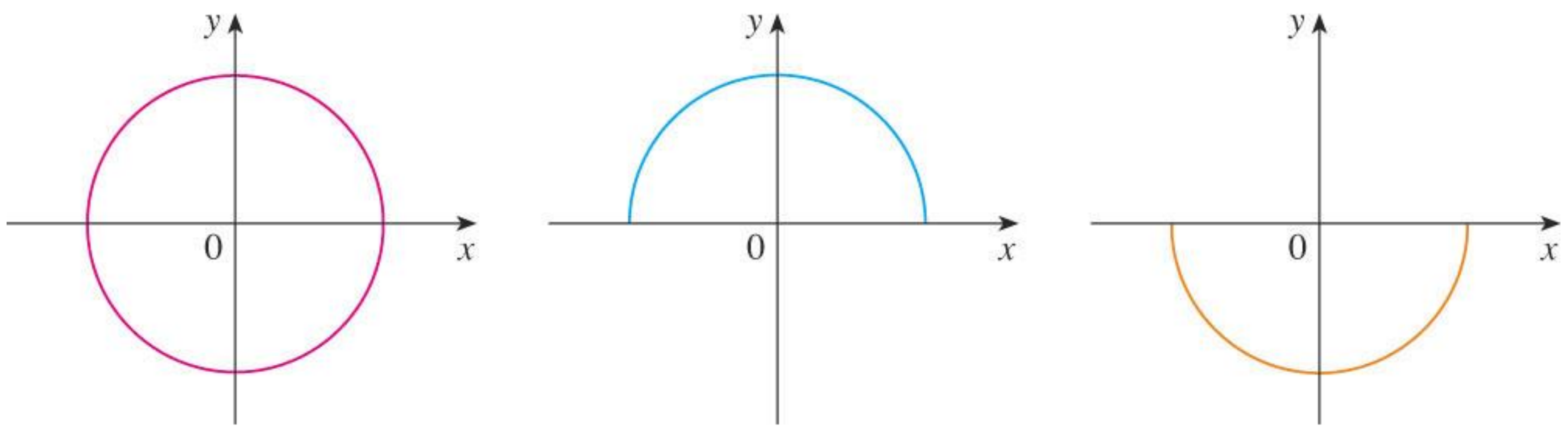
Answer: b

Suppose $h(x)=f(g(x))$ and $f(2)=3$, $g(2)=1$, $g'(2)=-1$, $f'(2)=2$, $f'(1)=5$. Find $h'(2)$.	
A	1
B	2
C	5
D	4
E	-5

Answer: e

IMPLICIT DIFFERENTIATION

- The graphs of *f* and *g* are the upper and lower semicircles of the circle
  - $x^2 + y^2 = 25.$



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

IMPLICIT DIFFERENTIATION METHOD

- Instead, we can use the method of implicit differentiation.
  - This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

IMPLICIT DIFFERENTIATION

Example 1

- a. If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

- b. Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

IMPLICIT DIFFERENTIATION

Example 1 a

- Remembering that *y* is a function of *x* and using the Chain Rule, we have:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 0$$

- Then, we solve this equation for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{x}{y}$

E. g. 1 b—Solution 1

- At the point (3, 4) we have $x = 3$ and $y = 4$.

- So, $\dfrac{dy}{dx} = -\dfrac{3}{4}$

  – Thus, an equation of the tangent to the circle at (3, 4) is: $y - 4 = -\frac{3}{4}(x - 3)$ or $3x + 4y = 25$.

2.7
Related Rates

In this section, we will learn:

How to compute the rate of change of one quantity in terms of that of another quantity.

STRATEGY

•It is useful to recall some of the problem-solving principles from Chapter 1 and adapt them to related rates in light of our experience in Examples 1–3.

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. (…, p.129)

RELATED RATES

Example 1

- Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s.
- How fast is the radius of the balloon increasing when the diameter is 50 cm?

Example 1

- The key thing to remember is that rates of change are derivatives.

  - In this problem, the volume and the radius are both functions of the time t.
  - The rate of increase of the volume with respect to time is the derivative dV / dt.
  - The rate of increase of the radius is dr / dt.

Example 1

- To connect dV/dt and dr/dt, first we relate V and r by the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Example 1

- To use the given information, we differentiate each side of the equation with respect to *t*.
  - To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Example 1

- Now, we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi^2}\frac{dV}{dt}$$

  - If we put r = 25 and dV / dt = 100 in this equation, we obtain:

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

  - The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

Example 1

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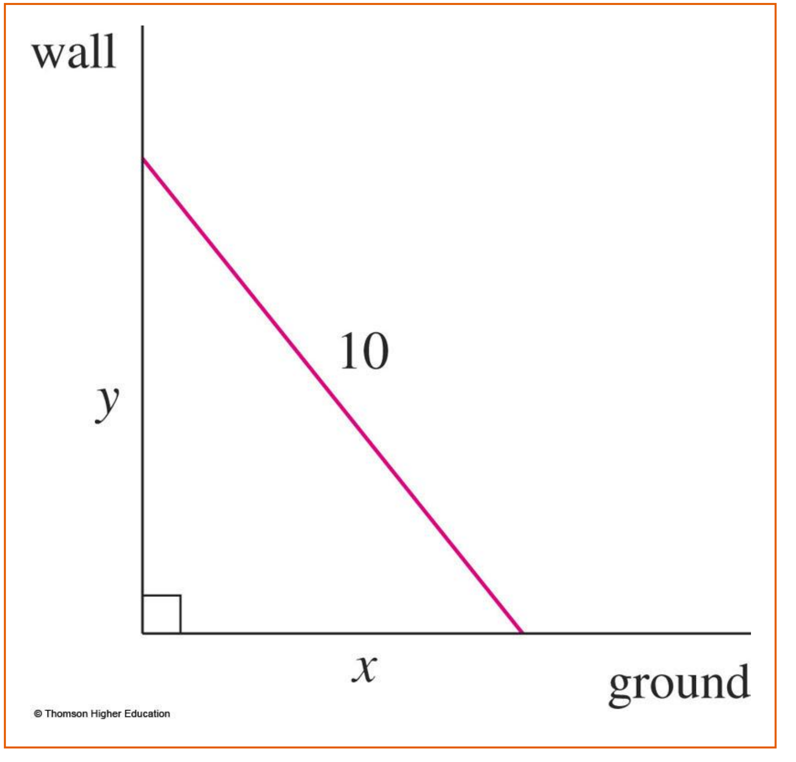
  - The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

Example 2

- A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

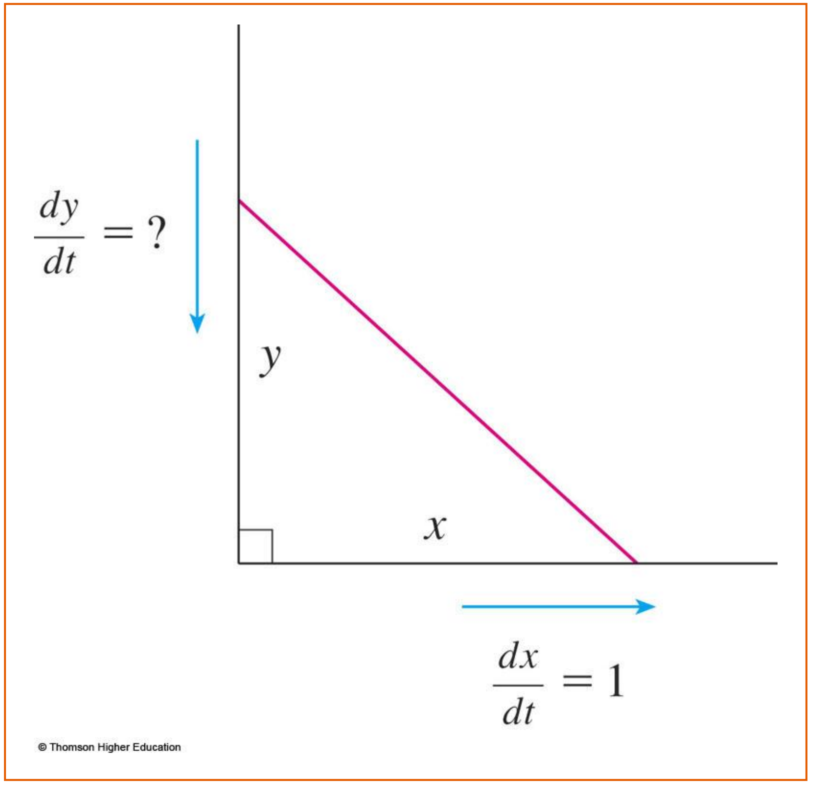
Example 2

- We first draw a diagram and label it as in the figure.
  - Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground.
  - Note that x and y are both functions of t (time, measured in seconds).



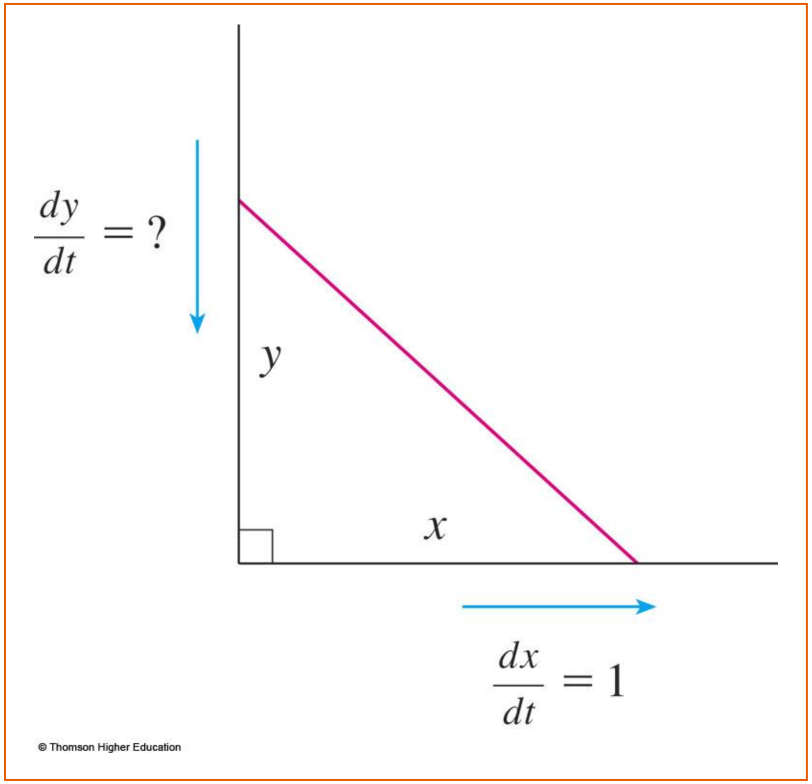
Example 2

- We are given that $dx / dt = 1$ ft/s and we are asked to find dy / dt when $x = 6$ ft.



Example 2

- In this problem, the relationship between x and y is given by the Pythagorean Theorem: $x^2 + y^2 = 100$



Example 2

- Differentiating each side with respect to *t* using the Chain Rule, we have:

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

  – Solving this equation for the desired rate, we obtain:

$$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$$

Example 2

- When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx / dt = 1$, we have:

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}\ ft/s$$

  - The fact that *dy* / *dt* is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of ¾ ft/s.
  - That is, the top of the ladder is sliding down the wall at a rate of ¾ ft/s.

2.8

Linear Approximations and Differentials

In this section, we will learn about:

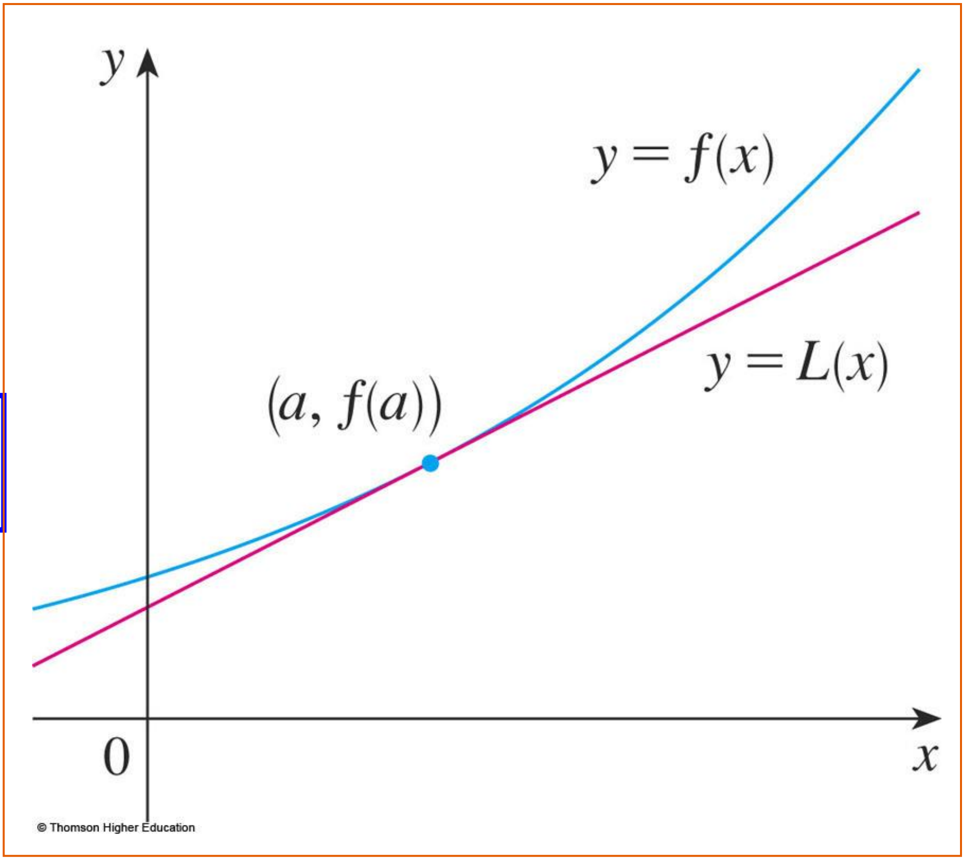
Linear approximations and differentials and their applications.

LINEAR APPROXIMATIONS

- We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

An equation of this tangent line is

$$\boxed{L(x) = y = f(a) + f'(a)(x - a)}$$



LINEAR APPROXIMATION

Equation 1

- The approximation

$$\boxed{f(x) \approx f(a) + f'(a)(x - a) = L(x)}$$

- is called the linear approximation of *f* at *a*.

LINEAR APPROXIMATIONS

Example 1

- Find the linearization of the function $f(x)=\sqrt{x+3}$ at *a* = 1 and use it to approximate the numbers

$$\sqrt{3.98} \text{ and } \sqrt{4.05}$$

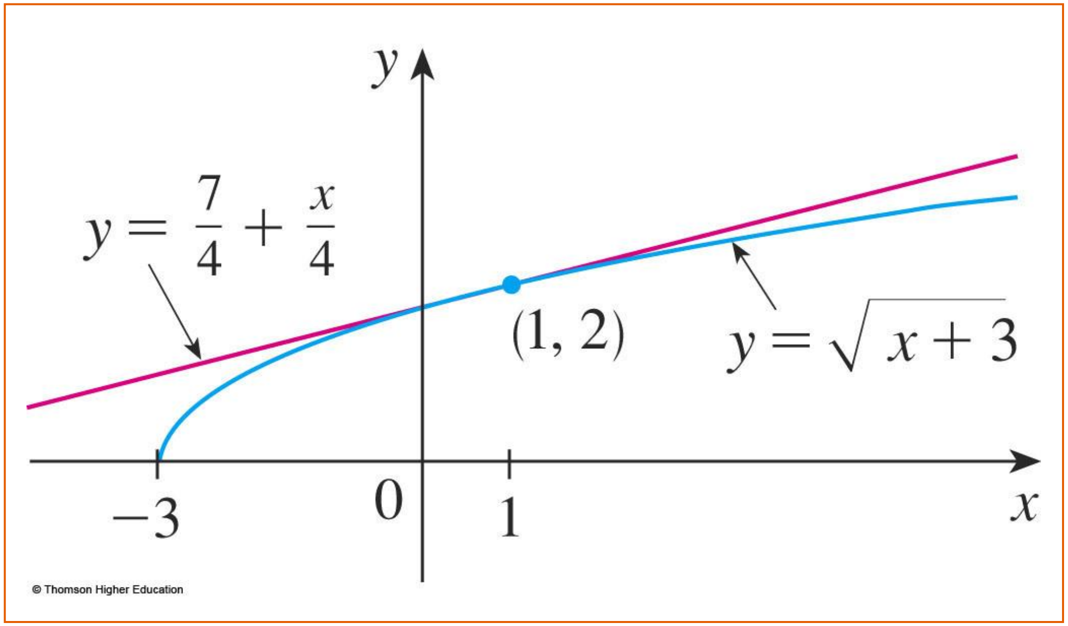
- Are these approximations overestimates or underestimates?

LINEAR APPROXIMATIONS

Example 1

- Putting these values into Equation 2, we see that the linearization is:

$$
\begin{aligned}
L(x) &= f(1) + f'(1)(x-1) \\
&= 2 + \tfrac{1}{4}(x-1) \\
&= \frac{7}{4} + \frac{x}{4}
\end{aligned}
$$



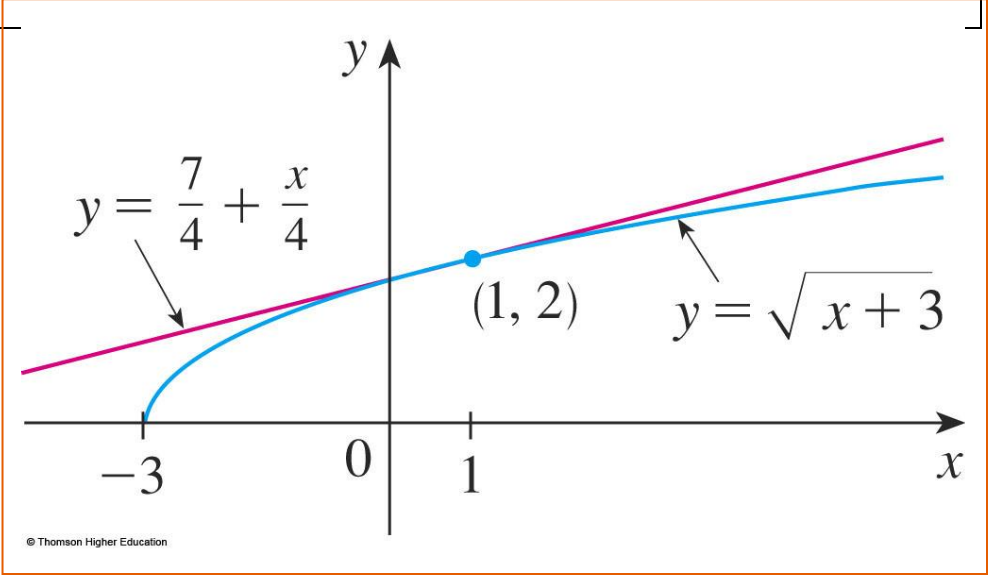
LINEAR APPROXIMATIONS

Example 1

- The corresponding linear approximation is:
- $$\boxed{\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}}$$ (when x is near 1)
- In particular, we have:

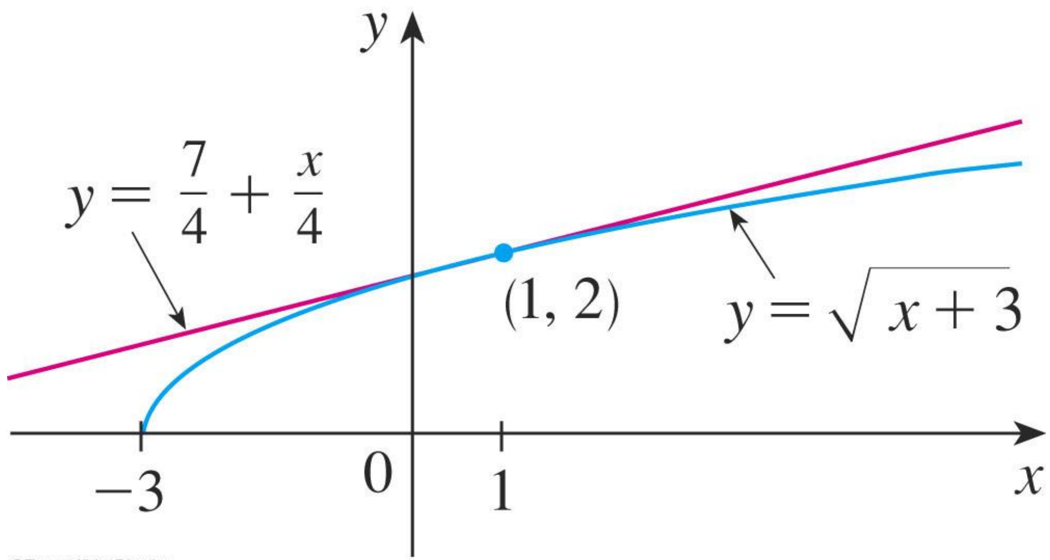
$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$



LINEAR APPROXIMATIONS

- Look at the table and the figure.
  - The tangent line approximation gives good estimates if *x* is close to 1.
  - However, the accuracy decreases when *x* is farther away from 1.



	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

DIFFERENTIALS

Example 4

- The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm.
-
- What is the maximum error in using this value of the radius to compute the volume of the sphere?

DIFFERENTIALS

Example 4

- If the radius of the sphere is r, then its volume is $V = 4/3\pi r^3$.
  - If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV.

Example 4

- This can be approximated by the differential
  - $dV = 4\pi r^2 dr$
- When $r = 21$ and $dr = 0.05$, this becomes:
- $$dV = 4\pi(21)^2\, 0.05 \approx 277$$
  - The maximum error in the calculated volume is about 277 cm^3.

RELATIVE ERROR

Note

- Relative error is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}$$

  - Thus, the relative error in the volume is about three times the relative error in the radius.

RELATIVE ERROR

Note

- In the example, the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume.
  - The errors could also be expressed as percentage errors of 0.24% in the radius and 0.7% in the volume.

Thanks