Introduction to Linear Algebra

Linear algebra plays an important role in machine learning and general mathematics.

- Linear system of equations
- Matrix algebra
- Eigenvalues and eigenvectors
- Vector spaces

Linear Algebra - Chapter 1

1 Systems of Linear Equations

OUR GOAL

- **Elementary Operations**
- **Gaussian Elimination**
- **Homogeneous Equations**

1.1. Solutions and Elementary Operations

coefficients **variables = unknowns**

- $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

is called a **linear equation**

- If $a_1s_1 + a_2s_2 + \ldots + a_ns_n = b$

➔ $(s_1, s_2, \ldots, s_n)$ is called solution of the equation

- A system may have:

  no solution $\mathbf{0}$

  unique solution $\mathbf{1}$

  an **infinite family of solutions** $\boldsymbol{\infty}$

Inconsistent	Consistent	
No solutions	Unique solution	Infinitely many solutions

Example 1

- $$\begin{cases} x+2y=1 \\ x+2y=3 \end{cases}$$ $$\begin{cases} x+y-z=1 \\ x+y+z=3 \end{cases}$$

no solution (0,2,1), (2,0,1) (t,2-t,1)

Inconsistent **Consistent**

(infinitely many solutions)

(t,2-t,1): **general solution,**
given in **parametric form**,
t is **parameter (t is arbitrary)**

Algebraic Method

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= -1 \\ 2x_1 - x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 + 5x_4 &= 2 \end{aligned} \Longrightarrow \left[\begin{array}{cccc|c} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{array}\right]$$

augmented matrix

coefficient matrix

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$

constant matrix

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Example 2

- Consider the system

$$\begin{cases} x - y = 1 \\ x + 2y = -2 \end{cases}$$

augmented matrix $\left\langle \begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & -2 \end{array} \right\rangle$

Example 2

$$\begin{cases} x - y = 1 \\ x + 2y = -2 \end{cases} \Leftrightarrow \begin{cases} x - y = 1 \\ 0x + 3y = -3 \end{cases} \Leftrightarrow \begin{cases} x - y = 1 \\ 0x + y = -1 \end{cases} \Leftrightarrow \begin{cases} x + 0y = 0 \\ 0x + y = -1 \end{cases}$$

$$\left\langle \begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & -2 \end{array} \right\rangle \to \left\langle \begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -3 \end{array} \right\rangle \to \left\langle \begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -1 \end{array} \right\rangle \to \left\langle \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \end{array} \right\rangle$$

$\Rightarrow$ Solution (0,-1)

Elementary Operations (phép biến đổi sơ cấp)

- **Interchange** two equations (type I)

$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow[interchange]{} \begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases}$$

- **Multiply** one equation by a **nonzero number** (type II)

$$\begin{cases} 2x + 3y = 5 \\ x - 2y = -1 \end{cases} \xrightarrow[multiplied\ by\ -2]{} \begin{cases} 2x + 3y = 5 \\ -2x + 4y = 2 \end{cases}$$

- **Add a multiple** of one equation to a different equation (type III)

$$\begin{cases} x - 2y = -1 \\ 2x + 3y = 5 \end{cases} \xrightarrow[add\ a\ multiple]{} \begin{cases} x - 2y = -1 \\ 0x + 7y = 7 \end{cases}$$

Elementary operations

- Interchange two rows

$$\begin{bmatrix} 0 & -2 & 3 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

- Multiply one row by a nonzero number

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 2 & 0 & 1 \end{bmatrix}$$

- Add a multiple of one row to different row

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{-2r_1 + r_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 0 & 2 & -3 \end{bmatrix}$$

A row-echelon matrix has 3 properties

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- All the zero rows are at the bottom
- The first nonzero entry from the left in each nonzero row is a 1, called the **leading 1** for that row
- Each leading 1 is **to the right** of all leading 1's in the rows above it

Row-echelon matrix

The row-echelon matrix has the “staircase” form

leading ones

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(for any choice in *-position)

Which matrices are in row-echelon form?

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 2 & * \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A reduced row-echelon matrix

- It is a row-echelon matrix
- Each leading 1 is the **only nonzero** entry in its column

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is a reduced row- echelon matrix?

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xcancel{\begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xcancel{\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

How to carry a matrix to (reduced) row-echelon form?

Gaussian Algorithm

Theorem. Every matrix can be brought to (reduced) row-echelon form by a series of elementary row operations

- Step 1. If all row are **zeros**, stop
- Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it *a*) and move the row containing *a* to the top position
- Step 3. Multiply that row by 1/a to creat the **leading 1**
- Step 4. By subtracting multiples of that row from the rows below it, make each entry below the leading 1 **zero**
- Step 5. Repeat step 1-4 on the matrix consisting of the remaining rows

Gaussian Algorithm

$$step\ 1: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow stop$$

leading one

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 2 & 2 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 4 & 7 & 5 & -1 \end{bmatrix} \xrightarrow{-4r_1+r_4} \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 3 & 7 & -1 \end{bmatrix}$$

step 2 **step 3** **step 4** **step 5**

Example

Carry the matrix

$$\begin{bmatrix} 2 & 6 & -2 & 2 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{bmatrix}$$

- to row-echelon matrix
- to reduced row-echelon matrix

$$
\begin{bmatrix} 2 & 6 & -2 & 2 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{bmatrix}
\xrightarrow{\frac{1}{2}r_1}
\begin{bmatrix} 1 & 3 & -1 & 1 \\ -2 & -3 & 11 & 4 \\ 3 & 11 & 3 & 0 \end{bmatrix}
\xrightarrow[-3r_1+r_3]{2r_1+r_2}
\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 3 & 9 & 6 \\ 0 & 2 & 6 & -3 \end{bmatrix}
$$

$$
\xrightarrow{\frac{1}{3}r_2}
\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 6 & -3 \end{bmatrix}
\xrightarrow{-2r_2+r_3}
\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -7 \end{bmatrix}
\xrightarrow{-\frac{1}{7}r_3}
\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

row-echelon matrix

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-r_3+r_1]{-2r_3+r_2} \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3r_2+r_1} \begin{bmatrix} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced row-echelon matrix

Gauss-Jordan Elimination

(for solving a system of linear equations)

- Step 1.Using elementary row operations, **augmented matrix →** **reduced row-echelon matrix**

$$\left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{cases} x + 0y - 10z = 0 \\ 0x + 1y + 3z = 0 \\ 0x + 0y + 0z = 1 \end{cases}$$

- Step 2. If a row [0 0 0…0 1] occurs, the system is **inconsistent**

reduced row echelon matrix

- Step 3. Otherwise, assign the **nonleading variables** as parameters, solve for the leading variables in terms of parameters

$$\left[\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} 1x + 0y - 10z = 0 \\ 0x + 1y + 3z = 0 \\ 0x + 0y + 0z = 0 \end{cases}$$

z=t (parameter)

z is nonleading variable

Solve the following system of equations

$x - 2y = 3$

$2x - 3y + 2z = 5$

$-3x + 7y + 2z = 10$

Solution.

Carry the augmented matrix to reduced row-echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 2 & -3 & 2 & 5 \\ -3 & 7 & 2 & 10 \end{array}\right] \xrightarrow[3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 19 \end{array}\right] \xrightarrow{-r_2+r_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

$$\xrightarrow{\frac{1}{20}r_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

inconsistent

Example 3: Solve the following system of equations.

$$x_1 - 2x_2 - x_3 + 3x_4 = 1$$
$$2x_1 - 4x_2 + x_3 = 5$$
$$x_1 - 2x_2 + 2x_3 - 3x_4 = 4$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

leading one

$$\to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

reduced row echelon matrix

x_2,x_4 are **<u>nonleading variables</u>**, so we set x_2=t and x_4=s (parameters) and then compute x_1, x_3

$$\begin{aligned} x_1 &= 2 + 2t - s \\ x_2 &= t \\ x_3 &= 1 + 2s \\ x_4 &= s \end{aligned}$$

Theorem 2

Suppose a system of m equations in **n variables** has a solution. If the **rank** of the augment matrix is **r** then the set of solutions involves exactly **n-r** parameters

leading one

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rankA=2

4(number of variables)- 2(rankA) =2
(two parameters : x_2=t, x_4=s)

1.3.Homogeneous Equations (phương trình thuần nhất)

- The system is called **homogeneous** if the constant matrix has all the entry are zeros

- Note that every homogeneous system **has at least one solution (0,0,…,0),** called **trivial solution**
- If a homogeneous system of linear equations has **nontrivial solution** then it has infinite family of solutions.

Example 1: Show that the following homogeneous system has nontrivial solutions.

$$x_1 - x_2 + 2x_3 + x_4 = 0$$
$$2x_1 + 2x_2 - x_4 = 0$$
$$3x_1 + x_2 + 2x_3 + x_4 = 0$$

Solution

The reduction of the augmented matrix to reduced row – echelon form is outined below.

$$\left\langle\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 & 0 \end{array}\right\rangle \rightarrow \left\langle\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -4 & -3 & 0 \\ 0 & 4 & -4 & -2 & 0 \end{array}\right\rangle \rightarrow \left\langle\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right\rangle$$

The leading variables are x_1, x_2 and x_4, so x_3 is assigned as a parameter – say $x_3 = t$. Then the general solution is

$$x_1 = -t$$

$$x_2 = t$$

$$x_3 = t$$

$$x_4 = 0$$

Hence, taking $t = 1$, we get a nontrivial solution: $x_1 = -1, x_2 = 1, x_3 = 1, x_4 = 0$.

Theorem 1

If a homogeneous system of linear equations has **more variables than equations**, then it has nontrivial solution (in fact, infinitely many)

Note that the converse of theorem 1 is not true

The rank of a matrix

- **The rank of the matrix A**, rankA = the *number of leading ones* in the reduced row-echelon form of A.

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Rank } 2$$

Summary

Elementary Operations

Gaussian Elimination

Homogeneous Equations

THANKS