Chapter 5: COUNTING

Rules when submitting:

1. File format: pdf. (Instruction: https://youtu.be/TFG-mSPBx0I)
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1. Suppose that a "word" is any string of seven letters of the alphabet, with repeated letters allowed.

a) How many words are there?
Ans: 26^7.

b) How many words end with the letter T?
Ans: 26^6.

c) How many words begin with R and end with T?
Ans: 26^5.

d) How many words begin with A or B?
Ans: $2 \cdot 26^6$.

e) How many words begin with AAB in some order?
Ans: $3 \cdot 26^4$.

f) How many words have no vowels?
Ans: 21^7.

h) How many words have exactly one vowel?
Ans: $5 \cdot 7 \cdot 21^6$.

2. Let us consider all bit strings of length 12.

a) How many begin with 110?
Ans: 2^9.

b) How many begin with 11 and end with 10?
Ans: 2^8.

c) How many begin with 11 or end with 10?
Ans: $2 \cdot 2^{10} - 2^8$.

3. How many permutations of the seven letters A,B,C,D,E,F,G are there?
Ans: 7!.

4. How many permutations of the seven letters A,B,C,D,E,F,G have E in the first position?
Ans: 6!.

5. How many permutations of the seven letters A,B,C,D,E,F,G have A immediately to the left of E?
Ans: 6!.

6. How many permutations of the seven letters A,B,C,D,E,F,G neither begin nor end with A?
Ans: $5 \cdot 6!$.

7. How many permutations of the seven letters A,B,C,D,E,F,G do not have the vowels next to each other?
Ans: $7! - 2 \cdot 6!$.

8. Let A be the set of all bit strings of length 10.

a) How many bit strings of length 10 are there?
Ans: 2^{10}.

b) How many bit strings of length 10 begin with 1101?
Ans: 2^6.

9. A club with 20 women and 17 men needs to choose three different members to be president, vice president, and treasurer.
(a) In how many ways is this possible?
(b) In how many ways is this possible if women will be chosen as president and vice president and a man as treasurer?
Ans: (a) $37 \cdot 36 \cdot 35$. (b) $20 \cdot 19 \cdot 17$.

10. Suppose $|A| = 4$ and $|B| = 10$. Find the number of functions $f: A \to B$.
    Ans: 10^4.

11. Suppose $|A| = 4$ and $|B| = 10$. Find the number of 1-1 functions $f: A \to B$.
    Ans: $P(10,4)$.

12. Suppose $|A| = 10$ and $|B| = 4$. Find the number of 1-1 functions $f: A \to B$.
    Ans: 0.

13. A be the set of all strings of decimal digits of length five. For example 00312 and 19483 are strings in A.

a) Find $|A|$.
   Ans: 10^5.

b) How many strings in A begin with 774?
   Ans: 10^2.

c) How many strings in A have exactly one 5?
   Ans: $5 \cdot 9^4$.

14. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that begin and end with the same letter.
    Ans: $26 \cdot 26^6$.

Chapter 7.

15. Describe each sequence recursively. Include initial conditions and assume that the sequences begin with a_1.

a) $a_n = 5^n$.
   Ans: $a_n = 5a_{n-1}, a_1 = 5$.

b) The Fibonacci numbers.
   Ans: $a_n = a_{n-1} + a_{n-2}$, $a_1 = a_2 = 1$.

c) 0,1,0,1,0,1,….
   Ans: $a_n = a_{n-2}$, $a_1 = 0$, $a_2 = 1$.

d) $a_n = 1 + 2 + 3 + ... + n$.
Ans: $a_n = a_{n-1} + n$, $a_1 = 1$.

e) $a_n = n!$.
Ans: $a_n = na_{n-1}$, $a_1 = 1$.

f) $a_n =$ the number of bit strings of length n with an even number of 0s.
Ans: $a_n = a_{n-1} + 2^{n-2}$, $a_1 = 1$.

g) $a_n =$ the number of bit strings of length n that begin with 1.
Ans: $a_n = 2a_{n-1}$, $a_1 = 1$.

h) $a_n =$ the number of bit strings of length n that contain a pair of consecutive 0s.
Ans: $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$, $a_1 = 0$, $a_2 = 1$.

16. Verify that $a_n = 6$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
Ans: $4 \cdot 6 - 3 \cdot 6 = 1 \cdot 6 = 6$.

17. Verify that $a_n = 3^n$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
Ans: $4 \cdot 3^{n-1} - 3 \cdot 3^{n-2} = 4 \cdot 3^{n-1} - 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.

18. Verify that $a_n = 3^{n+4}$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
Ans: $4 \cdot 3^{n+3} - 3 \cdot 3^{n+2} = 4 \cdot 3^{n+3} - 3^{n+3} = 3 \cdot 3^{n+3} = 3^{n+4}$.

19. Find a recurrence relation with initial condition(s) satisfied by the sequence. Assume a_0 is the first term of the sequence.

a) $a_n = 2^n$.
Ans: $a_n = 2a_{n-1}$, $a_0 = 1$.

b) $a_n = 2^n + 1$.
Ans: $a_n = 2a_{n-1} - 1$, $a_0 = 2$.

c) $a_n = (-1)^n$.
Ans: $a_n = -a_{n-1}$, $a_0 = 1$.

20. You take a job that pays \$25,000 annually.

(a) How much do you earn n years from now if you receive a three percent raise each year?

(b) How much do you earn n years from now if you receive a five percent raise each year?

Ans: (a) $25{,}000 \cdot 1.03^n$. (b) $25{,}000 \cdot 1.05^n$.

21. The solutions to $a_n = -3a_{n-1} + 18a_{n-2}$ have the form $a_n = c \cdot 3^n + d \cdot (-6)^n$. Which of the following are solutions to the given recurrence relation?

(a) $a_n = 3^{n+1} + (-6)^n$.

(b) $a_n = 5(-6)^n$.

(c) $a_n = 3c - 6d$.

(d) $a_n = 3^{n-2}$.

Ans: (a) Yes. (b) Yes. (c) No. (d) Yes.

22. Suppose $f(n) = 3f(n/2) + 1$, $f(1) = 1$. Find $f(8)$.

Ans: 40.

23. Suppose $f(n) = f(n/3) + 2n$, $f(1) = 1$. Find $f(27)$.

Ans: 79.

24. Suppose $f(n) = 2f(n/2)$, $f(8) = 2$. Find $f(1)$.

Ans: 1/4.