Bayesian Learning - Lecture 7

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Lecture overview

- Random number generation
- Monte Carlo simulation
- Gibbs sampling

Monte Carlo sampling

- If $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(N)}$ is an *iid* sequence from a distribution $p(\theta)$, then

$$\begin{aligned} \frac{1}{N}\sum_{t=1}^{N}\theta^{(t)} &\rightarrow E(\theta) \\ \frac{1}{N}\sum_{t=1}^{N}g(\theta^{(t)}) &\rightarrow E[g(\theta)] \end{aligned}$$

  where $g(\theta)$ is some well-behaved function.
- Easy to compute tail probabilities $\Pr(\theta \leq c)$ by letting

$$g(\theta) = I\left(\theta \leq c\right)$$

  and

$$\frac{1}{N}\sum_{t=1}^{N}g(\theta^{(t)}) = \frac{\#\ \theta\text{-draws smaller than } c}{N}.$$

Direct sampling by the inverse CDF method

- How to simulate from a distribution?
- Let $f(x)$ be the density function of a stochastic variable. CDF: $F(x)$. Inverse CDF method:
  1. Generate u from the uniform distribution on $[0, 1]$.
  2. Compute $x = F^{-1}(u)$.
- Example 1: Exponential distribution:

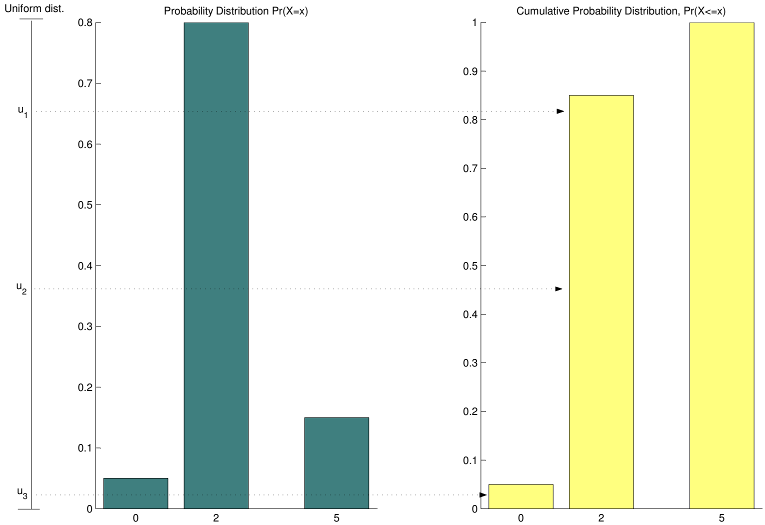
$$u = F(x) = 1 - \exp(-\lambda x)$$

Inverting gives

$$x = -\ln(1 - u)/\lambda$$

But $1 - u$ is also uniformly distributed on [0,1]. Thus: if $x = -(\ln u)/\lambda$ where $u \sim \mathit{Unif}(0, 1)$, then $x \sim \mathit{Expon}(\lambda)$.

Inverse CDF method, discrete case



Direct sampling by the inverse CDF method

- Example 2: Cauchy distribution:

$$\begin{aligned} f(x) &= \frac{1}{\pi}\frac{1}{1+x^2} \\ u &= F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x) \end{aligned}$$

  Inverting ...

$$x = \tan[\pi(u - 1/2)].$$

- We can also use relations between distribution to sample from distributions.
- Cauchy-example, cont. If y and z are independent $N(0,1)$ variables, then $z = \frac{y}{z} \sim Cauchy$.
- Example: Chi-square. If $x_1, ..., x_v \overset{iid}{\sim} N(0,1)$, then $y = \sum_{i=1}^{v} x_i^2 \sim \chi_v^2$.

Gibbs sampling

- Easily implemented methods for sampling from multivariate distributions, $p(\theta_1, ..., \theta_k)$.
- Requirements: Easily sampled full conditional posteriors:
  - $p(\theta_1|\theta_2, \theta_3 ..., \theta_k)$
  - $p(\theta_2|\theta_1, \theta_3, ..., \theta_k)$
  - $\vdots$
  - $p(\theta_k|\theta_1, \theta_2, ..., \theta_{k-1})$
- Started out in the early 80's in the image analysis literature.

The Gibbs sampling algorithm

A: Choose initial values $\theta_2^{(0)}, \theta_3^{(0)}, ..., \theta_n^{(0)}$.

B: B_1 Draw $\theta_1^{(1)}$ from $p(\theta_1|\theta_2^{(0)}, \theta_3^{(0)}, ..., \theta_n^{(0)})$

B_2 Draw $\theta_2^{(1)}$ from $p(\theta_2|\theta_1^{(1)}, \theta_3^{(0)}, ..., \theta_n^{(0)})$

$\vdots$

B_n Draw $\theta_n^{(1)}$ from $p(\theta_n|\theta_1^{(1)}, \theta_2^{(1)}, ..., \theta_{n-1}^{(1)})$

C: Repeat Step B N times.

Gibbs sampling, cont.

- The Gibbs draws $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(N)}$ are dependent, but arithmetic means converge to expected values

$$\begin{aligned} \frac{1}{N}\sum_{t=1}^{N}\theta_j^{(t)} &\rightarrow E(x_j) \\ \frac{1}{N}\sum_{t=1}^{N}g(\theta^{(t)}) &\rightarrow E[g(\theta)] \end{aligned}$$

- More generally, the Gibbs sequence $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(N)}$ converges in distribution to the target posterior $p(\theta_1, ..., \theta_k)$.
- $\theta_j^{(1)}, ..., \theta_j^{(N)}$ converge to the marginal distribution of θ_j, $p(\theta_j)$.

Gibbs sampling for normal model with non-conjugate prior

- Normal model with semi-conjugate prior

$$\mu \sim N(\mu_0, \tau_0^2)$$
$$\sigma^2 \sim Inv - \chi^2(\nu_0, \sigma_0^2)$$

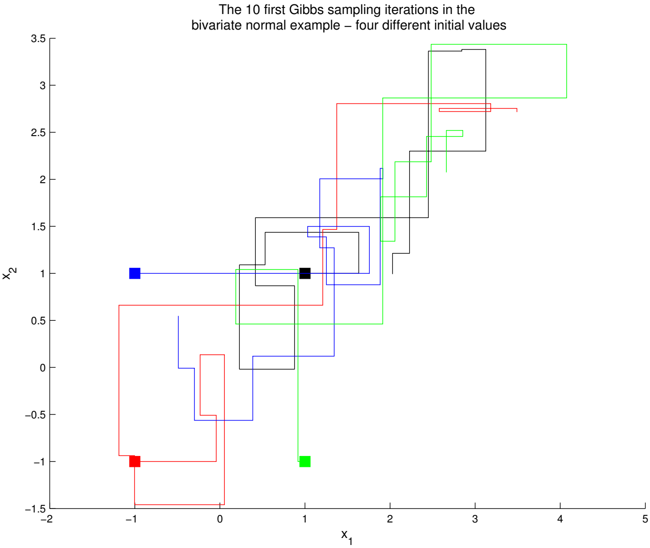
- Conditional posteriors

$$\mu|\sigma^2, x \sim N\left(\mu_n, \tau_n^2\right)$$
$$\sigma^2|\mu, x \sim Inv - \chi^2\left(\nu_n, \frac{\nu_0\sigma_0^2 + \sum_{i=1}^n (x_i - \mu)^2}{n + \nu_0}\right)$$

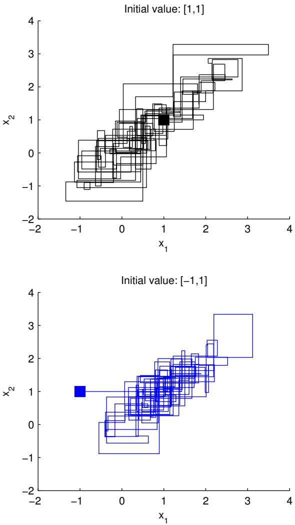
Gibbs sampling multivariate normal

- Bivariate normal:
  - Joint distribution
$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$
  - Full conditional posteriors:
$$\begin{aligned} \theta_1|\theta_2 &\sim N[\mu_1 + \rho(\theta_2 - \mu_2), 1-\rho^2] \\ \theta_2|\theta_1 &\sim N[\mu_2 + \rho(\theta_1 - \mu_1), 1-\rho^2] \end{aligned}$$

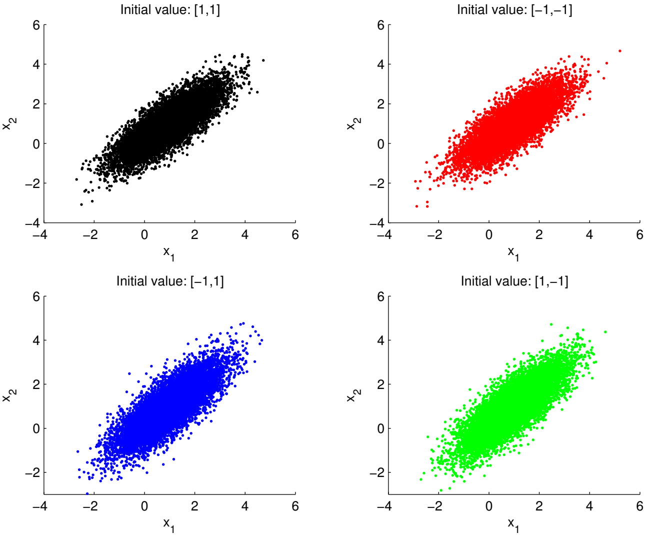
Gibbs sampling - Bivariate normal



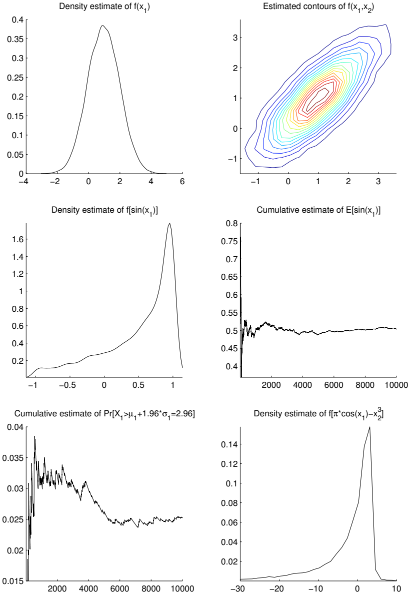
Gibbs sampling - Bivariate normal



Gibbs sampling - Bivariate normal



Gibbs sampling - Bivariate normal



Autoregressive processes (AR)

- AR(p) process

$$x_t = \mu + \phi_1(x_{t-1} - \mu) + ... + \phi_p(x_{t-p} - \mu) + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2).$$

- Random walk prior:

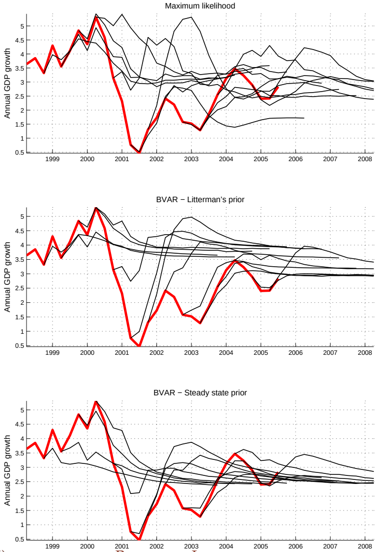
$$\begin{aligned} E(\phi_1) &= 1 \\ E(\phi_j) &= 0 \text{ for } j = 2, ..., p. \\ StDev(\phi_j) &= \frac{\psi}{j}. \end{aligned}$$

  Note how the prior shrinks longer lags more heavily toward zero.
- $\mu = E(x_t)$ is the unconditional mean or steady-state of the process. 'where the system goes to if the shocks (ε_t) are turned off'.
- μ is important as long-run forecasts (quickly) approach the steady state.
- Prior: $\mu \sim N(\theta_\mu, \psi_\mu^2)$, independent of $\phi^{'}$s and σ.

Gibbs sampling for AR processes

- The posterior can be simulated by Gibbs sampling:
  - $\mu|\phi, \sigma^2, x \sim$ Normal
  - $\phi|\mu, \sigma^2, x \sim$ Multivariate Normal
  - $\sigma^2|\mu, \phi, x \sim$ Scaled Inverse χ^2
- Everything above can easily be extended to vector processes (VARs).
- Example: Swedish GDP and inflation forecasts.

GDP growth forecasts from Bayesian VARs



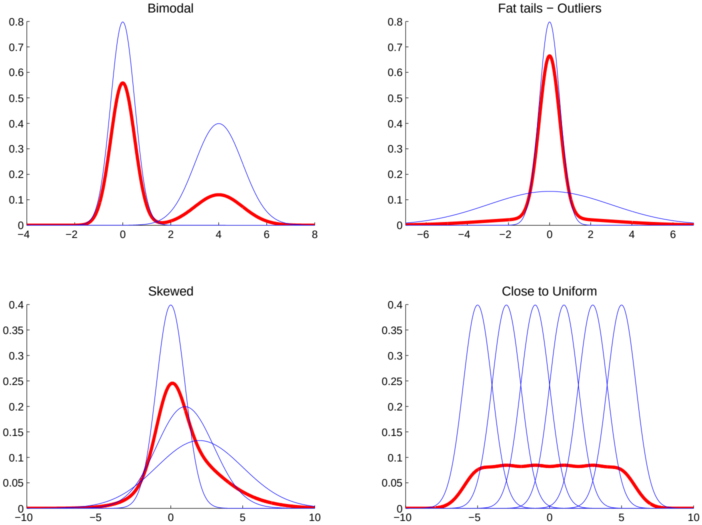
Data augmentation - Mixture distributions

- Two-component mixture of normals [MN(2)]

$$p(x) = \pi \cdot \phi(x|\mu_1, \sigma_1^2) + (1 - \pi) \cdot \phi(x|\mu_2, \sigma_2^2),$$

  where $\phi(x|\mu, \sigma^2)$ denotes the PDF of a normal variate x with mean μ and variance σ^2.
- Simulate from a MN(2):
  - Simulate an indicator $I \sim Bern(\pi)$.
  - If $I = 0$, simulate x from $N(\mu_1, \sigma_1^2)$
  - If $I = 1$, simulate x from $N(\mu_2, \sigma_2^2)$.

Illustration of mixture distributions



Mixture distributions, cont.

- Not easy to estimate directly - the likelihood is a product of sums.
- **Assume** that we knew which of the two densities each observation came from.

$$I_i = \begin{cases} 0 \text{ if } x_i \text{ came from Density 1} \\ 1 \text{ if } x_i \text{ came from Density 2} \end{cases}.$$

- Armed with knowledge of $I_1, ..., I_n$ it is now easy to estimate π, $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ by separating the sample according to the I's.
- But we do **not** know $I_1, ..., I_n$!

Mixture distributions, cont.

- Gibbs sampling to the rescue. Assume: Prior $\pi \sim Beta(\alpha_1, \alpha_2)$. Conjugate prior for (μ_j, σ_j^2), see Lecture 3. $n_2 = \sum_{i=1}^{n} I_i$ and $n_1 = n - n_2$.
- Algorithm:
  - $\pi \mid I, x \sim Beta(\alpha_1 + n_1, \alpha_2 + n_2)$
  - $\sigma_1^2 \mid \mu_1, I, x \sim Inv\text{-}\chi^2$ and $\mu_1 | I, \sigma^2, x \sim N$
  - $\sigma_2^2 \mid \mu_2, I, x \sim Inv\text{-}\chi^2$ and $\mu_2 | I, \sigma^2, x \sim N$
  - $I_i \mid \pi, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2, x \sim Bern(\theta_i)$, $i = 1, ..., n$,

$$\theta_i = \frac{(1-\pi)\phi(x_i; \mu_2, \sigma_2^2)}{\pi\phi(x_i; \mu_1, \sigma_1^2) + (1-\pi)\phi(x_i; \mu_2, \sigma_2^2)}.$$

Mixture distributions, cont.

- Generalization k-component mixture of normals

$$p(x) = \sum_{j=1}^{k} \pi_j \phi(x; \mu_j, \sigma_j^2),$$

where $\sum_{j=1}^{k} \pi_j = 1$.

- Gibbs sampling with multi-class indicators ($I_i = j$ if observation i comes from density j):
  - $(\pi_1, ..., \pi_k) \mid I, x \sim \mathit{Dirichlet}(\alpha_1 + n_1, \alpha_2 + n_2, ..., \alpha_k + n_k)$
  - $\sigma_j^2 \mid \mu_j, I, x \sim \mathit{Inv}\text{-}\chi^2$ and $\mu_j | I, \sigma_j^2, x \sim N$, $\quad$ *for* $j = 1, ..., k$,
  - $I_i \mid \pi, \mu, \sigma^2, x \sim \mathit{Multinomial}(\theta_{i1}, ..., \theta_{ik})$, *for* $i = 1, ..., n$,

$$\theta_{ij} = \frac{\pi_j \phi(x_i; \mu_j, \sigma_j^2)}{\sum_{r=1}^{k} \pi_r \phi(x_i; \mu_r, \sigma_r^2)}.$$

- More generally: Gibbs sampling is very powerful for missing data problems.

Data augmentation - Probit regression

- Probit model:
$$\Pr(y_i = 1 \mid x_i) = \Phi(x_i'\beta)$$
- Random utility formulation of the probit:
$$\begin{aligned} u_i &\sim N(x_i'\beta, 1) \\ y_i &= \begin{cases} 1 & \text{om } u_i > 0 \\ 0 & \text{om } u_i \leq 0 \end{cases}. \end{aligned}$$
- Check: $\Pr(y_i = 1 \mid x_i) = \Pr(u_i > 0) = 1 - \Pr(u_i \leq 0) = 1 - \Pr(u_i - x_i'\beta < -x_i'\beta) = 1 - \Phi(-x_i'\beta) = \Phi(x_i'\beta)$.
- If $u = (u_1, ..., u_n)$ were observed, then β could be analyzed by traditional linear regression. But, u is not observed. Gibbs sampling to the rescue!

Gibbs sampling for the Probit regression

- Simulate from joint posterior $p(u, \beta|y)$ iterating between the full conditional posteriors:
  - $p(\beta|u, y)$, which is multivariate normal (this is just a linear regression)
  - $p(u_i|\beta, y), \; i = 1, ..., n$.
- The full conditional posterior distribution of u_i is:

$$
\begin{aligned}
p(u_i|\beta, y) &\propto p(y_i|\beta, u_i) p(u_i|\beta) \\
&= \begin{cases} N(u_i|x_i'\beta, 1) & \text{truncated to } u_i \in (-\infty, 0] \text{ if } y_i = 0 \\ N(u_i|x_i'\beta, 1) & \text{truncated to } u_i \in (0, \infty) \text{ if } y_i = 1 \end{cases}
\end{aligned}
$$

- Collect the β-draws. A histogram of these draws approximates $p(\beta|y) = \int p(u, \beta|y) du$.

Improving the efficiency of the Gibbs sampler

- *Efficient blocking*. Correlated parameters should be included in the same updating block.
- *Reparametrization*. Convergence can improve dramatically in alternative parametrizations.
- *Data augmentation*. Bring in latent (unobserved) variables that make the full conditional posteriors more easily sampled (Probit, Mixture models etc). Downside: Typically increases the autocorrelation between draws.
- *Parameter expansion*. Introducing (non-sense) parameters in the model may break the dependence between the original parameters (Example probit).