Bayesian Learning - Lecture 2

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Lecture overview

- The Poisson model
- Conjugate priors
- Prior elicitation - how to come up with a prior.
- Non-informative priors

Poisson model

- **Model**:

$$y_1, ..., y_n | \theta \overset{iid}{\sim} Pois(\theta)$$

- **Likelihood** from iid Poisson sample $y = (y_1, ..., y_n)$

$$p(y|\theta) = \left[\prod_{i=1}^{n} p(y_i|\theta)\right] \propto \theta^{(\sum_{i=1}^{n} y_i)} \exp(-\theta n),$$

- ***Prior***:

$$p(\theta) \propto \theta^{\alpha-1} \exp(-\theta\beta) \propto Gamma(\alpha, \beta)$$

which contains the info: $\alpha - 1$ counts in β observations.

Poisson model, cont.

- ***Posterior***

$$
\begin{aligned}
p(\theta|y_1,...,y_n) &\propto \left[\prod_{i=1}^n p(y_i|\theta)\right] p(\theta) \\
&\propto \theta^{\sum_{i=1}^n y_i} \exp(-\theta n)\theta^{\alpha-1}\exp(-\theta\beta) \\
&= \theta^{\alpha+\sum_{i=1}^n y_i - 1}\exp[-\theta(\beta+n)],
\end{aligned}
$$

which is proportional to the $Gamma(\alpha + \sum_{i=1}^n y_i, \beta + n)$ distribution.

- Prior-to-Posterior mapping:

$$
\begin{aligned}
\text{Model: } & y_1,...,y_n|\theta \overset{iid}{\sim} Pois(\theta) \\
\text{Prior: } & \theta \sim Gamma(\alpha,\beta) \\
\text{Posterior: } & \theta|y_1,...,y_n \sim Gamma(\alpha + \textstyle\sum_{i=1}^n y_i, \beta + n).
\end{aligned}
$$

Poisson example - Bomb hits in London

$n = 576$, $\sum_{i=1}^{n} y_i = 229 \cdot 0 + 211 \cdot 1 + 93 * 2 + 35 * 3 + 7 * 4 + 1 \cdot 5 = 537$.

Average number of hits per region$=\bar{y} = 537/576 \approx 0.9323$.

$$p(\theta|y) \propto \theta^{\alpha+537-1} \exp[-\theta(\beta+576)]$$

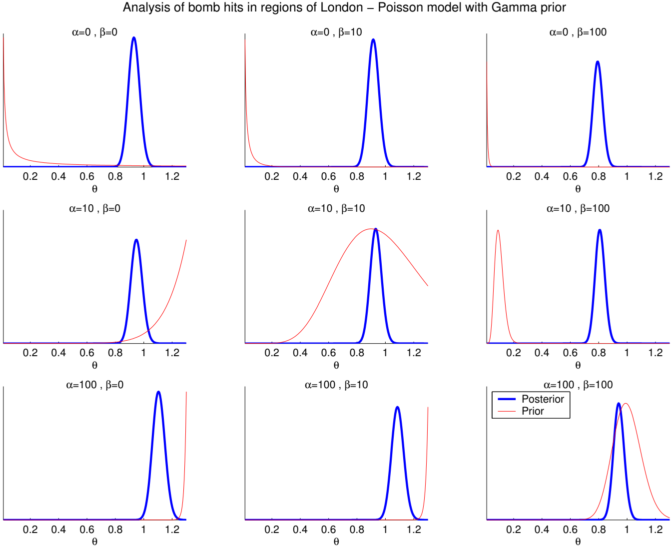
$$E(\theta|y) = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} \approx \bar{y} \approx 0.9323,$$

and

$$SD(\theta|y) = \left(\frac{\alpha + \sum_{i=1}^{n} y_i}{(\beta+n)^2}\right)^{1/2} = \frac{(\alpha + \sum_{i=1}^{n} y_i)^{1/2}}{(\beta+n)} \approx \frac{(537)^{1/2}}{576} \approx 0.0402.$$

if α and β are small compared to $\sum_{i=1}^{n} y_i$ and n.

Poisson bomb hits in London



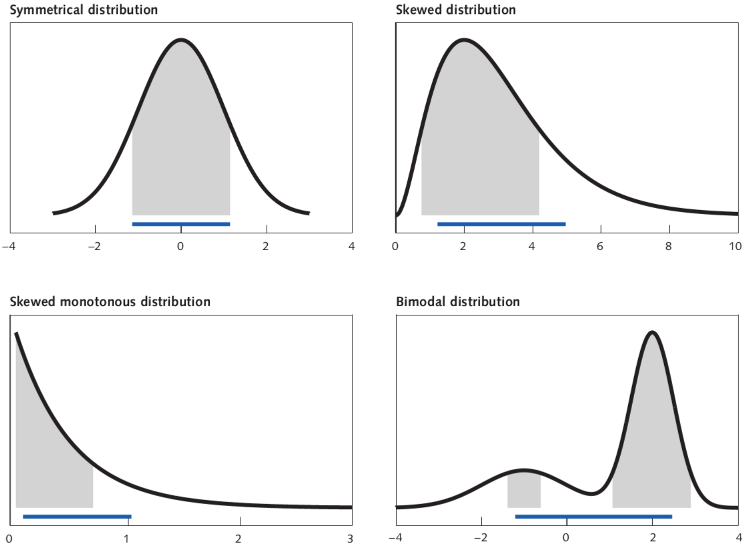
Poisson example - posterior intervals

- **Bayesian 95% interval**: the probability that the unknown parameter θ lies in the interval is 0.95. What a relief!
- Approximate 95% **credible interval** for θ (for small α and β):

$$E(\theta|y) \pm 1.96 \cdot SD(\theta|y) = [0.8535; 1.0111]$$

- An exact 95% equal-tail interval is $[0.8550;\ 1.0125]$ (assuming $\alpha = \beta = 0$)
- **Highest Posterior Density** (**HPD**) interval contains the θ values with highest pdf.
- An exact Highest Posterior Density (HPD) interval is $[0.8525;\ 1.0144]$. Obtained numerically, assuming $\alpha = \beta = 0$.

Illustration of different interval types



Conjugate priors

- Normal likelihood: Normal prior$\to$Normal posterior. (posterior belongs to the same distribution family as prior)
- Bernoulli likelihood: Beta prior$\to$Beta posterior.
- Poisson likelihood: Gamma prior$\to$Gamma posterior.
- **Conjugate priors**: A prior is conjugate to a model (likelihood) if the prior and posterior belong to the same distributional family.
- Formal definition: Let $\mathcal{F} = \{p(y|\theta), \theta \in \Theta\}$ be a class of sampling distributions. A family of distributions $\mathcal{P}$ is conjugate for $\mathcal{F}$ if

$$p(\theta) \in \mathcal{P} \Rightarrow p(\theta|x) \in \mathcal{P}$$

  holds for all $p(y|\theta) \in \mathcal{F}$.

Prior elicitation

- The prior should be determined (elicited) by an **expert**. Typically, expert$\neq$statistician.
- Elicit the prior on a **quantity that she knows well** (maybe log odds $\ln\frac{\theta}{1-\theta}$ when the model is $Bern(\theta)$). The statistician can always compute the implied prior on other quantities after the elicitation.
- Elicit the prior by asking the expert probabilistic questions:
  - $E(\theta) =?$
  - $SD(\theta) =?$
  - $Pr(\theta < c) =?$
  - $Pr(y > c) =?$
- **Show the expert some consequences** of her elicitated prior. If she does not agree with these consequences, iterate the above steps until she is happy.
- **Beware of psychological effects**, such as anchoring.

Prior elicitation - AR(p) example

- **Autoregressive process** or order p

$$y_t = \phi_1(y_{t-1} - \mu) + ... + \phi_p(y_{t-p} - \mu) + \varepsilon_t, \ \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

- Informative prior on the unconditional mean: $\mu \sim N(\mu_0, \tau_0^2)$. Usually, μ_0 and τ_0^2 can be specified accurately.
- "Noninformative" prior on σ^2: $p(\sigma^2) \propto 1/\sigma^2$
- Assume for simplicity that all ϕ_i, $i = 1, ..., p$ are independent a priori, and $\phi_i \sim N(\mu_i, \psi_i)$
- Prior on $\phi = (\phi_1, ..., \phi_p)$ centered on persistent AR(1) process: $\mu_1 = 0.8, \mu_2 = ... = \mu_p = 0$
- Prior variance of the ϕ_i decay towards zeros: $Var(\phi_i) = \frac{c}{i^\lambda}$, so that "longer" lags are more likely to be zero a priori. λ is a parameter that can be used to determine the rate of decay.

Different types of prior information

- Real **expert information**. Combo of previous studies and experience.
- Vague prior information, or even **noninformative priors**.
- **Reporting priors**. Easy to understand the information they contain.
- **Smoothness priors**. Regularization. Shrinkage. Big thing in modern statistics/machine learning.

Non-informative priors

- ... do not exist!
- ... may be **improper** and still lead to proper posterior
- **Regularization priors.**
- Ideal: Present the posterior distributions for all possible priors.
- Practical communication - **Reference priors**.
- Model the prior in terms of a few **hyperparameters**.

NON-INFORMATIVE PRIORS

- **Subjective consensus**: when extreme priors give essentially the same posterior.

$$p(\theta|y) \to N\left(\hat{\theta}, J^{-1}_{\hat{\theta},\mathbf{x}}\right) \text{ for all } p(\theta) \text{ as } n \to \infty,$$

where $J_{\theta,\mathbf{x}}$ is the **observed information**

$$J_{\theta,\mathbf{x}} = -\frac{\partial^2 \ln L(\theta;\mathbf{x})}{\partial\theta^2}|_{\theta=\hat{\theta}}$$

- A common non-informative prior is **Jeffreys' prior**

$$p(\theta) = |I_\theta|^{1/2},$$

where I_θ is the **Fisher information**

$$I_\theta = E_{\mathbf{x}|\theta}\left(J_{\theta,\mathbf{x}}\right)$$

Jeffreys' prior for Bernoulli trial data

$$x_1, ..., x_n|\theta \overset{iid}{\sim} Bern(\theta).$$

$$\ln p(\mathbf{x}|\theta) = s \ln \theta + f \ln(1-\theta)$$

$$\frac{d \ln p(\mathbf{x}|\theta)}{d\theta} = \frac{s}{\theta} - \frac{f}{(1-\theta)}$$

$$\frac{d^2 \ln p(\mathbf{x}|\theta)}{d\theta^2} = -\frac{s}{\theta^2} - \frac{f}{(1-\theta)^2}$$

$$I(\theta) = \frac{E_{\mathbf{x}|\theta}(s)}{\theta^2} + \frac{E_{\mathbf{x}|\theta}(f)}{(1-\theta)^2} = \frac{n\theta}{\theta^2} + \frac{n(1-\theta)}{(1-\theta)^2} = \frac{n}{\theta(1-\theta)}$$

Thus, the Jeffreys' prior is

$$p(\theta) = |I(\theta)|^{1/2} \propto \theta^{-1/2}(1-\theta)^{-1/2} \propto Beta(\theta|1/2, 1/2).$$

Properties of Jeffreys prior

- **Invariant** to 1:1 transformations of θ. Doesn't matter which parametrization we derive the prior, it always contains the same info.
- Two models with identical likelihood functions (up to constant) can yield different Jeffreys' prior. Jeffreys' prior **does not respect the likelihood principle**. The crux of the matter is the expectation with respect to the sampling distribution.
- Jeffreys' prior may be a very complicated (**non-conjugate**) distribution.
- **Problematic in multivariate problems**. Dubious results in many standard models.