Bayesian Learning - Lecture 12

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Overview

- Bayesian variable selection
- Summary and review of course material

Bayesian variable selection

- Linear regression:

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p + \varepsilon.$$

- Which variables have non-zero coefficient? Example of hypotheses:

$$\begin{aligned} H_0 &: \beta_0 = \beta_1 = ... = \beta_p = 0 \\ H_1 &: \beta_1 = 0 \\ H_2 &: \beta_1 = \beta_2 = 0 \end{aligned}$$

- Introduce **variable selection indicators** $\mathcal{I} = (I_1, ..., I_p)$.
- Example: $\mathcal{I} = (1, 1, 0)$ means that $\beta_1 \neq 0$ and $\beta_2 \neq 0$, but $\beta_3 = 0$, so x_3 drops out of the model.

Bayesian variable selection, cont.

- Model inference, just crank the Bayesian machine:

$$p(\mathcal{I}|\mathbf{y}, \mathbf{X}) \propto p(\mathbf{y}|\mathbf{X}, \mathcal{I}) \cdot p(\mathcal{I})$$

- The prior $p(\mathcal{I})$ is typically taken to be $I_1, ..., I_p|\theta \overset{iid}{\sim} Bernoulli(\theta)$.
- θ is the **prior inclusion probability**.

- Challenge: Computing the **marginal likelihood** for each model ($\mathcal{I}$)

$$p(\mathbf{y}|\mathbf{X}, \mathcal{I}) = \int p(\mathbf{y}|\mathbf{X}, \mathcal{I}, \beta) p(\beta|\mathbf{X}, \mathcal{I}) d\beta$$

Bayesian variable selection, cont.

- Let $\beta_{\mathcal{I}}$ denote the **non-zero** coefficients under $\mathcal{I}$.
- Prior:

$$\beta_{\mathcal{I}}|\sigma^2 \sim N\left(0, \sigma^2\Omega_{\mathcal{I},0}^{-1}\right)$$
$$\sigma^2 \sim Inv-\chi^2\left(\nu_0, \sigma_0^2\right)$$

- **Marginal likelihood**

$$p(\mathbf{y}|\mathbf{X}, \mathcal{I}) \propto \left|\mathbf{X}_{\mathcal{I}}'\mathbf{X}_{\mathcal{I}} + \Omega_{\mathcal{I},0}^{-1}\right|^{-1/2} |\Omega_{\mathcal{I},0}|^{1/2} \left(\nu_0\sigma_0^2 + RSS_{\mathcal{I}}\right)^{-(\nu_0+n-1)/2}$$

where $\mathbf{X}_{\mathcal{I}}$ is the covariate matrix for the subset given by $\mathcal{I}$.

- $RSS_{\mathcal{I}}$ is (almost) the residual sum of squares under model implied by $\mathcal{I}$

$$RSS_{\mathcal{I}} = \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}_{\mathcal{I}}\left(\mathbf{X}_{\mathcal{I}}'\mathbf{X}_{\mathcal{I}} + \Omega_{\mathcal{I},0}\right)^{-1}\mathbf{X}_{\mathcal{I}}'\mathbf{y}$$

Bayesian variable selection via Gibbs sampling

- But there are 2^p model combinations to go through! Ouch!
- ... but most will have essentially zero posterior probability. Phew!

- Simulate from the joint posterior distribution:

$$p(\beta, \sigma^2, \mathcal{I}|\mathbf{y},\mathbf{X}) = p(\beta, \sigma^2|\mathcal{I}, \mathbf{y},\mathbf{X})p(\mathcal{I}|\mathbf{y}, \mathbf{X}).$$

- Simulate from $p(\mathcal{I}|\mathbf{y})$ using Gibbs sampling:
  - Draw $I_1|\mathcal{I}_{-1}, \mathbf{y}, \mathbf{X}$
  - Draw $I_2|\mathcal{I}_{-2}, \mathbf{y}, \mathbf{X}$
  - ...
  - Draw $I_p|\mathcal{I}_{-p}, \mathbf{y}, \mathbf{X}$

- Only need to compute $Pr(I_i = 0|\mathcal{I}_{-i}, \mathbf{y}, \mathbf{X})$ and $Pr(I_i = 1|\mathcal{I}_{-i}, \mathbf{y}, \mathbf{X})$.
- Automatic model averaging, all in one simulation run.
- If needed, simulate from $p(\beta, \sigma^2|\mathcal{I}, \mathbf{y},\mathbf{X})$ for each draw of $\mathcal{I}$.

Pseudo code for Bayesian variable selection

0 Initialize $\mathcal{I}^{(0)} = (I_1^{(0)}, I_2^{(0)} ..., I_p^{(0)})$

1 Simulate σ^2 and β from [Note: $\nu_n, \sigma_n^2, \mu_n, \Omega_n$ all depend on $\mathcal{I}^{(0)}$]

- $\sigma^2 | \mathcal{I}^{(0)}, \mathbf{y}, \mathbf{X} \sim Inv - \chi^2 \left(\nu_n, \sigma_n^2 \right)$
- $\beta | \sigma^2, \mathcal{I}^{(0)}, \mathbf{y}, \mathbf{X} \sim N \left[\mu_n, \sigma^2 \Omega_n^{-1} \right]$

2.1 Simulate $I_1 | \mathcal{I}_{-1}, \mathbf{y}, \mathbf{X}$ by [define $\mathcal{I}_{prop}^{(0)} = (1 - I_1^{(0)}, I_2^{(0)} ..., I_p^{(0)})$]

- compute marginal likelihoods: $p(\mathbf{y} | \mathbf{X}, \mathcal{I}^{(0)})$ and $p(\mathbf{y} | \mathbf{X}, \mathcal{I}_{prop}^{(0)})$
- Simulate $I_1^{(1)} \sim Bernoulli(\kappa)$ where

$$\kappa = \frac{p(\mathbf{y} | \mathbf{X}, \mathcal{I}^{(0)}) \cdot p(\mathcal{I}^{(0)})}{p(\mathbf{y} | \mathbf{X}, \mathcal{I}^{(0)}) \cdot p(\mathcal{I}^{(0)}) + p(\mathbf{y} | \mathbf{X}, \mathcal{I}_{prop}^{(0)}) \cdot p(\mathcal{I}_{prop}^{(0)})}$$

2.2 Simulate $I_2 | \mathcal{I}_{-2}, \mathbf{y}, \mathbf{X}$ as in Step 2.1, but $\mathcal{I}^{(0)} = (I_1^{(1)}, I_2^{(0)}, ..., I_p^{(0)})$

⋮

2.p Simulate $I_p | \mathcal{I}_{-p}, \mathbf{y}, \mathbf{X}$ as in Step 2.1, but $\mathcal{I}^{(0)} = (I_1^{(1)}, I_2^{(1)}, ..., I_p^{(0)})$

3 Repeat Steps 1-2 many times.

Simple general Bayesian variable selection

- The previous algorithm only works when we can integrate out all the model parameters to obtain

$$p(\mathcal{I}|\mathbf{y},\mathbf{X}) = \int p(\beta,\sigma^2,\mathcal{I}|\mathbf{y},\mathbf{X})d\beta d\sigma$$

- MH - propose β and $\mathcal{I}$ jointly from the proposal distribution

$$q(\beta_p|\beta_c,\mathcal{I}_p)q(\mathcal{I}_p|\mathcal{I}_c)$$

- Main difficulty: how to propose the non-zero elements in β_p?
- Simple approach:
  - Approximate posterior with all variables in the model: $\beta|\mathbf{y},\mathbf{X} \overset{approx}{\sim} N\left[\hat{\beta}, J_{\mathbf{y}}^{-1}(\hat{\beta})\right]$
  - Propose β_p from $N\left[\hat{\beta}, J_{\mathbf{y}}^{-1}(\hat{\beta})\right]$, conditional on the zero restrictions implied by $\mathcal{I}_p$. Formulas are available.