Bayesian Learning - Lecture 6

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Lecture overview

- Classification
- Naive Bayes
- Logistic regression
- Normal approximation of posterior

Bayesian classification

- **Classification: output is a discrete label**. Examples:
  - binary (0-1). Spam/Ham.
  - Multi-class. $(c = 1, 2, ..., C)$. $\{iPhone, Android, Windows, Other\}$.
- **Bayesian classification**

$$\underset{c \in \mathcal{C}}{\operatorname{argmax}}\, p(c|\mathbf{x})$$

  where $\mathbf{x} = (x_1, ..., x_p)$ is a covariate/feature vector.
- **Discriminative models** - model $p(c|\mathbf{x})$ directly.
- Examples: logistic regression, support vector machines.
- **Generative models** - Use Bayes' theorem

$$p(c|\mathbf{x}) \propto p(\mathbf{x}|c)p(c)$$

  and model class-conditional distribution $p(\mathbf{x}|c)$ and prior $p(c)$.
- Examples: discriminant analysis, naive Bayes.

Naive Bayes

- By Bayes' theorem
$$p(c|\mathbf{x}) \propto p(\mathbf{x}|c)p(c)$$
- $p(c)$ can be estimated by Multinomial-Dirichlet analysis.
- $p(\mathbf{x}|c)$ can be $N(\theta_c, \Sigma_c)$ or mixture of normals (see last module).
- $p(\mathbf{x}|c)$ can be very high-dimensional and hard to estimate.
- Even with binary features, the outcome space of $p(\mathbf{x}|c)$ can be huge.
- Naive Bayes: **features are assumed independent**
$$p(\mathbf{x}|c) = \prod_{j=1}^{n} p(x_j|c)$$
- Naive Bayes solution
$$p(c|\mathbf{x}) \propto \left[\prod_{j=1}^{n} p(x_j|c)\right] p(c)$$

Classification with logistic regression

- Response is assumed to be **binary** ($y = 0$ or 1).
- Example: Spam ($y = 1$) or Ham ($y = 0$). Covariates: \$-symbols, etc.
- **Logistic regression**

$$\Pr(y_i = 1 \mid x_i) = \frac{\exp(x_i'\beta)}{1 + \exp(x_i'\beta)}.$$

- Likelihood

$$p(y|X,\beta) = \prod_{i=1}^{n} \frac{[\exp(x_i'\beta)]^{y_i}}{1 + \exp(x_i'\beta)}.$$

- Prior $\beta \sim N(0, \lambda^{-1} I)$. Posterior is non-standard.
- Alternative: **Probit regression** (see Lab 3)

$$Pr(y_i = 1|x_i) = \Phi(x_i^{'}\beta)$$

- **Multi-class** ($c = 1, 2, ..., C$) logistic regression

$$\Pr(y_i = c \mid x_i) = \frac{\exp(x_i'\beta_c)}{\sum_{k=1}^{C} \exp(x_i'\beta_k)}$$

Large sample approximate posterior

- **Taylor expansion of log-posterior** around the posterior mode $\theta = \tilde{\theta}$:

$$\ln p(\theta|y) = \ln p(\tilde{\theta}|y) + \frac{\partial \ln p(\theta|y)}{\partial\theta}|_{\theta=\tilde{\theta}}(\theta - \tilde{\theta})$$
$$+ \frac{1}{2!}\frac{\partial^2 \ln p(\theta|y)}{\partial\theta^2}|_{\theta=\tilde{\theta}}(\theta - \tilde{\theta})^2 + \ldots$$

- From the definition of the posterior mode:

$$\frac{\partial \ln p(\theta|y)}{\partial\theta}|_{\theta=\tilde{\theta}} = 0$$

- So, in large samples (where we can ignore higher order terms):

$$p(\theta|y) \approx p(\tilde{\theta}|y)\exp\left(-\frac{1}{2}J_{\mathbf{y}}(\tilde{\theta})(\theta - \tilde{\theta})^2\right)$$

  where $J_{\mathbf{y}}(\tilde{\theta}) = -\frac{\partial^2 \ln p(\theta|y)}{\partial\theta^2}|_{\theta=\tilde{\theta}}$ is the observed information.
- Approximate posterior

$$\theta|y \overset{approx}{\sim} N\left[\tilde{\theta}, J_{\mathbf{y}}^{-1}(\tilde{\theta})\right]$$

Example: gamma posterior

- Poisson model: $\theta|y_1,...,y_n \sim Gamma(\alpha+\sum_{i=1}^n y_i, \beta+n)$

$$\log p(\theta|y_1,...,y_n) \propto (\alpha+\sum_{i=1}^n y_i - 1)\log\theta - \theta(\beta+n)$$

- First derivative of log density

$$\frac{\partial \ln p(\theta|y)}{\partial\theta} = \frac{\alpha+\sum_{i=1}^n y_i - 1}{\theta} - (\beta+n)$$

$$\tilde{\theta} = \frac{\alpha+\sum_{i=1}^n y_i - 1}{\beta+n}$$

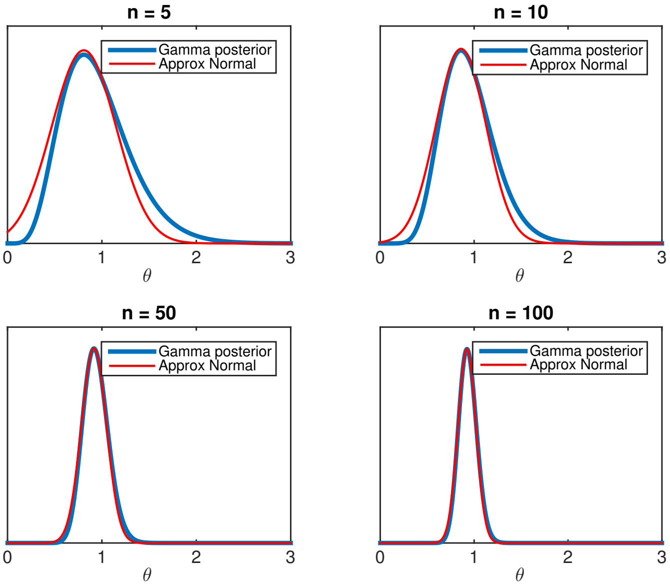
- Second derivative at mode $\tilde{\theta}$

$$\frac{\partial^2 \ln p(\theta|y)}{\partial\theta^2}\Big|_{\theta=\tilde{\theta}} = -\frac{\alpha+\sum_{i=1}^n y_i - 1}{\left(\frac{\alpha+\sum_{i=1}^n y_i - 1}{\beta+n}\right)^2} = -\frac{(\beta+n)^2}{\alpha+\sum_{i=1}^n y_i - 1}$$

- So, the normal approximation is

$$N\left[\frac{\alpha+\sum_{i=1}^n y_i - 1}{\beta+n}, \frac{\alpha+\sum_{i=1}^n y_i - 1}{(\beta+n)^2}\right]$$

Example: gamma posterior



Normal approximation of posterior

- $\theta|y \overset{approx}{\sim} N\left[\tilde{\theta}, J_{\mathbf{y}}^{-1}(\tilde{\theta})\right]$ works also when θ is a vector.
- How to compute $\tilde{\theta}$ and $J_{\mathbf{y}}(\tilde{\theta})$?
- Standard **optimization routines** may be used. (`optim.r`).
  - **Input**: an expression proportional to $\log p(\theta|y)$ and initial values.
  - **Output**: $\log p(\tilde{\theta}|y)$, $\tilde{\theta}$ and Hessian matrix $(-J_{\mathbf{y}}(\tilde{\theta}))$.
- **Re-parametrization** may improve normal approximation. [Don't forget the **Jacobian**!]
  - If $\theta \geq 0$ use $\phi = \log(\theta)$.
  - If $0 \leq \theta \leq 1$, use $\phi = \ln[\theta/(1-\theta)]$.
- **Heavy tailed approximation**: $\theta|y \overset{approx}{\sim} t_v\left[\tilde{\theta}, J_{\mathbf{y}}^{-1}(\tilde{\theta})\right]$ for suitable degrees of freedom v.

Example: gamma posterior - reparam.

- Poisson model revisited. Reparameterize to $\phi = \log(\theta)$.
- Use change-of-variables formula from a basic probability course

$$\log p(\phi|y_1, ..., y_n) \propto (\alpha + \sum_{i=1}^n y_i - 1)\phi - \exp(\phi)(\beta + n) + \phi$$

- Taking first and second derivatives and evaluating at $\tilde{\phi}$ gives

$$\tilde{\phi} = \log\left(\frac{\alpha + \sum_{i=1}^n y_i - 1}{\beta + n}\right) \text{ and } \frac{\partial^2 \ln p(\phi|y)}{\partial \phi^2}\Big|_{\phi=\tilde{\phi}} = \alpha + \sum_{i=1}^n y_i - 1$$

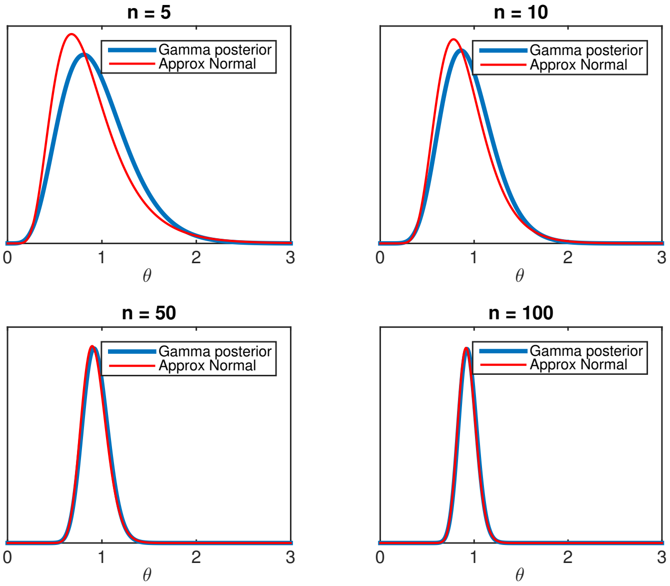
- So, the normal approximation for $p(\phi|y_1, ...y_n)$ is

$$\phi = \log(\theta) \sim N\left[\log\left(\frac{\alpha + \sum_{i=1}^n y_i - 1}{\beta + n}\right), \frac{1}{\alpha + \sum_{i=1}^n y_i - 1}\right]$$

which means that $p(\theta|y_1, ...y_n)$ is log-normal:

$$\theta|y \sim LN\left[\log\left(\frac{\alpha + \sum_{i=1}^n y_i - 1}{\beta + n}\right), \frac{1}{\alpha + \sum_{i=1}^n y_i - 1}\right]$$

EXAMPLE: GAMMA POSTERIOR - REPARAMETERIZED



Normal approximation of posterior

- Even if the posterior of θ is approx normal, **interesting functions** of $g(\theta)$ may not be (e.g. predictions).
- But approximate posterior of $g(\theta)$ can be obtained by **simulating** from $N\left[\tilde{\theta}, J_{\mathbf{y}}^{-1}(\tilde{\theta})\right]$.
- **Example**: Posterior of Gini coefficient.
  - Model: $x_1, ..., x_n|\mu, \sigma^2 \sim LN(\mu, \sigma^2)$.
  - Let $\phi = \log(\sigma^2)$. And $\theta = (\mu, \phi)$.
  - Joint posterior $p(\mu, \phi)$ may be approximately normal: $\theta|y \overset{approx}{\sim} N\left[\tilde{\theta}, J_{\mathbf{y}}^{-1}(\tilde{\theta})\right]$.
  - Simulate $\theta^{(1)}, ..., \theta^{(N)}$ from $N\left[\tilde{\theta}, J_{\mathbf{y}}^{-1}(\tilde{\theta})\right]$. Compute $\sigma^{(1)}, ..., \sigma^{(N)}$.
  - Compute $G^{(i)} = 2\Phi\left(\sigma^{(i)}/\sqrt{2}\right)$ for $i = 1, ..., N$.