Bayesian Learning - Lecture 3

Mattias Villani

Division of Statistics
Department of Computer and Information Science
Linköping University

Lecture overview

- Multiparameter models
- Marginalization
- Normal model with unknown variance
- Bayesian analysis of multinomial data
- Bayesian analysis of multivariate normal data

Marginalization

- Models usually contains several parameter $\theta_1, \theta_2, ...$ Examples: $x_i \overset{iid}{\sim} N(\theta, \sigma^2)$; multiple regression ...
- The Bayesian computes the joint posterior distribution

$$p(\theta_1, \theta_2, ..., \theta_p|y) \propto p(y|\theta_1, \theta_2, ..., \theta_p)p(\theta_1, \theta_2, ..., \theta_p).$$

... or in vector form:

$$p(\theta) \propto p(y|\theta)p(\theta).$$

- Complicated to graph the joint posterior.
- Some of the parameters may not be of direct interest (nuisance parameters), but are nevertheless needed in the model.
- No problem: just integrate them out (marginalize with respect to, average over) all nuisance parameters.
- Example: $\theta = (\theta_1, \theta_2)'$, where θ_2 is a nuisance. We are interested in the marginal posterior of θ_1

$$p(\theta_1|y) = \int p(\theta_1, \theta_2|y)d\theta_2 = \int p(\theta_1|\theta_2, y)p(\theta_2|y)d\theta_2.$$

Normal model with unknown variance - Uniform prior

- Model:

$$x_1, \ldots, x_n \overset{iid}{\sim} N(\theta, \sigma^2)$$

- Prior

$$p(\theta, \sigma^2) \propto (\sigma^2)^{-1}$$

- Posterior:

$$\theta|\sigma^2, \mathbf{x} \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$
$$\sigma^2|\mathbf{x} \sim \text{Inv} - \chi^2(n-1, s^2),$$

where

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

is the usual sample variance.

Normal model with unknown variance - Uniform prior, cont.

- Simulating the posterior of the normal model with non-informative prior:
  1. Draw $X \sim \chi^2(n-1)$
  2. Compute $\sigma^2 = \frac{(n-1)s^2}{X}$ (this a draw from Inv-$\chi^2(n-1, s^2)$)
  3. Draw a θ from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ conditional on the previous draw σ^2
  4. Repeat step 1-3 many times.
- The sampling is implemented in the R program NormalNonInfoPrior.R
- We may derive the marginal posterior analytically as

$$\theta|\mathbf{x} \sim t_{n-1}\left(\bar{x}, \frac{s^2}{n}\right).$$

Multinomial model with Dirichlet prior

- *Data*: $y = (y_1, \ldots y_K)$, where y_k counts the number of observations in the kth category. $\sum_{k=1}^{K} y_k = n$. Example: brand choices.
- Multinomial model:

$$p(y|\theta) \propto \prod_{k=1}^{K} \theta_k^{y_k}, \text{ where } \sum_{k=1}^{K} \theta_j = 1.$$

- *Conjugate prior*: $\text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$

$$p(\theta) \propto \prod_{k=1}^{K} \theta_j^{\alpha_j - 1}.$$

- Moments of $\theta = (\theta_1, \ldots, \theta_K)' \sim \textit{Dirichlet}(\alpha_1, \ldots, \alpha_K)$

$$\mathrm{E}(\theta_k) = \frac{\alpha_k}{\sum_{j=1}^{K} \alpha_j}$$

$$\mathrm{V}(\theta_k) = \frac{\mathrm{E}(\theta_k)\left[1 - \mathrm{E}(\theta_k)\right]}{1 + \sum_{k=1}^{K} \alpha_k}$$

- Note that $\sum_{k=1}^{K} \alpha_k$ is a precision parameter.

Multinomial model with Dirichlet prior, cont.

- 'Non-informative': $\alpha_1 = ... = \alpha_K = 1$ (uniform and proper).
- Simulating from the Dirichlet distribution:
  - Generate $x_1 \sim Gamma(\alpha_1, \beta), ..., x_K \sim Gamma(\alpha_K, \beta)$, independently. Any β will do as long it is the same for all x_i.
  - Compute $y_k = x_k / (\sum_{j=1}^{K} x_j)$.
  - $y = (y_1, ..., y_K)$ is a draw from the Dirichlet$(\alpha_1, ..., \alpha_K)$ distribution.
- *Prior-to-Posterior updating*:

$$
\begin{aligned}
\textit{Model}: \quad & y = (y_1, ...y_K) \sim \text{Multin}(n; \theta_1, ..., \theta_K) \\
\textit{Prior}: \quad & \theta = (\theta_1, ..., \theta_K) \sim \text{Dirichlet}(\alpha_1, ..., \alpha_K) \\
\textit{Posterior}: \quad & \theta | y \sim \text{Dirichlet}(\alpha_1 + y_1, ..., \alpha_K + y_K).
\end{aligned}
$$

Example: market shares

- A recent survey among consumer smartphones owners in the U.S. showed that among the 513 respondents:
  - 180 owned an iPhone
  - 230 owned an Android phone
  - 62 owned a Blackberry phone
  - 41 owned some other mobile phone.
- Previous survey: iPhone 30%, Android 30%, Blackberry 20% and Other 20%.
- Pr(Android has largest share | Data)
- Prior: $\alpha_1 = 15, \alpha_2 = 15, \alpha_3 = 10$ and $\alpha_4 = 10$ (prior info is equivalent to a survey with only 50 respondents)
- Posterior: $(\theta_1, \theta_2, \theta_3, \theta_4)|\mathbf{y} \sim \text{Dirichlet}(179, 261, 72, 51)$

R CODE FOR MARKET SHARE EXAMPLE

```r
# Setting up data and prior
y <- c(180, 230, 62, 41)  # The cell phone survey data (K=4)
alpha <- c(15, 15, 10, 10)  # Dirichlet prior hyperparameters
nIter <- 100  # Number of posterior draws

# Defining a function that simulates from a Dirichlet distribution
SimDirichlet <- function(nIter, param) {
    nCat <- length(param)
    thetaDraws <- as.data.frame(matrix(NA, nIter, nCat))  # Storage.
    for (j in 1:nCat) {
        thetaDraws[, j] <- rgamma(nIter, param[j], 1)
    }
    for (i in 1:nIter) {
        thetaDraws[i, ] = thetaDraws[i, ]/sum(thetaDraws[i, ])
    }
    return(thetaDraws)
}

# Posterior sampling from Dirichlet posterior
thetaDraws <- SimDirichlet(nIter, y + alpha)
```

R CODE FOR MARKET SHARE EXAMPLE, CONT

```r
# Posterior mean and standard deviation of Androids share (in %)
message(mean(100 * thetaDraws[, 2]))
```

```
## 43.4717612280017
```

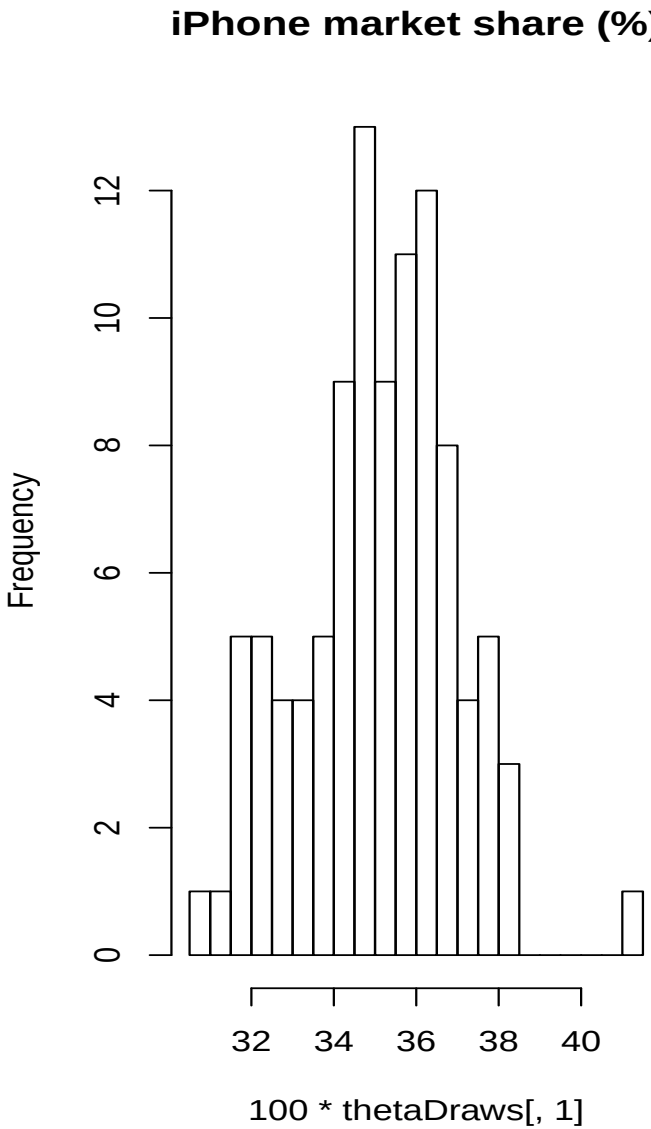
```r
message(sd(100 * thetaDraws[, 2]))
```

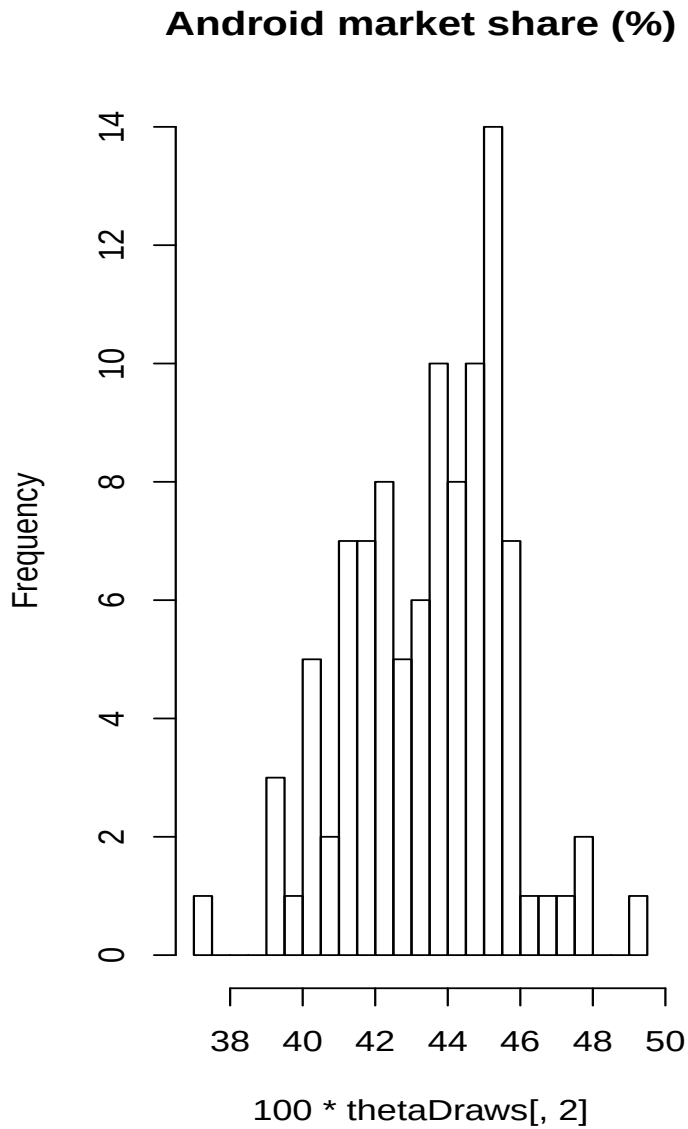
```
## 2.12255952634333
```

```r
# Computing the posterior probability that Android is the largest
PrAndroidLargest <- sum(thetaDraws[, 2] > max(thetaDraws[, c(1, 3, 4)]))/nIter
message(paste("Pr(Android has the largest market share) = ", PrAndroidLargest))
```

```
## Pr(Android has the largest market share) = 0.86
```

R CODE FOR MARKET SHARE EXAMPLE, CONT





Multivariate normal - known covariance matrix

- Model:

$$y_1, ..., y_n \overset{iid}{\sim} N_p(\mu, \Sigma)$$

where Σ is a known covariance matrix.

- Density

$$p(y|\mu, \Sigma) = |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right)$$

- Likelihood:

$$p(y_1, ..., y_n|\mu, \Sigma) \propto |\Sigma|^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)'\Sigma^{-1}(y_i-\mu)\right)$$

$$= |\Sigma|^{-n/2} \exp\left(-\frac{1}{2} tr\Sigma^{-1} S_\mu\right)$$

where $S_\mu = \sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)'$.

Multivariate normal - known covariance matrix, cont.

- Prior:

$$\mu \sim N_p(\mu_0, \Lambda_0)$$

- Posterior:

$$\mu|y \sim N(\mu_n, \Lambda_n)$$

where

$$\begin{aligned} \mu_n &= (\Lambda_0^{-1} + n\Sigma^{-1})^{-1}(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y}) \\ \Lambda_n^{-1} &= \Lambda_0^{-1} + n\Sigma^{-1} \end{aligned}$$

- Note how the posterior mean is (matrix) weighted average of prior and data information.
- Noninformative prior: let the precision go to zero: $\Lambda_0^{-1} \to 0$.

Multivariate normal - Conjugate prior

- Conjugate prior is Normal-IW$(\mu_0, \kappa_0, \Lambda_0, \nu_0)$

$$\Sigma \sim Inv - Wishart(\Lambda_0, \nu_0)$$
$$\mu|\Sigma \sim N(\mu_0, \kappa_0^{-1}\Sigma)$$

- Density:

$$|\Sigma|^{-[(\nu_0+p)/2+1]} \exp\left(-\frac{1}{2}tr(\Lambda_0\Sigma^{-1}) - \frac{\kappa_0}{2}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\right)$$

- Posterior is Normal-IW$(\mu_n, \kappa_n, \Lambda_n, \nu_n)$

$$\mu_n = \frac{\kappa_0}{\kappa_0+n}\mu_0 + \frac{n}{\kappa_0+n}\bar{y}$$
$$\kappa_n = \kappa_0 + n$$
$$\nu_n = \nu_0 + n$$
$$\Lambda_n = \Lambda_0 + S + \frac{\kappa_0 n}{\kappa_0+n}(\bar{y}-\mu_0)(\bar{y}-\mu_0)'$$