

BAYESIAN LEARNING - LECTURE 3

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LECTURE OVERVIEW

- ▶ Multiparameter models
- ▶ Marginalization
- ▶ Normal model with unknown variance
- ▶ Bayesian analysis of multinomial data
- ▶ Bayesian analysis of multivariate normal data

MARGINALIZATION

- ▶ Models with multiple parameters $\theta_1, \theta_2, \dots$
- ▶ Examples: $x_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$; multiple regression ...
- ▶ **Joint posterior distribution**

$$p(\theta_1, \theta_2, \dots, \theta_p | y) \propto p(y | \theta_1, \theta_2, \dots, \theta_p) p(\theta_1, \theta_2, \dots, \theta_p).$$

... or in vector form:

$$p(\theta) \propto p(y | \theta) p(\theta).$$

- ▶ Complicated to graph the joint posterior.
- ▶ Some of the parameters may not be of direct interest (**nuisance**).
- ▶ Integrate out (**marginalize**) all nuisance parameters.
- ▶ Example: $\theta = (\theta_1, \theta_2)'$, θ_2 is a nuisance. **Marginal posterior** of θ_1

$$p(\theta_1 | y) = \int p(\theta_1, \theta_2 | y) d\theta_2 = \int p(\theta_1 | \theta_2, y) p(\theta_2 | y) d\theta_2.$$

NORMAL MODEL WITH UNKNOWN VARIANCE - UNIFORM PRIOR

► Model

$$x_1, \dots, x_n \stackrel{iid}{\sim} N(\theta, \sigma^2)$$

► Prior

$$p(\theta, \sigma^2) \propto (\sigma^2)^{-1}$$

► Posterior

$$\begin{aligned}\theta | \sigma^2, \mathbf{x} &\sim N\left(\bar{x}, \frac{\sigma^2}{n}\right) \\ \sigma^2 | \mathbf{x} &\sim \text{Inv} - \chi^2(n-1, s^2),\end{aligned}$$

where

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$$

is the usual sample variance.

NORMAL MODEL WITH UNKNOWN VARIANCE - UNIFORM PRIOR

- ▶ **Simulating** the posterior of the normal model with non-informative prior:
 1. Draw $X \sim \chi^2(n-1)$
 2. Compute $\sigma^2 = \frac{(n-1)s^2}{X}$ (this a draw from $\text{Inv-}\chi^2(n-1, s^2)$)
 3. Draw a θ from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ conditional on the previous draw σ^2
 4. Repeat step 1-3 many times.
- ▶ The sampling is implemented in the R program `NormalNonInfoPrior.R`
- ▶ We may derive the marginal posterior analytically as

$$\theta|\mathbf{x} \sim t_{n-1}\left(\bar{x}, \frac{s^2}{n}\right).$$

NORMAL MODEL - NORMAL PRIOR

- ▶ Model

$$y_1, \dots, y_n | \theta, \sigma^2 \stackrel{iid}{\sim} N(\theta, \sigma^2)$$

- ▶ Conjugate prior

$$\theta | \sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{\kappa_0}\right)$$

$$\sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)$$

NORMAL MODEL WITH NORMAL PRIOR

► Posterior

$$\theta|y, \sigma^2 \sim N\left(\mu_n, \frac{\sigma^2}{\kappa_n}\right)$$
$$\sigma^2|y \sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2).$$

where

$$\begin{aligned}\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\ \kappa_n &= \kappa_0 + n \\ \nu_n &= \nu_0 + n \\ \nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2.\end{aligned}$$

NORMAL MODEL WITH NORMAL PRIOR

► Posterior

$$\theta|y, \sigma^2 \sim N\left(\mu_n, \frac{\sigma^2}{\kappa_n}\right)$$
$$\sigma^2|y \sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2).$$

where

$$\begin{aligned}\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\ \kappa_n &= \kappa_0 + n \\ \nu_n &= \nu_0 + n \\ \nu_n \sigma_n^2 &= \nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2.\end{aligned}$$

► Marginal posterior

$$\theta \sim t_{\nu_n}(\mu_n, \sigma_n^2/\kappa_n)$$

MULTINOMIAL MODEL WITH DIRICHLET PRIOR

- ▶ *Data*: $y = (y_1, \dots, y_K)$, where y_k counts the number of observations in the k th category. $\sum_{k=1}^K y_k = n$. Example: brand choices.
- ▶ **Multinomial model**:

$$p(y|\theta) \propto \prod_{k=1}^K \theta_k^{y_k}, \text{ where } \sum_{k=1}^K \theta_k = 1.$$

- ▶ **Conjugate prior**: $\text{Dirichlet}(\alpha_1, \dots, \alpha_K)$

$$p(\theta) \propto \prod_{j=1}^K \theta_j^{\alpha_j - 1}.$$

- ▶ Moments of $\theta = (\theta_1, \dots, \theta_K)' \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_K)$

$$\begin{aligned} E(\theta_k) &= \frac{\alpha_k}{\sum_{j=1}^K \alpha_j} \\ V(\theta_k) &= \frac{E(\theta_k) [1 - E(\theta_k)]}{1 + \sum_{k=1}^K \alpha_k} \end{aligned}$$

- ▶ Note that $\sum_{k=1}^K \alpha_k$ is a precision parameter.

MULTINOMIAL MODEL WITH DIRICHLET PRIOR

- ▶ 'Non-informative': $\alpha_1 = \dots = \alpha_K = 1$ (uniform and proper).
- ▶ **Simulating** from the Dirichlet distribution:
 - ▶ Generate $x_1 \sim \text{Gamma}(\alpha_1, 1), \dots, x_K \sim \text{Gamma}(\alpha_K, 1)$.
 - ▶ Compute $y_k = x_k / (\sum_{j=1}^K x_j)$.
 - ▶ $y = (y_1, \dots, y_K)$ is a draw from the $\text{Dirichlet}(\alpha_1, \dots, \alpha_K)$ distribution.
- ▶ **Prior-to-Posterior updating:**

Model: $y = (y_1, \dots, y_K) \sim \text{Multin}(n; \theta_1, \dots, \theta_K)$

Prior : $\theta = (\theta_1, \dots, \theta_K) \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_K)$

Posterior : $\theta|y \sim \text{Dirichlet}(\alpha_1 + y_1, \dots, \alpha_K + y_K)$.

EXAMPLE: MARKET SHARES

- ▶ A recent survey among consumer smartphones owners in the U.S. showed that among the 513 respondents:
 - ▶ 180 owned an iPhone
 - ▶ 230 owned an Android phone
 - ▶ 62 owned a Blackberry phone
 - ▶ 41 owned some other mobile phone.
- ▶ Previous survey: iPhone 30%, Android 30%, Blackberry 20% and Other 20%.
- ▶ $\Pr(\text{Android has largest share} \mid \text{Data})$
- ▶ Prior: $\alpha_1 = 15, \alpha_2 = 15, \alpha_3 = 10$ and $\alpha_4 = 10$ (prior info is equivalent to a survey with only 50 respondents)
- ▶ Posterior: $(\theta_1, \theta_2, \theta_3, \theta_4) \mid \mathbf{y} \sim \text{Dirichlet}(195, 245, 72, 51)$

R CODE FOR MARKET SHARE EXAMPLE

```
# Setting up data and prior
y <- c(180,230,62,41) # The cell phone survey data (K=4)
alpha <- c(15,15,10,10) # Dirichlet prior hyperparameters
nIter <- 1000 # Number of posterior draws

# Defining a function that simulates from a Dirichlet distribution
SimDirichlet <- function(nIter, param){
  nCat <- length(param)
  thetaDraws <- as.data.frame(matrix(NA, nIter, nCat)) # Storage.
  for (j in 1:nCat){
    thetaDraws[,j] <- rgamma(nIter,param[j],1)
  }
  for (i in 1:nIter){
    thetaDraws[i,] = thetaDraws[i,]/sum(thetaDraws[i,])
  }
  return(thetaDraws)
}

# Posterior sampling from Dirichlet posterior
thetaDraws <- SimDirichlet(nIter,y + alpha)
```

R CODE FOR MARKET SHARE EXAMPLE, CONT

```
# Posterior mean and standard deviation of Androids share (in %)
message(mean(100*thetaDraws[,2]))

## 43.4732151708756

message(sd(100*thetaDraws[,2]))

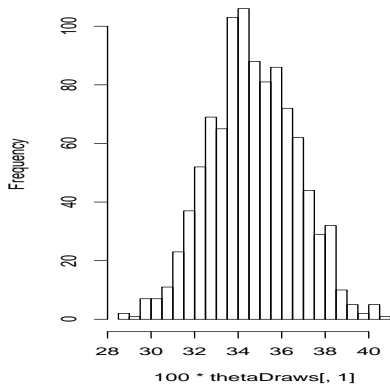
## 2.1364073162011

# Computing the posterior probability that Android is the largest
PrAndroidLargest <- sum(thetaDraws[,2] > max(thetaDraws[,c(1,3,4)]))/nIter
message(paste('Pr(Android has the largest market share) = ', PrAndroidLargest))

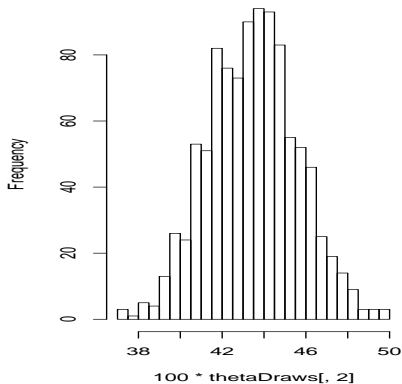
## Pr(Android has the largest market share) = 0.892
```

R CODE FOR MARKET SHARE EXAMPLE, CONT

iPhone market share (%)



Android market share (%)



MULTIVARIATE NORMAL - KNOWN Σ

► Model

$$y_1, \dots, y_n \stackrel{iid}{\sim} N_p(\mu, \Sigma)$$

where Σ is a known covariance matrix.

► Density

$$p(y|\mu, \Sigma) = |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(y - \mu)' \Sigma^{-1}(y - \mu)\right)$$

► Likelihood

$$\begin{aligned} p(y_1, \dots, y_n|\mu, \Sigma) &\propto |\Sigma|^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^n (y_i - \mu)' \Sigma^{-1}(y_i - \mu)\right) \\ &= |\Sigma|^{-n/2} \exp\left(-\frac{1}{2} \text{tr} \Sigma^{-1} S_\mu\right) \end{aligned}$$

where $S_\mu = \sum_{i=1}^n (y_i - \mu)(y_i - \mu)'$.

MULTIVARIATE NORMAL - KNOWN Σ

► Prior

$$\mu \sim N_p(\mu_0, \Lambda_0)$$

► Posterior

$$\mu|y \sim N(\mu_n, \Lambda_n)$$

where

$$\begin{aligned}\mu_n &= (\Lambda_0^{-1} + n\Sigma^{-1})^{-1}(\Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{y}) \\ \Lambda_n^{-1} &= \Lambda_0^{-1} + n\Sigma^{-1}\end{aligned}$$

- Note how the posterior mean is (matrix) weighted average of prior and data information.
- **Noninformative prior:** let the precision go to zero: $\Lambda_0^{-1} \rightarrow 0$.