6.S183 (IAP 2026) Problem Set 1

due: 9 Jan 2025 23:59

Collaboration Policy: In the case of written homework assignments, your assignment must represent your own individual work. Although you may discuss homework problems with other students, please write up your own solutions.
Submission: Gradescope (`https://www.gradescope.com/courses/1203177`).

Question 1 (3 points)

a. Prove that for any distribution over random variables X and Y we have

$$\operatorname{argmin}_f \mathop{\mathbb{E}}_{x,y} \left[\|f(y) - x\|^2 \right] = \mathbb{E}[x \mid y].$$

Hint: Solve the optimization problem pointwise for fixed y.

b. Let μ be the density function of a data distribution, so that $\mu(x_0) \geq 0$ for all $x_0 \in \mathbb{R}^n$ and $\int_{\mathcal{K}} \mu(x_0) dx_0 = 1$. Consider the following loss function for fixed σ.

$$\mathcal{L}_\sigma(\epsilon_\sigma) = \mathop{\mathbb{E}}_{x_0 \sim \mu,\, \epsilon \sim N(0, I_n)} \left[\|\epsilon_\sigma(x_0 + \sigma\epsilon) - \epsilon\|^2 \right] \tag{1}$$

Write down the exact minimizer $\epsilon^*_\sigma(x_\sigma)$ of (1) in terms of the data distribution $\mu(x_0)$ and input x_σ. Hint: use part a, Bayes' rule, as well as the probability density function of $N(x_0, \sigma^2 I_n)$.

c. Let x_0 be a random variable distributed according to some continuous density $p_0(x)$. Define $x_t := x_0 + \sigma(t)\epsilon$ for some function $\sigma : [0, 1] \to \mathbb{R}_{\geq 0}$ and let $p_t(x)$ be the probability density associated with x_t.

Show that $v(x, t) = \mathbb{E}[\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} \epsilon | x_t = x]$ and $p_t(x)$ satisfy the *transport equation* governing the evolution of the density p_t subject to a velocity field v:

$$\partial_t p_t(x) + \nabla \cdot (v(x, t) p_t(x)) = 0$$

Hint: Consider the time derivative of the *characteristic function* $g(t, k) = \mathbb{E}[e^{ik \cdot x_t}]$. You may also use the fact from Fourier analysis that

$$\int e^{ik \cdot x} \nabla \cdot f(x) dx = -ik \cdot \int e^{ik \cdot x} f(x) dx.$$

and, for continuous functions f and g, $\int e^{ik \cdot x} f(x) dx = \int e^{ik \cdot x} g(x) dx$ iff $f = g$.

Question 2 (3 points)

We will write code to train a diffusion model for a toy 2D dataset. See the provided Jupyter notebook for instructions. You can open the notebook in Google Colab (for this assignment a CPU-only instance is sufficient) `https://colab.research.google.com/drive/1gNZkGePLZmH9uGdjhroxkYe_GXA5KDSy`. For submission, export the notebook as a PDF (ensure all plots and figures are visible!) and upload via gradescope.