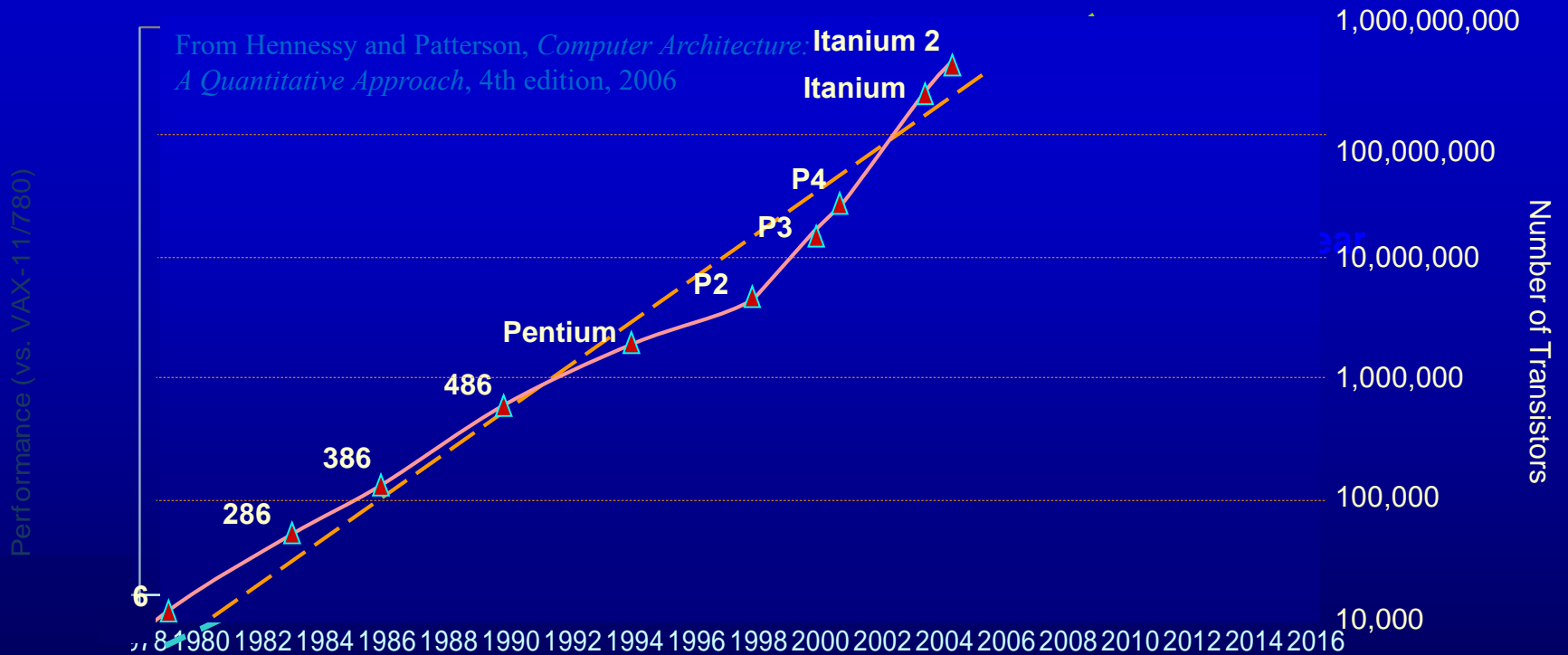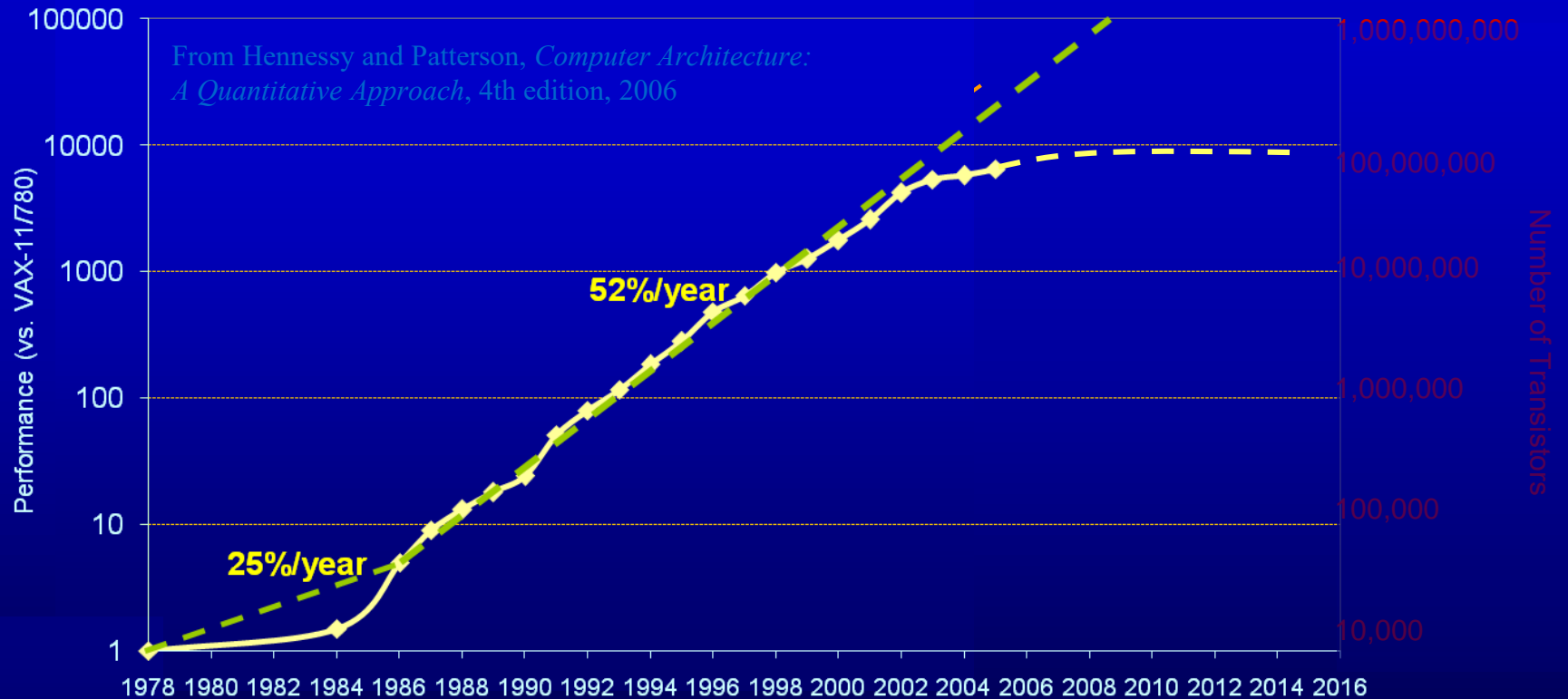# 6.035

Parallelization

Outline

- **Why Parallelism**

- Parallel Execution

- Parallelizing Compilers

- Dependence Analysis

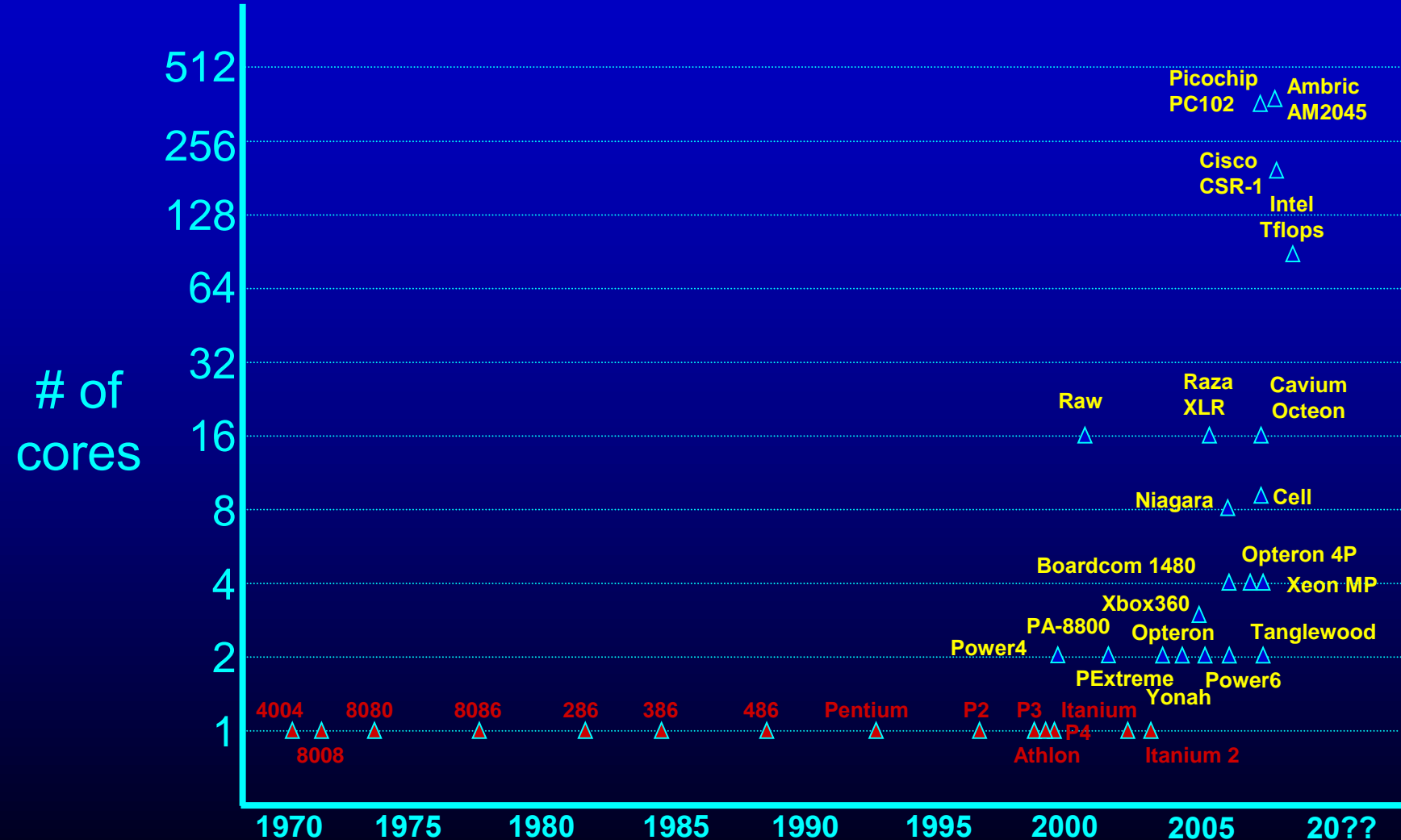- Increasing Parallelization Opportunities

Moore's Law

From Hennessy and Patterson, *Computer Architecture: A Quantitative Approach*, 4th edition, 2006

Performance (vs. VAX-11/780)

Number of Transistors

286
386
486
Pentium
P2
P3
P4
Itanium
Itanium 2

1,000,000,000
100,000,000
10,000,000
1,000,000
100,000
10,000

1978 1980 1982 1984 1986 1988 1990 1992 1994 1996 1998 2000 2002 2004 2006 2008 2010 2012 2014 2016

Uniprocessor Performance (SPECint)

From Hennessy and Patterson, *Computer Architecture: A Quantitative Approach*, 4th edition, 2006

52%/year

25%/year

Multicores Are Here!

Issues with Parallelism

- Amdhal's Law
  - Any computation can be analyzed in terms of a portion that must be executed sequentially, Ts, and a portion that can be executed in parallel, Tp. Then for n processors:
  - T(n) = Ts + Tp/n
  - T(∞) = Ts, thus maximum speedup (Ts + Tp) /Ts

- Load Balancing
  - The work is distributed among processors so that *all* processors are kept busy when parallel task is executed.

- Granularity
  - The size of the parallel regions between synchronizations or the ratio of computation (useful work) to communication (overhead).

Outline

- Why Parallelism

- **Parallel Execution**

- Parallelizing Compilers

- Dependence Analysis

- Increasing Parallelization Opportunities

Types of Parallelism

- Instruction Level Parallelism (ILP)

  → Scheduling and Hardware

- Task Level Parallelism (TLP)

  → Mainly by hand

- Loop Level Parallelism (LLP) or Data Parallelism

  → Hand or Compiler Generated

- Pipeline Parallelism

  → Hardware or Streaming

- Divide and Conquer Parallelism

  → Recursive functions

Why Loops?

- 90% of the execution time in 10% of the code
  - Mostly in loops

- If parallel, can get good performance
  - Load balancing

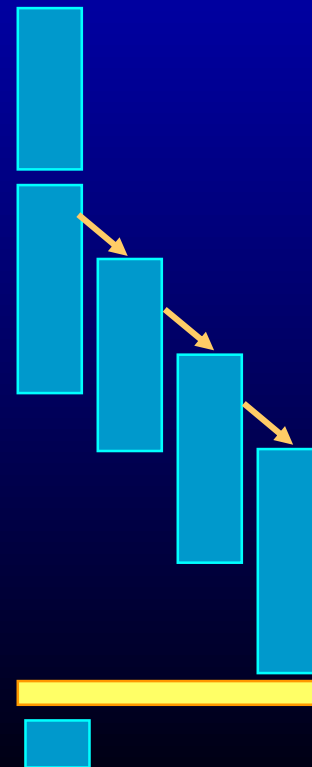- Relatively easy to analyze

Programmer Defined Parallel Loop

- FORALL
  - No "loop carried dependences"
  - Fully parallel

- FORACROSS
  - Some "loop carried dependences"
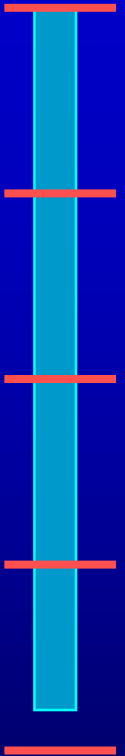
Parallel Execution

- Example

```
FORPAR I = 0 to N
   A[I] = A[I] + 1
```

- Block Distribution: Program gets mapped into

```
Iters = ceiling(N/NUMPROC);
FOR P = 0 to NUMPROC-1
   FOR I = P*Iters to MIN((P+1)*Iters, N)
     A[I] = A[I] + 1
```

- SPMD (Single Program, Multiple Data) Code

```
If(myPid == 0) {
   …
   Iters = ceiling(N/NUMPROC);
}
Barrier();
FOR I = myPid*Iters to MIN((myPid+1)*Iters, N)
   A[I] = A[I] + 1
Barrier();
```

Parallel Execution

- Example
```
FORPAR I = 0 to N
  A[I] = A[I] + 1
```

- Block Distribution: Program gets mapped into
```
Iters = ceiling(N/NUMPROC);
FOR P = 0 to NUMPROC-1
  FOR I = P*Iters to MIN((P+1)*Iters, N)
    A[I] = A[I] + 1
```

- Code fork a function
```
Iters = ceiling(N/NUMPROC);
FOR P = 0 to NUMPROC – 1 { ParallelExecute(func1, P); }
BARRIER(NUMPROC);
void func1(integer myPid)
{
  FOR I = myPid*Iters to MIN((myPid+1)*Iters, N)
    A[I] = A[I] + 1
}
```

Parallel Thread Basics

- Create separate threads
  - Create an OS thread
    - (hopefully) it will be run on a separate core
  - pthread_create(&thr, NULL, &entry_point, NULL)
  - Overhead in thread creation
    - Create a separate stack
    - Get the OS to allocate a thread
- Thread pool
  - Create all the threads (= num cores) at the beginning
  - Keep N-1 idling on a barrier, while sequential execution
  - Get them to run parallel code by each executing a function
  - Back to the barrier when parallel region is done

Outline

- Why Parallelism

- Parallel Execution

- **Parallelizing Compilers**

- Dependence Analysis

- Increasing Parallelization Opportunities

Parallelizing Compilers

- Finding FORALL Loops out of FOR loops

- Examples

```
FOR I = 0 to 5
  A[I] = A[I] + 1


FOR I = 0 to 5
  A[I] = A[I+6] + 1


For I = 0 to 5
  A[2*I] = A[2*I + 1] + 1
```
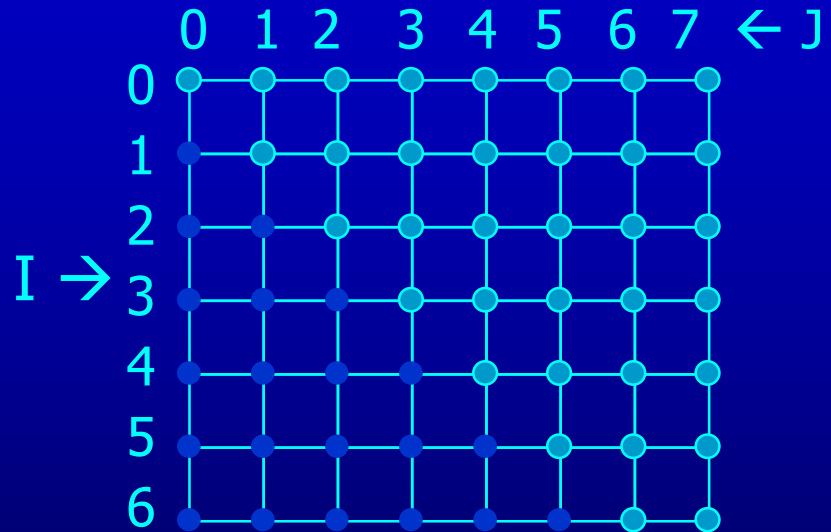
Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1
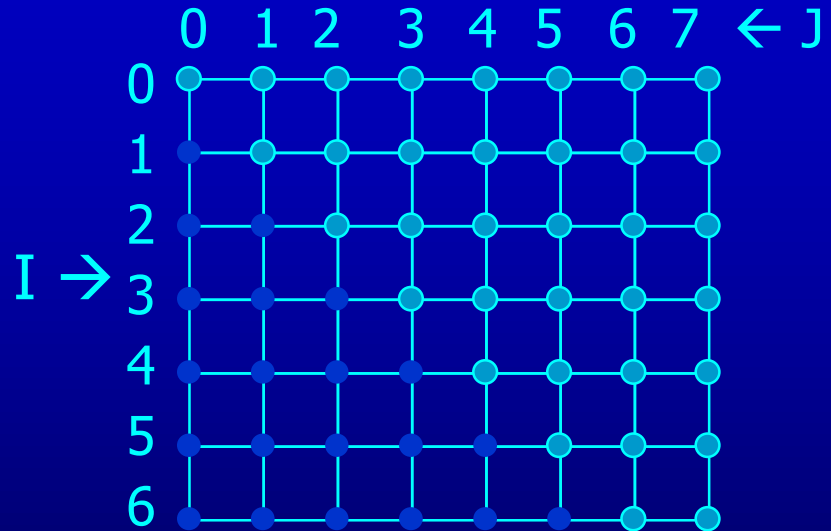
```
FOR I = 0 to 6
  FOR J = I to 7
```

```
        0  1  2  3  4  5  6  7  ← J
      0 ●──●──●──●──●──●──●──●
      1 ●──●──●──●──●──●──●──●
      2 ●──●──●──●──●──●──●──●
I → 3 ●──●──●──●──●──●──●──●
      4 ●──●──●──●──●──●──●──●
      5 ●──●──●──●──●──●──●──●
      6 ●──●──●──●──●──●──●──●
```

- Iterations are represented as coordinates in iteration space
  - $\bar{i} = [i_1, i_2, i_3, ..., i_n]$

Iteration Space

- N deep loops → N-dimensional discrete iteration space
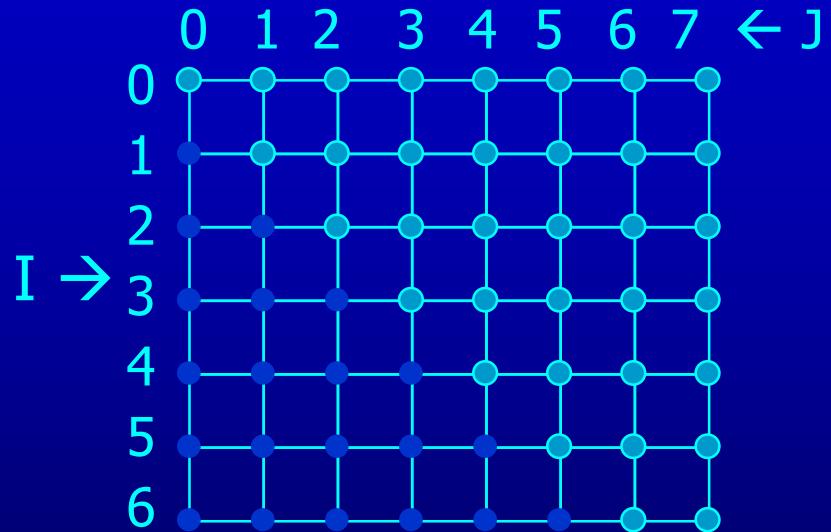 - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
  FOR J = I to 7
```


- Iterations are represented as coordinates in iteration space
- Sequential execution order of iterations → Lexicographic order
  [0,0], [0,1], [0,2], …, [0,6], [0,7],
       [1,1], [1,2], …, [1,6], [1,7],
              [2,2], …, [2,6], [2,7],
                     ………
                            [6,6], [6,7],

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
  FOR J = I to 7
```


- Iterations are represented as coordinates in iteration space
- Sequential execution order of iterations → Lexicographic order
- Iteration $\bar{i}$ is lexicograpically less than $\bar{j}$, $\bar{i} < \bar{j}$ iff
  there exists c s.t. $i_1 = j_1$, $i_2 = j_2$, ... $i_{c-1} = j_{c-1}$ and $i_c < j_c$

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
  FOR J = I to 7
```
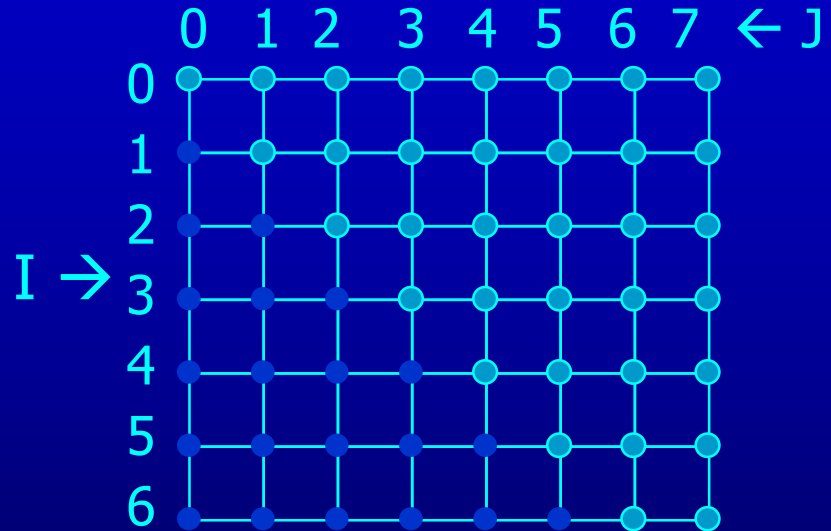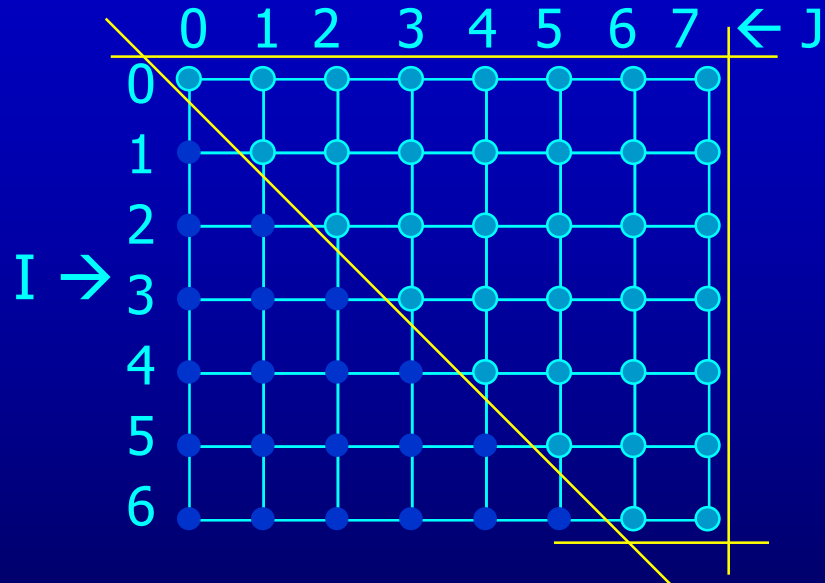

- An affine loop nest
  - Loop bounds are integer linear functions of constants, loop constant variables and outer loop indexes
  - Array accesses are integer linear functions of constants, loop constant variables and loop indexes

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
   FOR J = I to 7
```


- Affine loop nest → Iteration space as a set of linear inequalities
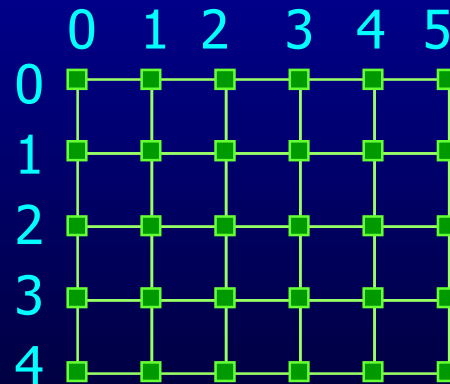
$$0 \leq I$$
$$I \leq 6$$
$$I \leq J$$
$$J \leq 7$$

Data Space

- M dimensional arrays → M-dimensional discrete cartesian space
  - a hypercube

`Integer A(10)`

`Float B(5, 6)`

Dependences

- True dependence

  ```
  a  =
     =  a
  ```

- Anti dependence

  ```
     =  a
  a  =
  ```

- Output dependence

  ```
  a  =
  a  =
  ```

- Definition:
  Data dependence exists for a dynamic instance i and j iff
  - either i or j is a write operation
  - i and j refer to the same variable
  - i executes before j

- How about array accesses within loops?

Outline

- Why Parallelism

- Parallel Execution

- Parallelizing Compilers

- **Dependence Analysis**

- Increasing Parallelization Opportunities

Array Accesses in a loop

```
FOR I = 0 to 5
   A[I] = A[I] + 1
```
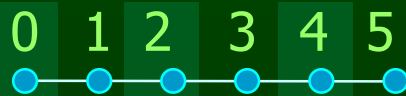
Iteration Space

0   1   2   3   4   5

Data Space

0   1   2   3   4   5   6   7   8   9   10 11 12

Array Accesses in a loop

```
FOR I = 0 to 5
A[I] = A[I] + 1
```

Iteration Space

0 1 2 3 4 5

Data Space

0 1 2 3 4 5 6 7 8 9 10 11 12

= A[I]

A[I]

= A[I]

A[I]

= A[I]

A[I]

= A[I]

A[I]

= A[I]

A[I]

= A[I]

A[I]
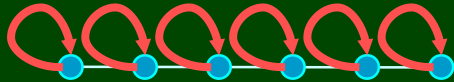
Array Accesses in a loop
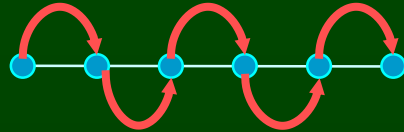
FOR I = 0 to 5
A[I+1] = A[I] + 1

Iteration Space

0 1 2 3 4 5

Data Space

0 1 2 3 4 5 6 7 8 9 10 11 12

= A[I]

A[I+1]

= A[I]

A[I+1]

= A[I]

A[I+1]

= A[I]

A[I+1]

= A[I]

A[I+1]

= A[I]

A[I+1]
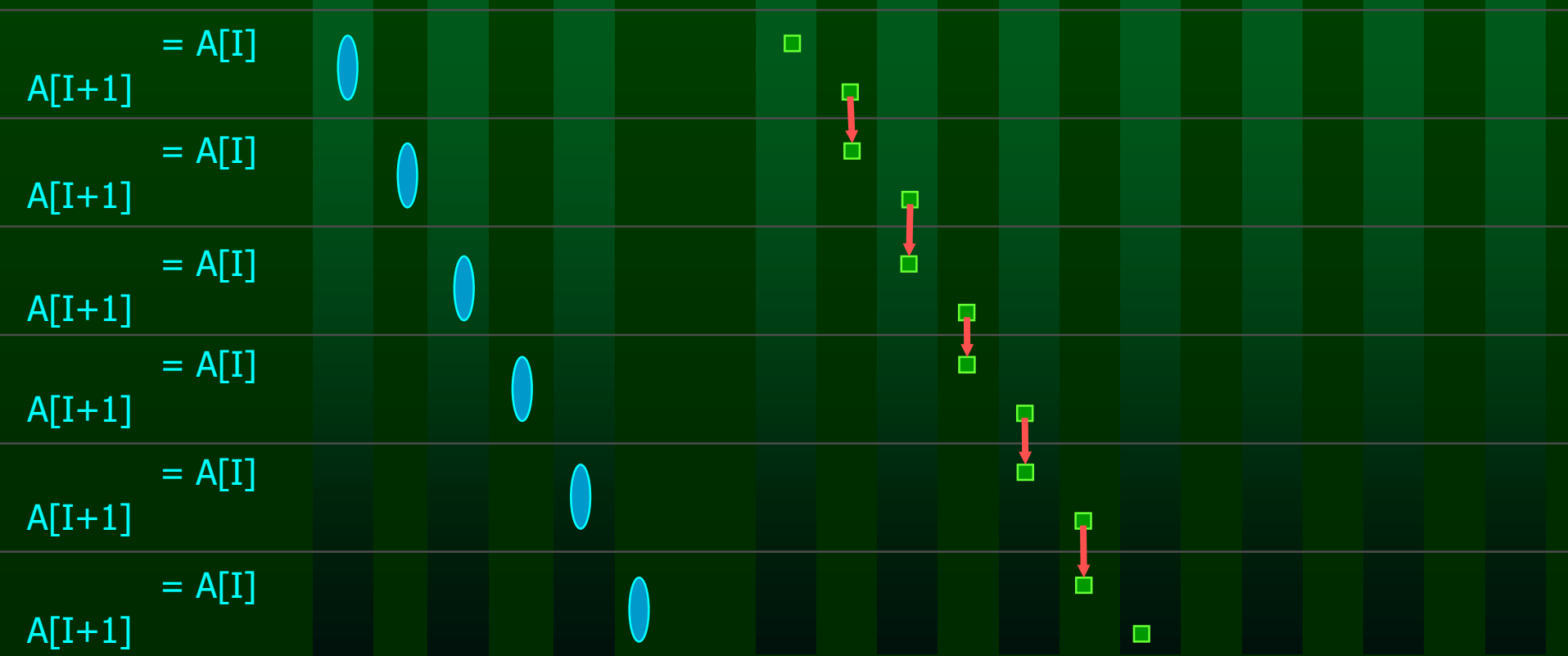
Array Accesses in a loop
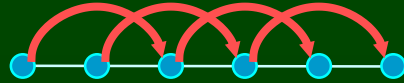
FOR I = 0 to 5
A[I] = A[I+2] + 1

Iteration Space
0 1 2 3 4 5

Data Space
0 1 2 3 4 5 6 7 8 9 10 11 12

= A[I+2]

A[I]

= A[I+2]

A[I]

= A[I+2]

A[I]

= A[I+2]

A[I]

= A[I+2]

A[I]

= A[I+2]

A[I]
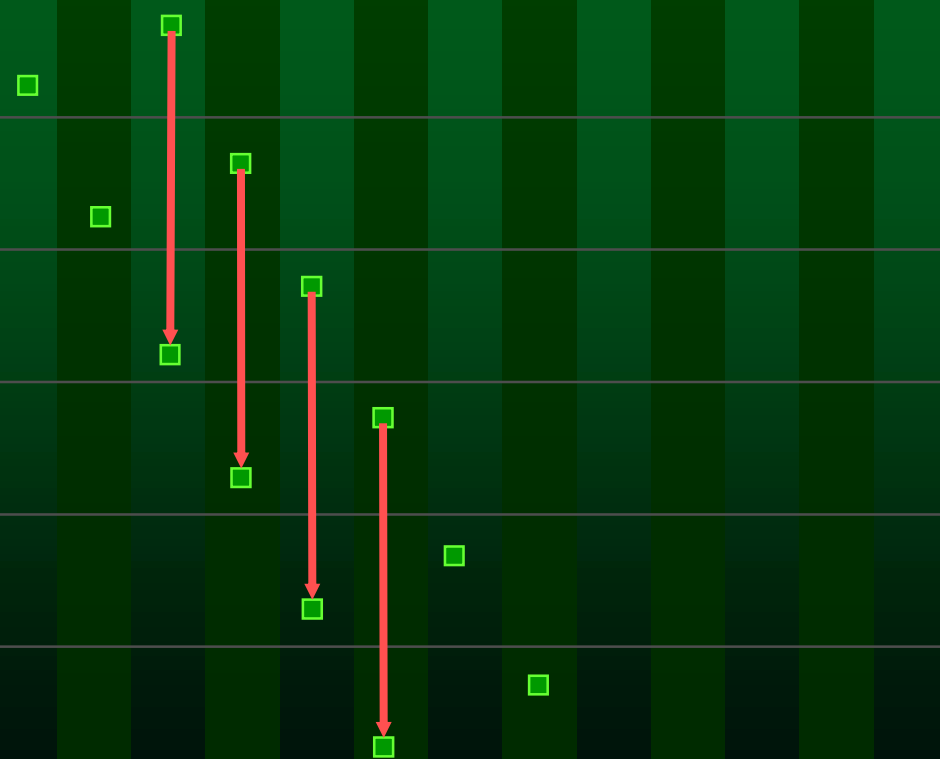
Array Accesses in a loop
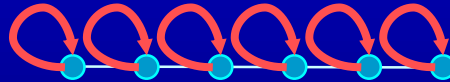
```
FOR I = 0 to 5
    A[2*I] = A[2*I+1] + 1
```
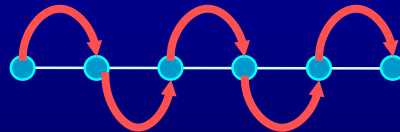

Iteration Space

Data Space
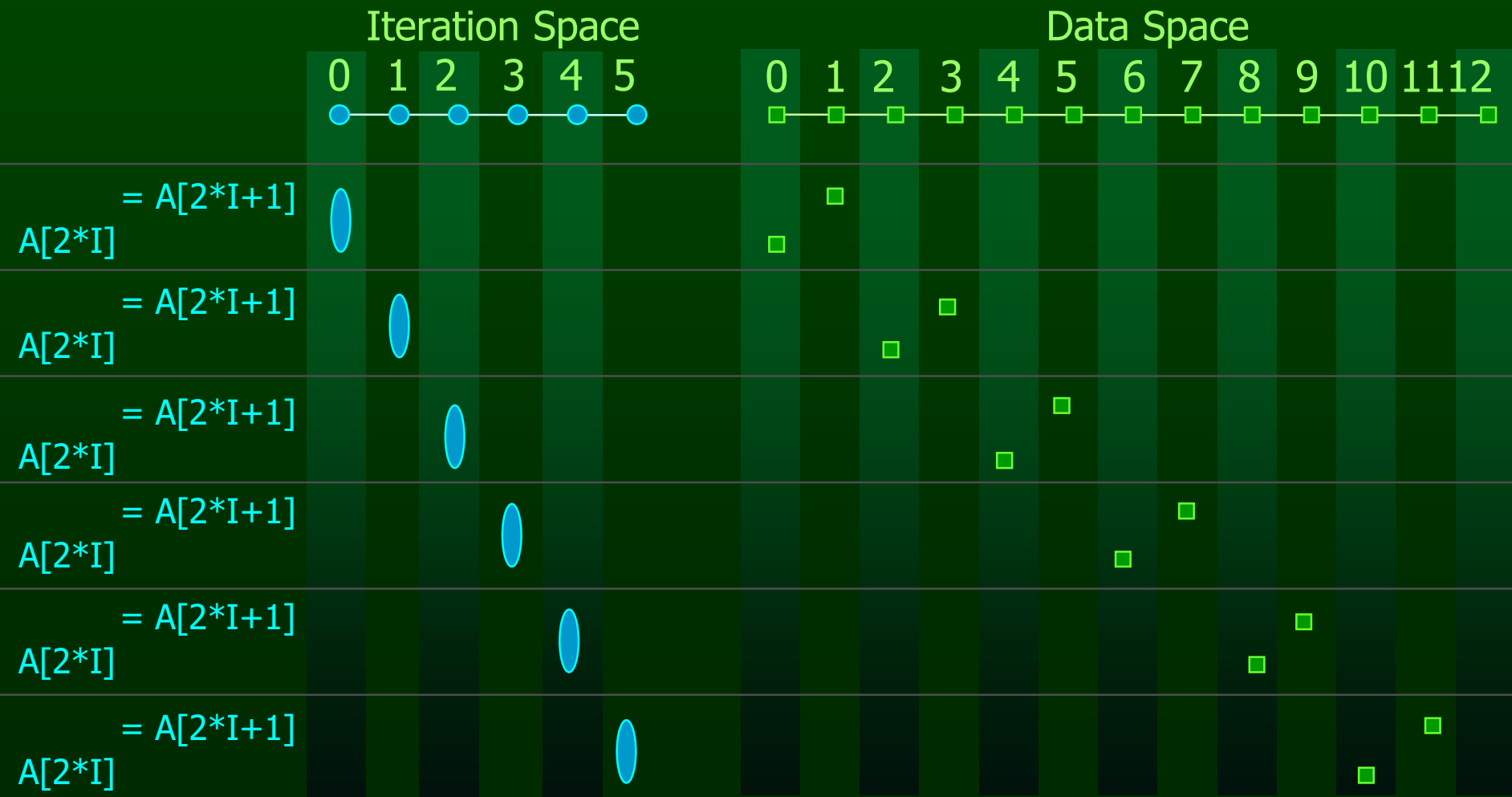
Distance Vectors

- A loop has a distance d if there exist a data dependence from iteration i to j and d = j-i

$$dv = [0]$$


```
FOR I = 0 to 5
    A[I] = A[I] + 1
```

$$dv = [1]$$


```
FOR I = 0 to 5
    A[I+1] = A[I] + 1
```

$$dv = [2]$$


```
FOR I = 0 to 5
    A[I] = A[I+2] + 1
```
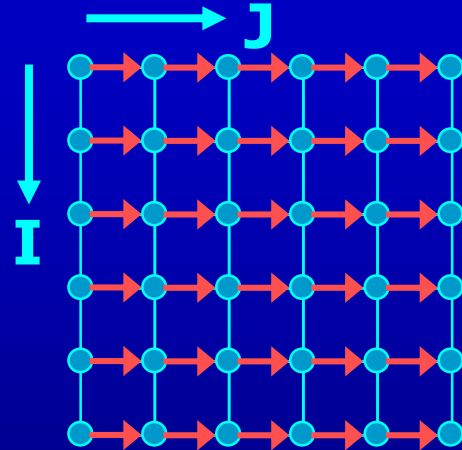
$$dv = [1], \quad [2] \quad \ldots \quad = \quad [*]$$


```
FOR I = 0 to 5
    A[I] = A[0] + 1
```
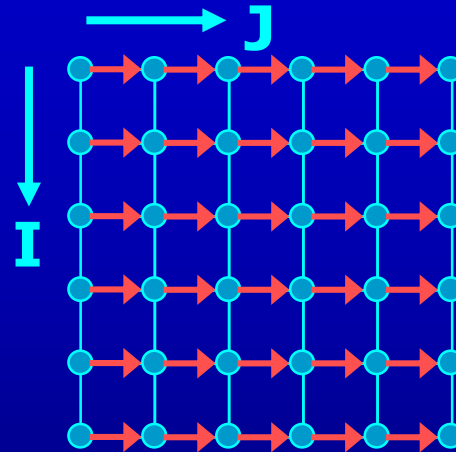
Multi-Dimensional Dependence

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I, J-1] + 1
```

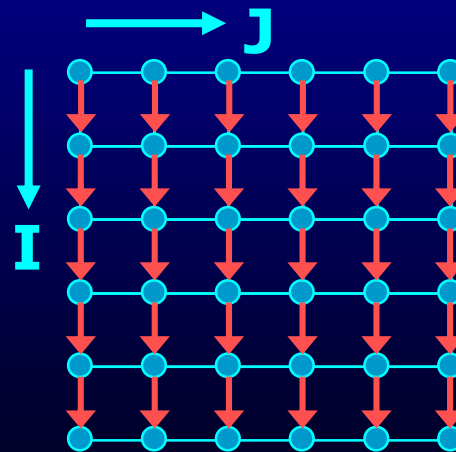$$dv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Multi-Dimensional Dependence

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I, J-1] + 1
```

$$dv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$


```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I+1, J] + 1
```
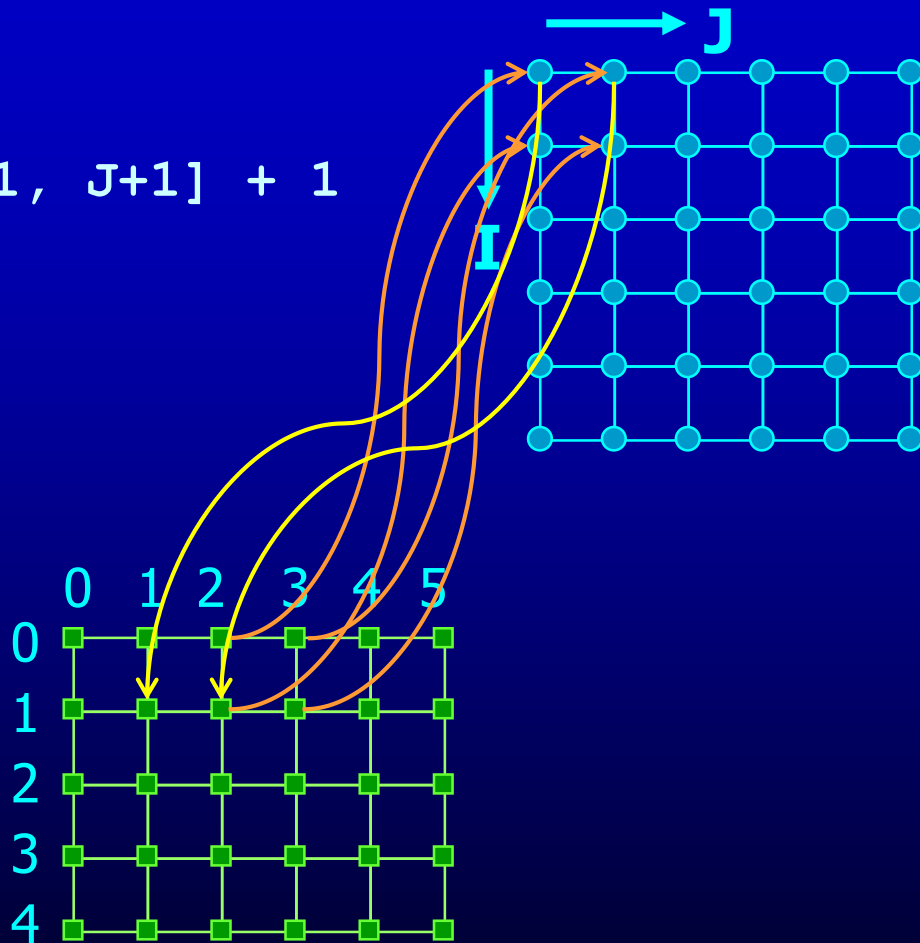
$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Outline

- Dependence Analysis
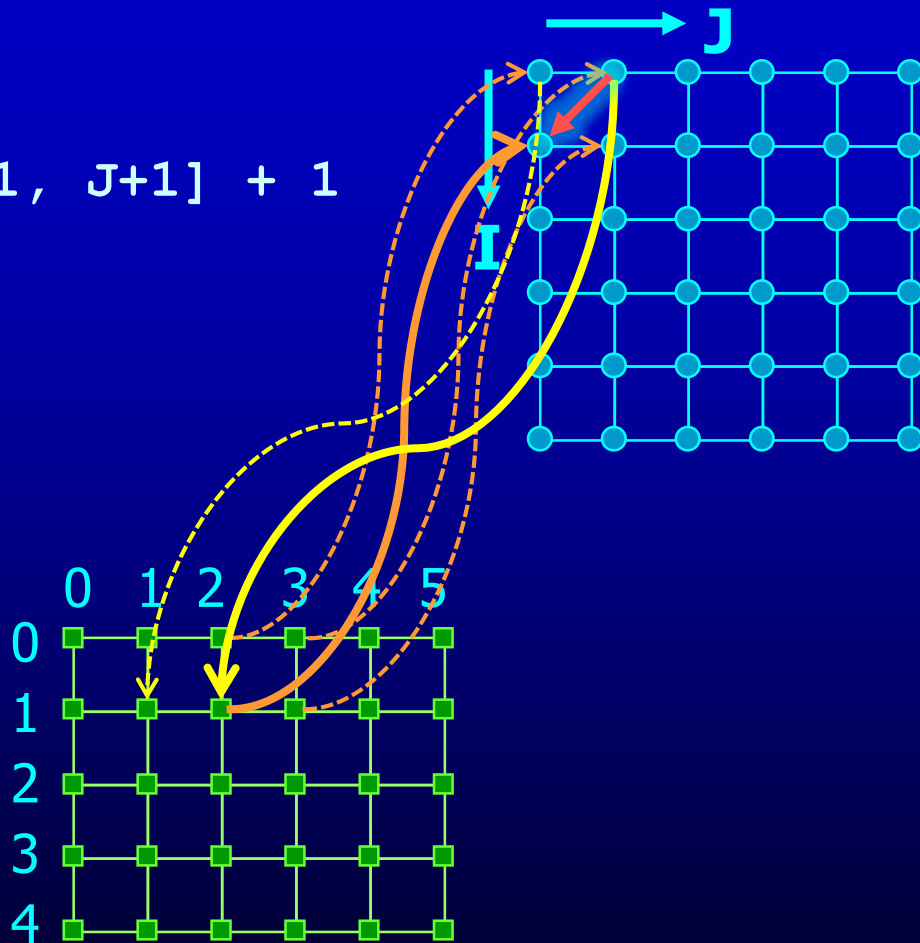- Increasing Parallelization Opportunities

What is the Dependence?

```
FOR I = 1 to n
   FOR J = 1 to n
    A[I, J] = A[I-1, J+1] + 1
```

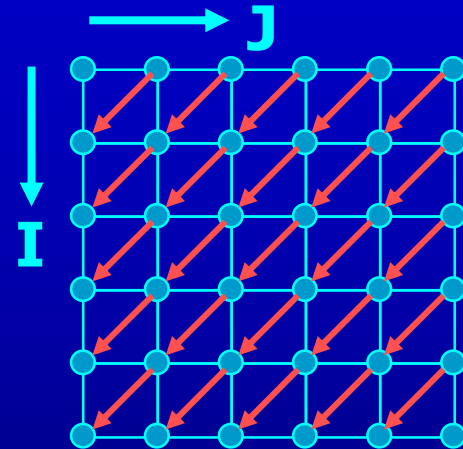# What is the Dependence?

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I-1, J+1] + 1
```

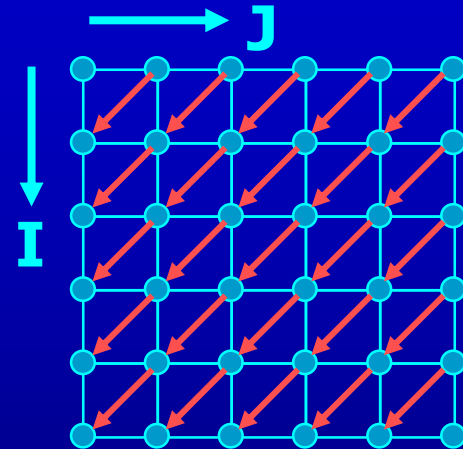# What is the Dependence?

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I-1, J+1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

What is the Dependence?

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I-1, J+1] + 1
```



```
FOR I = 1 to n
  FOR J = 1 to n
    B[I] = B[I-1] + 1
```

What is the Dependence?

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I-1, J+1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
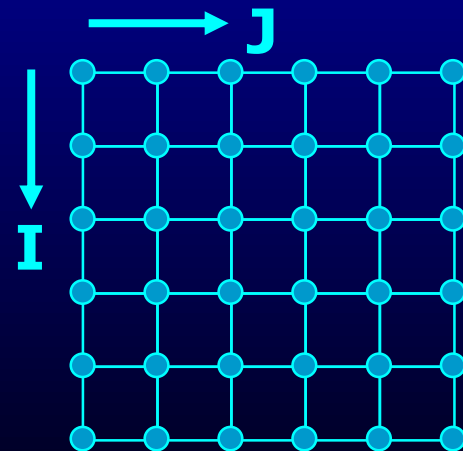
```
FOR I = 1 to n
  FOR J = 1 to n
    B[I] = B[I-1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \ldots = \begin{bmatrix} 1 \\ * \end{bmatrix}$$
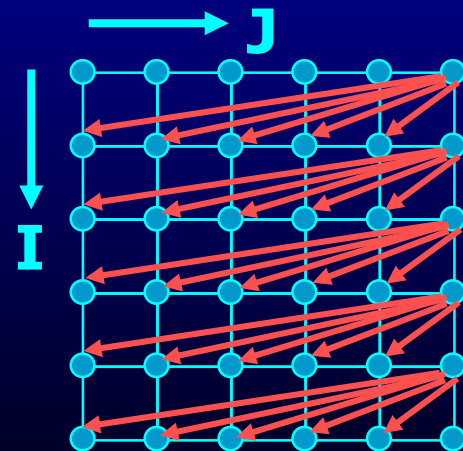
What is the Dependence?

```
FOR i = 1 to N-1
  FOR j = 1 to N-1
   A[i,j] = A[i,j-1] + A[i-1,j];
```
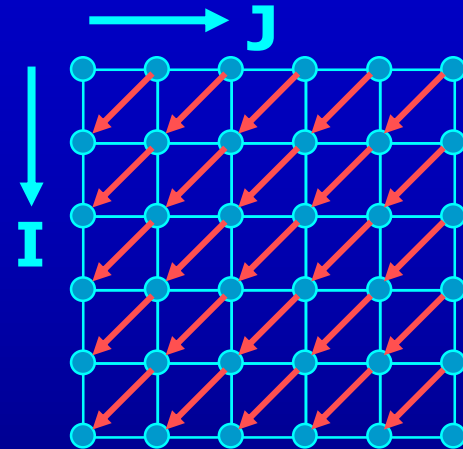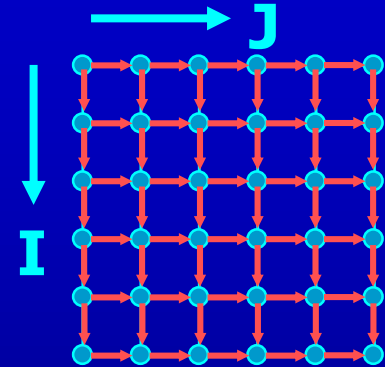
$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Recognizing FORALL Loops

- Find data dependences in loop
  - For every pair of array acceses to the same array

    If the first access has at least one dynamic instance (an iteration) in which it refers to a location in the array that the second access also refers to in at least one of the later dynamic instances (iterations).

    Then there is a data dependence between the statements
  - (Note that same array can refer to itself – output dependences)

- Definition
  - Loop-carried dependence:
    dependence that crosses a loop boundary

- If there are no loop carried dependences → parallelizable

Data Dependence Analysis

- I: Distance Vector method
- II: Integer Programming

Distance Vector Method

- The ith loop is parallelizable for all dependence $d = [d_1, \ldots, d_i, \ldots d_n]$

  either

      one of $d_1, \ldots, d_{i-1}$ is > 0

  or

      all $d_1, \ldots, d_i = 0$

Is the Loop Parallelizable?

$dv = \begin{bmatrix} 0 \end{bmatrix}$ **Yes**

```
FOR I = 0 to 5
    A[I] = A[I] + 1
```

$dv = \begin{bmatrix} 1 \end{bmatrix}$ **No**

```
FOR I = 0 to 5
    A[I+1] = A[I] + 1
```
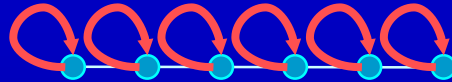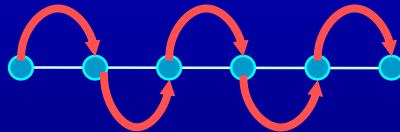
$dv = \begin{bmatrix} 2 \end{bmatrix}$ **No**

```
FOR I = 0 to 5
    A[I] = A[I+2] + 1
```
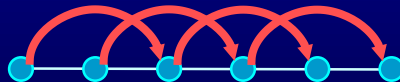
$dv = \begin{bmatrix} * \end{bmatrix}$ **No**

```
FOR I = 0 to 5
    A[I] = A[0] + 1
```

Are the Loops Parallelizable?

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I, J-1] + 1
```
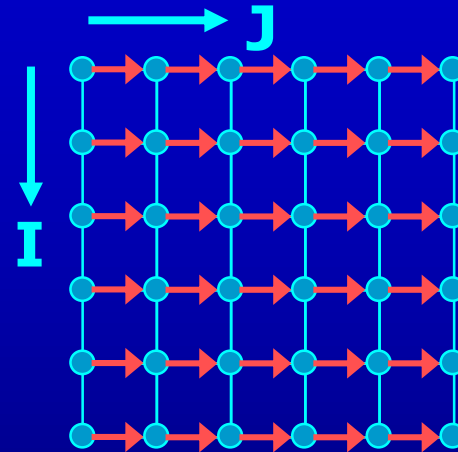
$$dv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ **Yes**
No

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I+1, J] + 1
```

$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ **No**
Yes

Are the Loops Parallelizable?

```
FOR I = 1 to n
  FOR J = 1 to n
    A[I, J] = A[I-1, J+1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

No
Yes
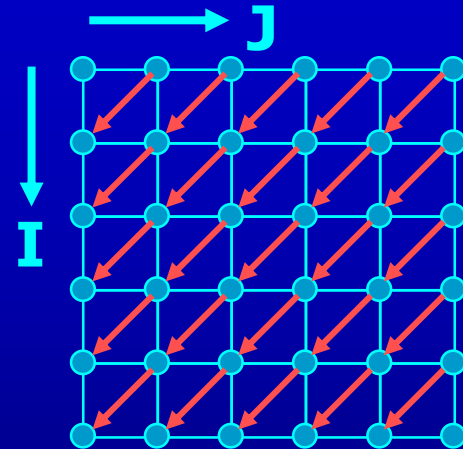
```
FOR I = 1 to n
  FOR J = 1 to n
    B[I] = B[I-1] + 1
```

$$dv = \begin{bmatrix} 1 \\ * \end{bmatrix}$$

No
Yes
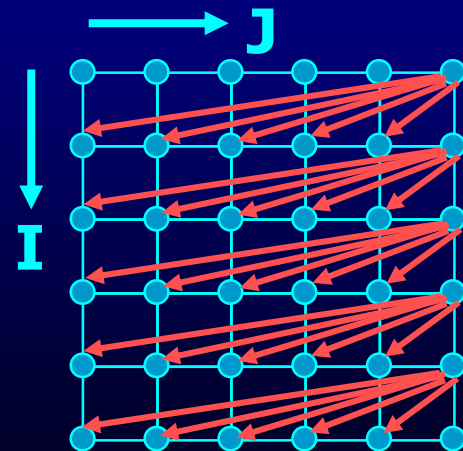
Integer Programming Method

- Example

```
FOR I = 0 to 5
   A[I+1] = A[I] + 1
```

- Is there a loop-carried dependence between A[I+1] and A[I]
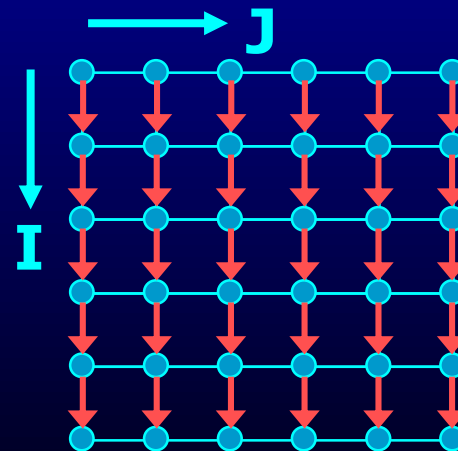 - Are there two distinct iterations i_w and i_r such that $A[i_w+1]$ is the same location as $A[i_r]$
  - $\exists$ integers i_w, i_r $0 \leq i_w, i_r \leq 5$ $i_w \neq i_r$ $i_w + 1 = i_r$

- Is there a dependence between A[I+1] and A[I+1]
  - Are there two distinct iterations i_1 and i_2 such that $A[i_1+1]$ is the same location as $A[i_2+1]$
  - $\exists$ integers i_1, i_2 $0 \leq i_1, i_2 \leq 5$ $i_1 \neq i_2$ $i_1 + 1 = i_2 + 1$

Integer Programming Method

```
FOR I = 0 to 5
    A[I+1] = A[I] + 1
```

- Formulation

    - $\exists$ an integer vector $\bar{\imath}$ such that $\hat{A}\,\bar{\imath} \le \bar{b}$ where $\hat{A}$ is an integer matrix and $\bar{b}$ is an integer vector

Iteration Space

```
FOR I = 0 to 5
    A[I+1] = A[I] + 1
```

- N deep loops → n-dimensional discrete cartesian space

- Affine loop nest → Iteration space as a set of linear inequalities
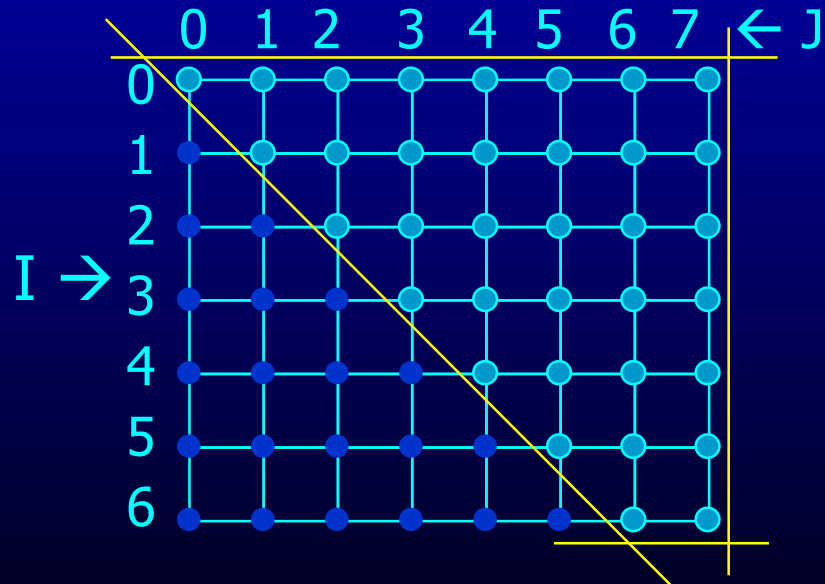
  $0 \leq I$
  $\quad I \leq 6$
  $I \leq J$
  $\quad J \leq 7$

Integer Programming Method

```
FOR I = 0 to 5
    A[I+1] = A[I] + 1
```

- Formulation

  - $\exists$ an integer vector $\bar{\imath}$ such that $\hat{A}\,\bar{\imath} \leq \bar{b}$ where $\hat{A}$ is an integer matrix and $\bar{b}$ is an integer vector

- Our problem formulation for A[i] and A[i+1]
  - $\exists$ integers i_w, i_r $0 \leq i_w, i_r \leq 5$ $i_w \neq i_r$ $i_w + 1 = i_r$
  - $i_w \neq i_r$ is not an affine function
    - divide into 2 problems
    - Problem 1 with $i_w < i_r$ and problem 2 with $i_r < i_w$
    - If either problem has a solution $\rightarrow$ there exists a dependence
  - How about $i_w + 1 = i_r$
    - Add two inequalities to single problem
      $i_w + 1 \leq i_r$, and $i_r \leq i_w + 1$

Integer Programming Formulation

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- Problem 1

$0 \leq i_w$

$i_w \leq 5$

$0 \leq i_r$

$i_r \leq 5$

$i_w < i_r$

$i_w + 1 \leq i_r$

$i_r \leq i_w + 1$

Integer Programming Formulation

```
FOR I = 0 to 5
   A[I+1] = A[I] + 1
```

- Problem 1

$$0 \leq i_w \quad \rightarrow \quad -i_w \leq 0$$

$$i_w \leq 5 \quad \rightarrow \quad i_w \leq 5$$

$$0 \leq i_r \quad \rightarrow \quad -i_r \leq 0$$

$$i_r \leq 5 \quad \rightarrow \quad i_r \leq 5$$

$$i_w < i_r \quad \rightarrow \quad i_w - i_r \leq -1$$

$$i_w + 1 \leq i_r \quad \rightarrow \quad i_w - i_r \leq -1$$

$$i_r \leq i_w + 1 \quad \rightarrow \quad -i_w + i_r \leq 1$$

Integer Programming Formulation

$\hat{A}$ $\quad$ $\bar{b}$

- Problem 1

					$\hat{A}$		$\bar{b}$
$0 \leq i_w$	$\rightarrow$	$-i_w \leq 0$			-1	0	0
$i_w \leq 5$	$\rightarrow$	$i_w \leq 5$			1	0	5
$0 \leq i_r$	$\rightarrow$	$-i_r \leq 0$			0	-1	0
$i_r \leq 5$	$\rightarrow$	$i_r \leq 5$			0	1	5
$i_w < i_r$	$\rightarrow$	$i_w - i_r \leq -1$			1	-1	-1
$i_w + 1 \leq i_r$	$\rightarrow$	$i_w - i_r \leq -1$			1	-1	-1
$i_r \leq i_w + 1$	$\rightarrow$	$-i_w + i_r \leq 1$			-1	1	1

- and problem 2 with $i_r < i_w$

Generalization

- An affine loop nest

```
FOR i₁ = f_l1(c₁…c_k) to I_u1(c₁…c_k)
  FOR i₂ = f_l2(i₁,c₁…c_k) to I_u2(i₁,c₁…c_k)

    ……

      FOR i_n = f_ln(i₁…i_{n-1},c₁…c_k) to I_un(i₁…i_{n-1},c₁…c_k)
          A[f_a1(i₁…i_n,c₁…c_k), f_a2(i₁…i_n,c₁…c_k),…,f_am(i₁…i_n,c₁…c_k)]
```

- Solve 2*n problems of the form

  - $i_1 = j_1, \; i_2 = j_2, \;…… \; i_{n-1} = j_{n-1}, \; i_n < j_n$
  - $i_1 = j_1, \; i_2 = j_2, \;…… \; i_{n-1} = j_{n-1}, \; j_n < i_n$
  - $i_1 = j_1, \; i_2 = j_2, \;…… \; i_{n-1} < j_{n-1}$
  - $i_1 = j_1, \; i_2 = j_2, \;…… \; j_{n-1} < i_{n-1}$

    ………………
  - $i_1 = j_1, \; i_2 < j_2$
  - $i_1 = j_1, \; j_2 < i_2$
  - $i_1 < j_1$
  - $j_1 < i_1$

Outline

- Why Parallelism

- Parallel Execution

- Parallelizing Compilers

- Dependence Analysis

- **Increasing Parallelization Opportunities**

Increasing Parallelization Opportunities

- Scalar Privatization

- Reduction Recognition

- Induction Variable Identification

- Array Privatization

- Loop Transformations

- Granularity of Parallelism

- Interprocedural Parallelization

Scalar Privatization

- Example

```
FOR i = 1 to n
  X = A[i] * 3;
  B[i] = X;
```

- Is there a loop carried dependence?
- What is the type of dependence?

Privatization

- Analysis:
  - Any anti- and output- loop-carried dependences

- Eliminate by assigning in local context

```
FOR i = 1 to n
    integer Xtmp;
    Xtmp = A[i] * 3;
    B[i] = Xtmp;
```

- Eliminate by expanding into an array

```
FOR i = 1 to n
    Xtmp[i] = A[i] * 3;
    B[i] = Xtmp[i];
```

Privatization

- Need a final assignment to maintain the correct value after the loop nest

- Eliminate by assigning in local context

```
FOR i = 1 to n
   integer Xtmp;
   Xtmp = A[i] * 3;
   B[i] = Xtmp;
   if(i == n) X = Xtmp
```

- Eliminate by expanding into an array

```
FOR i = 1 to n
   Xtmp[i] = A[i] * 3;
   B[i] = Xtmp[i];
X = Xtmp[n];
```

Another Example

- How about loop-carried true dependences?

- Example
  ```
  FOR i = 1 to n
     X = X + A[i];
  ```

- Is this loop parallelizable?

Reduction Recognition

- Reduction Analysis:
  - Only associative operations
  - The result is never used within the loop

- Transformation

```
Integer Xtmp[NUMPROC];
Barrier();
FOR i = myPid*Iters to MIN((myPid+1)*Iters, n)
   Xtmp[myPid] = Xtmp[myPid] + A[i];
Barrier();
If(myPid == 0) {
  FOR p = 0 to NUMPROC-1
   X = X + Xtmp[p];
  …
```

Induction Variables

- Example
  ```
  FOR i = 0 to N
    A[i] = 2^i;
  ```

- After strength reduction
  ```
  t = 1
  FOR i = 0 to N
    A[i] = t;
    t = t*2;
  ```

- What happened to loop carried dependences?

- Need to do opposite of this!
  - Perform induction variable analysis
 - Rewrite IVs as a function of the loop variable

Array Privatization

- Similar to scalar privatization

- However, analysis is more complex
  - Array Data Dependence Analysis:
    Checks if two iterations access the same location
  - Array Data Flow Analysis:
    Checks if two iterations access the same value

- Transformations
  - Similar to scalar privatization
  - Private copy for each processor or expand with an additional dimension

Loop Transformations

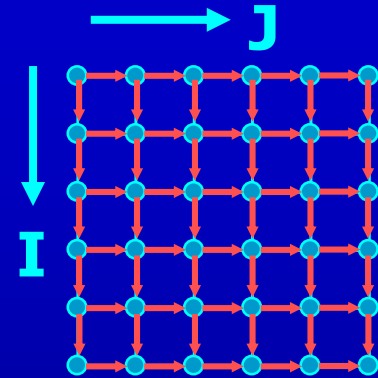- A loop may not be parallel as is
- Example

```
FOR i = 1 to N-1
  FOR j = 1 to N-1
    A[i,j] = A[i,j-1] + A[i-1,j];
```

J

I
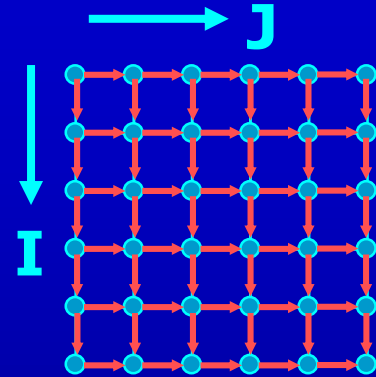
Loop Transformations
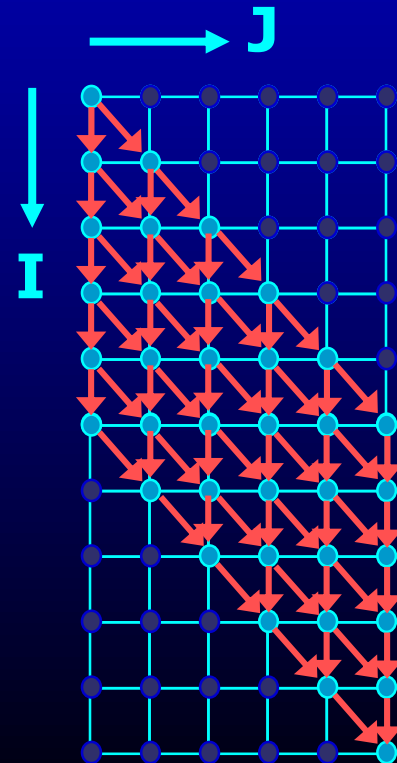
- A loop may not be parallel as is
- Example

```
FOR i = 1 to N-1
  FOR j = 1 to N-1
    A[i,j] = A[i,j-1] + A[i-1,j];
```

- After loop Skewing

$$\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$$

```
FOR i = 1 to 2*N-3
  FORPAR j = max(1,i-N+2) to min(i, N-1)
    A[i-j+1,j] = A[i-j+1,j-1] + A[i-j,j];
```
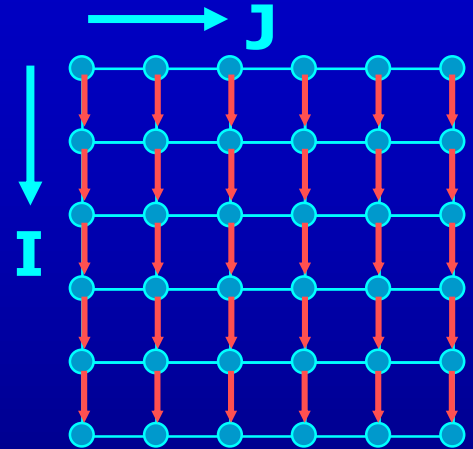
Granularity of Parallelism

- Example

```
FOR i = 1 to N-1
   FOR j = 1 to N-1
     A[i,j] = A[i,j] + A[i-1,j];
```

- Gets transformed into

```
FOR i = 1 to N-1
   Barrier();
   FOR j = 1+ myPid*Iters to MIN((myPid+1)*Iters, n-1)
     A[i,j] = A[i,j] + A[i-1,j];
   Barrier();
```

- Inner loop parallelism can be expensive
  – Startup and teardown overhead of parallel regions
  – Lot of synchronization
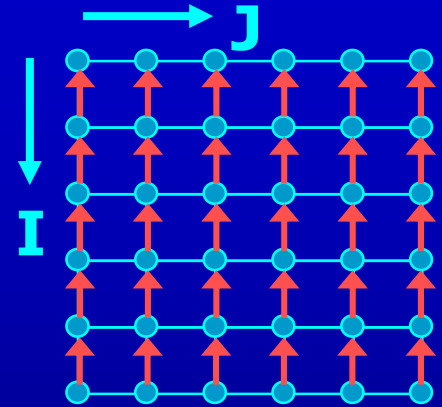 – Can even lead to slowdowns

Granularity of Parallelism

- Inner loop parallelism can be expensive

- Solutions
  - Don't parallelize if the amount of work within the loop is too small

  or

  - Transform into outer-loop parallelism

Outer Loop Parallelism
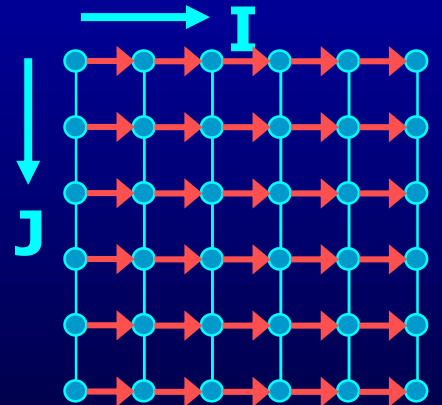
- Example

```
FOR i = 1 to N-1
    FOR j = 1 to N-1
        A[i,j] = A[i,j] + A[i-1,j];
```

- After Loop Transpose

```
FOR j = 1 to N-1
    FOR i = 1 to N-1
        A[i,j] = A[i,j] + A[i-1,j];
```

- Get mapped into

```
Barrier();
FOR j = 1+ myPid*Iters to MIN((myPid+1)*Iters, n-1)
    FOR i = 1 to N-1
        A[i,j] = A[i,j] + A[i-1,j];
Barrier();
```

Unimodular Transformations

- Interchange, reverse and skew
- Use a matrix transformation

$$I_{new} = A \; I_{old}$$

- Interchange
$$\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$$

- Reverse
$$\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$$

- Skew
$$\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$$

Legality of Transformations

- Unimodular transformation with matrix A is valid iff.
  For all dependence vectors v
  the first non-zero in Av is positive

- Example

```
FOR i = 1 to N-1
   FOR j = 1 to N-1
     A[i,j] = A[i,j] + A[i-1,j];
```

$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Interchange $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ✔

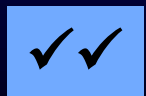- Reverse $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ ✘

- Skew $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✔✔

Interprocedural Parallelization

- Function calls will make a loop unparallelizatble
  - Reduction of available parallelism
  - A lot of inner-loop parallelism

- Solutions
  - Interprocedural Analysis
  - Inlining

Interprocedural Parallelization

- Issues
  - Same function reused many times
  - Analyze a function on each trace → Possibly exponential
  - Analyze a function once → unrealizable path problem

- Interprocedural Analysis
  - Need to update all the analysis
  - Complex analysis
  - Can be expensive

- Inlining
  - Works with existing analysis
  - Large code bloat → can be very expensive

```
HashSet h;
for i = 1 to n
    int v = compute(i);
    h.insert(i);

    Are iterations independent?
    Can you still execute the loop in parallel?
    Do all parallel executions give same result?
```

Summary

- Multicores are here
    - Need parallelism to keep the performance gains
    - Programmer defined or compiler extracted parallelism

- Automatic parallelization of loops with arrays
    - Requires Data Dependence Analysis
    - Iteration space & data space abstraction
    - An integer programming problem

- Many optimizations that'll increase parallelism