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SEB SCHOOL OF ECONOMICS AND BUSINESS

3. Hypotheses Testing

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3.1 Confidence Intervals of Regression Coefficients

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Motivation

The basic task of econometrics is to confront the existing economic theories with the economic reality by using the methods of statistical inference.

P. Samuelson, *Econometrica*, 1954 (22)

Statistical Inference

Null hypothesis → **Alternative hypothesis**

$$H_0 : \beta_j = \beta_0 \quad ; \quad H_1 : \beta_j \neq \beta_0$$

$$H_0 : \beta_j \leq \beta_0 \quad ; \quad H_1 : \beta_j > \beta_0$$

$$H_0 : \beta_j \geq \beta_0 \quad ; \quad H_1 : \beta_j < \beta_0$$

if it has one value

Simple hypothesis ↔ **Composite hypothesis**

if it has more values

One-tailed test ↔ **Two-tailed test**

the value is only lower or higher ($>$, $<$)

the value is either lower or higher ($=$, $\neq$)

Statistical Inference

In social sciences we want to reject the null hypothesis

Goal: REJECT H_0

What to put into the null hypothesis?

H_o is not rejected! → **H_o is rejected!**

Computer software includes (direct) testing of the following hypothesis:

$$H_0 : \beta_j = \beta_0 \quad ; \quad H_1 : \beta_j \neq \beta_0$$

Simple hypothesis! **Two-tailed test!**

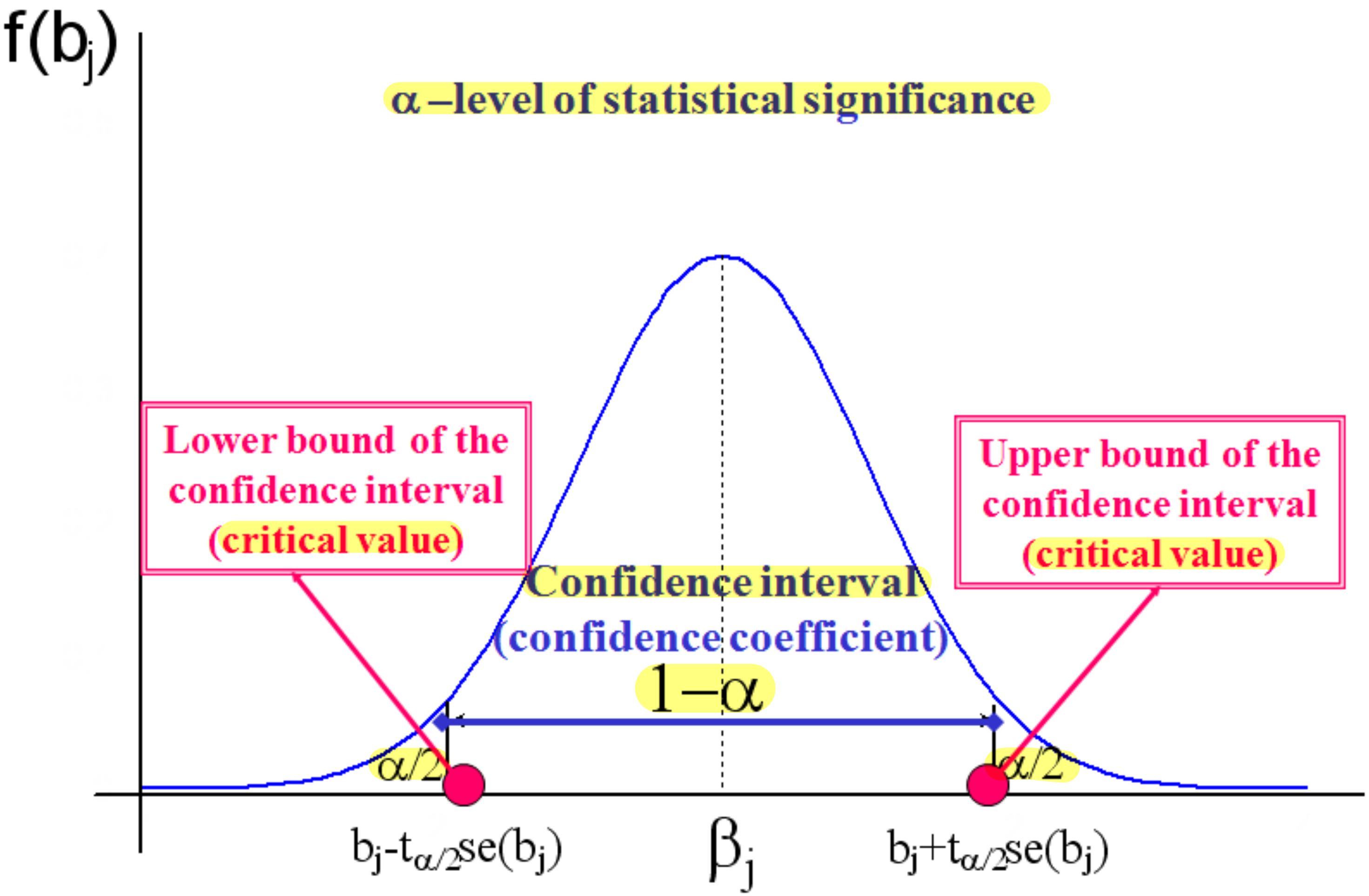
Calculation of Confidence Intervals

$$t = \frac{b_j - \beta_j}{\sqrt{\operatorname{var}(b_j)}} = \frac{b_j - \beta_j}{se(b_j)} \sim t_{(n-k)}$$

$$\Pr(-t_{\alpha/2} \leq \frac{b_j - \beta_j}{se(b_j)} \leq t_{\alpha/2}) = 1 - \alpha$$

$$\Pr(b_j - t_{\alpha/2} se(b_j) \leq \beta_j \leq b_j + t_{\alpha/2} se(b_j)) = 1 - \alpha$$

Defining the Confidence Intervals



Interpreting the Confidence Intervals

DECISION RULE:

A If the value of regression coefficient at the null hypothesis, i.e. β_0, is within the calculated confidence interval, we do not reject H_0.

B If the value of regression coefficient β_0 is not included in the calculated confidence interval, we reject H_0.

Interpreting the Confidence Intervals

- The probability that, based on sample data, an interval is constructed that will include β_j, is equal to $(1 - \alpha)$.
- This interval is random, i.e. changing from one sample to another. => with other samples your results may differ
- The findings are valid on average, i.e. for a large number of confidence intervals.
- The calculated confidence interval can not be interpreted as if it contains the true value with probablity $(1 - \alpha)$. The probability that an already constructed interval contains β_j is either 1 or 0.

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3.2 Testing for Statistical Significance of Regression Coefficients

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Hypotheses about Individual Values of β_j

Essential elements:

① **test statistic** (sample t-test statistic)

② **sample distribution of the test statistic taking into account H_0** (student's t-distribution)

↳ this distribution works only for H_0.

$$t = \frac{b_j - \beta_0}{\sqrt{\mathrm{var}(b_j)}} = \frac{b_j - \beta_0}{se(b_j)} \underset{H_0}{\sim} t_{(n-k)}$$

$$\Pr(-t_{\alpha/2} \le \frac{b_j - \beta_0}{se(b_j)} \le t_{\alpha/2}) = 1 - \alpha$$

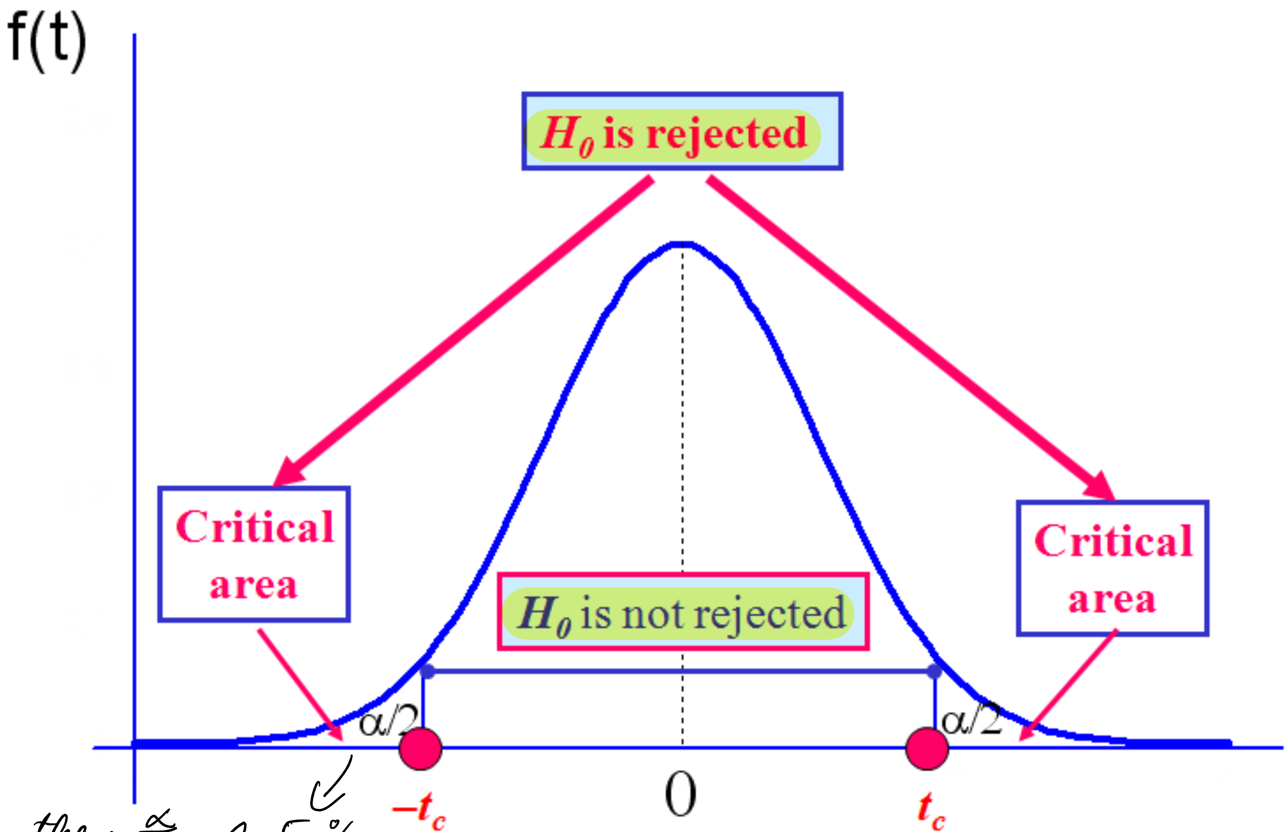
Hypotheses testing

Essential elements of testing:

$\boxed{t \underset{H_0}{\sim} t_{n-k}}$

Slide 10

Hypotheses about Individual Values of β_j



if $\alpha = 5\%$, then $\frac{\alpha}{2} = 2,5\%$

If the value of t–statistic is in the critical area,

H_0 is rejected at the level of statistical significance α!

Hypotheses about Individual Values of β_j

Hypothesis testing scheme about the values of regression coefficients based on *t*–statistic

Hypothesis type	H_0	H_1	H_0 is rejected, if
Two-tailed test	$\beta_j = \beta_{j0}$	$\beta_j \neq \beta_{j0}$	$\lvert t \rvert \geq t_{\alpha/2}$
One-tailed test	$\beta_j \leq \beta_{j0}$	$\beta_j > \beta_{j0}$ right tail	$t \geq t_{\alpha}$
One-tailed test	$\beta_j \geq \beta_{j0}$	$\beta_j < \beta_{j0}$ left tail	$t \leq -t_{\alpha}$

Type I Error ! ⟷ **Type II Error !**

Rule of thumb:

H_0 is rejected at $\alpha = 0.05$ or below

- **$t = 2$ or above; two-tailed test**
- **$t = 1.65$ or above; one-tailed test**

Exact Level of Statistical Significance

***p*–value** or **exact level of statistical significance** is the *smallest* statistical significance level that enables *rejection of the null hypothesis* based on a *given data sample*.

E.g. $p = \Pr\left(|t| \geq t_{\alpha/2}\right)$ for the two-tailed test.

The calculated *p*–value is *different* for *every data sample*.

Most often used rule of thumb: H_o is rejected, if it holds:

$$p \leq (\alpha = 0.05) \Rightarrow$$

we don t need to reject the null hypothesis on lower α in this class

In general: *p*–value is the probability of obtaining test statistic value at least as extreme as the one actually observed, under the assumption that the *null hypothesis is correct*.

Hypotheses testing

Essential elements of testing:

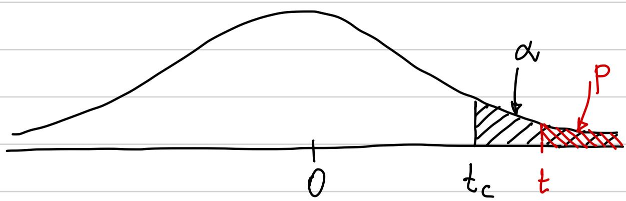
$$\boxed{t \underset{H_0}{\sim} t_{n-k}}$$

Slide 10

Test statistic vs. p-value

Slide 13

An example of one-tailed t-test in the right tail of the distribution:



H_0 is rejected: $t \geq t_c$ or $p \leq \alpha$. !!

Meaning of Hypotheses Testing

Rejection of the null hypothesis means:

- The difference between the regression coefficient estimate and the value β_0, tested in the null hypothesis, is statistically significant (statistically significantly different from 0).
- It is unlikely that the difference was established (purely) by chance. This probability is equal to the level of statistical significance of the test α, or $\alpha/2$ for the two-tailed test.
- If the value of regression coefficient, tested in the null hypothesis, is equal to zero ($\beta_0 = 0$), then rejection of the null hypothesis means that the regression coefficient estimate is statistically significantly different from 0 (the estimated regression coefficient is "statistically significant").

↳ meaning is the same, we use this term to reduce words in explanation

Meaning of Hypotheses Testing

Statistical significance ⟷ Economic / theoretical acceptability

The value of p does not tell you if the variables have explanatory power

- A statistically significant effect of an explanatory variable on the dependent variable, measured by the value of the t–statistic, does not yet mean the theoretical acceptability of the given explanatory variable.

- High value of the t–statistic does not mean that this explanatory variable is more important than another explanatory variable with a lower value of t–statistic. We can not make conclusions about this based on the value b_j either. For this purpose we use the standardised regression coefficients.
  ↳ to find out which variable has more explanatory

- We have to be aware that the value of t–statistic depends on several already mentioned determinants. This has to be taken into account when doing statistical inference.

"Beta" coefficients or standardised regression coefficients

Slide 15

1) Standardize x_j and y:

$$x_{ji}^{s} = \frac{x_{ji} - \bar{x}_j}{sd(x_j)} \; ; \; \forall j = 1, \dots, k$$

$$y_i^{s} = \frac{y_i - \bar{y}}{sd(y)}$$

now they are comparable
$$\begin{cases} Var(x_j^{s}) = 1 \\ Var(y^{s}) = 1 \end{cases}$$

2) Estimate:

$$\widehat{y_i^{s}} = \underbrace{b_1}_{=0} + \underline{beta_2} \cdot x_{2i}^{s} + \underline{beta_3} \cdot x_{3i}^{s} + \dots + \underline{beta_k} \cdot x_{ki}^{s}$$

standardised or "beta" coefficients

3) Compare $|beta_j|$.

Higher value of $|beta_j|$ means that the variable x_j has more explanatory power with explaining the variability of y.

Statistical Inference

Conclusion based on sample data	Actual state of nature (on the population)	
	H_0 is true	H_0 is false
H_0 is rejected	False conclusion Pr = α (Type I Error)	Correct conclusion Pr = $1-\beta$ (Power of the test)
H_0 is not rejected	Correct conclusion Pr = $1-\alpha$	False conclusion Pr = β (Type II Error)

Due to the presence of Type II Error, the null hypothesis is either rejected or not rejected, but not accepted. If the null hypothesis is rejected, a conclusion is made from the alternative hypothesis. The alternative hypothesis is neither accepted nor rejected, except on those occasions, when its distribution is explicitly known.

Type 1 error α - the null is true and you reject it

Type 2 error β - the null is not true and you don't reject it.

Statistical inference — Slide 16

- We can commit either the Type I error or the Type II error.

- Our goal is to:
  - minimize the probability of Type I error ($\alpha \leq 0.05$) and at the same time to
  - minimize the probability of Type II error or to maximize the power of the test ($1-\beta \geqslant 0.8$).

- Challenges: ↗ when α goes down β goes up
  - there is a trade-off between α and β in practice and
  - in order to regulate β, we would have to know the actual state of nature (1 sample!).

- Reality: in practice we focus on α and at the same time we try to ensure appropriate power of the test $(1-\beta)$.

- Power of the test depends on:
  1) Type of the test;
  2) Difference between the actual and the tested value (b_j vs. β_j) (+)
     - if the values are 3 and 7 power of the test is higher than if the values are 3 and 3,1
  3) Measurement error (−);
  4) Sample size (+)
     - A larger sample leads to higher power of the test.

!

- As we could commit the Type II error, we never accept H_0, only reject.
- If H_0 is rejected, we make a conclusion from H_1.
- H_1 is neither accepted nor rejected, as its distribution is usually not known explicitly (not the same as that of H_0).

We conclude that...

Statistical Inference in Econometrics

Goal: Application of the *linear regression model* for determining the validity of theory based on a sample of data.

The model is used for testing hypotheses about a given data generating process (DGP), where we consider only linear restrictions (for now).

The hypothesis has to be testable:
– we are able to write it down in terms of mathematical notation;
– we are able to test it within our model.

Statistical Inference in Econometrics

Nested and non-nested models: models are nested, if one can be obtained by applying a restriction on the other.

M_1 and M_3 are nested (you say $\beta_3 = 0$)

$$M_1 = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$$
$$M_2 = (\beta_1, 0, \beta_3, \beta_4, \beta_5)$$
$$M_3 = (\beta_1, \beta_2, 0, \beta_4, \beta_5)$$

M_1 and M_2 are nested (you say $\beta_2 = 0$).

M_2 and M_3 are not nested $\Rightarrow$ it is impossible to get the same model by applying the restriction on the other.

Restricted models and non-restricted models:

– the restricted model ("null" model) is the model that has the null hypothesis already incorporated;

– the non-restricted model ("alternative" model) is the base model, i.e. the model from the alternative hypothesis.

Testing Linear Combinations of b

12P1T

$$H_0 : \mathbf{c}^T \boldsymbol{\beta} = r \quad ; \quad H_1 : \mathbf{c}^T \boldsymbol{\beta} \neq r$$

β - vector of the values you are testing

c - vector of the parameter of the linear combination

r - hypothesized value (scalar)

$$t = \frac{\mathbf{c}^T \mathbf{b} - r}{se(\mathbf{c}^T \mathbf{b})} = \frac{\mathbf{c}^T \mathbf{b} - r}{\sqrt{\mathrm{var}(\mathbf{c}^T \mathbf{b})}}$$

$$\mathrm{var}(\mathbf{c}^T \mathbf{b}) = \mathbf{c}^T ((\mathrm{var-cov}(\mathbf{b}))\mathbf{c}$$

Testing Linear Combinations of b

there will not be more than 3 parameters on the test

$$\text{var}-\text{cov}(\mathbf{b}) = s_e^2\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}$$

if there are 2 parameters

$$var\left(ab_j + cb_l\right) = a^2\, var\left(b_j\right) + c^2\, var\left(b_l\right) + 2ac\, cov\left(b_j, b_l\right)$$

if there are 3 parameters

$$var\left(ab_j + cb_l + db_m\right) = a^2\, var\left(b_j\right) + c^2\, var\left(b_l\right) + d^2\, var\left(b_m\right) +$$

$$+2ac\, cov\left(b_j, b_l\right) + 2ad\, cov\left(b_j, b_m\right) + 2cd\, cov\left(b_l, b_m\right)$$

Trinity of Tests in Econometrics

1. **Wald Test (*F*–test)**
2. **Lagrange Multiplier Test (*LM*–test)**
3. **Likelihood Ratio Test (*LR*–test)**

Fundaments of *F*–test and *LM*–test

BASE MODEL

RSS_B

$$RSS_B = \mathbf{e}_B^T \mathbf{e}_B$$

RESTRICTED MODEL

RSS_R

$$RSS_R = \mathbf{e}_R^T \mathbf{e}_R$$

$$RSS_R \geq RSS_B$$

$$R_R^2 \leq R_B^2$$

the unexplained sum of squares is greater in the restricted model than in the base model => because of the restrictions

Are the differences in *RSS* statistically significant? - test

If the restriction is valid, then the RSS doesn't increase a lot.

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Wald Test (*F*–test)

this formula is always valid

$$F = \frac{(RSS_R - RSS_B)/m}{RSS_B/(n-k_B)} \sim F_{(m,\,n-k_B)}$$

Degrees of freedom:

$m = (n - k_R) - (n - k_B) = k_B - k_R$

if you're using this formula you need to be careful => see next slide

$$F = \frac{((1-R_R^2)-(1-R_B^2))/m}{(1-R_B^2)/(n-k_B)} = \frac{(R_B^2 - R_R^2)/m}{(1-R_B^2)/(n-k_B)} \sim F_{(m,\,n-k_B)}$$

I. $F < F_c$ **at given statistical significance level;**
H_0 **is not rejected**

II. $F \geq F_c$ **at given statistical significance level;**
H_0 **is rejected**

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Wald Test (F–test)

Izpit

Comparable multiple determination coefficient of the restricted model, when the dependent variable in the restricted model is defined differently than in the base (unrestricted) model:

$$R_R^2 \quad \Rightarrow \quad R_R^{*2} = 1 - \frac{RSS_R}{TSS_B}$$

$$M_B: y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

$$\beta_3 = 1$$

$$y = \beta_1 + \beta_2 x_2 + x_3 + u$$

$$y - x_3 = \beta_1 + \beta_2 x_2 + u \quad (Me)$$

$\hookrightarrow Mr$

Wald Test (*F*–test)

APPLICATION OF THE *F*–STATISTIC (WALD TEST)

❶ **Testing hypotheses on particular values of regression coefficients or linear combinations of regression coefficients (equivalent to the *t*–statistic)**

Testing hypotheses on regression coefficients or their linear combinations

With one restriction (in Ho), $m=1$: $F=t^2$. Slide 25

$$F = \frac{(RSS_R - RSS_B)/m}{RSS_B/(n-k_B)} \sim F_{(m,\, n-k_B)}$$

Wald Test (*F*–test)

❷ Simultaneous testing of multiple hypotheses on regression coefficients

A Testing statistical significance of model reduction "General to Simple"

$$H_o : \beta_l = 0 \\ \beta_{l+1} = 0 \\ \vdots \\ \beta_{l+(m-1)} = 0$$

H_1 : At least one value is not equal to 0

$$F = \frac{(RSS_R - RSS_B)/m}{RSS_B/(n-k_B)} \sim F_{(m, n-k_B)}$$

Wald Test (*F*–test)

B Testing statistical significance of model expansion "Simple to General"

$$H_o : \beta_{k+1} = 0$$
$$\beta_{k+2} = 0$$
$$\vdots$$
$$\beta_{k+m} = 0$$

H_1 : At least one value is not equal to 0

$$\boxed{F = \frac{(RSS_B - RSS_N)/m}{RSS_N/(n-k_N)} \sim F_{(m,\, n-k_N)}}$$

Simultaneous testing of multiple hypotheses on regression coefficients

Slides 26-27

A) Model reduction

Base model (M_B)

RSS_B, k_B

vs.

Restricted model (M_R)

RSS_R, k_R

("SHORTER MODEL")

B) Model expansion

Base model (M_B)

RSS_B, k_B

("SHORTER MODEL")

vs.

"New" (expanded) model (M_N)

RSS_N, k_N

Lagrange Multiplier (LM) Test

$$y_i = \beta_1 + \beta_2 x_{2i} + \cdots + \beta_k x_{ki} + \beta_{k+1} x_{k+1,i} + \cdots + \beta_{k+m} x_{k+m,i} + u_i$$

1

$$H_o : \beta_{k+1} = 0; \quad \beta_{k+2} = 0; \quad \ldots; \quad \beta_{k+m} = 0$$

H_1 : At least one value is not equal to 0

2

Estimate the restricted regression model and calculate e_i

$$y_i = g_1 + g_2 x_{2i} + \cdots + g_k x_{ki} + e_i \quad \Rightarrow \quad e_i$$

we put all of the restrictions in the model to calculate the residual.

Lagrange Multiplier (LM) Test

any regression which is not the main regression => it is used for testing

3 **Estimate the auxiliary regression:**

$$\hat{e}_i = g_1 + g_2 x_{2i} + \cdots + g_k x_{ki} + g_{k+1} x_{k+1,i} + \cdots + g_{k+m} x_{k+m,i} \quad \Rightarrow \quad R^2$$

parameter of the model

t statistic

4 $LM = nR^2 \sim \chi^2_m$

5 $LM \geq \chi^2_m$

H_o **is rejected**

Likelihood Ratio (LR) Test

BASE MODEL

$\ln L_B$

RESTRICTED MODEL

$\ln L_R$

$$\ln L = -\frac{n}{2}\left[\ln(2\pi) + \ln\left(\frac{RSS}{n}\right) + 1\right]$$

Are the differences in ln L statistically significant?

↳ can we reject the null hypothesis?

$$LR = -2\left(\ln L_R - \ln L_B\right) \sim \chi^2_m$$

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PART 2

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