Multiple regression model revisited

PRM: $y_i = \underbrace{\beta_1 + \beta_2 x_{2i} + \ldots + \beta_j x_{ji} + \ldots + \beta_k x_{ki}}_{E(y \mid x_2, \ldots, x_k)} + u_i$

SRM: $y_i = \underbrace{b_1 + b_2 x_{2i} + \ldots + b_j x_{ji} + \ldots + b_k x_{ki}}_{\hat{y}} + e_i$

Slide 14

An estimator of β, i.e. the formula to calculate b, should somehow minimize "e".

$$e_i = y_i - \hat{y}_i$$

Infinite number of possibilities:

1) $\min_b \sum_i e_i$ ✗

2) $\min_b \sum_i |e_i|$ ✗

3) $\min_b \sum_i e_i^2$ ✓ $\longmapsto$ Least squares estimator

etc.

It is convenient to employ matrix notation:

Slide 17

$$\underset{n\times 1}{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underbrace{\underset{n\times 1}{x_1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \underset{n\times 1}{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} \quad \cdots \quad \underset{n\times 1}{x_k} = \begin{bmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kn} \end{bmatrix}}$$

$$\underset{n\times k}{X} = \begin{bmatrix} 1 & x_{21} & \cdots & x_{k1} \\ 1 & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2n} & \cdots & x_{kn} \end{bmatrix}$$

$$\underset{n\times 1}{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$\underset{k\times 1}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$$

$$\underset{n\times 1}{y} = \underbrace{X \cdot b}_{\substack{(n\times k)(k\times 1) \\ n\times 1}} + \underset{n\times 1}{e} \quad \text{and} \quad e = y - X \cdot b$$

$\mapsto$ Slide 18

Remember that: **Slide 18**

$$e^T e = [e_1\ e_2\ \dots\ e_n] \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} =$$

$$= e_1^2 + e_2^2 + \dots + e_n^2 = \sum_i e_i^2$$

System of normal equations:

$$(X^TX)^{-1} / \quad X^T y = (X^TX) b$$ **Slide 19**

$$(X^TX)^{-1} X^T y = \underbrace{(X^TX)^{-1} X^T X}_{I} b$$

$$b = (X^TX)^{-1} X^T y$$

THE LEAST SQUARES ESTIMATOR

Remember that: **Slide 20**

$$X^T e = 0 \iff \sum_i x_{ji} \cdot e_i = 0, \forall j = 1, \dots, k$$

$$\hat{y}^T e = 0 \iff \sum_i \hat{y}_i \cdot e_i = 0$$

PDF, p. 1

E1 $e_i = y_i - \hat{y}_i = y_i - (b_1 + b_2 x_{2i} + b_3 x_{3i})$

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - b_1 - b_2 x_{2i} - b_3 x_{3i})^2 = S(b_1, b_2, b_3)$$

(compound function)

CRITERION FUNCTION

min

First-order conditions (FOCs):

$$\frac{\partial S}{\partial b_1} = 2\sum_{i=1}^{n} (y_i - b_1 - b_2 x_{2i} - b_3 x_{3i})(-1) = 0 \quad | : 2$$

$$\frac{\partial S}{\partial b_2} = 2\sum_{i=1}^{n} (y_i - b_1 - b_2 x_{2i} - b_3 x_{3i})(-x_{2i}) = 0 \quad | : 2$$

$$\frac{\partial S}{\partial b_3} = 2\sum_{i=1}^{n} (y_i - b_1 - b_2 x_{2i} - b_3 x_{3i})(-x_{3i}) = 0 \quad | : 2$$

System of normal equations:

$\sum_{i=1}^{n} b_1$

$$b_1 n + b_2 \sum_{i=1}^{n} x_{2i} + b_3 \sum_{i=1}^{n} x_{3i} = \sum_{i=1}^{n} y_i$$
$$b_1 \sum_{i=1}^{n} x_{2i} + b_2 \sum_{i=1}^{n} x_{2i}^2 + b_3 \sum_{i=1}^{n} x_{2i} x_{3i} = \sum_{i=1}^{n} x_{2i} y_i$$
$$b_1 \sum_{i=1}^{n} x_{3i} + b_2 \sum_{i=1}^{n} x_{2i} x_{3i} + b_3 \sum_{i=1}^{n} x_{3i}^2 = \sum_{i=1}^{n} x_{3i} y_i$$

or

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_{2i} & \sum_{i=1}^{n} x_{3i} \\ \sum_{i=1}^{n} x_{2i} & \sum_{i=1}^{n} x_2^2 & \sum_{i=1}^{n} x_{2i} x_{3i} \\ \sum_{i=1}^{n} x_{3i} & \sum_{i=1}^{n} x_{2i} x_{3i} & \sum_{i=1}^{n} x_{3i}^2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_{2i} y_i \\ \sum_{i=1}^{n} x_{3i} y_i \end{bmatrix}$$

$$X^T X \cdot b = X^T y$$

$$(X^TX)^{-1} \cdot / \quad X^T y = (X^TX) b$$

$$(X^TX)^{-1} X^T y = \underbrace{(X^TX)^{-1} X^TX}_{I} b$$

$$b = (X^TX)^{-1} X^T y$$

THE LEAST SQUARES ESTIMATOR

E2

PDF, p. 1

$$X = \begin{array}{ccc} "x_1" & x_2 & x_3 \\ \end{array}$$

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -4 \\ 1 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 5 \\ -4 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

$n \times k$ — 8×3 $\qquad$ $n \times 1$ — 8×1

$$X^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & -1 & 1 & -1 & -2 & 0 \\ -1 & -1 & 2 & -4 & 3 & 0 & 2 & -1 \end{bmatrix}$$

$k \times n$

3×8

$$\underbrace{X^T X}_{\underbrace{(3\times 8)(8\times 3)}_{3\times 3}} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$\underbrace{X^T y}_{\substack{k\times 1 \\ 3\times 1}} = \begin{bmatrix} 12 \\ -12 \\ -27 \end{bmatrix}$$

matrix of cofactors

$$(X^T X)^{-1} = \frac{1}{\det(X^T X)} \cdot [c_{ij}]^T_{k\times k}$$

$$(X^T X)^{-1} = \frac{1}{\underset{\det(X^T X) \neq 0}{3456}} \cdot \begin{bmatrix} 432 & 0 & 0 \\ 0 & 288 & 0 \\ 0 & 0 & 96 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/36 \end{bmatrix}$$

$$b = \underbrace{(X^T X)^{-1} \cdot X^T y}_{\underbrace{(3\times 3)\cdot(3\times 1)}_{3\times 1}} = \begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/36 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ -12 \\ -27 \end{bmatrix}$$

$$\underline{\underline{b = \begin{bmatrix} 1.5 \\ -1 \\ -0.75 \end{bmatrix}}} \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}$$