Tutorial, Part 2

Example 1

<u>Derivation</u> of b done at the lecture.

<u>Application</u> of $\boxed{b = (X^T X)^{-1} X^T y}$:

System of normal equations (expressions 6 and 7):

$$
\begin{aligned}
\overset{n}{10}\, b_1 &+ \overset{\Sigma x_2 = n\cdot\overline{x_2}}{(10\cdot 0.4)}\, b_2 + \overset{\Sigma x_3 = n\cdot\overline{x_3}}{(10\cdot 0.8)}\, b_3 = \overset{\Sigma y = n\cdot\overline{y}}{10\cdot 2.4} \\
4\, b_1 &+ 10\, b_2 - 2\, b_3 = 40 \\
8\, b_1 &- 2\, b_2 + 12\, b_3 = -2
\end{aligned}
$$

$$
X^T X = \begin{bmatrix} 10 & 4 & 8 \\ 4 & 10 & -2 \\ 8 & -2 & 12 \end{bmatrix} \qquad X^T y = \begin{bmatrix} 24 \\ 40 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

symmetry

Determinant:

$$
\begin{aligned}
\det(X^T X) &= 10\cdot 10\cdot 12 + 4(-2)8 + 8\cdot 4(-2) - \\
&\quad - 8\cdot 10\cdot 8 - 10(-2)(-2) - 4\cdot 4\cdot 12 = \\
&= 200
\end{aligned}
$$

Cofactors:

$$c_{11} = \begin{vmatrix} 10 & -2 \\ -2 & 12 \end{vmatrix} = 116$$

$$c_{12} = \begin{vmatrix} 4 & -2 \\ 8 & 12 \end{vmatrix} = 64$$

$$c_{13} = \begin{vmatrix} 4 & 10 \\ 8 & -2 \end{vmatrix} = -88$$

$$c_{21} = \underbrace{c_{12}}_{\text{symmetry}} = 64$$

$$c_{22} = \begin{vmatrix} 10 & 8 \\ 8 & 12 \end{vmatrix} = 56$$

$$c_{23} = \begin{vmatrix} 10 & 4 \\ 8 & -2 \end{vmatrix} = -52$$

$$c_{31} = \underbrace{c_{13}}_{\text{symmetry}} = -88$$

$$c_{32} = \underbrace{c_{23}}_{\text{symmetry}} = -52$$

$$c_{33} = \begin{vmatrix} 10 & 4 \\ 4 & 10 \end{vmatrix} = 84$$

$$(X^TX)^{-1} = \underset{\neq 0}{\frac{1}{200}} \cdot \begin{bmatrix} 116 & -64 & -88 \\ -64 & 56 & +52 \\ -88 & +52 & 84 \end{bmatrix}^T =$$

$$= \begin{bmatrix} 0.58 & -0.32 & -0.44 \\ -0.32 & 0.28 & 0.26 \\ -0.44 & 0.26 & 0.42 \end{bmatrix}$$

$$b = (X^TX)^{-1} \cdot X^Ty =$$

$$= \begin{bmatrix} 0.58 & -0.32 & -0.44 \\ -0.32 & 0.28 & 0.26 \\ -0.44 & 0.26 & 0.42 \end{bmatrix} \cdot \begin{bmatrix} 24 \\ 40 \\ -2 \end{bmatrix} = \underline{\underline{\begin{array}{c} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \end{array}}} \begin{array}{l} b_1 \\ b_2 \\ b_3 \end{array}$$

Example 2

Properties on **Slide 20**. Proofs:

$$\underset{\text{criterion function}}{\underset{\min}{S(b_1, b_2, b_3)}} = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 =$$

$$= \sum_{i=1}^n (\underbrace{y_i - b_1 - b_2x_{2i} - b_3x_{3i}}_{\text{composite function}})^2$$

- $$\frac{\partial S}{\partial b_1} = -2\sum_{i=1}^n (\underbrace{y_i - b_1 - b_2x_{2i} - b_3x_{3i}}_{e_i}) = 0$$

$$-2\sum_{i=1}^n e_i = 0$$

$$\underline{\underline{\sum_{i=1}^n e_i = 0}} \quad \text{P4}$$

INTERCEPT

- $$\frac{\partial S}{\partial b_2} = -2\sum_{i=1}^{n}\underbrace{(y_i - b_1 - b_2 x_{2i} - b_3 x_{3i})}_{e_i} x_{2i} = 0$$

$$-2\sum_{i=1}^{n} e_i x_{2i} = 0$$
$$\sum_{i=1}^{n} e_i x_{2i} = 0$$

$$\frac{\partial S}{\partial b_3} = -2\sum_{i=1}^{n}\underbrace{(y_i - b_1 - b_2 x_{2i} - b_3 x_{3i})}_{e_i} x_{3i} = 0$$

$$-2\sum_{i=1}^{n} e_i x_{3i} = 0$$
$$\sum_{i=1}^{n} e_i x_{3i} = 0$$

Generalize: $\sum_{i=1}^{n} e_i x_{ji} = 0$ or $X^T e = 0$ P1 ALWAYS

- $$\sum_{i=1}^{n} \hat{y}_i \cdot e_i = \sum_{i=1}^{n} (b_1 + b_2 x_{2i} + b_3 x_{3i}) \cdot e_i =$$
$$= \sum_{i=1}^{n} b_1 e_i + \sum_{i=1}^{n} b_2 x_{2i} e_i + \sum_{i=1}^{n} b_3 x_{3i} e_i =$$
$$= b_1 \underbrace{\sum_{i=1}^{n} e_i}_{=0\ [P4]} + b_2 \underbrace{\sum_{i=1}^{n} x_{2i} e_i}_{=0\ [P1]} + b_3 \underbrace{\sum_{i=1}^{n} x_{3i} e_i}_{=0\ [P1]} = 0$$

or $\hat{y}^T e = 0$ P2 ALWAYS

- $$y_i = b_1 + b_2 x_{2i} + \ldots + b_k x_{ki} + e_i \quad / \Sigma_i$$

$$\Sigma_i y_i = n b_1 + b_2 \Sigma_i x_{2i} + \ldots + b_k \Sigma_i x_{ki} + \underbrace{\Sigma_i e_i}_{=0\ [P4]} \quad / : n$$

$$\underbrace{\frac{1}{n} \Sigma_i y_i} = b_1 + b_2 \underbrace{\frac{1}{n} \Sigma_i x_{2i}} + \ldots + b_k \underbrace{\frac{1}{n} \Sigma_i x_{ki}}$$

$$\bar{y} = b_1 + b_2 \cdot \bar{x}_2 + \ldots + b_k \cdot \bar{x}_k$$

$$\underline{\underline{\bar{y} = \bar{\hat{y}}}} \quad P3$$

INTERCEPT

Finally, we calculate **a** and **b**:

$$\Sigma_{i=1}^{10} e_i = 0 \qquad (P4)$$

$$0.5 + b = 0$$

$$\underline{\underline{b = -0.5}}$$

$$\Sigma_{i=1}^{10} \hat{y}_i \cdot e_i = 0 \qquad (P2)$$

$$-7.375 + 1 \cdot a = 0$$

$$\underline{\underline{a = 7.375}}$$

Tutorial, Part 3

Example 1

We continue E1 from Tutorial, Part 2.

Variance estimators:

- $s_e^2 = \frac{y^T y - y^T X b}{n-k} = \frac{\sum_{i=1}^{n} y_i^2 - b^T X^T y}{n-k}$

$$y^T X \cdot b = b^T \cdot X^T y = [2\ 3\ -1] \cdot \begin{bmatrix} 24 \\ 40 \\ -2 \end{bmatrix} = 170$$

$$s_e^2 = \frac{184 - 170}{10-3} = \underline{\underline{2}}$$

variance estimate of the disturbances

- $\text{var-cov}(b) = s_e^2 \cdot (X^T X)^{-1} =$

$$= 2 \cdot \begin{bmatrix} 0.58 & -0.32 & -0.44 \\ -0.32 & 0.28 & 0.26 \\ -0.44 & 0.26 & 0.42 \end{bmatrix} =$$

variance estimates

$$= \underline{\underline{\begin{bmatrix} 1.16 & -0.64 & -0.88 \\ -0.64 & 0.56 & 0.52 \\ -0.88 & 0.52 & 0.84 \end{bmatrix}}}$$

variance-covariance matrix estimate of the regression coefficient estimates

Hypothesis testing:

- $H_0: \beta_1 = 0$
  $H_1: \beta_1 \neq 0$
  two-tailed test

$$t_1 = \frac{b_1 - \beta_1}{\sqrt{var(b_1)}} = \frac{2-0}{\sqrt{1.16}} = \underline{\underline{1.86}}$$

$$t_c(m = \underbrace{n-k}_{10-3} = 7, \alpha/2 = 0.025) = 2.365$$

$|t_1| < t_c$, do not reject H_0 at $\alpha = 0.05$

- $H_0: \beta_2 = 0$
  $H_1: \beta_2 \neq 0$

$$t_2 = \frac{b_2 - \beta_2}{\sqrt{var(b_2)}} = \frac{3-0}{\sqrt{0.56}} = \underline{\underline{4.01}}$$

$$t_c(m = n-k = 7, \alpha/2 = 0.025) = 2.365$$

$|t_2| > t_c$, reject H_0 at $\alpha = 0.05$ and conclude from $\underline{H_1}$
What, in this case?

- $H_0: \beta_3 = 0$
  $H_1: \beta_3 \neq 0$

$$t_3 = \frac{b_3 - \beta_3}{\sqrt{var(b_3)}} = \frac{-1-0}{\sqrt{0.84}} = \underline{\underline{-1.05}}$$

$$t_c(m = n-k = 7, \alpha/2 = 0.025) = 2.365$$

$|t_3| < t_c$, do not reject H_0 at $\alpha = 0.05$

Measures of fit of the model:

- $s_e = \sqrt{s_e^2} = \sqrt{2} = \underline{\underline{1,414}}$ std. error of regression
- $KV = \frac{s_e}{\bar{y}} = \frac{1.414}{2.4} = \underline{\underline{0.589}}$ estimate of the coef. of variation
- $\overset{\text{total}}{TSS} = \overset{\text{explained}}{ESS} + \overset{\text{residual}}{RSS}$

  $\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y}_i)^2 + \sum_{i=1}^{n}(\underbrace{y_i - \hat{y}_i}_{e_i})^2$

  $RSS = s_e^2 \cdot (n-k) = 2 \cdot (10-3) = \underline{\underline{14}}$

  $ESS = b^T X^T y - n\bar{y}^2 = 170 - 10 \cdot 2.4^2 = \underline{\underline{112.4}}$

  $TSS = ESS + RSS = \underline{\underline{126.4}}$

- $R^2 = \frac{ESS}{TSS} = \frac{112.4}{126.4} = \underline{\underline{0.889}}$ estimate of the multiple determination coeff.

  $\bar{R}^2 = R^2_{adj} = 1 - (1-R^2) \cdot \frac{n-1}{n-k} =$
  $= 1 - (1 - 0.889) \cdot \frac{9}{7} = \underline{\underline{0.857}}$

  adjusted R^2

- $H_0: \beta_j = 0, \; \forall j = 2,3$
  $H_1: \beta_j \neq 0, \; \exists j = 2,3$

  $F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} = \frac{0,889/2}{(1-0.889)/7} = \underline{\underline{28.1}}$

  $F_c(m_1 = k-1 = 2, m_2 = n-k = 7, \alpha = 0.05) = 4.74$

  $F > F_c$, reject H_0 at $\alpha = 0.05$ and conclude from $\underline{H_1}$

  What, in this case?

Elasticity estimates:

- $\hat{\varepsilon}_{\bar{Y},\bar{X}_2} = b_2 \cdot \frac{\overline{X_2}}{\bar{Y}} = 3 \cdot \frac{0.4}{2.4} = \underline{\underline{0.5}}$
- $\hat{\varepsilon}_{\bar{Y},\bar{X}_3} = b_3 \cdot \frac{\overline{X_3}}{\bar{Y}} = -1 \cdot \frac{0.8}{2.4} = \underline{\underline{-0.33}}$

Representation of the model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \qquad \text{(PRM)}$$

$$\hat{Y}_i = \overset{b_1}{2} + \overset{b_2}{3} X_{2i} \overset{b_3}{-1} X_3 \qquad \text{(SRM)}$$

t_j: (1.86) (4.01) (−1.05) [3 possibilities]

$n = 10$ $\quad R^2 = 0.889$ $\quad \bar{R}^2 = 0.857$

$s_e = 1.414$ $\quad F = 28.1$