Example: The file `consumption2.dta` contains data for the USA in the period 1960–1990, namely the values for household consumption (*C*) and disposable income (*Y*). Both variables are expressed in billion USD at prices from the year 1987.

a) Write down and explain the Koyck assumption and with its help derive an autoregressive model from the infinite distributed-lag model.
b) Estimate the autoregressive model from the previous point and, based on the estimates, write down the estimated distributed-lag model. Based on your calculations, explain the impact and long-term multiplier, as well as one intermediate and cumulative multiplier.
c) Check with an appropriate test whether there is first-order autocorrelation present in the analyzed autoregressive model.
d) Write down and explain the partial adjustment model. Estimate the derived autoregressive model and explain the results.
e) Write down and explain the adaptive expectations model. Estimate the derived autoregressive model and explain the results.

Koyck model (geometric scheme of distributed lags)

When estimating the infinite distributed-lag model, we can help ourselves with a variety of assumptions about the distribution of the effects of the explanatory variable on the dependent variable. Koyck assumed that all the effects of the lagged explanatory variable have the same sign and are in time decreasing in a geometric sequence. This means that the values of the regression coefficients of the lagged explanatory variable are geometrically decreasing by moving into the past (i.e. by increasing the lag length). This assumption can be written as follows:

$$\beta_k = \beta_0 \lambda^k; \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where the parameter λ, also referred to as the rate of decline (decay), occupies the values between 0 and 1. The values of the regression coefficients for lags are dependent on the regression coefficient β_0, which represents the effect of the explanatory variable on the dependent variable contemporaneously, and the rate of decline λ. The closer the value of λ is to one (zero), the slower (faster) is the decline of the effects of a lagged explanatory variable on the dependent variable.

Let us now write down the sum of all regression coefficients β_k:

$$\sum_{k=0}^{\infty} \beta_k = \sum_{k=0}^{\infty} \beta_0 \lambda^k = \beta_0 \left(1 + \lambda + \lambda^2 + \lambda^3 + \ldots\right). \tag{2}$$

The sum of the infinite geometric series (the expression in parenthesis) equals $1/(1 - \lambda)$, so that the sum of the regression coefficients, which represents the long-term multiplier in the infinite distributed-lag model, can be calculated by the following formula:

$$\sum_{k=0}^{\infty} \beta_k = \beta_0 \frac{1}{1-\lambda}. \tag{3}$$

Considering expression (1), we can write the infinite distributed-lag model as follows:

$$y_t = \alpha + \beta_0 x_t + \beta_0 \lambda x_{t-1} + \beta_0 \lambda^2 x_{t-2} + \ldots + u_t. \tag{4}$$

Estimating parameters of such a model with the least squares method is of course not possible, because the model is not linear in parameters, and additionally, infinitely many parameters would have to be estimated. Therefore, Koyck proposed a solution, the transformation of the model, which we present below.

We first lag model (4) for one time unit:

$$y_{t-1} = \alpha + \beta_0 x_{t-1} + \beta_0 \lambda x_{t-2} + \beta_0 \lambda^2 x_{t-3} + \ldots + u_{t-1}. \tag{5}$$

Then we multiply expression (5) with the rate of decline λ and subtract the obtained result from expression (4):

$$\begin{aligned} \lambda y_{t-1} &= \lambda \alpha + \beta_0 \lambda x_{t-1} + \beta_0 \lambda^2 x_{t-2} + \beta_0 \lambda^3 x_{t-3} + \ldots + \lambda u_{t-1}, \\ y_t - \lambda y_{t-1} &= \alpha(1-\lambda) + \beta_0 x_t + (u_t - \lambda u_{t-1}), \end{aligned} \tag{6}$$

or, by some rearrangement:

$$y_t = \alpha(1-\lambda) + \beta_0 x_t + \lambda y_{t-1} + (u_t - \lambda u_{t-1}). \tag{7}$$

With the help of the Koyck transformation we changed the infinite distributed-lag model into an autoregressive model, where the lagged dependent variable acts as an explanatory variable.

Expression (7) can be generalised as:

$$y_t = \gamma_1 + \gamma_2 x_t + \gamma_3 y_{t-1} + v_t, \tag{8}$$

wherein the following holds:

$$\gamma_1 = \alpha(1-\lambda); \qquad \gamma_2 = \beta_0; \qquad \gamma_3 = \lambda; \qquad v_t = u_t - \lambda u_{t-1}. \tag{9}$$

We call the regression coefficient β_0 the short-term or impact multiplier, which tells us how much on average the dependent variable in the current unit of time changes, if the explanatory variable in the same unit of time increases by one unit of measurement. The expression $\beta_0 \frac{1}{1-\lambda}$ represents the long-term multiplier and tells us by how many units on average the dependent variable changes in the long run, due to the change of the explanatory variable by one unit of measurement.

If we compare the infinite distributed-lags model in general and the model (7), we can see two key simplifications, which occur due to the Koyck transformation:

– in the model (7) only three parameters need to be estimated (α, β in λ), while in the infinite distributed-lag model there are infinitely many regression coefficients,

– furthermore, the problem of multicollinearity is gone, because we replaced variables x_t x_{t-1} x_{t-2} … with the lagged dependent variable.

In addition to these simplifications we need to take into account the following circumstances when estimating the autoregressive model derived based on the Koyck transformation:

– Given that the variable y_t and consequently the variable y_{t-1} are random variables, a random explanatory variable appears in model (7). As we know from the theory, one of the assumptions of the classical linear regression model requires the explanatory variable to be non-random, or at least independent of the random variable. Therefore, in the case of the autoregressive model it is necessary to determine whether that condition is met.

– In the autoregressive model the random variable v_t appears, defined as $u_t - \lambda u_{t-1}$, which means that its characteristics depend on the assumptions that hold for the random variable u_t. It turns out that the values of the random variable v_t are autocorellated, although the values of the initial random variable u_t are uncorrelated. This means that we have to pay particular attention to the problem of autocorrelation when estimating the autoregressive model.

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Partial adjustment (stock adjustment) model

The autoregressive model obtained based on the Koyck transformation is a result of purely algebraic derivations, which means that it is not founded based on economic theory. We can eliminate this disadvantage by taking into account the so-called partial adjustment. We proceed from a model in which the desired (equilibrium) or long-term level of the dependent variable is the linear function of an explanatory variable. In this case we get:

$$C_t^* = \beta_0 + \beta_1 Y_t + u_t . \qquad (10)$$

Long-term or equilibrium level of personal consumption, e.g., is therefore a linear function of disposable income. Since long-term values of personal consumption C^* are not directly known, we use the partial adjustment model:

$$C_t - C_{t-1} = \delta(C_t^* - C_{t-1}), \qquad 0 < \delta \le 1. \qquad (11)$$

The change in consumption in time unit *t* equals a share δ (called the adjustment coefficient) of the desired change in consumption during this time unit. If we insert expression (11) into expression (10), we get:

$$C_t - C_{t-1} = \delta(\beta_0 + \beta_1 Y_t + u - C_{t-1})\,; \tag{12}$$

which we can properly rearrange:

$$C_t = \delta\beta_0 + \delta\beta_1 Y_t + (1-\delta)C_{t-1} + \delta u_t\,. \tag{13}$$

In a general form we can write expression (13) as:

$$y_t = \gamma_1 + \gamma_2 x_t + \gamma_3 y_{t-1} + v_t\,, \tag{14}$$

wherein the following holds:

$$\gamma_1 = \delta\beta_0; \qquad \gamma_2 = \delta\beta_1; \qquad \gamma_3 = (1-\delta); \qquad v_t = \delta u_t\,. \tag{15}$$

Model (10) represents the long-term equlibrium consumption function, while the autoregressive model (14) represents the short-term consumption function. The regression coeffcicient β_1 represents the long-term marginal propensity to consume, whereas γ_2 represents the short-term marginal propensity to consume. From the expression (15) it follows that β_1 can be calculated by dividing the short-term marginal propensity to consume with the adjustment coefficient δ. The coefficient tells us what share of the gap between the desired and the actual consumption is eliminated in each time unit.

Adaptive expectations model (error learning model)

When using the adaptive expectations model we will stem from the model in which the dependent variable is a linear function of the expected long-term (equilibrium) level of the explanatory variable. In this case we could write:

$$C_t = \beta_0 + \beta_1 Y_t^* + u_t\,. \tag{16}$$

Personal consumption, e.g., is therefore a linear function of the expected long-term or equilibrium level of disposable income. As expected disposable income Y^* is not directly observed, we use the adaptive expectations model:

$$Y_t^* - Y_{t-1}^* = \gamma(Y_t - Y_{t-1}^*), \qquad 0 < \gamma \le 1. \tag{17}$$

Expectations regarding disposable income are in each unit of time “corrected” by a certain proportion (called the expectation coefficient) of the gap between the actual value of disposable income in the current time unit and its previous expected value.

We can write down the model (17) as:

$$Y_t^* = \gamma Y_t + (1-\gamma)Y_{t-1}^*, \tag{18}$$

which means that the expected value of disposable income in a given unit of time equals the weighted arithmetic average of the actual value of disposable income in this time unit and its expected value in the previous time unit, wherein γ and $1-\gamma$ are the respective weights.

In the event that γ equals 1, $Y_t^* = Y_t$ holds, which means that the expectations regarding the disposable income are realized promptly and completely. If we have the other extreme that $\gamma = 0$, $Y_t^* = Y_{t-1}^*$ holds, which means that expectations are static and will not change.

If we insert expression (18) into equation (16), we get:

$$C_t = \beta_0 + \beta_1(\gamma Y_t + (1-\gamma)Y_{t-1}^*) + u_t = \beta_0 + \beta_1\gamma Y_t + \beta_1(1-\gamma)Y_{t-1}^* + u_t. \tag{19}$$

We write model (16) in the previous time unit:

$$C_{t-1} = \beta_0 + \beta_1 Y_{t-1}^* + u_{t-1} \tag{20}$$

and express the expected value of disposable income Y_{t-1}^*:

$$Y_{t-1}^* = \frac{C_{t-1} - \beta_0 + u_{t-1}}{\beta_1}. \tag{21}$$

We insert it into equation (19) and with some derivation we get:

$$C_t = \gamma\beta_0 + \gamma\beta_1 Y_t + (1-\gamma)C_{t-1} + u_t - (1-\gamma)u_{t-1}. \tag{22}$$

In a general form we can write expression (22) as:

$$y_t = \xi_1 + \xi_2 x_t + \xi_3 y_{t-1} + v_t, \tag{23}$$

wherein the following holds:

$$\xi_1 = \gamma\beta_0; \qquad \xi_2 = \gamma\beta_1; \qquad \xi_3 = (1-\gamma); \qquad \xi_t = u_t - (1-\gamma)u_{t-1}. \tag{24}$$

Based on the theory of consumption we define the expected long-term disposable income as the permanent disposable income. Accordingly, the regression coefficient β_1 in model (16) represents the marginal propensity to consume out of disposable income, while the regression coefficient ξ_2 in the autoregressive model (23) represents the marginal propensity to consume out of current income. From expression (24) it follows that β_1 can be calculated by dividing the marginal propensity to consume out of current income by the expectation coefficient γ.

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Computer printout of the results in Stata:

```
. tsset year
       time variable:  year, 1960 to 1990
               delta:  1 unit

. regress c y

      Source |       SS       df       MS              Number of obs =      31
-------------+------------------------------           F(  1,    29) = 8830.35
       Model |  10704439.9     1  10704439.9           Prob > F      =  0.0000
    Residual |  35154.7449    29  1212.23258           R-squared     =  0.9967
-------------+------------------------------           Adj R-squared =  0.9966
       Total |  10739594.7    30  357986.489           Root MSE      =  34.817

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |   .9224173   .0098161    93.97   0.000     .9023412    .9424935
       _cons |  -39.55193   24.45265    -1.62   0.117    -89.56321    10.45935
------------------------------------------------------------------------------

. regress c y l.c

      Source |       SS       df       MS              Number of obs =      30
-------------+------------------------------           F(  2,    27) = 5761.50
       Model |  9871307.09     2  4935653.55           Prob > F      =  0.0000
    Residual |  23129.8458    27  856.660955           R-squared     =  0.9977
-------------+------------------------------           Adj R-squared =  0.9975
       Total |  9894436.94    29  341187.481           Root MSE      =  29.269

------------------------------------------------------------------------------
           c |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |   .4555839   .1359322     3.35   0.002      .176674    .7344938
             |
           c |
         L1. |   .5210732   .1502645     3.47   0.002     .2127558    .8293905
             |
       _cons |  -17.89849   23.59184    -0.76   0.455    -66.30495    30.50797
------------------------------------------------------------------------------

. predict ec, resid
(1 missing value generated)

. regress ec l.ec, nocons

      Source |       SS       df       MS              Number of obs =      29
-------------+------------------------------           F(  1,    28) =   12.12
       Model |  6969.43374     1  6969.43374           Prob > F      =  0.0017
    Residual |  16098.1117    28  574.932559           R-squared     =  0.3021
-------------+------------------------------           Adj R-squared =  0.2772
       Total |  23067.5454    29    795.4326           Root MSE      =  23.978

------------------------------------------------------------------------------
          ec |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          ec |
         L1. |   .5491917    .157737     3.48   0.002      .226082    .8723014
------------------------------------------------------------------------------

. scalar rho=_b[l.ec]
. display rho
.54919173
```

```
. regress ec y l.c l.ec

      Source |       SS       df       MS              Number of obs =      29
-------------+------------------------------           F(  3,    25) =    4.17
       Model |  7694.22168     3  2564.74056           Prob > F      =  0.0159
    Residual |  15371.1754    25  614.847017           R-squared     =  0.3336
-------------+------------------------------           Adj R-squared =  0.2536
       Total |  23065.3971    28  823.764182           Root MSE      =  24.796

------------------------------------------------------------------------------
          ec |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |   .1350957   .1243967     1.09   0.288     -.121104    .3912955
             |
           c |
         L1. |  -.1487069   .1369076    -1.09   0.288    -.4306733    .1332596
             |
          ec |
         L1. |   .6015586   .1701725     3.53   0.002     .2510817    .9520355
             |
       _cons |  -11.54498    21.9328    -0.53   0.603    -56.71643    33.62646
------------------------------------------------------------------------------

. scalar lm=e(N)*e(r2)

. display lm, invchi2tail(1, 0.05), chi2tail(1, lm)
9.6739036 3.8414588 .00186904

. qui regress c y l.c
. estat bgodfrey, nomiss0

Breusch-Godfrey LM test for autocorrelation
---------------------------------------------------------------------------
    lags(p)  |          chi2               df                 Prob > chi2
-------------+-------------------------------------------------------------
       1     |          9.674               1                   0.0019
---------------------------------------------------------------------------
                        H0: no serial correlation

. estat bgodfrey

Breusch-Godfrey LM test for autocorrelation
---------------------------------------------------------------------------
    lags(p)  |          chi2               df                 Prob > chi2
-------------+-------------------------------------------------------------
       1     |          9.797               1                   0.0017
---------------------------------------------------------------------------
                        H0: no serial correlation
```

■