University of Ljubljana

SEB SCHOOL OF ECONOMICS AND BUSINESS

6. Distributed–lag Models

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Basic definitions

- This topic concerns econometric analysis of **time series**.
- So far, the regression coefficients β_j only represented *contemporaneous changes* in y based on the increases in x_j.
- In practice, it often takes time for a change in an explanatory variable, Δx_j, to reflect on the dependent variable, Δy.
- This required time is usually more than one time unit (hour, day, week, month, year, etc.). This is especially the case with *higher-frequency data* (such as daily or hourly).

Basic definitions

There are several **groups of reasons** why these effects are not immediate:

- ❖ **Psychological** reasons, such as *consumer behaviour* (e.g. increased **income** results in increased **consumption**, but the effects are distributed over time based on habits and financial limitations of individuals);
- ❖ **Technological** reasons, such as *technological processes* (e.g. increased **investment** results in enhanced **production**, but this happens only gradually due to technological limitations, such as installing equipment and training);
- ❖ **Institutional** reasons, such as *contracts* and *deadlines* (e.g. increased **interest rate** results in increased **deposits**, but this happens only gradually due to fixed deposits that cannot be converted without cost).

Basic definitions

- When these effects are *not* immediate (contemporaneous), the usual static analysis ($x_{jt} \rightarrow y_t$) is *not* appropriate.
- We need to introduce **dynamic analysis** into our modelling framework, i.e. we need to take into account **time lags**.
- These are lagged values of **either** the explanatory variables x_j **or** of the dependent variable y as an explanatory variable (capturing the dynamic nature of all relevant x_j).
- Regression coefficients of those lagged variables capture the adjustment process of y with respect to x_j in time.
- This results in a *class of models* called in the literature the **distributed–lag models**.

Basic definitions

We distinguish between:

- **Finite (lag) distributed–lag models**, where the effect of Δx_{jt} on y_t completes after a certain finite number of time units.

  Also: *models of lagged explanatory variables*.

- **Infinite (lag) distributed–lag models**, where Δx_{jt} affects the current and all future values of y_t without time limitation.

  Also: *autoregression models*

  (analitical solutions of the model include among the explanatory variables also a lagged dependent variable, y_{t-1}).

Finite (lag) distributed–lag models

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \ldots + \beta_s x_{t-s} + u_t$$

Not clear in advance, how many lags to take into account.

Adding lags of an explanatory variable decreases degrees of freedom and increases multicollinearity.

We need to pay attention to changes in sign and large changes in values of regression coefficients.

Infinite (lag) distributed–lag models

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \ldots + u_t$$

t	β_0	Short-run (impact) multiplier
$t+1$	β_1	Intermediate (interim) multiplier
$t+2$	β_2	Intermediate (interim) multiplier
$t+3$	β_3	Intermediate (interim) multiplier
$\vdots$	$\vdots$	$\vdots$
Total	$\sum_{k=0}^{\infty} \beta_k$	Long-run (total) multiplier

Note: Subtotals $\beta_0 + \beta_1$, $\beta_0 + \beta_1 + \beta_2$, $\beta_0 + \beta_1 + \beta_2 + \beta_3$ etc. represent the cumulative multipliers for time periods $t+1$, $t+2$, $t+3$ etc.

Infinite (lag) distributed–lag models

- There is an obvious issue with this kind of models.
- Namely, one cannot estimate an *infinite* number of parameters (regression coefficients) based on a *finite* sample of (time series) data.
- For an *analytical solution* of the model, we have to assume a certain distribution of effects of x_j on y.
- This is usually done in literature in *three ways*, resulting in **three** infinite (lag) distributed–lag models:
  - Koyck model;
  - Partial adjustment model;
  - Adaptive expectations model.

Koyck model

A Geometric scheme of distributed lags

Koyck assumption:

- The effects of all x_{t-k} have the same sign;
- The effects decrease in time at a geometric sequence.

$$\beta_k = \beta_0 \lambda^k ; \qquad k = 0, 1, 2, \dots ; \qquad 0 < \lambda < 1 .$$

λ – rate of decline (decay)

$(1 - \lambda)$ – speed of adjustment

- $\lambda = 0$: no lagged effects over time;
- $\lambda = 1$: no decrease of effects over time.

Koyck model

Long-run (total) effect:

$$\sum_{k=0}^{\infty} \beta_k = \sum_{k=0}^{\infty} \beta_0 \lambda^k = \beta_0 \left(1 + \lambda + \lambda^2 + \lambda^3 + \ldots\right) = \beta_0 \frac{1}{1-\lambda}$$

$$\boxed{s = \frac{a_1}{1-q}}$$

sum of an infinite geometric series

Koyck model

Koyck transformation of the model:

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \ldots + u_t$$

$$\Big\downarrow \; \beta_k = \beta_0 \lambda^k$$

$$\left.\begin{aligned} y_t &= \alpha + \beta_0 x_t + \beta_0 \lambda x_{t-1} + \beta_0 \lambda^2 x_{t-2} + \beta_0 \lambda^3 x_{t-3} + \ldots + u_t \quad / \cdot\lambda \;\&\; \mathrm{L}.[\cdot] \\ \lambda y_{t-1} &= \lambda\alpha + \beta_0 \lambda x_{t-1} + \beta_0 \lambda^2 x_{t-2} + \beta_0 \lambda^3 x_{t-3} + \ldots + \lambda u_{t-1} \end{aligned}\right) -$$

$$y_t - \lambda y_{t-1} = \alpha(1-\lambda) + \beta_0 x_t + (u_t - \lambda u_{t-1})$$

Koyck model

$$y_t = \alpha(1-\lambda) + \beta_0 x_t + \lambda y_{t-1} + (u_t - \lambda u_{t-1})$$

$$y_t = \gamma_1 + \gamma_2 x_t + \gamma_3 y_{t-1} + v_t$$

Autoregression model!

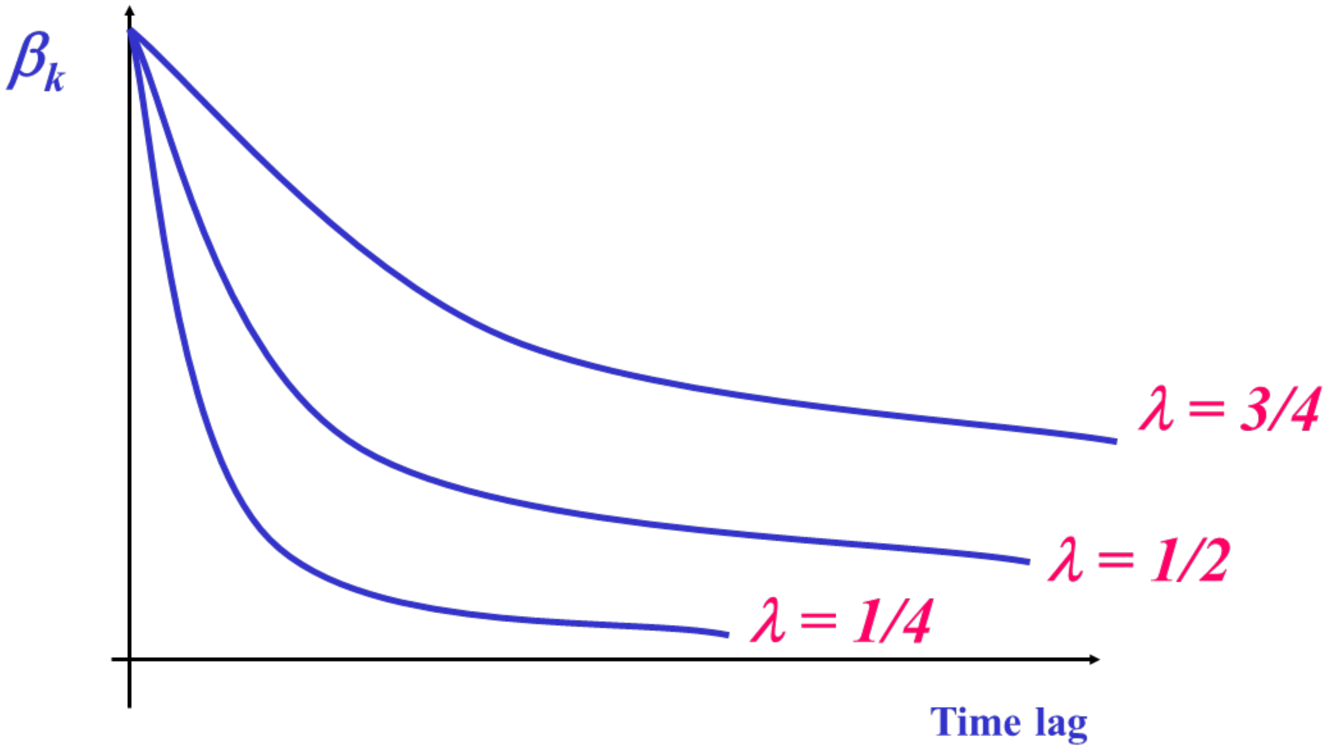
Koyck model

Median lag:

$$-\frac{\log 2}{\log \lambda} \Longleftrightarrow \frac{\ln 0.5}{\ln \lambda}$$

Median lag is the average necessary time for the realization of half of the long-run effect of x_j on y.

Koyck model



Partial adjustment model

B Partial adjustment model (Stock adjustment model)

We model long-run (equilibrium) level of y:

$$y_t^* = \beta_0 + \beta_1 x_t + u_t$$

It is not directly observed, but we can assume a gradual adjustment process between the the short-run level y_t and the long-run level y_t^*:

$$\underbrace{y_t - y_{t-1}}_{\text{Actual change}} = \delta(\underbrace{y_t^* - y_{t-1}}_{\text{Desired change}})\ ; \quad 0 < \delta < 1.$$

Partial adjustment model

δ – coefficient of adjustment

- $\delta = 0$: no adjustment over time;
- $\delta = 1$: immediate full adjustment.

Transformation of the model starts by rewriting the adjustment process:

$$y_t = \delta y_t^* - \delta y_{t-1} + y_{t-1}$$

$$y_t = \delta y_t^* + (1-\delta) y_{t-1}$$

Partial adjustment model

Then we plug in $y_t^* = \beta_0 + \beta_1 x_t + u_t$:

$$y_t = \delta(\beta_0 + \beta_1 x_t + u_t) + (1-\delta) y_{t-1}$$

$$y_t = \delta\beta_0 + \delta\beta_1 x_t + (1-\delta) y_{t-1} + \delta u_t$$

$$y_t = \gamma_1 + \gamma_2 x_t + \gamma_3 y_{t-1} + v_t$$

Autoregression model!

Adaptive expectations model

C Adaptive expectations model (Error learning model)

Now, the explanatory variable takes the expected long-run or equilibrium level x_t^*:

$$y_t = \beta_0 + \beta_1 x_t^* + u_t$$

x_t^* is not directly observed, but we can assume a gradual adjustment process of expectations about it:

$$\underbrace{x_t^* - x_{t-1}^*}_{\text{Current expectations – Past expectations}} = \gamma(x_t - x_{t-1}^*); \quad 0 < \gamma < 1.$$

Adaptive expectations model

γ – coefficient of expectation

- $\gamma = 0$: expectations are static;
- $\gamma = 1$: expectations realize immediately in full.

The difference between current and past expectations is thus a share of the gap between past expectations and its realization.

Transformation of the model starts by rearranging the adjustment process of expectations:

$$x_t^* = \gamma x_t + (1-\gamma) x_{t-1}^*$$

Adaptive expectations model

Now we plug current expectations $x_t^* = \gamma x_t + (1-\gamma)x_{t-1}^*$ into our initial model:

$$y_t = \beta_0 + \beta_1 x_t^* + u_t$$

$$y_t = \beta_0 + \beta_1 \gamma x_t + \beta_1 (1-\gamma) x_{t-1}^* + u_t$$

x_{t-1}^* is still not observed, so we derive it from lagged initial model:

$$y_{t-1} = \beta_0 + \beta_1 x_{t-1}^* + u_{t-1} \quad \Rightarrow \quad x_{t-1}^* = \frac{y_{t-1} - \beta_0 - u_{t-1}}{\beta_1}$$

Adaptive expectations model

Finally, we plug the expression just derived for x^*_{t-1} into our previous expression for y_t:

$$y_t = \beta_0 + \beta_1 \gamma x_t + \beta_1 (1-\gamma) \frac{y_{t-1} - \beta_0 - u_{t-1}}{\beta_1} + u_t$$

$$y_t = \beta_0 + \beta_1 \gamma x_t + (1-\gamma) y_{t-1} - \beta_0 (1-\gamma) - (1-\gamma) u_{t-1} + u_t$$

$$y_t = \beta_0 \gamma + \beta_1 \gamma x_t + (1-\gamma) y_{t-1} + \left[u_t - (1-\gamma) u_{t-1} \right]$$

$$y_t = \xi_1 + \xi_2 x_t + \xi_3 y_{t-1} + v_t$$

Greek letter Xi

Autoregression model!

Infinite (lag) distributed–lag models

WARNINGS

! Technically, in all three models the same function is estimated (based on sample data, the same values of regression coefficients are obtained).

! Theoretical assumptions of the three models are completely different. The regression coefficients are interpreted based on the assumptions of the estimated model.

! It is highly likely that the models exhibit (at least) first-order autocorrelation (proper tests should be applied).

! By introducing a lagged dependent variable as a regressor we include a stochastic regressor – endogeneity (instrumental variables estimator should be applied).

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