2020春季学期“数理逻辑”课程作业四参考答案

1. 证明以下公式永真:

(1) $\forall x A \leftrightarrow A[t/x]$,

证: $\forall x A \leftrightarrow A[t/x]$永真

$\Leftrightarrow$ 对于任意模型(M,σ), $M \vDash_\sigma \forall x A \leftrightarrow A[t/x]$

$\Leftrightarrow$ 对于任意模型(M,σ), $(\forall x A \leftrightarrow A[t/x])_{M[\sigma]} = T$

$\Leftrightarrow$ 对于任意模型(M,σ), $B_{\leftrightarrow}(\forall x A_{M[\sigma]}, A[t/x]_{M[\sigma]}) = T$

$\Leftrightarrow$ 对于任意模型(M,σ), $B_{\leftrightarrow}(A_{M[\sigma[x:=a]]}, A[t/x]_{M[\sigma]}) = T$, 对于任何$a \in M$ (i)

$A_{M[\sigma[x:=a]]} = T$

$\Rightarrow A[t/x]_{M[\sigma]} = A_M[\sigma[x := a]] = T$, $t = a$ (ii)

$A_{M[\sigma[x:=a]]} = F$

$\Rightarrow A[t/x]_{M[\sigma]} = A_M[\sigma[x := a]] = F$, $t = a$ (iii)

由ii和iii易见i成立

$\therefore$原命题得证　　　　　　$Q.E.D.$

(2) $A[t/x] \to \exists x A$.

证: $A[t/x] \to \exists x A$永真

$\Leftrightarrow$ 对于任意模型(M,σ), $M \vDash_\sigma A[t/x] \to \exists x A$

$\Leftrightarrow$ 对于任意模型(M,σ), $(A[t/x] \to \exists x A)_{M[\sigma]} = T$

$\Leftrightarrow$ 对于任意模型(M,σ), $B_{\to}(A[t/x]_{M[\sigma]}, \exists x A_{M[\sigma]}) = T$ (i)

$A[t/x]_{M[\sigma]} = T$

$\Rightarrow A_{M[\sigma[x:=t_{M[\sigma]}]]} = T$

$\Rightarrow A_{M[\sigma[x:=a]]} = T$, 对于某个$a \in M$

$\Rightarrow \exists x A_{M[\sigma]} = T$

$\therefore$ i成立

$\therefore$原命题得证　　　　　　$Q.E.D.$

2. 在G系统中证明下列序贯:

(1) $\vdash \forall x(P(x) \to Q(x)) \ \to \ (\exists x P(x) \to \exists x Q(x))$,

证：

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(P(x) \to Q(x)), P(y) \ \vdash \ \exists x Q(x), P(y)}{\forall x(P(x) \to Q(x)), P(y) \ \vdash \ \exists x Q(x)}\exists R}{\forall x(P(x) \to Q(x)), \exists x P(x) \ \vdash \ \exists x Q(x)}\exists L}{\forall x(P(x) \to Q(x)) \ \vdash \ \exists x P(x) \to \exists x Q(x)} \to R}{\vdash \forall x(P(x) \to Q(x)) \ \to \ (\exists x P(x) \to \exists x Q(x))} \to R$$

(2) $\vdash \exists x(P(x) \to Q(x)) \ \to \ (\forall x P(x) \to \exists x Q(x))$.

证：

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists x(P(x) \to Q(x)), \forall x P(x), P(t) \ \vdash \ \exists x Q(x), P(t)}{\exists x(P(x) \to Q(x)), \forall x P(x), P(t) \ \vdash \ \exists x Q(x)}\exists R}{\exists x(P(x) \to Q(x)), \forall x P(x) \ \vdash \ \exists x Q(x)}\forall L}{\exists x(P(x) \to Q(x)) \ \vdash \ (\forall x P(x) \to \exists x Q(x))} \to R}{\vdash \exists x(P(x) \to Q(x)) \ \to \ (\forall x P(x) \to \exists x Q(x))} \to R$$

3. 在G系统中证明下列序贯:

(1) **Axiom 4** $\vdash \forall x A(x) \to A[a/x]$,

证：

$$\dfrac{\dfrac{\forall x A(x), A[t/x] \ \vdash \ A[t/x]}{\forall x A(x) \ \vdash \ A[t/x]}\forall L}{\vdash \forall x A(x) \to A[t/x]} \to R$$

(2) **Axiom 5** $\vdash \forall x(A \to B(x)) \to (A \to \forall x B(x))$,

证：

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(A \to B(x)), A \ \vdash \ A, B(y) \qquad \forall x(A \to B(x)), A, B(y) \ \vdash \ B(y)}{\forall x(A \to B(x)), A, A \to B(y) \ \vdash \ B(y)} \to L}{\dfrac{\forall x(A \to B(x)), A, (A \to B(x))[y/x] \ \vdash \ B(y)}{\forall x(A \to B(x)), A \ \vdash \ B(y)}\forall L}}{\forall x(A \to B(x)), A \ \vdash \ \forall x B(x)}\forall R}{\forall x(A \to B(x)) \ \vdash \ A \to \forall x B(x)} \to R}{\vdash \forall x(A \to B(x)) \to (A \to \forall x B(x))} \to R$$

4. 证明：$\vdash A(y) \ \rightarrow \ \forall x((x \doteq y) \rightarrow A(x))$

证：

$$\cfrac{\cfrac{\cfrac{A(y), z \doteq y \ \vdash \ A(z))}{A(y) \ \vdash \ (z \doteq y) \rightarrow A(z))} \rightarrow R}{A(y) \ \vdash \ \forall x((x \doteq y) \rightarrow A(x))} \forall R}{\vdash A(y) \ \rightarrow \ \forall x((x \doteq y) \rightarrow A(x))} \rightarrow R$$

下面证明$A(y), z \doteq y \ \vdash \ A(z))$有效

$A(y), z \doteq y \ \vdash \ A(z))$有效

$\Leftrightarrow \models (A(y) \wedge z \doteq y) \rightarrow A(z)$

$\Leftrightarrow$ 对于任意模型(M, σ), $M \vDash_\sigma (A(y) \wedge z \doteq y) \rightarrow A(z)$

$\Leftrightarrow$ 对于任意模型(M, σ), $((A(y) \wedge z \doteq y) \rightarrow A(z))_{M[\sigma]} = T$

$\Leftrightarrow$ 对于任意模型(M, σ), $B_{\rightarrow}((A(y) \wedge z \doteq y)_{M[\sigma]}, A(z)_{M[\sigma]}) = T \ (i)$

$(A(y) \wedge z \doteq y)_{M[\sigma]} = T$

$\Rightarrow A(y)_{M[\sigma]} = T$ and $(z \doteq y)_{M[\sigma]} = T$

$\Rightarrow A(z)_{M[\sigma]} = A(y)_{M[\sigma]} = T$

$\therefore (i)$成立

$\therefore$原命题得证 $\qquad Q.E.D.$