Politecnico di Milano
STATISTICS (079086), 2009/2010, Prof. A.Barchielli
Problem set n. 5

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Exercise 1 The lifetime X of an electronic component is exponentially distributed with mean θ. Let $X_1, \dots, X_n$ be a sample from X.

a) Indicate the distribution of the random variable $T = \frac{2}{\theta}\sum_{i=1}^n X_i$.

b) Using T, give the form of the bilateral confidence interval for θ.

c) Let $n = 10$, $\sum_{i=1}^{10} X_i = 3.7$. Calculate the 90% interval. [(0.2356, 0.6789)]

Exercise 2 Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{U}[0, \theta]$, with θ positive unknown parameter.

1. Find the Method of Moments estimator $\tilde{\theta}$ of θ; calculate its mean value and its variance [$\tilde{\theta} = 2\bar{X}$; $E[\tilde{\theta}] = \theta$; $Var(\tilde{\theta}) = \frac{1}{3n}\theta^2$].
2. Find the MLE estimator $\widehat{\theta}$ of θ; calculate its density, its mean value and its variance [$\widehat{\theta} = X_{[n]} = \max(X_1, \dots, X_n)$; $f_{X_{[n]}}(x) = \frac{n}{\theta^n}x^{n-1}\mathbf{1}_{[0,\theta]}(x)$; $E[\widehat{\theta}] = \frac{n}{n+1}\theta$; $Var(\widehat{\theta}) = \theta^2\frac{n}{(n+2)(n+1)^2}$].
3. Verify that $\widehat{\theta}$ is asymptotically unbiased for θ.
4. Starting by $\widehat{\theta}$, construct an unbiased estimator T of θ and calculate its MSE. [$T = \frac{n+1}{n}X_{[n]}$; $MSE_T[\theta] = Var(T) = \frac{1}{n(n+2)}\theta^2$]
5. Which estimator do you prefer, between T and $\tilde{\theta}$? [T, because it has smaller SME]

Exercise 3 Let $X_1, \dots, X_n$ be a random sample extracted from a population with a $\Gamma(1/\lambda, 2)$ density, which is given by

$$f(x, \lambda) = \frac{x}{\lambda^2}\mathrm{e}^{-\frac{x}{\lambda}}\mathbf{1}_{(0,\infty)}(x),$$

where $\lambda > 0$ is an unknown parameter.

1. Determine an estimator $\widehat{\lambda}$ of λ with the maximum likelihood method.
2. By using the table of the main distributions, find the expressions of the mean and the variance of $\widehat{\lambda}$.
3. Compute the expression of the characteristic $k(\lambda) = P_\lambda(X_1 > 2)$ and give an MLE $\widehat{k}$ for it.
4. By using the properties of the maximum likelihood estimators, we get that the asymptotic distribution of $\frac{\widehat{\lambda}-\lambda}{\lambda/\sqrt{2n}}$ is approximately standard normal for large n. Determine an asymptotic confidence interval of approximate level 95% for λ, if $n = 200$ and $\widehat{\lambda} = 2.0$.

Solution.

1. $$L_\lambda(x_1, \dots, x_n) = \prod_{j=1}^n f(x_j, \lambda) = \frac{\prod_{j=1}^n x_j}{\lambda^{2n}}\mathrm{e}^{-\sum_{j=1}^n x_j/\lambda}, \qquad \ln L_\lambda = -2n\ln\lambda + \sum_{j=1}^n \ln x_j - \frac{n\overline{x}}{\lambda},$$

$$\frac{\partial \ln L_\lambda}{\partial\lambda} = -\frac{2n}{\lambda} + \frac{n\overline{x}}{\lambda^2} = \frac{2n}{\lambda^2}\left(\frac{\overline{x}}{2} - \lambda\right) \geq 0 \text{ if and only if } \lambda \leq \frac{\overline{x}}{2} \qquad \Rightarrow \qquad \widehat{\lambda} = \frac{\overline{X}}{2}.$$

2. From the properties of the sample mean and from the tables we have $\mathbb{E}_\lambda\left[\overline{X}\right] = 2\lambda$, $\mathrm{Var}_\lambda\left[\overline{X}\right] = \frac{2\lambda^2}{n}$. Then, $\mathbb{E}_\lambda\left[\widehat{\lambda}\right] = \lambda$, $\mathrm{Var}_\lambda\left[\widehat{\lambda}\right] = \frac{\lambda^2}{2n}$.

3. The characteristic $k(\lambda)$ is given by

$$k(\lambda) = P_\lambda(X_1 > 2) = \int_2^\infty \frac{x}{\lambda^2}\,\mathrm{e}^{-\frac{x}{\lambda}}\,\mathrm{d}x = -\frac{x}{\lambda}\,\mathrm{e}^{-\frac{x}{\lambda}}\Big|_2^\infty - \int_2^\infty \left(-\frac{1}{\lambda}\,\mathrm{e}^{-\frac{x}{\lambda}}\right)\mathrm{d}x$$
$$= -\left(1 + \frac{x}{\lambda}\right)\mathrm{e}^{-\frac{x}{\lambda}}\Big|_2^\infty = \mathrm{e}^{-2/\lambda}\left(1 + \frac{2}{\lambda}\right).$$

By the invariance property, its MLE is

$$\widehat{k} = k(\widehat{\lambda}) = \mathrm{e}^{-2/\widehat{\lambda}}\left(1 + \frac{2}{\widehat{\lambda}}\right).$$

4. Starting from the approximate pivot quantity $\frac{\widehat{\lambda}-\lambda}{\widehat{\lambda}/\sqrt{2n}}$ we get $P_\lambda\left(\left|\frac{\widehat{\lambda}-\lambda}{\widehat{\lambda}/\sqrt{2n}}\right| \le z_{1-\alpha/2}\right) \simeq 1-\alpha$, which gives the approximate CI with extrema $\widehat{\lambda} \pm z_{1-\alpha/2}\,\frac{\widehat{\lambda}}{\sqrt{2n}}$. For us $\widehat{\lambda} = 2.0$, $\sqrt{2n} = 20$, $z_{.975} = 1.960$ and we get the numerical CI $(1.804, 2.196)$.