Due date: 16/12/2009 at 10.30

Return the homework personally to the teacher or send it in PDF format by e-mail from the official Politecnico account.

1. Given a set of nodes N and a set of undirected arcs A, let A' be the set of directed arcs obtained giving both directions to the arcs of A, that is if $\{i,j\} \in A$ then both (i,j) and (j,i) belong to A'. Let us denote by $\mathcal{F}$ a family of subset of nodes, such that N and $\emptyset$ do not belong to $\mathcal{F}$. Consider the following optimization problem using a set of variables $x_{ij} (i<j)$ for each undirected arc $\{i,j\} \in A$ and a set of variables y_{ij} for each directed arc $(i,j) \in A'$:

$$\begin{aligned}
&\min \textstyle\sum_{\{i,j\}\in A} c_{ij} x_{ij} && && \\
&\textstyle\sum_{\{i,j\}\in A: i\in S, j\notin S} x_{ij} \geq 1 && && \forall S \in \mathcal{F} \\
&y_{ij} + y_{ji} - k x_{ij} \leq 0 && && \forall (i,j) \in A \\
&\textstyle\sum_{(ji)\in BS(i)} y_{ji} - \sum_{(ij)\in FS(i)} y_{ij} = \begin{cases} -(n-1) & \text{if } i=1 \\ 1 & \text{otherwise} \end{cases} && && \forall i \in N \\
&x_{ij} \geq 0 && && \forall \{ij\} \in A \\
&y_{ij} \geq 0 && && \forall (ij) \in A'
\end{aligned}$$

where $FS(i)$ and $BS(i)$ denote the outgoing and incoming arcs in node i and k is a given constant value.

Write the dual.

2. Consider the following Linear Programming problem:

$$\begin{aligned}
&\max x_1 + 2x_2 && \\
(a)\quad &-2x_1 - x_2 && \leq -3 \\
(b)\quad &-x_1 + 3x_2 && \leq 24 \\
(c)\quad &x_1 - x_2 && \leq 6 \\
(d)\quad &2x_1 + x_2 && \leq 29 \\
&x_1, x_2 \geq 0 &&
\end{aligned}$$

Prove that solution $\bar{x} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$ is optimal.

Variant If we interpret the problem as a production mix, where x_1 and x_2 represent the number of produced items of type 1 and type 2, respectively, and $(a),(b),(c),(d)$ represent the constraints on resource availability, consider the introduction of three possible new products $3,4,5$. The profit and resource consumption for the new products is summarized in the following table:

	3	**4**	**5**
(a)	1	-2	-1
(b)	2	2	4
(c)	-1	-2	-4
(d)	-1	2	4
profit	1	2	2

Which product could improve the solution if introduced in the production? Why?

3. Consider a matrix A with m rows and n columns and a vector b with m elements, prove that either the polyhedron $Ax \leq b$ is not empty, or the system $\{yb < 0, yA = 0, y \geq 0\}$ has a solution.

4. A manufacturer produces 4 types of device in 4 factories. Factories have different specializations and they can produce only a subset of device types, namely: factory 1 can produce devices 1, 2, and 3, factory 2 can produce only devices 2 and 3, factory 3 can produce devices 1 and 4 and factory 4 can produce device 3 and 4. specialized, that it any factory can produce any device. The production capacity of each factory is 250, 180, 300 and 100 items per day independently from the type. The market asks for 200 devices of type 1, 150 of type 2, 350 of type 3 and 100 of type 4 per day. Formulate the problem of distribute the production among the factories so that the demand is maximally met, in terms of network flows.