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Calcolo Scientifico per l'Informatica
- Laboratory Class -
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Davide Fugazza
fugazza@elet.polimi.it

Outline

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1. Conditioning and Stability
2. Linear Algebra: Direct Methods
3. Homework

Exercise 1 (From [2])

Consider the problem of evaluating the function $f(q)$

$$f(q) = \frac{1}{q}\left[1 - (1-2q)^{-\frac{3}{2}}\right],$$

for small and positive values of q.

1. Why is it not possible to accurately evaluate $f(q)$ in floating-point arithmetic? Justify the answer.
2. Is $f(q)$'s calculation by means of a series expansion a good approach?

Exercise 2

Consider the following integral with $n \in N$:

$$I_n = \int_0^1 \frac{x^n}{x+5}\,dx.$$

1. Evaluate I_{50}, using the recursive relation

$$I_n = \frac{1}{n} - 5I_{n-1}$$

   Comment and justify the results.

2. Propose a different recursive relation to increase the achievable accuracy.

Hint: Note that:

- $I_n > 0 \quad , \quad I_{n+1} < I_n$
- $\lim_{n\to+\infty} I_n = 0 \quad ,$
  $I_n < \int_0^1 x^n\,dx = 1/n.$
- $I_0 = \int_0^1 \frac{1}{x+5}\,dx = ln(6/5).$

Try by inverting the recurrency
...
$\lim_{n\to+\infty} I_{n+1} \approx I_n$, thus
$I_n \approx 1/6n$
...

Exercise 3 (From [2])

Given the following succession:

$$\begin{cases} x_{n+1} = 2^n \left[\sqrt{1 + \dfrac{x_n}{2^{n-1}}} - 1 \right] \\ x_0 > -1 \end{cases}$$

for which $\lim_{n \to +\infty} x_n = log(1 + x_0)$.

1. Calculate x_1, x_2, $\cdots x_{71}$ and give reason of the obtained results.
2. Transform the succession in an equivalent sequence to make its implementation converge to the theoretical limit.

Hint: Multiplying and dividing for the same quantity...

Exercise 4 (From [2])

Given the following integral:

$$I = \int_0^1 \frac{x^\alpha}{x+8}\,dx \quad , \qquad \forall \alpha \geq 0,$$

1. Give an overestimation for $\alpha = 40$.
2. Calculate the integral through MATLAB's *Symbolic Tool*.
3. Calculate I by applying the Fundamental Theorem of Integral Calculus.
4. Calculate I by the following recursive formula $(n = 1, \cdots, \alpha)$:

$$\begin{cases} A_n = -8A_{n-1} + \dfrac{1}{n} \\ A_0 = log\left(\dfrac{9}{8}\right) \end{cases}$$

5. Write a stable recursive formula for the calculus of I.

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Introduction to Linear Systems

Consider a linear system $(n \times n)$ in its matricial form:

$$A\underline{x} = \underline{b}.$$

where $A \in \mathbb{R}^{n\times n}$ and $\underline{b} \in \mathbb{R}^n$. If A is non-singular, the solution is given by:

$$\underline{x} = A^{-1}\,\underline{b}.$$

Note: the system is a linear *linear transformation* $\underline{b} = A\underline{x}$, where $\underline{b}$ can be viewed as the result of the linear combination of the columns of the matrix A $(\underline{A}_i)$ by means of the coefficients x_i (the components of vector $\underline{x}$):

$$x_1\underline{A}_1 + x_2\underline{A}_2 + x_3\underline{A}_3 + \cdots + x_n\underline{A}_n = \underline{b}.$$

Linear Systems: Direct Methods

The solution is obtained in a finite number of operations:

- Cramer's Rule
- Gaussian Elimination
- LU decomposition
- Cholesky decomposition

In MATLAB avoid:

```
>> x = A^(-1)*b
```

Use instead the `mldivide` operator (`backslash`):

```
>> x = A\b
```

See `help mldivide`.

Gaussian Elimination

Exercise 5

Given the following systems:

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 2 & 10 & 4 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -1 & -3 & 0 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

1. Apply step-by-step the gaussian elimination algorithm (reduce A_i to triangular form and then back-substitution). Comment the results.
2. Verify the solutions in MATLAB.

Outline

Homework

Exercise H2.1 (From [2])

The succession:

$$1, \frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^n}, \dots$$

can be generated with the following recursive relations:

$$\begin{cases} p_n = \frac{10}{3} p_{n-1} - p_{n-2} \\ p_0 = 1 \\ p_1 = \frac{1}{3} \end{cases} \qquad \begin{cases} p_n = \frac{1}{3} p_{n-1} \\ p_0 = 1 \end{cases}$$

1. Implement the two relations to generate the earlier 100 terms of the series.
2. Study the stability of the two algorithms and justify the results obtained before.

Hint: Use a direct analysis of error propagation

Homework

Exercise H2.2 (From [5])

Consider the following system:

$$A = \left[\begin{array}{cc} \epsilon & 1 \\ 1 & 1 \end{array} \right] = \left[\begin{array}{c} x_1 \\ x_1 \end{array} \right] = \left[\begin{array}{c} 1 \\ 2 \end{array} \right]$$

The exact solution, for any $\epsilon \neq 1$, is:

$$x_1 = \frac{1}{1-\epsilon}, \quad x_2 = \frac{1-2\epsilon}{1-\epsilon}$$

1. For ϵ very small and positive, what happens when we solve the system using the standard versus the pivoting algorithms?

Sol: without pivoting you will obtain $x_2 \approx 1$, and $x_1 \approx 0 \cdots$

Homework

Exercise H2.3 (From [2])

Given the following matrix:

$$A = \begin{bmatrix} 40 & 8 & 3 & 4 & 5 \\ 0 & 6 & 1 & 1 & 1 \\ 0 & 0 & 13 & 3 & 2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 9 & 8 \end{bmatrix}$$

1. Construct the lower triangular matrix L_1, with all ones on the main diagonal, such that $L_1 A = U$, with U upper triangular.
2. Can the Cholesky decomposition be applied?

Hint: `help chol`.