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Calcolo Scientifico per l'Informatica
- Laboratory Class -

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Davide Fugazza
fugazza@elet.polimi.it

Outline

Outline

Exercise 1

Consider the following matrix:

$$A = \begin{bmatrix} \beta + \alpha & \alpha \\ \alpha & \beta + \alpha \end{bmatrix},$$

where α and β are real and positive numbers.

1. Calculate the condition number of A in infinity-norm and 2-norm. $\kappa(A)$ varies as a function of α/β: For which values it become very large (small)?
2. From the obtained results at the previous point would it be reasonable to conclude that for a linear system with a coefficient matrix symmetric and positive definite is always well-conditioned?

Exercise 2 (From [5])

Consider the following block matrix:

$$A_4 = \left[\begin{array}{c|c|c|c} B_4 & I_4 & 0 & 0 \\ \hline I_4 & B_4 & I_4 & 0 \\ \hline 0 & I_4 & B_4 & I_4 \\ \hline 0 & 0 & I_4 & B_4 \end{array}\right],$$

where

$$B_4 = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix}, \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Calculate the LU decomposition of A.
2. Create the sparsity plot for the obtained decomposition.

Hint: see `help diag` for the matrix construction and `help spy` for the sparsity plot.

Outline

Iterative Methods

For very large, *sparse* and *non-structured* matrices A, *direct methods* produce the so-called *fill-in* phenomenon. *Iterative methods* are instead the correct choice: A sequence of iterations is produced to approximate the solution

$$x^* = \lim_{k \to +\infty} x^{(k)},$$

which is not reached in a finite number of iteration steps as it happens for direct methods. A *tolerance* criterion will be applied: $\left\| x^{(k+1)} - x^{(k)} \right\| \leq \epsilon$.
Among them:

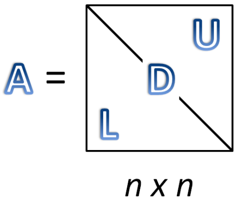
- Jacobi iteration method
- Gauss-Seidel iteration method
- Richardson iteration method
- Successive OverRelaxation method (SOR)
- Gradient methods

Jacobi iteration method

$$\underline{x}^{(k)} = B\underline{x}^{(k-1)} + \underline{g}$$

where

$$B = -D^{-1}(L+U)$$
$$\underline{g} = D^{-1}\underline{b}$$



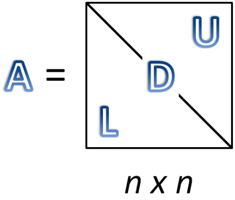

```
>> help jacob
   JACOB Calculates the solution of the linear system Ax=b by means
   of the Jacobi iteration method.
   [xv,iter]=jacob(A,b,x0,nmax,tol) calculates the solution of the
   linear system Ax=b with tolerance tol, starting from vector x0
   and with a number of iterations less than or equal to nmax.
   The function returns the vector xv, approximation of the solution
   and the number of the performed iterations.
```

Gauss-Seidel iteration method

$$\underline{x}^{(k)} = B\underline{x}^{(k-1)} + \underline{g}$$

where

$$B = -(D+L)^{-1}U\,,$$
$$\underline{g} = (D+L)^{-1}\underline{b}$$




```
>> help gausseid
   GAUSSEID Calculates the solution of the linear system Ax=b by means
   of the Gauss-Seidel iteration method.
   [xv,iter]=gausseid(A,b,x0,nmax,tol) calculates the solution of the
   linear system Ax=b with tolerance tol, starting from vector x0
   and with a number of iterations less than or equal to nmax.
   The function returns the vector xv, approximation of the solution
   and the number of the performed iterations.
```

Rapidity of convergence

- A sufficient condition for the convergence of the iteration method $\underline{x}^{(k)} = B\underline{x}^{(k-1)} + \underline{g}$ is that there exists a norm of B such that $\|B\| < 1$.
- Given the iteration method $\underline{x}^{(k)} = B\underline{x}^{(k-1)} + \underline{g}$, necessary and sufficient condition for its convergence is that $\rho(B) < 1$.
- If the matrix A of the linear system is strongly diagonal dominant the methods of Jacoby and Gauss-Seidel converge for all initial vector.
- If the matrix A of the linear system is symmetric and positive definite then the method of Gauss-Seidel converges for all initial vector.

Error estimation at the k-th iteration

$$\left\|\underline{x}^* - \underline{x}^{(k)}\right\| \leq \|B\|^k \left\|\underline{x}^* - \underline{x}^{(0)}\right\|$$

$$\left\|\underline{x}^* - \underline{x}^{(k)}\right\| \leq \frac{\|B\|^k}{1 - \|B\|} \left\|\underline{x}^{(1)} - \underline{x}^{(0)}\right\|$$

$$\left\|\underline{x}^* - \underline{x}^{(k)}\right\| \leq \frac{\|B\|}{1 - \|B\|} \left\|\underline{x}^{(k)} - \underline{x}^{(k-1)}\right\|$$

Richardson iteration method

Given the approximation $\underline{x}$ of the solution of the linear system $A\underline{x} = \underline{b}$ and the residual vector $\underline{r} = \underline{b} - A\underline{x}$, we can construct an iterative method for the residual correction of the form (stationary Richardson iterative method):

$$\underline{x}^{(k)} = \underline{x}^{(k-1)} + \alpha\, P^{-1}\, \underline{r}^{(k-1)}$$

where P is a suitable (non-singular) preconditioning matrix and $\alpha > 0$ is real positive number, known as the relaxation parameter for the given iteration method. The iteration matrix thus results:

$$B(\alpha) = I - \alpha\, P^{-1}\, A$$

```
>> help richpre
    RICHPRE Calculates the solution of the linear system Ax=b by means
    of the preconditioned Richardson iteration method.
    [xv,iter]=richpre(A,b,x0,nmax,tol,alpha,P) calculates the solution
    of the linear system Ax=b with tolerance tol, starting from vector
    x0 and with a number of iterations less than or equal to nmax. P
    is the precondition matrix and alpha is the weight.
```

Richardson iteration method

$$\underline{x}^{(k)} = \underline{x}^{(k-1)} + \alpha\, P^{-1}\, \underline{r}^{(k-1)}$$

$$B(\alpha) = I - \alpha\, P^{-1}\, A$$

The properties of convergence depend on α (and also on P).
Given the matrix A of the linear system which is *symmetric and positive definite*:

- The method converges for $0 < \alpha < 2/\lambda_{max}$. Where λ_{max} is the maximum eigenvalue of $P^{-1}A$.
- It exists a value of α that minimizes the spectral radius of $B(\alpha)$:

$$\alpha_{opt} = \frac{2}{\lambda_{max} + \lambda_{min}}.$$

Exercise 3 (From [2])

Consider the following linear system $Ax = b$:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

1. Apply the Jacobi method to calculate the solution with a tolerance of 10^{-5} and 10^{-8}. Does it converge?
2. Calculate the iteration matrix B_J for the Jacobi method. Evaluate the spectral radius $\rho(B_J)$ and $(B_J)^3$. Relying on the results, justify the answer at the previous point.
3. Does Gauss-Seidel method converge for the given system? Why?

Hint: Try also to calculate the eigenvalues of B_J...

Exercise 4 (From [2])

Given the linear system obtained with the following instructions:

```
>> A = diag(8*ones(8,1))+diag(2*ones(7,1),1)+diag(2*ones(7,1),-1);
>> b = A*ones(8,1);
```

1. Analyze the convergence of Jacobi and Gauss-Seidel methods.
2. Estimate the rapidity of convergence: How many iterations are needed to obtained a solution with tolerance of 10^{-12} with both the methods?
3. Experimentally verify the results obtained at the previous point.
4. Find the value of α such that the stationary Richardson iterative method (with no preconditioning) converges and the value α_{opt} which minimizes the spectral radius of the iteration matrix.

Exercise 5 (From [2])

Given the linear system obtained with the following instructions:

```
>> B = rand(5) + diag(10*ones(5,1));
>> A = B*B';
```

1. Analyze the convergence of Jacobi and Gauss-Seidel methods.
2. Given the (stationary) preconditioned Richardson iterative method, with a diagonal precondition matrix P, whose diagonal is the same one of A, numerically determine the values of the relaxation parameter $\alpha \in (0, 2)$ for which the method is convergent and, in particular, the optimal value α_{opt}.

Outline

Homework

Exercise H4.1

Calculate by hand the condition number and the determinant of the following diagonal matrix A:

```
>> A = diag(0.1*ones(15,1));
>> A = triu(rand(15),14)*1e5 + diag(ones(15,1));
```

1. Comment the obtained results: is in general the determinant a good measure of the condition number of the matrix?

Exercise H4.2

Consider the matrices obtained with the following commands:

- `>> A = round(3*rand(20)+5*eye(20));`
- `>> A = fix(3*rand(20)); A = A'*A;`

1. *Without performing any calculation* evaluate which iterative method is applicable for the solution of the system $Ax = b$. Justify the answer.

Homework

Exercise H4.3

Consider the following:

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

1. Study the convergence of Jacoby and Gauss-Seidel methods in the solution of the system $Ax = b$.
2. Calculate the spectral radius of the iteration matrices of both the methods.
3. How many iteration are needed to obtain the solution with a tolerance of 10^{-8}?

Homework

Exercise H4.4 (From [2])

Given the linear system $Ax = b$ with $A = D - (E + F)$, where D is the matrix formed by the diagonal of A, E is the lower and F the upper triangular part of A with the signs changed.
Let $N = E + F$. For a given $x^{(0)}$, consider the following iterative method:

$$\epsilon D x^{(k+1)} = (\epsilon - 1) D x^{(k)} + N x^{(k)} + b, \quad (k \geq 0, \epsilon \in \mathbb{R}^{+})$$

1. Suppose that Jacobi method converges, which are the conditions to be posed on ϵ such that also the proposed method converges?

Hint: Remember that for a given matrix B with eigenvalues λ_{Bi}, the matrix $(B - \alpha I)$ will have the correspondent $\lambda_i = \cdots$

Homework

Exercise H4.5

Given the linear system $Ax = b$ with:

$$A = \begin{bmatrix} 50 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1. Calculate $\|A\|_\infty$ and $\|A\|_1$.
2. Is A symmetric and positive definite?
3. Which one among all the studied methods can be used in the solution of $Ax = b$? If Jacoby method is suitable for the purpose, give an estimation of the convergence rapidity and give an overestimation of the error after the 4^{th} iteration, assuming a null initial vector. Experimentally validate the results.
4. Which is the meaning of the system $Ax = b$?
5. Convert the system to diagonal form.
6. Calculate the LU decomposition. Experimentally validate the answer.