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Calcolo Scientifico per l'Informatica
- Laboratory Class -
3

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Outline

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Introduction to Linear Systems

Consider a linear system $(n \times n)$ in its matricial form:

$$A\underline{x} = \underline{b}.$$

where $A \in \mathbb{R}^{n \times n}$ and $\underline{b} \in \mathbb{R}^{n}$. If A is non-singular, the solution is given by:

$$\underline{x} = A^{-1}\,\underline{b}.$$

Note: the system is a linear *linear transformation* $\underline{b} = A\underline{x}$, where $\underline{b}$ can be viewed as the result of the linear combination of the columns of the matrix A $(\underline{A}_i)$ by means of the coefficients x_i (the components of vector $\underline{x}$):

$$x_1\underline{A}_1 + x_2\underline{A}_2 + x_3\underline{A}_3 + \cdots + x_n\underline{A}_n = \underline{b}.$$

Linear Systems: Direct Methods

The solution is obtained in a finite number of operations:

- Cramer's Rule
- Gaussian Elimination
- LU decomposition
- Cholesky decomposition

In MATLAB avoid:

```
>> x = A^(-1)*b
```

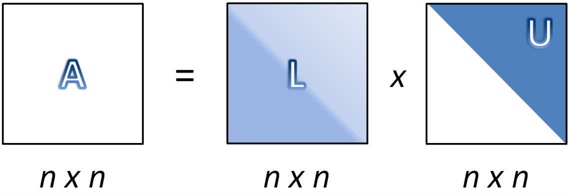
Use instead the `mldivide` operator (`backslash`):

```
>> x = A\b
```

See `help mldivide`.

Outline

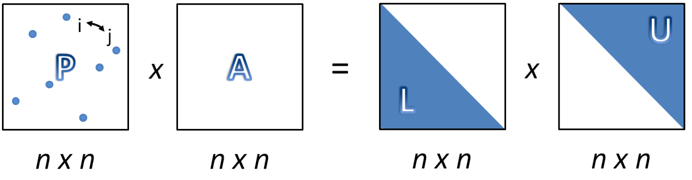
Linear Systems in MATLAB




```
>> help lu
 LU      LU factorization.
    [L,U] = LU(A) stores an upper triangular matrix in U and a
    "psychologically lower triangular matrix" (i.e. a product of
    lower triangular and permutation matrices) in L, so that A = L*U.
    A can be rectangular.

    [L,U,P] = LU(A) returns unit lower triangular matrix L, upper
    triangular matrix U, and permutation matrix P so that P*A = L*U.
```

Linear Systems in MATLAB

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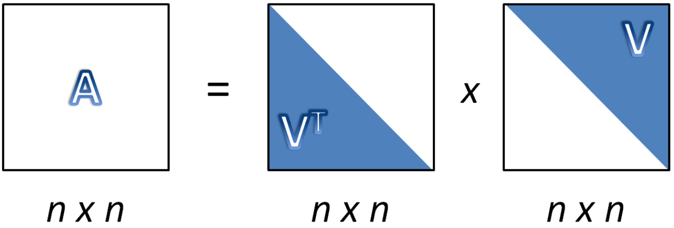
Linear Systems in MATLAB




```
>> help chol
 CHOL   Cholesky factorization.
    CHOL(A) uses only the diagonal and upper triangle of A. The lower
    triangle is assumed to be the (complex conjugate) transpose of the
    upper triangle.  If A is positive definite, then R = CHOL(A)
    produces an upper triangular R so that R'*R = A. If A is not
    positive definite, an error message is printed.
```

Exercise 1 (From [2])

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 11 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

1. Construct the L_1 matrix of the first step of Gaussian elimination that removes 3 under the first pivot. Perform the multiplication $L_1 A L_1^T$; Comment the result.
2. Construct the L_2 matrix which makes A triangular and verify that $L_2 L_1 A L_1^T L_2^T = D$, with D diagonal.
3. Calculate the inverse of the matrices L_i $(i = 1, 2)$, without using the command `inv` in MATLAB. Verify that $A = LDL^T$ with $L = (L_2 L_1)^{-1}$.

Exercise 2 (From [2])

Consider the matrix obtained with the following command:

```
A = pascal(4).
```

1. Calculate the decomposition $PA = LU$. Determine the upper triangular matrix $\overline{U}$ with unitary main diagonal and the diagonal matrix D such that $PA = LD\overline{U}$.
2. Calculate the decomposition $A = LU$, without pivoting. What about the computational cost for the resolution of the generic system $Ax = b$?
3. Experimentally verify that A is positive definite.

Hint: see `help pascal`, `help lu` and `help sparse`.

Exercise 3 (From [2])

Consider the following linear system $Ax = b$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha & 4 & 2 & 1 \\ 27 & 9 & \beta & 1 \\ 10^{-4} & 16 & 4 & 10^4 \end{bmatrix}, \quad b = A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

1. For which values of α and β does the LU decomposition of A exist with no pivoting?
2. Given $\alpha = 1$ and $\beta = 21$ find a permutation matrix P such that the $PA = LU$ decomposition exists.
3. Solve the system by means of the `lu` MATLAB command with the conditions given at point two.
4. Knowing that the exact solution of the system is $x = [1\,1\,1\,1]^T$, what can be inferred from the results obtained at the previous point?

Outline

The condition number

Given the system $Ax = b$ and a perturbation on the right-hand side term δb

$$A(x + \delta x) = b + \delta b$$

The condition number is defined as:

$$\kappa(A) = \|A\| \left\|A^{-1}\right\|$$

The higher its value the stronger the system solution sensitivity to relatively small changes in b or in A. We can give an *estimation* of the condition number as:

$$\kappa(A) \geq \frac{\|\delta x\|}{\|x\|} \frac{\|b\|}{\|\delta b\|}$$

It roughly indicates the number of digits of accuracy lost relative to the machine accuracy ($e.g.$ a $\kappa(A)$ of $\approx 1e6$ means about 10 digits of accuracy in MATLAB).

The condition number

```
>> help cond
 COND   Condition number with respect to inversion.
    COND(X) returns the 2-norm condition number (the ratio of the
    largest singular value of X to the smallest).  Large condition
    numbers indicate a nearly singular matrix.
    COND(X,P) returns the condition number of X in P-norm:
       NORM(X,P) * NORM(INV(X),P).
    where P = 1, 2, inf, or 'fro'.
```

```
>> help rcond
 RCOND  LAPACK reciprocal condition estimator.
    RCOND(X) is an estimate for the reciprocal of the condition of
    X in the 1-norm obtained by the LAPACK condition estimator. If
    X is well conditioned, RCOND(X) is near 1.0. If X is badly
    conditioned, RCOND(X) is near EPS.
```

See also `help condest` and `help norm`.

Exercise 4 (From [2])

Given the Hilbert matrix H_n of order n:

$$H_n = \begin{bmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ \cdots & \cdots & \cdots & \cdots \\ 1/n & 1/(n+1) & \cdots & 1/(2n-1) \end{bmatrix} \in \Re^{n \times n}, \quad n = 10, 20, 40$$

and let $Cond(\cdot)$ the conditioning in the 2-norm of a matrix.

1. Give an estimation of $Cond(H_n)$.
2. Calculate $Cond(H_{10})$ by means of the decomposition $A = UDV^T$.
3. Calculate $Cond(H_{10})$ by means of the eigenvalues of A.
4. Calculate $Cond(H_n)$ by means of `cond` MATLAB command and compare that value with the ones obtained at the previous points.
5. How large can you take n before the error in the solution of the generic linear system $H_n x = b$ is 100% ($i.e.$ there are no significant digits in the solution)?

Exercise 5 (From [2])

Calculate the condition number of the following matrix:

$$A = \begin{bmatrix} 1.001 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1.001 & 1 & 1 \\ 1 & 1.001 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1.001 \end{bmatrix}$$

by means of

1. `cond` MATLAB command.
2. The definition of condition number.
3. An approximate method implemented in MATLAB.

Outline

Homework

Exercise H3.1 (From [2])

Consider the following matrix A_n, defined as:

$$A = \begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 1 & 2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix},$$

1. Verify that A_n is symmetric positive-definite for $n = 10$.
2. Which is the form of the matrix V_{10} such that $A_{10} = V_{10}^T V_{10}$? Justify.
3. Demonstrate that A_n is symmetric and positive-definite $\forall n$.

Hint: see `help diag` for the matrix construction and for example `help all` for symmetry verification.

Homework

Exercise H3.2

Given the matrices $A_n = Pascal(n) \in \Re^{n\times n}$, with $n = 10, 20, 40$. Let $Cond(\cdot)$ the conditioning in the 2-norm of a matrix.

1. Give an estimate of $Cond(A_n)$.
2. Compare and comment the results with those obtained by means of `rcond`.

Exercise H3.3

Consider the matrix A:

$$A = \begin{bmatrix} 4 & -6 & 2 \\ 0 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

1. Evaluate its 1-norm and its infinity norm.
2. Evaluate its spectral radius.