Making the constant step size independant of initial bias :-

When just had the step size $\alpha_n = \alpha = \text{constant}$,

we have, $\boxed{Q_{n+1} = Q_n + \alpha[R_n - Q_n]}$

which expanded to

eq(1) — $\boxed{Q_n = (1-\alpha)^n Q_1 + \sum_{i=1}^{n} \alpha(1-\alpha)^{n-i} R_i}$

here Q_1 has a non-zero coefficient ($\because 0 \le \alpha < 1$)
hence there is an initial bias.

but if we change the step size to β, where:-

$$\beta_n = \frac{\alpha}{\bar{o}_n}, \quad n = \text{step no.}$$

and $\bar{o}_n = \bar{o}_{n-1} + \alpha(1 - \bar{o}_{n-1})$ $\forall\, n \ge 0$

with $\bar{o}_0 = 0$

will give us the benefits of step size at the same time, remove initial bias (Q_1)

So, $Q_{n+1} = Q_n + \beta_n(R_n - Q_n) = Q_n + \frac{\alpha}{\bar{o}_n}(R_n - Q_n)$

$$= \frac{\alpha}{\bar{o}_n} R_n + Q_n \frac{(1-\alpha)\bar{o}_{n-1}}{\bar{o}_n}$$

$$= \frac{\alpha}{\bar{o}_n} R_n + \left(\frac{\alpha}{\bar{o}_{n-1}} R_{n-1} + Q_{n-1}\frac{(1-\alpha)\bar{o}_{n-2}}{\bar{o}_{n-1}}\right)\frac{(1-\alpha)\bar{o}_{n-1}}{\bar{o}_n}$$

[using telescopic and $\bar{o}_n - \alpha = \bar{o}_{n-1}(1-\alpha)$]

$$= \frac{\alpha}{\bar{O}_n} R_n + \frac{\alpha(1-\alpha) R_n}{\bar{O}_n} + Q_{n-1} \frac{(1-\alpha)^2 \bar{O}_{n-2}}{\bar{O}_n}$$

$$= \frac{\alpha}{\bar{O}_n} R_n + \frac{\alpha(1-\alpha)}{\bar{O}_n} R_{n-1} + \left[\frac{\alpha}{\bar{O}_{n-2}} R_{n-2} + Q_{n-2} \frac{(1-\alpha)\bar{O}_{n-3}}{\bar{O}_{n-2}} \right] \frac{(1-\alpha)^2 \bar{O}_{n-2}}{\bar{O}_n}$$

$$= \frac{\alpha R_n}{\bar{O}_n} + \frac{\alpha(1-\alpha)}{\bar{O}_n} R_{n-1} + \frac{\alpha(1-\alpha)^2 R_{n-2}}{\bar{O}_n} + Q_{n-2} \frac{(1-\alpha)^3 \bar{O}_{n-3}}{\bar{O}_n}$$

$$= \frac{\alpha}{\bar{O}_n} \left[\sum_{i=0}^{n-1} R_{n-i} (1-\alpha)^i \right] + \underbrace{Q_1 \frac{\bar{O}_0 (1-\alpha)^n}{\bar{O}_n}}_{= \text{zero}, \ \because \ \bar{O}_0 = 0} \rightarrow \text{following the pattern.}$$

$$\Rightarrow \boxed{\frac{\alpha}{\bar{O}_n} \left[\sum_{i=0}^{n-1} R_{n-i} (1-\alpha)^i \right] = Q_{n+1}} \text{ — eq (2)} \rightarrow \text{no initial bias here}$$

So,

$$\boxed{Q_{n+1} = \frac{\alpha}{\bar{O}_n}\left[\sum_{i=0}^{n-1} R_{n-i}(1-\alpha)^i\right]}$$

Here there is no effect of $Q_1 \rightarrow$ initial value.

Now, we need to show that Q_{n+1} is indeed a weighted-average.

So,

$$\frac{\alpha}{\bar{O}_n}\sum_{i=0}^{n-1}(1-\alpha)^i = 1 \rightarrow \text{To prove} \quad \text{— eq (3)}$$

we had

$$\bar{O}_n = \bar{O}_{n-1}(1-\alpha) + \alpha$$

$$= (1-\alpha)^2\bar{O}_{n-2} + (1-\alpha)\alpha + \alpha$$

$$= (1-\alpha)^3\bar{O}_{n-3} + (1-\alpha)^2\alpha + (1-\alpha)\alpha + \alpha$$

$$= (1-\alpha)^n\underbrace{\bar{O}_0}_{=0\ (\text{zero})} + \alpha\left(\sum_{i=0}^{n-1}(1-\alpha)^i\right)$$

$$\Rightarrow \frac{\bar{O}_n}{\alpha} = \sum_{i=0}^{n-1}(1-\alpha)^i \Rightarrow \boxed{\bar{O}_n = \alpha\sum_{i=0}^{n-1}(1-\alpha)^i} = \boxed{1-(1-\alpha)^n}$$

$\therefore$ in eq (3):-

$$\frac{\alpha\sum_{i=0}^{n-1}(1-\alpha)^i}{\alpha\sum_{i=0}^{n-1}(1-\alpha)^i} = 1 \quad \text{Hence proved.}$$

and Q_{n+1} is indeed exponential recency weighted

$\therefore$ weight of R_i is $\dfrac{\alpha(1-\alpha)^{n-i}}{(1-(1-\alpha)^n)}$ $\rightarrow$ this weight

is exponentially decreasing as 'n' increases

$\therefore$ $\underline{0 < (1-\alpha) < 1}$

as n increases, $n-i$ increases and

$\therefore$ $(1-\alpha)^{n-i}$ decreases.

Now we need to prove convergence, so :-

I) $\sum_{n=1}^{\infty} \beta_n = \infty$ and II) $\sum_{n=1}^{\infty} \beta_n^2 < \infty$

for I) $\beta_n = \dfrac{\alpha}{O_n} = \dfrac{\alpha}{1-(1-\alpha)^n}$, now $0 \leq \alpha < 1$

so, $0 \leq 1-\alpha < 1 \Rightarrow 0 \leq (1-\alpha)^n < 1 \quad \forall n \in Int.$

so, $0 \leq 1-(1-\alpha)^n < 1, \forall n \in Int$

$\Rightarrow \dfrac{1}{1-(1-\alpha)^n} > 1 \Rightarrow \boxed{\dfrac{\alpha}{1-(1-\alpha)^n} > \alpha} \quad \forall n \in Int$

and we know $\sum_{n=1}^{\infty} \alpha = \infty$, $\because \alpha =$ a constant

so, $\sum_{n=1}^{\infty} \dfrac{\alpha}{1-(1-\alpha)^n} = \sum_{n=1}^{\infty} (\geq \alpha) = \underline{\infty}$.