# Reasoning about types and code

## What we functional programmers need to know

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# Reasoning about types and code

Practitioners of functional programming need to know how to...

- ...reason about types:
  - design the required custom data types for the given application
  - derive an equivalent simpler type when possible
  - use type constructions to create data types with required properties
- ...reason about code:
  - verify that given implementations satisfy the required laws (e.g. monad)
  - derive lawful custom implementations of important typeclasses
  - verify that certain functions are computationally equivalent
  - derive an equivalent simpler code when possible

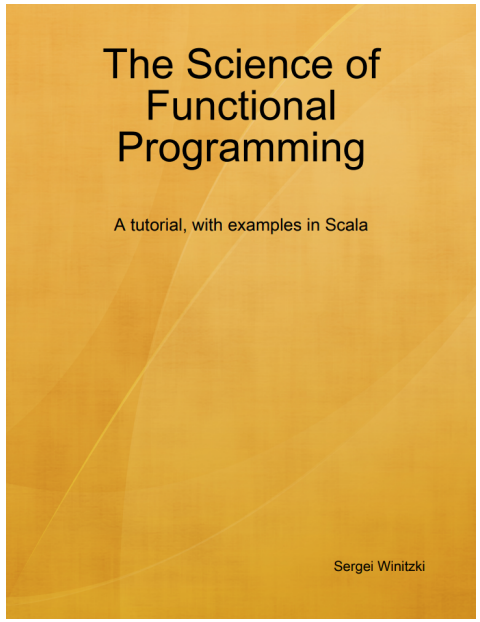
This requires a *very limited* amount of mathematics (polynomials, monoids)

# A new book: *The Science of Functional programming*

I am working on a new book,
*The Science of Functional Programming: A tutorial, with examples in Scala*
https://github.com/winitzki/sofp (free as in GNU FSF)

The book will explain (with examples and exercises):

- techniques of reasoning about types and type constructors
- techniques for symbolic calculations with code
- deriving and verifying laws symbolically (as equations for functions)
- real-life motivations for (and applications of) these techniques



# Examples of reasoning tasks I

1. Can we compute a value of type `Either[Z, R => A]` given a value of type `R => Either[Z, A]`? And conversely? (`A`, `R`, `Z` are type parameters.)

```
def f[Z, R, A](r: R => Either[Z, A]): Either[Z, R => A] = ???
def g[Z, R, A](e: Either[Z, R => A]): R => Either[Z, A] = ???
```

- It turns out that `f` cannot be implemented, but `g` can be:

```
def g[Z, R, A](e: Either[Z, R => A]): R => Either[Z, A] = {
  r => e.map(f => f(r))
}
```

# Examples of reasoning tasks II

2. How to use `for`/`yield` with `Either[Z, A]` and `Future[A]` together?

```
val result = for { // This code will not compile; we want to have...
  a <- Future(...) // ... a computation that takes time and may fail,
  b <- Either(...) // a computation whose result may be unavailable,
  c <- Future(...) // a computation that takes time and may fail.
} yield ??? // Continue computations when results are available.
```

   Should `result` have type `Either[Z,Future[A]]` or `Future[Either[Z,A]]`?
   How to combine `Either` with `Future` so that we can use `flatMap`?

- It turns out that `Either[Z,Future[A]]` is wrong (cannot implement `flatMap` correctly) but `Future[Either[Z,A]]` is the correct solution

# Examples of reasoning tasks III

3. Can we implement `flatMap` for the type constructor `Option[(A, A, A)]`?

```
def flatMap[A, B](fa: Option[(A, A, A)])(f: A => Option[(B, B, B)])
    : Option[(B, B, B)] = ???
```

- It turns out that `flatMap` *can* be implemented but fails the laws

# Examples of reasoning tasks IV

4. What type describes a chain of `Future[A]` operations that, on any failure, will automatically execute specified cleanups in reverse order?

- The type `Future[(A, Unit => Unit)]` has the required properties
- The "cleanup" operations `Unit => Unit` form a monoid and can be merged into a single cleanup executing in reverse order

```
type Tx[A] = (A, Unit => Unit)
def flatMap[A,B](task: Future[Tx[A]])(f: A => Future[Tx[B]])
            (implicit ec: ExecutionContext): Future[Tx[B]] =
  task.flatMap { case (a, cleanup1) => f(a).transform(
     { case (b, cleanup2) => (b, cleanup1 |+| cleanup2) },
     { ex => cleanup1(()); ex }
  )
```

# Examples of reasoning tasks V

5. Different people define a "free monad" via different sets of case classes. Are these definitions equivalent? What is the difference?
6. How to define a free monad generated by a `Pointed` functor (i.e. when the functor already has the `pure` method)? Will that type have better performance than the standard free monad generated by a functor?

# Short notation for types

- A concise mathematical notation for types and type constructors:

| Scala syntax | Type notation |
|---|---|
| type parameter `A` | $A$ |
| tuple type `(A, B)` or `case class C(a: A, b: B)` | $A \times B$ |
| disjunctive type `Either[A,B]` or `trait/case classes` | $A + B$ |
| function type `A => B` | $A \to B$ |
| `Unit` or an equivalent named type | $\mathbb{1}$ |
| primitive type (`Int`, `String`, etc.) | Int, String, ... |
| `Nothing` (the void type) | $\mathbb{0}$ |
| type constructor, `type P[A] = Option[(A,A,A)]` | $P^A \triangleq \mathbb{1} + A \times A \times A$ |
| type constructor as a parameter, `P[_]` | $P^\bullet$ |
| function with type parameter, `def f[A]: A => (A, A)` | $f^A : A \to A \times A$ |

This notation proved convenient for reasoning about equivalence of types:

$$(A + B) \times C \cong A \times C + B \times C \quad , \qquad A \to B \to C \cong A \times B \to C \quad ,$$

$$\mathbb{0} + A \cong A \quad , \qquad A + B \to C \cong (A \to C) \times (B \to C) \quad .$$

# Short notation for code

- A concise mathematical notation for code:

| Scala syntax | Code notation |
|---|---|
| variable, or function argument `x: A` | $x^{:A}$ |
| tuple value `(a, b)` | $a \times b$ or $a^{:A} \times b^{:B}$ |
| value of disjunctive type `Left[A, B](x)` | $x^{:A} + \mathbb{0}^{:B}$ |
| nameless function `{ x: A => expr }` | $x^{:A} \to \text{expr}$ |
| the `Unit` value, `()` | $1$ |
| `def f[A, B]: A => B => A = x => _ => x` | $f^{:A \to B \to A} \triangleq x^{:A} \to \_^{:B} \to x$ |
| the identity function, `identity[A]` | $\text{id}^A$ or $\text{id}^{:A \to A}$ |
| *forward* composition, `f andThen g` | $f \fatsemi g$ or $f^{:A \to B} \fatsemi g^{:B \to C}$ |
| argument chaining, `x.pipe(f)` (Scala 2.13) | $x \triangleright f$ where $x^{:A}$ and $f^{:A \to B}$ |
| lifted function, `p.map(f)` where `p: P[A]` | $p \triangleright f^{\uparrow P}$ where $p^{:P^A}$ and $f^{:A \to B}$ |

This notation proved convenient for reasoning about equational laws:

$$f^{\uparrow P} \fatsemi g^{\uparrow P} \fatsemi h^{\uparrow P} = (f \fatsemi g \fatsemi h)^{\uparrow P} \quad , \qquad x \triangleright f \triangleright g = x \triangleright f \fatsemi g \quad .$$

# Example I. Free monad generated by a type constructor

A free monad generated by a type constructor `F[_]` is...

1. From a blog post by Gabriel Gonzalez (2012), translated into Scala:

```
sealed trait Free1[F[_], T]
case class Pure[F[_], T](t: T)                    extends Free1[F, T]
case class Flatten[F[_], T](f: F[Free1[F, T]]) extends Free1[F, T]
```

2. From a talk given by Rúnar Bjarnason (2014):

```
sealed trait Free2[F[_], T]
case class Pure[F[_], T](t: T) extends Free2[F, T]
case class Bind[F[_], T, A](f: F[A], g: A => Free2[F, T])
     extends Free2[F, T]
```

3. From a talk given by Kelley Robinson (2016):

```
sealed trait Free3[F[_], T]
case class Pure[F[_], T](t: T)          extends Free3[F, T]
case class Suspend[F[_], T](f: F[T])    extends Free3[F, T]
case class FlatMap[F[_], T, A](f: Free3[F, A], g: A => Free3[F, T])
     extends Free3[F, T]
```

Are these definitions different? How to reason about them?

# Example I. Writing the definitions in the type notation

A free monad generated by a type constructor `F[_]` is...

1. From G. Gonzalez: $\text{Free}_1^{F,T} \triangleq T + F^{\text{Free}_1^{F,T}}$

```
sealed trait Free1[F[_], T]
case class Pure[F[_], T](t: T)                    extends Free1[F, T]
case class Flatten[F[_], T](f: F[Free1[F, T]]) extends Free1[F, T]
```

2. From R. Bjarnason: $\text{Free}_2^{F,T} \triangleq T + \exists A.\, F^A \times (A \to \text{Free}_2^{F,T})$

```
sealed trait Free2[F[_], T]
case class Pure[F[_], T](t: T) extends Free2[F, T]
case class Bind[F[_], T, A](f: F[A], g: A => Free2[F, T])
     extends Free2[F, T]
```

3. From K. Robinson: $\text{Free}_3^{F,T} \triangleq T + F^T + \exists A.\, \text{Free}_3^{F,A} \times (A \to \text{Free}_3^{F,T})$

```
sealed trait Free3[F[_], T]
case class Pure[F[_], T](t: T)          extends Free3[F, T]
case class Suspend[F[_], T](f: F[T])    extends Free3[F, T]
case class FlatMap[F[_], T, A](f: Free3[F, A], g: A => Free3[F, T])
     extends Free3[F, T]
```

# Example I. Three definitions of the free monad

What is the difference between definitions 1 and 2?

- Definition $\text{Free}_1^{F,T} \triangleq T + F^{\text{Free}_1^{F,T}}$ assumes that $F^\bullet$ is a functor
- Definition $\text{Free}_2^{F,T} \triangleq T + \exists A.\, F^A \times (A \to \text{Free}_2^{F,T})$ works for any $F^\bullet$
  - if $F^\bullet$ is a functor, the Yoneda identity gives $F^B \cong \exists A.\, F^A \times (A \to B)$
  - then $T + \exists A.\, F^A \times (A \to \text{Free}_2^{F,T}) \cong T + F^{\text{Free}_2^{F,T}}$
- So, definitions 1 and 2 are equivalent when $F^\bullet$ is a functor

What is the difference between definitions 2 and 3?

- Can we replace $\text{Free}_3^{F,A}$ by $F^A$ in definition 3? Yes.

$$
\begin{aligned}
& T + F^T + \exists A.\, \underline{\text{Free}_3^{F,A}} \times (A \to \text{Free}_3^{F,T}) \\
&= T + \underline{F^T} + \exists A.\, \left(A + F^A + \exists B.\, \text{Free}_3^{F,B} \times (B \to \text{Free}_3^{F,A})\right) \times (A \to \text{Free}_3^{F,T}) \\
&= T + \underline{\exists A.\, F^A \times (A \to T)} + \underline{\exists A.\, A \times (A \to \text{Free}_3^{F,T})} + \exists A.\, F^A \times (A \to \text{Free}_3^{F,T}) \\
&\qquad + \exists A.\, \exists B.\, \underline{\text{Free}_3^{F,B} \times (B \to \text{Free}_3^{F,A}) \times (A \to \text{Free}_3^{F,T})} \quad .
\end{aligned}
$$

By the monad's associativity law, we reduce $\text{Free}_3^{F,B} \times (B \to \text{Free}_3^{F,A}) \times (A \to \text{Free}_3^{F,T})$ to $\text{Free}_3^{F,B} \times (B \to \text{Free}_3^{F,A} \times (A \to \text{Free}_3^{F,T}))$. By inductive assumption, this is $\text{Free}_3^{F,B} \times (B \to \text{Free}_3^{F,T})$ and can be replaced by $F^B \times (B \to \text{Free}_3^{F,A})$.

# Example II. The selection monad transformer

- The selection monad is defined as `Sel[A] = (A => R) => A`
  - This is related to the "search monad", `(A => Boolean) => Option[A]`
    - ★ but *not* related to the "search monad" in Haskell's monad-dijkstra
- The paper "Monad Transformers for Backtracking Search" https://arxiv.org/pdf/1406.2058.pdf defines the selection monad transformer `SelT[A] = (A => M[R]) => M[A]`. The author writes,

  *The proof that the selection monad satisfies the monad laws was found using an ad-hoc computer program written by Martin Escardó [...] The author has verified by hand that the selection monad transformer preserves the unit laws, however the proof for the associativity law appears to be unmanageable. The author is currently working on a formally verified proof that the selection monad transformer preserves the monad laws.*

- The proof was too hard because (most likely) the author tried to write symbolic calculations in Haskell syntax, which is not well adapted to proofs. (This would have been worse if using Scala syntax.)
  - Doing a formally verified proof is *much harder* than proving by hand!
    - ★ (and you *already* need to know how to prove it by hand)
- In the short code notation, the proof takes 5 lines of calculations

# Example II. Proof of the associativity law

The selection monad transformer's type is $T^{M,A} \triangleq (A \to M^R) \to M^A$
Define Kleisli functions $f : A \to T^{M,B} \triangleq A \to (B \to M^R) \to M^B$
Use a trick: flip arguments, $f : (B \to M^R) \to A \to M^B$ and $g : (C \to M^R) \to B \to M^C$
Define Kleisli composition, $f \diamond_T g : (C \to M^R) \to A \to M^C$ using the known $\diamond_M$ as

$$f \diamond_T g \triangleq k^{:C \to M^R} \to f(g(k) \diamond_M k) \diamond_M g(k)$$

The operation $\diamond_M$ already satisfies the associativity law, $(p \diamond_M q) \diamond_M r = p \diamond_M (q \diamond_M r)$
Prepare the result of $g \diamond_T h$ to be used later:

$$(g \diamond_T h)(k) = \underline{g(h(k) \diamond_M k) \diamond_M h(k)}$$

Now check associativity law for the flipped Kleisli composition $\diamond_T$:

$$\begin{aligned}
&(f \diamond_T g) \diamond_T h = k \to (\underline{f \diamond_T g})(h(k) \diamond_M k) \diamond_M h(k) \\
&= k \to f(g(\underline{h(k) \diamond_M k}) \diamond_M \underline{h(k) \diamond_M k}) \diamond_M g(\underline{h(k) \diamond_M k}) \diamond_M h(k) \quad , \\
&f \diamond_T (g \diamond_T h) = k \to f((\underline{g \diamond_T h})(k) \diamond_M k) \diamond_M (\underline{g \diamond_T h})(k) \\
&= k \to f(\underline{g(h(k) \diamond_M k) \diamond_M h(k)} \diamond_M k) \diamond_M g(\underline{h(k) \diamond_M k) \diamond_M h(k)} \quad .
\end{aligned}$$

So we have proved the associativity law, $(f \diamond_T g) \diamond_T h = f \diamond_T (g \diamond_T h)$
In a similar way, can prove the laws for another transformer, $T^{M,A} \triangleq (M^A \to R) \to M^A$

# Summary

- Mastering the type/code reasoning takes about 6 months of practice
  - and you will be able to use FP much more effectively in actual coding
  - the special notation helped me a lot
- Lots of explanations, examples, and exercises in the upcoming book
- Current progress: chapters 1–7 ready with 2 proofreading passes
- Chapters 8 and 13 are in progress
- Source (LaTeX) for the book: https://github.com/winitzki/sofp