

Chapter 8: Applicative functors and profunctors

Part 2: Their laws and structure

Sergei Winitzki

Academy by the Bay

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Deriving the `ap` operation from `map2`

Can we avoid having to define map_n separately for each n ?

- Use curried arguments, $\text{fmap}_2 : (A \Rightarrow B \Rightarrow Z) \Rightarrow F^A \Rightarrow F^B \Rightarrow F^Z$
- Set $A \equiv (B \Rightarrow Z)$ and apply fmap_2 to the identity $\text{id}^{(B \Rightarrow Z) \Rightarrow (B \Rightarrow Z)}$: obtain $\text{ap}^{[B, Z]} : F^{B \Rightarrow Z} \Rightarrow F^B \Rightarrow F^Z \equiv \text{fmap}_2(\text{id})$
- The functions `fmap2` and `ap` are computationally equivalent:

$$\text{fmap}_2 f^{A \Rightarrow B \Rightarrow Z} = \text{fmap } f \circ \text{ap}$$

$$\begin{array}{ccc} & \text{fmap } f & \\ & \nearrow & \\ F^A & & F^{B \Rightarrow Z} \\ & \searrow \text{fmap}_2 (f^{A \Rightarrow B \Rightarrow Z}) & \searrow \text{ap} \\ & & (F^B \Rightarrow F^Z) \end{array}$$

- The functions `fmap3`, `fmap4` etc. can be defined similarly:

$$\text{fmap}_3 f^{A \Rightarrow B \Rightarrow C \Rightarrow Z} = \text{fmap } f \circ \text{ap} \circ \text{fmap}_{F^B \Rightarrow ?} \text{ap}$$

$$\begin{array}{ccccc} & \text{fmap } f & & \text{ap}^{[B, C \Rightarrow Z]} & \\ & \nearrow & & \longrightarrow & \\ F^A & & F^{B \Rightarrow C \Rightarrow Z} & & (F^B \Rightarrow F^{C \Rightarrow Z}) \\ & \searrow \text{fmap}_3 (f^{A \Rightarrow B \Rightarrow C \Rightarrow Z}) & & \searrow \text{fmap}_{F^B \Rightarrow ?} \text{ap}^{[C, Z]} & \\ & & & & (F^B \Rightarrow F^C \Rightarrow F^Z) \end{array}$$

- Using the infix syntax will get rid of $\text{fmap}_{F^B \Rightarrow ?} \text{ap}$ (see example code)
 - ▶ Note the pattern: a natural transformation is equivalent to a lifting

Deriving the `zip` operation from `map2`

- The types $A \Rightarrow B \Rightarrow C$ and $A \times B \Rightarrow C$ are equivalent (curry/uncurry)
- Uncurry `fmap2` to `fmap2` : $(A \times B \Rightarrow C) \Rightarrow F^A \times F^B \Rightarrow F^C$
- Compute `fmap2(f)` with $f = \text{id}^{A \times B \Rightarrow A \times B}$, expecting to obtain a simpler natural transformation:

$$\text{zip} : F^A \times F^B \Rightarrow F^{A \times B}$$

- This is quite similar to `zip` for lists:

`List(1, 2).zip(List(10, 20)) = List((1, 10), (2, 20))`

- The functions `zip` and `fmap2` are computationally equivalent:

$$\begin{aligned}\text{zip} &= \text{fmap2}(\text{id}) \\ \text{fmap2}(f^{A \times B \Rightarrow C}) &= \text{zip} \circ \text{fmap } f\end{aligned}$$

$$\begin{array}{ccc} & F^{A \times B} & \\ \text{zip} \nearrow & & \searrow \text{fmap } f^{A \times B \Rightarrow C} \\ F^A \times F^B & \xrightarrow{\quad \text{fmap2}(f^{A \times B \Rightarrow C}) \quad} & F^C \end{array}$$

- The functor F is **zipable** if such a `zip` exists (with appropriate laws)
 - ▶ The same pattern: a natural transformation is equivalent to a lifting

* Equivalence of the operations `ap` and `zip`

- Set $A \equiv B \Rightarrow C$, get $\text{zip}^{[B \Rightarrow C, B]} : F^{B \Rightarrow C} \times F^B \Rightarrow F^{(B \Rightarrow C) \times B}$
- Use `eval` : $(B \Rightarrow C) \times B \Rightarrow C$ and $\text{fmap}(\text{eval}) : F^{(B \Rightarrow C) \times B} \Rightarrow F^C$
- Uncurry: $\text{app}^{[B, C]} : F^{B \Rightarrow C} \times F^B \Rightarrow F^C \equiv \text{zip} \circ \text{fmap}(\text{eval})$
- The functions `zip` and `app` are computationally equivalent:
 - ▶ use $\text{pair} : (A \Rightarrow B \Rightarrow A \times B) = a^A \Rightarrow b^B \Rightarrow a \times b$
 - ▶ use $\text{fmap}(\text{pair}) \equiv \text{pair}^\uparrow$ on an fa^{F^A} , get $(\text{pair}^\uparrow fa) : F^{B \Rightarrow A \times B}$; then

$$\text{zip}(fa \times fb) = \text{app}\left((\text{pair}^\uparrow fa) \times fb\right)$$

$$\text{app}^{[B, C]} = \text{zip}^{[B \Rightarrow C, B]} \circ \text{fmap}(\text{eval})$$

$$F^{B \Rightarrow C} \times F^B \begin{array}{c} \xrightarrow{\text{zip}} F^{(B \Rightarrow C) \times B} \\ \xrightarrow{\text{app}^{[B, C]}} F^C \\ \searrow \text{fmap}(\text{eval}) \end{array}$$

- Rewrite this using curried arguments: $\text{fzip}^{[A, B]} : F^A \Rightarrow F^B \Rightarrow F^{A \times B}$; $\text{ap}^{[B, C]} : F^{B \Rightarrow C} \Rightarrow F^B \Rightarrow F^C$; then $\text{ap } f = \text{fzip } f \circ \text{fmap}(\text{eval})$.
- Now $\text{fzip } p^{F^A} q^{F^B} = \text{ap}(\text{pair}^\uparrow p) q$, hence we may omit the argument q : $\text{fzip} = \text{pair}^\uparrow \circ \text{ap}$. With explicit types: $\text{fzip}^{[A, B]} = \text{pair}^\uparrow \circ \text{ap}^{[B, A \Rightarrow B]}$.

Motivation for applicative laws. Naturality laws for `map2`

Treat `map2` as a replacement for a monadic block with independent effects:

<pre>for { x ← cont1 y ← cont2 } yield g(x, y)</pre>	<pre>map2 (cont1, cont2) { (x, y) ⇒ g(x, y) }</pre>
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- Main idea: Formulate the monad laws in terms of `map2` and `pure`

Naturality laws: Manipulate data in one of the containers

<pre>for { x ← cont1.map(f) y ← cont2 } yield g(x, y)</pre>	<pre>for { x ← cont1 y ← cont2 } yield g(f(x), y)</pre>
---	---

and similarly for `cont2` instead of `cont1`; now rewrite in terms of `map2`:

- **Left naturality** for `map2`:

```
map2(cont1.map(f), cont2)(g)  
= map2(cont1, cont2){ (x, y) ⇒ g(f(x), y) }
```

- **Right naturality** for `map2`:

```
map2(cont1, cont2.map(f))(g)  
= map2(cont1, cont2){ (x, y) ⇒ g(x, f(y)) }
```

Associativity and identity laws for `map2`

Inline two generators out of three, in two different ways:

```
for {
  x ← cont1
  (y, z) ← for {
    yy ← cont2
    zz ← cont3
  } yield (yy, zz)
} yield g(x, y, z)

for {
  (x, y) ← for {
    xx ← cont1
    yy ← cont2
  } yield (xx, yy)
  z ← cont3
} yield g(x, y, z)
```

Write this in terms of `map2` to obtain the **associativity law** for `map2`:

```
map2(cont1, map2(cont2, cont3)((_,_)) { case(x,(y,z)) ⇒ g(x,y,z) })
= map2(map2(cont1, cont2)((_,_)), cont3) { case((x,y),z) ⇒ g(x,y,z) }
```

Empty context precedes a generator, or follows a generator:

```
for { x ← pure(a)
      y ← cont
    } yield g(x, y)

for {
  y ← cont
} yield g(a, y)
```

Write this in terms of `map2` to obtain the **identity laws** for `map2` and `pure`:

```
map2(pure(a), cont)(g) = cont.map { y ⇒ g(a, y) }
map2(cont, pure(b))(g) = cont.map { x ⇒ g(x, b) }
```

Deriving the laws for `zip`: naturality law

- The laws for `map2` in a short notation; here $f \otimes g \equiv \{a \times b \Rightarrow f(a) \times g(b)\}$

$$\text{fmap2} \left(g^{A \times B \Rightarrow C} \right) \left(f^\uparrow q_1 \times q_2 \right) = \text{fmap2} \left((f \otimes \text{id}) \circ g \right) (q_1 \times q_2)$$

$$\text{fmap2} \left(g^{A \times B \Rightarrow C} \right) \left(q_1 \times f^\uparrow q_2 \right) = \text{fmap2} \left((\text{id} \otimes f) \circ g \right) (q_1 \times q_2)$$

$$\text{fmap2} (g_{1.23}) (q_1 \times \text{fmap2} (\text{id}) (q_2 \times q_3)) = \text{fmap2} (g_{12.3}) (\text{fmap2} (\text{id}) (q_1 \times q_2) \times q_3)$$

$$\text{fmap2} \left(g^{A \times B \Rightarrow C} \right) \left(\text{pure } a^A \times q_2^{F^B} \right) = (b \Rightarrow g(a \times b))^\uparrow q_2$$

$$\text{fmap2} \left(g^{A \times B \Rightarrow C} \right) \left(q_1^{F^A} \times \text{pure } b^B \right) = (a \Rightarrow g(a \times b))^\uparrow q_1$$

- Express `map2` through `zip`:

$$\text{fmap}_2 g^{A \times B \Rightarrow C} \left(q_1^{F^A} \times q_2^{F^B} \right) \equiv \left(\text{zip} \circ g^\uparrow \right) (q_1 \times q_2)$$

$$\text{fmap}_2 g^{A \times B \Rightarrow C} \equiv \text{zip} \circ g^\uparrow$$

- Combine the two naturality laws into one by using two functions f_1, f_2 :

$$(f_1^\uparrow \otimes f_2^\uparrow) \circ \text{fmap2 } g = \text{fmap2} \left((f_1 \otimes f_2)^\uparrow \circ g \right)$$

$$(f_1^\uparrow \otimes f_2^\uparrow) \circ \text{zip} \circ g^\uparrow = \text{zip} \circ (f_1 \otimes f_2)^\uparrow \circ g^\uparrow$$

- The **naturality law** for `zip` then becomes: $(f_1^\uparrow \otimes f_2^\uparrow) \circ \text{zip} = \text{zip} \circ (f_1 \otimes f_2)^\uparrow$

Deriving the laws for `zip`: associativity law

- Express `map2` through `zip` and substitute into the associativity law:

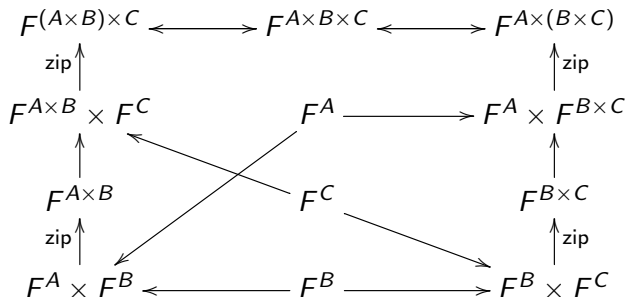
$$g_{1.23}^{\uparrow}(\text{zip}(q_1 \times \text{zip}(q_2 \times q_3))) = g_{12.3}^{\uparrow}(\text{zip}(\text{zip}(q_1 \times q_2) \times q_3))$$

- The arbitrary function g is preceded by transformations of the tuples,

$$a \times (b \times c) \equiv (a \times b) \times c \quad (\text{type isomorphism})$$

- Assume that the isomorphism transformations are applied as needed, then we may formulate the **associativity law** for `zip` more concisely:

$$\text{zip}(\text{zip}(q_1 \times q_2) \times q_3) \cong \text{zip}(q_1 \times \text{zip}(q_2 \times q_3))$$



Deriving the laws for `zip`: identity laws

- Identity laws seem to be complicated, e.g. the left identity:

$$g^\uparrow (\text{zip} (\text{pure } a \times q)) = (b \Rightarrow g (a \times b))^\uparrow q$$

- Replace `pure` by an *equivalent* “wrapped unit” method `wu: F[Unit]`

$$\text{wu}^{F^1} \equiv \text{pure}(1); \quad \text{pure}(a^A) = (1 \Rightarrow a)^\uparrow \text{wu}$$

Then the left identity law can be simplified using left naturality:

$$g^\uparrow (\text{zip} (((1 \Rightarrow a)^\uparrow \text{wu}) \times q)) = g^\uparrow (((1 \Rightarrow a) \otimes \text{id})^\uparrow \text{zip} (\text{wu} \times q))$$

- Denote $\phi^{B \Rightarrow 1 \times B} \equiv b \Rightarrow 1 \times b$ and $\beta_a^{1 \times B \Rightarrow A \times B} \equiv (1 \Rightarrow a) \otimes \text{id}$; then the function $b \Rightarrow g (a \times b)$ can be expressed more simply as $\phi \circ \beta_a \circ g$, and the identity law becomes

$$g^\uparrow (\beta_a^\uparrow \text{zip} (\text{wu} \times q)) = (\beta_a \circ g)^\uparrow (\text{zip} (\text{wu} \times q)) = (\phi \circ \beta_a \circ g)^\uparrow q = (\beta_a \circ g)^\uparrow (\phi^\uparrow q)$$

Omitting the common prefix $(\beta_a \circ g)^\uparrow$, we obtain the **left identity law**:

$$\text{zip} (\text{wu} \times q) = \phi^\uparrow q$$

- Note that ϕ^\uparrow is an isomorphism between F^B and $F^{1 \times B}$
 - ★ Assume that this isomorphism is applied as needed, then we may write

$$\text{zip} (\text{wu} \times q) \cong q$$

- Similarly, the **right identity law** can be written as $\text{zip} (q \times \text{wu}) \cong q$

Similarity between applicative laws and monoid laws

- Define infix syntax for `zip` and write $\text{zip}(p \times q) \equiv p \bowtie q$
- Then the associativity and identity laws may be written as

$$q_1 \bowtie (q_2 \bowtie q_3) \cong (q_1 \bowtie q_2) \bowtie q_3$$

$$(w u \bowtie q) \cong q$$

$$(q \bowtie w u) \cong q$$

These are the laws of a monoid (with some assumed transformations)

- Naturality law for `zip` written in the infix syntax:

$$f_1^\uparrow q_1 \bowtie f_2^\uparrow q_2 = (f_1 \otimes f_2)^\uparrow (q_1 \bowtie q_2)$$

- `wu` has no laws; the naturality for `pure` follows automatically
- The laws are simplest when formulated in terms of `zip` and `wu`
 - ▶ Naturality for `zip` will usually follow from parametricity
 - ★ A third naturality law for `map2` follows from defining `map2` through `zip`!
- “Zippable” functors have only the associativity and naturality laws
- Applicative functors are a strict superset of monadic functors
 - ▶ There are applicative functors that *cannot* be monads
 - ▶ Applicative functor implementation may disagree with the monad

A third naturality law for `map2`

- There must be one more naturality law for `map2`
- Transform the result of a `map2`:

<pre>(for { x ← cont1 y ← cont2 } yield g(x, y)).map(f)</pre>	<pre>for { x ← cont1 y ← cont2 } yield f(g(x, y))</pre>
---	---

- Write this in terms of `map2`, obtain a third naturality law:

```
map2(cont1, cont2)(g).map(f)  
= map2(cont1, cont2)(g andThen f)
```

$$\text{fmap2}(g) \circ f^\uparrow = \text{fmap2}(g \circ f)$$

$$f^\uparrow(\text{fmap2}(g)(p \times q)) = \text{fmap2}(g \circ f)(p \times q)$$

- This law automatically follows if we define `map2` through `zip`:

$$\text{fmap2}(g) \circ f^\uparrow = \text{zip} \circ g^\uparrow \circ f^\uparrow = \text{zip} \circ (g \circ f)^\uparrow$$

- Note: We always have one naturality law per type parameter

Applicative operation `ap` as a “lifting”

- Consider `ap` as a “lifting” since it has type $F^{A \Rightarrow B} \Rightarrow (F^A \Rightarrow F^B)$
- A “lifting” should obey the identity and the composition laws
 - An “identity” value of type $F^{A \Rightarrow A}$, mapped to $\text{id}^{F^A \Rightarrow F^A}$ by `ap`
 - ★ A good candidate for that value is $\text{id}_\odot \equiv \text{pure}(\text{id}^{A \Rightarrow A})$
 - A “composition” of an $F^{A \Rightarrow B}$ and an $F^{B \Rightarrow C}$, yielding an $F^{A \Rightarrow C}$
 - ★ We can use `map2` to implement this composition, denoted $g \odot h$:

$$g^{F^{A \Rightarrow B}} \odot h^{F^{B \Rightarrow C}} \equiv \text{fmap2}(p^{A \Rightarrow B} \times q^{B \Rightarrow C} \Rightarrow p \circ q)(g, h)$$

- What are the laws that follow for $g \odot h$ from the `map2` laws?

$$\text{id}_\odot \odot h = h; \quad g \odot \text{id}_\odot = g$$

$$g^{F^{A \Rightarrow B}} \odot (h^{F^{B \Rightarrow C}} \odot k^{F^{C \Rightarrow D}}) = (g \odot h) \odot k$$

$$\left((x^{B \Rightarrow C} \Rightarrow f^{A \Rightarrow B} \circ x)^\uparrow g^{F^{B \Rightarrow C}} \right) \odot h^{F^{C \Rightarrow D}} = (x^{B \Rightarrow D} \Rightarrow f^{A \Rightarrow B} \circ x)^\uparrow (g \odot h)$$

$$g^{F^{A \Rightarrow B}} \odot \left((x^{B \Rightarrow C} \Rightarrow x \circ f^{C \Rightarrow D})^\uparrow h^{F^{B \Rightarrow C}} \right) = (x^{A \Rightarrow C} \Rightarrow x \circ f^{C \Rightarrow D})^\uparrow (g \odot h)$$

- The first 3 laws are the identity & associativity laws of a *category*
 - ★ The morphism type is $A \rightsquigarrow B \equiv F^{A \Rightarrow B}$, the composition is \odot
- The last 2 laws are naturality laws, connecting `fmap` and \odot
- Therefore `ap` is a functor’s “lifting” of morphisms from two categories

Deriving the category laws for (id_\odot, \odot)

The five laws for id_\odot and \odot follow from the five `map2` laws

- Consider $\text{id}_\odot \odot h$ and substitute the definition of \odot via `map2`, cf. slide 7:
 $\text{id}_\odot \odot h = \text{fmap2} (p \times q \Rightarrow p \circ q) (\text{pure}(\text{id}) \times h) = (b \Rightarrow \text{id} \circ b)^\uparrow h = h$
- The law $g \odot \text{id}_\odot = g$ is derived similarly
- Associativity law: $g \odot (h \odot k) = \text{fmap2}(\circ) (g \times \text{fmap2}(\circ) (h \times k))$ The 3rd naturality law gives: $\text{fmap2}(\circ) (h \times k) = (\circ)^\uparrow (\text{fmap2}(\text{id}) (h \times k))$, and then:

$$\begin{aligned} g \odot (h \odot k) &= \text{fmap2} (x \times (y \times z) \Rightarrow x \circ y \circ z) (g \times \text{fmap2}(\text{id}) (h \times k)) \\ (g \odot h) \odot k &= \text{fmap2} ((x \times y) \times z \Rightarrow x \circ y \circ z) (\text{fmap2}(\text{id}) (g \times h) \times k) \end{aligned}$$

Now the associativity law for `fmap2` yields $g \odot (h \odot k) = (g \odot h) \odot k$

- Derive naturality laws for \odot from the three `map2` naturality laws:
 $((x \Rightarrow f \circ x)^\uparrow g) \odot h = \text{fmap2}(\circ) ((x \Rightarrow f \circ x)^\uparrow g \times h) =$
 $\text{fmap2} (x \times y \Rightarrow f \circ x \circ y) (g \times h) = (x \Rightarrow f \circ x)^\uparrow (\text{fmap2}(\circ) (g \times h)) =$
 $(x \Rightarrow f \circ x)^\uparrow (g \odot h)$
- The law is $g \odot (x \Rightarrow x \circ f)^\uparrow h = (x \Rightarrow x \circ f)^\uparrow (g \odot h)$ is derived similarly

Deriving the functor laws for ap

Now that we established the laws for \odot , we have ap laws:

$$\text{ap}^{[B,Z]} : F^{B \Rightarrow Z} \Rightarrow F^B \Rightarrow F^Z = \text{fmap}_2 \left(\text{id}^{(B \Rightarrow Z) \Rightarrow (B \Rightarrow Z)} \right)$$

Identity law: $\text{ap}(\text{id}_{\odot}) = \text{id}^{F^A \Rightarrow F^A}$

- Derivation: $\text{ap}(\text{id}_{\odot}^{F^A \Rightarrow A})(q^{F^A}) = \text{fmap}_2(\text{id}^{(A \Rightarrow A) \Rightarrow A \Rightarrow A})(\text{pure}(\text{id}^{A \Rightarrow A}))(q^{F^A}) = \text{fmap}_2(f \times x \Rightarrow f(x))(\text{pure}(\text{id}) \times q) = (x \Rightarrow \text{id}(x))^{\uparrow} q = \text{id}^{\uparrow} q = q$
- Easier derivation: first, express ap via \odot using the isomorphisms

$$A \cong 1 \Rightarrow A; \quad F^A \cong F^{1 \Rightarrow A}$$

Then $\text{ap}(p^{F^B \Rightarrow Z})(q^{F^B}) \cong q^{F^{1 \Rightarrow B}} \odot p^{F^{B \Rightarrow Z}}$ and so $\text{ap}(\text{id}_{\odot})(q) \cong q \odot \text{id}_{\odot} = q$

Composition law: $\text{ap}(g \odot h) = \text{ap}(g) \circ \text{ap}(h)$

- Derivation: use $\text{ap } p \, q \cong q \odot p$ to get $\text{ap}(g \odot h)(q) \cong q \odot (g \odot h)$ while $(\text{ap}(g) \circ \text{ap}(h)) \, q = \text{ap}(h)(\text{ap}(g)(q)) \cong \text{ap}(h)(q \odot g) \cong (q \odot g) \odot h$

Constructions of applicative functors

- All monadic constructions still hold for applicative functors
 - Additionally, there are some non-monadic constructions
- 1 $F^A \equiv 1$ (constant functor) and $F^A \equiv A$ (identity functor)
 - 2 $F^A \equiv G^A \times H^A$ for any applicative G^A and H^A
 - ▶ but $G^A + H^A$ is in general *not* applicative
 - 3 $F^A \equiv A + G^A$ for any applicative G^A (**free pointed** over G)
 - 4 $F^A \equiv A + G^{F^A}$ (recursive) for *any* functor G^A (**free monad** over G)
 - 5 $F^A \equiv H^A \Rightarrow A$ for *any* contrafunctor H^A

Constructions that do not correspond to monadic ones:

- 6 $F^A \equiv Z$ (constant functor, Z a monoid)
 - 7 $F^A \equiv Z + G^A$ for any applicative G^A and monoid Z
 - 8 $F^A \equiv G^{H^A}$ when both G and H are applicative
- Applicative that disagrees with its monad: $F^A \equiv 1 + (1 \Rightarrow A \times F^A)$
 - Examples of non-applicative functors: $F^A \equiv (P \Rightarrow A) + (Q \Rightarrow A)$,
 $F^A \equiv (A \Rightarrow P) \Rightarrow Q$, $F^A \equiv (A \Rightarrow P) \Rightarrow 1 + A$

All non-parameterized exp-poly types are monoids

- Using known monoid constructions (Chapter 7), we can implement $X + Y$, $X \times Y$, $X \Rightarrow Y$ as monoids when X and Y are monoids
- All primitive types have at least one monoid instance:
 - ▶ `Int`, `Float`, `Double`, `Char`, `Boolean` are “numeric” monoids
 - ▶ `Seq[A]`, `Set[A]`, `Map[K,V]` are set-like monoids
 - ▶ `String` is equivalent to a sequence of integers; `Unit` is a trivial monoid
- Therefore, all exponential-polynomial types without type parameters are monoids in at least one way
- Example of an exponential-polynomial type without type parameters:
 $\text{Int} + \text{String} \times \text{String} \times (\text{Int} \Rightarrow \text{Bool}) + (\text{Bool} \times \text{String} \Rightarrow 1 + \text{String})$
- Example of a non-monoid type with type parameters: $A \Rightarrow B$

By constructions 1, 2, 6, 7, *all* polynomial F^A with monoidal coefficients are applicative: write $F^A = Z_1 + A \times (Z_2 + A \times \dots)$ with some monoids Z_i

- Examples: $F^A = 1 + A \times A$ (this F^A cannot be a monad!)
- $F^A = A + A \times A \times Z$ where Z is a monoid (this F^A is a monad)

Previous examples of non-applicative functors are all *non-polynomial*

Definition and constructions of applicative contrafunctors

- The applicative functor laws, if formulated via `zip` and `wu`, do not use `map` and therefore can be formulated for contrafunctors
- Define an **applicative contrafunctor** C^A as having `zip` and `wu`:

$$\text{zip} : C^A \times C^B \Rightarrow C^{A \times B}; \quad \text{wu} : C^1$$

- Identity and associativity laws must hold for `zip` and `wu`
 - ▶ Note: applying `contramap` to the function $a \times b \Rightarrow a$ will yield some $C^A \Rightarrow C^{A \times B}$, but this will *not* give a valid implementation of `zip`!
- Naturality must hold for `zip`, but with `contramap` instead of `map`
 - ▶ There are no corresponding `pure` or `contraap`! But have $\forall A : C^A$

Applicative contrafunctor constructions:

- 1 $C^A \equiv Z$ (constant functor, Z a monoid)
 - 2 $C^A \equiv G^A \times H^A$ for any applicative contrafunctors G^A and H^A
 - 3 $C^A \equiv G^A + H^A$ for any applicative contrafunctors G^A and H^A
 - 4 $C^A \equiv H^A \Rightarrow G^A$ for *any functor* H^A and applicative contrafunctor G^A
 - 5 $C^A \equiv G^{H^A}$ if a functor G^A and contrafunctor H^A are both applicative
- All exponential-polynomial contrafunctors with monoidal coefficients are applicative! (These constructions cover all exp-poly cases.)

Definition and laws of profunctors

- **Profunctors** have the type parameter in both contravariant and covariant positions; they can have neither `map` nor `contramap`
- Examples of profunctors: $P^A \equiv 1 + \text{Int} \times A \Rightarrow A$; $P^A \equiv A + (A \Rightarrow \text{String})$
- Example of non-profunctor: a GADT, $F^A \equiv \text{String}^{F^{\text{Int}}} + \text{Int}^{F^1}$

```
sealed trait F[A]
final case class F1(s: String) extends F[Int]
final case class F2(i: Int) extends F[Unit]
```

- Rigorously: P^A is a profunctor if a type function $Q^{X,Y}$ exists which is a contrafunctor in X and a functor in Y , and such that $P^A \equiv Q^{A,A}$
- Profunctors have `xmap` of type $(A \Rightarrow B) \times (B \Rightarrow A) \Rightarrow (P^A \Rightarrow P^B)$
- Identity law: `xmap(id, id) = id`
- Composition law: `xmap(f1, g1) ∘ xmap(f2, g2) = xmap(f1 ∘ f2, g2 ∘ g1)`
- ▶ both `xmap` and the laws follow from the functor and contrafunctor laws
- All exp-poly type constructors are profunctors since the type parameter is always in either a covariant or a contravariant position

Definition and constructions of applicative profunctors

- Definition of **applicative profunctor**: has `zip` and `wu` with the laws
 - ▶ There is no corresponding `ap`! But have `pure` : $A \Rightarrow P^A$

Applicative profunctors admit all previous constructions, and in addition:

- 1 $P^A \equiv G^A \times H^A$ for any applicative profunctors G^A and H^A
- 2 $P^A \equiv Z + G^A$ for any applicative profunctor G^A and monoid Z
- 3 $P^A \equiv A + G^A$ for any applicative profunctor G^A
- 4 $P^A \equiv F^A \Rightarrow Q^A$ for *any functor* F^A and applicative profunctor Q^A
 - ▶ Non-working construction: $P^A \equiv H^A \Rightarrow A$ for a profunctor H^A
- 5 $P^A \equiv G^{H^A}$ for a functor G^A and a profunctor H^A , both applicative

Commutative applicative functors

- The monoidal operation \oplus can be **commutative** w.r.t. its arguments:

$$x \oplus y = y \oplus x$$

- Applicative operation **zip** can be **commutative** w.r.t. its arguments:

$$(a \times b \Rightarrow b \times a)^\uparrow (fa \bowtie fb) = fb \bowtie fa$$

or $fa \bowtie fb \cong fb \bowtie fa$, implicitly using the isomorphism $a \times b \Rightarrow b \times a$

- Applicative functor is commutative if the second effect is independent of the first effect (not only of the first value)
- Examples:
 - ▶ List is commutative; applicative parsers are not
 - ▶ If defined through the monad instance, **zip** is usually not commutative
 - ▶ All polynomial functors with *commutative* monoidal coefficients are commutative applicative functors
- Most applicative constructions preserve commutativity
- The same applies to applicative contrafunctors and profunctors
- Commutativity makes proving associativity easier:

$$(fa \bowtie fb) \bowtie fc \cong fc \bowtie (fb \bowtie fa)$$

so it's sufficient to swap fa and fc and show equivalence

Categorical overview of “regular” functor classes

The “liftings” show the types of category’s morphisms

class name	lifting’s name and type signature	category’s morphism
functor	$\text{fmap} : (A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$	$A \Rightarrow B$
filterable	$\text{fmapOpt} : (A \Rightarrow 1 + B) \Rightarrow F^A \Rightarrow F^B$	$A \Rightarrow 1 + B$
monad	$\text{flm} : (A \Rightarrow F^B) \Rightarrow F^A \Rightarrow F^B$	$A \Rightarrow F^B$
applicative	$\text{ap} : F^{A \Rightarrow B} \Rightarrow F^A \Rightarrow F^B$	$F^{A \Rightarrow B}$
contrafunctor	$\text{contrafmap} : (B \Rightarrow A) \Rightarrow F^A \Rightarrow F^B$	$B \Rightarrow A$
profunctor	$\text{xmap} : (A \Rightarrow B) \times (B \Rightarrow A) \Rightarrow F^A \Rightarrow F^B$	$(A \Rightarrow B) \times (B \Rightarrow A)$
contra-filterable	$\text{contrafmapOpt} : (B \Rightarrow 1 + A) \Rightarrow F^A \Rightarrow F^B$	$B \Rightarrow 1 + A$
Not yet considered:		
comonad	$\text{coflm} : (F^A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$	$F^A \Rightarrow B$

Need to define each category’s composition and identity morphism

Then impose the category laws, the naturality laws, and the functor laws

- Obtained a systematic picture of the “regular” type classes
- Some classes (e.g. contra-applicative) aren’t covered by this scheme
- Some of the possibilities (e.g. “contramonad”) don’t actually work out

Exercises

- ➊ Show that `pure` will be automatically a natural transformation when it is defined using `wu` as shown in the slides.
- ➋ Use naturality of `pure` to show that $\text{pure } f \odot \text{pure } g = \text{pure } (f \circ g)$
- ➌ Show that $F^A \equiv (A \Rightarrow Z) \Rightarrow (1 + A)$ is a functor but not applicative.
- ➍ Show that P^S is a monoid if S is a monoid and P is any applicative functor, contrafunctor, or profunctor.
- ➎ Implement an applicative instance for $F^A = 1 + \text{Int} \times A + A \times A \times A$.
- ➏ Using applicative constructions, show without lengthy proofs that $F^A = G^A + H^{G^A}$ is applicative if G and H are applicative functors.
- ➐ Explicitly implement contrafunctor construction 2 and prove the laws.
- ➑ For any contrafunctor H^A , construction 5 says that $F^A \equiv H^A \Rightarrow A$ is applicative. Implement the code of `zip(fa, fb)` for this construction.
- ➒ Show that the recursive functor $F^A \equiv 1 + G^{A \times F^A}$ is applicative if G^A is applicative and wu_F is defined recursively as $0 + \text{pure}_G (1 \times \text{wu}_F)$.
- ➓ Explicitly implement profunctor construction 5 and prove the laws.
- ➑ Prove rigorously that all exponential-polynomial type constructors are profunctors.
- ➒ Implement profunctor and applicative instances for $P^A \equiv A + Z \times G^A$ where G^A is a given applicative profunctor and Z is a monoid.
- ➓ Show that, for any profunctor P^A , one can implement a function of type $A \Rightarrow P^B \Rightarrow P^{A \times B}$ but not of type $A \Rightarrow P^B \Rightarrow P^A$.