Correlation coefficient of X and Y

Karl Pearson's Coefficient of Correlation:

Correlation co-efficient between two variables X and Y, is a numerical measure of linear relationship between them, usually denoted by $\rho(X,Y)$ or ρ_{XY} is defined by

$$\rho(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{Cov(X,Y)}{\sqrt{V(X)}\cdot\sqrt{V(Y)}}$$

where $\sigma_{XY} = Cov(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$

$$\sigma_X^2 = V(X) = E\left[(X-E(X))\right]^2$$

$$\sigma_Y^2 = V(Y) = E\left[(Y-E(Y))\right]^2$$

Note:-

1. $\rho(X,Y)$ provides a measure of linear relationship between X and Y.
   For non-linear relationship, however, it is not suitable.
2. Karl Pearson's Correlation Co-efficient is also called Product-moment correlation co-efficient
3. $-1 \le \rho(X,Y) \le 1$
   * If $\rho = -1$, the correlation is perfect and -ve
   * If $\rho = 1$, the correlation is perfect and +ve

4. Correlation co-efficient is independent of Change of Origin and Scale

i.e if $U = \frac{X-a}{h}$, $V = \frac{Y-b}{K}$ then

$$\rho(U,V) = \rho(X,Y)$$

=x=

Problems–

The j.p.m.f of (X,Y) is given below:

Y \ X	−1	1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$

find
$E(X)$
$E(X^2)$
$V(X) = \sigma_X^2$
$E(Y)$
$E(Y^2)$
$V(Y) = \sigma_Y^2$
$E(XY) =$

Find the correlation co-efficient between X and Y

Solution:–

Y \ X	−1	1	$P_2(y)$
0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$
1	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{2}$
$P_1(x)$	$\frac{3}{8}$	$\frac{5}{8}$	1

From table

$$P_1(x) = \sum_{y=0}^{1} P(x,y) = P(x,0) + P(x,1)$$

$$P_1(-1) = P(-1,0) + P(-1,1) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

lly $$P_1(1) = \frac{5}{8}$$

$$\sum P_1(x) = P_1(-1) + P_1(1) = \frac{3}{8} + \frac{5}{8} = 1$$

$$P_2(y) = \sum_{x=-1}^{1} P(x,y) = P(-1,y) + P(1,y)$$

$$P_2(0) = P(-1,0) + P(1,0) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$P_2(1) = P(-1,1) + P(1,1) = \frac{2}{8} + \frac{2}{8} = \frac{1}{2}$$

$$\sum P_2(y) = P_2(0) + P_2(1) = \frac{1}{2} + \frac{1}{2} = 1$$

Now $$E(x) = \sum_{-1}^{1} x\, P_1(x)$$

$$= -1\, P_1(-1) + 1\, P_1(1)$$

$$= -1\left(\frac{3}{8}\right) + 1 \cdot \frac{5}{8} = \frac{2}{8} = \frac{1}{4}$$

$$E(x^2) = \sum_{-1}^{1} x^2 P_1(x)$$

$$= (-1)^2 P_1(-1) + 1^2 P_1(1)$$

$$= 1\left(\frac{3}{8}\right) + 1 \cdot \frac{5}{8} = 1$$

$$V(X) = E(x^2) - [E(x)]^2$$

$$= 1 - \left(\frac{1}{4}\right)^2$$

$$= 1 - \frac{1}{16}$$

$$\sigma_x^2 = V(x) = \frac{15}{16}$$

$$\sigma_x = \sqrt{\frac{15}{16}}$$

$$E[Y] = \sum_{y=0}^{1} y P_2(y)$$

$$= 0 P_2(0) + 1 P_2(1)$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

$$E[Y^2] = \sum_{y=0}^{1} y^2 P_2(y)$$

$$= 0^2 P_2(0) + 1^2 P_2(1)$$

$$= \frac{1}{2}$$

$$V(Y) = \sigma_Y^2 = E[Y^2] - \{E[Y]\}^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E[XY] = \sum_{x=-1}^{1} \sum_{y=0}^{1} xy\, p(x,y)$$

$$= \sum_{x=-1}^{1} \sum_{y=0}^{1} xy\, p(x,y)$$

$$= \sum_{x=-1}^{1} \left[x(0) \cdot p(x,0) + x(1)\, p(x,1) \right]$$

$$= \sum_{x=-1}^{1} x\, p(x,1)$$

$$= (-1)\, p(-1,1) + 1\, p(1,1)$$

$$= -1 \frac{2}{8} + \frac{2}{8}$$

$$= 0$$

$$\operatorname{cov}(X,Y) = E[XY] - E[X] \cdot E[Y]$$

$$= 0 - \frac{1}{4} \times \frac{1}{2} = -\frac{1}{8}$$

$$\therefore \sigma_{XY} = \operatorname{cov}(X,Y) = -\frac{1}{8}$$

Correlation co-efficient

$$\rho(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)} \cdot \sqrt{V(Y)}} = \frac{-\frac{1}{8}}{\sqrt{\frac{15}{16}} \cdot \sqrt{\frac{1}{4}}}$$

$$= \frac{-\frac{1}{8}}{\sqrt{\frac{15}{64}}}$$

$$= \frac{-\frac{1}{8}}{\frac{\sqrt{15}}{8}}$$

$$\rho = -\frac{1}{\sqrt{15}} = -0.2582$$

Problem:- 2

Two r.v X and Y have the j.p.d.f

$$f(x,y) = \begin{cases} 2-x-y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$$

Find correlation co-efficient between X & Y.

Solution:-

Given $f(x,y) = \begin{cases} 2-x-y, & 0<x<1, 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$

M. P.d.f. of X

$$f_1(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 (2-x-y)\,dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= \left[2 - x - \frac{1}{2}\right]$$

$$= \frac{3}{2} - x$$

<u>M.P. d.f of Y</u> :-

$$f_2(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$= \left(2x - \frac{x^2}{2} - xy\right)_0^1$$

$$= \left(2 - \frac{1}{2} - y\right)$$

$$= \frac{3}{2} - y$$

$\therefore \quad f_1(x) = \frac{3}{2} - x$

$f_2(y) = \frac{3}{2} - y$

To find $E(X)$, $E(Y)$

$$E(X) = \int_0^1 x f_1(x)\,dx$$

$$= \int_0^1 x\left(\frac{3}{2} - x\right)dx$$

$$= \int_0^1 \frac{3}{2}x\,dx - \int_0^1 x^2\,dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2}\right) - \left(\frac{1}{3}\right)$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

Ily $E(Y) = \frac{5}{12}$

To find $E(XY)$

$$E(XY) = \int_0^1 \int_0^1 xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 xy\,(2-x-y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 \left(2xy - x^2y - xy^2\right)dx\,dy$$

$$= \int_0^1 \left[\int_0^1 (2xy - x^2 y - xy^2) \, dy \right] dx$$

$$= \int_0^1 \left[2x \left(\frac{y^2}{2} \right)_0^1 - x^2 \left(\frac{y^2}{2} \right)_0^1 - x \left(\frac{y^3}{3} \right)_0^1 \right] dx$$

$$= \int_0^1 \left(x - \frac{x^2}{2} - \frac{x}{3} \right) dx$$

$1 - \frac{1}{3}$

$$= \int_0^1 \left(\frac{2}{3} x - \frac{x^2}{2} \right) dx$$

$$= \frac{2}{3} \left(\frac{x^2}{2} \right)_0^1 - \left(\frac{x^3}{6} \right)_0^1$$

$$= \frac{\cancel{2}}{3} \left(\frac{1}{\cancel{2}} \right) - \frac{1}{6}$$

$$= \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}$$

$$\therefore \; E(XY) = \frac{1}{6}$$

$$\therefore \; E(X) = \frac{5}{12}, \quad E(Y) = \frac{5}{12}, \quad E(XY) = \frac{1}{6}$$

$\sigma_{xy} = cov(x,y) = E(xy) - E(x)\cdot E(y)$

$$cov(x,y) = E(xy) - E(x)E(y)$$

$$= \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12}$$

$$= \frac{1}{6} - \frac{25}{144}$$

$$= \frac{24-25}{144} = -\frac{1}{144}$$

To find Variance of x and y :-

$$V(x) = \sigma_x^2 = E(x^2) - [E(x)]^2$$

Now $E(x^2) = \int_0^1 x^2 f_1(x)\,dx$

$$= \int_0^1 x^2\left(\frac{3}{2} - x\right)dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 - x^3\right)dx$$

$$= \frac{\cancel{3}}{2}\left(\frac{x^3}{\cancel{3}}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4} = \frac{1}{4}$$

lly $E(y^2) = \frac{1}{4}$.

$$V(x) = \sigma_x^2 = E(x^2) - [E(x)]^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2$$

$$= \frac{1}{4} - \frac{25}{144}$$

$$= \frac{36 - 25}{144}$$

$$V(x) = \sigma_x^2 = \frac{11}{144} \Rightarrow \sigma_x = \sqrt{V(x)}$$

$$\sigma_x = \sqrt{\frac{11}{144}}$$

lly $\sigma_y = \sqrt{\frac{11}{144}}$

correlation co-efficient

$$\rho(x,y) = \frac{Cov(x,y)}{\sqrt{V(x)} \cdot \sqrt{V(y)}}$$

$$= \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}} \cdot \sqrt{\frac{11}{144}}} = \frac{-\frac{1}{\cancel{144}}}{\frac{11}{\cancel{144}}}$$

$$f = \frac{-1}{11}$$

$$= x =$$