-: Normal Distribution:-

Normal Distribution:-

The continuous Random Variable X is said to have a normal distribution, if its p.d.f is defined by

$$f(x) = c\, e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} \text{ for } -\infty < x < \infty.$$

It involves two parameter a & b, the constant c is determined by the condition that total area under the curve is unity

To find the value c :-

$$\text{P.d.f } f(x) = c\, e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}, \quad -\infty < x < \infty$$

We know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} c\, e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx = 1$$

put $\frac{x-a}{b} = z$

$$x - a = bz$$
$$dx = b\, dz$$

$$\Rightarrow c\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, b\, dz = 1$$

$$\Rightarrow cb\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz = 1$$

$\left(\because e^{-\frac{z^2}{2}} \text{ is even function}\right)$

$$\Rightarrow 2cb\int_{0}^{\infty} e^{-\frac{z^2}{2}}\, dz = 1$$

put $\frac{z^2}{2} = t$

$z^2 = 2t$

$z = \sqrt{2t}$

$dz = \sqrt{2}\cdot\frac{1}{2\sqrt{t}}\,dt$

$= \frac{1}{\sqrt{2t}}\,dt$

$\Gamma(n) = \int_0^{\infty} e^{-x}\cdot x^{n-1}\,dx$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Rightarrow 2cb\int_0^{\infty} e^{-t}\,\frac{1}{\sqrt{2}\cdot\sqrt{t}}\,dt = 1$$

$$\Rightarrow \frac{2cb}{\sqrt{2}}\int_0^{\infty} e^{-t}\cdot t^{-\frac{1}{2}}\,dt = 1$$

$$\Rightarrow \sqrt{2}\,cb\int_0^{\infty} e^{-t}\,t^{\frac{1}{2}-1}\,dt = 1$$

$$\Rightarrow \sqrt{2}\,cb\,\Gamma(\tfrac{1}{2}) = 1$$

$$\Rightarrow \sqrt{2}\,cb\,\sqrt{\pi} = 1$$

$$\Rightarrow c = \frac{1}{b\sqrt{2\pi}}$$

$$\boxed{\therefore c = \frac{1}{\sqrt{2\pi}\cdot b}}$$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}, \quad -\infty < x < \infty$$

Mean of the Normal Distribution:-

$$E(X) = \text{Mean} = \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi} \cdot b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$$

put $\frac{x-a}{b} = z$

$x - a = bz$, $x = a + bz$

$dx = b\,dz$

$$= \frac{1}{\sqrt{2\pi} \cdot \cancel{b}} \int_{-\infty}^{\infty} (bz + a)\, e^{-\frac{z^2}{2}} \cancel{b}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} b z\, e^{-\frac{z^2}{2}} dz + a \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \right]$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = \begin{cases} 2 \int_0^a f(x)\,dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd} \end{cases}}$$

$$= \frac{1}{\sqrt{2\pi}} \left[2a \int_0^{\infty} e^{-\frac{z^2}{2}} dz \right]$$

$$= \frac{\sqrt{2}\, a}{\sqrt{\pi}} \int_0^{\infty} e^{-\frac{z^2}{2}} dz$$

put $\frac{z^2}{2}=t$

$$dz = \frac{1}{\sqrt{2t}}\,dt$$

$$= \frac{\cancel{\sqrt{2}}\,a}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t}\cdot t^{-1/2}}{\cancel{\sqrt{2}}}\,dt$$

$$= \frac{a}{\sqrt{\pi}} \int_0^\infty e^{-t}\cdot t^{\frac{1}{2}-1}\,dt$$

$$= \frac{a}{\cancel{\sqrt{\pi}}}\cdot\cancel{\sqrt{\pi}} \qquad (\because \Gamma(\tfrac{1}{2})=\sqrt{\pi})$$

$$\mu = a$$

$$\therefore\ f(x) = \frac{1}{\sqrt{2\pi}\cdot b}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{b}\right)^2} \qquad -\infty<x<\infty$$

Variance of the normal distribution:-

Note: 1. $V(x) = E(x^2) - [E(x)]^2$

2. $V(x) = E[(x-\mu)^2]$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu \int_{-\infty}^{\infty} x f(x)dx +$$

$$\mu^2 \int_{-\infty}^{\infty} f(x)\,dx$$

$$= E(x^2) - 2\mu \cdot E(x) + \mu^2 (1)$$

$$= E(x^2) - 2\mu \cdot \mu + \mu^2$$

$$= E(x^2) - 2\mu^2 + \mu^2$$

$$= E(x^2) - \mu^2 = E(x^2) - [E[x]]^2$$

$$= \times =$$

$$V(x) = E((x-\mu)^2)$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi} \cdot b} e^{-\frac{1}{2}\left(\frac{x-\mu}{b}\right)^2} dx$$

put put $\frac{x-\mu}{b} = z$

$x - \mu = bz \Rightarrow x = bz + \mu$

$dx = b\,dz$

$$= \frac{1}{\sqrt{2\pi}\cdot \cancel{b}} \int_{-\infty}^{\infty} b^2 z^2 e^{-\frac{z^2}{2}} \cdot \cancel{b}\, dz$$

$$= \frac{2b^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

put $\frac{z^2}{2} = t$

$$z^2 = 2t$$

$$dz = \frac{1}{\sqrt{2t}} dt$$

$$= \frac{2b^2}{\sqrt{2\pi}} \int_0^{\infty} 2t\, e^{-t} \frac{1}{\sqrt{2t}} dt$$

$$= \frac{2b^2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2t}\, e^{-t} dt$$

$$= \frac{2b^2\sqrt{2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} \cdot t^{3/2 - 1} dt$$

$$= \frac{2b^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$\Gamma(n+1) = n\Gamma(n)$ if $n > 0$

$$= \frac{2b^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{\cancel{2} b^2}{\sqrt{\pi}} \frac{1}{\cancel{2}} \Gamma\left(\frac{1}{2}\right)$$

$$\sigma^2 = \frac{b^2}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

$$\sigma^2 = b^2$$

$$\therefore \quad \sigma = b$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty}$$

this p.d.f $f(x)$ is called Normal Distribution and it is denoted by $X \sim N(\mu, \sigma^2)$, read as X follows normal distribution with parameters μ & σ^2

Normal Distribution:-

A c.r.v X is said to have a normal distribution with parameters μ and σ^2, if its p.d.f is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

The c.d.f of X is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

=*=

Note:-

1. The graph of $f(x)$ is bell-shaped curve and is symmetric about the line $x = \mu$

* The Top of the bell is directly above μ
* For large values of σ, the curve tends to flatten out and

* For smaller value of σ, it has a sharp peak

Standard Normal Distribution

If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{x-\mu}{\sigma}$ is known as standard Normal distribution with mean $E(Z) = 0$, $V(Z) = 1$ and we write $Z \sim N(0,1)$, its p.d.f is given by

$$g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$$