Generating Functions

1. Moment Generating Function: (M.G.F)

The moment generating function (m.g.f) of a r.v x is denoted by $M_x(t)$ and it is defined as

$$M_x(t) = E\left[e^{tx}\right]$$

$$\therefore M_x(t) = E\left[e^{tx}\right]$$

$$= E\left[1 + \frac{t}{1!}x + \frac{t^2}{2!}x^2 + \cdots + \frac{t^r}{r!}x^r + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \cdots + \frac{t^r}{r!}E(x^r) + \cdots$$

$$\therefore M_x(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' \quad \text{--- (1)}$$

where

$E(x) = \mu_1' = 1^{st}$ moment about origin

$E(x^2) = \mu_2' = 2^{nd}$ " " "

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$E(x^r) = \mu_r' = r^{th}$ moment about origin

$\therefore M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$

$\therefore$ The co-efficient of $\frac{t^r}{r!}$ in $M_X(t)$ is μ_r'

$\therefore \mu_r' = E(X^r) = r^{th}$ moment about origin

Moment Generating Function about mean

The m.g.f of X about mean $\mu = E(X) = \mu_1'$ is defined as

$$M_{X-\mu}(t) = E\left[e^{t(x-\mu)} \right]$$

$$= E\left[1 + \frac{t}{1!}(x-\mu) + \frac{t^2}{2!}(x-\mu)^2 + \cdots + \frac{t^r}{r!}(x-\mu)^r + \cdots \right]$$

Here $E(x-\mu) = \mu_1 = 1^{st}$ moment about mean

$E((x-\mu)^2) = \mu_2 = 2^{nd}$ moment about mean

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$E((x-\mu)^r) = \mu_r = r^{th}$ moment about mean

$$= 1 + \frac{t}{1!} E(x-\mu) + \frac{t^2}{2!} E((x-\mu)^2) + \cdots$$

$$+ \frac{t^r}{r!} E[(x-\mu)^r] + \cdots$$

$$M_{x-\mu}(t) = 1 + \frac{t}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \cdots + \frac{t^r}{r!}\mu_r + \cdots$$

$$\therefore M_{x-\mu}(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r.$$

$\therefore$ The co-efficient of $\frac{t^r}{r!}$ in $M_{x-\mu}(t)$ is μ_r.

$$\therefore \mu_r = E\left[(x-\mu)^r\right]$$

$$\mu_1 = E[x-\mu] = 0$$

$$\mu_2 = E((x-\mu)^2)$$

$$= E\left[x^2 + \mu^2 - 2x\mu\right]$$

$$= E(x^2) + \mu^2 - 2\mu E(x)$$

$$= E(x^2) + \mu^2 - 2\mu \cdot \mu$$

$$= E(x^2) + \mu^2 - 2\mu^2$$

$$= E(x^2) - \mu^2$$

$$= E(x^2) - [E(x)]^2$$

$$= \sigma^2$$

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Note!-

1. If x is DRV with p.m.f p(x) then

$$M_x(t) = E[e^{tx}] = \sum_x e^{tx} p(x)$$

2. If x is CRV with p.d.f f(x) then

$$M_x(t) = E[e^{tx}] = \int_x e^{tx} \cdot f(x)\,dx$$

3. Moment generating function $M_x(t)$ is used to calculate the higher moment.

Moments using MGF :-

Differentiating eq (1) w.r.to t and then putting t = 0

$$\mu_1' = \frac{d}{dt}[M_x(t)]_{t=0}$$

$$\mu_2' = \frac{d^2}{dt^2}[M_x(t)]_{t=0}$$

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$$\mu_r' = \frac{d^r}{dt^r}[M_x(t)]_{t=0}$$

Theorems:-

1. $M_{ax}(t) = M_x(at)$, where 'a' is a constant

Proof:- By definition $M_x(t) = E[e^{tx}]$

$$M_{ax}(t) = E[e^{tax}]$$
$$= E[e^{atx}]$$
$$= M_x(at)$$

$$\therefore \quad M_{ax}(t) = M_x(at)$$

2. The m.g.f of the sum of n independent r.v is equal to the product of their respective moment generating functions, i.e

$$M_{x_1+x_2+\cdots+x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$$

Proof:-

$$M_{x_1+x_2+\cdots+x_n}(t) = E[e^{t(x_1+x_2+\cdots+x_n)}]$$
$$= E[e^{tx_1+tx_2+\cdots+tx_n}]$$
$$= E[e^{tx_1} \cdot e^{tx_2} \cdots e^{tx_n}]$$
$$= E[e^{tx_1}]\, E[e^{tx_2}] \cdots E[e^{tx_n}]$$
$$= M_{x_1}(t)\; M_{x_2}(t) \cdots M_{x_n}(t)$$

$$\therefore M_{x_1+x_2+\cdots+x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$$

Effect of change of origin and scale on MGF :

Let a r.v x be transformed to a new variable U by changing both the origin and scale in x as $\frac{x-a}{h}$ where a & h are constants

the m.g.f of U (about origin) is given by

$$M_U(t) = E[e^{tU}]$$

$$= E\left[e^{t\left(\frac{x-a}{h}\right)}\right]$$

$$= E\left[e^{\frac{tx}{h} - \frac{at}{h}}\right]$$

$$= E\left[e^{\frac{tx}{h}} \cdot e^{-\frac{at}{h}}\right]$$

$$M_U(t) = e^{-\frac{at}{h}} E\left[e^{\frac{tx}{h}}\right]$$

$$M_U(t) = M_{\frac{x-a}{h}}(t) = e^{-\frac{at}{h}} E\left[e^{\frac{tx}{h}}\right]$$

Note:- If $Y = ax+b$ then

$$x = \frac{Y-b}{a}$$

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(ax+b)}]$$

$$= E[e^{tax+tb}]$$

$$= E[e^{atx} \cdot e^{tb}]$$

$$= e^{tb} E[e^{atx}]$$

$$M_Y(t) = e^{tb} M_x(at)$$

$$\therefore M_Y(t) = e^{tb} M_x(at) \text{ where } Y = ax+b.$$

Problem:- If x represent the outcome when a fair die is tossed, find the m.g.f of x and hence find $E(x)$, $Var(x)$

Solution: When a fair die is tossed

$$P(x=x) = \frac{1}{6}, \quad x = 1,2,3,4,5,6$$

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{6} e^{tx} p(x)$$

$$= e^{t} \cdot p(1) + e^{2t} p(2) + e^{3t} \cdot p(3) + e^{4t} p(4) + e^{5t} \cdot p(5) + e^{6t} p(6)$$

$$= e^{t} \frac{1}{6} + e^{2t} \cdot \frac{1}{6} + e^{3t} \cdot \frac{1}{6} + e^{4t} \frac{1}{6} + e^{5t} \cdot \frac{1}{6} + e^{6t} \cdot \frac{1}{6}$$

$$M_x(t) = \frac{1}{6}\left[e^{t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right]$$

$$E(X) = \frac{d}{dt}\left[M_x(t)\right]_{t=0}$$

$$= \frac{1}{6}\left[e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}\left[1 + 2 + 3 + 4 + 5 + 6\right]$$

$$= \frac{1}{\cancel{6}}\left[\frac{\cancel{6}(7)}{2}\right]$$

$$= \frac{7}{2}$$

$$E(X^2) = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}$$

$$= \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}\left[1+4+9+16+25+36\right]$$

$$= \frac{91}{6}$$

$$V(X) = E(X^2) - \left[E(X)\right]^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{35}{12} \checkmark$$

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Find the m.g.f of the random variable X whose probability function $P(X=x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots$ and hence find its mean.

<u>Solution:</u> By definition,

$$M_X(t) = E\left[e^{tx}\right]$$

$$= \sum_{x=1}^{\infty} e^{tx} P(X=x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2^x}\right)$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1} \quad \left(\because (1-x)^{-1} = 1+x+x^2+\cdots\right)$$

$$= \frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{\cancel{2}}\left(\frac{\cancel{2}}{2-e^t}\right) = \frac{e^t}{2-e^t}.$$

$$\therefore M_X(t) = \frac{e^t}{2-e^t}$$

$$\therefore \mu_1' = \frac{d}{dt}\left[M_x(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - (e^t)^2 + (e^t)^2}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}$$

$$= \frac{2e^0}{(2-e^0)^2}$$

$$= \frac{2}{(1)^2} = 2$$

$$\therefore E(x) = \text{Mean} = \mu_1' = 2$$

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If the moments of a r.v X are defined by $E(X^r) = \mu_r' = 0.6$, $r = 1, 2 \ldots$ show that $P(X=0) = 0.4$, $P(X=1) = 0.6$ and $P(X \geq 2) = 0$.

<u>Solution:</u>—

We know that

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

where $\mu_r' = E(X^r) = 0.6$

$$\therefore M_X(t) = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6)$$

$$= 1 + 0.6\left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

$$= 1 + 0.6\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots - 1\right)$$

$$= 1 + 0.6\left(e^t - 1\right)$$

$$= 1 + 0.6e^t - 0.6$$

$$= 0.4 + 0.6e^t \quad \text{—— } (1)$$

But by definition,

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{r=0}^{\infty} e^{tx} p(x=x)$$

$$M_X(t) = p(x=x) + e^{t} p(x=1) + e^{2t} p(x=2) + e^{3t} p(x=3) + --- + \quad \dots (2)$$

From equation (1) & (2), we have

$$0.4 + 0.6e^{t} = p(x=0) + e^{t} p(x=1) + e^{2t} p(x=2) + e^{3t} p(x=3) + ---$$

Equating the co-efficients of like terms on both sides

$$p(x=0) = 0.4$$

$$p(x=1) = 0.6$$

$$p(x=2) = p(x=3) = \dots = 0$$

$$\Rightarrow p(x \geq 2) = 0.$$