Bivariate Random Variable :-

Bivariate Random variable :-

Let S be the sample space associated with a random experiment. Let R be the real line. If $(X,Y) : S \longrightarrow R \times R$,

i.e $(X,Y)(s) = (X(s), Y(s)) \ \forall s \in S$

then the pair (X,Y) is known as a Bivariate Random Variable.

Note:- 1. If X and Y are both discrete r.v then (X,Y) is a Bivariate DRV.

2. If X and Y are both continuous r.v then (X,Y) is a Bivariate CRV.

Bivariate Discrete Random Variable

Joint Probability mass function:-

Let (X,Y) be a BDRV, which takes the values (x_i, y_j) for $i = 1,2,3 \ldots m$, $j = 1,2,3 \ldots n$

Let $P(x_i, y_j) = P(X = x_i, Y = y_j) \ \forall i$ and j

then $p(x_i, y_j) \geq 0 \ \forall$ i and j

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i, y_j) = 1.$$

The function $p(x,y)$ is known as joint probability mass function (j.p.m.f) of (X,Y)

Marginal Probability mass function:-

Let (X,Y) be a BDRV with j.p.m.f given by (x,y). The marginal p.m.f of X and Y are given by

$$P_1(x_i) = \sum_{j=1}^{n} p(x_i, y_j) \text{ for } i = 1,2,3 \cdots m \text{ and}$$

$$P_2(y_j) = \sum_{i=1}^{m} p(x_i, y_j) \text{ for } j = 1,2,3 \cdots n.$$

respectively.

Note:- X and Y are independent $\iff$

$$p(x_i, y_j) = P_1(x_i) \cdot P_2(y_j) \ \forall (i,j)$$

Conditional Probability Mass function:-

Let (X,Y) be a BDRV with j.p.m.f given by $P(x,y)$. Then
The conditional probability mass function of X and $Y = y_j$ given by

$$p_{1/2}(x_i|y_j) = \frac{P(x_i, y_j)}{P_2(y_j)} \text{ for } i = 1,2 \cdots m$$

The conditional Probability mass function of Y and $X = x_i$ given by

$$P_{2/1}(y_j|x_i) = \frac{P(x_i, y_j)}{P_1(x_i)} \text{ for } j = 1,2 \cdots n.$$

Problem:- A fair coin is tossed three times. Let X be a random variable that takes the value '0' if the first toss is a tail. takes the value '1' if the first toss is a head. and Y be a random variable that defines the total number of heads in the three tosses. Then

i) Determine the joint, marginal and conditional mass functions of X and Y.

ii) Are X and Y independent?

Solution :-

i) The sample space and values of X and Y are given in the following table.

Out comes in S.S	Value of X	Value of Y
HHH	1	3
HHT	1	2
HTH	1	2
HTT	1	1
THH	0	2
THT	0	1
TTH	0	1
TTT	0	0

Here X takes the values 0 and 1

Y takes the values 0, 1, 2, 3.

Then j.p.m.f of (x, y) is computed below:

$$P(0,0) = P(X=0, Y=0) = P\{(TTT)\} = \frac{1}{8}$$

$$P(0,1) = P(X=0, Y=1) = P(\{THT, TTH\}) = \frac{2}{8} = \frac{1}{4}$$

$$P(0,2) = P(X=0, Y=2) = P(\{THH\}) = \frac{1}{8}$$

$$P(0,3) = P(X=0, Y=3) = 0$$

$P(1,0) = P(X=1, Y=0) = 0$

$P(1,1) = P(X=1, Y=1) = P(\{HTT\}) = \frac{1}{8}$

$P(1,2) = P(X=1, Y=2) = P(\{HHT, HTH\}) = \frac{2}{8} = \frac{1}{4}$

$P(1,3) = P(X=1, Y=3) = P(\{HHH\}) = \frac{1}{8}$

The m.p.m.f of X is given by

$$P_1(x) = \sum_{y=0}^{3} P(x,y)$$

$$P_1(0) = P(0,0) + P(0,1) + P(0,2) + P(0,3)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 = \frac{1}{2}$$

$$P_1(1) = P(1,0) + P(1,1) + P(1,2) + P(1,3)$$

$$= 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2}$$

The m.p.m.f of Y is given by

$$P_2(y) = \sum_{x=0}^{1} P(x,y)$$

$$P_2(0) = P(0,0) + P(1,0) = \frac{1}{8} + 0 = \frac{1}{8}$$

$$P_2(1) = P(0,1) + P(1,1) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P_2(2) = P(0,2) + P(1,2) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$P_2(3) = P(0,3) + P(1,3) = 0 + \frac{1}{8} = \frac{1}{8}$$

The conditional p.m.f of x given by Y is computed below.

$$P_{1/2}(x|y) = \frac{P(x,y)}{P_2(y)},$$

$$P_{\frac{1}{2}}(0|0) = \frac{P(0,0)}{P_2(0)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1.$$

$$P_{\frac{1}{2}}(0|1) = \frac{P(0,1)}{P_2(1)} = \frac{\frac{2}{8}}{\frac{3}{8}} = \frac{2}{3}$$

$$P_{\frac{1}{2}}(0|2) = \frac{P(0,2)}{P_2(2)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}$$

$$P_{\frac{1}{2}}(0|3) = \frac{P(0,3)}{P_2(3)} = \frac{0}{\frac{1}{8}} = 0$$

$$P_{\frac{1}{2}}(1|0) = \frac{P(1,0)}{P_2(0)} = \frac{0}{\frac{1}{8}} = 0$$

$$P_{\frac{1}{2}}(1|1) = \frac{P(1,1)}{P_2(1)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}$$

$$P_{\frac{1}{2}}(1|2) = \frac{P(1,2)}{P_2(2)} = \frac{\frac{2}{8}}{\frac{3}{8}} = \frac{2}{3}$$

$$P_{\frac{1}{2}}(1|3) = \frac{P(1,3)}{P_2(3)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1.$$

The conditional p.m.f of Y given x is computed below.

$$P_{2|1}(y/x) = \frac{P(x,y)}{P_1(x)}.$$

$$P_{2|1}(0|0) = \frac{P(0,0)}{P_1(0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{2|1}(1|0) = \frac{P(0,1)}{P_1(0)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P_{2|1}(2|0) = \frac{P(0,2)}{P_1(0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{2|1}(3|0) = \frac{P(0,3)}{P_1(0)} = \frac{0}{\frac{1}{2}} = 0$$

$$P_{2|1}(0|1) = \frac{P(1,0)}{P_1(1)} = \frac{0}{\frac{1}{2}} = 0$$

$$P_{2|1}(1|1) = \frac{P(1,1)}{P_1(1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{2|1}(2|1) = \frac{P(1,2)}{P_1(1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P_{2|1}(3|1) = \frac{P(1,3)}{P_1(1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

(ii) Here $P(0,0) = \frac{1}{8}$, $P_1(0) = \frac{1}{2}$, $P_2(0) = \frac{1}{8}$

$\because$ $P(0,0) \neq P_1(0) \cdot P_2(0)$.

$\therefore$ X, Y are not Independent.

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Problem :-2 :

The j. P. m. f of (X, Y) is given by

$$p(x,y) = \begin{cases} k(2x+y), & \text{for } x = 1, 2;\ y = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

where k is constant.

a. Find the Value of k.
b. Find marginal and conditional P. m. fs.
c. Are X and Y Independent.?

Solution:-

a) Since $p(x,y)$ is a p.m.f,

$$\sum_{x=1}^{2}\sum_{y=1}^{2} p(x,y)=1$$

$$\Rightarrow \sum_{x=1}^{2}\sum_{y=1}^{2} k(2x+y)=1$$

$$\Rightarrow k\left[\sum_{x=1}^{2}\left[(2x+1)+(2x+2)\right]\right]=1$$

$$\Rightarrow k\left[(2(1)+1)+(2(1)+2)+(2(2)+1)+(2(2)+2)\right]=1$$

$$\Rightarrow k\left[3+4+5+6\right]=1$$

$$\Rightarrow k\left[18\right]=1$$

$$k=\frac{1}{18}$$

$\therefore p(x,y)=\frac{1}{18}(2x+y)$ when $x=1,2$; $y=1,2$

b) The m.p.m.f of x is given by

$$p_1(x)=\sum_{y=1}^{2} p(x,y)$$

$$=\frac{1}{18}\sum_{y=1}^{2}(2x+y)$$

$$=\frac{1}{18}\left[2x+1+2x+2\right]=\frac{1}{18}\left[4x+3\right]$$

$$\therefore P_1(x) = \frac{4x+3}{18}, \quad x = 1, 2$$

The m.p.m.f of Y is given by

$$P_2(y) = \sum_{x=1}^{2} P(x,y)$$

$$= \frac{1}{18} \sum_{x=1}^{2} (2x+y)$$

$$= \frac{1}{18} [2(1)+y+2(2)+y]$$

$$= \frac{1}{18} [2y+6] = \frac{y+3}{9}$$

$$\therefore P_2(y) = \frac{y+3}{9} \text{ for } y = 1, 2$$

The c.p.m.f of X given Y is given by

$$P_{\frac{1}{2}}(x|y) = \frac{P(x,y)}{P_2(y)}$$

$$= \frac{\frac{1}{\cancel{18}_2}(2x+y)}{\frac{y+3}{\cancel{9}}}$$

$$= \frac{2x+y}{2y+6}$$

The c.p.m.f of Y given X is given by

$$P_{\frac{2}{1}}(y|x) = \frac{P(x,y)}{P_1(x)} = \frac{\frac{1}{\cancel{18}}(2x+y)}{\frac{4x+3}{\cancel{18}}} = \frac{2x+y}{4x+3}$$

c) $P_1(x) \cdot P_2(y) = \frac{4x+3}{18} \cdot \frac{y+3}{9} \neq P(x,y)$

thus X and Y are not independent.