Problems based on CRV

If X is a CRV with p.d.f

$$f(x) = \begin{cases} Kx, & 0 \le x < 1 \\ K, & 1 \le x < 2 \\ -K(x-3), & 2 \le x < 3 \\ 0, & \text{Otherwise} \end{cases}$$

i) Determine K

ii) Compute $P(X \le 1.5)$ —

Solution:—

We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} Kx\,dx + \int_{1}^{2} K\,dx + \int_{2}^{3} -K(x-3)\,dx = 1$$

$$\Rightarrow K\left(\frac{x^2}{2}\right)_0^1 + K(x)_1^2 - K\left(\frac{x^2}{2} - 3x\right)_2^3 = 1$$

$$\Rightarrow K\left[\frac{1}{2}\right] + K[2-1] - K\left[\frac{9}{2} - 9 - \frac{4}{2} + 6\right] = 1$$

$$\Rightarrow \frac{K}{2} + K - K\left[\frac{5}{2} - 3\right] = 1$$

$$\Rightarrow \frac{3}{2}K - K\left[\frac{-1}{2}\right] = 1$$

$$\Rightarrow \frac{3}{2}K + \frac{K}{2} = 1$$

$\Rightarrow \frac{2 \cancel{4}K}{\cancel{x}} = 1$

$\Rightarrow K = \frac{1}{2}$

(ii) compute $P(x \leq 1.5)$

$$P(x \leq 1.5) = \int_{-\infty}^{1.5} f(x)dx$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{1.5} f(x)dx$$

$$= K\int_{0}^{1} x\,dx + \int_{1}^{1.5} K\,dx$$

$$= K\left(\frac{x^2}{2}\right)_0^1 + K(x)_1^{1.5}$$

$$= K\left(\frac{1}{2}\right) + K(1.5-1)$$

$$= K\left(\frac{1}{2}\right) + K(0.5)$$

$$= K\left(\frac{1}{2} + 0.5\right)$$

$$= K\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= K$$

$$= \frac{1}{2} \quad (\because K = \tfrac{1}{2})$$

Find the mean and variance of the random variable x, whose p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ 0, & \text{Otherwise} \end{cases}$$

<u>solution:-</u>

$$E(x) = \mu = \int_0^2 x f(x)\,dx$$

$$= \int_0^2 x \cdot \left(\tfrac{1}{2}\right) dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^2$$

$$= \frac{1}{2}\left(\frac{4}{2} - 0\right) = \frac{1}{2}\left(\frac{\cancel{4}^{2}}{\cancel{2}}\right) = 1$$

$$\therefore E(X) = \mu = 1$$

$$E(x^2) = \int_0^2 x^2 f(x)\,dx$$

$$= \int_0^2 x^2 \left(\tfrac{1}{2}\right) dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^2 = \frac{1}{2}\left(\frac{\cancel{8}^{4}}{3}\right) = \frac{4}{3}$$

$$\therefore \text{Variance } V(X) = \sigma^2 = E[X^2] - [E(X)]^2$$

$$= \frac{4}{3} - 1^2 = \frac{1}{3} ✓$$

Find the mean of the r.v x whose p.d.f is given by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{Otherwise} \end{cases}$$

Solution:-

$$E[x] = \int_0^\infty x\, f(x)dx$$

$$= \int_0^\infty x\, e^{-x}$$

$$= \left[x\left(\frac{e^{-x}}{-1}\right) - 1\, \frac{e^{-x}}{(-1)^2} \right]_0^\infty$$

$$= 1$$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where $v_1 = \int v\,dx$, $v_2 = \int v_1 dx$, $v_3 = \int v_2 dx$, ...

$u' = \frac{du}{dx}$, $u'' = \frac{du'}{dx} = \frac{d^2u}{dx^2}$

Properties

1. If x is a r.v and a & b are constants then $E[ax+b] = aE[x]+b$.

Proof:- Let x be C.R.V with p.d.f $f(x)$

then $$E[ax+b] = \int_{-\infty}^{\infty} (ax+b)\cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} ax\, f(x)dx + \int_{-\infty}^{\infty} b f(x)dx$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\,E[x] + b\cdot(1)$$

$$= a\,E[x] + b$$

2. $V(x) = E[x^2] - [E(x)]^2$

<u>proof</u>:-

By definition of Variance

$$V(x) = \sigma^2 = E[(x - E(x))^2]$$

$$= E[x^2 + [E(x)]^2 - 2x\,E(x)]$$

$$= E[x^2] + E([E(x)]^2) - E[2x\,E(x)]$$

We know that $E(\text{const}) = \text{const}$

$$= E[x^2] + [E(x)]^2 - 2E(x)\cdot E(x)$$

$$= E[x^2] + [E(x)]^2 - 2[E(x)]^2$$

$$= E[x^2] - [E(x)]^2$$

$= \times =$

If p.d.f

$$f(x) = \begin{cases} Kx^3, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

i) Find K

ii) Find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$

iii) $F(x) = P(X \le x)$ [Hint $\int_0^x f(x)\,dx$

If the p.d.f of X is given by

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x \le 1 \\ \frac{1}{2}, & 1 < x \le 2 \\ \frac{(3-x)}{2}, & 2 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected value of $x^3 - 5x + 3$

[Hint

$$E[x^3 - 5x + 3] = E[x^3] - 5E(x) + 3 \quad \text{--- (1)}$$

To find $E[x^3] = \int_{-\infty}^{\infty} x^3 f(x)\,dx$

$$= \int_{-\infty}^{0} x^3 f(x)\,dx + \int_0^2 x^3 f(x)\,dx + \int_2^3 x^3 f(x)\,dx \;]$$

$$= \int_0^2 x^3 f(x)\,dx + \int_2^3 x^3 f(x)\,dx$$

Hy To find $E(x)$ ✓ ⊥