Covariance :-

If X and Y are two r.v then the covariance between them is defined by

$$Cov(X,Y) = E[(X-E(X))(Y-E(Y))]$$

$$= E[(XY - XE(Y) - YE(X) + E(X)\cdot E(Y)]$$

$$= E(XY) - E(X)E(Y) - \cancel{E(Y)E(X)} + \cancel{E(X)\cdot E(Y)}$$

$$= E(XY) - E(X)\cdot E(Y)$$

Note:-

1. If X and Y are independent then
$$\text{Covariance}(X,Y) = Cov(X,Y) = 0$$

2. $Cov(aX, bY) = ab\, Cov(X,Y)$
where a and b are constant.

3. $Cov(X+a, Y+b) = Cov(X,Y)$.

4. * If X, Y are discrete then $E(X) = \sum x P_1(x)$
$E(Y) = \sum y P_2(y)$

* If X, Y are continuous then
$$E(X) = \int_{-\infty}^{\infty} x f_1(x)\,dx, \quad E(Y) = \int_{-\infty}^{\infty} y f_2(y)\,dy.$$

The j.p.d.f of X and Y is given by

$$f(x,y) = \begin{cases} 2-x-y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find i) m.p.d.fs of X and Y. ✓

ii) c.p.d.fs of X and Y ✓

iii) $V(X)$, $V(Y)$.

iv) $Cor(X,Y)$

Solution:-

i) $f_1(x) = \int_0^1 f(x,y)\,dy$

$$= \int_0^1 (2-x-y)\,dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= 2 - x - \frac{1}{2} = \frac{3}{2} - x.$$

$$\therefore f_1(x) = \begin{cases} \frac{3}{2} - x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

Ily $f_2(y) = \begin{cases} \frac{3}{2} - y, & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f_{\frac{1}{2}}(x|y) = \dfrac{f(x,y)}{f_2(y)}$

$= \dfrac{2-x-y}{\frac{3}{2}-y}, \quad 0<x,\ y<1$

$f_{\frac{2}{1}}(y|x) = \dfrac{f(x,y)}{f_1(x)}$

$= \dfrac{2-x-y}{\frac{3}{2}-x}, \quad 0<x,\ y<1$

(iii) $E(x) = \int_0^1 x\, f_1(x)\, dx$

$= \int_0^1 x\left(\frac{3}{2}-x\right) dx$

$= \int_0^1 \left(\frac{3}{2}x - x^2\right) dx$

$= \left(\frac{3}{2}\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{3}{2}\cdot\frac{1}{2} - \frac{1}{3}$

$= \frac{3}{4} - \frac{1}{3}$

$= \frac{9-4}{12} = \frac{5}{12}$

$E(x^2) = \int_0^1 x^2 f_1(x)\, dx$

$$= \int_0^1 x^2\left(\frac{3}{2}-x\right)dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 - x^3\right)dx$$

$$= \left(\frac{3}{2}\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

Similarly $V(Y) = \frac{11}{144}$

(iv) $$E(XY) = \int_0^1\int_0^1 xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\,dx\,dy$$

$$= \int_0^1 \left(2x\frac{y^2}{2} - x^2\frac{y^2}{2} - x\frac{y^3}{3}\right)_0^1 dx$$

$$= \int_0^1 \left(x - \frac{x^2}{2} - \frac{x}{3}\right)dx$$

$$= \int_0^1 \left(\frac{2}{3}x - \frac{x^2}{2}\right)dx$$

$$= \left[\frac{2}{3} \frac{x^2}{\cancel{2}} - \frac{x^3}{6} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}$$

$$\operatorname{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12} = -\frac{1}{144}$$

$= x =$