Problems based on conditional probability

1. Consider a family with two children. Assume that each child is likely to be a boy as it is to be a girl. What is the conditional probability that both children are boys, given that

   i) the older child is a boy

   ii) at least one of the child is a boy

Solution:

Sample space $S = \{(b,b),(b,g),(g,b)(g,g)\}$

Define events:

A = event is older child is a boy

$= \{(b,b)(b,g)$

$n(A) = 2$ then $P(A) = \frac{2}{4} = \frac{1}{2}$

B = event is younger child is boy

$= \{(b,b)(g,b)\}$

$n(B) = 2$ then $P(B) = \frac{2}{4} = \frac{1}{2}$

then $A \cap B$ = event is both children are boys

$= \{(b, b)\}$

$$P(A \cap B) = \frac{1}{4}$$

$A \cup B = \{(b, b), (b, g) (g, b)\}$

= at least one of the child is boys

$$P(A \cup B) = \frac{3}{4}$$

i) $P(A \cap B | A) = \frac{P((A \cap B) \cap A)}{P(A)}$

$$= \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{\cancel{4}_2} \times \frac{\cancel{2}}{1} = \frac{1}{2}$$

(ii) $P(A\cap B \mid A\cup B) = \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)}$

$$= \frac{P(A\cap B)}{P(A\cup B)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{3}$$

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A fair dice is thrown twice. Let A, B, and C denote the following events

A : First toss is odd

B : Second toss is even

C : Sum of numbers is 7.

(i) Find $P(A)$, $P(B)$, $P(C)$ ✓

(ii) Show that A, B, C are pair wise independent.

(iii) Show that A, B, C are not independent.

Solution:-

The sample space $S = \{ (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)$
$(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)$
$(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)$
$(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)$
$(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)$
$(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) \}$

$\therefore \; n(S) = 36$

i) A = event of getting First toss is odd

$= \{ (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)$
$(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)$
$(5,1) (5,2) (5,3) (5,4) (5,5) (5,6) \}$

$n(A) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

Let B = event of getting Second toss is even

$= \{ (1,2) (1,4) (1,6)$
$(2,2) (2,4) (2,6)$
$(3,2) (3,4) (3,6)$
$(4,2) (4,4) (4,6)$
$(5,2) (5,4) (5,6)$
$(6,2) (6,4) (6,6) \}$

$n(B) = 18$

$$P(B) = \frac{n(B)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

Let C = event of getting sum of numbers is 7

$= \{ (1,6)\ (6,1)$
$(2,5)\ (5,2)$
$(3,4)\ (4,3) \}$

$n(C) = 6$

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii)

$A \cap B = \{ (1,2)(1,4)(1,6)$
$(3,2)(3,4)(3,6)$
$(5,2)(5,4)(5,6) \}$

$n(A \cap B) = 9$

$$\therefore P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

We have $P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

then $P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(A \cap B) = P(A) \cdot P(B)$

$\therefore$ A, B are independent

$$B \cap C = \{(1,6),(5,2)(3,4)\}$$

$$n(B \cap C) = 3$$

$$P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{3}{36} = \frac{1}{2}$$

We have $P(B) = \frac{1}{2}$

$P(C) = \frac{1}{6}$

$$\therefore \quad P(B) \cdot P(C) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

$$\therefore \quad P(B \cap C) = P(B) \cdot P(C)$$

$\therefore$ B, C are independent.

$$C \cap A = A \cap C = \{(1,6)(5,2)(3,4)\}$$

$$n(A \cap C) = 3$$

$$P(A \cap C) = \frac{n(A \cap C)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

We have $P(A) = \frac{1}{2}$,

$P(C) = \frac{1}{6}$

$$\therefore \quad P(A) \cdot P(C) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

$$\therefore \quad P(A \cap C) = P(A) \cdot P(B)$$

$\therefore$ A, C are independent

$\therefore$ A, B, C are pair wise Independent.

(iii) $A \cap B \cap C = \{(1,6)(3,4)(5,2)\}$

$n(A \cap B \cap C) = 3$

$\therefore P(A \cap B \cap C) = \dfrac{n(A \cap B \cap C)}{n(S)} = \dfrac{3}{36} = \dfrac{1}{12}$

We have $P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

$P(C) = \frac{1}{6}$

$P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{6} = \frac{1}{24}$

$\therefore P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

$\therefore$ A, B, C are not independent

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