Bivariate continuous Random Variable:-

Joint Probability Density Function:-

Let (X, Y) be a bivariate CRV.

Let $P(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d f(x,y)\,dx\,dy$.

for some real numbers a, b, c, d such that $a < b$, $c < d$. then

i, $f(x,y) \ge 0 \;\forall (x,y)$ and

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

the function $f(x,y)$ is known as the joint probability density function of the BCRV (X, Y).

Marginal Probability density function:-

Let (X, Y) be a bivariate CRV with j.p.d.f $f(x,y)$. The marginal p.d.f of X & Y are given by

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad \text{respectively}$$

Note:-

X and Y are independent $\Longleftrightarrow$
$f(x,y) = f_1(x)\cdot f_2(y)$.

Conditional Probability density function:-

Let (X,Y) be a bivariate CRV with j.p.d.f $f(x,y)$. Let $f_1(x)$, $f_2(y)$ be the m.p.d.f of X and Y respectively.

The conditional p.d.f of X given Y are given by

$$f_{\frac{1}{2}}(x|y) = \frac{f(x,y)}{f_2(y)}$$

The conditional p.d.f of Y given X are given by

$$f_{\frac{2}{1}}(y|x) = \frac{f(x,y)}{f_1(x)}$$

Cumulative distribution function:-

The cumulative distribution of a BRV (X,Y) is defined by

* If (X,Y) is BDRV with j.p.m.f $p(x,y)$.

$$F(x,y) = P(X \le x, Y \le y)$$
$$= \sum_{t \le x} \sum_{s \le y} p(t,s).$$

* If (X,Y) is BCRV with j.p.d.f $f(x,y)$.

$$F(x,y) = P(X \le x, Y \le y)$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

Properties of Cumulative distribution functions:-

1. $0 \le F(x,y) \le 1.$
2. $F(\infty,\infty) = 1, \quad F(-\infty,-\infty) = 0.$
3 $P(a < X \le b, Y \le d) = F(b,d) - F(a,d)$
$P(X \le b, c < Y \le d) = F(b,d) - F(b,c)$
4. $P(a < X \le b, c < Y \le d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

Note:-

If (X,Y) is a BCRV with c.d.f $F(x,y)$, then its j.p.d.f is given

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \, \partial y}$$

Example :- 1

The j.p.d.f of (X,Y) is given by

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } 0<x<\infty, \\ & 0<y<\infty. \\ 0 & \text{otherwise.} \end{cases}$$

a) Find marginal p.d.f of X and Y

b) Are X and Y independent ?

Solution :- The m.p.d.f of X is given by

$$f_1(x) = \int_0^\infty f(x,y)\,dy$$

$$= \int_0^\infty e^{-(x+y)}\,dy$$

$$= \int_0^\infty e^{-x} \cdot e^{-y}\,dy$$

$$= e^{-x} \cdot \left[\frac{e^{-y}}{-1}\right]_0^\infty = e^{-x}(1) = e^{-x}.$$

$$= \begin{cases} e^{-x}, & \text{for } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The m. p. d. f of Y is given by

$$f_2(y) = \int_0^\infty f(x,y)\,dx$$

$$= \int_0^\infty e^{-(x+y)}\, dx$$

$$= e^{-y} \int_0^\infty e^{-x}\, dx$$

$$= e^{-y}(1)$$

$$= e^{-y}$$

$$\therefore f_2(y) = \begin{cases} e^{-y}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

b) $\because f_1(x) \cdot f_2(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f(x,y)$

$\therefore$ X, Y are independent.

<u>Ex:-2</u>: The j.p.d.f of (X,Y) is given by

$$f(x,y) = \begin{cases} x\, e^{-x(y+1)}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

a) Determine marginal and conditional p.d.fs

b) Are X and Y independent?

<u>Solution</u>:-

The m.p.d.f of X is given by

$$f_1(x) = \int_0^\infty f(x,y)\,dy$$

$$= \int_0^\infty x\, e^{-x(y+1)}\,dy$$

$$= x e^{-x} \int_0^\infty e^{-xy}\,dy$$

$$= \cancel{x} e^{-x} \left[\frac{e^{-xy}}{-\cancel{x}} \right]_0^\infty$$

$$= e^{-x} [1] = e^{-x}.$$

$\therefore \quad f_1(x) = e^{-x} \text{ for } 0 < x < \infty$

The m.p.d.f of Y is given by

$$f_2(y) = \int_0^\infty f(x,y)\,dx$$

$$= \int_0^\infty x\, e^{-x(y+1)}\,dx$$

$$= \left[x \frac{e^{-x(y+1)}}{-(y+1)} \right]_0^\infty - \int_0^\infty \frac{e^{-x(y+1)}}{-(y+1)} dx$$

$$= \int_0^\infty \frac{e^{-x(y+1)}}{y+1} dx$$

$$= \frac{1}{y+1} \int_0^\infty e^{-x(y+1)} dx$$

$$= \frac{1}{y+1} \left[\frac{e^{-x(y+1)}}{-(y+1)} \right]_0^\infty$$

$$= \frac{1}{(y+1)^2} (1) = \frac{1}{(y+1)^2}$$

$$\therefore f_2(y) = \frac{1}{(y+1)^2}, \quad 0 < y < \infty$$

The conditional P.d.f of X given Y is given by

$$f_{\frac{1}{2}}(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$= \frac{x e^{-x(y+1)}}{\frac{1}{(y+1)^2}}$$

$$= x(y+1)^2 e^{-x(y+1)}$$

$$\therefore f_{\frac{1}{2}}(x|y) = x(y+1)^2 e^{-x(y+1)} \quad \text{for } 0<x<\infty$$

The conditional probability of Y given X is given by

$$f_{\frac{2}{1}}(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{x e^{-x(y+1)}}{e^{-x}}$$

$$= x e^{-xy}$$

$$\therefore f_{2/1}(y|x) = x e^{-xy} \quad \text{for } 0<x<\infty$$

$$f_1(x) \cdot f_2(y) = e^{-x} \frac{1}{(y+1)^2} \neq f(x,y)$$

Hence, X and Y are not independent.

The j.p.d.f of (X,Y) is given by

$$f(x,y) = kx^3 y, \quad 0<x<2, \; 0<y<1$$

a) Find k
b) Find the m. p.d.fs of X and Y
c) Are X and Y independent ?

Solution:-

a) We have $\int_0^2 \int_0^1 f(x,y)\,dx\,dy = 1$

$$\int_0^2 \int_0^1 K x^3 y \,dx\,dy = 1$$

$$\Rightarrow K \int_0^2 x^3 \left(\frac{y^2}{2}\right)_0^1 dx = 1$$

$$\Rightarrow \frac{K}{2} \int_0^2 x^3\,dx = 1$$

$$\Rightarrow \frac{K}{2} \left(\frac{x^4}{4}\right)_0^2 = 1$$

$$\Rightarrow K = \frac{1}{2}$$

(b) $\therefore f(x,y) = \frac{1}{2} x^3 y$

The m.p.d.f of x is given by

$$f_1(x) = \int_0^1 f(x,y)\,dy$$

$$= \frac{1}{2} \int_0^1 x^3 y\,dy$$

$$= \frac{1}{2} x^3 \left(\frac{y^2}{2}\right)_0^1 = \frac{x^3}{4}$$

$\therefore f_1(x) = \frac{x^3}{4}$ for $0 < x < 2$

The m.p.d.f of Y is given by

$$f_2(y) = \int_0^2 f(x,y)\,dx$$

$$= \frac{1}{2} \int_0^2 x^3 y \,dx$$

$$= \frac{1}{2} y \left(\frac{x^4}{4} \right)_0^2$$

$$= \frac{1}{2} y \left(\frac{16}{4} - 0 \right) = 2y$$

$\therefore f_2(y) = 2y$ for $0 < y < 1$

c) $f_1(x) \cdot f_2(y) = \frac{x^3}{4} \cdot 2y = \frac{x^3 y}{2} = f(x,y) \cdot \forall (x,y)$

$\because f_1(x) f_2(y) = f(x,y) \quad \forall (x,y)$

$\therefore$ X, Y are independent.