Bivariate Normal Distribution:-

Let X and Y be two normally correlated variables with correlation co-efficient ρ

Let $E(X) = \mu_1$, $V(X) = \sigma_x^2 = \sigma_1^2$

$E[Y] = \mu_2$, $V(Y) = \sigma_Y^2 = \sigma_2^2$

The Bivariate CRV of (X,Y) is said to follow bivariate normal distribution with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$, if its p.d.f is given by

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}$$

$-\infty < x, y, \mu_1, \mu_2 < \infty$

$\sigma_1 > 0$,

$\sigma_2 > 0$

$-1 < \rho < 1$

and denoted by $(X,Y) \sim BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ and read as (X,Y) follows BND with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ and ρ

Note:-

1. m.p.d.f of x and y

* $f_1(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$ for $-\infty < x < \infty$

* $f_2(y) = \frac{1}{\sqrt{2\pi}\cdot\sigma_2} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$ for $-\infty < y < \infty$

2. If $(X, Y) \sim BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

then $X \sim N(\mu_1, \sigma_1^2$

$Y \sim N(\mu_2, \sigma_2^2)$

3. c.p.d.f of x given y is given by

$$f_{\frac{1}{2}}(x/y) = \frac{1}{\sigma_1\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left\{(x-\mu_1) - \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right\}^2}$$

* The univariate normal distribution mean

$$E[X|Y=y] = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2) \text{ \&}$$

$$V[X|Y=y) = \sigma_1^2(1-\rho^2)$$

Thus, the c.p.d.f of X for fixed Y is given by

$$\underline{(X|Y=y)} \sim N[\underline{\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)}, \underline{\sigma_1^2(1-\rho^2)}$$

Ily the c.p.d.f of Y for fixed $X=x$ is given by

$$f_{2|1}\left(\frac{y}{x}\right)=\frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}}\,e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[(y-\mu_2)-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2}$$

$$-\infty<y<\infty$$

thus c.p.d.f of Y for fixed x is given by

$$(Y|X=x)\sim N\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\sigma_2^2(1-\rho^2)\right)$$

where $E[Y|X=x]=\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)$

$$V[Y|X=x]=\sigma_2^2(1-\rho^2)$$

$$E[X|Y=y]=\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$$

$$V(X|Y=y]=\sigma_1^2(1-\rho^2)$$

If $(X,Y)\sim BN(5,10,1,25,\rho)$ where $\rho>0$. find ρ when $P(4<Y<16|X=5)=0.954$

solution:-

Given

$$(X,Y)\sim BN(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$$

$$(X,Y)\sim BN(5,10,1,25,\rho)$$

We know that $(Y|X=x) \sim N(\mu, \sigma^2)$

Here $\mu = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1)$, $\sigma^2 = \sigma_2^2(1-\rho^2)$

$\mu = 10 + \rho \frac{5}{1}(x-5)$ | $\sigma^2 = 25(1-\rho^2)$

$= 10 + \rho\, 5(5-5)$

($\because$ Given $x=5$)

$\mu = 10$

$\therefore (Y|X=5) \sim N(10, 25(1-\rho^2))$

Given

$P(4 < Y < 16 | X=5) = 0.954$

Let $z = \frac{Y-\mu}{\sigma} = \frac{Y-10}{5\sqrt{1-\rho^2}} \sim N(0,1)$

$P\left(\frac{4-10}{\sigma} < Z < \frac{16-10}{\sigma}\right) = 0.954$

$P\left(\frac{-6}{\sigma} < Z < \frac{6}{\sigma}\right) = 0.954$

$2P\left(0 < Z < \frac{6}{\sigma}\right) = 0.954$

$$P\left(0 < Z < \frac{6}{\sigma}\right) = 0.477$$

From standard normal table,

$$\frac{6}{\sigma} = 2$$

$$\therefore \sigma = 3$$

$$\sigma^2 = 9$$

$$\sigma^2 = 25(1-\rho^2)$$

$$9 = 25(1-\rho^2)$$

$$\frac{9}{25} = 1-\rho^2$$

$$\rho^2 = 1 - \frac{9}{25}$$

$$\rho^2 = \frac{16}{25} \Rightarrow \rho = \frac{4}{5}$$

$$= 0.8$$

Find $\operatorname{cov}(X,Y)$ for the jointly Normal distrib

$$f(x,y) = \frac{1}{2\pi\sqrt{3}} \exp\left[-\frac{[(2x-y)^2 + 2xy]}{6}\right]$$

$$-\infty < x, y < \infty$$

Solution:-

$(x,y) \sim BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]} \quad \text{— (1)}$$

Given

$$f(x,y) = \frac{1}{2\pi\sqrt{3}} \exp\left[-\frac{\{(2x-y)^2 + 2xy\}}{6}\right]$$

$$= \frac{1}{2\pi\sqrt{3}} \exp\left[-\frac{\{4x^2+y^2-2xy\}}{6}\right] \quad \text{— (2)}$$

Comparing (1) & (2)

we get $\mu_1 = \mu_2 = 0$ then

(1) becomes

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} - 2\rho\frac{xy}{\sigma_1\sigma_2}\right]} \quad \text{— (3)}$$

Comparing (2), (3)

$$\sigma_1\sigma_2\sqrt{1-\rho^2} = \sqrt{3}, \quad \sigma_1^2(1-\rho^2) = \frac{3}{4}, \quad \sigma_2^2(1-\rho^2) = 3$$

and $\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} = \frac{1}{3}$

$\sigma_1 \sigma_2 \sqrt{1-\rho^2} = \sqrt{3}$

SOBS

$\sigma_1^2 \sigma_2^2 (1-\rho^2) = 3$

$\sigma_1^2 (3) = 3$

$\sigma_1^2 = 1$

$\sigma_1 = 1$

$\sigma_1^2 (1-\rho^2) = \frac{3}{4}$

$1 (1-\rho^2) = \frac{3}{4}$

$1 - \frac{3}{4} = \rho^2$

$\rho^2 = \frac{1}{4}$

$\rho = \frac{1}{2}$

=x=

Determine the parameters of the BND

$$f(x,y) = c\, e^{-\left[\frac{16(x-2)^2 - 12(x-2)(y+3) + 9(y+3)^2}{216}\right]}$$

If variable x and Y are connected by the equation $ax + bY + c = 0$. Show that the correlation between them is -1 if signs of a and b are same and $+1$ if signs of a and b are different.

Solutions- Given $ax + bY + c = 0$ — (1)

$\Rightarrow E[ax + bY + c] = E[0] = 0$

$\Rightarrow aE[x] + bE[Y] + c = 0$ — (2)

① ②

$$ax - aE(X) + bY - bE(Y) + c - c = 0$$

$$a(X - E(X)) + b(Y - E(Y)) = 0$$

$$a(X - E(X)) = -b(Y - E(Y))$$

$$X - E(X) = -\frac{b}{a}(Y - E(Y))$$

$$\therefore \operatorname{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$$

$$= E\left[-\frac{b}{a}(Y - E(Y))(Y - E(Y))\right]$$

$$= -\frac{b}{a}E\left[\{Y - E(Y)\}^2\right]$$

$$\operatorname{Cov}(X,Y) = -\frac{b}{a}\sigma_Y^2$$

$$\sigma_X^2 = E\left[(X - E(X))^2\right]$$

$$= E\left[\left[-\frac{b}{a}(Y - E(Y))\right]^2\right]$$

$$= \frac{b^2}{a^2}E\left[(Y - E(Y))^2\right]$$

$$= \frac{b^2}{a^2}\sigma_Y^2$$

$$\sigma_x = \pm \frac{b}{a} \sigma_y$$

$$= \left|\frac{b}{a}\right| \sigma_y$$

$$\text{Correlation } \rho(x,y) = \frac{cov(x,y)}{\sigma_x \cdot \sigma_y}$$

$$= \frac{-\frac{b}{a} \sigma_y^2}{\left|\frac{b}{a}\right| \sigma_y \cdot \sigma_y}$$

$$= \frac{-\frac{b}{a} \cancel{\sigma_y^2}}{\left|\frac{b}{a}\right| \cancel{\sigma_y^2}}$$

$$= \frac{-\frac{b}{a}}{\left|\frac{b}{a}\right|}$$

if signs of a & b are same then correlation $(x,y) = -1$

if " " " different " $(x,y) = 1$

=x=

If $X \sim N(\mu, \sigma^2)$, $(Y/x) \sim N(x, \sigma^2)$
Show that $(X, Y) \sim BN(\mu, \mu, \sigma^2, 2\sigma^2; \rho)$

<u>Solution</u>:- Given

$X \sim N(\mu, \sigma^2)$ then

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{—— (1)}$$

$(Y/x) \sim N(x, \sigma^2)$ then

$$g(y/x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{y-x}{\sigma}\right)^2} \quad \text{—— (2)}$$

$$\therefore h(x,y) = g(y/x) \cdot f(x)$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{y-x}{\sigma}\right)^2\right]}$$

consider $\left(\frac{y-x}{\sigma}\right)^2 = \left(\frac{y-\mu+\mu-x}{\sigma}\right)^2$

$$= \left(\frac{y-\mu}{\sigma}\right)^2 + \left(\frac{x-\mu}{\sigma}\right)^2 - 2\left(\frac{x-\mu}{\sigma}\right)\left(\frac{y-\mu}{\sigma}\right)$$

$$\therefore h(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{y-\mu}{\sigma}\right)^2 + \left(\frac{x-\mu}{\sigma}\right)^2 - 2\left(\frac{x-\mu}{\sigma}\right)\left(\frac{y-\mu}{\sigma}\right)\right]}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left[2\left(\frac{x-\mu}{\sigma}\right)^2 - 2\left(\frac{x-\mu}{\sigma}\right)\left(\frac{y-\mu}{\sigma}\right) + \left(\frac{y-\mu}{\sigma}\right)^2\right]}$$

The Bivariate normal p.d.f is given by — ③

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}$$

— ④

Comparing eq ② & ③

$\sigma_1\sigma_2\sqrt{1-\rho^2} = \sigma^2$, $\sigma_1^2(1-\rho^2) = \frac{1}{2}\sigma^2$

$\frac{\sigma_1\sigma_2(1-\rho^2)}{\rho} = \sigma^2$, $\sigma_2^2(1-\rho^2) = \sigma^2$,

$\mu_1 = \mu_2 = \mu$

On solving, we get

$\rho^2 = \frac{1}{2}$, $\sigma_2^2 = 2\sigma^2$, $\sigma_1^2 = \sigma^2$

thus $(X,Y) \sim BN\left(\mu, \mu, \sigma^2, 2\sigma^2, \frac{1}{\sqrt{2}}\right)$

$= \times =$