Problems based CRV (1)

1. If a r.v has the probability density function $f(x)$ as

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

find the probabilities that it will take on a value (i) between 1 and 3
(ii) greater than 0.5.

Solution:-

(i) The probability that a variable takes a value between 1 and 3 is given by

$$P(1 \le x \le 3) = \int_1^3 f(x)\,dx$$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3 \qquad \left[\because \int e^{-ax}\,dx = \frac{e^{-ax}}{-a} + c\right]$$

$$= 2\left[\frac{e^{-6}}{-2} - \frac{e^{-2}}{(-2)}\right]$$

$$= \frac{2}{2}\left[e^{-2} - e^{-6}\right]$$

$$= e^{-2} - e^{-6}$$

(ii) The probability that a variable takes a value greater than 0.5 is.

$$P(x \geq 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{\infty} 2 \cdot e^{-2x}\,dx.$$

$$= \cancel{2}\left(\frac{e^{-2x}}{-\cancel{2}}\right)_{0.5}^{\infty}$$

$$= -\left(e^{-2(\infty)} - e^{-2(0.5)}\right)$$

$$(\because e^{-\infty} = 0)$$

$$= -\left(0 - e^{-1}\right)$$

$$= e^{-1}.$$

$$\therefore P(x \geq 0.5) = e^{-1}$$

$= \times =$.

2. A continuous r.v has the p.d.f

$$f(x) = \begin{cases} K x e^{-\lambda x}, & x \geq 0,\ \lambda > 0. \\ 0, & \text{otherwise} \end{cases}$$

(i) Determine K.

(ii) Find Mean

(iii) Variance.

Solution:-

(i) $\because$ the total probability is unity

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e $$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} K x e^{-\lambda x}\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} K x e^{-\lambda x}\,dx = 1$$

$$\Rightarrow K\left[\int_{0}^{\infty} x e^{-2x}\,dx\right] = 1$$

$$\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\right.$$

where $v_1 = \int v\,dx$ $\quad u' = \frac{du}{dx}$

$v_2 = \int v_1\,dx$ $\quad u'' = \frac{du'}{dx} = \frac{d}{dx}\left(\frac{du}{dx}\right)$

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$$\left. = \frac{d^2u}{dx^2}\right]$$

$$\Rightarrow K\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\cdot\left(\frac{e^{-\lambda x}}{(-\lambda)^2}\right)\right]_0^\infty = 1$$

$$\Rightarrow K\left[\frac{e^{-\lambda(0)}}{\lambda^2}\right] = 1$$

$$\Rightarrow K\left[\frac{1}{\lambda^2}\right] = 1$$

$$\therefore \underline{\underline{K = \lambda^2}}$$

(ii) mean $= \mu = E(x) = \int_0^\infty x\, f(x)\, dx$

$$= \int_0^\infty x \cdot K \cdot x \cdot e^{-\lambda x}\, dx$$

$$= \lambda^2 \int_0^\infty x^2 e^{-\lambda x}\, dx$$

$$= \lambda^2\left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x\left(\frac{e^{-\lambda x}}{(-\lambda)^2}\right) + 2\left(\frac{e^{-\lambda x}}{(-\lambda)^3}\right)\right]_0^\infty$$

$$= \cancel{\lambda^2}\left[\frac{2\, e^{-\lambda(0)}}{\lambda^3}\right]$$

$$E(x) = \frac{2}{\lambda}$$

$$E(x^2) = \int_0^\infty x^2 f(x)\, dx$$

$$= \lambda^2 \int_0^\infty x^2 \cdot x \cdot e^{-\lambda x}\, dx$$

$$= \lambda^2 \int_0^\infty x^3 e^{-\lambda x}\, dx$$

$$= \lambda^2\left[x^3 \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \frac{e^{-\lambda x}}{(-\lambda)^2} + 6x \frac{e^{-\lambda x}}{(-\lambda)^3} - 6\frac{e^{-\lambda x}}{(-\lambda)^4}\right]_0^\infty$$

$$= \cancel{\lambda^2}\left[6\,\frac{e^{-\lambda(0)}}{\lambda^{4}{}^{2}}\right]$$

$$= \frac{6}{\lambda^2}$$

$\therefore$ Variance $= V(x) = \sigma^2$

$$= E(x^2) - [E(x)]^2$$

$$= \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$= \frac{6-4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow x =$$