Higher-order Cheeger Inequalities

Spectral Graph Theory

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March 31, 2022

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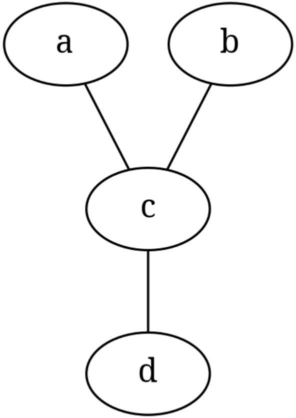

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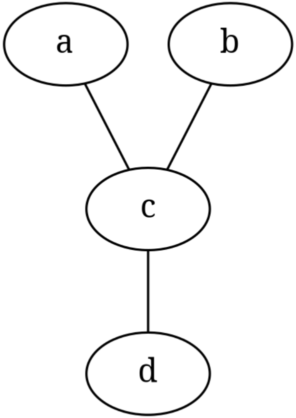
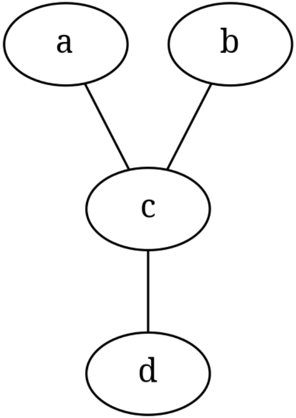


Laplacian Matrix



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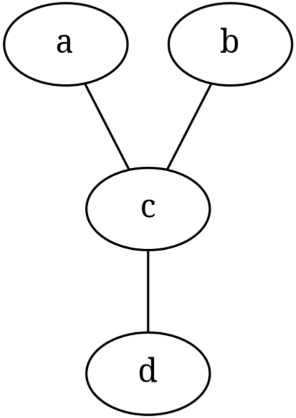

$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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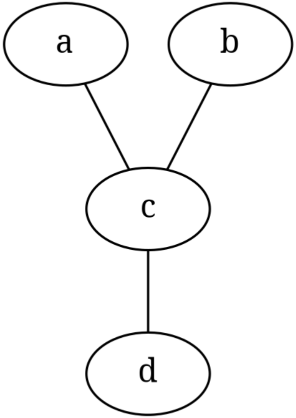

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$$D = \begin{array}{c c} \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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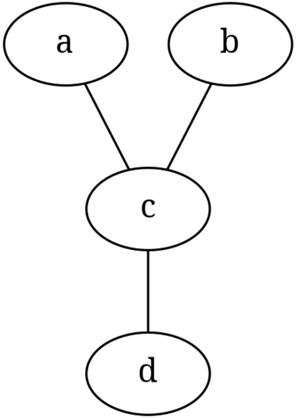

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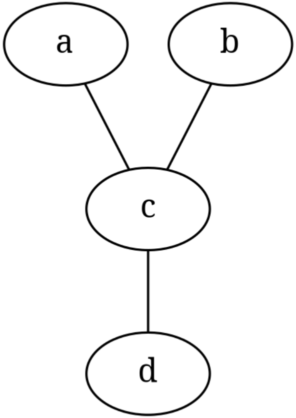

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Normalized: $\mathcal{L} = D^{-1/2} L D^{-1/2}$

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Eigenvalues of L	$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$
Eigenvectors of L	$f_1 \quad f_2 \quad \ldots \quad f_n$

Eigenvalues of Laplacian

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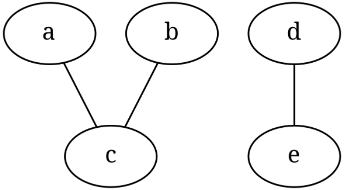

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Eigenvectors as an Optimization Problem

Relationship between λ_2 and arbitrary $f \in \mathbb{R}^n$ (i.e $f : V \to \mathbb{R}$).

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Courant-Fischer Theorem:

$$\lambda_2 = \min_{\mathbf{1} \perp f \in \mathbb{R}^n} \frac{\sum_{\{x,y\} \in E} (f(x) - f(y))^2}{\sum_{v \in V} f(v)^2}$$

f_2 is the vector which minimizes the above.

Graph Drawing Problem

For a general graph, how can we draw it? (map $V \to \mathbb{R}^2$)

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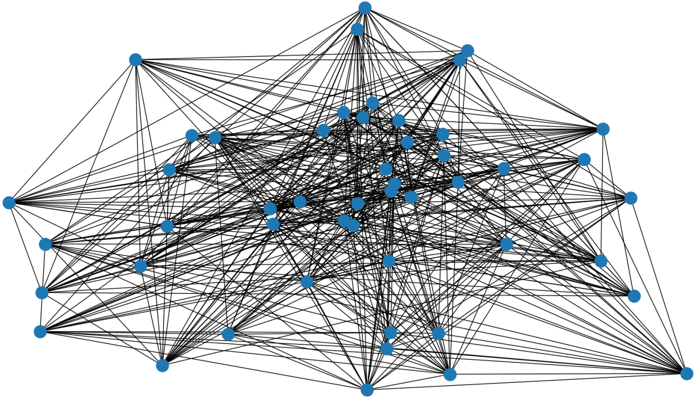

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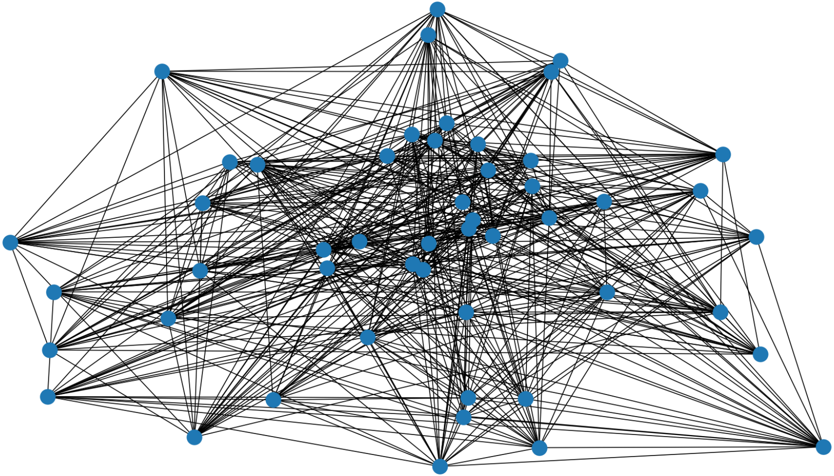

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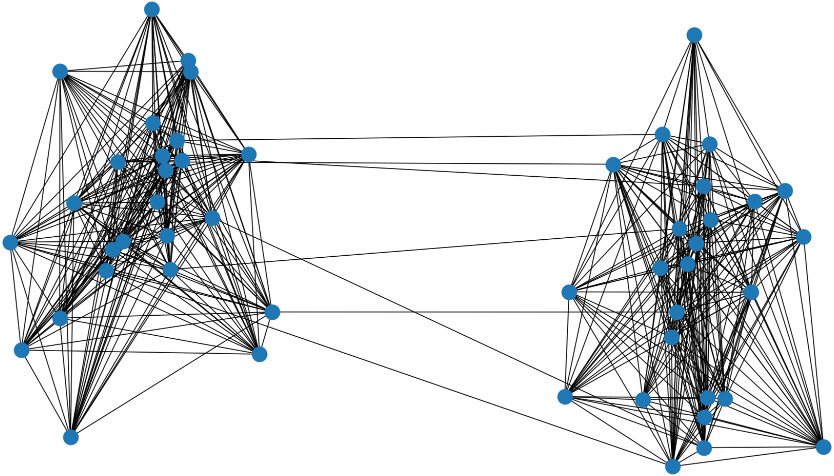

Algebraic Connectivity

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$$\lambda_2 \approx 0 \iff G \text{ is } \textit{nearly} \text{ disconnected?}$$

Cheeger Constant

$$h = \min_{S \subseteq V} \max\{\phi(S), \phi(S^c)\}$$

where $\phi(S) = \dfrac{\{\#\text{ edges leaving } S\}}{\{\#\text{ vertices in } S\}}$

h tells us how 'hard' the Graph is to bisect.

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Higher-order Cheeger Inequalities

Does this generalize for $k > 2$?

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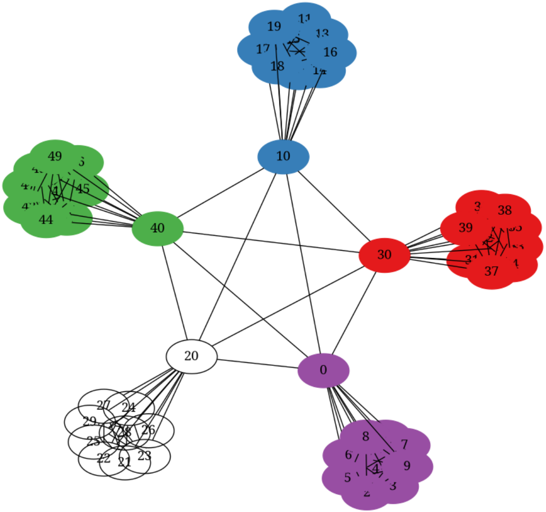
$$\frac{\lambda_k}{2} \leq h_k \leq O(k^2)\sqrt{\lambda_k}$$

[Lee, Oveis Gharan, Trevisan'14]

k-clustering Problem



Proof Overview ($k = 3$)

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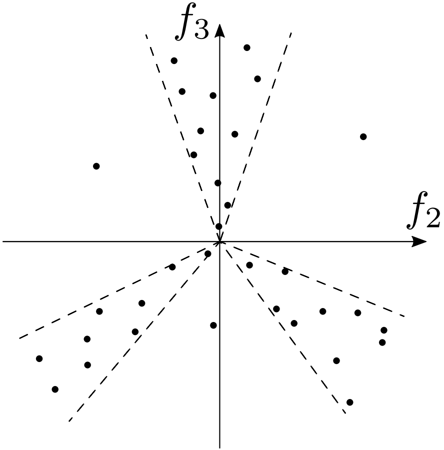

Proof Overview

Goal: Split graph into k clusters to form upper bound on h_k

Spectral embedding: $F(v) = (f_2(v), f_3(v), \ldots, f_k(v))$

1. Regions of diameter Δ contain less than $\frac{1}{k(1-\Delta^2)}$ fraction of vertices

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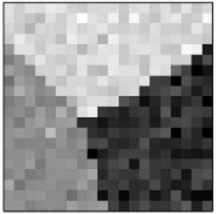
3. Form the clustering from vertices within each region

Applications of Spectral Clustering



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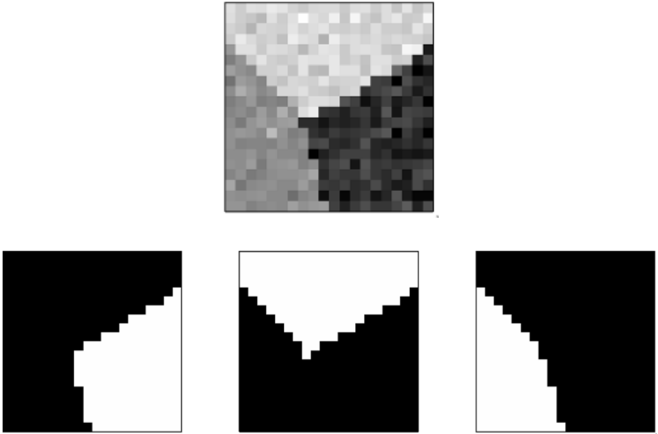

[Shi, Malik'00]

Thank you for listening.

Fan Chung (1997)
Spectral Graph Theory

James Lee, Shayan Oveis Gharan, Luca Trevisan (2014)
Multi-way spectral partitioning and higher-order Cheeger inequalities
Journal of the ACM