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Answer Key 4

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Lec 07:

1 (1) $\because \left|f_n(x)-|x|\right| = \left|\frac{1}{n^2(\sqrt{x^2+\frac{1}{n^2}}+|x|)}\right| \le \frac{1}{n}$

$\therefore \lim\limits_{n\to\infty}\sup\limits_{x\in\mathcal{X}}\left|f_n(x)-|x|\right| = 0$

$\therefore$ uniformly convergent

(2) $\because \sup\limits_{x\in\mathcal{X}}\left|f_n(x)-0\right| = \frac{1}{4}$

$\therefore$ not uniformly convergent

(3) $\because \left|f_n(x)-0\right| \le \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \le \frac{1}{n+1}$

$\therefore \lim\limits_{n\to\infty}\sup\limits_{x\in\mathcal{X}}\left|f_n(x)-0\right| = 0$

$\therefore$ uniformly convergent

(4) $\because$ if $n>100$, $\lim\limits_{n\to\infty}\sup\limits_{x\in\mathcal{X}}\left|f_n(x)-0\right| = \lim\limits_{n\to\infty}\frac{\ln n}{n} = 0$

$\therefore$ uniformly convergent

2 (1) $S(x) = \begin{cases} x, & x\in[0,1) \\ 0, & x=1 \end{cases}$ is not continuous

$\therefore$ not uniformly convergent

(2) denote $a_n(x) = \frac{x^2}{\left(1+x^2\right)^n},\ b_n(x) = (-1)^n$

$\because$ if $n > 2$, $\sup\limits_{x \in \mathcal{X}} |a_n(x) - 0| < \frac{1}{n-1}$

$\therefore a_n(x) \xrightarrow{\mathcal{X}} 0$

$\because a_n(x)$ is about n monotonically decreasing and $\sum\limits_{n=1}^{\infty} b_n(x)$ is uniformly bounded

$\therefore$ uniformly convergent

(3) $\because \left| \frac{\sin nx}{\sqrt[3]{n^4+x^4}} \right| \leq \frac{1}{n^{\frac{4}{3}}}$

$\therefore$ uniformly convergent

(4) $\because \left| \frac{x}{1+n^4x^2} \right| \leq \frac{1}{n^2}$

$\therefore$ uniformly convergent

(5) denote $a_n(x) = \frac{1}{\sqrt{n+x}}$, $b_n(x) = \sin nx \sin x$

$\because \sin nx \sin x = \frac{\cos(n-1)x - \cos(n+1)x}{2}$

$\therefore \sum\limits_{n=1}^{\infty} b_n(x)$ is uniformly bounded

$\because a_n(x)$ is about n monotonically decreasing and $a_n(x) \xrightarrow{\mathcal{X}} 0$

$\therefore$ uniformly convergent

(6) $\because \left| \frac{(-1)^n\left(1-e^{-nx}\right)}{n^2+x^2} \right| \leq \left| \frac{1}{n^2} \right|$

$\therefore$ uniformly convergent

3 $\because \left| \frac{\ln(1+nx)}{nx^n} \right| \leq \frac{1}{x^{n-1}} \leq \frac{1}{\alpha^{n-1}}$

$\therefore$ uniformly convergent $\square$

4 $\because f_0(x)$ is continuous over $[0, a]$

$\therefore \exists A$ s.t. $|f(x)| < A$

$\because f_n(x) = \int_0^x f_{n-1}(t)\,\mathrm{d}t$

$\therefore \left|f_n(x)\right| \leq \frac{Ax^n}{n!} \leq \frac{Aa^n}{n!}$

$\therefore f_n(x) \xrightarrow{\mathcal{X}} 0$ $\square$

5 if $\sum\limits_{n=1}^{\infty} \left|f_n(x)\right|$ uniformly convergent

$\because \left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} \left|f_i(x)\right|$

$\therefore \forall \epsilon > 0,\ \exists N > 0,\ \text{when } m,n > N,\ \forall x \in \mathcal{X},\ \left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} \left|f_i(x)\right| < \epsilon$

$\therefore \sum_{n=1}^{\infty} f_n(x)$ uniformly convergent □

but the inverse is not true, for example $f_n(x) = \frac{(-1)^n x}{n}$ □

Lec 08:

1 denote $a_n = \max(|\varphi_n(a)|, |\varphi_n(b)|)$

$\because \varphi_n(x)$ is about x monotonous over $[a, b]$

$\therefore |\varphi_n(x)| \leq a_n \leq |\varphi_n(a)| + |\varphi_n(b)|$

$\because \sum_{n=1}^{\infty} \left|\varphi_n(a)\right|,\ \sum_{n=1}^{\infty} \left|\varphi_n(b)\right|$ is absolutely convergent

$\therefore$ uniformly convergent □

2 $\forall a, b,\ 0 < a < b,\ x \in [a, b]$

$\because 0 < \frac{n}{\mathrm{e}^{xn}} \leq \frac{n}{\mathrm{e}^{an}}$ and $\sum_{n=1}^{\infty} \frac{n}{\mathrm{e}^{an}}$ is convergent

$\therefore \sum_{n=1}^{\infty} n\mathrm{e}^{-nx}$ is uniformly convergent over $(0, +\infty)$

$\because n\mathrm{e}^{-nx}$ is continuous

$\therefore$ continuous □

3 denote $a_n(x) = \frac{\sin nx}{n^3}$

$\because \left|a_n(x)\right| \leq \frac{1}{n^3}$

$\therefore \sum_{n=1}^{\infty} a_n(x)$ is uniformly convergent

$\because a_n(x)$ is continuous $\therefore f(x)$ is continuous

$\because \left|a_n'(x)\right| \leq \frac{1}{n^2} \therefore \sum_{n=1}^{\infty} a_n'(x)$ is uniformly convergent

$\therefore f'(x) = \sum_{n=1}^{\infty} a_n'(x)$

$\because a_n'(x)$ is continuous $\therefore f'(x)$ is continuous □

4 for $n>1,\ \forall m\in\mathbb{N}\because \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(e^x\right)^{-\ln n}=(-\ln n)^m\left(e^x\right)^{-\ln n}=(-\ln n)^m\frac{1}{n^x}$

$\therefore \forall\alpha>1,$ when $x\geq\alpha,\ \left|\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right)\right|\leq(\ln n)^m\frac{1}{n^\alpha}$

$\because \sum\limits_{n=1}^{\infty}(\ln n)^m\frac{1}{n^\alpha}$ is convergent and $(\ln n)^m\frac{1}{n^x}$ is continuous

$\therefore \sum\limits_{n=1}^{\infty}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right)$ uniformly convergent

$\therefore \zeta^{(n)}(x)$ is continuous $\square$

5 $\because \left|\frac{\sin(2^n\pi x)}{2^n}\right|\leq\frac{1}{2^n}\therefore$ uniformly convergent

$\because \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(2^n\pi x)}{2^n}=\pi\cos\left(2^n\pi x\right)$

$$\lim_{n\to+\infty}\pi\cos\left(2^n\pi x\right)=\begin{cases}\text{not exists},\ x\neq\frac{m}{2^k},\ m,k\in\mathbb{Z}\\ \pi,\ x=\frac{m}{2^k},\ m,k\in\mathbb{Z}\end{cases}\neq 0$$

$\therefore$ can't doing derivation at every formula $\square$

6 if $|x|=1,\ f(x)=\int_{-\pi}^{\pi}\frac{1-x^2}{1+x^2-2x\cos\theta}\,\mathrm{d}\theta=0$

if $|x|<1,\ f(x)=\int_{-\pi}^{\pi}1+2\sum\limits_{n=1}^{\infty}x^n\cos n\theta\,\mathrm{d}\theta$

$\because |x^n\cos n\theta|\leq|x^n|,\ \sum\limits_{n=1}^{\infty}|x^n|$ is convergent

$\therefore \sum\limits_{n=1}^{\infty}x^n\cos n\theta$ is about θ uniformly convergent

$\therefore f(x)=2\pi+2\sum\limits_{n=1}^{\infty}\int_{-\pi}^{\pi}x^n\cos n\theta\,\mathrm{d}\theta=2\pi$

if $|x|>1,\ f(x)=-2\pi$

in conclusion

$$f(x)=\begin{cases}0,\ |x|=1\\ 2\pi,\ |x|<1\\ -2\pi,\ |x|>1\end{cases}$$