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Answer Key 8

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Lec 16

1 (1) $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \mathrm{e}^{x\sqrt{1-\cos^2 x}}(-\sin x) - \mathrm{e}^{x\sqrt{1-\sin^2 x}}(\cos x) + \int_{\sin x}^{\cos x} \sqrt{1-y^2}\mathrm{e}^{x\sqrt{1-y^2}}\,\mathrm{d}y$

$$= \int_{\sin x}^{\cos x} \sqrt{1-y^2}\mathrm{e}^{x\sqrt{1-y^2}}\,\mathrm{d}y - \mathrm{e}^{x|\sin x|}\sin x - \mathrm{e}^{x|\cos x|}\cos x$$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \int_{x^2}^{x^2} f(x,s)\,\mathrm{d}s + \int_0^x 2xf(t,x^2)\,\mathrm{d}t = \int_0^x 2xf(t,x^2)\,\mathrm{d}t$

2 $F(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{x\cos\theta}(\mathrm{e}^{ix\sin\theta}+\mathrm{e}^{-ix\sin\theta})}{2}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{x\mathrm{e}^{i\theta}}+\mathrm{e}^{x\mathrm{e}^{-i\theta}}}{2}\,\mathrm{d}\theta$

$F'(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{i\theta}\frac{\mathrm{e}^{x\mathrm{e}^{i\theta}}}{2}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-i\theta}\frac{\mathrm{e}^{x\mathrm{e}^{-i\theta}}}{2}\,\mathrm{d}\theta = 0$

$\therefore F(x) = \text{const}, \quad F(x) = F(0) = 1$

3 $I = -\int_0^1 \sin(\ln x)\int_a^b x^y\,\mathrm{d}y\,\mathrm{d}x = -\int_a^b\int_0^1 \sin(\ln x)x^y\,\mathrm{d}x\,\mathrm{d}y$

$= \int_a^b \frac{1}{(y+1)^2+1}\,\mathrm{d}y = \arctan(b+1) - \arctan(a+1)$

4 $F(x) = \frac{1}{h^2}\int_0^h \left[\int_0^h f(x+\xi+\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi = \frac{1}{h^2}\int_x^{x+h}\left[\int_\xi^{\xi+h} f(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi$

$F'(x) = \frac{1}{h^2}\left[\int_{x+h}^{x+2h} f(\eta)\,\mathrm{d}\eta - \int_x^{x+h} f(\eta)\,\mathrm{d}\eta\right]$

$F''(x) = \frac{1}{h^2}\{f(x+2h) - f(x+h) - [f(x+h) - f(x)]\}$

$= \frac{f(x+2h)-2f(x+h)+f(h)}{h^2}$ $\square$

Lec 17

1 (1) $\because \left|\frac{\cos xy}{x^2+y^2}\right| \le \frac{1}{a^2+y^2},\ \int_0^\infty \frac{1}{a^2+y^2}\,\mathrm{d}y$ converges

$\therefore$ uniformly converges

(2) $\because \int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$ uniformly converges, $\mathrm{e}^{-\alpha x}$ is about x monotonically decreasing and $\left|\mathrm{e}^{-\alpha x}\right| \le 1$

$\therefore$ uniformly converges

(3) $\because f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}},\ x > 0 \\ 0,\ x = 0 \end{cases}$

$\therefore$ not uniformly converges

(4) $\int_0^1 \frac{1}{x^y} \sin\frac{1}{x}\,\mathrm{d}x = \int_1^\infty x^{y-2} \sin x\,\mathrm{d}x$

apply second mean value theorem for definite integrals, $\forall N \in \mathbb{N}^*,\ \exists y = 2 - \frac{1}{(2N+1)\pi}$

$\int_{2N\pi}^{(2N+1)\pi} x^{y-2} \sin x\,\mathrm{d}x \le 2[(2N+1)\pi]^{y-2} = 2[(2N+1)\pi]^{-\frac{1}{(2N+1)\pi}} > 1$

$\therefore$ not uniformly converges

(5) $\forall \epsilon > 0,\ \exists \delta < \frac{1}{4}\epsilon^2$ s.t. $\int_{y-\delta}^{y} \frac{\sin xy}{\sqrt{y-x}}\,\mathrm{d}x \le \int_{y-\delta}^{y} \frac{1}{\sqrt{y-x}}\,\mathrm{d}x = 2\sqrt{\delta} \le \epsilon$

$\therefore$ uniformly converges

(6) $\int_0^1 x^{p-1} \ln^2 x\,\mathrm{d}x = \int_{-\infty}^0 \mathrm{e}^{px} x^2\,\mathrm{d}x$

$\because \forall N > 0,\ \exists p = \frac{1}{\sqrt[3]{N+1}}$ s.t. $\int_{-\sqrt[3]{N+1}}^{-\sqrt[3]{N}} \mathrm{e}^{px} x^2\,\mathrm{d}x > \frac{\mathrm{e}^{-\sqrt[3]{N+1}p}}{3} = \frac{\mathrm{e}^{-1}}{3}$

$\therefore$ not uniformly converges

2 $\because F(u) \le \int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x = A$

$\therefore F(u)$ is bounded

$\forall \epsilon > 0,\ \exists K(\epsilon) > \frac{1}{\epsilon}$, s.t. $\int_{-\infty}^{-K} |f(x)|\,\mathrm{d}x + \int_K^{+\infty} |f(x)|\,\mathrm{d}x < \frac{\epsilon}{3}$

$\exists \delta = \frac{1}{3K^2A}$ when $|u_1 - u_2| < \delta$

$|F(u_2) - F(u_1)| = \left|\int_{-\infty}^{+\infty} f(x)(\cos u_2 x - \cos u_1 x)\,\mathrm{d}x\right|$

$\le \frac{2}{3}\epsilon + 2|\int_{-K}^{+K} f(x) \sin\frac{u_2+u_1}{2}x \sin\frac{u_2-u_1}{2}x\,\mathrm{d}x|$

$\leq \frac{2}{3}\epsilon + \frac{1}{3KA}\int_{-K}^{+K} \left|f(x)\sin\frac{u_2+u_1}{2}x\right| \mathrm{d}x$

$\leq \epsilon$

$\therefore$ uniformly continuous $\square$

3 $\because \int_0^{+\infty} f(t)\,\mathrm{d}t$ uniformly converges, e^{-xt} is about x monotonically decreasing and $\left|\mathrm{e}^{-xt}\right| \leq 1\ (x \geq 0)$

$\therefore \int_0^{+\infty} \mathrm{e}^{-xt} f(t)\,\mathrm{d}t$ uniformly converges

$\therefore \lim\limits_{x\to 0}\int_0^{+\infty} \mathrm{e}^{-xt} f(t)\,\mathrm{d}t = \int_0^{+\infty} f(t)\,\mathrm{d}t$ $\square$