Peking University

College of Engineering

Answer Key

Mathematical Analysis (3)

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Contents

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Lec 01

1 Use reduction to absurdity. Suppose $\lim_{n\to\infty} a_n \neq 0$ or doesn't exist.

$\therefore \exists\varepsilon>0, \forall N_1>0,\ \exists n>N_1,\ |a_n|>3\varepsilon$. For such $\varepsilon,\ \exists N_2>0,\ \forall n>N_2,\ |a_{2n}+2a_n|<\varepsilon$

$\therefore \exists N>N_2,\ |a_N|>3\varepsilon$ and $|a_{2N}+2a_N|<\varepsilon$

$\therefore\ |a_{2N}|>5\varepsilon$. similarly, $|a_{4N}|>9\varepsilon$ and then $|a_{2^pN}|>\left(2^{p+1}+1\right)\varepsilon$ for $p\in\mathbb{N}$, which contradict the boundedness of a_n $\square$

2 Let $a_n = \frac{2}{3}+b_n$. So $\lim_{n\to\infty}(b_{2n}+2b_n)=0$. According to the conclusion of previous problem, $\lim_{n\to\infty} b_n = 0$, and $\lim_{n\to\infty} a_n = \frac{2}{3}$ is evident. $\square$

3 (1) $\because x_{n+2}=1+\frac{1}{x_{n+1}}=1+\frac{1}{1+\frac{1}{x_n}}=1+\frac{x_n}{1+x_n}$ and using mathematical induction: $x_n>0$

$\therefore$ for $n\geqslant 3,\ 1<x_n<2$

$\therefore 1\leqslant \varliminf_{n\to\infty} x_n \leqslant \varlimsup_{n\to\infty} x_n \leqslant 2$ $\square$

(2) $\because$ for $n\geqslant 3,\ \left|\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}\right| = \left|-\frac{1}{x_n+1}\right| < \frac{1}{2}$

$\therefore x_n$ is a Cauchy sequence

Calculate the positive fixed point of equation $x^*=1+\frac{1}{x^*}$

$\therefore \lim_{n\to\infty} x_n = x^* = \frac{1+\sqrt{5}}{2}$

5 (1) $\because \lim_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$

$= \varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n \leqslant \varlimsup_{n\to\infty}(x_n+y_n) \leqslant \varlimsup_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$

$= \lim_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$ $\square$

(2) The proof for $\varlimsup_{n\to\infty} y_n = \pm\infty$ is direct. Suppose $\varlimsup_{n\to\infty} y_n = A$.

$\forall\varepsilon_1>0, \exists N_1(\varepsilon_1)>0,\ \forall n>N_1(\varepsilon_1),\ y_n<A+\varepsilon_1$, and $\exists\left\{y_{n_k^{\varepsilon_1}}\right\},\ y_{n_k^{\varepsilon_1}}>A-\varepsilon_1$

$\forall\varepsilon_2>0,\ \exists N_2(\varepsilon_2)>0,\ \forall n>N_2(\varepsilon_2),\ |x_n-x^*|<\varepsilon_2,\ x^*=\lim_{n\to\infty} x_n$

$\forall\varepsilon>0,\ \exists\varepsilon_1>0,\ \varepsilon_2>0,\ \forall\delta_1\in(0,\varepsilon_1),\ \delta_2\in(-\varepsilon_2,\ \varepsilon_2)$, s.t. $0\leqslant\frac{\delta_1}{x^*+\delta_2}-\frac{A\delta_2}{x^*(x^*+\delta_2)}<\varepsilon$, and $0\leqslant\frac{\delta_1}{x^*}+\frac{(A-\delta_1)\delta_2}{x^*(x^*+\delta_2)}<\varepsilon$

$\therefore$ for $\forall\varepsilon>0,\ \exists N=\max\left(N_1(\varepsilon_1),\ N_2(\varepsilon_2)\right),\ \forall n>N,\ (x_ny_n)>x^*A+\varepsilon$

and $\exists\left\{y_{n_k^{\varepsilon_1}}x_{n_k^{\varepsilon_1}}\right\},\ y_{n_k^{\varepsilon_1}}x_{n_k^{\varepsilon_1}}>x^*A-\varepsilon$ $\square$

6 (1) $\sup_{k\geqslant n} a_k = 1,\ \inf_{k\geqslant 2n} a_k = \inf_{k\geqslant 2n+1} a_k = a_{2n+1} = \frac{1}{2^{-2n-1}-1}$

$\therefore \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = 1,\ \varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -1$

(2) $\sup_{k\geqslant 2n} a_k = \sup_{k\geqslant 2n-1} a_k = a_{2n} = 1 + \frac{1}{2n}$
$\inf_{k\geqslant 2n} a_k = \inf_{k\geqslant 2n+1} a_k = a_{2n+1} = -1 - \frac{1}{2n+1}$

$\therefore \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = 1,\ \varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -1$

(3) $|a_n| = \frac{1}{n} \to 0$
$\therefore \varlimsup_{n\to\infty} a_n = \varliminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0$

(4) For a period of $n = 0 \sim 9 \bmod 10$, maximum a_n is $\sin\frac{2\pi}{5}$, minimum a_n is $-\sin\frac{2\pi}{5}$
$\therefore \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = \sin\frac{2\pi}{5},\ \varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -\sin\frac{2\pi}{5}$

Lec 02

1 Suppose $\varlimsup_{n\to\infty} na_n > 0$. Then just suppose $\varlimsup_{n\to\infty} na_n \geqslant 1$
$\therefore \exists \left\{a_{n_k}\right\},\ a_{n_k} \geqslant \frac{1}{n_k}$
Then $\exists \left\{a_{n_{k_l}}\right\},\ n_{k_{l+1}} \geqslant 2n_{k_l}$
$\therefore \sum_{n=1}^{\infty} a_n \geqslant \sum_{l=2}^{\infty} \left(\frac{n_{k_l}}{2}\frac{1}{n_{k_l}}\right) = \infty$
$\therefore \varlimsup_{n\to\infty} na_n = 0$
$\because \varliminf_{n\to\infty} na_n \geqslant 0$
$\therefore \lim_{n\to\infty} na_n = 0$

3 (1) $\because 0 \leqslant \frac{1}{(5n-4)(5n+1)} \leqslant \frac{1}{n^2}$
$\therefore$ absolutely convergent

(2) $\because \lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$
$\therefore$ divergent

(3) $\because 0 \leqslant \frac{1}{2^n} + \frac{1}{3^n} \leqslant \frac{1}{2^{n-1}}$
$\therefore$ absolutely convergent

(4) $\because 0 \leqslant \frac{1}{(3n-2)(3n+1)} \leqslant \frac{1}{n^2}$
$\therefore$ absolutely convergent

(5) $\because \lim_{n\to\infty} a_n = 1 \neq 0$
$\therefore$ divergent

4 (1) $\forall\varepsilon>0,\ \exists N>\log_{|q|}\frac{\varepsilon(1-|q|)}{A},\ \forall n_1,\ n_2>N$

$\sum\limits_{n=n_1}^{n_2}|a_nq^n|\leqslant A\sum\limits_{n=n_1}^{n_2}|q|^n=A|q|^{n_1}\frac{1-|q|^{n_2-n_1+1}}{1-|q|}<\varepsilon$

$\therefore$ absolutely convergent

(2) $\because a_n=\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}>\frac{1}{3n}$

$\therefore \exists\varepsilon=1>0,\ \forall N>0,\ \exists n_1>N,\ n_2=3n_1+10>N$

$\sum\limits_{n=n_1}^{n_2}a_n>\sum\limits_{n=n_1}^{n_2}\frac{1}{3n}>\frac{2n_2}{3}\frac{1}{3n_2}>\frac{2}{9}$

$\therefore$ divergent

5 Let $b_k=\sum\limits_{n=n_{k-1}+1}^{n_k}a_n$

$\therefore \forall\varepsilon>0,\ \exists K>0,\ \forall k_1,k_2>K,\ \left|\sum\limits_{k=k_1}^{k_2}b_k\right|<\varepsilon$

$\therefore \exists N>n_{K+1}+1,\ \forall n_1,n_2>N,\ \left|\sum\limits_{n=n_1}^{n_2}a_n\right|<\left|\sum\limits_{k=K}^{\infty}b_k\right|\leqslant\varepsilon$

$\therefore$ absolutely convergent

2

Lec 03

1 (2) $\ln(1+\frac{1}{n})=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+o(\frac{1}{n^4})$

$\therefore a_n=\frac{1}{12n^2}+o(\frac{1}{n^2})$

$\therefore \lim\limits_{n\to\infty}\frac{a_n}{\frac{1}{12n^2}}=1$

$\therefore$ absolutely convergent

(3) $\ln[(1+\frac{1}{n})^n]=n\ln(1+\frac{1}{n})=1-\frac{1}{2n}+o(\frac{1}{n})$

$\therefore (1+\frac{1}{n})^n=\mathrm{e}^{1-\frac{1}{2n}+o(\frac{1}{n})}=\mathrm{e}-\frac{\mathrm{e}}{2n}+o(\frac{1}{n})$

$\therefore \lim\limits_{n\to\infty}\frac{a_n}{(\frac{\mathrm{e}}{2n})^p}=1$

$$\begin{cases} p>1: \text{absolutely convergent} \\ p\leqslant 1: \text{divergent} \end{cases}$$

2 (1) $\lim\limits_{n\to\infty}\sqrt[n]{(1-\frac{1}{n})^{n^2}}=\lim\limits_{n\to\infty}(1-\frac{1}{n})^n=\frac{1}{\mathrm{e}}<1$

$\therefore$ absolutely convergent

(2) $\frac{a_{n+1}}{a_n} = \frac{x}{1+x^n}$

$$\begin{cases} x < 1: \lim\limits_{n\to\infty} \frac{x}{1+x^n} = x < 1, \text{absolutely convergent} \\ x = 1: \lim\limits_{n\to\infty} \frac{x}{1+x^n} = \frac{1}{2}, \text{absolutely convergent} \\ x > 1: \lim\limits_{n\to\infty} \frac{x}{1+x^n} = 0, \text{absolutely convergent} \end{cases}$$

3 (1) $\int_2^\infty 3^{-x^{\frac{1}{2}}}\,\mathrm{d}x = \int_{\sqrt{2}}^\infty 2t\cdot 3^{-t}\,\mathrm{d}t$

$\therefore$ absolutely convergent

(2) $\frac{1}{a^{\ln x}} = \frac{1}{x^{\ln a}}$

$$\begin{cases} a > \mathrm{e}: \ln a > 1, \text{absolutely convergent} \\ a \leqslant \mathrm{e}: \ln a \leqslant 1, \text{divergent} \end{cases}$$

4 (1) $\lim\limits_{n\to\infty} a_n = \frac{1}{2}$

$\therefore$ divergent

(4) $\lim\limits_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = 1$

$\therefore$ divergent

(5) For $n > 1000, \ln(n+1) > 2$

$\therefore \frac{1}{[\ln(n+1)]^n} < \frac{1}{2^n}$

$\therefore$ absolutely convergent

6 (3) $\int \frac{1}{x\ln x(\ln\ln x)}\,\mathrm{d}x = \ln\ln\ln x + C$

$\therefore$ for $\sigma \leqslant 0$, divergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}} \leqslant 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n(\ln n)^{1+\sigma}}$$

$$\leqslant 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{2^{k-1}[\ln(2^{k-1})]^{1+\sigma}}$$

$$\leqslant 10 + \sum_{k=2}^{\infty} \frac{1}{[\ln(2^{k-1})]^{1+\sigma}}$$

$$= 10 + \sum_{k=2}^{\infty} \frac{1}{(k-1)^{1+\sigma}}$$

$\therefore$ for $\sigma > 0$, $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}}$ absolutely convergent

Also $\because$ for $\forall p > 0, \exists X > 0, \forall x > X, (\ln x)^p > \ln \ln x$

$\therefore$ for $\sigma > 0$, absolutely convergent

(4) Let $p = 1$, $\int_2^{\infty} \frac{1}{x \ln x (\ln \ln x)^q} \mathrm{d}x = \int_{\ln 2}^{\infty} \frac{\mathrm{d}t}{t(\ln t)^q}$

$\therefore$ similar to the condition in previous problem

In conclusion:

$$\begin{cases} p > 1 : \text{absolutely convergent} \\ p = 1 : \begin{cases} q > 1 : \text{absolutely convergent} \\ q \leqslant 1 : \text{divergent} \end{cases} \\ p < 1 : \text{divergent} \end{cases}$$

Lec 04

2 (1) $(k^2 - 1)a_{k^2-1} = \frac{1}{k^2-1}$, $k^2 a_{k^2} = 1$

$\exists \varepsilon = \frac{1}{2}, \forall N > 0, \exists k > N, \left| 1 - \frac{1}{k^2-1} \right| > \varepsilon$

$\therefore \lim\limits_{n \to \infty} a_n$ doesn't exist $\square$

(2) Let $b_k = \left[\frac{1}{k^2} + \sum\limits_{n=k^2+1}^{(k+1)^2-1} \frac{1}{n^2} \right]$

Evidently b_k is absolutely convergent

Use conclusion of Lec 02 Problem 05

$\therefore$ absolutely convergent $\square$

3 (1) Evidently $\lim\limits_{n \to \infty} x_n$ exists. Let $x_n \to A$

$\therefore \lim\limits_{n \to \infty} \frac{1 - \frac{x_n}{x_{n+1}}}{\frac{x_{n+1} - x_n}{A}} = 1$

$\because \sum\limits_{n=1}^{\infty} \frac{x_{n+1} - x_n}{A} = 1 - \frac{x_1}{A}$, converge

$\therefore$ absolutely convergent $\square$

(2) $\exists\varepsilon=\frac{1}{2},\forall N>0,\ \exists n_1,\ n_2>N,\ \sum\limits_{n=n_1}^{n_2}(1-\frac{x_n}{x_{n+1}})>\frac{x_{n_2}-x_{n_1}}{x_{n_2}}>\varepsilon$

$\therefore$ divergent $\square$

4 (1) Let $b_k=a_{3k+1}+a_{3k+2}+a_{3k+3}$

$|b_k|$ monotonically decreases to 0 and $\operatorname{sgn}\left(\frac{b_k}{b_{k-1}}\right)=-1$

Use the conclusion of Lec 04 Problem 05, a_n converges

$\because |a_n|=\frac{1}{n}$

$\therefore$ conditionally convergent

(2) For $a\neq 0$, $n\sqrt{1+\frac{a^2}{n^2}}=n+\frac{a^2}{2n}+o(\frac{1}{n})$

$\therefore a_n=(-1)^n\sin\left(\frac{\pi a^2}{2n}\right)+o\left(\frac{1}{n}\right)$

$|a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}})=-1$

$\therefore a_n$ converges

$\because \lim\limits_{n\to\infty}\frac{\sin\frac{\pi a^2}{2n}}{\frac{1}{100n}}>1$

$\therefore$ In conclusion:

$$\begin{cases} a\neq 0: \text{conditionally convergent} \\ a=0: \text{absolutely convergent} \end{cases}$$

(3) $\ln\left(1+\frac{(-1)^n}{n^p}\right)=\frac{(-1)^n}{n^p}-\frac{1}{n^{2p}}+o\left(\frac{1}{n^{2p}}\right)$

For $p>0$, $\left|\frac{(-1)^n}{n^p}\right|$ monotonically decreases to 0 and $\operatorname{sgn}\left(\frac{\frac{(-1)^n}{n^p}}{\frac{(-1)^{n-1}}{(n-1)^p}}\right)=-1$

$\sum\limits_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges when $p>\frac{1}{2}$, diverges when $p\leqslant\frac{1}{2}$

$\sum\limits_{n=1}^{\infty}\frac{1}{n^{p}}$ converges when $p>1$, diverges when $p\leqslant 1$

For $p\leqslant 0$, evidently diverge

$\therefore$ In conclusion:

$$\begin{cases} p \leqslant \frac{1}{2} : \text{divergent} \\ \frac{1}{2} < p \leqslant 1 : \text{conditionally convergent} \\ p > 1 : \text{absolutely convergent} \end{cases}$$

(4) Let $b_k = |a_{2k-1}| + |a_{2k}|$, $0 < b_k < \frac{1}{2^{k-1}}$

$\therefore$ b_k converges

Use conclusion of Lec 02 Problem 05, $|a_n|$ converges

$\therefore$ absolutely convergent

(5) $\sum\limits_{n=1}^{2N} a_n < e - \sum\limits_{n=1}^{N} \frac{1}{2n}$

$\because -\sum\limits_{n=1}^{N} \frac{1}{2n} \to -\infty$

$\therefore$ divergent

(6) $\because$ $|a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore$ a_n converges

$\because |a_n| > \frac{1}{n}$

$\therefore$ conditionally convergent

(7) $\because \int_2^\infty x^3 2^{-x} \, \mathrm{d}x$ converges

$\therefore$ absolutely convergent

(8) $\because$ $|a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore$ a_n converges

$\because |a_n| > \frac{1}{20n}$

$\therefore$ conditionally convergent

(9) $\because$ $|a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore$ a_n converges

$\because \lim\limits_{n\to\infty} \frac{a_n}{\frac{1}{n}} = x$

$\therefore$ In conclusion:

$$\begin{cases} x \neq 0 : \text{conditionally convergent} \\ x = 0 : \text{absolutely convergent} \end{cases}$$

(10) Let $b_k = a_{2k-1} + a_{2k} = \frac{2}{k-1}$

$\therefore$ divergent

3

Lec 04

5 Let b_k equal to the sum of k^{th} set of successive a_n which have the same sign

If n_0 is in the k_0^{th} set, denote $k(n_0) = k_0$

$\because \sum\limits_{k=1}^{\infty} b_k$ convergent, $\lim\limits_{k\to\infty} b_k = 0$

$\therefore \forall \varepsilon > 0,\ \exists K > 0,\ \forall k_1,\ k_2 > K,\ |\sum\limits_{k=k_1}^{k_2} b_k| < \varepsilon,\ |b_{k_1}| + |b_{k_2}| < \varepsilon$

$\because \exists N,\ k(N) > K,\ \forall n_1,\ n_2 > N,\ |\sum\limits_{n=n_1}^{n_2} a_n| \leqslant \varepsilon + |b_{k(n_1)}| + |b_{k(n_2)}| < 2\varepsilon$

$\therefore$ convergent

For $a_n = \frac{(-1)^{[\sqrt{n}]}}{n}$, let $b_k = (-1)^k \sum\limits_{n=(k-1)^2+1}^{k^2} \frac{1}{n}$

$\therefore |b_k| < \frac{2k}{(k-1)^2}$

$\therefore |b_k|$ monotonically decreases to 0 and $\text{sgn}(\frac{b_k}{b_{k-1}}) = -1$

$\therefore b_k$ converges But $|a_n| = \frac{1}{n}$

$\therefore$ conditionally convergent

8 $\forall \varepsilon > 0, \exists N > 0, \forall n_1, n_2 > N$, $\max\{|\sum\limits_{n=n_1}^{n_2} n(a_n - a_{n-1})|, |n_1 a_{n_1-1}|,\ |n_2 a_{n_2}|\} < \varepsilon$

$\therefore |\sum\limits_{n=n_1}^{n_2-1} a_n| = |\sum\limits_{n=n_1}^{n_2} n(a_n - a_{n-1}) + n_1 a_{n_1-1} - n_2 a_{n_2}| < 3\varepsilon$

$\therefore$ convergent $\square$

Lec 05

3 (1) $x_{n+1}-x_n=\frac{\ln(n+1)}{n+1}-\frac{1}{2}\left[\ln^2(n+1)-\ln^2 n\right]=\frac{\ln(n+1)}{n+1}-\int_n^{n+1}\frac{\ln x}{x}\,\mathrm{d}x$

$\frac{\ln x}{x}$ is monotonically decreasing over $[\mathrm{e},\ +\infty)$

if $n>3,\ \frac{\ln(n+1)}{n+1}<\int_n^{n+1}\frac{\ln x}{x}\,\mathrm{d}x<\frac{\ln n}{n}$

$\therefore x_{n+1}<x_n,\ x_n$ monotonically decreasing

$$x_n=\sum_{k=1}^{n}\frac{\ln k}{k}-\frac{1}{2}(\ln n)^2=\sum_{k=1}^{n}\frac{\ln k}{k}-\int_1^n\frac{\ln x}{x}\,\mathrm{d}x$$

$$=\sum_{k=1}^{2}\frac{\ln k}{k}-\int_1^3\frac{\ln x}{x}\,\mathrm{d}x+\sum_{k=3}^{n}\frac{\ln k}{k}-\int_3^n\frac{\ln x}{x}\,\mathrm{d}x>\frac{\ln 2}{2}-\ln^2 3$$

$\therefore$ convergence

(2) $x_{n+1}-x_n=\frac{1}{\sqrt{n+1}}-2\sqrt{n+1}-2\sqrt{n}<0$

$\therefore x_n$ monotonically decreasing

$$\sqrt{n}=\sqrt{n}-\sqrt{n-1}+\sqrt{n-1}-\sqrt{n-2}+\cdots+\sqrt{2}-\sqrt{1}+1$$

$$=\frac{1}{\sqrt{n}+\sqrt{n-1}}+\frac{1}{\sqrt{n-1}+\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{2}+\sqrt{1}}+1$$

$$<\frac{1}{2}\left(\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{1}}\right)+1$$

$$x_n>1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-\left(\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{1}}+2\right)=-2+\frac{1}{\sqrt{n}}>-2$$

$\therefore$ convergence

4 $\sum_{n=0}^{\infty}|x|^n=\frac{1}{1-|x|},\ \sum_{n=0}^{\infty}|y|^n=\frac{1}{1-|y|}$ (both absolutely convergent)

$\therefore \sum_{n=1}^{\infty}(x^{n-1}+x^{n-2}y+\cdots+y^{n-1})=\sum_{n=0}^{\infty}x^n\sum_{n=0}^{\infty}y^n=\frac{1}{(1-x)(1-y)}$ □

5 $\lim_{n\to\infty}\sqrt[n]{n!}=\infty$

$\therefore$ radius of convergence is ∞

$\therefore \sum_{n=0}^{\infty}\frac{x^n}{n!}=\mathrm{e}^x,\ \sum_{n=0}^{\infty}\frac{y^n}{n!}=\mathrm{e}^y$ both absolutely convergent

$\therefore \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{y^n}{n!}=\mathrm{e}^{x+y}=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}$ □

Lec 06

1 (1) Let $p_n = q_n = 1,\ \prod_{n=1}^{\infty}(p_n + q_n) = \prod_{n=1}^{\infty} 2 = \infty$

$\therefore$ divergent

(2) $\prod_{n=1}^{\infty} p_n,\ \prod_{n=1}^{\infty} q_n$ converge

$\Rightarrow \sum_{n=1}^{\infty} \ln p_n,\ \sum_{n=1}^{\infty} \ln q_n$ converge

$\Rightarrow \sum_{n=1}^{\infty} (\ln p_n + \ln q_n)$ converges

$\Rightarrow \sum_{n=1}^{\infty} \ln(p_n q_n)$ converges

$\Rightarrow \prod_{n=1}^{\infty} p_n q_n$ converges

(3) Let $q_n = p_n$ and use conclusion of previous problem

$\therefore$ convergent

(4) $\prod_{n=1}^{\infty} q_n$ converges

$\Rightarrow \sum_{n=1}^{\infty} \ln q_n$ converges

$\Rightarrow -\sum_{n=1}^{\infty} \ln q_n$ converges

$\Rightarrow \prod_{n=1}^{\infty} \frac{1}{q_n}$ converges

Use conclusion of Lec 06 Prob 1(2), $\prod_{n=1}^{\infty} \frac{p_n}{q_n}$ converges

2 Denote $T_n = \prod_{k=1}^{n}(1 + u_k),\ S_n = \sum_{k=1}^{n} u_k,\ S'_n = \sum_{k=1}^{n}(u_k)^2$

$\therefore S_{2n} = \sum_{k=1}^{n} \frac{1}{k} \to \infty,\ S'_{2n} > 2\sum_{k=1}^{n} \frac{1}{k} \to \infty$

$\because \lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} u_{2n} = 0$

$\therefore S_n,\ S'_n$ diverges

$\because (1 + u_{2k-1})(1 + u_{2k}) = 1 - \frac{1}{k^{\frac{3}{2}}}$

$\therefore T_{2n}$ converges, let A denote its limit

$\therefore \forall\varepsilon > 0, \exists N_1 > 0, \forall n > N_1,\ |T_{2n} - A| < \varepsilon$

And $\lim\limits_{n\to\infty} \frac{T_{2n+1}}{T_{2n}} = u_{2n+1} + 1 = 1$

$\therefore$ for $\varepsilon > 0,\ \exists N_2 > 0,\ \forall n > N_2,\ |T_{2n+1} - T_{2n}| < \varepsilon$

$\therefore \forall\varepsilon > 0,\ \exists N = \max\{2N_1 + 10,\ 2N_2 + 10\} > 0,\ \forall n > N,\ |T_n - A| < 2\varepsilon$

$\therefore T_n$ converges $\square$

3 (1) $\lim\limits_{n\to\infty} \frac{\ln[(\frac{n^2-1}{n^2+1})^p]}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{p\ln(1-\frac{2}{n^2+1})}{\frac{1}{n^2}} = -2p$

$\therefore$ convergent

(2) $\lim\limits_{n\to\infty} \frac{\ln\sqrt[n]{1+\frac{1}{n}}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}\ln(1+\frac{1}{n})}{\frac{1}{n^2}} = 1$

$\therefore$ convergent

(3) $\lim\limits_{n\to\infty} \frac{\ln\sqrt[n]{\ln(n+x)-\ln n}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}\ln\ln(1+\frac{x}{n})}{\frac{1}{n}} = -\infty$

$\therefore$ divergent

(4) $\lim\limits_{n\to\infty} \frac{\ln\frac{n^2-4}{n^2-1}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{\ln\left(1-\frac{3}{n^2-1}\right)}{\frac{1}{n^2}} = -3$

$\therefore$ convergent

(5) $\ln a^{\frac{(-1)^n}{n}} = \frac{(-1)^n}{n}\ln a$

$\because \frac{1}{n}$ monotonically decreases to 0

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}\ln a$ converges

$\therefore$ convergent

(6) $\because \prod\limits_{k=1}^{n} \sqrt{\frac{k+1}{k+2}} = \sqrt{\frac{2}{n+2}} \to 0$

$\therefore$ divergent

5 Due to convergence $\lim\limits_{n\to\infty} a_n = 0$

$\tan(\frac{\pi}{4} + x) = 1 + Ax + o(x),\ A = \tan'(\frac{\pi}{4}) > 0$

$\therefore \lim\limits_{n\to\infty} \frac{|\ln[\tan(\frac{\pi}{4}+a_n)]|}{|a_n|} = A$

$\therefore \sum\limits_{n=1}^{\infty} \ln[\tan(\frac{\pi}{4} + a_n)]$ converges

$\therefore$ convergent $\square$

4

Lec 07

1 (1) $\because \left|f_n(x) - |x|\right| = \left|\frac{1}{n^2(\sqrt{x^2+\frac{1}{n^2}}+|x|)}\right| \leq \frac{1}{n}$

$\therefore \lim\limits_{n\to\infty} \sup\limits_{x\in\mathcal{X}} \left|f_n(x) - |x|\right| = 0$

$\therefore$ uniformly convergent

(2) $\because \sup\limits_{x\in\mathcal{X}} \left|f_n(x) - 0\right| = \frac{1}{4}$

$\therefore$ not uniformly convergent

(3) $\because \left|f_n(x) - 0\right| \leq \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \leq \frac{1}{n+1}$

$\therefore \lim\limits_{n\to\infty} \sup\limits_{x\in\mathcal{X}} \left|f_n(x) - 0\right| = 0$

$\therefore$ uniformly convergent

(4) $\because$ if $n > 100$, $\lim\limits_{n\to\infty} \sup\limits_{x\in\mathcal{X}} \left|f_n(x) - 0\right| = \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

$\therefore$ uniformly convergent

2 (1) $S(x) = \begin{cases} x, & x \in [0,1) \\ 0, & x = 1 \end{cases}$ is not continuous

$\therefore$ not uniformly convergent

(2) denote $a_n(x) = \frac{x^2}{(1+x^2)^n}$, $b_n(x) = (-1)^n$

$\because$ if $n > 2$, $\sup\limits_{x\in\mathcal{X}} |a_n(x) - 0| < \frac{1}{n-1}$

$\therefore a_n(x) \xrightarrow{\mathcal{X}} 0$

$\because a_n(x)$ is about n monotonically decreasing and $\sum\limits_{n=1}^{\infty} b_n(x)$ is uniformly bounded

$\therefore$ uniformly convergent

(3) $\because \left|\frac{\sin nx}{\sqrt[3]{n^4+x^4}}\right| \leq \frac{1}{n^{\frac{4}{3}}}$

$\therefore$ uniformly convergent

(4) $\because \left|\frac{x}{1+n^4x^2}\right| \leq \frac{1}{n^2}$

$\therefore$ uniformly convergent

(5) denote $a_n(x) = \frac{1}{\sqrt{n+x}},\ b_n(x) = \sin nx \sin x$

$\because \sin nx \sin x = \frac{\cos(n-1)x - \cos(n+1)x}{2}$

$\therefore \sum\limits_{n=1}^{\infty} b_n(x)$ is uniformly bounded

$\because a_n(x)$ is about n monotonically decreasing and $a_n(x) \xrightarrow{\mathcal{X}} 0$

$\therefore$ uniformly convergent

(6) $\because \left|\frac{(-1)^n\left(1-e^{-nx}\right)}{n^2+x^2}\right| \leq \left|\frac{1}{n^2}\right|$

$\therefore$ uniformly convergent

3 $\because \left|\frac{\ln(1+nx)}{nx^n}\right| \leq \frac{1}{x^{n-1}} \leq \frac{1}{\alpha^{n-1}}$

$\therefore$ uniformly convergent $\square$

4 $\because f_0(x)$ is continuous over $[0, a]$

$\therefore \exists A$ s.t. $|f(x)| < A$

$\because f_n(x) = \int_0^x f_{n-1}(t)\,\mathrm{d}t$

$\therefore \left|f_n(x)\right| \leq \frac{Ax^n}{n!} \leq \frac{Aa^n}{n!}$

$\therefore f_n(x) \xrightarrow{\mathcal{X}} 0$ $\square$

5 if $\sum\limits_{n=1}^{\infty} \left|f_n(x)\right|$ uniformly convergent

$\because \left|\sum\limits_{i=n}^{m} f_i(x)\right| \leq \sum\limits_{i=n}^{m} \left|f_i(x)\right|$

$\therefore \forall \epsilon > 0,\ \exists N > 0,$ when $m, n > N,\ \forall x \in \mathcal{X},\ \left|\sum\limits_{i=n}^{m} f_i(x)\right| \leq \sum\limits_{i=n}^{m} \left|f_i(x)\right| < \epsilon$

$\therefore \sum\limits_{n=1}^{\infty} f_n(x)$ uniformly convergent $\square$

but the inverse is not true, for example $f_n(x) = \frac{(-1)^n x}{n}$ $\square$

Lec 08

1 denote $a_n = \max(|\varphi_n(a)|, |\varphi_n(b)|)$

$\because \varphi_n(x)$ is about x monotonous over $[a, b]$

$\therefore |\varphi_n(x)| \leq a_n \leq |\varphi_n(a)| + |\varphi_n(b)|$

$\because \sum\limits_{n=1}^{\infty} \left|\varphi_n(a)\right|,\ \sum\limits_{n=1}^{\infty} \left|\varphi_n(b)\right|$ is absolutely convergent

$\therefore$ uniformly convergent $\square$

2 $\forall a, b,\ 0 < a < b,\ x \in [a, b]$

$\because 0 < \frac{n}{\mathrm{e}^{xn}} \leq \frac{n}{\mathrm{e}^{an}}$ and $\sum\limits_{n=1}^{\infty} \frac{n}{\mathrm{e}^{an}}$ is convergent

$\therefore \sum\limits_{n=1}^{\infty} n\mathrm{e}^{-nx}$ is uniformly convergent over $(0, +\infty)$

$\because n\mathrm{e}^{-nx}$ is continuous

$\therefore$ continuous $\square$

3 denote $a_n(x) = \frac{\sin nx}{n^3}$

$\because \left|a_n(x)\right| \leq \frac{1}{n^3}$

$\therefore \sum\limits_{n=1}^{\infty} a_n(x)$ is uniformly convergent

$\because a_n(x)$ is continuous $\therefore f(x)$ is continuous

$\because \left|a_n'(x)\right| \leq \frac{1}{n^2} \therefore \sum\limits_{n=1}^{\infty} a_n'(x)$ is uniformly convergent

$\therefore f'(x) = \sum\limits_{n=1}^{\infty} a_n'(x)$

$\because a_n'(x)$ is continuous $\therefore f'(x)$ is continuous $\square$

4 for $n > 1,\ \forall m \in \mathbb{N} \because \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(e^x\right)^{-\ln n} = (-\ln n)^m \left(e^x\right)^{-\ln n} = (-\ln n)^m \frac{1}{n^x}$

$\therefore \forall \alpha > 1,$ when $x \geq \alpha,\ \left|\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right)\right| \leq (\ln n)^m \frac{1}{n^\alpha}$

$\because \sum\limits_{n=1}^{\infty} (\ln n)^m \frac{1}{n^\alpha}$ is convergent and $(\ln n)^m \frac{1}{n^x}$ is continuous

$\therefore \sum\limits_{n=1}^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left(\frac{1}{n^x}\right)$ uniformly convergent

$\therefore \zeta^{(n)}(x)$ is continuous $\square$

5 $\because \left|\frac{\sin(2^n \pi x)}{2^n}\right| \le \frac{1}{2^n} \therefore$ uniformly convergent

$\because \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(2^n \pi x)}{2^n} = \pi \cos(2^n \pi x)$

$$\lim_{n\to+\infty} \pi \cos(2^n \pi x) = \begin{cases} \text{not exists}, \ x \ne \frac{m}{2^k}, \ m, k \in \mathbb{Z} \\ \pi, \ x = \frac{m}{2^k}, \ m, k \in \mathbb{Z} \end{cases} \ne 0$$

$\therefore$ can't doing derivation at every formula $\square$

6 if $|x| = 1, \ f(x) = \int_{-\pi}^{\pi} \frac{1-x^2}{1+x^2-2x\cos\theta} \,\mathrm{d}\theta = 0$

if $|x| < 1, \ f(x) = \int_{-\pi}^{\pi} 1 + 2\sum\limits_{n=1}^{\infty} x^n \cos n\theta \,\mathrm{d}\theta$

$\because |x^n \cos n\theta| \le |x^n|, \ \sum\limits_{n=1}^{\infty} |x^n|$ is convergent

$\therefore \sum\limits_{n=1}^{\infty} x^n \cos n\theta$ is about θ uniformly convergent

$\therefore f(x) = 2\pi + 2\sum\limits_{n=1}^{\infty} \int_{-\pi}^{\pi} x^n \cos n\theta \,\mathrm{d}\theta = 2\pi$

if $|x| > 1, \ f(x) = -2\pi$

in conclusion

$$f(x) = \begin{cases} 0, \ |x| = 1 \\ 2\pi, \ |x| < 1 \\ -2\pi, \ |x| > 1 \end{cases}$$

5

Lec 09

1 (1) $\varliminf\limits_{n\to\infty} \sqrt[n]{\frac{n!}{2^n}} = +\infty$

$\therefore R = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(2) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{[3+(-1)^n]^n}} = \frac{1}{4}$

$\therefore R = \frac{1}{4}$

$\because \sum\limits_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1}$ converge and $|x| = \frac{1}{4}$: $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverge

$\therefore$ convergence region: $(-\frac{1}{4}, \frac{1}{4})$

(3) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n+1}{\ln(n+1)}} = 1$

$\therefore R = 1$

$\sum\limits_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

$\frac{\ln(n+1)}{n+1}$ monotonically decreases to 0 when $n > 3$

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln(n+1)}{n+1}$ converges

$\therefore$ divergent when $x = 1$, convergent when $x = -1$

$\therefore$ convergence region: $[-1, 1)$

(4) $|x| = 1$: convergent

$|x| > 1$: $\lim\limits_{n\to\infty} \frac{|x|^{n^2}}{2^n} = +\infty \neq 0$

$\therefore$ convergence region: $[-1, 1]$

(5) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{3^n+(-2)^n}} = \frac{1}{3} \lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{1+(-\frac{2}{3})^n}} = \frac{1}{3}$

$\therefore R = \frac{1}{3}$

$x + 1 = \frac{1}{3}$: $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum\limits_{n=1}^{\infty} \frac{(-\frac{2}{3})^n}{n}$ converges $\Rightarrow$ diverges

$x + 1 = -\frac{1}{3}$: $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, $\sum\limits_{n=1}^{\infty} \frac{(\frac{2}{3})^n}{n}$ converges $\Rightarrow$ converges

$\therefore$ convergence region: $[-\frac{4}{3}, -\frac{2}{3})$

(6) $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{(1+\frac{1}{n})^{n^2}}} = \frac{1}{\mathrm{e}}$

$\therefore R = \frac{1}{\mathrm{e}}$

$\lim\limits_{n\to\infty} \frac{(1+\frac{1}{n})^{n^2}}{\mathrm{e}^n} = \lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x^2}}}{\mathrm{e}^{\frac{1}{x}}} = \mathrm{e}^{\lim\limits_{x\to 0} \ln \frac{(1+x)^{\frac{1}{x^2}}}{\mathrm{e}^{\frac{1}{x}}}} = \mathrm{e}^{\lim\limits_{x\to 0} \frac{\ln(1+x)-x}{x^2}} = \mathrm{e}^{-\frac{1}{2}} \neq 0$

$\therefore$ convergence region: $(-\frac{1}{\mathrm{e}}, \frac{1}{\mathrm{e}})$

(7) Just let $a \geqslant b$

$\varlimsup\limits_{n\to\infty} \sqrt[n]{a^n + b^n} = a \varlimsup\limits_{n\to\infty} \sqrt[n]{1 + (\frac{b}{a})^n} = a$

$\therefore R = a$

$\lim\limits_{n\to\infty} \frac{a^n}{a^n+b^n} \geqslant \frac{1}{2}$

$\therefore$ convergence region: $(-a, a)$

(8) $\varlimsup\limits_{n\to\infty} \sqrt[n]{n2^n} = 2$

$\therefore R = 2$

$|x| = \sqrt{2}$: $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

$\therefore$ convergence region: $[-\sqrt{2}, \sqrt{2}]$

(9) $1 \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)(2n)!!}{(2n-1)!!}} \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)(2n)!!(2n-1)!!}{(2n-1)!!(2n-1)!!}} \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)!}{(2n-1)!}}$
$= \lim\limits_{n\to\infty} \sqrt[2n-1]{2n(2n+1)} = 1$

$\therefore R = 1$

$|x| = 1$: $\frac{(2n-1)!!}{(2n+1)(2n)!!}$ monotonically decreases to 0

$\therefore$ convergent

$\therefore$ convergence region: $[-1, 1]$

2 (1) $|x| < \sqrt{A} \Rightarrow x^2 < A \Rightarrow$ convergent

$|x| > \sqrt{A} \Rightarrow x^2 > A \Rightarrow$ divergent

$\therefore R = \sqrt{A}$

(2) $\frac{1}{R} = \varlimsup\limits_{n\to\infty} \sqrt[n]{|a_n + b_n|} \leqslant \varlimsup\limits_{n\to\infty} \sqrt[n]{2\max(|a_n|, |b_n|)} = \max(\frac{1}{A}, \frac{1}{B})$

$\therefore R \geqslant \min(A, B)$

(3) $\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n b_n|} \leqslant \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\, \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \frac{1}{AB}$

$\therefore R \geqslant AB$

3 Let $A_m(x) = \sum\limits_{n=1}^{m} a_n x^n,\ B_m(x) = \sum\limits_{n=1}^{m} b_n x^n,\ S_m(x) = \sum\limits_{n=1}^{m} a_n b_{m-n+1} x^{m+1},$

$R_m(x) = A_m(x)B_m(x) - \sum\limits_{n=1}^{m} S_n(x),\ M = \sup\limits_n[\max(|a_n|, |b_n|)]$

$0 < x < 1$: $|R_m(x)| < M^2 \frac{m^2-m}{2} x^{m+2}$

$\therefore \lim\limits_{m\to\infty} R_m(x) = 0,\ 0 < x < 1$

$\because [0, 1]$ is in the uniformly convergence region of $A_m(x), B_m(x), \sum\limits_{n=1}^{m} S_n(x)$

$\therefore \lim\limits_{m\to\infty} R_m(x)$ is also continuous in $[0, 1]$

$\therefore \lim\limits_{m\to\infty} R_m(1) = 0$

$\therefore \sum\limits_{n=1}^{\infty} S_n(x) = \lim\limits_{m\to\infty} A_m B_m = AB$ □

4 Apparently $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent in $[0, r)$

$\therefore \int_0^x \lim\limits_{m\to\infty} \sum\limits_{n=0}^{m} a_n t^n \,\mathrm{d}t = \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$

$\because \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is uniformly convergent in $[0, r]$ and $\frac{a_n x^{n+1}}{n+1}$ is continuous

$\therefore \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is continuous in $[0, r]$

$\therefore \lim\limits_{x\to r^-} \int_0^x \lim\limits_{m\to\infty} \sum\limits_{n=0}^{m} a_n t^n \,\mathrm{d}t$ exists and is equal to $\sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ □

$f(x) = -\frac{\ln(1-x)}{x} = \sum\limits_{n=0}^{\infty} \frac{x^n}{n+1}$

$\because r = \underline{\lim_{n\to\infty}} \sqrt[n]{n+1} = 1$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ convergent

$\therefore \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^1 \frac{\ln(1-x)}{x} \,\mathrm{d}x$ □

Lec 10

1 Convergence in $(-\infty,\infty)$ is apparent.

$f^{(2n+1)}(0)=\sum\limits_{m=1}^{\infty}\frac{(-1)^n(2^m)^{2n+1}}{m!}$

$\therefore f(x)\sim\sum\limits_{n=0}^{\infty}\sum\limits_{m=1}^{\infty}\frac{(-1)^n(2^m x)^{2n+1}}{m!(2n+1)!}\sim\sum\limits_{n=0}^{\infty}(\mathrm{e}^{2^{2n+1}}-1)\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\because \varlimsup\limits_{n\to\infty}\sqrt[2n+1]{\frac{(2n+1)!}{\mathrm{e}^{2^{2n+1}}}}=0$

$\therefore R=0$

$\therefore$ divergent when $x\neq 0$

2 $f^{(n)}(0)=\frac{n!}{a^{n+1}}$

$f(x)\sim\sum\limits_{n=0}^{\infty}\frac{x^n}{a^{n+1}}$

$\varlimsup\limits_{n\to\infty}\sqrt[n]{|a|^{n+1}}=|a|$

$|\frac{a^n}{a^{n+1}}|=\frac{1}{|a|}\nrightarrow 0$

$\therefore$ convergence region: $(-|a|,|a|)$

3 $f^{(n)}(2)=\frac{(-1)^{n-1}(n-1)!}{2^n}$

$f(x)\sim\ln 2+\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}(x-2)^n}{2^n n}$

$\varlimsup\limits_{n\to\infty}\sqrt[n]{2^n n}=2$

$\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges and $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges

$\therefore$ convergence region: $(0,4]$

4 (1) $f^{(n)}(0)=(\frac{\sin x}{x})^{(n-1)}\big|_{x=0}$

$\because \sin x\sim\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\therefore \frac{\sin x}{x}\sim\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$

$f^{(2n+1)}(0)=(\frac{\sin x}{x})^{(2n)}\big|_{x=0}=\frac{(-1)^n(2n)!}{(2n+1)!}=\frac{(-1)^n}{2n+1}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}$

$R = \varliminf\limits_{n\to\infty} \sqrt[2n+1]{(2n+1)!(2n+1)} = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(2) $\cos t^2 \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}$

$\therefore f^{(4n+1)}(0) = (\cos t^2)^{(4n)}\Big|_{t=0} = \frac{(-1)^n (4n)!}{(2n)!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}$

$R = \varliminf\limits_{n\to\infty} \sqrt[4n+1]{(2n)!(4n+1)} = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(3) Let $x = \tan t$

$\arctan \frac{2x}{1-x^2} = -\arctan(\tan 2t) = -2\arctan x \sim -2\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\varliminf\limits_{n\to\infty} \sqrt[2n+1]{2n+1} = 1$

$|x| = 1$: $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

$\therefore$ convergence region: $[-1, 1]$ (with definition: $\arctan(\pm\infty) = \pm\pi$)

(4) $f^{(1)}(0) = \frac{1}{\sqrt{1+x^2}} \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} x^{2n}$

$\therefore f^{(2n+1)}(0) = (\frac{1}{\sqrt{1+x^2}})^{(2n)}\Big|_{x=0} = \frac{(-1)^n (2n-1)!!(2n)!}{(2n)!!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!(2n+1)} x^{2n+1}$

$\varliminf\limits_{n\to\infty} \sqrt[2n+1]{\frac{(2n)!!(2n+1)}{(2n-1)!!}} = 1$ (see Lec 09 Prob 1(9))

$\frac{(2n-1)!!}{(2n)!!(2n+1)}$ monotonically decreases to 0

$\therefore$ converges when $|x| = 1$

$\therefore$ convergence region: $[-1, 1]$

5 (1) $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$

$\therefore \sum\limits_{n=1}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x - 1, x \in \mathbb{R}$

(2) $\int_0^x \ln(1+x)\,\mathrm{d}x = (1+x)\ln(1+x) - x$

$\because$ convergence radius of $\ln(1+x)$'s Maclaurin series is 1

$\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$ converges at $|x| = 1$

According to Lec 09 Prob 4, $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n(n+1)} = (1+x)\ln(1+x) - x$ in $[-1, 1]$

(define $(1+x)\ln(1+x) - x = 1$ at $x = -1$)

(3) $\int_0^x f(x)\,\mathrm{d}x \sim \sum\limits_{n=1}^{\infty} nx^n \sim \frac{x}{(x-1)^2}$

$\because$ Convergence radius is 1 and $f(x)$ diverges at $|x| = 1$

$\therefore f(x) = \left[\frac{x}{(x-1)^2}\right]' = \frac{1+x}{(1-x)^3}$ in $(-1, 1)$

(4) Convergence region is $\mathbb{R}$

$\therefore \int_0^x f(x)\,\mathrm{d}x = \sum\limits_{n=1}^{\infty} \frac{x^{2n+1}}{n!} = x(\mathrm{e}^{x^2} - 1)$

$\therefore f(x) = \left[x(\mathrm{e}^{x^2} - 1)\right]' = (2x^2 + 1)\mathrm{e}^{x^2} - 1$

(5) Let $A = \sum\limits_{n=1}^{\infty} \frac{2n-1}{2^n}$

$\therefore A = 2A - A = 1 + \sum\limits_{n=1}^{\infty} \frac{2n+1}{2^n} - \sum\limits_{n=1}^{\infty} \frac{2n-1}{2^n} = 3$

(6) $\sum\limits_{n=0}^{\infty} \frac{n^2+1}{2^n n!} = \mathrm{e}^{\frac{1}{2}} + \sum\limits_{n=1}^{\infty} \frac{n}{2^n(n-1)!} = \mathrm{e}^{\frac{1}{2}} + \frac{1}{2}\sum\limits_{n=1}^{\infty} \frac{(\frac{1}{2})^{n-1}}{(n-1)!} + \frac{1}{4}\sum\limits_{n=2}^{\infty} \frac{(\frac{1}{2})^{n-2}}{(n-2)!} = \frac{7}{4}\mathrm{e}^{\frac{1}{2}}$

6

Lec 12

3 $\because f'(x)$ monotonically increases in $[0, 2\pi]$

$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = -\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin nx\,\mathrm{d}x$

$= -\frac{1}{n\pi}\sum\limits_{i=0}^{n-1} \int_{\frac{2i}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x)\sin nx\,\mathrm{d}x$

$$= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\left[\int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi} f'(x)\sin nx\,\mathrm{d}x + \int_{\frac{2i+1}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x)\sin nx\,\mathrm{d}x\right]$$

$$= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi}\left[f'(x) - f'(x+\tfrac{\pi}{n})\right]\sin nx\,\mathrm{d}x \geq 0 \qquad \square$$

Lec 13

1 $\lim\limits_{p\to\infty}\int_{-\pi}^{\pi} f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \lim\limits_{p\to\infty}\int_0^{\pi}[f(t)-f(-t)]\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_0^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_{\delta}^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_0^{\delta}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$\because \frac{f(t)-f(-t)}{2\sin\frac{t}{2}} \in \mathcal{R}[\delta,\pi]$

$\therefore \lim\limits_{p\to\infty}\int_{\delta}^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t = 0$

$\because f(t)$ is continuous and has unilateral derivative at $t=0$

$\therefore \exists\delta>0,\ M>0$ s.t. when $t\in(0,\delta),\ -M < \frac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt < M$

$\therefore 0 = \lim\limits_{\delta\to0} -M\delta \leq \lim\limits_{\delta\to0}\lim\limits_{p\to\infty}\int_0^{\delta}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t \leq \lim\limits_{\delta\to0} M\delta = 0$

$\therefore \lim\limits_{p\to\infty}\int_{-\pi}^{\pi} f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t = \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t \qquad \square$

2 (1) $f(x) \backsim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n\cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = (-1)^n\frac{4}{n^2}$

$f(x) = \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

(2) $f(x) \backsim \sum\limits_{n=1}^{\infty} b_n\sin nx$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = -\frac{4}{\pi n^3} + \frac{2(-1)^n}{\pi}\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)$

$f(x) = \sum\limits_{n=1}^{\infty}\left[\frac{4(-1)^n}{\pi n^3} - \frac{4}{\pi n^3} - \frac{2\pi(-1)^n}{n}\right]\sin nx$

(3) $f(x) \backsim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,\mathrm{d}x = \frac{8\pi^2}{3}$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,\mathrm{d}x = \frac{4}{n^2}$

$b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx\,\mathrm{d}x = -\frac{4\pi}{n}$

$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)$

(4) from (1) $f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$

$f(\pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

3 $a_0 = \frac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{\pi}$

$a_n = \frac{(-1)^n (\mathrm{e}^{\pi} - \mathrm{e}^{-\pi})}{\pi(n^2+1)}$

$b_n = -\frac{(-1)^n n (\mathrm{e}^{\pi} - \mathrm{e}^{-\pi})}{\pi(n^2+1)}$

$f(x) = \frac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)} (\cos nx - n \sin nx) \right]$

if $x = \pi$, $\frac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2} \right) = \frac{1}{2}(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi})$

$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi(\mathrm{e}^{\pi} + \mathrm{e}^{-\pi})}{2(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi})} - \frac{1}{2}$

4 $\because f(x) \in \mathcal{R}$, apply bessel's inequality

$\therefore \sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ convergent

$\because \frac{|a_n|}{n} \leq \frac{1}{2}(a_n^2 + \frac{1}{n^2})$ and $\frac{|b_n|}{n} \leq \frac{1}{2}(b_n^2 + \frac{1}{n^2})$

$\therefore \sum_{n=1}^{\infty} \frac{a_n}{n}$ and $\sum_{n=1}^{\infty} \frac{b_n}{n}$ convergent □

7

Lec 14

1 $e^{\cos x}\cos(\sin x) = e^{\cos x}\frac{e^{i\sin x}+e^{-i\sin x}}{2} = \frac{1}{2}\left(e^{e^{ix}} + e^{e^{-ix}}\right) = \sum\limits_{n=0}^{\infty}\frac{1}{2}\frac{(e^{ix})^n+(e^{-ix})^n}{n!} = \sum\limits_{n=0}^{\infty}\frac{1}{2}\frac{e^{inx}+e^{-inx}}{n!}$

$= \sum\limits_{n=0}^{\infty}\frac{\cos(nx)}{n!}$

2 Let $f(x+c) = \frac{a_0'}{2}+\sum\limits_{n=1}^{\infty}a_n'\cos(nx)+b_n'\sin(nx) = \frac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos[n(x+c)]+b_n\sin[n(x+c)] =$

$\frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nc)+b_n\sin(nc)]\cos(nx) + [b_n\cos(nc) - a_n\sin(nc)]\sin(nx)$

$\therefore a_0' = a_0,\ a_n' = a_n\cos(nc)+b_n\sin(nc),\ b_n' = b_n\cos(nc) - a_n\sin(nc)$

3 $\pi - x = \sum\limits_{n=1}^{\infty}\frac{2}{n}\sin(nx),\ x\in(0,2\pi)$

$\therefore \frac{1}{2\pi}\int_0^{2\pi}f(x)(\pi-x)\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)\,\mathrm{d}x$

$\because \sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)$ is controlled by $\sum\limits_{n=1}^{\infty}\frac{2}{n}b_n$ and Lec 13 Prob 04's conclusion only needs $f(x)$ to be integrable.

$\therefore \sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)$ uniformly converges

$\therefore \frac{1}{2\pi}\int_0^{2\pi}f(x)(\pi-x)\,\mathrm{d}x = \frac{1}{2\pi}\sum\limits_{n=1}^{\infty}\int_0^{2\pi}\frac{2}{n}b_n\sin^2(nx)\,\mathrm{d}x = \sum\limits_{n=1}^{\infty}\frac{b_n}{n}$ $\square$

4 (1) Apply periodic extension to $f(x)$ and set $f(2n\pi) = 0$, which won't change Fourier series

$\because \sigma_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi}f(x+t)\left(\frac{\sin(\frac{nt}{2})}{\sin\frac{t}{2}}\right)^2\mathrm{d}x,\ \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(\frac{nt}{2})}{\sin\frac{t}{2}}\right)^2\mathrm{d}x = 1,\ |f(x+t)| \leqslant \frac{\pi}{2}$

$\therefore |\sigma_n(x)| \leqslant \frac{\pi}{2}$ $\square$

(2) Due to pointwise convergence, $\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n} = \begin{cases} f(x),\ 0<x<2\pi \\ 0,\ x=0 \end{cases}$

$\therefore \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right| < \frac{\pi}{2},\ 0\leqslant x<2\pi$

$\because \left|\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}\right|$ is 2π-periodic

$\therefore \left|\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}\right| < \frac{\pi}{2} + 1,\ x \in \mathbb{R}$ $\square$

Lec 15

1 (1) $a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = -\frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x = -\frac{b_n'}{n}$

$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,\mathrm{d}x = \frac{a_n'}{n}$

(ignore discontinuous points of f') $\square$

(2) Lec 13 Prob 04's conclusion can be extended as $\sum\limits_{n=1}^{\infty}\left|\frac{a_n'}{n}\right|$ and $\sum\limits_{n=1}^{\infty}\left|\frac{b_n'}{n}\right|$ converges, thus the convergence of $\sum\limits_{n=1}^{\infty}(|a_n| + |b_n|)$ is straightforward $\square$

(3) $|\frac{a_0}{2}| + \sum\limits_{n=1}^{\infty}(|a_n| + |b_n|)$ converges $\Rightarrow \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ absolutely uniformly converges

$\because \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ pointwise converges to $f(x)$

$\therefore \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ uniformly converges to $f(x)$ $\square$

2 (1) Due to symmetry, $b_n = 0$

$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,\mathrm{d}x = \frac{4}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x\cos(nx)\,\mathrm{d}x = \begin{cases} 0,\ n = 1 \\ \frac{2}{\pi}\frac{(-1)^{n+1}-1}{n^2-1},\ n \neq 1 \end{cases}$$

$f(x) = \frac{2}{\pi} - \sum\limits_{n=1}^{\infty} \frac{4}{\pi(4n^2-1)}\cos 2nx$

$\because$ the Fourier series is controlled by $\sum\limits_{n=2}^{\infty} \frac{4}{\pi(n^2-1)}$

$\therefore$ uniformly converges

$\because f(x)$ continuous

$\therefore$ the Fourier series pointwise thus uniformly converges to $f(x)$

(2) Due to symmetry, $a_n = 0$

$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,\mathrm{d}x = \frac{4[1-(-1)^n]}{n^3\pi}$

$f(x) = \sum\limits_{n=1}^{\infty} \frac{8}{\pi(2n-1)^3}\sin(2n-1)x$

$\because$ the Fourier series is controlled by $\sum\limits_{n=1}^{\infty} \frac{8}{n^3\pi}$

$\therefore$ uniformly converges

$\because f(x)$ continuous

$\therefore$ the Fourier series pointwise thus uniformly converges to $f(x)$

3 Let $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin(nx)$, $f'(x) = \sum\limits_{n=1}^{\infty} a'_n \cos(nx)$, $f''(x) = \sum\limits_{n=1}^{\infty} b''_n \sin(nx)$, $f'''(x) = \sum\limits_{n=1}^{\infty} a'''_n \cos(nx)$

$a'_n = \frac{2}{\pi}\int_0^\pi f'(x)\cos(nx)\,\mathrm{d}x = \frac{2n}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x = nb_n$

$b''_n = \frac{2}{\pi}\int_0^\pi f''(x)\sin(nx)\,\mathrm{d}x = -\frac{2}{\pi}\int_0^\pi f'(x)\cos(nx)\,\mathrm{d}x = -na'_n = -n^2 b_n$

$a'''_n = \frac{2}{\pi}\int_0^\pi f'''(x)\cos(nx)\,\mathrm{d}x = \frac{2n}{\pi}\int_0^\pi f''(x)\sin(nx)\,\mathrm{d}x = nb''_n$

apply Lec 13 Prob 04's conclusion $\sum\limits_{n=1}^{\infty} \left|\frac{b''_n n}{n}\right|$ convergent

$\therefore \sum\limits_{n=1}^{\infty} |b_n|$, $\sum\limits_{n=1}^{\infty} |a'_n|$, $\sum\limits_{n=1}^{\infty} |b''_n|$ convergent

$\therefore$ the Fourier series of $f(x)$, $f'(x)$, $f''(x)$ are uniformly convergent relatively to the original function and their coefficients are results of term-by-term differentiation of $f(x)$

$\therefore$ the Fourier series of $f(x)$ is 2$^{\text{nd}}$ order termwise differentiable

$\because a'_n = nb_n$ and $\sum\limits_{n=1}^{\infty} a'^2_n$ converges

$\therefore \sum\limits_{n=1}^{\infty} n^2 b_n^2$ converges $\square$

8

Lec 16

1 (1) $\frac{\mathrm{d}}{\mathrm{d}x}F(x)=\mathrm{e}^{x\sqrt{1-\cos^2 x}}(-\sin x)-\mathrm{e}^{x\sqrt{1-\sin^2 x}}(\cos x)+\int_{\sin x}^{\cos x}\sqrt{1-y^2}\mathrm{e}^{x\sqrt{1-y^2}}\,\mathrm{d}y$

$$=\int_{\sin x}^{\cos x}\sqrt{1-y^2}\mathrm{e}^{x\sqrt{1-y^2}}\,\mathrm{d}y-\mathrm{e}^{x|\sin x|}\sin x-\mathrm{e}^{x|\cos x|}\cos x$$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}F(x)=\int_{x^2}^{x^2}f(x,s)\,\mathrm{d}s+\int_0^x 2xf(t,x^2)\,\mathrm{d}t=\int_0^x 2xf(t,x^2)\,\mathrm{d}t$

2 $F(x)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{x\cos\theta}(\mathrm{e}^{ix\sin\theta}+\mathrm{e}^{-ix\sin\theta})}{2}\,\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{x\mathrm{e}^{i\theta}}+\mathrm{e}^{x\mathrm{e}^{-i\theta}}}{2}\,\mathrm{d}\theta$

$F'(x)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{i\theta}\frac{\mathrm{e}^{x\mathrm{e}^{i\theta}}}{2}\,\mathrm{d}\theta+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\theta}\frac{\mathrm{e}^{x\mathrm{e}^{-i\theta}}}{2}\,\mathrm{d}\theta=0$

$\therefore F(x)=\text{const},\ \ F(x)=F(0)=1$

3 $I=-\int_0^1\sin(\ln x)\int_a^b x^y\,\mathrm{d}y\,\mathrm{d}x=-\int_a^b\int_0^1\sin(\ln x)x^y\,\mathrm{d}x\,\mathrm{d}y$

$=\int_a^b\frac{1}{(y+1)^2+1}\,\mathrm{d}y=\arctan(b+1)-\arctan(a+1)$

4 $F(x)\ =\frac{1}{h^2}\int_0^h\left[\int_0^h f(x+\xi+\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi=\frac{1}{h^2}\int_x^{x+h}\left[\int_\xi^{\xi+h}f(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi$

$F'(x)\ =\frac{1}{h^2}\left[\int_{x+h}^{x+2h}f(\eta)\,\mathrm{d}\eta-\int_x^{x+h}f(\eta)\,\mathrm{d}\eta\right]$

$F''(x)=\frac{1}{h^2}\{f(x+2h)-f(x+h)-[f(x+h)-f(x)]\}$

$=\frac{f(x+2h)-2f(x+h)+f(h)}{h^2}$ □

Lec 17

1 (1) $\because\left|\frac{\cos xy}{x^2+y^2}\right|\le\frac{1}{a^2+y^2},\ \int_0^\infty\frac{1}{a^2+y^2}\,\mathrm{d}y$ converges

$\therefore$ uniformly converges

(2) $\because\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x=\frac{\pi}{2}$ uniformly converges, $\mathrm{e}^{-\alpha x}$ is about x monotonically decreasing and $\left|\mathrm{e}^{-\alpha x}\right|\le 1$

$\therefore$ uniformly converges

(3) $\because f(x)=\begin{cases}\frac{1}{\sqrt{2\pi}},\ x>0\\ 0,\ x=0\end{cases}$

$\therefore$ not uniformly converges

(4) $\int_0^1 \frac{1}{x^y}\sin\frac{1}{x}\,\mathrm{d}x = \int_1^{\infty} x^{y-2}\sin x\,\mathrm{d}x$

apply second mean value theorem for definite integrals, $\forall N \in \mathbb{N}^*,\ \exists y = 2 - \frac{1}{(2N+1)\pi}$

$\int_{2N\pi}^{(2N+1)\pi} x^{y-2}\sin x\,\mathrm{d}x \le 2[(2N+1)\pi]^{y-2} = 2[(2N+1)\pi]^{-\frac{1}{(2N+1)\pi}} > 1$

$\therefore$ not uniformly converges

(5) $\forall \epsilon > 0,\ \exists \delta < \frac{1}{4}\epsilon^2$ s.t. $\int_{y-\delta}^{y} \frac{\sin xy}{\sqrt{y-x}}\,\mathrm{d}x \le \int_{y-\delta}^{y} \frac{1}{\sqrt{y-x}}\,\mathrm{d}x = 2\sqrt{\delta} \le \epsilon$

$\therefore$ uniformly converges

(6) $\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x = \int_{-\infty}^{0} \mathrm{e}^{px}x^2\,\mathrm{d}x$

$\because \forall N > 0,\ \exists p = \frac{1}{\sqrt[3]{N+1}}$ s.t. $\int_{-\sqrt[3]{N+1}}^{-\sqrt[3]{N}} \mathrm{e}^{px}x^2\,\mathrm{d}x > \frac{\mathrm{e}^{-\sqrt[3]{N+1}p}}{3} = \frac{\mathrm{e}^{-1}}{3}$

$\therefore$ not uniformly converges

2 $\because |F(u)| \le \int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x = A$

$\therefore F(u)$ is bounded

$\forall \epsilon > 0,\ \exists K(\epsilon) > \frac{1}{\epsilon}$, s.t. $\int_{-\infty}^{-K} |f(x)|\,\mathrm{d}x + \int_{K}^{+\infty} |f(x)|\,\mathrm{d}x < \frac{\epsilon}{3}$

$\exists \delta = \frac{1}{3K^2A}$ when $|u_1 - u_2| < \delta$

$|F(u_2) - F(u_1)| = \left|\int_{-\infty}^{+\infty} f(x)(\cos u_2x - \cos u_1x)\,\mathrm{d}x\right|$

$\le \frac{2}{3}\epsilon + 2|\int_{-K}^{+K} f(x)\sin\frac{u_2+u_1}{2}x\sin\frac{u_2-u_1}{2}x\,\mathrm{d}x|$

$\le \frac{2}{3}\epsilon + \frac{1}{3KA}\int_{-K}^{+K} |f(x)\sin\frac{u_2+u_1}{2}x|\,\mathrm{d}x$

$\le \epsilon$

$\therefore$ uniformly continuous $\square$

3 $\because \int_0^{+\infty} f(t)\,\mathrm{d}t$ uniformly converges, e^{-xt} is about x monotonically decreasing and $\left|\mathrm{e}^{-xt}\right| \le 1\ (x \ge 0)$

$\therefore \int_0^{+\infty} \mathrm{e}^{-xt}f(t)\,\mathrm{d}t$ uniformly converges

$\therefore \lim\limits_{x\to 0}\int_0^{+\infty} \mathrm{e}^{-xt}f(t)\,\mathrm{d}t = \int_0^{+\infty} f(t)\,\mathrm{d}t$ $\square$

9

Lec 18

1 (1) $\int_0^{+\infty} \frac{\sin^4 x}{x^2}\,\mathrm{d}x = -\int_0^{+\infty} \sin^4 x\,\mathrm{d}\frac{1}{x} = \int_0^{+\infty} \frac{4\sin^3 x\cos x}{x}\,\mathrm{d}x = \int_0^{+\infty} \frac{2\sin^2 x\sin 2x}{x}\,\mathrm{d}x = \int_0^{+\infty} \frac{\sin 2x}{x}\,\mathrm{d}x - \int_0^{+\infty} \frac{\sin 2x\cos 2x}{x}\,\mathrm{d}x = \frac{\pi}{2} - \frac{1}{2}\int_0^{+\infty} \frac{\sin 4x}{x}\,\mathrm{d}x = \frac{\pi}{4}$

(3) $\int_0^{+\infty} (\frac{\sin ax}{x})^2\,\mathrm{d}x = -\int_0^{+\infty} \sin^2 ax\,\mathrm{d}\frac{1}{x} = \int_0^{+\infty} \frac{2a\sin ax\cos ax}{x}\,\mathrm{d}x = \int_0^{+\infty} \frac{a\sin 2ax}{x}\,\mathrm{d}x = \frac{\pi|a|}{2}$

(4) $\frac{1-\mathrm{e}^x}{x} = \int_0^1 \mathrm{e}^{-xt}\,\mathrm{d}t$

$\therefore \int_0^{+\infty} \frac{1-\mathrm{e}^x}{x}\cos x\,\mathrm{d}x = \int_0^{+\infty}\int_0^1 \mathrm{e}^{-xt}\cos x\,\mathrm{d}t\,\mathrm{d}x = \int_0^1\int_0^{+\infty} \mathrm{e}^{-xt}\cos x\,\mathrm{d}x\,\mathrm{d}t$

using Lec 18 Example 03, let $\alpha = t,\ \beta = 1$

$\therefore \int_0^1\int_0^{+\infty} \mathrm{e}^{-xt}\cos x\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{1}{t^2+1}\,\mathrm{d}t^2 = \frac{\ln 2}{2}$

2 (1) $I_n(a) = \int_0^{+\infty} \frac{\mathrm{d}x}{(x^2+a^2)^n}$

$\therefore I_n'(a) = -2anI_{n+1}(a)$

$\because I_1(a) = \frac{\pi}{2a}$

$\therefore$ assume that $I_n(a) = \frac{\pi(2n-3)!!}{2^n(n-1)!a^{2n-1}}$ and prove it using mathematical induction

(2) $\int_0^{+\infty} \frac{\mathrm{e}^{-ax^2}-\mathrm{e}^{-bx^2}}{x}\,\mathrm{d}x = \int_0^{+\infty}\int_a^b x\mathrm{e}^{-x^2y}\,\mathrm{d}y\,\mathrm{d}x = \int_a^b\int_0^{+\infty} x\mathrm{e}^{-x^2y}\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \frac{1}{2y}\,\mathrm{d}y = \frac{1}{2}\ln\frac{b}{a}$

(3) $I_n(a) = \int_0^{+\infty} \mathrm{e}^{-ax^2}x^{2n}\,\mathrm{d}x$

$\therefore I_n'(a) = -I_{n+1}(a)$

$\because I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$\therefore I_n(a) = (-1)^n(\frac{1}{2}\sqrt{\frac{\pi}{a}})^{(n)} = \frac{\sqrt{\pi}(2n-1)!!}{2^{n+1}a^{n+\frac{1}{2}}}$

(4) $I_n(a) = \int_0^1 x^{a-1}(\ln x)^n\,\mathrm{d}x$

$\therefore I_n'(a) = I_{n+1}(a)$

$\therefore I_n(a) = [I_1(a)]^{(n-1)} = (-\frac{1}{a^2})^{(n-1)} = \frac{(-1)^n n!}{a^{n+1}}$

Lec 19

1 (1) $\int_0^1 \sqrt{x^3(1-\sqrt{x})}\,\mathrm{d}x\ (t=\sqrt{x}, x=t^2)$

$= 2\int_0^1 t^4(1-t)^{\frac{1}{2}}\,\mathrm{d}t = 2\beta(5,\frac{3}{2}) = \frac{4!2^6}{11!!}$

$= \frac{512}{3465}$

(3) $\int_0^{+\infty} \frac{\mathrm{d}x}{1+x^4} = \int_0^{+\infty} \frac{\mathrm{d}x^{\frac{1}{4}}}{1+x} = \int_0^{+\infty} \frac{1}{4}\frac{x^{-\frac{3}{4}}}{1+x}\,\mathrm{d}x\ (x=\frac{1}{1-t}-1=\frac{t}{1-t},\ t=1-\frac{1}{1+x}=\frac{x}{1+x})$

$= \frac{1}{4}\int_0^1 (1-t)^{-\frac{1}{4}}t^{-\frac{3}{4}}\,\mathrm{d}t = \frac{1}{4}\beta(\frac{3}{4},\frac{1}{4})$

$= \frac{\sqrt{2}\pi}{4}$

(4) $\int_0^{\pi} \frac{\mathrm{d}\theta}{\sqrt{3-\cos\theta}}\ (x=\frac{1-\cos\theta}{2},\ \cos\theta=1-2x)$

$= \int_0^1 \frac{\mathrm{d}x}{\sqrt{2}\sqrt{x(1-x^2)}}\ (t=x^2,\ x=\sqrt{t})$

$= \frac{1}{2\sqrt{2}}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t$

$= \frac{\sqrt{2}}{4}\beta(\frac{1}{4},\frac{1}{2})$

(5) $\int_0^{+\infty} \frac{x^{m-1}\,\mathrm{d}x}{2+x^n}$

$= 2^{\frac{m}{n}-1}\int_0^{+\infty} \frac{x^{m-1}\,\mathrm{d}x}{1+x^n}\ (x=(t-1)^{\frac{1}{n}})$

$= \frac{2^{\frac{m}{n}-1}}{|n|}\int_0^1 \frac{(\frac{1-x}{x})^{\frac{m}{n}-1}\,\mathrm{d}x}{x} = \frac{2^{\frac{m}{n}-1}}{|n|}\beta(1-\frac{m}{n},\frac{m}{n})$

$= \frac{2^{\frac{m}{n}-1}}{|n|}\frac{\pi}{\sin\frac{m}{n}\pi}$

(6) $\int_0^{+\infty} \frac{\cosh 2qu}{(\cosh u)^{2p}}\,\mathrm{d}u\ (\mathrm{e}^u=t,\ u=\ln t)$

$= 2^{2p-1}\int_1^{+\infty} \frac{\frac{1}{t}(t^{2q}+t^{-2q})}{\frac{1}{t^{2p}}(1+t^2)^{2p}}\,\mathrm{d}t\ (x=t^2,\ t=\sqrt{x})$

$= 2^{2p-2}\int_1^{+\infty} \frac{x^{p-1}(x^q+x^{-q})}{(1+x)^{2p}}\,\mathrm{d}x\ (1+x=\frac{1}{t},\ t=\frac{1}{1+x})$

$= 2^{2p-2}\int_{\frac{1}{2}}^1 t^{2p}(\frac{1-t}{t})^{p-1}[(\frac{1-t}{t})^q+(\frac{1-t}{t})^{-q}]\frac{1}{t^2}\,\mathrm{d}t$ (the two integral terms are equal)

$= 2^{2p-2}\beta(p-q,p+q)$

(7) $\int_{-1}^1 \frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\,\mathrm{d}x\ (x=\tan\theta)$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\cos\theta+\sin\theta)^{2m}(\cos\theta-\sin\theta)^{2n}}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}\,\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+\sin 2\theta)^m(1-\sin 2\theta)^n}{\cos 2\theta}\,\mathrm{d}\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+\sin\theta)^m(1-\sin\theta)^n}{\cos\theta}\,\mathrm{d}\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+\sin\theta)^m(1-\sin\theta)^n}{\cos^2\theta}\,\mathrm{d}\sin\theta$

$= \frac{1}{2}\int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}\,\mathrm{d}x \ (1+x=2t,\ t=\frac{1+x}{2})$

$= \int_0^1 (2t)^{m-1}[2(1-t)]^{n-1}\,\mathrm{d}t$

$= 2^{m+n-2}\beta(m,n)$

2 $\beta(r,p)\beta(r+p,q) = \frac{\Gamma(r)\Gamma(p)\Gamma(r+p)\Gamma(q)}{\Gamma(r+p)\Gamma(r+p+q)} = \frac{\Gamma(r)\Gamma(q)\Gamma(r+q)\Gamma(p)}{\Gamma(r+q)\Gamma(r+p+q)} = \beta(r,q)\beta(r+q,p)$ $\square$

3 $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)^{\cos 2\alpha}\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\frac{\sin(\theta+\frac{\pi}{4})}{\cos(\theta+\frac{\pi}{4})}\right]^{\cos 2\alpha}\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}(\tan\theta)^{\cos 2\alpha}\,\mathrm{d}\theta$

$= \int_0^{\frac{\pi}{2}}(\sin t)^{\cos 2\alpha}(\cos t)^{-\cos 2\alpha}\,\mathrm{d}\theta = \frac{1}{2}\beta(\frac{1+\cos 2\alpha}{2}, \frac{1-\cos 2\alpha}{2}) = \frac{1}{2}\beta(\cos^2\alpha, \sin^2\alpha)$

$= \frac{\pi}{2\sin(\pi\cos^2\alpha)}$

10

P12

5

- $\because \|f\|'' \le \|f\|,\ \|f\|' \le (b-a)\|f\|$

  $\therefore$ if in the sense of $\|\ \|$ $f_n \to f$, then in the sense of $\|\ \|',\ \|\ \|''$ $f_n \to f$
- $\exists f_n \in C^k[a,b],\ f \in C^k[a,b]$, in the sense of $\|\ \|$ $f_n \nrightarrow f$, but in the sense of $\|\ \|',\ \|\ \|''$ $f_n \to f$
- norm $\|\ \|'$ is not equal to norm $\|\ \|''$ $(k=0)$

P18

4 (i) $\because$ integral is linear

$\therefore I$ is linear

(ii) for $f_n(x) = \begin{cases} 1 & -n < x < n \\ x+n+1 & -n-1 \leqslant x \leqslant -n \\ n+1-x & n \leqslant x \leqslant n+1 \\ 0 & \text{other} \end{cases}$

$f_n \in C_c(R),\ \|f_n\| = 1,\ \lim\limits_{n\to+\infty} I(f_n) = +\infty$

$\therefore I$ is not bounded

P26

5 suppose $m = n = 1$ construct a discontinuous function $y_i = \begin{cases} \ln|x|, \ x \neq 0 \\ 0, \ x = 0 \end{cases}$ and evidently the preimage of a compact set is also compact

$\therefore$ negative $\square$

11

P65

5 (i) $|f(\boldsymbol{x}) - f(\boldsymbol{y})| = 0 \Rightarrow |\boldsymbol{x} - \boldsymbol{y}| = 0$

(ii) $|\,\mathrm{D}f \cdot \mathrm{d}\boldsymbol{x}| = |\,\mathrm{d}f(\boldsymbol{x})| \geqslant c|\,\mathrm{d}\boldsymbol{x}|$

$\therefore$ the absolute values of $\mathrm{D}f$'s eigenvalues are not smaller than c

(iii) $\because |f(\boldsymbol{x}) - f(\boldsymbol{0})| \geqslant c|\boldsymbol{x}|$

$\therefore \lim\limits_{|\boldsymbol{x}| \to +\infty} |f(\boldsymbol{x})| = +\infty$

$\therefore \forall \boldsymbol{\xi} \in \mathbb{R}^n,\ \exists \boldsymbol{x}_0 \in \mathbb{R}^n$ (minimum point) s.t. $\mathrm{d}|f(\boldsymbol{x}) - \boldsymbol{\xi}|^2\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} = 2[f(\boldsymbol{x}_0) - \boldsymbol{\xi}]^{\mathrm{T}} \cdot \mathrm{D}f(\boldsymbol{x}_0) \cdot \mathrm{d}\boldsymbol{x} = 0$

$\because \mathrm{D}f \neq 0$

$\therefore f(\boldsymbol{x}_0) = \boldsymbol{\xi}$ $\square$

P79

3 S is bounded

for $\forall \varepsilon > 0$, the accumulation points can be covered by a finite set of closed rectangles with total volume $V_0 < \frac{\varepsilon}{2}$

$\because$ if there are infinite points left in S, there will be an arbitrarily small region containing infinite points in S outside previous rectangles, which contradicts to the fact that all accumulation points are already covered.

$\therefore$ the left points could be covered by another finite set of closed rectangles with total volume $V_1 < \frac{\varepsilon}{2}$ $\square$

P88

4 Negative

$f_n(x) = \frac{1}{x^2+\frac{1}{n}}$ is integrable on $[0,1]$

$\lim\limits_{n\to\infty} f_n(x) = \frac{1}{x^2} = f(x)$ is not integrable on $[0,1]$ $\square$

12

P117

2 (1) $\{U_\alpha\}$ is a open cover of $M,\ \forall U_\alpha,\ \exists f : U_\alpha \mapsto V_\alpha,\ (x, y, \sqrt{x^2+y^2}) \mapsto (x, y)$

$\because$ f is a bijection, f is continuous, the inverse function f^{-1} is continuous

$\therefore$ f is a homeomorphism

$\therefore$ M is a two-dimensional manifold

(2) Assume that the entire cone is a two-dimensional manifold

for open cover $U_\alpha \ni (0,0,0),\ \exists$ homeomorphism $\varphi : U_\alpha \mapsto V_\alpha$

$\because V_\alpha \setminus \{\varphi(0,0,0)\}$ is connected, but $U_\alpha \setminus \{(0,0,0)\}$ is not

$\therefore$ contradict with two homeomorphic spaces share the same topological properties $\square$

6 Assume that $\{U_\alpha\}$ is the open cover of M, for $\forall U_\alpha,\ \exists \varphi_\alpha : U_\alpha \mapsto V_\alpha$

$\varphi_\alpha : (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-k}, f(x_1, \cdots, x_n))$

$$\mathrm{D}\varphi_\alpha = \begin{bmatrix} E_{(n-k)\times(n-k)} & 0_{(n-k)\times k} \\ \mathrm{D}f_{k\times(1\sim(n-k))} & \mathrm{D}f_{k\times((n-k+1)\sim n)} \end{bmatrix}$$

$\because \forall x_0 \in U_\alpha$ $\mathrm{rank}(\mathrm{D}f(x_0)) = k$, assume that $\mathrm{rank}(\mathrm{D}f(x_0)_{k\times((n-k+1)\sim n)}) = k$

$\therefore \det \mathrm{D}\varphi_\alpha \neq 0$

according to implicit function theorem, $\exists$ open set $U \ni x_0$, open set $V \ni \varphi_\alpha(x_0)$ s.t. $\varphi_\alpha : U \mapsto V$ is a diffeomorphism

$\varphi_\alpha : U \cap M \mapsto V \cap \{(y_1, \cdots, y_n) \in \mathbb{R}^n | y_{n-k+1} = \cdots = y_n = 0\}$

$\therefore M$ is a (n-k)-dimensional differential manifold $\square$

P128

4 $\omega \wedge \cdot$ is a linear operation which maps 0 to 0

$\therefore$ Apparently M_ω is a linear subspace of V $\square$

let the first q vectors of $\{e_1 \cdots e_n\}$ be the complete orthonormal basis of M_ω

$\because$ for $\omega = \sum\limits_{1 \leqslant i_1 < \cdots < i_p \leqslant n} \omega_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$ and for $e_k \in \{e_1, \cdots, e_q\}$, $\omega \wedge e_k = 0$

$\therefore \delta_{i_1 \cdots i_p k} = 0$ for $\forall i_1 \cdots i_p \in \{i_1, \cdots, i_p | \omega_{i_1 \cdots i_p} \neq 0\}$

$\therefore k$ must be a common index of all non-zero $\omega_{i_1 \cdots i_p}$

$\therefore q \leqslant p$ $\square$

$(\Rightarrow)$: under the previous setting, let $q = p$

$\therefore \{1, 2, \cdots, p\}$ are all common indices of all non-zero $\omega_{i_1 \cdots i_p}$

$\therefore$ only $\omega_{1 \cdots p} \neq 0 \Rightarrow \omega = \omega_{1 \cdots p} e_1 \wedge \cdots \wedge e_p$

$(\Leftarrow)$: let $\omega = v_1 \wedge \cdots \wedge v_q$

apparently $\{v_1 \cdots v_p\}$ are linearly independent

$\therefore \{v_1 \cdots v_p\}$ can be the basis of M_ω $\square$

13

P148

10 Let $\eta = \sum\limits_{i=1}^{n} \frac{(-1)^{i-1}}{n} x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}\hat{x}_i \wedge \cdots \wedge \mathrm{d}x_n$ and verify that $\mathrm{d}\eta = \omega$ $\square$

11 Let ω be $k-$form

$$\therefore \mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta = 0 \qquad \square$$

12 Let ω be $k-$form and $\eta = \mathrm{d}\varphi$

$$\mathrm{d}[(-1)^k \omega \wedge \varphi] = (-1)^k \,\mathrm{d}\omega \wedge \varphi + \omega \wedge \mathrm{d}\varphi = \omega \wedge \eta \qquad \square$$