Peking University

Answer Key 12

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P117

2 (1) $\{U_\alpha\}$ is a open cover of $M,\ \forall U_\alpha,\ \exists f: U_\alpha \mapsto V_\alpha,\ (x, y, \sqrt{x^2+y^2}) \mapsto (x, y)$

$\because f$ is a bijection, f is continuous, the inverse function f^{-1} is continuous

$\therefore f$ is a homeomorphism

$\therefore$ M is a two-dimensional manifold

(2) Assume that the entire cone is a two-dimensional manifold

for open cover $U_\alpha \ni (0,0,0),\ \exists$ homeomorphism $\varphi: U_\alpha \mapsto V_\alpha$

$\because V_\alpha \setminus \{\varphi(0,0,0)\}$ is connected, but $U_\alpha \setminus \{(0,0,0)\}$ is not

$\therefore$ contradict with two homeomorphic spaces share the same topological properties $\square$

6 Assume that $\{U_\alpha\}$ is the open cover of M, for $\forall U_\alpha,\ \exists \varphi_\alpha: U_\alpha \mapsto V_\alpha$

$\varphi_\alpha: (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{n-k}, f(x_1, \cdots, x_n))$

$$\mathrm{D}\varphi_\alpha = \begin{bmatrix} E_{(n-k)\times(n-k)} & 0_{(n-k)\times k} \\ \mathrm{D}f_{k\times(1\sim(n-k))} & \mathrm{D}f_{k\times((n-k+1)\sim n)} \end{bmatrix}$$

$\because \forall x_0 \in U_\alpha$ $\mathrm{rank}(\mathrm{D}f(x_0)) = k$, assume that $\mathrm{rank}(\mathrm{D}f(x_0)_{k\times((n-k+1)\sim n)}) = k$

$\therefore \det \mathrm{D}\varphi_\alpha \neq 0$

according to implicit function theorem, $\exists$ open set $U \ni x_0$, open set $V \ni \varphi_\alpha(x_0)$ s.t.

- $\varphi_\alpha : U \mapsto V$ is a diffeomorphism
- $\varphi_\alpha : U \cap M \mapsto V \cap \{(y_1, \cdots, y_n) \in \mathbb{R}^n | y_{n-k+1} = \cdots = y_n = 0\}$

$\therefore M$ is a (n-k)-dimensional differential manifold $\square$

P128

4 $\omega \wedge \cdot$ is a linear operation which maps 0 to 0

$\therefore$ Apparently M_ω is a linear subspace of V $\square$

let the first q vectors of $\{e_1 \cdots e_n\}$ be the complete orthonormal basis of M_ω

$\because$ for $\omega = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} \omega_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$ and for $e_k \in \{e_1, \cdots, e_q\},\ \omega \wedge e_k = 0$

$\therefore \delta_{i_1 \cdots i_p k} = 0$ for $\forall i_1 \cdots i_p \in \{i_1, \cdots, i_p | \omega_{i_1 \cdots i_p} \neq 0\}$

$\therefore k$ must be a common index of all non-zero $\omega_{i_1 \cdots i_p}$

$\therefore q \leqslant p$ $\square$

$(\Rightarrow)$: under the previous setting, let $q = p$

$\therefore \{1, 2, \cdots, p\}$ are all common indices of all non-zero $\omega_{i_1 \cdots i_p}$

$\therefore$ only $\omega_{1 \cdots p} \neq 0 \Rightarrow \omega = \omega_{1 \cdots p} e_1 \wedge \cdots \wedge e_p$

$(\Leftarrow)$: let $\omega = v_1 \wedge \cdots \wedge v_q$

apparently $\{v_1 \cdots v_p\}$ are linearly independent

$\therefore \{v_1 \cdots v_p\}$ can be the basis of M_ω $\square$