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Answer Key 7

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Lec 14

1 $\mathrm{e}^{\cos x}\cos(\sin x) = \mathrm{e}^{\cos x}\frac{\mathrm{e}^{i\sin x}+\mathrm{e}^{-i\sin x}}{2} = \frac{1}{2}\left(\mathrm{e}^{\mathrm{e}^{ix}} + \mathrm{e}^{\mathrm{e}^{-ix}}\right) = \sum\limits_{n=0}^{\infty}\frac{1}{2}\frac{(\mathrm{e}^{ix})^n+(\mathrm{e}^{-ix})^n}{n!} = \sum\limits_{n=0}^{\infty}\frac{1}{2}\frac{\mathrm{e}^{inx}+\mathrm{e}^{-inx}}{n!}$
$= \sum\limits_{n=0}^{\infty}\frac{\cos(nx)}{n!}$

2 Let $f(x+c) = \frac{a_0'}{2} + \sum\limits_{n=1}^{\infty} a_n'\cos(nx) + b_n'\sin(nx) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n\cos[n(x+c)] + b_n\sin[n(x+c)] =$
$\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} [a_n\cos(nc) + b_n\sin(nc)]\cos(nx) + [b_n\cos(nc) - a_n\sin(nc)]\sin(nx)$

$\therefore\ a_0' = a_0,\ a_n' = a_n\cos(nc) + b_n\sin(nc),\ b_n' = b_n\cos(nc) - a_n\sin(nc)$

3 $\pi - x = \sum\limits_{n=1}^{\infty}\frac{2}{n}\sin(nx),\ x\in(0,2\pi)$

$\therefore\ \frac{1}{2\pi}\int_0^{2\pi} f(x)(\pi - x)\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)\,\mathrm{d}x$

$\because\ \sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)$ is controlled by $\sum\limits_{n=1}^{\infty}\frac{2}{n}b_n$ and Lec 13 Prob 04's conclusion only needs $f(x)$ to be integrable.

$\therefore\ \sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)$ uniformly converges

$\therefore\ \frac{1}{2\pi}\int_0^{2\pi} f(x)(\pi - x)\,\mathrm{d}x = \frac{1}{2\pi}\sum\limits_{n=1}^{\infty}\int_0^{2\pi}\frac{2}{n}b_n\sin^2(nx)\,\mathrm{d}x = \sum\limits_{n=1}^{\infty}\frac{b_n}{n}$ $\square$

4 (1) Apply periodic extension to $f(x)$ and set $f(2n\pi) = 0$, which won't change Fourier

series

$\because \sigma_n(x) = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{\sin\left(\frac{nt}{2}\right)}{\sin\frac{t}{2}}\right)^2 \mathrm{d}x,\ \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\left(\frac{nt}{2}\right)}{\sin\frac{t}{2}}\right)^2 \mathrm{d}x = 1,\ |f(x+t)| \leqslant \frac{\pi}{2}$

$\therefore |\sigma_n(x)| \leqslant \frac{\pi}{2}$ □

(2) Due to pointwise convergence, $\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n} = \begin{cases} f(x),\ 0 < x < 2\pi \\ 0,\ x = 0 \end{cases}$

$\therefore \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right| < \frac{\pi}{2},\ 0 \leqslant x < 2\pi$

$\because \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right|$ is 2π-periodic

$\therefore \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right| < \frac{\pi}{2} + 1,\ x \in \mathbb{R}$ □

Lec 15

1 (1) $a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = -\frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,\mathrm{d}x = -\frac{b_n'}{n}$

$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,\mathrm{d}x = \frac{a_n'}{n}$

(ignore discontinuous points of f') □

(2) Lec 13 Prob 04's conclusion can be extended as $\sum\limits_{n=1}^{\infty}\left|\frac{a_n'}{n}\right|$ and $\sum\limits_{n=1}^{\infty}\left|\frac{b_n'}{n}\right|$ converges, thus the convergence of $\sum\limits_{n=1}^{\infty}(|a_n| + |b_n|)$ is straightforward □

(3) $|\frac{a_0}{2}| + \sum\limits_{n=1}^{\infty}(|a_n| + |b_n|)$ converges $\Rightarrow \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ absolutely uniformly converges

$\because \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ pointwise converges to $f(x)$

$\therefore \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ uniformly converges to $f(x)$ □

2 (1) Due to symmetry, $b_n = 0$

$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x = \frac{4}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos(nx)\,\mathrm{d}x = \begin{cases} 0, \ n = 1 \\ \frac{2}{\pi}\frac{(-1)^{n+1}-1}{n^2-1}, \ n \neq 1 \end{cases}$$

$$f(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2-1)} \cos 2nx$$

$\because$ the Fourier series is controlled by $\sum\limits_{n=2}^{\infty} \frac{4}{\pi(n^2-1)}$

$\therefore$ uniformly converges

$\because$ $f(x)$ continuous

$\therefore$ the Fourier series pointwise thus uniformly converges to $f(x)$

(2) Due to symmetry, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,\mathrm{d}x = \frac{4[1-(-1)^n]}{n^3\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi(2n-1)^3} \sin(2n-1)x$$

$\because$ the Fourier series is controlled by $\sum\limits_{n=1}^{\infty} \frac{8}{n^3\pi}$

$\therefore$ uniformly converges

$\because$ $f(x)$ continuous

$\therefore$ the Fourier series pointwise thus uniformly converges to $f(x)$

3 Let $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin(nx)$, $f'(x) = \sum\limits_{n=1}^{\infty} a'_n \cos(nx)$, $f''(x) = \sum\limits_{n=1}^{\infty} b''_n \sin(nx)$, $f'''(x) = \sum\limits_{n=1}^{\infty} a'''_n \cos(nx)$

$$a'_n = \frac{2}{\pi}\int_0^\pi f'(x)\cos(nx)\,\mathrm{d}x = \frac{2n}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x = nb_n$$

$$b''_n = \frac{2}{\pi}\int_0^\pi f''(x)\sin(nx)\,\mathrm{d}x = -\frac{2}{\pi}\int_0^\pi f'(x)\cos(nx)\,\mathrm{d}x = -na'_n = -n^2 b_n$$

$$a'''_n = \frac{2}{\pi}\int_0^\pi f'''(x)\cos(nx)\,\mathrm{d}x = \frac{2n}{\pi}\int_0^\pi f''(x)\sin(nx)\,\mathrm{d}x = nb''_n$$

apply Lec 13 Prob 04's conclusion $\sum\limits_{n=1}^{\infty} \left|\frac{b''_n n}{n}\right|$ convergent

$\therefore \sum\limits_{n=1}^{\infty} |b_n|$, $\sum\limits_{n=1}^{\infty} |a'_n|$, $\sum\limits_{n=1}^{\infty} |b''_n|$ convergent

$\therefore$ the Fourier series of $f(x),\ f'(x),\ f''(x)$ are uniformly convergent relatively to the original function and their coefficients are results of term-by-term differentiation of $f(x)$

$\therefore$ the Fourier series of $f(x)$ is 2^{nd} order termwise differentiable

$\because a'_n = nb_n$ and $\sum_{n=1}^{\infty} a'^2_n$ converges

$\because \sum_{n=1}^{\infty} n^2 b_n^2$ converges $\square$