Answer Key 1

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Lec 01:

1 Use reduction to absurdity. Suppose $\lim_{n\to\infty} a_n \neq 0$ or doesn't exist. So $\exists\varepsilon > 0, \forall N_1 > 0, \exists n > N_1, |a_n| > 3\varepsilon$. For such $\varepsilon, \exists N_2 > 0, \forall n > N_2, |a_{2n} + 2a_n| < \varepsilon$ So $\exists N > N_2, |a_N| > 3\varepsilon$ and $|a_{2N} + 2a_N| < \varepsilon$. So $|a_{2N}| > 5\varepsilon$. Similarly, $|a_{4N}| > 9\varepsilon$, and then $|a_{2^p N}| > \left(2^{p+1}+1\right)\varepsilon$ for $p \in \mathbb{N}$, which contradict the boundedness of a_n. □

2 Let $a_n = \frac{2}{3} + b_n$. So $\lim_{n\to\infty}(b_{2n} + 2b_n) = 0$. According to the conclusion of previous problem, $\lim_{n\to\infty} b_n = 0$, and $\lim_{n\to\infty} a_n = \frac{2}{3}$ is evident. □

3 1) $\because x_{n+2} = 1 + \frac{1}{x_{n+1}} = 1 + \frac{1}{1+\frac{1}{x_n}} = 1 + \frac{x_n}{1+x_n}$ and using mathematical induction: $x_n > 0$

$\therefore$ for $n \geqslant 3, 1 < x_n < 2$

$\therefore 1 \leqslant \varliminf_{n\to\infty} x_n \leqslant \varlimsup_{n\to\infty} x_n \leqslant 2$ □

2) $\because$ for $n \geqslant 3, \left|\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}\right| = \left|-\frac{1}{x_n+1}\right| < \frac{1}{2}$

$\therefore x_n$ is a Cauchy sequence

Calculate the positive fixed point of equation $x^* = 1 + \frac{1}{x^*}$

$\therefore \lim_{n\to\infty} x_n = x^* = \frac{1+\sqrt{5}}{2}$

5 1) $\because \lim_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$

$= \varliminf_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n \leqslant \varlimsup_{n\to\infty}(x_n + y_n) \leqslant \varlimsup_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$

$= \lim_{n\to\infty} x_n + \varlimsup_{n\to\infty} y_n$ □

2) The proof for $\varlimsup_{n\to\infty} y_n = \pm\infty$ is direct. Suppose $\varlimsup_{n\to\infty} y_n = A$.

$\forall \varepsilon_1 > 0, \exists N_1(\varepsilon_1) > 0, \forall n > N_1(\varepsilon_1), y_n < A + \varepsilon_1$, and $\exists \left\{ y_{n_k^{\varepsilon_1}} \right\}, y_{n_k^{\varepsilon_1}} > A - \varepsilon_1$
$\forall \varepsilon_2 > 0, \exists N_2(\varepsilon_2) > 0, \forall n > N_2(\varepsilon_2), |x_n - x^*| < \varepsilon_2, x^* = \lim_{n\to\infty} x_n$
$\forall \varepsilon > 0, \exists \varepsilon_1 > 0, \varepsilon_2 > 0, \forall \delta_1 \in (0, \varepsilon_1), \delta_2 \in (-\varepsilon_2, \varepsilon_2)$, s.t. $0 \leqslant \frac{\delta_1}{x^*+\delta_2} - \frac{A\delta_2}{x^*(x^*+\delta_2)} < \varepsilon$, and $0 \leqslant \frac{\delta_1}{x^*} + \frac{(A-\delta_1)\delta_2}{x^*(x^*+\delta_2)} < \varepsilon$
$\therefore$ for $\forall \varepsilon > 0, \exists N = \max\left(N_1(\varepsilon_1), N_2(\varepsilon_2)\right), \forall n > N, (x_n y_n) > x^* A + \varepsilon$
and $\exists \left\{ y_{n_k^{\varepsilon_1}} x_{n_k^{\varepsilon_1}} \right\}, y_{n_k^{\varepsilon_1}} x_{n_k^{\varepsilon_1}} > x^* A - \varepsilon$ $\square$

6 1) $\sup_{k\geqslant n} a_k = 1, \inf_{k\geqslant 2n} a_k = \inf_{k\geqslant 2n+1} a_k = a_{2n+1} = \frac{1}{2^{-2n-1}-1}$
$\therefore \overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = 1, \underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -1$

2) $\sup_{k\geqslant 2n} a_k = \sup_{k\geqslant 2n-1} a_k = a_{2n} = 1 + \frac{1}{2n}$
$\inf_{k\geqslant 2n} a_k = \inf_{k\geqslant 2n+1} a_k = a_{2n+1} = -1 - \frac{1}{2n+1}$
$\therefore \overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = 1, \underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -1$

3) $|a_n| = \frac{1}{n} \to 0$
$\therefore \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n = 0$

4) For a period of $n = 0 \sim 9 \bmod 10$, maximum a_n is $\sin\frac{2\pi}{5}$, minimum a_n is $-\sin\frac{2\pi}{5}$
$\therefore \overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = \sin\frac{2\pi}{5}, \underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -\sin\frac{2\pi}{5}$

Lec 02:

1 Suppose $\overline{\lim_{n\to\infty}} n a_n > 0$. Then just suppose $\overline{\lim_{n\to\infty}} n a_n \geqslant 1$
$\therefore \exists \left\{ a_{n_k} \right\}, a_{n_k} \geqslant \frac{1}{n_k}$
Then $\exists \left\{ a_{n_{k_l}} \right\}, n_{k_{l+1}} \geqslant 2 n_{k_l}$
$\therefore \sum_{n=1}^{\infty} a_n \geqslant \sum_{l=2}^{\infty} \left(\frac{n_{k_l}}{2} \frac{1}{n_{k_l}} \right) = \infty$
$\therefore \overline{\lim_{n\to\infty}} n a_n = 0$
$\because \underline{\lim_{n\to\infty}} n a_n \geqslant 0$
$\therefore \lim_{n\to\infty} n a_n = 0$

3 1) $\because 0 \leqslant \frac{1}{(5n-4)(5n+1)} \leqslant \frac{1}{n^2}$
$\therefore$ absolutely convergent

2) $\because \lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$
$\therefore$ divergent

3) $\because 0 \leqslant \frac{1}{2^n} + \frac{1}{3^n} \leqslant \frac{1}{2^{n-1}}$

$\therefore$ absolutely convergent

4) $\because 0 \leqslant \frac{1}{(3n-2)(3n+1)} \leqslant \frac{1}{n^2}$

$\therefore$ absolutely convergent

5) $\because \lim\limits_{n\to\infty} a_n = 1 \neq 0$

$\therefore$ divergent

4 1) $\forall\varepsilon > 0, \exists N > \log_{|q|} \frac{\varepsilon(1-|q|)}{A}, \forall n_1, n_2 > N$

$\sum\limits_{n=n_1}^{n_2} |a_n q^n| \leqslant A \sum\limits_{n=n_1}^{n_2} |q|^n = A|q|^{n_1} \frac{1-|q|^{n_2-n_1+1}}{1-|q|} < \varepsilon$

$\therefore$ absolutely convergent

2) $\because a_n = \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} > \frac{1}{3n}$

$\therefore \exists\varepsilon = 1 > 0, \forall N > 0, \exists n_1 > N, n_2 = 3n_1 + 10 > N$

$\sum\limits_{n=n_1}^{n_2} a_n > \sum\limits_{n=n_1}^{n_2} \frac{1}{3n} > \frac{2n_2}{3} \frac{1}{3n_2} > \frac{2}{9}$

$\therefore$ divergent

5 Let $b_k = \sum\limits_{n=n_{k-1}+1}^{n_k} a_n$

$$\therefore \forall\varepsilon > 0, \exists K > 0, \forall k_1, k_2 > K, \left|\sum_{k=k_1}^{k_2} b_k\right| < \varepsilon$$

$$\therefore \exists N > n_{K+1} + 1, \forall n_1, n_2 > N, \left|\sum_{n=n_1}^{n_2} a_n\right| < \left|\sum_{k=K}^{\infty} b_k\right| \leqslant \varepsilon$$

$\therefore$ absolutely convergent