PEKING UNIVERSITY

Answer Key 9

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Lec 18

1 (1) $\int_0^{+\infty} \frac{\sin^4 x}{x^2}\,\mathrm{d}x = -\int_0^{+\infty} \sin^4 x \,\mathrm{d}\frac{1}{x} = \int_0^{+\infty} \frac{4\sin^3 x\cos x}{x}\,\mathrm{d}x = \int_0^{+\infty} \frac{2\sin^2 x \sin 2x}{x}\,\mathrm{d}x = \int_0^{+\infty} \frac{\sin 2x}{x}\,\mathrm{d}x - \int_0^{+\infty} \frac{\sin 2x\cos 2x}{x}\,\mathrm{d}x = \frac{\pi}{2} - \frac{1}{2}\int_0^{+\infty} \frac{\sin 4x}{x}\,\mathrm{d}x = \frac{\pi}{4}$

(3) $\int_0^{+\infty} (\frac{\sin ax}{x})^2\,\mathrm{d}x = -\int_0^{+\infty} \sin^2 ax \,\mathrm{d}\frac{1}{x} = \int_0^{+\infty} \frac{2a\sin ax\cos ax}{x}\,\mathrm{d}x = \int_0^{+\infty} \frac{a\sin 2ax}{x}\,\mathrm{d}x = \frac{\pi|a|}{2}$

(4) $\frac{1-\mathrm{e}^x}{x} = \int_0^1 \mathrm{e}^{-xt}\,\mathrm{d}t$

$\therefore \int_0^{+\infty} \frac{1-\mathrm{e}^x}{x}\cos x\,\mathrm{d}x = \int_0^{+\infty}\int_0^1 \mathrm{e}^{-xt}\cos x\,\mathrm{d}t\,\mathrm{d}x = \int_0^1\int_0^{+\infty} \mathrm{e}^{-xt}\cos x\,\mathrm{d}x\,\mathrm{d}t$

using Lec 18 Example 03, let $\alpha = t,\ \beta = 1$

$\therefore \int_0^1\int_0^{+\infty} \mathrm{e}^{-xt}\cos x\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{1}{t^2+1}\,\mathrm{d}t^2 = \frac{\ln 2}{2}$

2 (1) $I_n(a) = \int_0^{+\infty} \frac{\mathrm{d}x}{(x^2+a^2)^n}$

$\therefore I_n'(a) = -2anI_{n+1}(a)$

$\because I_1(a) = \frac{\pi}{2a}$

$\therefore$ assume that $I_n(a) = \frac{\pi(2n-3)!!}{2^n(n-1)!a^{2n-1}}$ and prove it using mathematical induction

(2) $\int_0^{+\infty} \frac{\mathrm{e}^{-ax^2}-\mathrm{e}^{-bx^2}}{x}\,\mathrm{d}x = \int_0^{+\infty}\int_a^b x\mathrm{e}^{-x^2y}\,\mathrm{d}y\,\mathrm{d}x = \int_a^b\int_0^{+\infty} x\mathrm{e}^{-x^2y}\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \frac{1}{2y}\,\mathrm{d}y = \frac{1}{2}\ln\frac{b}{a}$

(3) $I_n(a) = \int_0^{+\infty} \mathrm{e}^{-ax^2}x^{2n}\,\mathrm{d}x$

$\therefore I_n'(a) = -I_{n+1}(a)$

$\because I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$\therefore I_n(a) = (-1)^n(\frac{1}{2}\sqrt{\frac{\pi}{a}})^{(n)} = \frac{\sqrt{\pi}(2n-1)!!}{2^{n+1}a^{n+\frac{1}{2}}}$

(4) $I_n(a) = \int_0^1 x^{a-1}(\ln x)^n \,\mathrm{d}x$

$\therefore I_n'(a) = I_{n+1}(a)$

$\therefore I_n(a) = [I_1(a)]^{(n-1)} = (-\frac{1}{a^2})^{(n-1)} = \frac{(-1)^n n!}{a^{n+1}}$

Lec 19

1 (1) $\int_0^1 \sqrt{x^3(1-\sqrt{x})}\,\mathrm{d}x \ (t=\sqrt{x}, x=t^2)$

$= 2\int_0^1 t^4(1-t)^{\frac{1}{2}}\,\mathrm{d}t = 2\beta(5,\frac{3}{2}) = \frac{4!2^6}{11!!}$

$= \frac{512}{3465}$

(3) $\int_0^{+\infty} \frac{\mathrm{d}x}{1+x^4} = \int_0^{+\infty} \frac{\mathrm{d}x^{\frac{1}{4}}}{1+x} = \int_0^{+\infty} \frac{1}{4}\frac{x^{-\frac{3}{4}}}{1+x}\,\mathrm{d}x \ (x = \frac{1}{1-t} - 1 = \frac{t}{1-t},\ t = 1 - \frac{1}{1+x} = \frac{x}{1+x})$

$= \frac{1}{4}\int_0^1 (1-t)^{-\frac{1}{4}}t^{-\frac{3}{4}}\,\mathrm{d}t = \frac{1}{4}\beta(\frac{3}{4},\frac{1}{4})$

$= \frac{\sqrt{2}\pi}{4}$

(4) $\int_0^{\pi} \frac{\mathrm{d}\theta}{\sqrt{3-\cos\theta}} \ (x = \frac{1-\cos\theta}{2},\ \cos\theta = 1-2x)$

$= \int_0^1 \frac{\mathrm{d}x}{\sqrt{2}\sqrt{x(1-x^2)}} \ (t=x^2,\ x=\sqrt{t})$

$= \frac{1}{2\sqrt{2}}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}\,\mathrm{d}t$

$= \frac{\sqrt{2}}{4}\beta(\frac{1}{4},\frac{1}{2})$

(5) $\int_0^{+\infty} \frac{x^{m-1}\,\mathrm{d}x}{2+x^n}$

$= 2^{\frac{m}{n}-1}\int_0^{+\infty} \frac{x^{m-1}\,\mathrm{d}x}{1+x^n} \ (x = (t-1)^{\frac{1}{n}})$

$= \frac{2^{\frac{m}{n}-1}}{|n|}\int_0^1 \frac{(\frac{1-x}{x})^{\frac{m}{n}-1}\,\mathrm{d}x}{x} = \frac{2^{\frac{m}{n}-1}}{|n|}\beta(1-\frac{m}{n},\frac{m}{n})$

$= \frac{2^{\frac{m}{n}-1}}{|n|}\frac{\pi}{\sin\frac{m}{n}\pi}$

(6) $\int_0^{+\infty} \frac{\cosh 2qu}{(\cosh u)^{2p}}\,\mathrm{d}u \ (\mathrm{e}^u = t,\ u = \ln t)$

$= 2^{2p-1}\int_1^{+\infty} \frac{\frac{1}{t}(t^{2q}+t^{-2q})}{\frac{1}{t^{2p}}(1+t^2)^{2p}}\,\mathrm{d}t \ (x=t^2,\ t=\sqrt{x})$

$= 2^{2p-2}\int_1^{+\infty} \frac{x^{p-1}(x^q+x^{-q})}{(1+x)^{2p}}\,\mathrm{d}x \ (1+x = \frac{1}{t},\ t = \frac{1}{1+x})$

$= 2^{2p-2}\int_{\frac{1}{2}}^{1} t^{2p}(\frac{1-t}{t})^{p-1}[(\frac{1-t}{t})^{q} + (\frac{1-t}{t})^{-q}]\frac{1}{t^2}\,\mathrm{d}t$ (the two integral terms are equal)

$= 2^{2p-2}\beta(p-q, p+q)$

(7) $\int_{-1}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\,\mathrm{d}x \ (x=\tan\theta)$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(\cos\theta+\sin\theta)^{2m}(\cos\theta-\sin\theta)^{2n}}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}\,\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(1+\sin 2\theta)^m(1-\sin 2\theta)^n}{\cos 2\theta}\,\mathrm{d}\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+\sin\theta)^m(1-\sin\theta)^n}{\cos\theta}\,\mathrm{d}\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+\sin\theta)^m(1-\sin\theta)^n}{\cos^2\theta}\,\mathrm{d}\sin\theta$

$= \frac{1}{2}\int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}\,\mathrm{d}x \ (1+x=2t,\ t=\frac{1+x}{2})$

$= \int_0^1 (2t)^{m-1}[2(1-t)]^{n-1}\,\mathrm{d}t$

$= 2^{m+n-2}\beta(m,n)$

2 $\beta(r,p)\beta(r+p,q) = \frac{\Gamma(r)\Gamma(p)\Gamma(r+p)\Gamma(q)}{\Gamma(r+p)\Gamma(r+p+q)} = \frac{\Gamma(r)\Gamma(q)\Gamma(r+q)\Gamma(p)}{\Gamma(r+q)\Gamma(r+p+q)} = \beta(r,q)\beta(r+q,p)$ □

3 $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)^{\cos 2\alpha}\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\frac{\sin(\theta+\frac{\pi}{4})}{\cos(\theta+\frac{\pi}{4})}\right]^{\cos 2\alpha}\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}(\tan\theta)^{\cos 2\alpha}\,\mathrm{d}\theta$

$= \int_0^{\frac{\pi}{2}}(\sin t)^{\cos 2\alpha}(\cos t)^{-\cos 2\alpha}\,\mathrm{d}\theta = \frac{1}{2}\beta(\frac{1+\cos 2\alpha}{2}, \frac{1-\cos 2\alpha}{2}) = \frac{1}{2}\beta(\cos^2\alpha, \sin^2\alpha)$

$= \frac{\pi}{2\sin(\pi\cos^2\alpha)}$