Peking University

College of Engineering

Mathematical Analysis (3)

# Answer Key

Circle

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# Contents

# 1

## Lec 01

1 Use reduction to absurdity. Suppose $\lim\limits_{n\to\infty} a_n \neq 0$ or doesn't exist. So $\exists\varepsilon > 0$, $\forall N_1 > 0, \exists n > N_1, |a_n| > 3\varepsilon$. For such $\varepsilon, \exists N_2 > 0, \forall n > N_2, |a_{2n} + 2a_n| < \varepsilon$ So $\exists N > N_2, |a_N| > 3\varepsilon$ and $|a_{2N} + 2a_N| < \varepsilon$. So $|a_{2N}| > 5\varepsilon$. Similarly, $|a_{4N}| > 9\varepsilon$, and then $|a_{2^pN}| > \left(2^{p+1}+1\right)\varepsilon$ for $p \in \mathbb{N}$, which contradict the boundedness of $a_n$. $\square$

2 Let $a_n = \frac{2}{3} + b_n$. So $\lim\limits_{n\to\infty}(b_{2n} + 2b_n) = 0$. According to the conclusion of previous problem, $\lim\limits_{n\to\infty} b_n = 0$, and $\lim\limits_{n\to\infty} a_n = \frac{2}{3}$ is evident. $\square$

3 (1) $\because x_{n+2} = 1 + \frac{1}{x_{n+1}} = 1 + \frac{1}{1+\frac{1}{x_n}} = 1 + \frac{x_n}{1+x_n}$ and using mathematical induction: $x_n > 0$

$\therefore$ for $n \geqslant 3,\ 1 < x_n < 2$

$\therefore 1 \leqslant \varliminf\limits_{n\to\infty} x_n \leqslant \varlimsup\limits_{n\to\infty} x_n \leqslant 2$ $\square$

(2) $\because$ for $n \geqslant 3$, $\left|\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}\right| = \left|-\frac{1}{x_n+1}\right| < \frac{1}{2}$

$\therefore x_n$ is a Cauchy sequence

Calculate the positive fixed point of equation $x^* = 1 + \frac{1}{x^*}$

$\therefore \lim\limits_{n\to\infty} x_n = x^* = \frac{1+\sqrt{5}}{2}$

5 (1) $\because \lim\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$

$= \varliminf\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n \leqslant \varlimsup\limits_{n\to\infty}(x_n + y_n) \leqslant \varlimsup\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$

$= \lim\limits_{n\to\infty} x_n + \varlimsup\limits_{n\to\infty} y_n$ $\square$

(2) The proof for $\varlimsup\limits_{n\to\infty} y_n = \pm\infty$ is direct. Suppose $\varlimsup\limits_{n\to\infty} y_n = A$.

$\forall\varepsilon_1 > 0, \exists N_1(\varepsilon_1) > 0,\ \forall n > N_1(\varepsilon_1),\ y_n < A + \varepsilon_1$, and $\exists\left\{y_{n_k^{\varepsilon_1}}\right\},\ y_{n_k^{\varepsilon_1}} > A - \varepsilon_1$

$\forall\varepsilon_2 > 0, \exists N_2(\varepsilon_2) > 0,\ \forall n > N_2(\varepsilon_2),\ |x_n - x^*| < \varepsilon_2,\ x^* = \lim\limits_{n\to\infty} x_n$

$\forall\varepsilon > 0, \exists\varepsilon_1 > 0,\ \varepsilon_2 > 0, \forall\delta_1 \in (0, \varepsilon_1),\ \delta_2 \in (-\varepsilon_2,\ \varepsilon_2)$, s.t. $0 \leqslant \frac{\delta_1}{x^*+\delta_2} - \frac{A\delta_2}{x^*(x^*+\delta_2)} < \varepsilon$, and $0 \leqslant \frac{\delta_1}{x^*} + \frac{(A-\delta_1)\delta_2}{x^*(x^*+\delta_2)} < \varepsilon$

$\therefore$ for $\forall\varepsilon > 0,\ \exists N = \max\left(N_1(\varepsilon_1),\ N_2(\varepsilon_2)\right),\ \forall n > N,\ (x_n y_n) > x^*A + \varepsilon$

and $\exists\left\{y_{n_k^{\varepsilon_1}} x_{n_k^{\varepsilon_1}}\right\},\ y_{n_k^{\varepsilon_1}} x_{n_k^{\varepsilon_1}} > x^*A - \varepsilon$ $\square$

6 (1) $\sup\limits_{k\geqslant n} a_k = 1,\ \inf\limits_{k\geqslant 2n} a_k = \inf\limits_{k\geqslant 2n+1} a_k = a_{2n+1} = \frac{1}{2^{-2n-1}-1}$

$\therefore \varlimsup\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\sup\limits_{k\geqslant n} a_k = 1,\ \varliminf\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\inf\limits_{k\geqslant n} a_k = -1$

(2) $\sup\limits_{k\geqslant 2n} a_k = \sup\limits_{k\geqslant 2n-1} a_k = a_{2n} = 1 + \frac{1}{2n}$

$\inf\limits_{k\geqslant 2n} a_k = \inf\limits_{k\geqslant 2n+1} a_k = a_{2n+1} = -1 - \frac{1}{2n+1}$

$\therefore \overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = 1,\ \ \underline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -1$

(3) $|a_n| = \frac{1}{n} \to 0$
$\therefore \overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} a_n = 0$

(4) For a period of $n = 0 \sim 9 \bmod 10$, maximum $a_n$ is $\sin\frac{2\pi}{5}$, minimum $a_n$ is $-\sin\frac{2\pi}{5}$
$\therefore \overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \sup_{k\geqslant n} a_k = \sin\frac{2\pi}{5},\ \underline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k = -\sin\frac{2\pi}{5}$

## Lec 02

1 Suppose $\overline{\lim_{n\to\infty}}\, na_n > 0$. Then just suppose $\overline{\lim_{n\to\infty}}\, na_n \geqslant 1$
$\therefore \exists \left\{a_{n_k}\right\}, a_{n_k} \geqslant \frac{1}{n_k}$
Then $\exists \left\{a_{n_{k_l}}\right\}, n_{k_{l+1}} \geqslant 2n_{k_l}$
$\therefore \sum_{n=1}^{\infty} a_n \geqslant \sum_{l=2}^{\infty} \left(\frac{n_{k_l}}{2}\frac{1}{n_{k_l}}\right) = \infty$
$\therefore \overline{\lim_{n\to\infty}}\, na_n = 0$
$\because \underline{\lim_{n\to\infty}}\, na_n \geqslant 0$
$\therefore \lim_{n\to\infty} na_n = 0$

3 (1) $\because 0 \leqslant \frac{1}{(5n-4)(5n+1)} \leqslant \frac{1}{n^2}$
$\therefore$ absolutely convergent

(2) $\because \lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$
$\therefore$ divergent

(3) $\because 0 \leqslant \frac{1}{2^n} + \frac{1}{3^n} \leqslant \frac{1}{2^{n-1}}$
$\therefore$ absolutely convergent

(4) $\because 0 \leqslant \frac{1}{(3n-2)(3n+1)} \leqslant \frac{1}{n^2}$
$\therefore$ absolutely convergent

(5) $\because \lim_{n\to\infty} a_n = 1 \neq 0$
$\therefore$ divergent

4 (1) $\forall \varepsilon > 0, \exists N > \log_{|q|} \frac{\varepsilon(1-|q|)}{A}, \forall n_1, n_2 > N$
$\sum_{n=n_1}^{n_2} |a_n q^n| \leqslant A \sum_{n=n_1}^{n_2} |q|^n = A|q|^{n_1}\frac{1-|q|^{n_2-n_1+1}}{1-|q|} < \varepsilon$
$\therefore$ absolutely convergent

(2) $\because a_n = \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} > \frac{1}{3n}$
$\therefore \exists \varepsilon = 1 > 0, \forall N > 0,\ \exists n_1 > N,\ n_2 = 3n_1 + 10 > N$
$\sum_{n=n_1}^{n_2} a_n > \sum_{n=n_1}^{n_2} \frac{1}{3n} > \frac{2n_2}{3}\frac{1}{3n_2} > \frac{2}{9}$

$\therefore$ divergent

5 Let $b_k = \sum\limits_{n=n_{k-1}+1}^{n_k} a_n$

$\therefore \forall \varepsilon > 0, \exists K > 0, \forall k_1, k_2 > K, \ \left|\sum\limits_{k=k_1}^{k_2} b_k\right| < \varepsilon$

$\therefore \exists N > n_{K+1} + 1, \ \forall n_1, n_2 > N, \ \left|\sum\limits_{n=n_1}^{n_2} a_n\right| < \left|\sum\limits_{k=K}^{\infty} b_k\right| \leqslant \varepsilon$

$\therefore$ absolutely convergent

# 2

**Lec 03**

1 (2) $\ln(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + o(\frac{1}{n^4})$

$\therefore a_n = \frac{1}{12n^2} + o(\frac{1}{n^2})$

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{\frac{1}{12n^2}} = 1$

$\therefore$ absolutely convergent

(3) $\ln[(1+\frac{1}{n})^n] = n\ln(1+\frac{1}{n}) = 1 - \frac{1}{2n} + o(\frac{1}{n})$

$\therefore (1+\frac{1}{n})^n = e^{1-\frac{1}{2n}+o(\frac{1}{n})} = e - \frac{e}{2n} + o(\frac{1}{n})$

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{(\frac{e}{2n})^p} = 1$

$\begin{cases} p > 1 : \text{absolutely convergent} \\ p \leqslant 1 : \text{divergent} \end{cases}$

2 (1) $\lim\limits_{n\to\infty} \sqrt[n]{(1-\frac{1}{n})^{n^2}} = \lim\limits_{n\to\infty} (1-\frac{1}{n})^n = \frac{1}{e} < 1$

$\therefore$ absolutely convergent

(2) $\frac{a_{n+1}}{a_n} = \frac{x}{1+x^n}$

$\begin{cases} x < 1 : \lim\limits_{n\to\infty} \frac{x}{1+x^n} = x < 1, \text{absolutely convergent} \\ x = 1 : \lim\limits_{n\to\infty} \frac{x}{1+x^n} = \frac{1}{2}, \text{absolutely convergent} \\ x > 1 : \lim\limits_{n\to\infty} \frac{x}{1+x^n} = 0, \text{absolutely convergent} \end{cases}$

3 (1) $\int_2^{\infty} 3^{-x^{\frac{1}{2}}} \, \mathrm{d}x = \int_{\sqrt{2}}^{\infty} 2t \cdot 3^{-t} \, \mathrm{d}t$

$\therefore$ absolutely convergent

(2) $\frac{1}{a^{\ln x}} = \frac{1}{x^{\ln a}}$

$$\begin{cases} a > e : \ln a > 1, \text{absolutely convergent} \\ a \leqslant e : \ln a \leqslant 1, \text{divergent} \end{cases}$$

4 (1) $\lim\limits_{n\to\infty} a_n = \frac{1}{2}$

$\therefore$ divergent

(4) $\lim\limits_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = 1$

$\therefore$ divergent

(5) For $n > 1000, \ln(n+1) > 2$

$\therefore \frac{1}{[\ln(n+1)]^n} < \frac{1}{2^n}$

$\therefore$ absolutely convergent

6 (3) $\int \frac{1}{x\ln x(\ln\ln x)}\,\mathrm{d}x = \ln\ln\ln x + C$

$\therefore$ for $\sigma \leqslant 0$, divergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}} \leqslant 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n(\ln n)^{1+\sigma}}$$

$$\leqslant 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{2^{k-1}[\ln(2^{k-1})]^{1+\sigma}}$$

$$\leqslant 10 + \sum_{k=2}^{\infty} \frac{1}{[\ln(2^{k-1})]^{1+\sigma}}$$

$$= 10 + \sum_{k=2}^{\infty} \frac{1}{(k-1)^{1+\sigma}}$$

$\therefore$ for $\sigma > 0$, $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}}$ absolutely convergent

Also $\because$ for $\forall p > 0, \exists X > 0, \forall x > X, (\ln x)^p > \ln\ln x$

$\therefore$ for $\sigma > 0$, absolutely convergent

(4) Let $p = 1$, $\int_2^{\infty} \frac{1}{x\ln x(\ln\ln x)^q}\,\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{\mathrm{d}t}{t(\ln t)^q}$

$\therefore$ similar to the condition in previous problem

In conclusion:

$$\begin{cases} p > 1 : \text{absolutely convergent} \\ p = 1 : \begin{cases} q > 1 : \text{absolutely convergent} \\ q \leqslant 1 : \text{divergent} \end{cases} \\ p < 1 : \text{divergent} \end{cases}$$

**Lec 04**

2 (1) $(k^2-1)a_{k^2-1}=\frac{1}{k^2-1},\ k^2a_{k^2}=1$

$\exists\varepsilon=\frac{1}{2},\forall N>0,\exists k>N,\left|1-\frac{1}{k^2-1}\right|>\varepsilon$

$\therefore\lim\limits_{n\to\infty}a_n$ doesn't exist $\square$

(2) Let $b_k=\left[\frac{1}{k^2}+\sum\limits_{n=k^2+1}^{(k+1)^2-1}\frac{1}{n^2}\right]$

Evidently $b_k$ is absolutely convergent

Use conclusion of Lec 02 Problem 05

$\therefore$ absolutely convergent $\square$

3 (1) Evidently $\lim\limits_{n\to\infty}x_n$ exists. Let $x_n\to A$

$\therefore\lim\limits_{n\to\infty}\frac{1-\frac{x_n}{x_{n+1}}}{\frac{x_{n+1}-x_n}{A}}=1$

$\because\sum\limits_{n=1}^{\infty}\frac{x_{n+1}-x_n}{A}=1-\frac{x_1}{A}$, converge

$\therefore$ absolutely convergent $\square$

(2) $\exists\varepsilon=\frac{1}{2},\forall N>0,\ \exists n_1,\ n_2>N,\ \sum\limits_{n=n_1}^{n_2}(1-\frac{x_n}{x_{n+1}})>\frac{x_{n_2}-x_{n_1}}{x_{n_2}}>\varepsilon$

$\therefore$ divergent $\square$

4 (1) Let $b_k=a_{3k+1}+a_{3k+2}+a_{3k+3}$

$|b_k|$ monotonically decreases to 0 and $\mathrm{sgn}(\frac{b_k}{b_{k-1}})=-1$

Use the conclusion of Lec 04 Problem 05, $a_n$ converges

$\because|a_n|=\frac{1}{n}$

$\therefore$ conditionally convergent

(2) For $a\neq0$, $n\sqrt{1+\frac{a^2}{n^2}}=n+\frac{a^2}{2n}+o(\frac{1}{n})$

$\therefore a_n=(-1)^n\sin(\frac{\pi a^2}{2n})+o(\frac{1}{n})$

$|a_n|$ monotonically decreases to 0 and $\mathrm{sgn}(\frac{a_n}{a_{n-1}})=-1$

$\therefore a_n$ converges

$\because\lim\limits_{n\to\infty}\frac{\sin\frac{\pi a^2}{2n}}{\frac{1}{100n}}>1$

$\therefore$ In conclusion:

$$\begin{cases} a \neq 0 : \text{conditionally convergent} \\ a = 0 : \text{absolutely convergent} \end{cases}$$

(3) $\ln\left(1 + \frac{(-1)^n}{n^p}\right) = \frac{(-1)^n}{n^p} - \frac{1}{n^{2p}} + o(\frac{1}{n^{2p}})$

For $p > 0$, $\left|\frac{(-1)^n}{n^p}\right|$ monotonically decreases to 0 and $\operatorname{sgn}\left(\frac{\frac{(-1)^n}{n^p}}{\frac{(-1)^{n-1}}{(n-1)^p}}\right) = -1$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges when $p > \frac{1}{2}$, diverges when $p \leqslant \frac{1}{2}$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$, diverges when $p \leqslant 1$

For $p \leqslant 0$, evidently diverge

$\therefore$ In conclusion:

$$\begin{cases} p \leqslant \frac{1}{2} : \text{divergent} \\ \frac{1}{2} < p \leqslant 1 : \text{conditionally convergent} \\ p > 1 : \text{absolutely convergent} \end{cases}$$

(4) Let $b_k = |a_{2k-1}| + |a_{2k}|$, $0 < b_k < \frac{1}{2^{k-1}}$

$\therefore b_k$ converges

Use conclusion of Lec 02 Problem 05, $|a_n|$ converges

$\therefore$ absolutely convergent

(5) $\sum\limits_{n=1}^{2N} a_n < e - \sum\limits_{n=1}^{N} \frac{1}{2n}$

$\because -\sum\limits_{n=1}^{N} \frac{1}{2n} \to -\infty$

$\therefore$ divergent

(6) $\because |a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because |a_n| > \frac{1}{n}$

$\therefore$ conditionally convergent

(7) $\because \int_2^{\infty} x^3 2^{-x} \,\mathrm{d}x$ converges

$\therefore$ absolutely convergent

(8) $\because |a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because |a_n| > \frac{1}{20n}$

$\therefore$ conditionally convergent

(9) $\because |a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because \lim\limits_{n\to\infty} \frac{a_n}{\frac{1}{n}} = x$

$\therefore$ In conclusion:

$\begin{cases} x \neq 0 : \text{conditionally convergent} \\ x = 0 : \text{absolutely convergent} \end{cases}$

(10) Let $b_k = a_{2k-1} + a_{2k} = \frac{2}{k-1}$

$\therefore$ divergent

# 3

**Lec 04**

5 Let $b_k$ equal to the sum of $k^{th}$ set of successive $a_n$ which have the same sign

If $n_0$ is in the $k_0^{th}$ set, denote $k(n_0) = k_0$

$\because \sum\limits_{k=1}^{\infty} b_k$ convergent, $\lim\limits_{k\to\infty} b_k = 0$

$\therefore \forall \varepsilon > 0, \exists K > 0, \forall k_1, k_2 > K, |\sum\limits_{k=k_1}^{k_2} b_k| < \varepsilon, |b_{k_1}| + |b_{k_2}| < \varepsilon$

$\because \exists N, k(N) > K, \forall n_1, n_2 > N, |\sum\limits_{n=n_1}^{n_2} a_n| \leqslant \varepsilon + |b_{k(n_1)}| + |b_{k(n_2)}| < 2\varepsilon$

$\therefore$ convergent

For $a_n = \frac{(-1)^{[\sqrt{n}]}}{n}$, let $b_k = (-1)^k \sum\limits_{n=(k-1)^2+1}^{k^2} \frac{1}{n}$

$\therefore |b_k| < \frac{2k}{(k-1)^2}$

$\therefore |b_k|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{b_k}{b_{k-1}}) = -1$

$\therefore b_k$ converges But $|a_n| = \frac{1}{n}$

$\therefore$ conditionally convergent

8 $\forall\varepsilon>0, \exists N>0, \forall n_1, n_2>N,$

$\max\{|\sum\limits_{n=n_1}^{n_2} n(a_n-a_{n-1})|, |n_1a_{n_1-1}|, |n_2a_{n_2}|\}<\varepsilon$

$\therefore |\sum\limits_{n=n_1}^{n_2-1} a_n| = |\sum\limits_{n=n_1}^{n_2} n(a_n-a_{n-1})+n_1a_{n_1-1}-n_2a_{n_2}|<3\varepsilon$

$\therefore$ convergent $\square$

## Lec 05

3 (1) $x_{n+1}-x_n=\frac{\ln(n+1)}{n+1}-\frac{1}{2}\left[\ln^2(n+1)-\ln^2 n\right]=\frac{\ln(n+1)}{n+1}-\int_n^{n+1}\frac{\ln x}{x}\,\mathrm{d}x$

$\frac{\ln x}{x}$ is monotonically decreasing over $[\mathrm{e},+\infty)$

if $n>3$, $\frac{\ln(n+1)}{n+1}<\int_n^{n+1}\frac{\ln x}{x}\,\mathrm{d}x<\frac{\ln n}{n}$

$\therefore x_{n+1}<x_n$, $x_n$ monotonically decreasing

$$x_n=\sum_{k=1}^{n}\frac{\ln k}{k}-\frac{1}{2}(\ln n)^2=\sum_{k=1}^{n}\frac{\ln k}{k}-\int_1^n\frac{\ln x}{x}\,\mathrm{d}x$$

$$=\sum_{k=1}^{2}\frac{\ln k}{k}-\int_1^3\frac{\ln x}{x}\,\mathrm{d}x+\sum_{k=3}^{n}\frac{\ln k}{k}-\int_3^n\frac{\ln x}{x}\,\mathrm{d}x>\frac{\ln 2}{2}-\ln^2 3$$

$\therefore$ convergence

(2) $x_{n+1}-x_n=\frac{1}{\sqrt{n+1}}-2\sqrt{n+1}-2\sqrt{n}<0$

$\therefore x_n$ monotonically decreasing

$$\sqrt{n}=\sqrt{n}-\sqrt{n-1}+\sqrt{n-1}-\sqrt{n-2}+\cdots+\sqrt{2}-\sqrt{1}+1$$

$$=\frac{1}{\sqrt{n}+\sqrt{n-1}}+\frac{1}{\sqrt{n-1}+\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{2}+\sqrt{1}}+1$$

$$<\frac{1}{2}\left(\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{1}}\right)+1$$

$$x_n>1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-\left(\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{1}}+2\right)=-2+\frac{1}{\sqrt{n}}>-2$$

$\therefore$ convergence

4 $\sum\limits_{n=0}^{\infty}|x|^n=\frac{1}{1-|x|}$, $\sum\limits_{n=0}^{\infty}|y|^n=\frac{1}{1-|y|}$ (both absolutely convergent)

$\therefore \sum\limits_{n=1}^{\infty}(x^{n-1}+x^{n-2}y+\cdots+y^{n-1})=\sum\limits_{n=0}^{\infty}x^n\sum\limits_{n=0}^{\infty}y^n=\frac{1}{(1-x)(1-y)}$ $\square$

5 $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$

$\therefore$ radius of convergence is $\infty$

$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x, \sum_{n=0}^{\infty} \frac{y^n}{n!} = \mathrm{e}^y$ both absolutely convergent

$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} \frac{y^n}{n!} = \mathrm{e}^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$ $\square$

## Lec 06

1 (1) Let $p_n = q_n = 1$, $\prod_{n=1}^{\infty}(p_n + q_n) = \prod_{n=1}^{\infty} 2 = \infty$

$\therefore$ divergent

(2) $\prod_{n=1}^{\infty} p_n, \prod_{n=1}^{\infty} q_n$ converge

$\Rightarrow \sum_{n=1}^{\infty} \ln p_n, \sum_{n=1}^{\infty} \ln q_n$ converge

$\Rightarrow \sum_{n=1}^{\infty} (\ln p_n + \ln q_n)$ converges

$\Rightarrow \sum_{n=1}^{\infty} \ln(p_n q_n)$ converges

$\Rightarrow \prod_{n=1}^{\infty} p_n q_n$ converges

(3) Let $q_n = p_n$ and use conclusion of previous problem

$\therefore$ convergent

(4) $\prod_{n=1}^{\infty} q_n$ converges

$\Rightarrow \sum_{n=1}^{\infty} \ln q_n$ converges

$\Rightarrow -\sum_{n=1}^{\infty} \ln q_n$ converges

$\Rightarrow \prod_{n=1}^{\infty} \frac{1}{q_n}$ converges

Use conclusion of Lec 06 Prob 1(2), $\prod_{n=1}^{\infty} \frac{p_n}{q_n}$ converges

2 Denote $T_n = \prod_{k=1}^{n}(1+u_k), S_n = \sum_{k=1}^{n} u_k, S'_n = \sum_{k=1}^{n}(u_k)^2$

$\therefore S_{2n} = \sum_{k=1}^{n} \frac{1}{k} \to \infty, S'_{2n} > 2\sum_{k=1}^{n} \frac{1}{k} \to \infty$

$\because \lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} u_{2n} = 0$

$\therefore S_n, S'_n$ diverges

$\because (1+u_{2k-1})(1+u_{2k}) = 1 - \frac{1}{k^{\frac{3}{2}}}$

$\therefore T_{2n}$ converges, let A denote its limit

$\therefore \forall \varepsilon > 0, \exists N_1 > 0, \forall n > N_1, |T_{2n} - A| < \varepsilon$

And $\lim_{n\to\infty} \frac{T_{2n+1}}{T_{2n}} = u_{2n+1} + 1 = 1$

$\therefore$ for $\varepsilon > 0, \exists N_2 > 0, \forall n > N_2, |T_{2n+1} - T_{2n}| < \varepsilon$

$\therefore \forall \varepsilon > 0, \exists N = \max\{2N_1 + 10, 2N_2 + 10\} > 0, \forall n > N, |T_n - A| < 2\varepsilon$

$\therefore T_n$ converges $\square$

3 (1) $\lim_{n\to\infty} \frac{\ln[(\frac{n^2-1}{n^2+1})^p]}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{p\ln(1-\frac{2}{n^2+1})}{\frac{1}{n^2}} = -2p$

$\therefore$ convergent

(2) $\lim_{n\to\infty} \frac{\ln \sqrt[n]{1+\frac{1}{n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}\ln(1+\frac{1}{n})}{\frac{1}{n^2}} = 1$

$\therefore$ convergent

(3) $\lim_{n\to\infty} \frac{\ln \sqrt[n]{\ln(n+x)-\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}\ln\ln(1+\frac{x}{n})}{\frac{1}{n}} = -\infty$

$\therefore$ divergent

(4) $\lim_{n\to\infty} \frac{\ln \frac{n^2-4}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln\left(1-\frac{3}{n^2-1}\right)}{\frac{1}{n^2}} = -3$

$\therefore$ convergent

(5) $\ln a^{\frac{(-1)^n}{n}} = \frac{(-1)^n}{n}\ln a$

$\because \frac{1}{n}$ monotonically decreases to 0

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\ln a$ converges

$\therefore$ convergent

(6) $\because \prod_{k=1}^{n} \sqrt{\frac{k+1}{k+2}} = \sqrt{\frac{2}{n+2}} \to 0$

$\therefore$ divergent

5 Due to convergence $\lim_{n\to\infty} a_n = 0$

$\tan(\frac{\pi}{4} + x) = 1 + Ax + o(x), A = \tan'(\frac{\pi}{4}) > 0$

$\therefore \lim_{n\to\infty} \frac{|\ln[\tan(\frac{\pi}{4}+a_n)]|}{|a_n|} = A$

$\therefore \sum_{n=1}^{\infty} \ln[\tan(\frac{\pi}{4} + a_n)]$ converges

$\therefore$ convergent $\square$

# 4

## Lec 07

1 (1) $\because \left|f_n(x) - |x|\right| = \left|\frac{1}{n^2(\sqrt{x^2+\frac{1}{n^2}}+|x|)}\right| \leq \frac{1}{n}$

$\therefore \lim_{n\to\infty} \sup_{x\in\mathcal{X}} \left|f_n(x) - |x|\right| = 0$

$\therefore$ uniformly convergent

(2) $\because \sup_{x\in\mathcal{X}} \left|f_n(x) - 0\right| = \frac{1}{4}$

$\therefore$ not uniformly convergent

(3) $\because \left|f_n(x) - 0\right| \leq \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \leq \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} \sup_{x\in\mathcal{X}} \left|f_n(x) - 0\right| = 0$

$\therefore$ uniformly convergent

(4) $\because$ if $n > 100$, $\lim_{n\to\infty} \sup_{x\in\mathcal{X}} \left|f_n(x) - 0\right| = \lim_{n\to\infty} \frac{\ln n}{n} = 0$

$\therefore$ uniformly convergent

2 (1) $S(x) = \begin{cases} x, & x \in [0,1) \\ 0, & x = 1 \end{cases}$ is not continuous

$\therefore$ not uniformly convergent

(2) denote $a_n(x) = \frac{x^2}{(1+x^2)^n}$, $b_n(x) = (-1)^n$

$\because$ if $n > 2$, $\sup\limits_{x\in\mathcal{X}} |a_n(x) - 0| < \frac{1}{n-1}$

$\therefore a_n(x) \xrightarrow{\mathcal{X}} 0$

$\because a_n(x)$ is about $n$ monotonically decreasing and $\sum\limits_{n=1}^{\infty} b_n(x)$ is uniformly bounded

$\therefore$ uniformly convergent

(3) $\because \left|\frac{\sin nx}{\sqrt[3]{n^4+x^4}}\right| \le \frac{1}{n^{\frac{4}{3}}}$

$\therefore$ uniformly convergent

(4) $\because \left|\frac{x}{1+n^4x^2}\right| \le \frac{1}{n^2}$

$\therefore$ uniformly convergent

(5) denote $a_n(x) = \frac{1}{\sqrt{n+x}}$, $b_n(x) = \sin nx \sin x$

$\because \sin nx \sin x = \frac{\cos(n-1)x - \cos(n+1)x}{2}$

$\therefore \sum\limits_{n=1}^{\infty} b_n(x)$ is uniformly bounded

$\because a_n(x)$ is about $n$ monotonically decreasing and $a_n(x) \xrightarrow{\mathcal{X}} 0$

$\therefore$ uniformly convergent

(6) $\because \left|\frac{(-1)^n\left(1-e^{-nx}\right)}{n^2+x^2}\right| \le \left|\frac{1}{n^2}\right|$

$\therefore$ uniformly convergent

3 $\because \left|\frac{\ln(1+nx)}{nx^n}\right| \le \frac{1}{x^{n-1}} \le \frac{1}{\alpha^{n-1}}$

$\therefore$ uniformly convergent $\square$

4 $\because f_0(x)$ is continuous over $[0, a]$

$\therefore \exists A$ s.t. $|f(x)| < A$

$\because f_n(x) = \int_0^x f_{n-1}(t)\,\mathrm{d}t$

$\therefore \left|f_n(x)\right| \le \frac{Ax^n}{n!} \le \frac{Aa^n}{n!}$

$\therefore f_n(x) \xrightarrow{\mathcal{X}} 0$ $\square$

5 if $\sum\limits_{n=1}^{\infty} \left|f_n(x)\right|$ uniformly convergent

$\because \left|\sum\limits_{i=n}^{m} f_i(x)\right| \le \sum\limits_{i=n}^{m} \left|f_i(x)\right|$

$\therefore \forall \epsilon > 0,\ \exists N > 0,\ \text{when } m, n > N,\ \forall x \in \mathcal{X},\ \left|\sum\limits_{i=n}^{m} f_i(x)\right| \leq \sum\limits_{i=n}^{m} \left|f_i(x)\right| < \epsilon$

$\therefore \sum\limits_{n=1}^{\infty} f_n(x)$ uniformly convergent $\square$

but the inverse is not true, for example $f_n(x) = \frac{(-1)^n x}{n}$ $\square$

**Lec 08**

1 denote $a_n = \max(|\varphi_n(a)|, |\varphi_n(b)|)$

$\because \varphi_n(x)$ is about $x$ monotonous over $[a, b]$

$\therefore |\varphi_n(x)| \leq a_n \leq |\varphi_n(a)| + |\varphi_n(b)|$

$\because \sum\limits_{n=1}^{\infty} \left|\varphi_n(a)\right|,\ \sum\limits_{n=1}^{\infty} \left|\varphi_n(b)\right|$ is absolutely convergent

$\therefore$ uniformly convergent $\square$

2 $\forall a, b,\ 0 < a < b,\ x \in [a, b]$

$\because 0 < \frac{n}{\mathrm{e}^{xn}} \leq \frac{n}{\mathrm{e}^{an}}$ and $\sum\limits_{n=1}^{\infty} \frac{n}{\mathrm{e}^{an}}$ is convergent

$\therefore \sum\limits_{n=1}^{\infty} n\mathrm{e}^{-nx}$ is uniformly convergent over $(0, +\infty)$

$\because n\mathrm{e}^{-nx}$ is continuous

$\therefore$ continuous $\square$

3 denote $a_n(x) = \frac{\sin nx}{n^3}$

$\because \left|a_n(x)\right| \leq \frac{1}{n^3}$

$\therefore \sum\limits_{n=1}^{\infty} a_n(x)$ is uniformly convergent

$\because a_n(x)$ is continuous $\therefore f(x)$ is continuous

$\because \left|a_n'(x)\right| \leq \frac{1}{n^2} \therefore \sum\limits_{n=1}^{\infty} a_n'(x)$ is uniformly convergent

$\therefore f'(x) = \sum\limits_{n=1}^{\infty} a_n'(x)$

$\because a_n'(x)$ is continuous $\therefore f'(x)$ is continuous $\square$

4 for $n > 1,\ \forall m \in \mathbb{N} \because \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(e^x\right)^{-\ln n} = (-\ln n)^m \left(e^x\right)^{-\ln n} = (-\ln n)^m \frac{1}{n^x}$

$\therefore \forall \alpha > 1,\ \text{when } x \geq \alpha,\ \left|\frac{\mathrm{d}^{\,m}}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right)\right| \leq (\ln n)^m \frac{1}{n^\alpha}$

$\because \sum\limits_{n=1}^{\infty} (\ln n)^m \frac{1}{n^\alpha}$ is convergent and $(\ln n)^m \frac{1}{n^x}$ is continuous

$\therefore \sum\limits_{n=1}^{\infty} \frac{\mathrm{d}^{\,m}}{\mathrm{d}x^m}\left(\frac{1}{n^x}\right)$ uniformly convergent

$\therefore \zeta^{(n)}(x)$ is continuous $\square$

5 $\because \left|\frac{\sin(2^n \pi x)}{2^n}\right| \leq \frac{1}{2^n} \therefore$ uniformly convergent

$\because \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(2^n\pi x)}{2^n} = \pi \cos\left(2^n \pi x\right)$

$$\lim_{n\to+\infty} \pi\cos\left(2^n\pi x\right) = \begin{cases} \text{not exists},\ x \neq \frac{m}{2^k},\ m,k \in \mathbb{Z} \\ \pi,\ x = \frac{m}{2^k},\ m,k\in\mathbb{Z} \end{cases} \neq 0$$

$\therefore$ can't doing derivation at every formula $\square$

6 if $|x| = 1,\ f(x) = \int_{-\pi}^{\pi} \frac{1-x^2}{1+x^2-2x\cos\theta}\,\mathrm{d}\theta = 0$

if $|x| < 1,\ f(x) = \int_{-\pi}^{\pi} 1 + 2\sum\limits_{n=1}^{\infty} x^n \cos n\theta\,\mathrm{d}\theta$

$\because |x^n \cos n\theta| \leq |x^n|,\ \sum\limits_{n=1}^{\infty} |x^n|$ is convergent

$\therefore \sum\limits_{n=1}^{\infty} x^n \cos n\theta$ is about $\theta$ uniformly convergent

$\therefore f(x) = 2\pi + 2\sum\limits_{n=1}^{\infty}\int_{-\pi}^{\pi} x^n\cos n\theta\,\mathrm{d}\theta = 2\pi$

if $|x| > 1,\ f(x) = -2\pi$

in conclusion

$$f(x) = \begin{cases} 0,\ |x| = 1 \\ 2\pi,\ |x| < 1 \\ -2\pi,\ |x| > 1 \end{cases}$$

# 5

## Lec 09

1 (1) $\varliminf\limits_{n\to\infty} \sqrt[n]{\frac{n!}{2^n}} = +\infty$

$\therefore R = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(2) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{[3+(-1)^n]^n}} = \frac{1}{4}$

$\therefore R = \frac{1}{4}$

$\because \sum\limits_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1}$ converge and $|x| = \frac{1}{4} : \sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverge

$\therefore$ convergence region: $(-\frac{1}{4}, \frac{1}{4})$

(3) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n+1}{\ln(n+1)}} = 1$

$\therefore R = 1$

$\sum\limits_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

$\frac{\ln(n+1)}{n+1}$ monotonically decreases to 0 when $n > 3$

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln(n+1)}{n+1}$ converges

$\therefore$ divergent when $x = 1$, convergent when $x = -1$

$\therefore$ convergence region: $[-1, 1)$

(4) $|x| = 1$: convergent

$|x| > 1$: $\lim\limits_{n\to\infty} \frac{|x|^{n^2}}{2^n} = +\infty \neq 0$

$\therefore$ convergence region: $[-1, 1]$

(5) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{3^n+(-2)^n}} = \frac{1}{3} \lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{1+(-\frac{2}{3})^n}} = \frac{1}{3}$

$\therefore R = \frac{1}{3}$

$x + 1 = \frac{1}{3}$: $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum\limits_{n=1}^{\infty} \frac{(-\frac{2}{3})^n}{n}$ converges $\Rightarrow$ diverges

$x + 1 = -\frac{1}{3}$: $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, $\sum\limits_{n=1}^{\infty} \frac{(\frac{2}{3})^n}{n}$ converges $\Rightarrow$ converges

$\therefore$ convergence region: $[-\frac{4}{3}, -\frac{2}{3})$

(6) $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{(1+\frac{1}{n})^{n^2}}} = \frac{1}{\mathrm{e}}$

$\therefore R = \frac{1}{\mathrm{e}}$

$$\lim_{n\to\infty} \frac{(1+\frac{1}{n})^{n^2}}{\mathrm{e}^n} = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x^2}}}{\mathrm{e}^{\frac{1}{x}}} = \mathrm{e}^{\lim\limits_{x\to 0} \ln \frac{(1+x)^{\frac{1}{x^2}}}{\mathrm{e}^{\frac{1}{x}}}} = \mathrm{e}^{\lim\limits_{x\to 0} \frac{\ln(1+x)-x}{x^2}} = \mathrm{e}^{-\frac{1}{2}} \neq 0$$

$\therefore$ convergence region: $(-\frac{1}{\mathrm{e}}, \frac{1}{\mathrm{e}})$

(7) Just let $a \geqslant b$

$$\varlimsup_{n\to\infty} \sqrt[n]{a^n+b^n} = a \lim_{n\to\infty} \sqrt[n]{1+(\frac{b}{a})^n} = a$$

$\therefore R = a$

$$\lim_{n\to\infty} \frac{a^n}{a^n+b^n} \geqslant \frac{1}{2}$$

$\therefore$ convergence region: $(-a, a)$

(8) $\varlimsup\limits_{n\to\infty} \sqrt[n]{n2^n} = 2$

$\therefore R = 2$

$|x| = \sqrt{2}$: $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

$\therefore$ convergence region: $[-\sqrt{2}, \sqrt{2}]$

(9) $1 \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)(2n)!!}{(2n-1)!!}} \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)(2n)!!(2n-1)!!}{(2n-1)!!(2n-1)!!}} \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)!}{(2n-1)!}}$
$= \lim\limits_{n\to\infty} \sqrt[2n-1]{2n(2n+1)} = 1$

$\therefore R = 1$

$|x| = 1$: $\frac{(2n-1)!!}{(2n+1)(2n)!!}$ monotonically decreases to 0

$\therefore$ convergent

$\therefore$ convergence region: $[-1, 1]$

2 (1) $|x| < \sqrt{A} \Rightarrow x^2 < A \Rightarrow$ convergent

$|x| > \sqrt{A} \Rightarrow x^2 > A \Rightarrow$ divergent

$\therefore R = \sqrt{A}$

(2) $\frac{1}{R} = \varlimsup\limits_{n\to\infty} \sqrt[n]{|a_n+b_n|} \leqslant \varlimsup\limits_{n\to\infty} \sqrt[n]{2\max(|a_n|, |b_n|)} = \max(\frac{1}{A}, \frac{1}{B})$

$\therefore R \geqslant \min(A, B)$

(3) $\frac{1}{R} = \varlimsup\limits_{n\to\infty} \sqrt[n]{|a_n b_n|} \leqslant \varlimsup\limits_{n\to\infty} \sqrt[n]{|a_n|} \varlimsup\limits_{n\to\infty} \sqrt[n]{|b_n|} = \frac{1}{AB}$

$\therefore R \geqslant AB$

3 Let $A_m(x) = \sum\limits_{n=1}^{m} a_n x^n,\ B_m(x) = \sum\limits_{n=1}^{m} b_n x^n,\ S_m(x) = \sum\limits_{n=1}^{m} a_n b_{m-n+1} x^{m+1},$

$R_m(x) = A_m(x)B_m(x) - \sum\limits_{n=1}^{m} S_n(x),\ M = \sup\limits_n[\max(|a_n|, |b_n|)]$

$0 < x < 1$: $|R_m(x)| < M^2 \frac{m^2-m}{2} x^{m+2}$

$\therefore \lim\limits_{m\to\infty} R_m(x) = 0,\ 0 < x < 1$

$\because [0,1]$ is in the uniformly convergence region of $A_m(x), B_m(x), \sum\limits_{n=1}^{m} S_n(x)$

$\therefore \lim\limits_{m\to\infty} R_m(x)$ is also continuous in $[0,1]$

$\therefore \lim\limits_{m\to\infty} R_m(1) = 0$

$\therefore \sum\limits_{n=1}^{\infty} S_n(x) = \lim\limits_{m\to\infty} A_m B_m = AB$ □

4 Apparently $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent in $[0, r)$

$\therefore \int_0^x \lim\limits_{m\to\infty} \sum\limits_{n=0}^{m} a_n t^n \,\mathrm{d}t = \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$

$\because \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is uniformly convergent in $[0, r]$ and $\frac{a_n x^{n+1}}{n+1}$ is continuous

$\therefore \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is continuous in $[0, r]$

$\therefore \lim\limits_{x\to r^-} \int_0^x \lim\limits_{m\to\infty} \sum\limits_{n=0}^{m} a_n t^n \,\mathrm{d}t$ exists and is equal to $\sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ □

$f(x) = -\frac{\ln(1-x)}{x} = \sum\limits_{n=0}^{\infty} \frac{x^n}{n+1}$

$\because r = \varliminf\limits_{n\to\infty} \sqrt[n]{n+1} = 1$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ convergent

$\therefore \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^1 \frac{\ln(1-x)}{x} \,\mathrm{d}x$ □

## Lec 10

1 Convergence in $(-\infty, \infty)$ is apparent.

$f^{(2n+1)}(0) = \sum\limits_{m=1}^{\infty} \frac{(-1)^n (2^m)^{2n+1}}{m!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \sum\limits_{m=1}^{\infty} \frac{(-1)^n (2^m x)^{2n+1}}{m!(2n+1)!} \sim \sum\limits_{n=0}^{\infty} (\mathrm{e}^{2^{2n+1}} - 1) \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\because \varliminf\limits_{n\to\infty} \sqrt[2n+1]{\frac{(2n+1)!}{\mathrm{e}^{2^{2n+1}}}} = 0$

$\therefore R = 0$

$\therefore$ divergent when $x \neq 0$

2 $f^{(n)}(0) = \frac{n!}{a^{n+1}}$

$f(x) \sim \sum\limits_{n=0}^{\infty} \frac{x^n}{a^{n+1}}$

$\varliminf\limits_{n\to\infty} \sqrt[n]{|a|^{n+1}} = |a|$

$|\frac{a^n}{a^{n+1}}| = \frac{1}{|a|} \nrightarrow 0$

$\therefore$ convergence region: $(-|a|, |a|)$

3 $f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$

$f(x) \sim \ln 2 + \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{2^n n}$

$\varliminf\limits_{n\to\infty} \sqrt[n]{2^n n} = 2$

$\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore$ convergence region: $(0, 4]$

4 (1) $f^{(n)}(0) = (\frac{\sin x}{x})^{(n-1)}\Big|_{x=0}$

$\because \sin x \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\therefore \frac{\sin x}{x} \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

$f^{(2n+1)}(0) = (\frac{\sin x}{x})^{(2n)}\Big|_{x=0} = \frac{(-1)^n (2n)!}{(2n+1)!} = \frac{(-1)^n}{2n+1}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}$

$R = \varliminf\limits_{n\to\infty} \sqrt[2n+1]{(2n+1)!(2n+1)} = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(2) $\cos t^2 \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}$

$\therefore f^{(4n+1)}(0) = (\cos t^2)^{(4n)}\big|_{t=0} = \frac{(-1)^n(4n)!}{(2n)!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}$

$R = \varliminf\limits_{n\to\infty} \sqrt[4n+1]{(2n)!(4n+1)} = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(3) Let $x = \tan t$

$\arctan \frac{2x}{1-x^2} = -\arctan(\tan 2t) = -2\arctan x \sim -2\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\varliminf\limits_{n\to\infty} \sqrt[2n+1]{2n+1} = 1$

$|x| = 1$: $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

$\therefore$ convergence region: $[-1, 1]$ ( with definition: $\arctan(\pm\infty) = \pm\pi$ )

(4) $f^{(1)}(0) = \frac{1}{\sqrt{1+x^2}} \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{(2n)!!}x^{2n}$

$\therefore f^{(2n+1)}(0) = (\frac{1}{\sqrt{1+x^2}})^{(2n)}\big|_{x=0} = \frac{(-1)^n(2n-1)!!(2n)!}{(2n)!!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{(2n)!!(2n+1)}x^{2n+1}$

$\varliminf\limits_{n\to\infty} \sqrt[2n+1]{\frac{(2n)!!(2n+1)}{(2n-1)!!}} = 1$ (see Lec 09 Prob 1(9))

$\frac{(2n-1)!!}{(2n)!!(2n+1)}$ monotonically decreases to 0

$\therefore$ converges when $|x| = 1$

$\therefore$ convergence region: $[-1, 1]$

5 (1) $\mathrm{e}^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$

$\therefore \sum\limits_{n=1}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x - 1, x \in \mathbb{R}$

(2) $\int_0^x \ln(1+x)\,\mathrm{d}x = (1+x)\ln(1+x) - x$

$\because$ convergence radius of $\ln(1+x)$'s Maclaurin series is 1

$\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$ converges at $|x| = 1$

According to Lec 09 Prob 4, $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n(n+1)} = (1+x)\ln(1+x) - x$ in $[-1,1]$ (define $(1+x)\ln(1+x) - x = 1$ at $x = -1$)

(3) $\int_0^x f(x)\,\mathrm{d}x \sim \sum\limits_{n=1}^{\infty} nx^n \sim \frac{x}{(x-1)^2}$

$\because$ Convergence radius is 1 and $f(x)$ diverges at $|x| = 1$

$\therefore f(x) = \left[\frac{x}{(x-1)^2}\right]' = \frac{1+x}{(1-x)^3}$ in $(-1,1)$

(4) Convergence region is $\mathbb{R}$

$\therefore \int_0^x f(x)\,\mathrm{d}x = \sum\limits_{n=1}^{\infty} \frac{x^{2n+1}}{n!} = x(\mathrm{e}^{x^2} - 1)$

$\therefore f(x) = [x(\mathrm{e}^{x^2} - 1)]' = (2x^2+1)\mathrm{e}^{x^2} - 1$

(5) Let $A = \sum\limits_{n=1}^{\infty} \frac{2n-1}{2^n}$

$\therefore A = 2A - A = 1 + \sum\limits_{n=1}^{\infty} \frac{2n+1}{2^n} - \sum\limits_{n=1}^{\infty} \frac{2n-1}{2^n} = 3$

(6) $\sum\limits_{n=0}^{\infty} \frac{n^2+1}{2^n n!} = \mathrm{e}^{\frac{1}{2}} + \sum\limits_{n=1}^{\infty} \frac{n}{2^n(n-1)!} = \mathrm{e}^{\frac{1}{2}} + \frac{1}{2}\sum\limits_{n=1}^{\infty} \frac{(\frac{1}{2})^{n-1}}{(n-1)!} + \frac{1}{4}\sum\limits_{n=2}^{\infty} \frac{(\frac{1}{2})^{n-2}}{(n-2)!} = \frac{7}{4}\mathrm{e}^{\frac{1}{2}}$

# 6

## Lec 12

3 $\because f'(x)$ monotonically increases in $[0, 2\pi]$

$$\begin{aligned}
\therefore a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = -\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin nx\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\int_{\frac{2i}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x)\sin nx\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\left[\int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi} f'(x)\sin nx\,\mathrm{d}x + \int_{\frac{2i+1}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x)\sin nx\,\mathrm{d}x\right] \\
&= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi} \left[f'(x) - f'(x+\tfrac{\pi}{n})\right]\sin nx\,\mathrm{d}x \geq 0
\end{aligned}$$ $\square$

**Lec 13**

1 $\lim\limits_{p\to\infty}\int_{-\pi}^{\pi} f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \lim\limits_{p\to\infty}\int_0^{\pi}[f(t)-f(-t)]\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_0^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_{\delta}^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_0^{\delta}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$\because \frac{f(t)-f(-t)}{2\sin\frac{t}{2}} \in \mathcal{R}[\delta,\pi]$

$\therefore \lim\limits_{p\to\infty}\int_{\delta}^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t = 0$

$\because f(t)$ is continuous and has unilateral derivative at $t=0$

$\therefore \exists\delta>0,\ M>0$ s.t. when $t\in(0,\delta)$, $-M < \frac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt < M$

$\therefore 0 = \lim\limits_{\delta\to 0} -M\delta \le \lim\limits_{\delta\to 0}\lim\limits_{p\to\infty}\int_0^{\delta}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t \le \lim\limits_{\delta\to 0} M\delta = 0$

$\therefore \lim\limits_{p\to\infty}\int_{-\pi}^{\pi} f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t = \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t$ □

2 (1) $f(x) \backsim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n\cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = (-1)^n\frac{4}{n^2}$

$f(x) = \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

(2) $f(x) \backsim \sum\limits_{n=1}^{\infty} b_n\sin nx$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = -\frac{4}{\pi n^3} + \frac{2(-1)^n}{\pi}\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)$

$f(x) = \sum\limits_{n=1}^{\infty}\left[\frac{4(-1)^n}{\pi n^3} - \frac{4}{\pi n^3} - \frac{2\pi(-1)^n}{n}\right]\sin nx$

(3) $f(x) \backsim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{8\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{4}{n^2}$

$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = -\frac{4\pi}{n}$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

(4) from (1) $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$

$$f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3 $a_0 = \frac{e^{\pi}-e^{-\pi}}{\pi}$

$a_n = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(n^2+1)}$

$b_n = -\frac{(-1)^n n(e^{\pi}-e^{-\pi})}{\pi(n^2+1)}$

$$f(x) = \frac{e^{\pi}-e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)}(\cos nx - n\sin nx)\right]$$

if $x = \pi$, $\frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2}\right) = \frac{1}{2}(e^{\pi} - e^{-\pi})$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi(e^{\pi}+e^{-\pi})}{2(e^{\pi}-e^{-\pi})} - \frac{1}{2}$$

4 $\because f(x) \in \mathcal{R}$, apply bessel's inequality

$\therefore \sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ convergent

$\because \frac{|a_n|}{n} \leq \frac{1}{2}(a_n^2 + \frac{1}{n^2})$ and $\frac{|b_n|}{n} \leq \frac{1}{2}(b_n^2 + \frac{1}{n^2})$

$\therefore \sum_{n=1}^{\infty} \frac{a_n}{n}$ and $\sum_{n=1}^{\infty} \frac{b_n}{n}$ convergent □

# 7

## Lec 14

1 $e^{\cos x}\cos(\sin x) = e^{\cos x}\frac{e^{i\sin x}+e^{-i\sin x}}{2} = \frac{1}{2}\left(e^{e^{ix}} + e^{e^{-ix}}\right) = \sum_{n=0}^{\infty} \frac{1}{2}\frac{(e^{ix})^n+(e^{-ix})^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2}\frac{e^{inx}+e^{-inx}}{n!}$

$= \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}$

2 Let $f(x+c)=\frac{a_0'}{2}+\sum\limits_{n=1}^{\infty}a_n'\cos(nx)+b_n'\sin(nx)=\frac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos[n(x+c)]+b_n\sin[n(x+c)]=\frac{a_0}{2}+\sum\limits_{n=1}^{\infty}[a_n\cos(nc)+b_n\sin(nc)]\cos(nx)+[b_n\cos(nc)-a_n\sin(nc)]\sin(nx)$

$\therefore a_0'=a_0,\ a_n'=a_n\cos(nc)+b_n\sin(nc),\ b_n'=b_n\cos(nc)-a_n\sin(nc)$

3 $\pi-x=\sum\limits_{n=1}^{\infty}\frac{2}{n}\sin(nx),\ x\in(0,2\pi)$

$\therefore \frac{1}{2\pi}\int_0^{2\pi}f(x)(\pi-x)\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi}\sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)\,\mathrm{d}x$

$\because \sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)$ is controlled by $\sum\limits_{n=1}^{\infty}\frac{2}{n}b_n$ and Lec 13 Prob 04's conclusion only needs $f(x)$ to be integrable.

$\therefore \sum\limits_{n=1}^{\infty}\frac{2}{n}b_n\sin^2(nx)$ uniformly converges

$\therefore \frac{1}{2\pi}\int_0^{2\pi}f(x)(\pi-x)\,\mathrm{d}x=\frac{1}{2\pi}\sum\limits_{n=1}^{\infty}\int_0^{2\pi}\frac{2}{n}b_n\sin^2(nx)\,\mathrm{d}x=\sum\limits_{n=1}^{\infty}\frac{b_n}{n}$ □

4 (1) Apply periodic extension to $f(x)$ and set $f(2n\pi)=0$, which won't change Fourier series

$\because \sigma_n(x)=\frac{1}{2n\pi}\int_{-\pi}^{\pi}f(x+t)\left(\frac{\sin\left(\frac{nt}{2}\right)}{\sin\frac{t}{2}}\right)^2\mathrm{d}x,\ \frac{1}{2n\pi}\int_{-\pi}^{\pi}\left(\frac{\sin\left(\frac{nt}{2}\right)}{\sin\frac{t}{2}}\right)^2\mathrm{d}x=1,\ |f(x+t)|\leqslant\frac{\pi}{2}$

$\therefore |\sigma_n(x)|\leqslant\frac{\pi}{2}$ □

(2) Due to pointwise convergence, $\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}=\begin{cases}f(x),\ 0<x<2\pi\\ 0,\ x=0\end{cases}$

$\therefore \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right|<\frac{\pi}{2},\ 0\leqslant x<2\pi$

$\because \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right|$ is $2\pi$-periodic

$\therefore \left|\sum\limits_{n=1}^{\infty}\frac{\sin(nx)}{n}\right|<\frac{\pi}{2}+1,\ x\in\mathbb{R}$ □

## Lec 15

1 (1) $a_n=\int_{-\pi}^{\pi}f(x)\cos(nx)\,\mathrm{d}x=-\frac{1}{n}\int_{-\pi}^{\pi}f'(x)\sin(nx)\,\mathrm{d}x=-\frac{b_n'}{n}$

$b_n=\int_{-\pi}^{\pi}f(x)\sin(nx)\,\mathrm{d}x=\frac{1}{n}\int_{-\pi}^{\pi}f'(x)\cos(nx)\,\mathrm{d}x=\frac{a_n'}{n}$

(ignore discontinuous points of $f'$) $\square$

(2) Lec 13 Prob 04's conclusion can be extended as $\sum\limits_{n=1}^{\infty}\left|\frac{a_n'}{n}\right|$ and $\sum\limits_{n=1}^{\infty}\left|\frac{b_n'}{n}\right|$ converges, thus the convergence of $\sum\limits_{n=1}^{\infty}(|a_n|+|b_n|)$ is straightforward $\square$

(3) $|\frac{a_0}{2}|+\sum\limits_{n=1}^{\infty}(|a_n|+|b_n|)$ converges $\Rightarrow \frac{a_0}{2}+\sum\limits_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]$ absolutely uniformly converges

$\because \frac{a_0}{2}+\sum\limits_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]$ pointwise converges to $f(x)$

$\therefore \frac{a_0}{2}+\sum\limits_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]$ uniformly converges to $f(x)$ $\square$

2 (1) Due to symmetry, $b_n=0$

$a_0=\frac{2}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x=\frac{4}{\pi}$

$a_n=\frac{2}{\pi}\int_0^{\pi}\sin x\cos(nx)\,\mathrm{d}x=\begin{cases}0, & n=1\\ \frac{2}{\pi}\frac{(-1)^{n+1}-1}{n^2-1}, & n\neq 1\end{cases}$

$f(x)=\frac{2}{\pi}-\sum\limits_{n=1}^{\infty}\frac{4}{\pi(4n^2-1)}\cos 2nx$

$\because$ the Fourier series is controlled by $\sum\limits_{n=2}^{\infty}\frac{4}{\pi(n^2-1)}$

$\therefore$ uniformly converges

$\because f(x)$ continuous

$\therefore$ the Fourier series pointwise thus uniformly converges to $f(x)$

(2) Due to symmetry, $a_n=0$

$b_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\sin(nx)\,\mathrm{d}x=\frac{4[1-(-1)^n]}{n^3\pi}$

$f(x)=\sum\limits_{n=1}^{\infty}\frac{8}{\pi(2n-1)^3}\sin(2n-1)x$

$\because$ the Fourier series is controlled by $\sum\limits_{n=1}^{\infty}\frac{8}{n^3\pi}$

$\therefore$ uniformly converges

$\because f(x)$ continuous

$\therefore$ the Fourier series pointwise thus uniformly converges to $f(x)$

3 Let $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin(nx)$, $f'(x) = \sum\limits_{n=1}^{\infty} a'_n \cos(nx)$, $f''(x) = \sum\limits_{n=1}^{\infty} b''_n \sin(nx)$, $f'''(x) = \sum\limits_{n=1}^{\infty} a'''_n \cos(nx)$

$a'_n = \frac{2}{\pi} \int_0^{\pi} f'(x) \cos(nx) \,\mathrm{d}x = \frac{2n}{\pi} \int_0^{\pi} f(x) \sin(nx) \,\mathrm{d}x = nb_n$

$b''_n = \frac{2}{\pi} \int_0^{\pi} f''(x) \sin(nx) \,\mathrm{d}x = -\frac{2}{\pi} \int_0^{\pi} f'(x) \cos(nx) \,\mathrm{d}x = -na'_n = -n^2 b_n$

$a'''_n = \frac{2}{\pi} \int_0^{\pi} f'''(x) \cos(nx) \,\mathrm{d}x = \frac{2n}{\pi} \int_0^{\pi} f''(x) \sin(nx) \,\mathrm{d}x = nb''_n$

apply Lec 13 Prob 04's conclusion $\sum\limits_{n=1}^{\infty} \left| \frac{b''_n n}{n} \right|$ convergent

$\therefore \sum\limits_{n=1}^{\infty} |b_n|$, $\sum\limits_{n=1}^{\infty} |a'_n|$, $\sum\limits_{n=1}^{\infty} |b''_n|$ convergent

$\therefore$ the Fourier series of $f(x)$, $f'(x)$, $f''(x)$ are uniformly convergent relatively to the original function and their coefficients are results of term-by-term differentiation of $f(x)$

$\therefore$ the Fourier series of $f(x)$ is $2^{\text{nd}}$ order termwise differentiable

$\because a'_n = nb_n$ and $\sum\limits_{n=1}^{\infty} a'^2_n$ converges

$\therefore \sum\limits_{n=1}^{\infty} n^2 b_n^2$ converges $\square$

# 8

**Lec 16**

1 (1) $\frac{\mathrm{d}}{\mathrm{d}x} F(x) = \mathrm{e}^{x\sqrt{1-\cos^2 x}}(-\sin x) - \mathrm{e}^{x\sqrt{1-\sin^2 x}}(\cos x) + \int_{\sin x}^{\cos x} \sqrt{1-y^2}\mathrm{e}^{x\sqrt{1-y^2}} \,\mathrm{d}y$

$$= \int_{\sin x}^{\cos x} \sqrt{1-y^2}\mathrm{e}^{x\sqrt{1-y^2}} \,\mathrm{d}y - \mathrm{e}^{x|\sin x|} \sin x - \mathrm{e}^{x|\cos x|} \cos x$$

(2) $\frac{\mathrm{d}}{\mathrm{d}x} F(x) = \int_{x^2}^{x^2} f(x,s) \,\mathrm{d}s + \int_0^x 2xf(t,x^2) \,\mathrm{d}t = \int_0^x 2xf(t,x^2) \,\mathrm{d}t$

2 $F(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{x\cos\theta}(\mathrm{e}^{ix\sin\theta} + \mathrm{e}^{-ix\sin\theta})}{2} \,\mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{x\mathrm{e}^{i\theta}} + \mathrm{e}^{x\mathrm{e}^{-i\theta}}}{2} \,\mathrm{d}\theta$

$F'(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{i\theta} \frac{\mathrm{e}^{x\mathrm{e}^{i\theta}}}{2} \,\mathrm{d}\theta + \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-i\theta} \frac{\mathrm{e}^{x\mathrm{e}^{-i\theta}}}{2} \,\mathrm{d}\theta = 0$

$\therefore F(x) = \text{const}$ $F(x) = F(0) = 1$

3 $I = -\int_0^1 \sin(\ln x) \int_a^b x^y \,\mathrm{d}y \,\mathrm{d}x = -\int_a^b \int_0^1 \sin(\ln x) x^y \,\mathrm{d}x \,\mathrm{d}y$

$= \int_a^b \frac{1}{(y+1)^2+1} \,\mathrm{d}y = \arctan(b+1) - \arctan(a+1)$

4 $F(x) = \frac{1}{h^2}\int_0^h\left[\int_0^h f(x+\xi+\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi = \frac{1}{h^2}\int_x^{x+h}\left[\int_\xi^{\xi+h} f(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi$

$F'(x) = \frac{1}{h^2}\left[\int_{x+h}^{x+2h} f(\eta)\,\mathrm{d}\eta - \int_x^{x+h} f(\eta)\,\mathrm{d}\eta\right]$

$F''(x) = \frac{1}{h^2}\{f(x+2h) - f(x+h) - [f(x+h) - f(x)]\}$

$= \frac{f(x+2h)-2f(x+h)+f(h)}{h^2}$ □

## Lec 17

1 (1) $\because \left|\frac{\cos xy}{x^2+y^2}\right| \le \frac{1}{a^2+y^2},\ \int_0^\infty \frac{1}{a^2+y^2}\,\mathrm{d}y$ converges

$\therefore$ uniformly converges

(2) $\because \int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$ uniformly converges, $\mathrm{e}^{-\alpha x}$ is about $x$ monotonically decreasing and $\left|\mathrm{e}^{-\alpha x}\right| \le 1$

$\therefore$ uniformly converges

(3) $\because f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}},\ x > 0 \\ 0,\ x = 0 \end{cases}$

$\therefore$ not uniformly converges

(4) $\int_0^1 \frac{1}{x^y}\sin\frac{1}{x}\,\mathrm{d}x = \int_1^\infty x^{y-2}\sin x\,\mathrm{d}x$

apply second mean value theorem for definite integrals, $\forall N \in \mathbb{N}^*,\ \exists y = 2 - \frac{1}{(2N+1)\pi}$

$\int_{2N\pi}^{(2N+1)\pi} x^{y-2}\sin x\,\mathrm{d}x \le 2[(2N+1)\pi]^{y-2} = 2[(2N+1)\pi]^{-\frac{1}{(2N+1)\pi}} > 1$

$\therefore$ not uniformly converges

(5) $\forall\epsilon > 0,\ \exists\delta < \frac{1}{4}\epsilon^2$ s.t. $\int_{y-\delta}^{y}\frac{\sin xy}{\sqrt{y-x}}\,\mathrm{d}x \le \int_{y-\delta}^{y}\frac{1}{\sqrt{y-x}}\,\mathrm{d}x = 2\sqrt{\delta} \le \epsilon$

$\therefore$ uniformly converges

(6) $\int_0^1 x^{p-1}\ln^2 x\,\mathrm{d}x = \int_{-\infty}^0 \mathrm{e}^{px}x^2\,\mathrm{d}x$

$\because \forall N > 0,\ \exists p = \frac{1}{\sqrt[3]{N+1}}$ s.t. $\int_{-\sqrt[3]{N+1}}^{-\sqrt[3]{N}} \mathrm{e}^{px}x^2\,\mathrm{d}x > \frac{\mathrm{e}^{-\sqrt[3]{N+1}p}}{3} = \frac{\mathrm{e}^{-1}}{3}$

$\therefore$ not uniformly converges

2 $\because |F(u)| \le \int_{-\infty}^{+\infty}|f(x)|\,\mathrm{d}x = A$

$\therefore F(u)$ is bounded

$\forall\epsilon > 0,\ \exists K(\epsilon) > \frac{1}{\epsilon}$, s.t. $\int_{-\infty}^{-K}|f(x)|\,\mathrm{d}x + \int_K^{+\infty}|f(x)|\,\mathrm{d}x < \frac{\epsilon}{3}$

$\exists\delta = \frac{1}{3K^2A}$ when $|u_1 - u_2| < \delta$

$|F(u_2) - F(u_1)| = \left|\int_{-\infty}^{+\infty} f(x)(\cos u_2 x - \cos u_1 x)\,\mathrm{d}x\right|$

$\leq \frac{2}{3}\epsilon + 2|\int_{-K}^{+K} f(x)\sin\frac{u_2+u_1}{2}x\sin\frac{u_2-u_1}{2}x\,\mathrm{d}x|$

$\leq \frac{2}{3}\epsilon + \frac{1}{3KA}\int_{-K}^{+K}|f(x)\sin\frac{u_2+u_1}{2}x|\,\mathrm{d}x$

$\leq \epsilon$

$\therefore$ uniformly continuous $\square$

3 $\because \int_0^{+\infty} f(t)\,\mathrm{d}t$ uniformly converges, $\mathrm{e}^{-xt}$ is about $x$ monotonically decreasing and $\left|\mathrm{e}^{-xt}\right| \leq 1$ $(x \geq 0)$

$\therefore \int_0^{+\infty} \mathrm{e}^{-xt} f(t)\,\mathrm{d}t$ uniformly converges

$\therefore \lim\limits_{x\to 0}\int_0^{+\infty} \mathrm{e}^{-xt} f(t)\,\mathrm{d}t = \int_0^{+\infty} f(t)\,\mathrm{d}t$ $\square$