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Answer Key 3

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Lec 04

5 Let b_k equal to the sum of k^{th} set of successive a_n which have the same sign

If n_0 is in the k_0^{th} set, denote $k(n_0) = k_0$

$\because \sum\limits_{k=1}^{\infty} b_k$ convergent, $\lim\limits_{k\to\infty} b_k = 0$

$\therefore \forall \varepsilon > 0, \exists K > 0, \forall k_1, k_2 > K, |\sum\limits_{k=k_1}^{k_2} b_k| < \varepsilon, |b_{k_1}| + |b_{k_2}| < \varepsilon$

$\because \exists N, k(N) > K, \forall n_1, n_2 > N, |\sum\limits_{n=n_1}^{n_2} a_n| \leqslant \varepsilon + |b_{k(n_1)}| + |b_{k(n_2)}| < 2\varepsilon$

$\therefore$ convergent

For $a_n = \frac{(-1)^{[\sqrt{n}]}}{n}$, let $b_k = (-1)^k \sum\limits_{n=(k-1)^2+1}^{k^2} \frac{1}{n}$

$\therefore |b_k| < \frac{2k}{(k-1)^2}$

$\therefore |b_k|$ monotonically decreases to 0 and $\text{sgn}(\frac{b_k}{b_{k-1}}) = -1$

$\therefore b_k$ converges But $|a_n| = \frac{1}{n}$

$\therefore$ conditionally convergent

8 $\forall \varepsilon > 0, \exists N > 0, \forall n_1, n_2 > N,$

$\max\{|\sum_{n=n_1}^{n_2} n(a_n - a_{n-1})|, |n_1 a_{n_1-1}|, |n_2 a_{n_2}|\} < \varepsilon$

$\therefore |\sum_{n=n_1}^{n_2-1} a_n| = |\sum_{n=n_1}^{n_2} n(a_n - a_{n-1}) + n_1 a_{n_1-1} - n_2 a_{n_2}| < 3\varepsilon$

$\therefore$ convergent $\square$

Lec 05

3 (1) $x_{n+1} - x_n = \frac{\ln(n+1)}{n+1} - \frac{1}{2}\left[\ln^2(n+1) - \ln^2 n\right] = \frac{\ln(n+1)}{n+1} - \int_n^{n+1} \frac{\ln x}{x}\,\mathrm{d}x$

$\frac{\ln x}{x}$ is monotonically decreasing over $[\mathrm{e}, +\infty)$

if $n > 3$, $\frac{\ln(n+1)}{n+1} < \int_n^{n+1} \frac{\ln x}{x}\,\mathrm{d}x < \frac{\ln n}{n}$

$\therefore x_{n+1} < x_n, x_n$ monotonically decreasing

$$x_n = \sum_{k=1}^{n} \frac{\ln k}{k} - \frac{1}{2}(\ln n)^2 = \sum_{k=1}^{n} \frac{\ln k}{k} - \int_1^n \frac{\ln x}{x}\,\mathrm{d}x$$

$$= \sum_{k=1}^{2} \frac{\ln k}{k} - \int_1^3 \frac{\ln x}{x}\,\mathrm{d}x + \sum_{k=3}^{n} \frac{\ln k}{k} - \int_3^n \frac{\ln x}{x}\,\mathrm{d}x > \frac{\ln 2}{2} - \ln^2 3$$

$\therefore$ convergence

(2) $x_{n+1} - x_n = \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} - 2\sqrt{n} < 0$

$\therefore x_n$ monotonically decreasing

$$\sqrt{n} = \sqrt{n} - \sqrt{n-1} + \sqrt{n-1} - \sqrt{n-2} + \cdots + \sqrt{2} - \sqrt{1} + 1$$

$$= \frac{1}{\sqrt{n}+\sqrt{n-1}} + \frac{1}{\sqrt{n-1}+\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{2}+\sqrt{1}} + 1$$

$$< \frac{1}{2}\left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{1}}\right) + 1$$

$$x_n > 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - \left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{1}} + 2\right) = -2 + \frac{1}{\sqrt{n}} > -2$$

$\therefore$ convergence

4 $\sum_{n=0}^{\infty} |x|^n = \frac{1}{1-|x|}, \sum_{n=0}^{\infty} |y|^n = \frac{1}{1-|y|}$ (both absolutely convergent)

$\therefore \sum_{n=1}^{\infty} (x^{n-1} + x^{n-2}y + \cdots + y^{n-1}) = \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} y^n = \frac{1}{(1-x)(1-y)}$ $\square$

5 $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$

$\therefore$ radius of convergence is ∞

$\therefore \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x, \sum\limits_{n=0}^{\infty} \frac{y^n}{n!} = \mathrm{e}^y$ both absolutely convergent

$\therefore \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} \sum\limits_{n=0}^{\infty} \frac{y^n}{n!} = \mathrm{e}^{x+y} = \sum\limits_{n=0}^{\infty} \frac{(x+y)^n}{n!}$ $\square$

Lec 06

1 (1) Let $p_n = q_n = 1, \prod\limits_{n=1}^{\infty} (p_n + q_n) = \prod\limits_{n=1}^{\infty} 2 = \infty$

$\therefore$ divergent

(2) $\prod\limits_{n=1}^{\infty} p_n, \prod\limits_{n=1}^{\infty} q_n$ converge

$\Rightarrow \sum\limits_{n=1}^{\infty} \ln p_n, \sum\limits_{n=1}^{\infty} \ln q_n$ converge

$\Rightarrow \sum\limits_{n=1}^{\infty} (\ln p_n + \ln q_n)$ converges

$\Rightarrow \sum\limits_{n=1}^{\infty} \ln(p_n q_n)$ converges

$\Rightarrow \prod\limits_{n=1}^{\infty} p_n q_n$ converges

(3) Let $q_n = p_n$ and use conclusion of previous problem

$\therefore$ convergent

(4) $\prod\limits_{n=1}^{\infty} q_n$ converges

$\Rightarrow \sum\limits_{n=1}^{\infty} \ln q_n$ converges

$\Rightarrow -\sum\limits_{n=1}^{\infty} \ln q_n$ converges

$\Rightarrow \prod\limits_{n=1}^{\infty} \frac{1}{q_n}$ converges

Use conclusion of Lec 06 Prob 1(2), $\prod\limits_{n=1}^{\infty} \frac{p_n}{q_n}$ converges

2 Denote $T_n = \prod\limits_{k=1}^{n} (1 + u_k), S_n = \sum\limits_{k=1}^{n} u_k, S'_n = \sum\limits_{k=1}^{n} (u_k)^2$

$\therefore S_{2n} = \sum\limits_{k=1}^{n} \frac{1}{k} \to \infty, S'_{2n} > 2\sum\limits_{k=1}^{n} \frac{1}{k} \to \infty$

$\because \lim\limits_{n\to\infty} u_{2n-1} = \lim\limits_{n\to\infty} u_{2n} = 0$

$\therefore S_n, S'_n$ diverges

$\because (1+u_{2k-1})(1+u_{2k}) = 1 - \frac{1}{k^{\frac{3}{2}}}$

$\therefore T_{2n}$ converges, let A denote its limit

$\therefore \forall \varepsilon > 0, \exists N_1 > 0, \forall n > N_1, |T_{2n} - A| < \varepsilon$

And $\lim\limits_{n\to\infty} \frac{T_{2n+1}}{T_{2n}} = u_{2n+1} + 1 = 1$

$\therefore$ for $\varepsilon > 0, \exists N_2 > 0, \forall n > N_2, |T_{2n+1} - T_{2n}| < \varepsilon$

$\therefore \forall \varepsilon > 0, \exists N = \max\{2N_1 + 10, 2N_2 + 10\} > 0, \forall n > N, |T_n - A| < 2\varepsilon$

$\therefore T_n$ converges $\square$

3 (1) $\lim\limits_{n\to\infty} \frac{\ln[(\frac{n^2-1}{n^2+1})^p]}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{p\ln(1-\frac{2}{n^2+1})}{\frac{1}{n^2}} = -2p$

$\therefore$ convergent

(2) $\lim\limits_{n\to\infty} \frac{\ln\sqrt[n]{1+\frac{1}{n}}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}\ln(1+\frac{1}{n})}{\frac{1}{n^2}} = 1$

$\therefore$ convergent

(3) $\lim\limits_{n\to\infty} \frac{\ln\sqrt[n]{\ln(n+x) - \ln n}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}\ln\ln(1+\frac{x}{n})}{\frac{1}{n}} = -\infty$

$\therefore$ divergent

(4) $\lim\limits_{n\to\infty} \frac{\ln\frac{n^2-4}{n^2-1}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{\ln\left(1-\frac{3}{n^2-1}\right)}{\frac{1}{n^2}} = -3$

$\therefore$ convergent

(5) $\ln a^{\frac{(-1)^n}{n}} = \frac{(-1)^n}{n}\ln a$

$\because \frac{1}{n}$ monotonically decreases to 0

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}\ln a$ converges

$\therefore$ convergent

(6) $\because \prod\limits_{k=1}^{n} \sqrt{\frac{k+1}{k+2}} = \sqrt{\frac{2}{n+2}} \to 0$

$\therefore$ divergent

5 Due to convergence $\lim_{n\to\infty} a_n = 0$

$\tan(\frac{\pi}{4} + x) = 1 + Ax + o(x), A = \tan'(\frac{\pi}{4}) > 0$

$\therefore \lim_{n\to\infty} \frac{|\ln[\tan(\frac{\pi}{4}+a_n)]|}{|a_n|} = A$

$\therefore \sum_{n=1}^{\infty} \ln[\tan(\frac{\pi}{4} + a_n)]$ converges

$\therefore$ convergent $\square$