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Answer Key 5

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Lec 09:

1 (1) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n!}{2^n}} = +\infty$

$\therefore R = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(2) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{[3+(-1)^n]^n}} = \frac{1}{4}$

$\therefore R = \frac{1}{4}$

$\because \sum\limits_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1}$ converge and $|x| = \frac{1}{4} : \sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverge

$\therefore$ convergence region: $(-\frac{1}{4}, \frac{1}{4})$

(3) $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n+1}{\ln(n+1)}} = 1$

$\therefore R = 1$

$\sum\limits_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

$\frac{\ln(n+1)}{n+1}$ monotonically decreases to 0 when $n > 3$

$\therefore \sum\limits_{n=1}^{\infty} \frac{(-1)^n \ln(n+1)}{n+1}$ converges

$\therefore$ divergent when $x = 1$, convergent when $x = -1$

$\therefore$ convergence region: $[-1,1)$

(4) $|x|=1$: convergent

$|x|>1$: $\lim\limits_{n\to\infty}\frac{|x|^{n^2}}{2^n}=+\infty\neq 0$

$\therefore$ convergence region: $[-1,1]$

(5) $\lim\limits_{n\to\infty}\sqrt[n]{\frac{n}{3^n+(-2)^n}}=\frac{1}{3}\lim\limits_{n\to\infty}\sqrt[n]{\frac{n}{1+(-\frac{2}{3})^n}}=\frac{1}{3}$

$\therefore R=\frac{1}{3}$

$x+1=\frac{1}{3}$: $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum\limits_{n=1}^{\infty}\frac{(-\frac{2}{3})^n}{n}$ converges $\Rightarrow$ diverges

$x+1=-\frac{1}{3}$: $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges, $\sum\limits_{n=1}^{\infty}\frac{(\frac{2}{3})^n}{n}$ converges $\Rightarrow$ converges

$\therefore$ convergence region: $[-\frac{4}{3},-\frac{2}{3})$

(6) $\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{(1+\frac{1}{n})^{n^2}}}=\frac{1}{\mathrm{e}}$

$\therefore R=\frac{1}{\mathrm{e}}$

$\lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})^{n^2}}{\mathrm{e}^n}=\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x^2}}}{\mathrm{e}^{\frac{1}{x}}}=\mathrm{e}^{\lim\limits_{x\to 0}\ln\frac{(1+x)^{\frac{1}{x^2}}}{\mathrm{e}^{\frac{1}{x}}}}=\mathrm{e}^{\lim\limits_{x\to 0}\frac{\ln(1+x)-x}{x^2}}=\mathrm{e}^{-\frac{1}{2}}\neq 0$

$\therefore$ convergence region: $(-\frac{1}{\mathrm{e}},\frac{1}{\mathrm{e}})$

(7) Just let $a\geqslant b$

$\lim\limits_{n\to\infty}\sqrt[n]{a^n+b^n}=a\lim\limits_{n\to\infty}\sqrt[n]{1+(\frac{b}{a})^n}=a$

$\therefore R=a$

$\lim\limits_{n\to\infty}\frac{a^n}{a^n+b^n}\geqslant\frac{1}{2}$

$\therefore$ convergence region: $(-a,a)$

(8) $\lim\limits_{n\to\infty}\sqrt[n]{n2^n}=2$

$\therefore R=2$

$|x|=\sqrt{2}$: $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges

$\therefore$ convergence region: $[-\sqrt{2},\sqrt{2}]$

(9) $1 \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)(2n)!!}{(2n-1)!!}} \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)(2n)!!(2n-1)!!}{(2n-1)!!(2n-1)!!}} \leqslant \lim\limits_{n\to\infty} \sqrt[2n-1]{\frac{(2n+1)!}{(2n-1)!}}$
$= \lim\limits_{n\to\infty} \sqrt[2n-1]{2n(2n+1)} = 1$

$\therefore R = 1$

$|x| = 1$: $\frac{(2n-1)!!}{(2n+1)(2n)!!}$ monotonically decreases to 0

$\therefore$ convergent

$\therefore$ convergence region: $[-1, 1]$

2 (1) $|x| < \sqrt{A} \Rightarrow x^2 < A \Rightarrow$ convergent

$|x| > \sqrt{A} \Rightarrow x^2 > A \Rightarrow$ divergent

$\therefore R = \sqrt{A}$

(2) $\frac{1}{R} = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n + b_n|} \leqslant \overline{\lim\limits_{n\to\infty}} \sqrt[n]{2\max(|a_n|, |b_n|)} = \max(\frac{1}{A}, \frac{1}{B})$

$\therefore R \geqslant \min(A, B)$

(3) $\frac{1}{R} = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n b_n|} \leqslant \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}\ \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|} = \frac{1}{AB}$

$\therefore R \geqslant AB$

3 Let $A_m(x) = \sum\limits_{n=1}^{m} a_n x^n,\ B_m(x) = \sum\limits_{n=1}^{m} b_n x^n,\ S_m(x) = \sum\limits_{n=1}^{m} a_n b_{m-n+1} x^{m+1},$

$R_m(x) = A_m(x)B_m(x) - \sum\limits_{n=1}^{m} S_n(x),\ M = \sup\limits_n[\max(|a_n|, |b_n|)]$

$0 < x < 1$: $|R_m(x)| < M^2 \frac{m^2-m}{2} x^{m+2}$

$\therefore \lim\limits_{m\to\infty} R_m(x) = 0,\ 0 < x < 1$

$\because [0, 1]$ is in the uniformly convergence region of $A_m(x), B_m(x), \sum\limits_{n=1}^{m} S_n(x)$

$\therefore \lim\limits_{m\to\infty} R_m(x)$ is also continuous in $[0, 1]$

$\therefore \lim\limits_{m\to\infty} R_m(1) = 0$

$\therefore \sum\limits_{n=1}^{\infty} S_n(x) = \lim\limits_{m\to\infty} A_m B_m = AB$ $\square$

4 Apparently $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent in $[0, r)$

$\therefore \int_0^x \lim\limits_{m\to\infty} \sum\limits_{n=0}^{m} a_n t^n \,\mathrm{d}t = \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$

$\because \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is uniformly convergent in $[0, r]$ and $\frac{a_n x^{n+1}}{n+1}$ is continuous

$\therefore \sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ is continuous in $[0, r]$

$\therefore \lim\limits_{x\to r^-} \int_0^x \lim\limits_{m\to\infty} \sum\limits_{n=0}^{m} a_n t^n \,\mathrm{d}t$ exists and is equal to $\sum\limits_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ $\square$

$f(x) = -\frac{\ln(1-x)}{x} = \sum\limits_{n=0}^{\infty} \frac{x^n}{n+1}$

$\because r = \varliminf\limits_{n\to\infty} \sqrt[n]{n+1} = 1$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ convergent

$\therefore \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^1 \frac{\ln(1-x)}{x} \,\mathrm{d}x$ $\square$

Lec 10:

1 Convergence in $(-\infty, \infty)$ is apparent.

$f^{(2n+1)}(0) = \sum\limits_{m=1}^{\infty} \frac{(-1)^n (2^m)^{2n+1}}{m!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \sum\limits_{m=1}^{\infty} \frac{(-1)^n (2^m x)^{2n+1}}{m!(2n+1)!} \sim \sum\limits_{n=0}^{\infty} (\mathrm{e}^{2^{2n+1}} - 1) \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\because \varliminf\limits_{n\to\infty} \sqrt[2n+1]{\frac{(2n+1)!}{\mathrm{e}^{2^{2n+1}}}} = 0$

$\therefore R = 0$

$\therefore$ divergent when $x \neq 0$

2 $f^{(n)}(0) = \frac{n!}{a^{n+1}}$

$f(x) \sim \sum\limits_{n=0}^{\infty} \frac{x^n}{a^{n+1}}$

$\varliminf\limits_{n\to\infty} \sqrt[n]{|a|^{n+1}} = |a|$

$|\frac{a^n}{a^{n+1}}| = \frac{1}{|a|} \nrightarrow 0$

$\therefore$ convergence region: $(-|a|, |a|)$

3 $f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$

$f(x) \sim \ln 2 + \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{2^n n}$

$\varlimsup\limits_{n\to\infty} \sqrt[n]{2^n n} = 2$

$\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore$ convergence region: $(0, 4]$

4 (1) $f^{(n)}(0) = (\frac{\sin x}{x})^{(n-1)}\Big|_{x=0}$

$\because \sin x \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\therefore \frac{\sin x}{x} \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

$f^{(2n+1)}(0) = (\frac{\sin x}{x})^{(2n)}\Big|_{x=0} = \frac{(-1)^n (2n)!}{(2n+1)!} = \frac{(-1)^n}{2n+1}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}$

$R = \varlimsup\limits_{n\to\infty} \sqrt[2n+1]{(2n+1)!(2n+1)} = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(2) $\cos t^2 \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}$

$\therefore f^{(4n+1)}(0) = (\cos t^2)^{(4n)}\Big|_{t=0} = \frac{(-1)^n (4n)!}{(2n)!}$

$\therefore f(x) \sim \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}$

$R = \varlimsup\limits_{n\to\infty} \sqrt[4n+1]{(2n)!(4n+1)} = +\infty$

$\therefore$ convergence region: $(-\infty, +\infty)$

(3) Let $x = \tan t$

$\arctan \frac{2x}{1-x^2} = -\arctan(\tan 2t) = -2\arctan x \sim -2\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\varlimsup\limits_{n\to\infty} \sqrt[2n+1]{2n+1} = 1$

$|x| = 1$: $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

$\therefore$ convergence region: $[-1,1]$ (with definition: $\arctan(\pm\infty)=\pm\pi$)

(4) $f^{(1)}(0)=\frac{1}{\sqrt{1+x^2}}\sim\sum\limits_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n)!!}x^{2n}$

$\therefore f^{(2n+1)}(0)=(\frac{1}{\sqrt{1+x^2}})^{(2n)}\Big|_{x=0}=\frac{(-1)^n(2n-1)!!(2n)!}{(2n)!!}$

$\therefore f(x)\sim\sum\limits_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{(2n)!!(2n+1)}x^{2n+1}$

$\varliminf\limits_{n\to\infty}\sqrt[2n+1]{\frac{(2n)!!(2n+1)}{(2n-1)!!}}=1$ (see Lec 09 Prob 1(9))

$\frac{(2n-1)!!}{(2n)!!(2n+1)}$ monotonically decreases to 0

$\therefore$ converges when $|x|=1$

$\therefore$ convergence region: $[-1,1]$

5 (1) $\mathrm{e}^x=\sum\limits_{n=0}^{\infty}\frac{x^n}{n!},x\in\mathbb{R}$

$\therefore\sum\limits_{n=1}^{\infty}\frac{x^n}{n!}=\mathrm{e}^x-1,x\in\mathbb{R}$

(2) $\int_0^x\ln(1+x)\,\mathrm{d}x=(1+x)\ln(1+x)-x$

$\because$ convergence radius of $\ln(1+x)$'s Maclaurin series is 1

$\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n(n+1)}$ converges at $|x|=1$

According to Lec 09 Prob 4, $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n(n+1)}=(1+x)\ln(1+x)-x$ in $[-1,1]$
(define $(1+x)\ln(1+x)-x=1$ at $x=-1$)

(3) $\int_0^x f(x)\,\mathrm{d}x\sim\sum\limits_{n=1}^{\infty}nx^n\sim\frac{x}{(x-1)^2}$

$\because$ Convergence radius is 1 and $f(x)$ diverges at $|x|=1$

$\therefore f(x)=\left[\frac{x}{(x-1)^2}\right]'=\frac{1+x}{(1-x)^3}$ in $(-1,1)$

(4) Convergence region is $\mathbb{R}$

$\therefore\int_0^x f(x)\,\mathrm{d}x=\sum\limits_{n=1}^{\infty}\frac{x^{2n+1}}{n!}=x(\mathrm{e}^{x^2}-1)$

$\therefore f(x)=[x(\mathrm{e}^{x^2}-1)]'=(2x^2+1)\mathrm{e}^{x^2}-1$

(5) Let $A=\sum\limits_{n=1}^{\infty}\frac{2n-1}{2^n}$

$$\therefore A = 2A - A = 1 + \sum_{n=1}^{\infty} \tfrac{2n+1}{2^n} - \sum_{n=1}^{\infty} \tfrac{2n-1}{2^n} = 3$$

(6) $\sum\limits_{n=0}^{\infty} \frac{n^2+1}{2^n n!} = \mathrm{e}^{\frac{1}{2}} + \sum\limits_{n=1}^{\infty} \frac{n}{2^n (n-1)!} = \mathrm{e}^{\frac{1}{2}} + \frac{1}{2} \sum\limits_{n=1}^{\infty} \frac{(\frac{1}{2})^{n-1}}{(n-1)!} + \frac{1}{4} \sum\limits_{n=2}^{\infty} \frac{(\frac{1}{2})^{n-2}}{(n-2)!} = \frac{7}{4} \mathrm{e}^{\frac{1}{2}}$