



PEKING UNIVERSITY

COLLEGE OF ENGINEERING

MATHEMATICAL ANALYSIS (3)

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## Answer Key

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☑ *Circle*

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# 1

## Lec 01

- 1 Use reduction to absurdity. Suppose  $\lim_{n \rightarrow \infty} a_n \neq 0$  or doesn't exist. So  $\exists \varepsilon > 0, \forall N_1 > 0, \exists n > N_1, |a_n| > 3\varepsilon$ . For such  $\varepsilon, \exists N_2 > 0, \forall n > N_2, |a_{2n} + 2a_n| < \varepsilon$ . So  $\exists N > N_2, |a_N| > 3\varepsilon$  and  $|a_{2N} + 2a_N| < \varepsilon$ . So  $|a_{2N}| > 5\varepsilon$ . Similarly,  $|a_{4N}| > 9\varepsilon$ , and then  $|a_{2^p N}| > (2^{p+1} + 1)\varepsilon$  for  $p \in \mathbb{N}$ , which contradict the boundedness of  $a_n$ .  $\square$
- 2 Let  $a_n = \frac{2}{3} + b_n$ . So  $\lim_{n \rightarrow \infty} (b_{2n} + 2b_n) = 0$ . According to the conclusion of previous problem,  $\lim_{n \rightarrow \infty} b_n = 0$ , and  $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$  is evident.  $\square$
- 3 (1)  $\because x_{n+2} = 1 + \frac{1}{x_{n+1}} = 1 + \frac{1}{1 + \frac{1}{x_n}} = 1 + \frac{x_n}{1+x_n}$  and using mathematical induction:  $x_n > 0$   
 $\therefore$  for  $n \geq 3, 1 < x_n < 2$   
 $\therefore 1 \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq 2$   $\square$
- (2)  $\because$  for  $n \geq 3, \left| \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \right| = \left| -\frac{1}{x_{n+1}} \right| < \frac{1}{2}$   
 $\therefore x_n$  is a Cauchy sequence  
 Calculate the positive fixed point of equation  $x^* = 1 + \frac{1}{x^*}$   
 $\therefore \lim_{n \rightarrow \infty} x_n = x^* = \frac{1+\sqrt{5}}{2}$
- 5 (1)  $\because \lim_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n$   
 $= \liminf_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \leq \overline{\lim}_{n \rightarrow \infty} (x_n + y_n) \leq \overline{\lim}_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n$   
 $= \lim_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n$   $\square$
- (2) The proof for  $\overline{\lim}_{n \rightarrow \infty} y_n = \pm\infty$  is direct. Suppose  $\overline{\lim}_{n \rightarrow \infty} y_n = A$ .  
 $\forall \varepsilon_1 > 0, \exists N_1(\varepsilon_1) > 0, \forall n > N_1(\varepsilon_1), y_n < A + \varepsilon_1$ , and  $\exists \left\{ y_{n_k^{\varepsilon_1}} \right\}, y_{n_k^{\varepsilon_1}} > A - \varepsilon_1$   
 $\forall \varepsilon_2 > 0, \exists N_2(\varepsilon_2) > 0, \forall n > N_2(\varepsilon_2), |x_n - x^*| < \varepsilon_2, x^* = \lim_{n \rightarrow \infty} x_n$   
 $\forall \varepsilon > 0, \exists \varepsilon_1 > 0, \varepsilon_2 > 0, \forall \delta_1 \in (0, \varepsilon_1), \delta_2 \in (-\varepsilon_2, \varepsilon_2), \text{ s.t. } 0 \leq \frac{\delta_1}{x^* + \delta_2} - \frac{A\delta_2}{x^*(x^* + \delta_2)} < \varepsilon$ ,  
 and  $0 \leq \frac{\delta_1}{x^*} + \frac{(A - \delta_1)\delta_2}{x^*(x^* + \delta_2)} < \varepsilon$   
 $\therefore$  for  $\forall \varepsilon > 0, \exists N = \max(N_1(\varepsilon_1), N_2(\varepsilon_2)), \forall n > N, (x_n y_n) > x^* A + \varepsilon$   
 and  $\exists \left\{ y_{n_k^{\varepsilon_1}} x_{n_k^{\varepsilon_1}} \right\}, y_{n_k^{\varepsilon_1}} x_{n_k^{\varepsilon_1}} > x^* A - \varepsilon$   $\square$
- 6 (1)  $\sup_{k \geq n} a_k = 1, \inf_{k \geq 2n} a_k = \inf_{k \geq 2n+1} a_k = a_{2n+1} = \frac{1}{2^{-2n-1}-1}$   
 $\therefore \overline{\lim}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup_{k \geq n} a_k = 1, \liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf_{k \geq n} a_k = -1$
- (2)  $\sup_{k \geq 2n} a_k = \sup_{k \geq 2n-1} a_k = a_{2n} = 1 + \frac{1}{2n}$   
 $\inf_{k \geq 2n} a_k = \inf_{k \geq 2n+1} a_k = a_{2n+1} = -1 - \frac{1}{2n+1}$

$$\therefore \overline{\lim}_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_k = 1, \quad \underline{\lim}_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_k = -1$$

$$(3) \quad |a_n| = \frac{1}{n} \rightarrow 0$$

$$\therefore \overline{\lim}_{n \rightarrow \infty} a_n = \underline{\lim}_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n = 0$$

$$(4) \quad \text{For a period of } n = 0 \sim 9 \bmod 10, \text{ maximum } a_n \text{ is } \sin \frac{2\pi}{5}, \text{ minimum } a_n \text{ is } -\sin \frac{2\pi}{5}$$

$$\therefore \overline{\lim}_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_k = \sin \frac{2\pi}{5}, \quad \underline{\lim}_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_k = -\sin \frac{2\pi}{5}$$

## Lec 02

1 Suppose  $\overline{\lim}_{n \rightarrow \infty} na_n > 0$ . Then just suppose  $\overline{\lim}_{n \rightarrow \infty} na_n \geq 1$

$$\therefore \exists \{a_{n_k}\}, a_{n_k} \geq \frac{1}{n_k}$$

$$\text{Then } \exists \{a_{n_{k_l}}\}, n_{k_{l+1}} \geq 2n_{k_l}$$

$$\therefore \sum_{n=1}^{\infty} a_n \geq \sum_{l=2}^{\infty} \left( \frac{n_{k_l}}{2} \frac{1}{n_{k_l}} \right) = \infty$$

$$\therefore \overline{\lim}_{n \rightarrow \infty} na_n = 0$$

$$\therefore \underline{\lim}_{n \rightarrow \infty} na_n \geq 0$$

$$\therefore \lim_{n \rightarrow \infty} na_n = 0$$

$$3 (1) \quad \because 0 \leq \frac{1}{(5n-4)(5n+1)} \leq \frac{1}{n^2}$$

$$\therefore \text{absolutely convergent}$$

$$(2) \quad \because \lim_{n \rightarrow \infty} a_n = \frac{1}{2} \neq 0$$

$$\therefore \text{divergent}$$

$$(3) \quad \because 0 \leq \frac{1}{2^n} + \frac{1}{3^n} \leq \frac{1}{2^{n-1}}$$

$$\therefore \text{absolutely convergent}$$

$$(4) \quad \because 0 \leq \frac{1}{(3n-2)(3n+1)} \leq \frac{1}{n^2}$$

$$\therefore \text{absolutely convergent}$$

$$(5) \quad \because \lim_{n \rightarrow \infty} a_n = 1 \neq 0$$

$$\therefore \text{divergent}$$

$$4 (1) \quad \forall \varepsilon > 0, \exists N > \log_{|q|} \frac{\varepsilon(1-|q|)}{A}, \forall n_1, n_2 > N$$

$$\sum_{n=n_1}^{n_2} |a_n q^n| \leq A \sum_{n=n_1}^{n_2} |q|^n = A|q|^{n_1} \frac{1-|q|^{n_2-n_1+1}}{1-|q|} < \varepsilon$$

$$\therefore \text{absolutely convergent}$$

$$(2) \quad \because a_n = \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} > \frac{1}{3n}$$

$$\therefore \exists \varepsilon = 1 > 0, \forall N > 0, \exists n_1 > N, n_2 = 3n_1 + 10 > N$$

$$\sum_{n=n_1}^{n_2} a_n > \sum_{n=n_1}^{n_2} \frac{1}{3n} > \frac{2n_2}{3} \frac{1}{3n_2} > \frac{2}{9}$$

∴ divergent

5 Let  $b_k = \sum_{n=n_{k-1}+1}^{n_k} a_n$

$$\therefore \forall \varepsilon > 0, \exists K > 0, \forall k_1, k_2 > K, \left| \sum_{k=k_1}^{k_2} b_k \right| < \varepsilon$$

$$\therefore \exists N > n_{K+1} + 1, \forall n_1, n_2 > N, \left| \sum_{n=n_1}^{n_2} a_n \right| < \left| \sum_{k=K}^{\infty} b_k \right| \leq \varepsilon$$

∴ absolutely convergent

## 2

### Lec 03

1 (2)  $\ln(1 + \frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + o(\frac{1}{n^4})$

$$\therefore a_n = \frac{1}{12n^2} + o(\frac{1}{n^2})$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{\frac{1}{12n^2}} = 1$$

∴ absolutely convergent

(3)  $\ln[(1 + \frac{1}{n})^n] = n \ln(1 + \frac{1}{n}) = 1 - \frac{1}{2n} + o(\frac{1}{n})$

$$\therefore (1 + \frac{1}{n})^n = e^{1 - \frac{1}{2n} + o(\frac{1}{n})} = e - \frac{e}{2n} + o(\frac{1}{n})$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{(\frac{e}{2n})^p} = 1$$

$$\begin{cases} p > 1 : \text{absolutely convergent} \\ p \leq 1 : \text{divergent} \end{cases}$$

2 (1)  $\lim_{n \rightarrow \infty} \sqrt[n]{(1 - \frac{1}{n})^{n^2}} = \lim_{n \rightarrow \infty} (1 - \frac{1}{n})^n = \frac{1}{e} < 1$

∴ absolutely convergent

(2)  $\frac{a_{n+1}}{a_n} = \frac{x}{1+x^n}$

$$\begin{cases} x < 1 : \lim_{n \rightarrow \infty} \frac{x}{1+x^n} = x < 1, \text{absolutely convergent} \\ x = 1 : \lim_{n \rightarrow \infty} \frac{x}{1+x^n} = \frac{1}{2}, \text{absolutely convergent} \\ x > 1 : \lim_{n \rightarrow \infty} \frac{x}{1+x^n} = 0, \text{absolutely convergent} \end{cases}$$

3 (1)  $\int_2^{\infty} 3^{-x^{\frac{1}{2}}} dx = \int_{\sqrt{2}}^{\infty} 2t \cdot 3^{-t} dt$

∴ absolutely convergent

$$(2) \frac{1}{a^{\ln x}} = \frac{1}{x^{\ln a}}$$

$$\begin{cases} a > e : \ln a > 1, \text{absolutely convergent} \\ a \leq e : \ln a \leq 1, \text{divergent} \end{cases}$$

$$4 (1) \lim_{n \rightarrow \infty} a_n = \frac{1}{2}$$

$\therefore$  divergent

$$(4) \lim_{n \rightarrow \infty} \frac{\frac{1}{n^{\frac{1}{n}}}}{\frac{1}{n}} = 1$$

$\therefore$  divergent

$$(5) \text{ For } n > 1000, \ln(n+1) > 2$$

$$\therefore \frac{1}{[\ln(n+1)]^n} < \frac{1}{2^n}$$

$\therefore$  absolutely convergent

$$6 (3) \int \frac{1}{x \ln x (\ln \ln x)} dx = \ln \ln \ln x + C$$

$\therefore$  for  $\sigma \leq 0$ , divergent

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}} &\leq 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n(\ln n)^{1+\sigma}} \\ &\leq 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{2^{k-1} [\ln(2^{k-1})]^{1+\sigma}} \\ &\leq 10 + \sum_{k=2}^{\infty} \frac{1}{[\ln(2^{k-1})]^{1+\sigma}} \\ &= 10 + \sum_{k=2}^{\infty} \frac{1}{(k-1)^{1+\sigma}} \end{aligned}$$

$$\therefore \text{ for } \sigma > 0, \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}} \text{ absolutely convergent}$$

$$\text{Also } \therefore \text{ for } \forall p > 0, \exists X > 0, \forall x > X, (\ln x)^p > \ln \ln x$$

$\therefore$  for  $\sigma > 0$ , absolutely convergent

$$(4) \text{ Let } p = 1, \int_2^{\infty} \frac{1}{x \ln x (\ln \ln x)^q} dx = \int_{\ln 2}^{\infty} \frac{dt}{t(\ln t)^q}$$

$\therefore$  similar to the condition in previous problem

In conclusion:

$$\begin{cases} p > 1 : \text{absolutely convergent} \\ p = 1 : \begin{cases} q > 1 : \text{absolutely convergent} \\ q \leq 1 : \text{divergent} \end{cases} \\ p < 1 : \text{divergent} \end{cases}$$

## Lec 04

2 (1)  $(k^2 - 1)a_{k^2-1} = \frac{1}{k^2-1}, k^2 a_{k^2} = 1$

$$\exists \varepsilon = \frac{1}{2}, \forall N > 0, \exists k > N, \left| 1 - \frac{1}{k^2-1} \right| > \varepsilon$$

$\therefore \lim_{n \rightarrow \infty} a_n$  doesn't exist □

(2) Let  $b_k = \left[ \frac{1}{k^2} + \sum_{n=k^2+1}^{(k+1)^2-1} \frac{1}{n^2} \right]$

Evidently  $b_k$  is absolutely convergent

Use conclusion of Lec 02 Problem 05

$\therefore$  absolutely convergent □

3 (1) Evidently  $\lim_{n \rightarrow \infty} x_n$  exists. Let  $x_n \rightarrow A$

$$\therefore \lim_{n \rightarrow \infty} \frac{1 - \frac{x_n}{x_{n+1}}}{\frac{x_{n+1} - x_n}{A}} = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{A} = 1 - \frac{x_1}{A}, \text{ converge}$$

$\therefore$  absolutely convergent □

(2)  $\exists \varepsilon = \frac{1}{2}, \forall N > 0, \exists n_1, n_2 > N, \sum_{n=n_1}^{n_2} \left( 1 - \frac{x_n}{x_{n+1}} \right) > \frac{x_{n_2} - x_{n_1}}{x_{n_2}} > \varepsilon$

$\therefore$  divergent □

4 (1) Let  $b_k = a_{3k+1} + a_{3k+2} + a_{3k+3}$

$$|b_k| \text{ monotonically decreases to 0 and } \operatorname{sgn}\left(\frac{b_k}{b_{k-1}}\right) = -1$$

Use the conclusion of Lec 04 Problem 05,  $a_n$  converges

$$\therefore |a_n| = \frac{1}{n}$$

$\therefore$  conditionally convergent

(2) For  $a \neq 0, n\sqrt{1 + \frac{a^2}{n^2}} = n + \frac{a^2}{2n} + o\left(\frac{1}{n}\right)$

$$\therefore a_n = (-1)^n \sin\left(\frac{\pi a^2}{2n}\right) + o\left(\frac{1}{n}\right)$$

$$|a_n| \text{ monotonically decreases to 0 and } \operatorname{sgn}\left(\frac{a_n}{a_{n-1}}\right) = -1$$

$\therefore a_n$  converges

$$\therefore \lim_{n \rightarrow \infty} \frac{\sin \frac{\pi a^2}{2n}}{\frac{1}{100n}} > 1$$



∴ In conclusion:

$$\begin{cases} a \neq 0 : \text{conditionally convergent} \\ a = 0 : \text{absolutely convergent} \end{cases}$$

$$(3) \ln \left( 1 + \frac{(-1)^n}{n^p} \right) = \frac{(-1)^n}{n^p} - \frac{1}{n^{2p}} + o\left(\frac{1}{n^{2p}}\right)$$

For  $p > 0$ ,  $\left| \frac{(-1)^n}{n^p} \right|$  monotonically decreases to 0 and  $\text{sgn} \left( \frac{\frac{(-1)^n}{n^p}}{\frac{(-1)^{n-1}}{(n-1)^p}} \right) = -1$

$\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$  converges when  $p > \frac{1}{2}$ , diverges when  $p \leq \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges when  $p > 1$ , diverges when  $p \leq 1$

For  $p \leq 0$ , evidently diverge

∴ In conclusion:

$$\begin{cases} p \leq \frac{1}{2} : \text{divergent} \\ \frac{1}{2} < p \leq 1 : \text{conditionally convergent} \\ p > 1 : \text{absolutely convergent} \end{cases}$$

$$(4) \text{ Let } b_k = |a_{2k-1}| + |a_{2k}|, 0 < b_k < \frac{1}{2^{k-1}}$$

∴  $b_k$  converges

Use conclusion of Lec 02 Problem 05,  $|a_n|$  converges

∴ absolutely convergent

$$(5) \sum_{n=1}^{2N} a_n < e - \sum_{n=1}^N \frac{1}{2n}$$

$$\because - \sum_{n=1}^N \frac{1}{2n} \rightarrow -\infty$$

∴ divergent

$$(6) \because |a_n| \text{ monotonically decreases to 0 and } \text{sgn}\left(\frac{a_n}{a_{n-1}}\right) = -1$$

∴  $a_n$  converges

$$\because |a_n| > \frac{1}{n}$$

∴ conditionally convergent

$$(7) \because \int_2^{\infty} x^3 2^{-x} dx \text{ converges}$$

∴ absolutely convergent

(8)  $\because |a_n|$  monotonically decreases to 0 and  $\text{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$  converges

$\because |a_n| > \frac{1}{20n}$

$\therefore$  conditionally convergent

(9)  $\because |a_n|$  monotonically decreases to 0 and  $\text{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$  converges

$\because \lim_{n \rightarrow \infty} \frac{a_n}{\frac{1}{n}} = x$

$\therefore$  In conclusion:

$\begin{cases} x \neq 0 : \text{conditionally convergent} \\ x = 0 : \text{absolutely convergent} \end{cases}$

(10) Let  $b_k = a_{2k-1} + a_{2k} = \frac{2}{k-1}$

$\therefore$  divergent

### 3

#### Lec 04

5 Let  $b_k$  equal to the sum of  $k^{th}$  set of successive  $a_n$  which have the same sign

If  $n_0$  is in the  $k_0^{th}$  set, denote  $k(n_0) = k_0$

$\because \sum_{k=1}^{\infty} b_k$  convergent,  $\lim_{k \rightarrow \infty} b_k = 0$

$\therefore \forall \varepsilon > 0, \exists K > 0, \forall k_1, k_2 > K, \left| \sum_{k=k_1}^{k_2} b_k \right| < \varepsilon, |b_{k_1}| + |b_{k_2}| < \varepsilon$

$\because \exists N, k(N) > K, \forall n_1, n_2 > N, \left| \sum_{n=n_1}^{n_2} a_n \right| \leq \varepsilon + |b_{k(n_1)}| + |b_{k(n_2)}| < 2\varepsilon$

$\therefore$  convergent

For  $a_n = \frac{(-1)^{[\sqrt{n}]}}{n}$ , let  $b_k = (-1)^k \sum_{n=(k-1)^2+1}^{k^2} \frac{1}{n}$

$\therefore |b_k| < \frac{2k}{(k-1)^2}$

$\therefore |b_k|$  monotonically decreases to 0 and  $\text{sgn}(\frac{b_k}{b_{k-1}}) = -1$

$\therefore b_k$  converges But  $|a_n| = \frac{1}{n}$

$\therefore$  conditionally convergent

$$8 \quad \forall \varepsilon > 0, \exists N > 0, \forall n_1, n_2 > N,$$

$$\max\left\{\left|\sum_{n=n_1}^{n_2} n(a_n - a_{n-1})\right|, |n_1 a_{n_1-1}|, |n_2 a_{n_2}|\right\} < \varepsilon$$

$$\therefore \left|\sum_{n=n_1}^{n_2-1} a_n\right| = \left|\sum_{n=n_1}^{n_2} n(a_n - a_{n-1}) + n_1 a_{n_1-1} - n_2 a_{n_2}\right| < 3\varepsilon$$

$\therefore$  convergent □

## Lec 05

$$3 \quad (1) \quad x_{n+1} - x_n = \frac{\ln(n+1)}{n+1} - \frac{1}{2} \left[ \ln^2(n+1) - \ln^2 n \right] = \frac{\ln(n+1)}{n+1} - \int_n^{n+1} \frac{\ln x}{x} dx$$

$\frac{\ln x}{x}$  is monotonically decreasing over  $[e, +\infty)$

$$\text{if } n > 3, \frac{\ln(n+1)}{n+1} < \int_n^{n+1} \frac{\ln x}{x} dx < \frac{\ln n}{n}$$

$\therefore x_{n+1} < x_n, x_n$  monotonically decreasing

$$\begin{aligned} x_n &= \sum_{k=1}^n \frac{\ln k}{k} - \frac{1}{2} (\ln n)^2 = \sum_{k=1}^n \frac{\ln k}{k} - \int_1^n \frac{\ln x}{x} dx \\ &= \sum_{k=1}^2 \frac{\ln k}{k} - \int_1^3 \frac{\ln x}{x} dx + \sum_{k=3}^n \frac{\ln k}{k} - \int_3^n \frac{\ln x}{x} dx > \frac{\ln 2}{2} - \ln^2 3 \end{aligned}$$

$\therefore$  convergence

$$(2) \quad x_{n+1} - x_n = \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} - 2\sqrt{n} < 0$$

$\therefore x_n$  monotonically decreasing

$$\begin{aligned} \sqrt{n} &= \sqrt{n} - \sqrt{n-1} + \sqrt{n-1} - \sqrt{n-2} + \cdots + \sqrt{2} - \sqrt{1} + 1 \\ &= \frac{1}{\sqrt{n} + \sqrt{n-1}} + \frac{1}{\sqrt{n-1} + \sqrt{n-2}} + \cdots + \frac{1}{\sqrt{2} + \sqrt{1}} + 1 \\ &< \frac{1}{2} \left( \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{1}} \right) + 1 \end{aligned}$$

$$x_n > 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - \left( \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{1}} + 2 \right) = -2 + \frac{1}{\sqrt{n}} > -2$$

$\therefore$  convergence

$$4 \quad \sum_{n=0}^{\infty} |x|^n = \frac{1}{1-|x|}, \sum_{n=0}^{\infty} |y|^n = \frac{1}{1-|y|} \quad (\text{both absolutely convergent})$$

$$\therefore \sum_{n=1}^{\infty} (x^{n-1} + x^{n-2}y + \cdots + y^{n-1}) = \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} y^n = \frac{1}{(1-x)(1-y)} \quad \square$$

$$5 \quad \lim_{n \rightarrow \infty} \sqrt[n]{n!} = \infty$$

$\therefore$  radius of convergence is  $\infty$

$$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \sum_{n=0}^{\infty} \frac{y^n}{n!} = e^y \text{ both absolutely convergent}$$

$$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} \frac{y^n}{n!} = e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$$

□

## Lec 06

$$1 \quad (1) \text{ Let } p_n = q_n = 1, \prod_{n=1}^{\infty} (p_n + q_n) = \prod_{n=1}^{\infty} 2 = \infty$$

$\therefore$  divergent

$$(2) \quad \prod_{n=1}^{\infty} p_n, \prod_{n=1}^{\infty} q_n \text{ converge}$$

$$\Rightarrow \sum_{n=1}^{\infty} \ln p_n, \sum_{n=1}^{\infty} \ln q_n \text{ converge}$$

$$\Rightarrow \sum_{n=1}^{\infty} (\ln p_n + \ln q_n) \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \ln(p_n q_n) \text{ converges}$$

$$\Rightarrow \prod_{n=1}^{\infty} p_n q_n \text{ converges}$$

$$(3) \text{ Let } q_n = p_n \text{ and use conclusion of previous problem}$$

$\therefore$  convergent

$$(4) \quad \prod_{n=1}^{\infty} q_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \ln q_n \text{ converges}$$

$$\Rightarrow - \sum_{n=1}^{\infty} \ln q_n \text{ converges}$$

$$\Rightarrow \prod_{n=1}^{\infty} \frac{1}{q_n} \text{ converges}$$

$$\text{Use conclusion of Lec 06 Prob 1(2), } \prod_{n=1}^{\infty} \frac{p_n}{q_n} \text{ converges}$$

2 Denote  $T_n = \prod_{k=1}^n (1 + u_k)$ ,  $S_n = \sum_{k=1}^n u_k$ ,  $S'_n = \sum_{k=1}^n (u_k)^2$

$$\therefore S_{2n} = \sum_{k=1}^n \frac{1}{k} \rightarrow \infty, S'_{2n} > 2 \sum_{k=1}^n \frac{1}{k} \rightarrow \infty$$

$$\therefore \lim_{n \rightarrow \infty} u_{2n-1} = \lim_{n \rightarrow \infty} u_{2n} = 0$$

$\therefore S_n, S'_n$  diverges

$$\therefore (1 + u_{2k-1})(1 + u_{2k}) = 1 - \frac{1}{k^{\frac{3}{2}}}$$

$\therefore T_{2n}$  converges, let A denote its limit

$$\therefore \forall \varepsilon > 0, \exists N_1 > 0, \forall n > N_1, |T_{2n} - A| < \varepsilon$$

$$\text{And } \lim_{n \rightarrow \infty} \frac{T_{2n+1}}{T_{2n}} = u_{2n+1} + 1 = 1$$

$$\therefore \text{for } \varepsilon > 0, \exists N_2 > 0, \forall n > N_2, |T_{2n+1} - T_{2n}| < \varepsilon$$

$$\therefore \forall \varepsilon > 0, \exists N = \max\{2N_1 + 10, 2N_2 + 10\} > 0, \forall n > N, |T_n - A| < 2\varepsilon$$

$\therefore T_n$  converges

□

$$3 \quad (1) \quad \lim_{n \rightarrow \infty} \frac{\ln[(\frac{n^2-1}{n^2+1})^p]}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{p \ln(1 - \frac{2}{n^2+1})}{\frac{1}{n^2}} = -2p$$

$\therefore$  convergent

$$(2) \quad \lim_{n \rightarrow \infty} \frac{\ln \sqrt[n]{1 + \frac{1}{n}}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \ln(1 + \frac{1}{n})}{\frac{1}{n^2}} = 1$$

$\therefore$  convergent

$$(3) \quad \lim_{n \rightarrow \infty} \frac{\ln \sqrt[n]{\ln(n+x) - \ln n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \ln \ln(1 + \frac{x}{n})}{\frac{1}{n}} = -\infty$$

$\therefore$  divergent

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\ln \frac{n^2-4}{n^2-1}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\ln(1 - \frac{3}{n^2-1})}{\frac{1}{n^2}} = -3$$

$\therefore$  convergent

$$(5) \quad \ln a^{\frac{(-1)^n}{n}} = \frac{(-1)^n}{n} \ln a$$

$\therefore \frac{1}{n}$  monotonically decreases to 0

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ln a$  converges

$\therefore$  convergent

$$(6) \because \prod_{k=1}^n \sqrt{\frac{k+1}{k+2}} = \sqrt{\frac{2}{n+2}} \rightarrow 0$$

$\therefore$  divergent

5 Due to convergence  $\lim_{n \rightarrow \infty} a_n = 0$

$$\tan\left(\frac{\pi}{4} + x\right) = 1 + Ax + o(x), A = \tan'\left(\frac{\pi}{4}\right) > 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{|\ln[\tan(\frac{\pi}{4} + a_n)]|}{|a_n|} = A$$

$$\therefore \sum_{n=1}^{\infty} \ln[\tan(\frac{\pi}{4} + a_n)] \text{ converges}$$

$\therefore$  convergent □

## 4

### Lec 07

$$1 \quad (1) \because |f_n(x) - |x|| = \left| \frac{1}{n^2(\sqrt{x^2 + \frac{1}{n^2}} + |x|)} \right| \leq \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \sup_{x \in \mathcal{X}} |f_n(x) - |x|| = 0$$

$\therefore$  uniformly convergent

$$(2) \because \sup_{x \in \mathcal{X}} |f_n(x) - 0| = \frac{1}{4}$$

$\therefore$  not uniformly convergent

$$(3) \because |f_n(x) - 0| \leq \frac{1}{n+1} \left(\frac{n}{n+1}\right)^n \leq \frac{1}{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} \sup_{x \in \mathcal{X}} |f_n(x) - 0| = 0$$

$\therefore$  uniformly convergent

$$(4) \because \text{if } n > 100, \lim_{n \rightarrow \infty} \sup_{x \in \mathcal{X}} |f_n(x) - 0| = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$\therefore$  uniformly convergent

$$2 \quad (1) \quad S(x) = \begin{cases} x, & x \in [0, 1) \\ 0, & x = 1 \end{cases} \quad \text{is not continuous}$$

$\therefore$  not uniformly convergent

$$(2) \text{ denote } a_n(x) = \frac{x^2}{(1+x^2)^n}, \quad b_n(x) = (-1)^n$$

$$\because \text{if } n > 2, \sup_{x \in \mathcal{X}} |a_n(x) - 0| < \frac{1}{n-1}$$

$$\therefore a_n(x) \xrightarrow{\mathcal{X}} 0$$

$$\because a_n(x) \text{ is about } n \text{ monotonically decreasing and } \sum_{n=1}^{\infty} b_n(x) \text{ is uniformly bounded}$$

$$\therefore \text{uniformly convergent}$$

$$(3) \because \left| \frac{\sin nx}{\sqrt[3]{n^4+x^4}} \right| \leq \frac{1}{n^{\frac{4}{3}}}$$

$$\therefore \text{uniformly convergent}$$

$$(4) \because \left| \frac{x}{1+n^4x^2} \right| \leq \frac{1}{n^2}$$

$$\therefore \text{uniformly convergent}$$

$$(5) \text{ denote } a_n(x) = \frac{1}{\sqrt{n+x}}, \quad b_n(x) = \sin nx \sin x$$

$$\because \sin nx \sin x = \frac{\cos(n-1)x - \cos(n+1)x}{2}$$

$$\therefore \sum_{n=1}^{\infty} b_n(x) \text{ is uniformly bounded}$$

$$\because a_n(x) \text{ is about } n \text{ monotonically decreasing and } a_n(x) \xrightarrow{\mathcal{X}} 0$$

$$\therefore \text{uniformly convergent}$$

$$(6) \because \left| \frac{(-1)^n(1-e^{-nx})}{n^2+x^2} \right| \leq \left| \frac{1}{n^2} \right|$$

$$\therefore \text{uniformly convergent}$$

$$3 \because \left| \frac{\ln(1+nx)}{nx^n} \right| \leq \frac{1}{x^{n-1}} \leq \frac{1}{\alpha^{n-1}}$$

$$\therefore \text{uniformly convergent} \quad \square$$

$$4 \because f_0(x) \text{ is continuous over } [0, a]$$

$$\therefore \exists A \text{ s.t. } |f(x)| < A$$

$$\because f_n(x) = \int_0^x f_{n-1}(t) dt$$

$$\therefore |f_n(x)| \leq \frac{Ax^n}{n!} \leq \frac{Aa^n}{n!}$$

$$\therefore f_n(x) \xrightarrow{\mathcal{X}} 0 \quad \square$$

$$5 \text{ if } \sum_{n=1}^{\infty} |f_n(x)| \text{ uniformly convergent}$$

$$\because \left| \sum_{i=n}^m f_i(x) \right| \leq \sum_{i=n}^m |f_i(x)|$$

$$\therefore \forall \epsilon > 0, \exists N > 0, \text{ when } m, n > N, \forall x \in \mathcal{X}, \left| \sum_{i=n}^m f_i(x) \right| \leq \sum_{i=n}^m |f_i(x)| < \epsilon$$

$$\therefore \sum_{n=1}^{\infty} f_n(x) \text{ uniformly convergent}$$

□

$$\text{but the inverse is not true, for example } f_n(x) = \frac{(-1)^n x}{n}$$

□

## Lec 08

$$1 \text{ denote } a_n = \max(|\varphi_n(a)|, |\varphi_n(b)|)$$

$$\therefore \varphi_n(x) \text{ is about } x \text{ monotonous over } [a, b]$$

$$\therefore |\varphi_n(x)| \leq a_n \leq |\varphi_n(a)| + |\varphi_n(b)|$$

$$\therefore \sum_{n=1}^{\infty} |\varphi_n(a)|, \sum_{n=1}^{\infty} |\varphi_n(b)| \text{ is absolutely convergent}$$

$$\therefore \text{uniformly convergent}$$

□

$$2 \forall a, b, 0 < a < b, x \in [a, b]$$

$$\therefore 0 < \frac{n}{e^{xn}} \leq \frac{n}{e^{an}} \text{ and } \sum_{n=1}^{\infty} \frac{n}{e^{an}} \text{ is convergent}$$

$$\therefore \sum_{n=1}^{\infty} ne^{-nx} \text{ is uniformly convergent over } (0, +\infty)$$

$$\therefore ne^{-nx} \text{ is continuous}$$

$$\therefore \text{continuous}$$

□

$$3 \text{ denote } a_n(x) = \frac{\sin nx}{n^3}$$

$$\therefore |a_n(x)| \leq \frac{1}{n^3}$$

$$\therefore \sum_{n=1}^{\infty} a_n(x) \text{ is uniformly convergent}$$

$$\therefore a_n(x) \text{ is continuous } \therefore f(x) \text{ is continuous}$$

$$\therefore |a'_n(x)| \leq \frac{1}{n^2} \therefore \sum_{n=1}^{\infty} a'_n(x) \text{ is uniformly convergent}$$

$$\therefore f'(x) = \sum_{n=1}^{\infty} a'_n(x)$$

$$\therefore a'_n(x) \text{ is continuous } \therefore f'(x) \text{ is continuous}$$

□

$$4 \text{ for } n > 1, \forall m \in \mathbb{N} \therefore \frac{d^m}{dx^m} \left( \frac{1}{n^x} \right) = \frac{d^m}{dx^m} (e^x)^{-\ln n} = (-\ln n)^m (e^x)^{-\ln n} = (-\ln n)^m \frac{1}{n^x}$$



$$\therefore \forall \alpha > 1, \text{ when } x \geq \alpha, \left| \frac{d^m}{dx^m} \left( \frac{1}{n^x} \right) \right| \leq (\ln n)^m \frac{1}{n^\alpha}$$

$$\therefore \sum_{n=1}^{\infty} (\ln n)^m \frac{1}{n^\alpha} \text{ is convergent and } (\ln n)^m \frac{1}{n^x} \text{ is continuous}$$

$$\therefore \sum_{n=1}^{\infty} \frac{d^m}{dx^m} \left( \frac{1}{n^x} \right) \text{ uniformly convergent}$$

$$\therefore \zeta^{(n)}(x) \text{ is continuous} \quad \square$$

$$5 \therefore \left| \frac{\sin(2^n \pi x)}{2^n} \right| \leq \frac{1}{2^n} \therefore \text{uniformly convergent}$$

$$\therefore \frac{d}{dx} \frac{\sin(2^n \pi x)}{2^n} = \pi \cos(2^n \pi x)$$

$$\lim_{n \rightarrow +\infty} \pi \cos(2^n \pi x) = \begin{cases} \text{not exists, } x \neq \frac{m}{2^k}, m, k \in \mathbb{Z} \\ \pi, x = \frac{m}{2^k}, m, k \in \mathbb{Z} \end{cases} \neq 0$$

$$\therefore \text{can't doing derivation at every formula} \quad \square$$

$$6 \text{ if } |x| = 1, f(x) = \int_{-\pi}^{\pi} \frac{1-x^2}{1+x^2-2x \cos \theta} d\theta = 0$$

$$\text{if } |x| < 1, f(x) = \int_{-\pi}^{\pi} 1 + 2 \sum_{n=1}^{\infty} x^n \cos n\theta d\theta$$

$$\therefore |x^n \cos n\theta| \leq |x^n|, \sum_{n=1}^{\infty} |x^n| \text{ is convergent}$$

$$\therefore \sum_{n=1}^{\infty} x^n \cos n\theta \text{ is about } \theta \text{ uniformly convergent}$$

$$\therefore f(x) = 2\pi + 2 \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} x^n \cos n\theta d\theta = 2\pi$$

$$\text{if } |x| > 1, f(x) = -2\pi$$

in conclusion

$$f(x) = \begin{cases} 0, & |x| = 1 \\ 2\pi, & |x| < 1 \\ -2\pi, & |x| > 1 \end{cases}$$

## 5

### Lec 09

$$1 \quad (1) \quad \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{2^n}} = +\infty$$

$$\therefore R = +\infty$$

$$\therefore \text{convergence region: } (-\infty, +\infty)$$

$$(2) \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{[3+(-1)^n]^n}} = \frac{1}{4}$$

$$\therefore R = \frac{1}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^{2n-1}}{2n-1} \text{ converge and } |x| = \frac{1}{4} : \sum_{n=1}^{\infty} \frac{1}{2n} \text{ diverge}$$

$$\therefore \text{convergence region: } (-\frac{1}{4}, \frac{1}{4})$$

$$(3) \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n+1}{\ln(n+1)}} = 1$$

$$\therefore R = 1$$

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1} \text{ diverges}$$

$$\frac{\ln(n+1)}{n+1} \text{ monotonically decreases to 0 when } n > 3$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n \ln(n+1)}{n+1} \text{ converges}$$

$$\therefore \text{divergent when } x = 1, \text{ convergent when } x = -1$$

$$\therefore \text{convergence region: } [-1, 1)$$

$$(4) |x| = 1: \text{convergent}$$

$$|x| > 1: \lim_{n \rightarrow \infty} \frac{|x|^{n^2}}{2^n} = +\infty \neq 0$$

$$\therefore \text{convergence region: } [-1, 1]$$

$$(5) \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3^n + (-2)^n}} = \frac{1}{3} \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{1 + (-\frac{2}{3})^n}} = \frac{1}{3}$$

$$\therefore R = \frac{1}{3}$$

$$x + 1 = \frac{1}{3}: \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, } \sum_{n=1}^{\infty} \frac{(-\frac{2}{3})^n}{n} \text{ converges} \Rightarrow \text{diverges}$$

$$x + 1 = -\frac{1}{3}: \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges, } \sum_{n=1}^{\infty} \frac{(\frac{2}{3})^n}{n} \text{ converges} \Rightarrow \text{converges}$$

$$\therefore \text{convergence region: } [-\frac{4}{3}, -\frac{2}{3})$$

$$(6) \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{(1+\frac{1}{n})^{n^2}}} = \frac{1}{e}$$

$$\therefore R = \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} \frac{(1+\frac{1}{n})^{n^2}}{e^n} = \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x^2}}}{e^{\frac{1}{x}}} = e^{\lim_{x \rightarrow 0} \ln \frac{(1+x)^{\frac{1}{x^2}}}{e^{\frac{1}{x}}}} = e^{\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2}} = e^{-\frac{1}{2}} \neq 0$$

$\therefore$  convergence region:  $(-\frac{1}{e}, \frac{1}{e})$

(7) Just let  $a \geq b$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a^n + b^n} = a \lim_{n \rightarrow \infty} \sqrt[n]{1 + (\frac{b}{a})^n} = a$$

$\therefore R = a$

$$\lim_{n \rightarrow \infty} \frac{a^n}{a^n + b^n} \geq \frac{1}{2}$$

$\therefore$  convergence region:  $(-a, a)$

$$(8) \lim_{n \rightarrow \infty} \sqrt[n]{n2^n} = 2$$

$\therefore R = 2$

$$|x| = \sqrt{2}: \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges}$$

$\therefore$  convergence region:  $[-\sqrt{2}, \sqrt{2}]$

$$(9) 1 \leq \lim_{n \rightarrow \infty} {}^{2n-1}\sqrt{\frac{(2n+1)(2n)!!}{(2n-1)!!}} \leq \lim_{n \rightarrow \infty} {}^{2n-1}\sqrt{\frac{(2n+1)(2n)!!(2n-1)!!}{(2n-1)!!(2n-1)!!}} \leq \lim_{n \rightarrow \infty} {}^{2n-1}\sqrt{\frac{(2n+1)!}{(2n-1)!}}$$

$$= \lim_{n \rightarrow \infty} {}^{2n-1}\sqrt{2n(2n+1)} = 1$$

$\therefore R = 1$

$$|x| = 1: \frac{(2n-1)!!}{(2n+1)(2n)!!} \text{ monotonically decreases to } 0$$

$\therefore$  convergent

$\therefore$  convergence region:  $[-1, 1]$

2 (1)  $|x| < \sqrt{A} \Rightarrow x^2 < A \Rightarrow$  convergent

$$|x| > \sqrt{A} \Rightarrow x^2 > A \Rightarrow \text{divergent}$$

$\therefore R = \sqrt{A}$

$$(2) \frac{1}{R} = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n + b_n|} \leq \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{2 \max(|a_n|, |b_n|)} = \max(\frac{1}{A}, \frac{1}{B})$$

$\therefore R \geq \min(A, B)$

$$(3) \frac{1}{R} = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n b_n|} \leq \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \frac{1}{AB}$$

$\therefore R \geq AB$

$$3 \text{ Let } A_m(x) = \sum_{n=1}^m a_n x^n, B_m(x) = \sum_{n=1}^m b_n x^n, S_m(x) = \sum_{n=1}^m a_n b_{m-n+1} x^{m+1},$$

$$R_m(x) = A_m(x)B_m(x) - \sum_{n=1}^m S_n(x), M = \sup_n [\max(|a_n|, |b_n|)]$$

$$0 < x < 1: |R_m(x)| < M^2 \frac{m^2-m}{2} x^{m+2}$$

$$\therefore \lim_{m \rightarrow \infty} R_m(x) = 0, 0 < x < 1$$

$$\therefore [0, 1] \text{ is in the uniformly convergence region of } A_m(x), B_m(x), \sum_{n=1}^m S_n(x)$$

$$\therefore \lim_{m \rightarrow \infty} R_m(x) \text{ is also continuous in } [0, 1]$$

$$\therefore \lim_{m \rightarrow \infty} R_m(1) = 0$$

$$\therefore \sum_{n=1}^{\infty} S_n(x) = \lim_{m \rightarrow \infty} A_m B_m = AB$$

□

$$4 \text{ Apparently } \sum_{n=0}^{\infty} a_n x^n \text{ is uniformly convergent in } [0, r)$$

$$\therefore \int_0^x \lim_{m \rightarrow \infty} \sum_{n=0}^m a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

$$\therefore \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \text{ is uniformly convergent in } [0, r] \text{ and } \frac{a_n x^{n+1}}{n+1} \text{ is continuous}$$

$$\therefore \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \text{ is continuous in } [0, r]$$

$$\therefore \lim_{x \rightarrow r^-} \int_0^x \lim_{m \rightarrow \infty} \sum_{n=0}^m a_n t^n dt \text{ exists and is equal to } \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

□

$$f(x) = -\frac{\ln(1-x)}{x} = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

$$\therefore r = \lim_{n \rightarrow \infty} \sqrt[n]{n+1} = 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ convergent}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^1 \frac{\ln(1-x)}{x} dx$$

□

## Lec 10

1 Convergence in  $(-\infty, \infty)$  is apparent.

$$f^{(2n+1)}(0) = \sum_{m=1}^{\infty} \frac{(-1)^n (2^m)^{2n+1}}{m!}$$

$$\therefore f(x) \sim \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n (2^m x)^{2n+1}}{m! (2n+1)!} \sim \sum_{n=0}^{\infty} (e^{2^{2n+1}} - 1) \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\therefore \lim_{n \rightarrow \infty} {}^{2n+1}\sqrt{\frac{(2n+1)!}{e^{2^{2n+1}}}} = 0$$

$$\therefore R = 0$$

$$\therefore \text{divergent when } x \neq 0$$

$$2 \quad f^{(n)}(0) = \frac{n!}{a^{n+1}}$$

$$f(x) \sim \sum_{n=0}^{\infty} \frac{x^n}{a^{n+1}}$$

$$\lim_{n \rightarrow \infty} {}^n\sqrt{|a|^{n+1}} = |a|$$

$$\left| \frac{a^n}{a^{n+1}} \right| = \frac{1}{|a|} \rightarrow 0$$

$$\therefore \text{convergence region: } (-|a|, |a|)$$

$$3 \quad f^{(n)}(2) = \frac{(-1)^{n-1} (n-1)!}{2^n}$$

$$f(x) \sim \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-2)^n}{2^n n}$$

$$\lim_{n \rightarrow \infty} {}^n\sqrt{2^n n} = 2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\therefore \text{convergence region: } (0, 4]$$

$$4 \quad (1) \quad f^{(n)}(0) = \left( \frac{\sin x}{x} \right)^{(n-1)} \Big|_{x=0}$$

$$\therefore \sin x \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\therefore \frac{\sin x}{x} \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$f^{(2n+1)}(0) = \left( \frac{\sin x}{x} \right)^{(2n)} \Big|_{x=0} = \frac{(-1)^n (2n)!}{(2n+1)!} = \frac{(-1)^n}{2n+1}$$

$$\therefore f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)! (2n+1)}$$

$$R = \lim_{n \rightarrow \infty} {}^{2n+1}\sqrt{(2n+1)! (2n+1)} = +\infty$$

$$\therefore \text{convergence region: } (-\infty, +\infty)$$

$$(2) \quad \cos t^2 \sim \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}$$

$$\therefore f^{(4n+1)}(0) = (\cos t^2)^{(4n)}|_{t=0} = \frac{(-1)^n (4n)!}{(2n)!}$$

$$\therefore f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}$$

$$R = \lim_{n \rightarrow \infty} \sqrt[4n+1]{(2n)!(4n+1)} = +\infty$$

$$\therefore \text{convergence region: } (-\infty, +\infty)$$

(3) Let  $x = \tan t$

$$\arctan \frac{2x}{1-x^2} = -\arctan(\tan 2t) = -2 \arctan x \sim -2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\lim_{n \rightarrow \infty} \sqrt[2n+1]{2n+1} = 1$$

$$|x| = 1: \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{ converges}$$

$$\therefore \text{convergence region: } [-1, 1] \text{ ( with definition: } \arctan(\pm\infty) = \pm\pi \text{ )}$$

$$(4) f^{(1)}(0) = \frac{1}{\sqrt{1+x^2}} \sim \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} x^{2n}$$

$$\therefore f^{(2n+1)}(0) = \left(\frac{1}{\sqrt{1+x^2}}\right)^{(2n)}|_{x=0} = \frac{(-1)^n (2n-1)!! (2n)!}{(2n)!!}$$

$$\therefore f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!! (2n+1)} x^{2n+1}$$

$$\lim_{n \rightarrow \infty} \sqrt[2n+1]{\frac{(2n)!! (2n+1)}{(2n-1)!!}} = 1 \text{ (see Lec 09 Prob 1(9))}$$

$$\frac{(2n-1)!!}{(2n)!! (2n+1)} \text{ monotonically decreases to 0}$$

$$\therefore \text{converges when } |x| = 1$$

$$\therefore \text{convergence region: } [-1, 1]$$

$$5 (1) e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$$

$$\therefore \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1, x \in \mathbb{R}$$

$$(2) \int_0^x \ln(1+x) dx = (1+x) \ln(1+x) - x$$

$$\therefore \text{convergence radius of } \ln(1+x) \text{'s Maclaurin series is 1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)} \text{ converges at } |x| = 1$$

According to Lec 09 Prob 4,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)} = (1+x) \ln(1+x) - x$  in  $[-1, 1]$   
 (define  $(1+x) \ln(1+x) - x = 1$  at  $x = -1$ )

$$(3) \int_0^x f(x) dx \sim \sum_{n=1}^{\infty} n x^n \sim \frac{x}{(x-1)^2}$$

$\therefore$  Convergence radius is 1 and  $f(x)$  diverges at  $|x| = 1$

$$\therefore f(x) = \left[ \frac{x}{(x-1)^2} \right]' = \frac{1+x}{(1-x)^3} \text{ in } (-1, 1)$$

(4) Convergence region is  $\mathbb{R}$

$$\therefore \int_0^x f(x) dx = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{n!} = x(e^{x^2} - 1)$$

$$\therefore f(x) = \left[ x(e^{x^2} - 1) \right]' = (2x^2 + 1)e^{x^2} - 1$$

$$(5) \text{ Let } A = \sum_{n=1}^{\infty} \frac{2n-1}{2^n}$$

$$\therefore A = 2A - A = 1 + \sum_{n=1}^{\infty} \frac{2n+1}{2^n} - \sum_{n=1}^{\infty} \frac{2n-1}{2^n} = 3$$

$$(6) \sum_{n=0}^{\infty} \frac{n^2+1}{2^n n!} = e^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{n}{2^n (n-1)!} = e^{\frac{1}{2}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(\frac{1}{2})^{n-1}}{(n-1)!} + \frac{1}{4} \sum_{n=2}^{\infty} \frac{(\frac{1}{2})^{n-2}}{(n-2)!} = \frac{7}{4} e^{\frac{1}{2}}$$

## 6

### Lec 12

3  $\therefore f'(x)$  monotonically increases in  $[0, 2\pi]$

$$\therefore a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = -\frac{1}{n\pi} \int_0^{2\pi} f'(x) \sin nx dx$$

$$= -\frac{1}{n\pi} \sum_{i=0}^{n-1} \int_{\frac{2i}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x) \sin nx dx$$

$$= -\frac{1}{n\pi} \sum_{i=0}^{n-1} \left[ \int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi} f'(x) \sin nx dx + \int_{\frac{2i+1}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x) \sin nx dx \right]$$

$$= -\frac{1}{n\pi} \sum_{i=0}^{n-1} \int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi} [f'(x) - f'(x + \frac{\pi}{n})] \sin nx dx \geq 0$$

□

### Lec 13

$$\begin{aligned}
1 \quad & \lim_{p \rightarrow \infty} \int_{-\pi}^{\pi} f(t) \frac{\cos \frac{t}{2} - \cos pt}{2 \sin \frac{t}{2}} dt \\
&= \lim_{p \rightarrow \infty} \int_0^{\pi} [f(t) - f(-t)] \frac{\cos \frac{t}{2} - \cos pt}{2 \sin \frac{t}{2}} dt \\
&= \frac{1}{2} \int_0^{\pi} [f(t) - f(-t)] \cot \frac{t}{2} dt - \lim_{p \rightarrow \infty} \int_0^{\pi} [f(t) - f(-t)] \frac{\cos pt}{2 \sin \frac{t}{2}} dt \\
&= \frac{1}{2} \int_0^{\pi} [f(t) - f(-t)] \cot \frac{t}{2} dt - \lim_{p \rightarrow \infty} \int_{\delta}^{\pi} [f(t) - f(-t)] \frac{\cos pt}{2 \sin \frac{t}{2}} dt - \lim_{p \rightarrow \infty} \int_0^{\delta} [f(t) - f(-t)] \frac{\cos pt}{2 \sin \frac{t}{2}} dt \\
&\because \frac{f(t) - f(-t)}{2 \sin \frac{t}{2}} \in \mathcal{R}[\delta, \pi] \\
&\therefore \lim_{p \rightarrow \infty} \int_{\delta}^{\pi} [f(t) - f(-t)] \frac{\cos pt}{2 \sin \frac{t}{2}} dt = 0 \\
&\because f(t) \text{ is continuous and has unilateral derivative at } t = 0 \\
&\therefore \exists \delta > 0, M > 0 \text{ s.t. when } t \in (0, \delta), -M < \frac{f(t) - f(-t)}{2 \sin \frac{t}{2}} \cos pt < M \\
&\therefore 0 = \lim_{\delta \rightarrow 0} -M\delta \leq \lim_{\delta \rightarrow 0} \lim_{p \rightarrow \infty} \int_0^{\delta} [f(t) - f(-t)] \frac{\cos pt}{2 \sin \frac{t}{2}} dt \leq \lim_{\delta \rightarrow 0} M\delta = 0 \\
&\therefore \lim_{p \rightarrow \infty} \int_{-\pi}^{\pi} f(t) \frac{\cos \frac{t}{2} - \cos pt}{2 \sin \frac{t}{2}} dt = \frac{1}{2} \int_0^{\pi} [f(t) - f(-t)] \cot \frac{t}{2} dt \quad \square
\end{aligned}$$

$$\begin{aligned}
2 \quad (1) \quad & f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\
& a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2\pi^2}{3} \\
& a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = (-1)^n \frac{4}{n^2} \\
& f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx
\end{aligned}$$

$$\begin{aligned}
(2) \quad & f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \\
& b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx = -\frac{4}{\pi n^3} + \frac{2(-1)^n}{\pi} \left( \frac{2}{n^3} - \frac{\pi^2}{n} \right) \\
& f(x) = \sum_{n=1}^{\infty} \left[ \frac{4(-1)^n}{\pi n^3} - \frac{4}{\pi n^3} - \frac{2\pi(-1)^n}{n} \right] \sin nx
\end{aligned}$$

$$\begin{aligned}
(3) \quad & f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \\
& a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{8\pi^2}{3} \\
& a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{4}{n^2} \\
& b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx dx = -\frac{4\pi}{n}
\end{aligned}$$



$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left( \frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right)$$

$$(4) \text{ from (1) } f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$f(\pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$3 \ a_0 = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$a_n = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(n^2+1)}$$

$$b_n = -\frac{(-1)^n n (e^{\pi} - e^{-\pi})}{\pi(n^2+1)}$$

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)} (\cos nx - n \sin nx) \right]$$

$$\text{if } x = \pi, \frac{e^{\pi} - e^{-\pi}}{\pi} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2} \right) = \frac{1}{2} (e^{\pi} - e^{-\pi})$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi(e^{\pi} + e^{-\pi})}{2(e^{\pi} - e^{-\pi})} - \frac{1}{2}$$

4  $\because f(x) \in \mathcal{R}$ , apply besel's inequality

$$\therefore \sum_{n=1}^{\infty} a_n^2 \text{ and } \sum_{n=1}^{\infty} b_n^2 \text{ convergent}$$

$$\therefore \frac{|a_n|}{n} \leq \frac{1}{2} (a_n^2 + \frac{1}{n^2}) \text{ and } \frac{|b_n|}{n} \leq \frac{1}{2} (b_n^2 + \frac{1}{n^2})$$

$$\therefore \sum_{n=1}^{\infty} \frac{a_n}{n} \text{ and } \sum_{n=1}^{\infty} \frac{b_n}{n} \text{ convergent}$$

□

## 7

### Lec 14

$$\begin{aligned} 1 \ e^{\cos x} \cos(\sin x) &= e^{\cos x} \frac{e^{i \sin x} + e^{-i \sin x}}{2} = \frac{1}{2} (e^{e^{ix}} + e^{e^{-ix}}) = \sum_{n=0}^{\infty} \frac{1}{2} \frac{(e^{ix})^n + (e^{-ix})^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2} \frac{e^{inx} + e^{-inx}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!} \end{aligned}$$

$$2 \text{ Let } f(x+c) = \frac{a'_0}{2} + \sum_{n=1}^{\infty} a'_n \cos(nx) + b'_n \sin(nx) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos[n(x+c)] + b_n \sin[n(x+c)] =$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nc) + b_n \sin(nc)] \cos(nx) + [b_n \cos(nc) - a_n \sin(nc)] \sin(nx)$$

$$\therefore a'_0 = a_0, \quad a'_n = a_n \cos(nc) + b_n \sin(nc), \quad b'_n = b_n \cos(nc) - a_n \sin(nc)$$

$$3 \quad \pi - x = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx), \quad x \in (0, 2\pi)$$

$$\therefore \frac{1}{2\pi} \int_0^{2\pi} f(x)(\pi - x) dx = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{2}{n} b_n \sin^2(nx) dx$$

$$\therefore \sum_{n=1}^{\infty} \frac{2}{n} b_n \sin^2(nx) \text{ is controlled by } \sum_{n=1}^{\infty} \frac{2}{n} b_n \text{ and Lec 13 Prob 04's conclusion only needs } f(x) \text{ to be integrable.}$$

$$\therefore \sum_{n=1}^{\infty} \frac{2}{n} b_n \sin^2(nx) \text{ uniformly converges}$$

$$\therefore \frac{1}{2\pi} \int_0^{2\pi} f(x)(\pi - x) dx = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{2}{n} b_n \sin^2(nx) dx = \sum_{n=1}^{\infty} \frac{b_n}{n} \quad \square$$

4 (1) Apply periodic extension to  $f(x)$  and set  $f(2n\pi) = 0$ , which won't change Fourier series

$$\therefore \sigma_n(x) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(x+t) \left( \frac{\sin(\frac{nt}{2})}{\sin \frac{t}{2}} \right)^2 dx, \quad \frac{1}{2n\pi} \int_{-\pi}^{\pi} \left( \frac{\sin(\frac{nt}{2})}{\sin \frac{t}{2}} \right)^2 dx = 1, \quad |f(x+t)| \leq \frac{\pi}{2}$$

$$\therefore |\sigma_n(x)| \leq \frac{\pi}{2} \quad \square$$

$$(2) \text{ Due to pointwise convergence, } \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \begin{cases} f(x), & 0 < x < 2\pi \\ 0, & x = 0 \end{cases}$$

$$\therefore \left| \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \right| < \frac{\pi}{2}, \quad 0 \leq x < 2\pi$$

$$\therefore \left| \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \right| \text{ is } 2\pi\text{-periodic}$$

$$\therefore \left| \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \right| < \frac{\pi}{2} + 1, \quad x \in \mathbb{R} \quad \square$$

## Lec 15

$$1 \quad (1) \quad a_n = \int_{-\pi}^{\pi} f(x) \cos(nx) dx = -\frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx = -\frac{b'_n}{n}$$

$$b_n = \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \cos(nx) dx = \frac{a'_n}{n}$$

(ignore discontinuous points of  $f'$ ) □

(2) Lec 13 Prob 04's conclusion can be extended as  $\sum_{n=1}^{\infty} \left| \frac{a'_n}{n} \right|$  and  $\sum_{n=1}^{\infty} \left| \frac{b'_n}{n} \right|$  converges, thus

the convergence of  $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$  is straightforward □

(3)  $\left| \frac{a_0}{2} \right| + \sum_{n=1}^{\infty} (|a_n| + |b_n|)$  converges  $\Rightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$  absolutely uniformly converges

$\therefore \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$  pointwise converges to  $f(x)$

$\therefore \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$  uniformly converges to  $f(x)$  □

2 (1) Due to symmetry,  $b_n = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos(nx) \, dx = \begin{cases} 0, & n = 1 \\ \frac{2}{\pi} \frac{(-1)^{n+1} - 1}{n^2 - 1}, & n \neq 1 \end{cases}$$

$$f(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2 - 1)} \cos 2nx$$

$\therefore$  the Fourier series is controlled by  $\sum_{n=2}^{\infty} \frac{4}{\pi(n^2 - 1)}$

$\therefore$  uniformly converges

$\therefore f(x)$  continuous

$\therefore$  the Fourier series pointwise thus uniformly converges to  $f(x)$

(2) Due to symmetry,  $a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(nx) \, dx = \frac{4[1 - (-1)^n]}{n^3 \pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi(2n-1)^3} \sin(2n-1)x$$

$\therefore$  the Fourier series is controlled by  $\sum_{n=1}^{\infty} \frac{8}{n^3 \pi}$

$\therefore$  uniformly converges

$\therefore f(x)$  continuous

$\therefore$  the Fourier series pointwise thus uniformly converges to  $f(x)$

3 Let  $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ ,  $f'(x) = \sum_{n=1}^{\infty} a'_n \cos(nx)$ ,  $f''(x) = \sum_{n=1}^{\infty} b''_n \sin(nx)$ ,  $f'''(x) =$

$$\sum_{n=1}^{\infty} a'''_n \cos(nx)$$

$$a'_n = \frac{2}{\pi} \int_0^{\pi} f'(x) \cos(nx) dx = \frac{2n}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = nb_n$$

$$b''_n = \frac{2}{\pi} \int_0^{\pi} f''(x) \sin(nx) dx = -\frac{2}{\pi} \int_0^{\pi} f'(x) \cos(nx) dx = -na'_n = -n^2 b_n$$

$$a'''_n = \frac{2}{\pi} \int_0^{\pi} f'''(x) \cos(nx) dx = \frac{2n}{\pi} \int_0^{\pi} f''(x) \sin(nx) dx = nb''_n$$

apply Lec 13 Prob 04's conclusion  $\sum_{n=1}^{\infty} \left| \frac{b''_n n}{n} \right|$  convergent

$$\therefore \sum_{n=1}^{\infty} |b_n|, \sum_{n=1}^{\infty} |a'_n|, \sum_{n=1}^{\infty} |b''_n| \text{ convergent}$$

$\therefore$  the Fourier series of  $f(x)$ ,  $f'(x)$ ,  $f''(x)$  are uniformly convergent relatively to the original function and their coefficients are results of term-by-term differentiation of  $f(x)$

$\therefore$  the Fourier series of  $f(x)$  is 2<sup>nd</sup> order termwise differentiable

$$\therefore a'_n = nb_n \text{ and } \sum_{n=1}^{\infty} a_n'^2 \text{ converges}$$

$$\therefore \sum_{n=1}^{\infty} n^2 b_n^2 \text{ converges}$$

□

## 8

### Lec 16

$$\begin{aligned} 1 \quad (1) \quad \frac{d}{dx} F(x) &= e^{x\sqrt{1-\cos^2 x}} (-\sin x) - e^{x\sqrt{1-\sin^2 x}} (\cos x) + \int_{\sin x}^{\cos x} \sqrt{1-y^2} e^{x\sqrt{1-y^2}} dy \\ &= \int_{\sin x}^{\cos x} \sqrt{1-y^2} e^{x\sqrt{1-y^2}} dy - e^{x|\sin x|} \sin x - e^{x|\cos x|} \cos x \end{aligned}$$

$$(2) \quad \frac{d}{dx} F(x) = \int_{x^2}^{x^2} f(x, s) ds + \int_0^x 2xf(t, x^2) dt = \int_0^x 2xf(t, x^2) dt$$

$$2 \quad F(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ix \cos \theta} (e^{ix \sin \theta} + e^{-ix \sin \theta})}{2} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{xe^{i\theta}} + e^{xe^{-i\theta}}}{2} d\theta$$

$$F'(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} \frac{e^{xe^{i\theta}}}{2} d\theta + \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} \frac{e^{xe^{-i\theta}}}{2} d\theta = 0$$

$$\therefore F(x) = \text{const } F(x) = F(0) = 1$$

$$\begin{aligned} 3 \quad I &= -\int_0^1 \sin(\ln x) \int_a^b x^y dy dx = -\int_a^b \int_0^1 \sin(\ln x) x^y dx dy \\ &= \int_a^b \frac{1}{(y+1)^2+1} dy = \arctan(b+1) - \arctan(a+1) \end{aligned}$$

$$\begin{aligned}
4 \quad F(x) &= \frac{1}{h^2} \int_0^h \left[ \int_0^h f(x + \xi + \eta) d\eta \right] d\xi = \frac{1}{h^2} \int_x^{x+h} \left[ \int_\xi^{\xi+h} f(\eta) d\eta \right] d\xi \\
F'(x) &= \frac{1}{h^2} \left[ \int_{x+h}^{x+2h} f(\eta) d\eta - \int_x^{x+h} f(\eta) d\eta \right] \\
F''(x) &= \frac{1}{h^2} \{f(x+2h) - f(x+h) - [f(x+h) - f(x)]\} \\
&= \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}
\end{aligned}$$

□

## Lec 17

- 1 (1)  $\because \left| \frac{\cos xy}{x^2+y^2} \right| \leq \frac{1}{a^2+y^2}, \int_0^\infty \frac{1}{a^2+y^2} dy$  converges  
 $\therefore$  uniformly converges
- (2)  $\because \int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$  uniformly converges,  $e^{-\alpha x}$  is about  $x$  monotonically decreasing  
and  $|e^{-\alpha x}| \leq 1$   
 $\therefore$  uniformly converges
- (3)  $\because f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & x > 0 \\ 0, & x = 0 \end{cases}$   
 $\therefore$  not uniformly converges
- (4)  $\int_0^1 \frac{1}{x^y} \sin \frac{1}{x} dx = \int_1^\infty x^{y-2} \sin x dx$   
apply second mean value theorem for definite integrals,  $\forall N \in \mathbb{N}^*, \exists y = 2 - \frac{1}{(2N+1)\pi}$   
 $\int_{2N\pi}^{(2N+1)\pi} x^{y-2} \sin x dx \leq 2[(2N+1)\pi]^{y-2} = 2[(2N+1)\pi]^{-\frac{1}{(2N+1)\pi}} > 1$   
 $\therefore$  not uniformly converges
- (5)  $\forall \epsilon > 0, \exists \delta < \frac{1}{4}\epsilon^2$  s.t.  $\int_{y-\delta}^y \frac{\sin xy}{\sqrt{y-x}} dx \leq \int_{y-\delta}^y \frac{1}{\sqrt{y-x}} dx = 2\sqrt{\delta} \leq \epsilon$   
 $\therefore$  uniformly converges
- (6)  $\int_0^1 x^{p-1} \ln^2 x dx = \int_{-\infty}^0 e^{px} x^2 dx$   
 $\because \forall N > 0, \exists p = \frac{1}{\sqrt[3]{N+1}}$  s.t.  $\int_{-\sqrt[3]{N+1}}^{-\frac{\sqrt[3]{N}}{\sqrt[3]{N+1}}} e^{px} x^2 dx > \frac{e^{-\frac{\sqrt[3]{N+1}p}}}{3} = \frac{e^{-1}}{3}$   
 $\therefore$  not uniformly converges
- 2  $\because |F(u)| \leq \int_{-\infty}^{+\infty} |f(x)| dx = A$   
 $\therefore F(u)$  is bounded  
 $\forall \epsilon > 0, \exists K(\epsilon) > \frac{1}{\epsilon}$ , s.t.  $\int_{-\infty}^{-K} |f(x)| dx + \int_K^{+\infty} |f(x)| dx < \frac{\epsilon}{3}$   
 $\exists \delta = \frac{1}{3K^2A}$  when  $|u_1 - u_2| < \delta$

$$\begin{aligned}
|F(u_2) - F(u_1)| &= \left| \int_{-\infty}^{+\infty} f(x)(\cos u_2 x - \cos u_1 x) dx \right| \\
&\leq \frac{2}{3}\epsilon + 2 \left| \int_{-K}^{+K} f(x) \sin \frac{u_2+u_1}{2} x \sin \frac{u_2-u_1}{2} x dx \right| \\
&\leq \frac{2}{3}\epsilon + \frac{1}{3KA} \int_{-K}^{+K} |f(x) \sin \frac{u_2+u_1}{2} x| dx \\
&\leq \epsilon
\end{aligned}$$

$\therefore$  uniformly continuous  $\square$

- 3  $\therefore \int_0^{+\infty} f(t) dt$  uniformly converges,  $e^{-xt}$  is about  $x$  monotonically decreasing and  $|e^{-xt}| \leq 1$  ( $x \geq 0$ )
- $\therefore \int_0^{+\infty} e^{-xt} f(t) dt$  uniformly converges
- $\therefore \lim_{x \rightarrow 0} \int_0^{+\infty} e^{-xt} f(t) dt = \int_0^{+\infty} f(t) dt$   $\square$

## 9

### Lec 18

- 1 (1)  $\int_0^{+\infty} \frac{\sin^4 x}{x^2} dx = - \int_0^{+\infty} \sin^4 x d\frac{1}{x} = \int_0^{+\infty} \frac{4 \sin^3 x \cos x}{x} dx = \int_0^{+\infty} \frac{2 \sin^2 x \sin 2x}{x} dx = \int_0^{+\infty} \frac{\sin 2x}{x} dx - \int_0^{+\infty} \frac{\sin 2x \cos 2x}{x} dx = \frac{\pi}{2} - \frac{1}{2} \int_0^{+\infty} \frac{\sin 4x}{x} dx = \frac{\pi}{4}$
- (3)  $\int_0^{+\infty} \left(\frac{\sin ax}{x}\right)^2 dx = - \int_0^{+\infty} \sin^2 ax d\frac{1}{x} = \int_0^{+\infty} \frac{2a \sin ax \cos ax}{x} dx = \int_0^{+\infty} \frac{a \sin 2ax}{x} dx = \frac{\pi|a|}{2}$
- (4)  $\frac{1-e^x}{x} = \int_0^1 e^{-xt} dt$
- $\therefore \int_0^{+\infty} \frac{1-e^x}{x} \cos x dx = \int_0^{+\infty} \int_0^1 e^{-xt} \cos x dt dx = \int_0^1 \int_0^{+\infty} e^{-xt} \cos x dx dt$
- using Lec 18 Example 03, let  $\alpha = t$ ,  $\beta = 1$
- $\therefore \int_0^1 \int_0^{+\infty} e^{-xt} \cos x dx dt = \frac{1}{2} \int_0^1 \frac{1}{t^2+1} dt^2 = \frac{\ln 2}{2}$
- 2 (1)  $I_n(a) = \int_0^{+\infty} \frac{dx}{(x^2+a^2)^n}$
- $\therefore I_n'(a) = -2anI_{n+1}(a)$
- $\therefore I_1(a) = \frac{\pi}{2a}$
- $\therefore$  assume that  $I_n(a) = \frac{\pi(2n-3)!!}{2^n(n-1)!a^{2n-1}}$  and prove it using mathematical induction
- (2)  $\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx = \int_0^{+\infty} \int_a^b x e^{-x^2 y} dy dx = \int_a^b \int_0^{+\infty} x e^{-x^2 y} dx dy = \int_a^b \frac{1}{2y} dy = \frac{1}{2} \ln \frac{b}{a}$
- (3)  $I_n(a) = \int_0^{+\infty} e^{-ax^2} x^{2n} dx$

$$\therefore I'_n(a) = -I_{n+1}(a)$$

$$\therefore I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\therefore I_n(a) = (-1)^n \left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)^{(n)} = \frac{\sqrt{\pi}(2n-1)!!}{2^{n+1}a^{n+\frac{1}{2}}}$$

$$(4) \quad I_n(a) = \int_0^1 x^{a-1} (\ln x)^n dx$$

$$\therefore I'_n(a) = I_{n+1}(a)$$

$$\therefore I_n(a) = [I_1(a)]^{(n-1)} = \left(-\frac{1}{a^2}\right)^{(n-1)} = \frac{(-1)^{n-1}n!}{a^{n+1}}$$

## Lec 19

$$1 \quad (1) \quad \int_0^1 \sqrt{x^3(1-\sqrt{x})} dx \quad (t = \sqrt{x}, x = t^2)$$

$$= 2 \int_0^1 t^4 (1-t)^{\frac{1}{2}} dt = 2\beta\left(5, \frac{3}{2}\right) = \frac{4!2^6}{11!!}$$

$$= \frac{512}{3465}$$

$$(3) \quad \int_0^{+\infty} \frac{dx}{1+x^4} = \int_0^{+\infty} \frac{dx}{1+x} = \int_0^{+\infty} \frac{1}{4} \frac{x^{-\frac{3}{4}}}{1+x} dx \quad (x = \frac{1}{1-t} - 1 = \frac{t}{1-t}, \quad t = 1 - \frac{1}{1+x} = \frac{x}{1+x})$$

$$= \frac{1}{4} \int_0^1 (1-t)^{-\frac{1}{4}} t^{-\frac{3}{4}} dt = \frac{1}{4} \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{\sqrt{2}\pi}{4}$$

$$(4) \quad \int_0^\pi \frac{d\theta}{\sqrt{3-\cos\theta}} \quad (x = \frac{1-\cos\theta}{2}, \quad \cos\theta = 1-2x)$$

$$= \int_0^1 \frac{dx}{\sqrt{2}\sqrt{x(1-x^2)}} \quad (t = x^2, \quad x = \sqrt{t})$$

$$= \frac{1}{2\sqrt{2}} \int_0^1 t^{-\frac{3}{4}} (1-t)^{-\frac{1}{2}} dt$$

$$= \frac{\sqrt{2}}{4} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$(5) \quad \int_0^{+\infty} \frac{x^{m-1} dx}{2+x^n}$$

$$= 2^{\frac{m}{n}-1} \int_0^{+\infty} \frac{x^{m-1} dx}{1+x^n} \quad (x = (t-1)^{\frac{1}{n}})$$

$$= \frac{2^{\frac{m}{n}-1}}{|n|} \int_0^1 \frac{\left(\frac{1-x}{x}\right)^{\frac{m}{n}-1} dx}{x} = \frac{2^{\frac{m}{n}-1}}{|n|} \beta\left(1 - \frac{m}{n}, \frac{m}{n}\right)$$

$$= \frac{2^{\frac{m}{n}-1}}{|n|} \frac{\pi}{\sin \frac{m}{n}\pi}$$

$$(6) \quad \int_0^{+\infty} \frac{\cosh 2qu}{(\cosh u)^{2p}} du \quad (e^u = t, \quad u = \ln t)$$

$$= 2^{2p-1} \int_1^{+\infty} \frac{\frac{1}{t}(t^{2q}+t^{-2q})}{t^{\frac{1}{2p}}(1+t^2)^{2p}} dt \quad (x = t^2, \quad t = \sqrt{x})$$

$$= 2^{2p-2} \int_1^{+\infty} \frac{x^{p-1}(x^q+x^{-q})}{(1+x)^{2p}} dx \quad (1+x = \frac{1}{t}, \quad t = \frac{1}{1+x})$$

$$\begin{aligned}
&= 2^{2p-2} \int_{\frac{1}{2}}^1 t^{2p} \left(\frac{1-t}{t}\right)^{p-1} \left[\left(\frac{1-t}{t}\right)^q + \left(\frac{1-t}{t}\right)^{-q}\right] \frac{1}{t^2} dt \quad (\text{the two integral terms are equal}) \\
&= 2^{2p-2} \beta(p-q, p+q) \\
(7) \quad &\int_{-1}^1 \frac{(1+x)^{2m-1} (1-x)^{2n-1}}{(1+x^2)^{m+n}} dx \quad (x = \tan \theta) \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\cos \theta + \sin \theta)^{2m} (\cos \theta - \sin \theta)^{2n}}{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+\sin 2\theta)^m (1-\sin 2\theta)^n}{\cos 2\theta} d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+\sin \theta)^m (1-\sin \theta)^n}{\cos \theta} d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+\sin \theta)^m (1-\sin \theta)^n}{\cos^2 \theta} d\sin \theta \\
&= \frac{1}{2} \int_{-1}^1 (1+x)^{m-1} (1-x)^{n-1} dx \quad (1+x = 2t, \quad t = \frac{1+x}{2}) \\
&= \int_0^1 (2t)^{m-1} [2(1-t)]^{n-1} dt \\
&= 2^{m+n-2} \beta(m, n)
\end{aligned}$$

$$2 \quad \beta(r, p) \beta(r+p, q) = \frac{\Gamma(r) \Gamma(p) \Gamma(r+p) \Gamma(q)}{\Gamma(r+p) \Gamma(r+p+q)} = \frac{\Gamma(r) \Gamma(q) \Gamma(r+q) \Gamma(p)}{\Gamma(r+q) \Gamma(r+p+q)} = \beta(r, q) \beta(r+q, p) \quad \square$$

$$\begin{aligned}
3 \quad &\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right)^{\cos 2\alpha} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[ \frac{\sin(\theta + \frac{\pi}{4})}{\cos(\theta + \frac{\pi}{4})} \right]^{\cos 2\alpha} d\theta = \int_0^{\frac{\pi}{2}} (\tan \theta)^{\cos 2\alpha} d\theta \\
&= \int_0^{\frac{\pi}{2}} (\sin t)^{\cos 2\alpha} (\cos t)^{-\cos 2\alpha} dt = \frac{1}{2} \beta\left(\frac{1+\cos 2\alpha}{2}, \frac{1-\cos 2\alpha}{2}\right) = \frac{1}{2} \beta(\cos^2 \alpha, \sin^2 \alpha) \\
&= \frac{\pi}{2 \sin(\pi \cos^2 \alpha)}
\end{aligned}$$

## 10

### P12

- 5
- $\because \|f\|'' \leq \|f\|, \|f\|' \leq (b-a)\|f\|$   
 $\therefore$  if in the sense of  $\| \| f_n \rightarrow f$ , then in the sense of  $\| \|', \| \|'' f_n \rightarrow f$
  - $\exists f_n \in C^k[a, b], f \in C^k[a, b]$ , in the sense of  $\| \| f_n \rightarrow f$ , but in the sense of  $\| \|', \| \|'' f_n \rightarrow f$
  - norm  $\| \|'$  is not equal to norm  $\| \|''$

### P18

- 4 (i)  $\because$  integral is linear

$\therefore I$  is linear

$$(ii) \text{ for } f_n(x) = \begin{cases} 1 & -n < x < n \\ x+n+1 & -n-1 \leq x \leq -n \\ n+1-x & n \leq x \leq n+1 \\ 0 & \text{other} \end{cases}$$



$$f_n \in C_c(R), \|f_n\| = 1, \lim_{n \rightarrow +\infty} I(f_n) = +\infty$$

$\therefore I$  is not bounded

## P26

5 suppose  $m = n = 1$  construct a discontinuous function  $y_i = \begin{cases} \ln|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$  and evidently the preimage of a compact set is also compact

$\therefore$  negative □

# 11

## P65

5 (i)  $|f(\mathbf{x}) - f(\mathbf{y})| = 0 \Rightarrow |\mathbf{x} - \mathbf{y}| = 0$

(ii)  $|\mathbf{D}f \cdot d\mathbf{x}| = |df(\mathbf{x})| \geq c|d\mathbf{x}|$

$\therefore$  the absolute values of  $\mathbf{D}f$ 's eigenvalues are not smaller than  $c$

(iii)  $\therefore |f(\mathbf{x}) - f(\mathbf{0})| \geq c|\mathbf{x}|$

$$\therefore \lim_{|\mathbf{x}| \rightarrow +\infty} |f(\mathbf{x})| = +\infty$$

$$\therefore \forall \boldsymbol{\xi} \in \mathbb{R}^n, \exists \mathbf{x}_0 \in \mathbb{R}^n \text{ s.t. } \left. d|f(\mathbf{x}) - \boldsymbol{\xi}|^2 \right|_{\mathbf{x}=\mathbf{x}_0} = 2[f(\mathbf{x}_0) - \boldsymbol{\xi}]^T \cdot \mathbf{D}f(\mathbf{x}_0) \cdot d\mathbf{x} = 0$$

$$\therefore \mathbf{D}f \neq 0$$

$$\therefore f(\mathbf{x}_0) = \boldsymbol{\xi}$$
□

## P79

3  $S$  is bounded

for  $\forall \varepsilon > 0$ , the accumulation points can be covered by a finite set of closed rectangles with total volume  $V_0 < \frac{\varepsilon}{2}$

$\therefore$  if there are infinite points left in  $S$ , there will be an arbitrarily small region containing infinite points in  $S$  outside previous rectangles, which contradicts to the fact that all accumulation points are already covered.

$\therefore$  the left points could be covered by another finite set of closed rectangles with total volume  $V_1 < \frac{\varepsilon}{2}$  □

**P88**

4 Negative

$f_n(x) = \frac{1}{x^2 + \frac{1}{n}}$  is integrable on  $[0, 1]$

$\lim_{n \rightarrow \infty} f_n(x) = \frac{1}{x^2} = f(x)$  is not integrable on  $[0, 1]$

□