Peking University

Answer Key 12

袁磊祺

December 25, 2019

P117

2 (1) 做 $f: R^2 \to M$

$$f(x,y) = (x, y, \sqrt{x^2+y^2})$$

则 f 为双射，且连续，f 的逆也连续，所以 f 是同胚。所以 M 是二维流形。

(2) 整个锥面不是二维流形。取 A 为满足 $0 \leqslant z < 1$,B 为满足 $-1 < z < 0$ 的点集。A，B 都是开集，且不相交。若存在从锥面到二维空间的同胚 f，则 $f(A), f(B)$ 为 R^2 中开集，且不相交，所以 $f(A) \bigcap \overline{f(B)} = \emptyset$, 但 $f(0) \in f(A)$ 且由于连续性，所以 $f(0) \in \overline{f(B)}$. 矛盾。

6 证：M 是闭的。设 $\overline{B_n}$ 为半径为 n 圆心在原点的闭球，$M_n = \overline{B_n} \bigcap M$, 则 M_n 为有限闭集，所以是紧的。对 $\forall x \in M_n$,, 由于 Df 的秩是 k，所以由隐函数定理，$\exists$开集$V_x \subseteq M$,开集$U_x \subseteq \Re^{n-k}$,使得$V_x \cong U_x$,V_x 构成 M_n 的一个开覆盖，由 M_n 的紧性，存在有限子覆盖仍记作 $\{V_x\}.G = \bigcup\limits_{n=1}^{\infty} \{V_x\}$ 为 M 的一个图册，所以 M 是 n-k 维流形。

P128

4 $\omega \wedge \cdot$ is a linear operation which maps 0 to 0

$\therefore$ Apparently M_ω is a linear subspace of V

$(\dim\{M_\omega\} < p)$: Let the first q vectors of $\{e_1 \cdots e_n\}$ be the complete orthonormal basis of M_ω

For $\omega = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant n} \omega_{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$ and $e_k \in \{e_1 \cdots e_q\}$

$\delta_{i_1 \cdots i_p k} = 0$ for $\forall i_1 \cdots i_p \in \{i_1 \cdots i_p | \omega_{i_1 \cdots i_p} \neq 0\}$

$\therefore$ k must be a common index of all non-zero $\omega_{i_1 \cdots i_p}$

$\therefore q \leqslant p$

$(\Rightarrow)$: Under the previous setting, let $q = p$

$\therefore \{1, 2, \cdots, p\}$ are all common indices of all non-zero $\omega_{i_1 \cdots i_p}$

$\therefore$ Only $\omega_{1 \cdots p} \neq 0 \Rightarrow \omega = \omega_{1 \cdots p} e_1 \wedge \cdots \wedge e_p$

$(\Leftarrow)$: Let $\omega = v_1 \wedge \cdots \wedge v_q$

Apparently $\{v_1 \cdots v_p\}$ are linearly independent

$\therefore \{v_1 \cdots v_p\}$ can be the basis of M_ω $\square$