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Answer Key 6

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November 24, 2019

Lec 12

3 $\because f'(x)$ monotonically increases in $[0, 2\pi]$

$$\begin{aligned}
\therefore a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \,\mathrm{d}x = -\frac{1}{n\pi}\int_0^{2\pi} f'(x)\sin nx\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\int_{\frac{2i}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x)\sin nx\,\mathrm{d}x \\
&= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\left[\int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi} f'(x)\sin nx\,\mathrm{d}x + \int_{\frac{2i+1}{n}\pi}^{\frac{2i+2}{n}\pi} f'(x)\sin nx\,\mathrm{d}x\right] \\
&= -\frac{1}{n\pi}\sum_{i=0}^{n-1}\int_{\frac{2i}{n}\pi}^{\frac{2i+1}{n}\pi}\left[f'(x) - f'(x+\frac{\pi}{n})\right]\sin nx\,\mathrm{d}x \geq 0
\end{aligned}$$

$\square$

Lec 13

1 $\lim\limits_{p\to\infty}\int_{-\pi}^{\pi} f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \lim\limits_{p\to\infty}\int_0^{\pi}[f(t)-f(-t)]\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_0^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\pi}[f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_\delta^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t - \lim\limits_{p\to\infty}\int_0^{\delta}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t$

$\because \frac{f(t)-f(-t)}{2\sin\frac{t}{2}} \in \mathcal{R}[\delta,\pi]$

$\therefore \lim\limits_{p\to\infty}\int_\delta^{\pi}[f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t = 0$

$\because f(t)$ is continuous and has unilateral derivative at $t=0$

$\therefore \exists \delta > 0,\ M > 0$ s.t. when $t \in (0,\delta)$, $-M < \frac{f(t)-f(-t)}{2\sin\frac{t}{2}}\cos pt < M$

$\therefore 0 = \lim\limits_{\delta\to 0} -M\delta \le \lim\limits_{\delta\to 0}\lim\limits_{p\to\infty}\int_0^\delta [f(t)-f(-t)]\frac{\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t \le \lim\limits_{\delta\to 0} M\delta = 0$

$\therefore \lim\limits_{p\to\infty}\int_{-\pi}^{\pi} f(t)\frac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}\,\mathrm{d}t = \frac{1}{2}\int_0^\pi [f(t)-f(-t)]\cot\frac{t}{2}\,\mathrm{d}t$ $\square$

2 (1) $f(x) \backsim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,\mathrm{d}x = \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,\mathrm{d}x = (-1)^n\frac{4}{n^2}$

$f(x) = \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

(2) $f(x) \backsim \sum\limits_{n=1}^{\infty} b_n \sin nx$

$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,\mathrm{d}x = -\frac{4}{\pi n^3} + \frac{2(-1)^n}{\pi}\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)$

$f(x) = \sum\limits_{n=1}^{\infty}\left[\frac{4(-1)^n}{\pi n^3} - \frac{4}{\pi n^3} - \frac{2\pi(-1)^n}{n}\right]\sin nx$

(3) $f(x) \backsim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos nx + b_n \sin nx]$

$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,\mathrm{d}x = \frac{8\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,\mathrm{d}x = \frac{4}{n^2}$

$b_n = \frac{1}{\pi}\int_0^\pi f(x)\sin nx\,\mathrm{d}x = -\frac{4\pi}{n}$

$f(x) = \frac{4\pi^2}{3} + \sum\limits_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$

(4) from (1) $f(x) = \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

$f(\pi) = \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \pi^2$

$\sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

3 $a_0 = \frac{\mathrm{e}^\pi - \mathrm{e}^{-\pi}}{\pi}$

$a_n = \frac{(-1)^n(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})}{\pi(n^2+1)}$

$b_n = -\frac{(-1)^n n(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})}{\pi(n^2+1)}$

$f(x) = \frac{\mathrm{e}^{\pi}-\mathrm{e}^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}(\cos nx - n\sin nx)\right]$

if $x = \pi$, $\frac{\mathrm{e}^{\pi}-\mathrm{e}^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}\right) = \frac{1}{2}(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi})$

$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi(\mathrm{e}^{\pi}+\mathrm{e}^{-\pi})}{2(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})} - \frac{1}{2}$

4 $\because f(x) \in \mathcal{R}$, apply bessel's inequality

$\therefore \sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ convergent

$\because \frac{|a_n|}{n} \le \frac{1}{2}(a_n^2 + \frac{1}{n^2})$ and $\frac{|b_n|}{n} \le \frac{1}{2}(b_n^2 + \frac{1}{n^2})$

$\therefore \sum_{n=1}^{\infty} \frac{a_n}{n}$ and $\sum_{n=1}^{\infty} \frac{b_n}{n}$ convergent $\square$