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Answer Key 2

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Lec 03

1 (2) $\ln(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + o(\frac{1}{n^4})$

$\therefore a_n = \frac{1}{12n^2} + o(\frac{1}{n^2})$

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{\frac{1}{12n^2}} = 1$

$\therefore$ absolutely convergent

(3) $\ln[(1+\frac{1}{n})^n] = n\ln(1+\frac{1}{n}) = 1 - \frac{1}{2n} + o(\frac{1}{n})$

$\therefore (1+\frac{1}{n})^n = e^{1-\frac{1}{2n}+o(\frac{1}{n})} = e - \frac{e}{2n} + o(\frac{1}{n})$

$\therefore \lim\limits_{n\to\infty} \frac{a_n}{(\frac{e}{2n})^p} = 1$

$$\begin{cases} p > 1 : \text{absolutely convergent} \\ p \leqslant 1 : \text{divergent} \end{cases}$$

2 (1) $\lim\limits_{n\to\infty} \sqrt[n]{(1-\frac{1}{n})^{n^2}} = \lim\limits_{n\to\infty} (1-\frac{1}{n})^n = \frac{1}{e} < 1$

$\therefore$ absolutely convergent

(2) $\frac{a_{n+1}}{a_n} = \frac{x}{1+x^n}$

$$\begin{cases} x<1: \lim\limits_{n\to\infty} \frac{x}{1+x^n} = x<1, \text{absolutely convergent} \\ x=1: \lim\limits_{n\to\infty} \frac{x}{1+x^n} = \frac{1}{2}, \text{absolutely convergent} \\ x>1: \lim\limits_{n\to\infty} \frac{x}{1+x^n} = 0, \text{absolutely convergent} \end{cases}$$

3 (1) $\int_2^\infty 3^{-x^{\frac{1}{2}}}\,\mathrm{d}x = \int_{\sqrt{2}}^\infty 2t\cdot 3^{-t}\,\mathrm{d}t$

$\therefore$ absolutely convergent

(2) $\frac{1}{a^{\ln x}} = \frac{1}{x^{\ln a}}$

$$\begin{cases} a>e: \ln a > 1, \text{absolutely convergent} \\ a\leqslant e: \ln a \leqslant 1, \text{divergent} \end{cases}$$

4 (1) $\lim\limits_{n\to\infty} a_n = \frac{1}{2}$

$\therefore$ divergent

(4) $\lim\limits_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = 1$

$\therefore$ divergent

(5) For $n > 1000, \ln(n+1) > 2$

$\therefore \frac{1}{[\ln(n+1)]^n} < \frac{1}{2^n}$

$\therefore$ absolutely convergent

6 (3) $\int \frac{1}{x\ln x(\ln\ln x)}\,\mathrm{d}x = \ln\ln\ln x + C$

$\therefore$ for $\sigma \leqslant 0$, divergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}} \leqslant 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n(\ln n)^{1+\sigma}}$$
$$\leqslant 10 + \sum_{k=2}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{2^{k-1}[\ln(2^{k-1})]^{1+\sigma}}$$
$$\leqslant 10 + \sum_{k=2}^{\infty} \frac{1}{[\ln(2^{k-1})]^{1+\sigma}}$$
$$= 10 + \sum_{k=2}^{\infty} \frac{1}{(k-1)^{1+\sigma}}$$

$\therefore$ for $\sigma > 0$, $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+\sigma}}$ absolutely convergent

Also $\because$ for $\forall p > 0, \exists X > 0, \forall x > X, (\ln x)^p > \ln \ln x$

$\therefore$ for $\sigma > 0$, absolutely convergent

(4) Let $p = 1$, $\int_2^{\infty} \frac{1}{x \ln x (\ln \ln x)^q} \,\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{\mathrm{d}t}{t (\ln t)^q}$

$\therefore$ similar to the condition in previous problem

In conclusion:

$$\begin{cases} p > 1 : \text{absolutely convergent} \\ p = 1 : \begin{cases} q > 1 : \text{absolutely convergent} \\ q \leqslant 1 : \text{divergent} \end{cases} \\ p < 1 : \text{divergent} \end{cases}$$

Lec 04

2 (1) $(k^2 - 1) a_{k^2 - 1} = \frac{1}{k^2 - 1}, \ k^2 a_{k^2} = 1$

$\exists \varepsilon = \frac{1}{2}, \forall N > 0, \exists k > N, \left| 1 - \frac{1}{k^2 - 1} \right| > \varepsilon$

$\therefore \lim\limits_{n \to \infty} a_n$ doesn't exist $\square$

(2) Let $b_k = \left[\frac{1}{k^2} + \sum\limits_{n = k^2 + 1}^{(k+1)^2 - 1} \frac{1}{n^2} \right]$

Evidently b_k is absolutely convergent

Use conclusion of Lec 02 Problem 05

$\therefore$ absolutely convergent $\square$

3 (1) Evidently $\lim\limits_{n \to \infty} x_n$ exists. Let $x_n \to A$

$\therefore \lim\limits_{n \to \infty} \frac{1 - \frac{x_n}{x_{n+1}}}{\frac{x_{n+1} - x_n}{A}} = 1$

$\because \sum\limits_{n=1}^{\infty} \frac{x_{n+1} - x_n}{A} = 1 - \frac{x_1}{A}$, converge

$\therefore$ absolutely convergent $\square$

(2) $\exists \varepsilon = \frac{1}{2}, \forall N > 0, \ \exists n_1, \ n_2 > N, \ \sum\limits_{n = n_1}^{n_2} (1 - \frac{x_n}{x_{n+1}}) > \frac{x_{n_2} - x_{n_1}}{x_{n_2}} > \varepsilon$

$\therefore$ divergent $\square$

4 (1) Let $b_k = a_{3k+1} + a_{3k+2} + a_{3k+3}$

$|b_k|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{b_k}{b_{k-1}}) = -1$

Use the conclusion of Lec 04 Problem 05, a_n converges

$\because |a_n| = \frac{1}{n}$

$\therefore$ conditionally convergent

(2) For $a \neq 0$, $n\sqrt{1 + \frac{a^2}{n^2}} = n + \frac{a^2}{2n} + o(\frac{1}{n})$

$\therefore a_n = (-1)^n \sin(\frac{\pi a^2}{2n}) + o(\frac{1}{n})$

$|a_n|$ monotonically decreases to 0 and $\operatorname{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because \lim\limits_{n\to\infty} \frac{\sin\frac{\pi a^2}{2n}}{\frac{1}{100n}} > 1$

$\therefore$ In conclusion:

$$\begin{cases} a \neq 0 : \text{conditionally convergent} \\ a = 0 : \text{absolutely convergent} \end{cases}$$

(3) $\ln\left(1 + \frac{(-1)^n}{n^p}\right) = \frac{(-1)^n}{n^p} - \frac{1}{n^{2p}} + o(\frac{1}{n^{2p}})$

For $p > 0$, $\left|\frac{(-1)^n}{n^p}\right|$ monotonically decreases to 0 and $\operatorname{sgn}\left(\frac{\frac{(-1)^n}{n^p}}{\frac{(-1)^{n-1}}{(n-1)^p}}\right) = -1$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges when $p > \frac{1}{2}$, diverges when $p \leqslant \frac{1}{2}$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$, diverges when $p \leqslant 1$

For $p \leqslant 0$, evidently diverge

$\therefore$ In conclusion:

$$\begin{cases} p \leqslant \frac{1}{2} : \text{divergent} \\ \frac{1}{2} < p \leqslant 1 : \text{conditionally convergent} \\ p > 1 : \text{absolutely convergent} \end{cases}$$

(4) Let $b_k = |a_{2k-1}| + |a_{2k}|$, $0 < b_k < \frac{1}{2^{k-1}}$

$\therefore b_k$ converges

Use conclusion of Lec 02 Problem 05, $|a_n|$ converges

$\therefore$ absolutely convergent

(5) $\sum\limits_{n=1}^{2N} a_n < e - \sum\limits_{n=1}^{N} \frac{1}{2n}$

$\because -\sum\limits_{n=1}^{N} \frac{1}{2n} \to -\infty$

$\therefore$ divergent

(6) $\because |a_n|$ monotonically decreases to 0 and $\mathrm{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because |a_n| > \frac{1}{n}$

$\therefore$ conditionally convergent

(7) $\because \int_2^\infty x^3 2^{-x}\,\mathrm{d}x$ converges

$\therefore$ absolutely convergent

(8) $\because |a_n|$ monotonically decreases to 0 and $\mathrm{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because |a_n| > \frac{1}{20n}$

$\therefore$ conditionally convergent

(9) $\because |a_n|$ monotonically decreases to 0 and $\mathrm{sgn}(\frac{a_n}{a_{n-1}}) = -1$

$\therefore a_n$ converges

$\because \lim\limits_{n\to\infty} \frac{a_n}{\frac{1}{n}} = x$

$\therefore$ In conclusion:

$$\begin{cases} x \neq 0 : \text{conditionally convergent} \\ x = 0 : \text{absolutely convergent} \end{cases}$$

(10) Let $b_k = a_{2k-1} + a_{2k} = \frac{2}{k-1}$

$\therefore$ divergent