(Q1b)

Given :

$$H_\delta(a) = \begin{cases} \frac{1}{2}a^2 & \text{if } |a| \leq \delta \\ \delta(|a| - \frac{1}{2}\delta) & \text{if } |a| > \delta \end{cases}$$

$\frac{1}{2}$

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1) When $|a| \leq \delta$: $H_\delta(a) = \frac{1}{2}a^2$

$$H'_\delta(a) = \cancel{2}\cdot\cancel{\tfrac{1}{2}}\, a = a \quad 1$$

$\frac{1}{2}\delta$

2) When a is +ve & $a > \delta$:

$$H_\delta(a) = \delta(|a| - \tfrac{1}{2}\delta) = \delta|a| - \delta^2$$

$$H'_\delta(a) = \delta \quad 2$$

3) When a is -ve & $a < -\delta$:

$$H_\delta(a) = \delta(|a| - \tfrac{1}{2}\delta) = \delta|a| - \ ^2$$

$$H'_\delta(a) = -\delta \quad 3$$

Thus $H'_\delta(a) = \begin{cases} a & \text{if } |a| \leq \delta \\ \delta & \text{if } a > \delta \\ -\delta & \text{if } a < -\delta \end{cases}$

substitute when $a = y - t$.

$$H'_\delta(y-t) = \begin{cases} y-t & \text{if } |y-t| \leq \delta \\ \delta & \text{if } (y-t) > \delta \\ -\delta & \text{if } (y-t) < -\delta \end{cases} \quad (4)$$

$$\frac{\partial L_\delta}{\partial w} = \frac{\partial H_\delta(y-t)}{\partial w} = \frac{\partial H_\delta(y-t)}{\partial (y-t)} \cdot \frac{\partial (y-t)}{\partial w}$$

$$\frac{\partial L_\delta}{\partial w} = \underbrace{H'_\delta(y-t)}_{\text{sub in } (4)} \cdot \frac{\partial (w^T x + \cancel{b} - \cancel{t})}{\partial w}$$

$$\boxed{\text{Answer: } \frac{\partial L_\delta}{\partial w} = H'_\delta(y-t) \cdot x \qquad \frac{\partial L_\delta}{\partial w} = \begin{cases} (y-t)x & \text{if } |y-t| \leq \delta \\ \delta x & \text{if } (y-t) > \delta \\ -\delta x & \text{if } (y-t) < -\delta \end{cases}}$$

$$\frac{\partial L_\delta}{\partial b} = \frac{\partial H_\delta(y-t)}{\partial b} = \frac{\partial H_\delta(y-t)}{\partial (y-t)} \cdot \frac{\partial (y-t)}{\partial b}$$

$$= H'_\delta(y-t) \cdot \frac{\partial (w^T x + \cancel{b} - \cancel{t})}{\cancel{\partial b}}$$

Answer: $\frac{\partial L_\delta}{\partial b} = H'_\delta(y-t)$

$$\frac{\partial L_\delta}{\partial b} = \begin{cases} (y-t) & \text{if } |y-t| \leq \delta \\ \delta & \text{if } (y-t) > \delta \\ -\delta & \text{if } (y-t) < -\delta \end{cases}$$

(Q2a) Let $\begin{cases} X, Y \text{ be the vectors of } \Sigma x^i, \Sigma y^i \\ A \text{ be the vector of } \Sigma a^i; \end{cases}$

$$J = \frac{1}{2}\sum_{i=1}^{N} a^{(i)}(y^i - w^T x^i)^2 + \frac{\lambda}{2}\|w\|^2$$

Convert to matrix form: $J = \frac{1}{2}(Y - Xw)^T A (Y - Xw) + \frac{\lambda}{2}\|w\|^2$

using rule $\|A\| = A \cdot A^T$: $J = \underbrace{\frac{1}{2}(Y - Xw)^T A (Y - Xw)}_{1} + \underbrace{\frac{\lambda}{2} w^T w}_{2}$

Note: We can find w^* by finding w^* where $\frac{\partial J}{\partial w} = 0$

$$\frac{\partial J}{\partial w} = \frac{\partial J}{\partial (Y - Xw)} \cdot \frac{\partial (Y - Xw)}{\partial w}$$ applied to 1 above

$$\frac{\partial J}{\partial w} = \frac{\partial \left[\frac{1}{2}(Y - Xw)^T A (Y - Xw)\right]}{\partial (Y - Xw)} \cdot \frac{\partial (Y - Xw)}{\partial w} + \frac{\partial (\frac{\lambda}{2} w^T w)}{\partial w}$$

$$\frac{\partial J}{\partial w} = (Y - Xw)^T A \cdot (-X) + \lambda w^T$$

set $\frac{\partial J}{\partial w}$ to 0 to find w^*:

$$(Y - Xw)^T A (-X) + \lambda w^T = 0$$

$$-Y^T A X + w^T X^T A X + \lambda w^T = 0$$

$$-Y^T A X + W^T(X^T A X + \lambda I) = 0$$ transpose all

since A is diagnol matrix $\therefore A^T = A$

$$\rightarrow (X^T A X + \lambda I)W - X^T A Y = 0$$

$$(X^T A X + \lambda I)W = X^T A Y$$

find solution of W: $\boxed{W^* = (X^T A X + \lambda I)^{-1} X^T A Y}$

Question 3)

Given for t^{th} iteration :

$$h_t \leftarrow \underset{h \in H}{\operatorname{argmin}} \sum_{i=1}^{N} w_i \mathbb{I}\{h(x^i) \neq t^i\},$$

2 sets of i

$$E = \{i : h_t(x^i) \neq t^i\} \text{ and } \mathbb{I} = 1$$
$$E^c = \{i : h_t(x^i) = t^i\} \text{ and } \mathbb{I} = -1$$

condition (1) from tips.

$$err_t = \frac{\sum_{i=1}^{N} w_i \mathbb{I}\{h_t x^i \neq t^i\}}{\sum_{i=1}^{N} w_i}$$

$$err'_t = \frac{\sum_{i=1}^{N} w'_i \mathbb{I}\{h_t x^i \neq t^i\}}{\sum_{i=1}^{N} w'_i}$$

$\leftarrow$ substitute (1) above, bottom split to 2 sets $i \in E^c$ & $i \in E$

$$err'_t = \frac{\sum_{i=1}^{N} w'_i \mathbb{I}\{h_t x^i \neq t^i\}}{\sum_{i \in E} w'_i + \sum_{i \in E^c} w'_i}$$

$\leftarrow$ top belong to $i \in E$ according to condition (1)

$$err'_t = \frac{\sum_{i \in E} w'_i \cdot (1)}{\sum_{i \in E} w'_i + \sum_{i \in E^c} w'_i}$$

substitute in $w'_i \leftarrow w_i \exp[-\alpha_t \cdot t^i \cdot h_t(x^i)]$

when $h_t(x^i) \neq t^i$, product of $t^i \cdot h_t(x^i)$ is $-$ve

when $h_t(x^i) = t^i$, product of t^i $h_t(x^i)$ is $+$ve after substitution :

$$err'_t = \frac{\sum_{i \in E} W_i \exp[-\alpha_t \cdot (-1)]}{\sum_{i \in E} W_i \exp[-\alpha_t \cdot (-1)] + \sum_{i \in E^c} W_i \exp[-\alpha_t (1)]}$$

$$err'_t = \frac{\sum_{i \in E} W_i \exp(\alpha_t)}{\sum_{i \in E} W_i \exp(\alpha_t) + \sum_{i \in E^c} W_i \exp(-\alpha_t)} \cdot \frac{\exp(\alpha_t)}{\exp(\alpha_t)}$$

Multiply top & bottom of err'_t by $\exp(\alpha_t)$

$$err'_t = \frac{\sum_{i \in E} W_i \exp(2\alpha_t) \quad 2}{\underbrace{\sum_{i \in E} W_i \exp(2\alpha_t)}_{2} + \underbrace{\sum_{i \in E^c} W_i}_{3}}$$

Note that given $\alpha_t = \frac{1}{2} \log \frac{1-err_t}{err_t}$

$$2\alpha_t = \log \frac{1-err_t}{err_t}$$

$$\exp(2\alpha_t) = \frac{1-err_t}{err_t}$$

2 : $\sum_{i \in E} W_i \exp(2\alpha_t) = \sum_{i \in E} W_i \left(\frac{1-err_t}{err_t}\right)$ 4

From tip given: $\frac{\sum_{i \in E} W_i}{\sum_{i=1}^{N} W_i} = err_t$

$$\therefore \frac{1-err_t}{err_t} = \frac{1-\frac{\sum_{i \in E} W_i}{\sum_{i=1}^{N} W_i}}{\frac{\sum_{i \in E} W_i}{\sum_{i=1}^{N} W_i}} = \frac{\left(\sum_{i=1}^{N} W_i - \sum_{i \in E} W_i\right) / \cancel{\sum_{i=1}^{N} W_i}}{\left(\sum_{i \in E} W_i\right) / \cancel{\sum_{i=1}^{N} W_i}}$$ cancel out

also note that top part: $\sum_{i=1}^{N} w_i - \sum_{i \in E} w_i = \sum_{i \in E^c} w_i$

So: $\dfrac{1-err_t}{err_t} = \dfrac{\sum_{i \in E_c} w_i}{\sum_{i \in E} w_i}$, substitute into 4

4 is:

$$-\sum_{i \in E} w_i \left(\frac{1-err_t}{err_t}\right) = -\cancel{\sum_{i \in E} w_i} \frac{\sum_{i \in E_c} w_i}{\cancel{\sum_{i \in E} w_i}} = \sum_{i \in E_c} w_i$$

(same as 3)

So

2 = 4 = 3 ∴ 2 = 3

so $\underbrace{\sum_{i \in E} w_i \exp(2\alpha_t)}_{2} = \underbrace{\sum_{i \in E^c} w_i}_{3}$

recall above

$$err'_t = \frac{\sum_{i \in E} w_i \exp(2\alpha_t) \quad 2}{\underbrace{\sum_{i \in E} w_i \exp(2\alpha_t)}_{2} + \underbrace{\sum_{i \in E^c} w_i}_{3}}$$

Thus: $err'_t = \dfrac{\sum_{i \in E_c} w_i}{\sum_{i \in E_c} w_i + \sum_{i \in E_c} w_i} = \dfrac{1}{2}$

Thus $\boxed{err'_t = \dfrac{\sum_{i=1}^{N} w'_i \mathbb{I}\{h_t x^i \neq t^i\}}{\sum_{i=1}^{N} w'_i} = \dfrac{1}{2}}$