Question 1 a)

$$E[Z] = E[(X-Y)^2] = E[X^2+Y^2+2XY] = E[X^2]+E[Y^2]-E[2XY] \quad (1)$$

$$E[Y] = E[X] = \int_0^1 x\,dx = \tfrac{1}{2}x^2\Big]_0^1 = \tfrac{1}{2} \quad (2)$$

$$E[Y^2] = E[X^2] = \int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{3} \quad (3)$$

$$E[Y^3] = E[X^3] = \int_0^1 x^3\,dx = \tfrac{1}{4}x^4\Big]_0^1 = \tfrac{1}{4} \quad (4)$$

$$E[Y^4] = E[X^4] = \int_0^1 x^4\,dx = \tfrac{1}{5}x^5\Big]_0^1 = \tfrac{1}{5} \quad (5)$$

Subsitute (2) (3) (4) (5) into (1)

$$E[Z] = E[X^2]+E[Y^2]-E[2XY]$$

$$E[Z] = E[X^2]+E[Y^2]-2E[X]E[Y]$$

$$E[Z] = \tfrac{1}{3}+\tfrac{1}{3}-2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$\boxed{E[Z] = \tfrac{1}{6}}$$

$$Var[Z] = E[Z^2]-E[Z]^2 \quad (1)$$

$$Z^2 = (X-Y)^4 = (X^2+Y^2-2XY)(X^2+Y^2-2XY) = X^4+6X^2Y^2-4X^3Y-4XY^3+Y^4 \quad (2)$$

substitute (2) into (1)

$$Var[Z] = E[X^4]+6E[X^2]E[Y^2]-4E[X^3]E[Y]-4E[X]E[Y^3]+E[Y^4] - E[Z]^2$$

$$Var[Z] = 2\times\tfrac{1}{5}+6\times\tfrac{1}{3}\times\tfrac{1}{3}-8\,\tfrac{1}{4}\times\tfrac{1}{2}-\tfrac{1}{36}$$

$$\boxed{Var[Z] = \tfrac{7}{180}}$$

Answer: The expection is $\frac{1}{6}$ and variance is $\frac{7}{180}$

Question 1b)

$$\because R = Z_1 + Z_2 + \cdots + Z_d = \sum_1^d Z_i$$

$$Z_i = (X_i - Y_i)^2$$

$$\therefore E[R] = \sum_1^d E[Z_i] = \boxed{d\,E[Z]} = \boxed{\frac{1}{6}d}$$

$$\therefore Var[R] = \sum_1^d Var[Z_i] = \boxed{d\,Var[Z]} = \boxed{\frac{7}{180}d}$$

Answer: The expectation is $dE[Z_i]$ which is $\frac{1}{6}d$.

The variance is $d\,Var[Z]$ which is $\frac{7}{180}d$.

Question 3a)

$$H(X) = \sum_x P(X) \log_2\left(\frac{1}{P(X)}\right) \quad (1)$$

Find limit: $\lim_{P(X) \to 0} P(X) \log_2\left(\frac{1}{P(X)}\right) = 0$

$$\therefore \sum_X P(X) \log_2\left(\frac{1}{P(X)}\right) \geq 0$$

$$\therefore H(X) \geq 0$$

QED

Question 3b)

$$KL(P||q) = \sum_x p(x)\log_2 \frac{p(x)}{q(x)} = -\sum_x p(x)\log_2 \frac{q(x)}{p(x)} = E\left[-\log_2 \frac{q(x)}{p(x)}\right] \quad ①$$

$\because \phi(E[x]) \leq E[\phi(x)]$ from Jensen's inequality

$$\therefore -\log_2 E\left[\frac{q(x)}{p(x)}\right] \leq E\left[-\log_2 \frac{q(x)}{p(x)}\right] \quad ②$$

$$-\log_2 E\left[\frac{q(x)}{p(x)}\right] = -\log_2 \sum_x \cancel{p(x)} \frac{q(x)}{\cancel{p(x)}} = -\log_2 \sum_x q(x) = -\log_2 1 = 0 \quad ③$$

substitute ③ into ②

$$E\left[-\log_2 \frac{q(x)}{p(x)}\right] \geq -\log_2 E\left[\frac{q(x)}{p(x)}\right]$$

$$E\left[-\log_2 \frac{q(x)}{p(x)}\right] \geq 0$$ substitute into ①

$$KL(P||q) = E\left[-\log_2 \frac{q(x)}{p(x)}\right]$$

$$\therefore KL(P||q) \geq 0$$

Question 3C)

(pg 3)

$$KL[P(x,y) \,||\, P(x)P(y)] \quad ①$$

$$= \sum_{x,y} P(x,y)\log\left[\frac{P(x,y)}{P(x)P(y)}\right]$$

$$= \sum_{x,y} P(x,y)\log\left[\frac{\cancel{P(x)}\,P(y|x)}{\cancel{P(x)}\,P(y)}\right]$$

$$= \sum_{x,y} P(x,y)\left[\log P(y|x) - \log P(y)\right]$$

$$= -\sum_y \log p(y) \sum_x p(x,y) + \sum_{x,y} P(x,y)\log p(y|x)$$

$$= -\sum_y \log P(y)\,P(y) - \sum_x P(x)\sum_y p(y|x)\log(y|x)$$

$$= H(y) - \sum_{x \in X} P(x)\,H(Y|X=x) = H(y) - H(y|X) = I(Y;x) \text{ by definition } ②$$

$\because$ ① = ②

$KL[P(x,y) || P(x)P(y)]$

$\therefore = I(Y;X)$