Compressed Sensing for Image Compression

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Problem Formulation

1. Image ($\boldsymbol{y}$) is $\boldsymbol{k}$-sparse in some basis ($\boldsymbol{\Psi}$)
2. Linear measurement model, $\boldsymbol{A}$
3. Minimize measurement error, penalizing cardinality

$$y = \Psi x, ||x||_0 \leq k$$

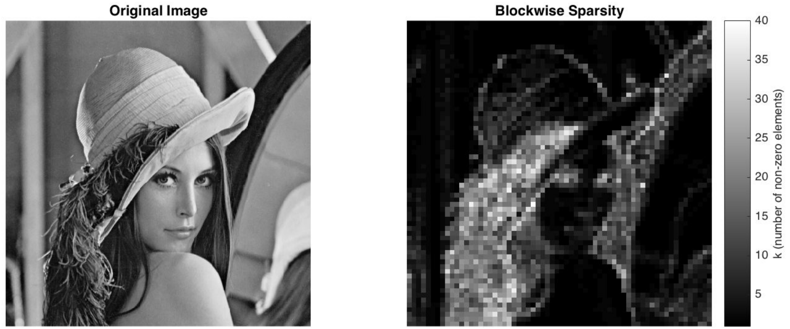
$$m = Ay = A\Psi x$$

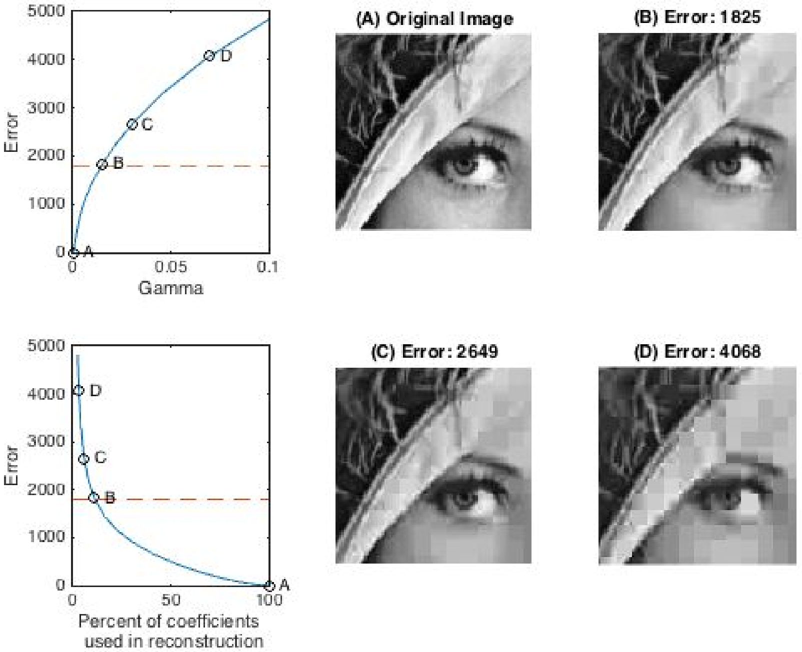
$$x^* = \arg\min_x ||A\Psi x - m||_2^2 + \alpha ||x||_0$$

A CS approach to JPEG

1. Image is 8x8 blockwise sparse according to local texture
2. Reformulate as cardinality *constraint*
3. Optimal algorithm:
   a. Break image into 8x8 blocks.
   b. For each block, compute DCT and zero out coefficients below a certain threshold.

In fact, this algorithm is provably optimal for *any* orthogonal decomposition.

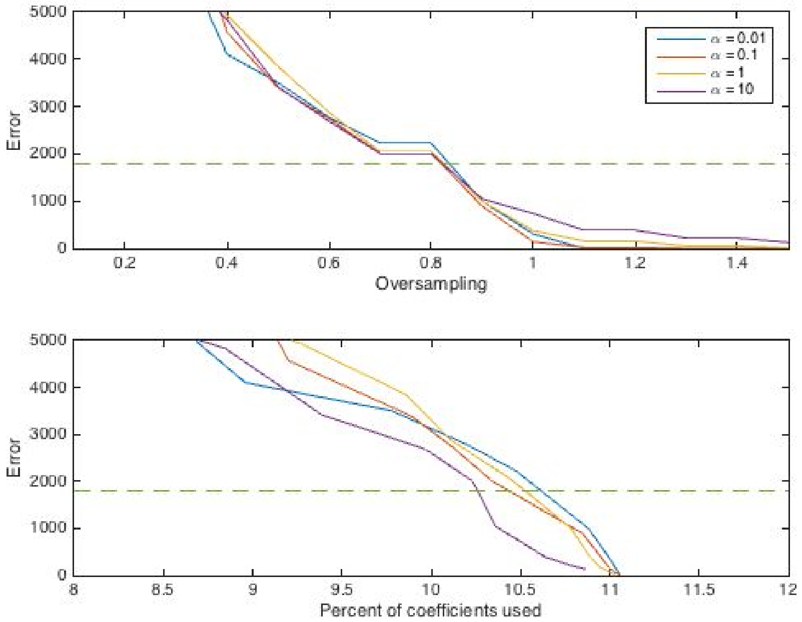




L1 Relaxation

$$x^* = \arg\min_x \|A\Psi x - m\|_2^2 + \alpha\|x\|_1$$

- $\boldsymbol{\Psi}$ = DCT basis; i.e. $\boldsymbol{\Psi x}$ = idct2($\boldsymbol{x}$)
- A = matrix of IID standard Gaussian random variables



CS cannot *overcompress*, but it can *undersample*.

What does undersampling look like?

Slight undersampling
(OSR = 0.9)

Severe undersampling,
weak sparsity penalization
(OSR = 0.1, α = 0.01)

Severe undersampling,
strong sparsity penalization
(OSR = 0.1, α = 10)

Theoretically….

Should be able to undersample by roughly three times the actual sparsity and still achieve faithful reconstruction[1].

@ OSR = 0.3, α = 0.01

- error ~ 6500 > 1800
- still completely recognizable!

[1]Stern, A., Rivenson, Y., and Javidi, B. "Optically compressed sensing using random aperture encoding" in Proc. of SPIE, 2008.

Reversed Huber Relaxation

$$x^* = \arg\min_x \|A\Psi x - m\|_2^2 + \rho^2\|x\|_2^2 + \alpha\|x\|_0$$
$$\approx \arg\min_x \|A\Psi x - m\|_2^2 + 2\alpha \sum_{i=1}^{n} B\left(\frac{\rho x_i}{\sqrt{\alpha}}\right)$$

- $\boldsymbol{\Psi}$ = DCT basis; i.e. $\boldsymbol{\Psi x}$ = idct2($\boldsymbol{x}$)
- $\boldsymbol{A}$ = matrix of IID standard Gaussian random variables
- $\boldsymbol{B}$ = “reversed Huber” function, `berhu`

$$B(\xi) \doteq \frac{1}{2}\min_{z\in[0,1]}\left(z + \frac{\xi^2}{z}\right) = \begin{cases} |\xi| & \text{if } |\xi| \le 1 \\ \frac{\xi^2+1}{2} & \text{otherwise} \end{cases}$$

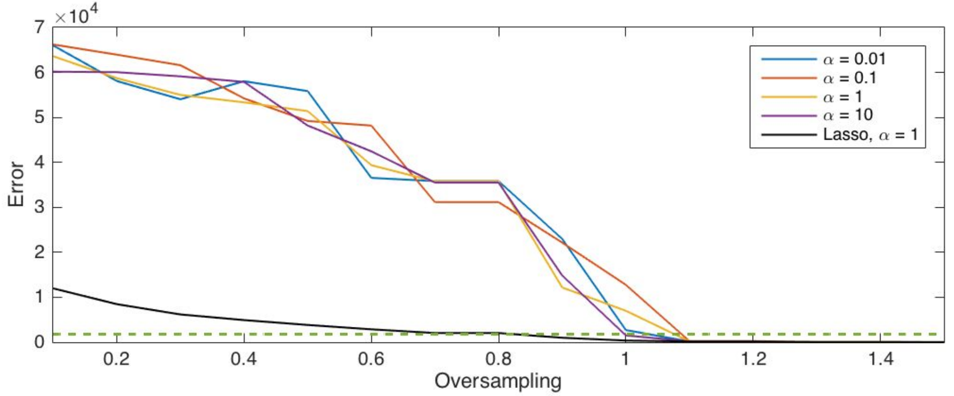
Oops…. Doesn't really work.

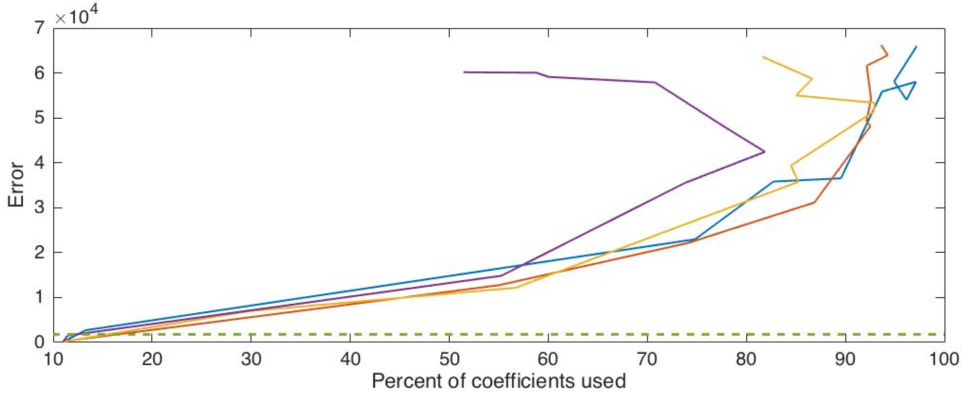
Problems:

- Very sensitive to undersampling
- Not getting more sparse as we undersample

Possible causes:

- Numerical errors
- Reversed Huber may simply not enforce sparsity as well as L1





Application: Image Denoising

- Additive Gaussian noise on the *measurements*
- Equivalent to corrupting blocks directly

$$m = Ay + \eta, \eta_i \sim \mathcal{N}(0, \sigma)$$

$$\begin{aligned} \hat{y} &= A^{-1}m \\ &= y + A^{-1}\eta \end{aligned}$$

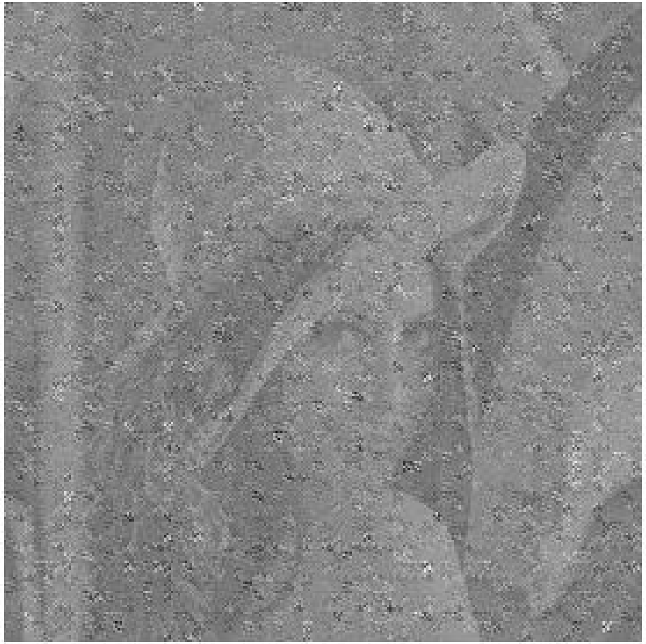
How robust is CS to such noise?

Intuition: noise is not sparse, and we observe each pixel many times.

Success!

L1 compressed sensing is surprisingly good at removing noise.

Noisy



Reconstructed