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CANTOR SET AND SOME OF ITS GENERALIZATIONS AS FRACTALS

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Abstract

The Cantor set is an example of an uncountable set with measure zero and has potential applications in various branches of mathematics such as topology, measure theory, dynamical systems, fractal geometry etc. It exhibits most of the characteristics of a fractal which are self similar in nature and have fractional dimension. In fact, the Cantor set is the simplest model of a fractal. In this paper, we have provided three types of generalization of the Cantor set depending on the process of its construction. Also, we discuss some characteristics of the fractal dimensions of these generalized Cantor sets.

1. Introduction

During the late eighteen century, mathematicians derived great satisfaction in producing sets with ever more strange and unusual properties, many of which are now recognized to be fractal in nature. George Cantor (1845-1918), a German mathematician at the

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Key Words : *Box counting dimension, Cantor set, Capacity dimension, Fractal.*

University of Halle carried out some fundamental work related to foundation of mathematics, which we now call as set theory. In 1883 he wrote a series of papers entitled " *Uber unedliche, lineare Punkimannigfaltigkeiten v Mathemati"-sche Annalen 21(1883) 545-591*" that contained the first systematic treatment of the point set topology of real line in which he raised some problems and attracted interest of researchers to the field of set theory. One of these is the classical Cantor set which he forwarded as an example of a perfect, nowhere dense subset of R.

Though Euclidean geometry is a very useful mathematical theory for modeling the world around us, still there are many shapes in nature for which Euclidean geometry fall short as a model. For example, the motion of a particle suspended in a fluid (Brownian motion), the length of a coastline of an island, the surface area of the human lung, the shapes of trees, clouds etc. For such an object, more closely one look at it, more irregular it appears. The study of such objects has resulted in a new area of mathematics called *Fractal Geometry*. Fractal geometry was popularized by the mathematician Benoit Mendelbrot, and it was he who coined the term fractal in 1977. Though the mathematical work of fractal geometry was first initiated by Cayley, Fatou and Julia in the late 19th and early 20th centuries, progress of research in this line was slow until the development of the electronic computer. Much of the current interest in fractals is a consequence of Mandelbrot's work. His computer simulations of maps of the complex plane have resulted in extremely complicated and beautiful fractals.

A fractal is a complex geometric shape with details down to the smallest spatial scales. Fractals are self similar, whereby a subset of a fractal, and any subset thereof, may resemble the fractal as a whole e.g. see Falconer (2003) and Kisner (2007). Consequently, a measure of the area of a fractal is often difficult to determine. For illustration let us begin with a simple example, the Sierpinski Carpet. The process of generation of the Sierpinski Carpet is that one starts with the unit box and divide it into nine equal boxes, and then remove the open central box. This process is repeated for each eight remaining sub boxes. The limiting set is a fractal which is a generalization of the two dimensional cantor set. Although it can be shown that the Sierpinski Carpet has zero area, it is still useful to make some kind of determination of its dimension.

The Cantor set, originally constructed for purely fascination, lately turned into near perfect models for a host of phenomena in the real world . from strange attractors of nonlinear dynamical systems to the distribution of galaxies in the universe. In modern terminology, the Cantor set is a classical example of a perfect set of the closed interval [0,1] that has the same cardinality as real line whose Lebesgue measure is zero. Since its advent, the Cantor set finds celebrated place in mathematical analysis and its applications, e.g. see Devaney (1992), Hutchinson (1981), Schoenfeld and Gruenhage (1975) and Mendes (1999) amongst others. For fundamental work relevant to study of Cantor set and Devil.s staircase, one may refer to Peitgen, et.al (2004). Devil.s staircase is the graphical representation of Cantor set in the form of Staircases, which has certain applications in dynamical systems, e.g. see Horiguchi and Morita (1984 [a,b]).

Moreover, Dovgoshey et.al. (2006) provided a systematic survey on the properties of the Cantor ternary map, which has its potential application in different branches of science and engineering. Recently, there have been considerable amount of work done to study the various properties of Cantor set. To mention a few one may refer to Shaver (2009), Lapiddus and Lu (2008), Beak (2012, 2004, 2002), Kumar et. al (2013).

The rest of the paper is organized as follows. In section-2, we provide a review of construction and properties of Cantor middle one-third set. Section-3 deals with concept of fractal dimension in box counting scheme. Finally, we have given extensive analysis for construction of our proposed generalized Cantor set as a generalization of classical Cantor middle one-third set in section-4.

2. Preliminaries

In this section, we recall the notion of construction of basic Cantor middle one-third set and review several definitions which are most important to construct our proposed generalized Cantor set.

Cantor Middle One-third Set : Let us consider the closed interval

$$F_0 = I = \{x \in R : 0 \leq x \leq 1\},$$

where R is the set of real numbers.

Now, remove from it the middle open one third interval

$$\Delta_1 = \left\{x \in R : \frac{1}{3} < x < \frac{2}{3}\right\}.$$

We get the remaining set as

$$F_1 = F_0 - \Delta_1 = \left\{x \in R : 0 \le x \le \frac{1}{3}\right\} \cup \left\{x \in R : \frac{2}{3} \le x \le 1\right\}$$

which is union of two closed intervals each of length 1/3. Again, from each of these two closed intervals, we remove the respective middle open intervals of length 1/9, that is

$$\Delta_2 = \left\{x : \frac{1}{9} < x < \frac{2}{9}\right\} \cup \left\{x : \frac{7}{9} < x < \frac{8}{9}\right\}.$$

After this removal, we get a smaller set that is the union of four closed intervals of length $\frac{1}{9}$ each of which we term as F_2 i.e.

$$\begin{aligned} F_2 &= F_0 - \{\Delta_1 \cup \Delta_2\} \\ &= \left\{x : 0 \le x \le \frac{1}{9}\right\} \cup \left\{x : \frac{2}{9} \le x \le \frac{3}{9}\right\} \cup \left\{x : \frac{6}{9} \le x \le \frac{7}{9}\right\} \cup \left\{x : \frac{8}{9} \le x \le 1\right\}. \end{aligned}$$

Continuing this way, in the n^{th}-step we remove a set Δ_n, which is union of 2^{n-1} disjoint open intervals each of length 3^{-n}. After this removal, we get the set F_n which is union of 2^n disjoint closed intervals of length 3^{-n} each.

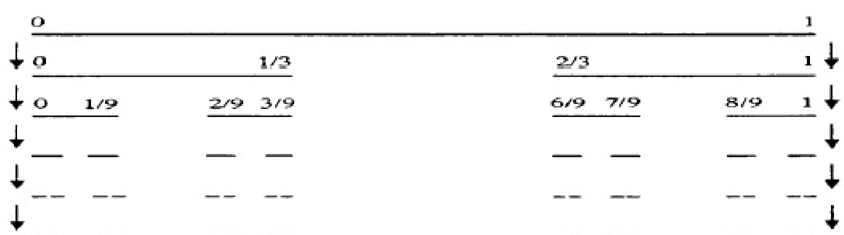

Figure 1 : Some initial steps of the construction of Cantor set

Definition 2.1 : The Cantor middle one-third set C is defined as

$$C = I - \bigcup_{n=1}^{\infty} \Delta_n = \bigcap_{n=0}^{\infty} F_n$$

where, F_{n+1} is constructed as above i.e. by trisecting F_n and removing the open middle onethird interval, F_0 being the closed real interval [0,1].

Proposition 2.1 : A number x in the closed interval $[0, 1]$ belongs to the Cantor 1/3 rd set C if and only if the triadic expansion of x can be expressible without the digit '1'.

Proof : Any number x in the closed interval $[0,1]$ can be expressible in the triadic notation as

$$x = \sum_{n=0}^{\infty} a_n 3^{-n} \quad \text{where} \quad a_n \in \{0,1,2\}.$$

The numbers (in triadic notation) whose first digit after decimal (i.e. digit in the 3^{-1} place) is 1 are greater than 0.1_3 and less than 0.2_3. Note that though, 0.1_3 contains the digit 1, it can be expressible as $0.0\overline{2}$ (since $0.0\overline{2} = 2.3^{-2} + 2.3^{-3} + \cdots\infty = (2.3^{-2}[1 + 3^{-1} + 3^{-2} + \cdots\infty] = 2.3^{-1}[2^{-1}.3] = 3^{-1} = 0.1_3)$. These are the numbers in the open interval $(1/3, 2/3)$ which we eliminated in the first step of construction of the Cantor middle third set C. Therefore, the first digit after decimal in the triadic expansion of any number in C can be expressible without the digit '1'.

The numbers whose triadic expansion contains 1 as the second digit after the decimal (the digit in the 3^{-2}-th place), that is the numbers of the form $0.01\cdots$ and $0.21\cdots$ are greater than 0.01_3 $(= 0.00\overline{2}_3)$ and less than $0.0\overline{2}_3$ or greater than 0.21_3 $(= 0.20\overline{2}_3)$ and less than 0.22_3. These are the numbers in the open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ which are removed in the second stage of construction of C.

Thus, in general, the numbers containing 1 in the kth position after decimal (3^{-k} th position) of its triadic notation are exactly those numbers which are eliminated in the k^{th} stage of construction of C. This means that in triadic notation the numbers of the Cantor set can be represented by using the digits 0 and 2.

Conversely, the numbers in $[0,1]$ whose triadic representation does not contain the digit 1 in its 3^{-1}-th place are the numbers in the closed intervals $[0, 1/3]$ and $[2/3, 1]$ except the numbers $1/3 = 0.1_3$ and 1. But these two can also be represented as $1/3 = 0.0\overline{2}_3$ and $1 = 0.\overline{2}_3$. Thus, the numbers in $[0,1]$ whose triadic representation does not contain 1 as its first digit after decimal are the numbers in F_1.

Similarly the numbers in $[0,1]$ whose first two digits after decimal (digits in 3^{-1} and 3^{-2}-th position) can be represented without using the digit 1 are the numbers in

$$F_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{9}{9} = 1\right].$$

Proceeding this way, one can show that the numbers in $[0,1]$ whose triadic representation can be expressible without using the digit 1 are the numbers in the Cantor middle third set C. $\square$

In view of the above proposition one can define Cantor middle one third set symbolically as follows:

Definition 2.2 : The Cantor middle third set C is the set of those real numbers x which can be represented as $x = \sum_{n=0}^{\infty} a_n 3^{-n}$, where $a_n \in \{0, 2\}$.

Proposition 2.2 : The Cantor middle one-third set C is closed, bounded and non-empty.

Proof : Each $F_n; n \in Z^+$ is closed, being union of closed sets. Also, the Cantor set C is closed being infinite intersection of $F_n; n \in Z^+$. Next, as subset of $[0, 1]$, C is bounded by 0 and 1. C is non-empty because it contains all the end points of each of the intervals constituting the set $F_n; n \in Z^+$.

Proposition 2.3 : The Cantor set C is uncountable.

Proof : Any number in the closed interval $[0, 1]$ can be represented in binary representation in a non terminating string (such as $\frac{1}{2} = 0.1_2 = 0.011_{\dots 2}$).

We define $f : [0, 1] \to C$ by $f(0.a_1, a_2, a_3, \cdots) = 0.b_1b_2b_3 \cdots$ where $b_i = \begin{cases} 0 & if \quad a_i = 0 \\ 2 & if \quad a_i = 1. \end{cases}$

Suppose,

$$f(0.a_1a_2a_3 \cdots) = f(0.a_1'a_2'a_3' \cdots)$$

$$\Rightarrow 0.b_1b_2b_3 \cdots = 0.b_1'b_2'b_3' \cdots$$

$$\Rightarrow b_i = b_i'; \quad \text{for each} \quad i = 1, 2, 3, \cdots$$

Now, $a_i = 0 \Rightarrow b_i = 0 \Rightarrow b_i' = 0 \Rightarrow a_i' = 0$; for each $i = 1, 2, 3, \cdots$.

Also $a_i = 1 \Rightarrow b_i = 2 \Rightarrow b_i' = 2 \Rightarrow a_i' = 1$; for each $i = 1, 2, 3, \cdots$.

Therefore, $0.a_1a_2a_3 \cdots = -.b_1'b_2'b_3' \cdots$.

This shows that f is one-one.

$$\therefore \quad f : [0, 1] \to \text{ Range } \ (f) \subseteq C \ \text{ is a bijective map.}$$

Hence, there exist a subset of C which is uncountable. Therefore C itself is uncountable.

Proposition 2.4 : The Cantor set C is compact.

Proof : The Cantor set C is a subset of the real line R which is closed and bounded. Therefore, by Hein-Borel Theorem C is compact.

Definition 2.3 : A subset A of the real line R is said to be totally disconnected (or has empty interior), if A contains no non-empty open intervals.

Proposition 2.5 : The Cantor middle one-third set C is totally disconnected.

Proof : Since the Lebesgue outer measure of an interval is its length, so, $m^*(F_0 = [0,1]) = 1$. Also, we have the recurrence relation

$$m^*(F_n) = \frac{2}{3} m^*(F_{n-1}) = \left(\frac{2}{3}\right)^n m^*(F_0) = \left(\frac{2}{3}\right)^n.$$

Therefore, $m^*(C) = \lim_{n\to\infty} m^*(F_n) = \lim_{n\to\infty} = \left(\frac{2}{3}\right)^n = 0$ and so, C does not contain any non-empty interval. Hence C is totally disconnected.

Definition 2.4 : A subset A of the set of real numbers R is called a perfect set if every point of A is a limit point of A.

Proposition 2.6 : The Cantor middle one-third set C is perfect.

Proof : Let $x \in C$ be arbitrary whose triadic expansion is

$$x = 0.a_1 a_2 a_3 \cdots a_n \cdots ; \quad \text{where} \quad a_i \in \{0, 2\}.$$

Now, let us consider $x_n = 0.a_1 a_2 a_3 \cdots a_n \cdots$, i.e. x_n agrees with x in the first n-places and $x_n \neq x$ and $x_n \neq x_m$ for all $n, m \in Z^+, n \neq m$. The $|x - x_n| \leq \sum_{k=n+1}^{\infty} \frac{2}{3^k} = 3^{-n} \to 0$ as $n \to \infty$.

This shows that x is a limit point and as x is arbitrary hence C is perfect.

Theorem 2.1 : Any subset E of the real line R which is compact, totally disconnected and perfect is homeomorphic to the Cantor middle one-third set C .

Proof : Let λ and Λ respectively be the lub (least upper bound) and glb (greatest lower bound) of the subset S of R. Consider the complement $S^c = [\lambda, \Lambda] - S$ of S. Clearly, S^c is union of countably many open intervals. Let $\mathcal{G}$ be the collection of all such open intervals. Also,$C^c = [0,1] - C$ is a countable collection of open intervals. We denote this collection of open intervals by $\mathcal{H}$. Now define $\phi = \mathcal{G} \to \mathcal{H}$ as follows:

Let $F_1 \in \mathcal{G}$ be the interval of maximal length. Then $\phi(F_1) = \left(\frac{1}{3}, \frac{2}{3}\right) \in C^c$. Next, let F_{21} and F_{22} be the open intervals in $\mathcal{G}$ such that F_{21} is left and F_{22} is right to F_1 and of maximal length among the members of $\mathcal{G}$ other than F_1, then

$$\phi(F_{21}) = \left(\frac{1}{9}, \frac{2}{9}\right) \quad \text{and} \quad \phi(F_{22}) = \left(\frac{7}{9}, \frac{8}{9}\right).$$

In this way one can define ϕ on the whole set $\mathcal{G}$, as $\mathcal{G}$ contains only finitely many sets of length greater than some fixed $\epsilon > 0$ and since any two intervals in $\mathcal{G}$ or in $\mathcal{H}$ have different end points. Construction of ϕ ensures that ϕ is bijective and order preserving in the sense that if F_α is to the left of F_β then $\phi(F_\alpha)$ is to the left of $\phi(F_\beta)$.

Now let us define $f : [\lambda, \Lambda] \to [0,1]$ as follows:
For $F \in \mathcal{G}$ let $f|_F : F \to \phi(F)$ be the unique linear increasing map, which maps F bijectively to $\phi(F)$. Since S and C are totally disconnected, they are nowhere dense and thus, there is a continuation $\xi : [\lambda, \Lambda] \to [0,1]$ given by :

$$\xi(x) = \sup\{f(y) : y \notin S, y \leq x\}.$$

Let $\psi = \xi|_S$, then $\psi : S \to C$ is monotone increasing, continuous bijective map. Now, we want to show that $\zeta = \psi^{-1}$ is continuous. Clearly ζ is again monotone increasing. Let $x \in C$ and $x_n \to x$. We need to show that $\zeta(x_n) \to \zeta(x)$. Since the sequence $\langle x_n \rangle$ contains a monotone subsequence, so without loss of generality assume that $\langle x_n \rangle$ itself is monotone increasing.
Now, $y = \lim\limits_{n\to\infty} \zeta(x_n) = \sup\limits_{n\geq 1} \zeta(x_n) \leq \zeta(x)$.
If possible, suppose $y < \zeta(x)$. Since S is closed we have $y \in S$ and $\zeta^{-1}(y) < x$. This implies that $y < x_n$ for large n. Therefore, by monotonicity $y < \zeta(x_n)$ which contradicts the fact $y = \lim\limits_{n\to\infty} \zeta(x_n)$. Thus, we must have $y = \zeta(x)$ i.e., $\zeta(x_n) \to \zeta(x)$.
Definition 2.5 : Any subset of the real line R which is compact, totally disconnected and perfect is called Cantor like set or simply Cantor set.

3. Concept of Fractal Dimension

The Cantor set is the prototype of a fractal. It is an object which appears self-similar under varying degrees of magnification. One of the typical features of fractal is its fractal dimension, which is essentially a measure of self similarity. It is sometimes referred to as similarity dimension. There are a number of non-equivalent ways of defining fractal dimension. One of the most popular way of computing dimension is capacity dimension. The capacity of a set was originally defined by Kolmogorov (1958) and is defined as follows:
Definition 3.1 : Let $N(K, r)$ be the minimum number of circular disks of radius r needed to cover the compact set K. The Kolmogorov dimension or capacity dimension $D(K)$ of the set K is defined as

$$D(K) = \lim_{r\to 0} \frac{\log N(K,r)}{\log(1/r)}. \tag{3.1}$$

From this definition it is clear that to compute the dimension of the set K, one need to find out the number $N(K,r)$, which is not an easy task for a sets like fractal. To avoid this difficulty some simplifications have been made as given below:

Let $N_{box}(K,r)$ be the minimum number of axis parallel square boxes of side length r needed to cover the set K.

Let, $N_{box}(K,r) = n$. We can replace each box with a disk of radius r, so we need n disks of radius r to cover K, i.e. $N(K,r) = n$. But the area of a circular disk of radius r is greater than that of a square box of length r, hence we have

$$N(K,r) \leq n = N_{box}(K,r). \tag{3.2}$$

Further, we suppose that $N(K,r) = m$. We can replace each disk of radius r by four square boxes each of length r. But, perhaps we can do better, therefore we may write

$$N_{box}(K,r) \leq 4m = 4N(K,r). \tag{3.3}$$

Now combining (3.2) and (3.3), we get

$$\begin{aligned}
&N(K,r) \leq N_{box}(K,r) \leq 4N(K,r) \\
&\Rightarrow \lim_{r\to 0} \frac{\log N(K,r)}{\log \frac{1}{r}} \leq \lim_{r\to 0} \frac{\log N_{box}(K,r)}{\log \frac{1}{r}} \leq \lim_{r\to 0} \frac{\log 4 + \log N(K,r)}{\log \frac{1}{r}} \\
&\Rightarrow D(K) \leq \lim_{r\to 0} \frac{\log N_{box}(K,r)}{\log \frac{1}{r}} \leq D(K)
\end{aligned}$$

$$\therefore \quad D(K) = \lim_{r\to 0} \frac{\log N_{box}(K,r)}{\log \frac{1}{r}}. \tag{3.4}$$

It should be noted that the above expression (3.4) is popularly known as box counting dimension of the set K and it is denoted by $D_B(K)$. This definition often gives sensible results for standard situations. For example, we need $N_{box}(K,r) = \frac{L}{r}$ numbers of boxes to cover a straight lie of length L, so that

$$D_B(K) = \lim_{r\to 0} \frac{\log(L) + \log(r^{-1})}{\log(r^{-1})} = 1.$$

By this scheme of computation of dimension of a compact set, we actually managed to overcome the difficulties of counting circular discs needed to cover the set K. Moreover, we refer to Sarkar and Choudhuri (1994) and Li et.al (2009) to interested readers to see how to compute fractal dimension of images with the help of box-counting dimension.

In rest of this paper we use box counting scheme for computing dimension of our proposed fractal sets.

4. Generalization of Cantor Middle One-third Set

In this section an attempt have been made to construct generalized Cantor set and discuss some properties related to their dimension. At this point, we want to mention that depending on the removal process in the construction of cantor set, here we have considered three different forms of generalization of the Cantor set C which are discussed below:

4.1 Generalization I

Let us consider the closed interval $F_0 = [0, 1]$. Now divide F_0 into n (≥ 3). equal parts and remove from it the central open interval of length $\frac{1}{n}$, i.e.

$$\Delta_1 = \left\{ x : \frac{n-1}{2n} < x < \frac{n+1}{2n} \right\}.$$

Then we get two closed intervals as

$$\begin{aligned} F_1 &= F_0 - \Delta_1 \\ &= \left\{ x : 0 \leq x \leq \frac{n-1}{2n} \right\} \cup \left\{ x : \frac{n+1}{2n} \leq x \leq 1 \right\} \\ &= L \cup R \end{aligned}$$

which is the generator of our proposed Cantor $\left(\frac{1}{n}\right)^{th}$ set which we have denoted as $C\left(\frac{1}{n}\right)$. In the above expression, $L = \left[0, \frac{(n-1)}{2n}\right]$ and $R = \left[\frac{(n+1)}{2n}, 1\right]$.

Next, we divide each of L and R into n equal parts and remove from each the central open interval of length $\frac{n-1}{2n^2}$ i.e.

$$\Delta_2 = \left\{ x : \frac{n^2 - 2n + 1}{4n^2} < x < \frac{n^2 - 1}{4n^2} \right\} \cup \left\{ x : \frac{3n^2 + 1}{4n^2} < x < \frac{3n^2 + 2n - 1}{4n^2} \right\}.$$

After this removal, we set the remaining part as F_2 i.e.

$$F_2 = F_0 - \{\Delta_1 \cup \Delta_2\}$$

which is union of four closed intervals, viz. LL, LR, RL, RR each of length $\left\{\frac{(n-1)}{2n}\right\}^2$.

Thus,

$$F_2 = LL \cup LR \cup RL \cup RR$$

where,

$$LL = \left[0, \frac{n^2-2n+1}{4n^2}\right],\ LR = \left[\frac{n^2-1}{4n^2}, \frac{n-1}{2n}\right],$$

$$RL = \left[\frac{n+1}{2n}, \frac{3n^2+1}{4n^2}\right] \text{ and } RR = \left[\frac{3n^2+2n-1}{4n^2}, 1\right].$$

Proceeding this way we get a sequence of closed intervals $\{F_n\}$ where F_k is union of 2^k numbers of closed intervals of length $\left\{\frac{(n-1)}{2n}\right\}^k$ each.

Now $C\left(\frac{1}{n}\right)$ is defined as

$$C\left(\frac{1}{n}\right) = F_0 - \left\{\bigcup_{i=1}^{\infty} \Delta_i\right\} = \bigcap_{k=0}^{\infty} F_k.$$

$$Geometrical\ representation\ of\ \ C\left(\frac{1}{10}\right):$$

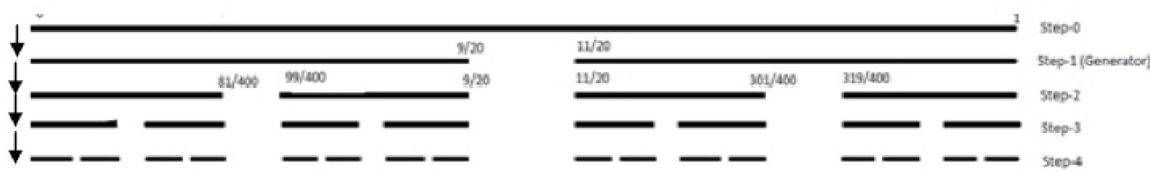

Figure -2: Some initial steps in the construction of $C\left(\frac{1}{10}\right)$

Theorem 4.1.1 : The set $C\left(\frac{1}{n}\right)$ is compact, totally disconnected and perfect sub set of $[0,1]$.

Proof : Clearly $C\left(\frac{1}{n}\right) \subseteq [0,1]$. For any two points $x, y \in C\left(\frac{1}{n}\right)$, we have $|x-y| \leq 1$ i.e. $C\left(\frac{1}{n}\right)$ is bounded. Also, it is closed as infinite union of closed sets is closed. So, it can be seen that $C\left(\frac{1}{n}\right)$ is compact, being closed and bounded subset of R.

To prove disconnectedness, we consider the recurrence relation

$$m^*(F_k) = \frac{n-1}{n} m^*(F_{k-1}); \quad k = 1, 2, 3, \cdots.$$

So,

$$m^*\left(C\left(\frac{1}{n}\right)\right) = \lim_{k\to\infty} m^*(F_k) = \lim_{k\to\infty} \left(\frac{n-1}{n}\right)^k m^*(F_0) = 0.$$

Therefore, $C\left(\frac{1}{n}\right)$ does not contain any non-empty interval which implies that $C\left(\frac{1}{n}\right)$ is totally disconnected.

Finally we show that $C\left(\frac{1}{n}\right)$ is perfect.

For this, we consider an arbitrary point x_0 of $C\left(\frac{1}{n}\right)$. Let ϵ be any positive real number. We can choose a sufficiently large natural number k such that $\left(\frac{n-1}{2n}\right)^k < \epsilon$. Then, the sequence $\langle x_k \rangle$ of points of the set $C\left(\frac{1}{n}\right)$ in the interval F_k are infinite in number, and all contained in the open interval $(x_0 - \epsilon, x_0 + \epsilon)$, which implies that

$$|x_k - x_0| < \epsilon \quad \text{as‘ } k \to \infty.$$

Therefore, x_0 is a limit point of $C\left(\frac{m}{n}\right)$. Now, as x_0 is arbitrary in $C\left(\frac{1}{n}\right)$, so, $C\left(\frac{1}{n}\right)$ is a perfect set. □

Remark 4.1.1 : In view of the Definition 2.5 and Theorem 4.1.1 it can be concluded that the constructed set $C\left(\frac{1}{n}\right)$ is a Cantor set.

4.1.2 Dimension and its Characteristics

To compute the dimension of the set $C\left(\frac{1}{n}\right)$ we proceeds as follows:

If we denote the minimum number of square boxes of side length ϵ needed to cover the compact set K by $N_{box}(K, \epsilon)$, then

$$N_{box}\left(C\left(\frac{1}{n}\right), \left(\frac{n-1}{2n}\right)^k\right) = 2^k; \qquad k = 0, 1, 2, \cdots$$

and therefore we have

$$D_B\left(C\left(\frac{1}{n}\right)\right) = \lim_{k\to\infty} \frac{\log 2^k}{\log\left(\frac{n-1}{2n}\right)^{-k}} = \frac{\log 2}{\log\left(\frac{2n}{n-1}\right)}; \quad n = 3, 4, 5, \cdots$$

The following Table-1 shows the box counting dimension of first few cantor sets of type $C\left(\frac{1}{n}\right)$ which are calculated using the above formula.

Table 1 : Box counting dimension of first few cantor sets of type $C\left(\frac{1}{n}\right)$

Cantor set of type $C\left(\frac{1}{n}\right)$	Box counting dimension $D_B\left(C\left(\frac{1}{n}\right)\right)$
$C\left(\frac{1}{3}\right)$	$D_B = 0.63093$

Cantor set of type $C\left(\frac{1}{n}\right)$	Box counting dimension $D_B\left(C\left(\frac{1}{n}\right)\right)$
$C\left(\frac{1}{4}\right)$	$D_B = 0.70669$
$C\left(\frac{1}{5}\right)$	$D_B = 0.75467$
$C\left(\frac{1}{6}\right)$	$D_B = 0.79174$
$C\left(\frac{1}{7}\right)$	$D_B = 0.81807$

Remark 4.1.2 : The set $C\left(\frac{1}{n}\right)$ for $n = 3$ is equivalent to the Cantor middle one-third set C, as in this case the generator is:

$$\begin{aligned} F_1 &= \left\{x : 0 \leq x \leq \frac{3-1}{2.3}\right\} \cup \left\{x : \frac{3+1}{2.3} \leq x \leq 1\right\} \\ &= \left\{x : 0 \leq x \leq \frac{1}{3}\right\} \cup \left\{x : \frac{2}{3} \leq x \leq 1\right\} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \end{aligned}$$

which is the generator of Cantor middle one third set C. Further,

$$D_B\left(C\left(\frac{1}{n}\right)\right)\Bigg]_{n=3} = \frac{\log 2}{\log\left(\frac{2.3}{3-1}\right)} = \frac{\log 2}{\log 3}$$

which agrees with the dimension of the Cantor middle one-third set C. This allow us to conclude that $C\left(\frac{1}{n}\right)$ is a generalization of the Cantor middle one-third set C. □

Theorem 4.1.2 : The sequence $\left\langle D_B\left(C\left(\frac{1}{n}\right)\right)\right\rangle_{n=3}^{\infty}$ of dimensions of the family of the Cantor set $\left\{C\left(\frac{1}{n}\right) : n = 3, 4, 5, \cdots\right\}$ is an increasing function of n and it asymptotically approaches to unity.

Proof : We have, $D_B\left(C\left(\frac{1}{n}\right)\right) = \frac{\log 2}{\log\left(\frac{2n}{n-1}\right)}; \quad n = 3, 4, 5, \cdots$

Differentiating with respect to n,

$$\frac{d}{dn} D_B\left(C\left(\frac{1}{n}\right)\right) = \log 2 \cdot \frac{\left(-\frac{n-1}{2n}\right)\left(\cdot \frac{-2}{(n-1)^2}\right)}{\left[\log\left(\frac{2n}{n-1}\right)\right]^2} = \frac{\log 2}{n(n-1)\left[\log \frac{2n}{n-1}\right]^2}.$$

Since $n \geq 3$, therefore we have $n(n-1) > 0$ and hence $\frac{d}{dn} D_B\left(C\left(\frac{1}{n}\right)\right) > 0$ which shows that$D_B\left(C\left(\frac{1}{n}\right)\right)$ is an increasing function of n.
Now, let us consider the limiting case of $D_B\left(C\left(\frac{1}{n}\right)\right)$

$$\lim_{n\to\infty} D_B\left(C\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{\log 2}{-\log\left(\frac{n-1}{2n}\right)} = \lim_{n\to\infty} \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{1}{2} - \frac{1}{2n}\right)} = 1.$$

□

The following figure is drawn by plotting $D_B\left(C\left(\frac{1}{n}\right)\right)$ for different values of n which graphically verifies the above mentioned theorem.

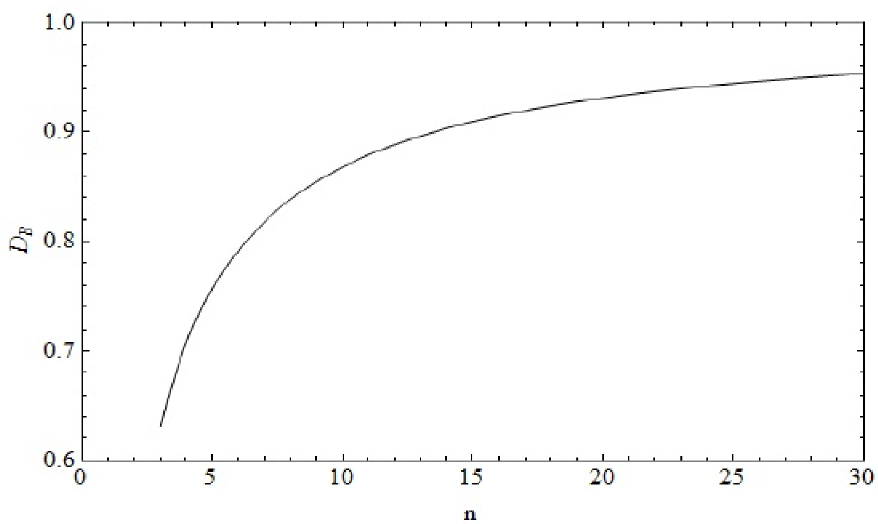

Figure 3 : Graphical representation of $D_B\left(C\left(\frac{1}{n}\right)\right)$ *vs n.*

Remark 4.1.3 : Theorem 4.1.2 shows that as n increases, the length $\left(\frac{1}{n}\right)$ of the removed open interval decreases and dimension of $C\left(\frac{1}{n}\right)$ increases i.e. the length of the removed interval is inversely proportional to the dimension of $C\left(\frac{1}{n}\right)$ and asymptotically approaches unity which is the topological dimension of the closed interval $[0,1]$. □

4.2 Generalization II

Next, we generalize the cantor one third set by removing the central open interval $\left(\frac{1}{n}, 1-\frac{1}{n}\right)$ of lenght $\left(1-\frac{2}{n}\right)$, i.e. we remove the open interval

$$\Delta_1 = \left\{x : \frac{1}{n} < x < 1 - \frac{1}{n}\right\}$$

from the closed interval $F_0 = [0, 1]$.

Note that in this method of removal, we vary the length of side intervals in terms of n and then remove the open interval in between, whereas in the generalization I, we vary the length of the central open interval in terms of n and then remove it.

After the removal of the central open interval Δ_1 we set the remaining part as F_1 i.e.

$$F_1 = F_0 - \Delta_1$$

which is union of two closed intervals $L = \left[0, \frac{1}{n}\right]$ and $R = \left[1 - \frac{1}{n}, 1\right]$. Thus, the generator of this method is

$$F_1 = L \cup R = \left[0, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1\right].$$

Next, we divide each of L and R into n equal parts and remove from them the central open intervals $\left(\frac{1}{n^2}, \frac{n-1}{n^2}\right)$ and $\left(1 - \frac{n-1}{n^2}, 1 - \frac{1}{n^2}\right)$ respectively i.e.

$$\Delta_2 = \left\{x : \frac{1}{n^2} < x < \frac{n-1}{n^2}\right\} \cup \left\{x : \frac{n^2 - n + 1}{n^2} < x < 1 - \frac{1}{n^2}\right\}.$$

The remaining part F_2 is the union of four closed intervals i.e.

$$F_2 = F_0 - \{\Delta_1 \cup \Delta_2\} = LL \cup LR \cup RL \cup RR$$

where

$$LL = \left[0, \frac{1}{n^2}\right], \;\; LR = \left[\frac{n-1}{n^2}, \frac{1}{n}\right], \;\; RL = \left[\frac{n-1}{n}, \frac{n^2 - n + 1}{n^2}\right] \text{ and } RR = \left[1 - \frac{1}{n^2}, 1\right].$$

Continuing this way we get a sequence of closed intervals $\{F_k\}$, where $F_0 = [0, 1]$ contains one closed interval of length $1, F_1$ contains 2 closed intervals each of length $\frac{1}{n}$, F_2 contains 2^2 closed intervals each of length n^{-2} i.e. in general, F_k contains 2^k numbers of closed intervals of length n^{-k} each.

The limiting set formed by this process is called the Cantor $\left(\frac{n-2}{n}\right)^{th}$ set and we denote it by $C\left(\frac{n-2}{n}\right)$; (where $n \geq 3$) i.e.

$$C\left(\frac{n-2}{n}\right) = F_0 - \left\{\bigcup_{i=1}^{\infty} \Delta_i\right\} = \bigcap_{k=0}^{\infty} F_k.$$

□

Geometrical representation of $C\left(\frac{3}{5}\right)$

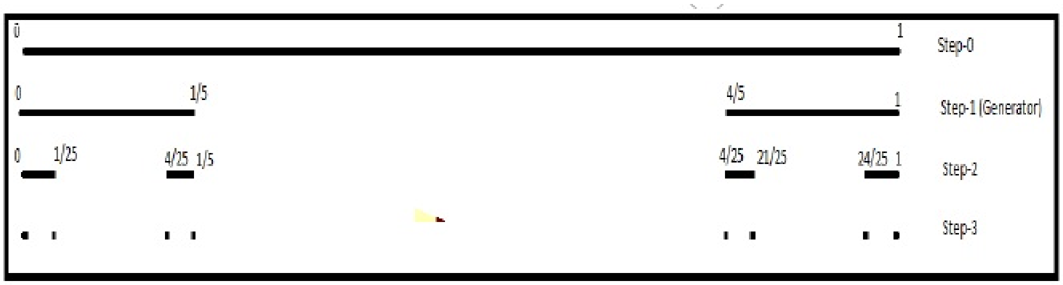

Figure – 4: Some initial stepsin the construction of $C\left(\frac{3}{5}\right)$

Theorem 4.2.1 : $C\left(\frac{n-2}{n}\right)$ is compact, totally disconnected and perfect sub set of $[0,1]$.
Proof : Clearly $C\left(\frac{n-2}{n}\right) \subseteq [0,1]$. For any two points $x, y \in C\left(\frac{n-2}{n}\right)$ we have, $|x-y| \leq 1$ i.e. $C\left(\frac{n-2}{n}\right)$ is bounded. Further, it is closed as infinite union of closed sets is closed. So, $C\left(\frac{n-2}{n}\right)$ is a compact set.
To prove the set as totally disconnected, we use the recurrence relation

$$m^*(F_k) = \frac{2}{n} m^*(F_{k-1}); \quad k = 1, 2, 3, \cdots .$$

We have

$$m^*\left(C\left(\frac{n-2}{n}\right)\right) = \lim_{k\to\infty} m^*(F_k) = \lim_{k\to\infty} \left(\frac{2}{n}\right)^k m^*(F_0) = \lim_{k\to\infty} \left(\frac{2}{n}\right) k = 0, \text{ since } n \geq 3.$$

Therefore, $C\left(\frac{n-2}{n}\right)$ does not contain any non-empty interval and hence $C\left(\frac{n-2}{n}\right)$ is totally disconnected.
Finally, we show that $C\left(\frac{n-2}{n}\right)$ is perfect. For this, we consider an arbitrary point x_0 of $C\left(\frac{n-2}{n}\right)$. Suppose, ϵ be any positive real number. Now, we can choose a sufficiently large natural number k such that $n^{-k} < \epsilon$. The sequence $\langle x_k \rangle$of points of the set $C\left(\frac{n-2}{n}\right)$ in the interval F_k are infinite in number, and all contained in the open interval $(x_0 - \epsilon, x_0 + \epsilon)$ which implies that

$$|x_k - x_0| < \epsilon \text{ as } k \to \infty.$$

Therefore, x_0 is a limit point of $C\left(\frac{n-2}{n}\right)$.
Now as x_0 is arbitrary $C\left(\frac{n-2}{n}\right)$ is a perfect set. $\square$

Remark 4.2.1 : In view of the Definition 2.5 and Theorem 4.2.1 it can be concluded that the constructed set $C\left(\frac{n-2}{n}\right)$ is a Cantor set.

4.2.2 Dimension of $C\left(\frac{n-2}{n}\right)$ and its Characteristics

If we indicate the number of square boxes needed to cover the set $C\left(\frac{n-2}{n}\right)$ by $N_{box}\left(C\left(\frac{n-2}{n}\right), \epsilon\right)$, then we have

$$N_{box}\left(C\left(\frac{n-2}{n}\right), n^{-k}\right) = 2^k; \quad k = 0, 1, 2, \cdots.$$

Hence,

$$\begin{aligned} D_B\left(C\left(\frac{n-2}{n}\right)\right) &= \lim_{k\to\infty} \frac{\log 2^k}{\log n^k} \\ &= \frac{\log 2}{\log n}; \qquad n = 3, 4, 5, \cdots \end{aligned}$$

Following table shows the dimension of first five Cantor sets of type $C\left(\frac{n-2}{n}\right)$.

Table 2 : Box counting dimension of first few Cantor sets of type $C\left(\frac{n-2}{n}\right)$

Cantor set of type $C\left(\frac{n-2}{n}\right)$	Box counting dimension $\left(C\left(\frac{n-2}{n}\right)\right)$
$C\left(\frac{1}{3}\right)$	$D_B = 0.63093$
$C\left(\frac{2}{4}\right)$	$D_B = 0.50000$
$C\left(\frac{3}{5}\right)$	$D_B = 0.43076$
$C\left(\frac{4}{6}\right)$	$D_B = 0.38685$
$C\left(\frac{5}{7}\right)$	$D_B = 0.35620$

Remark 4.2.2 : Here, we verify that the set $C\left(\frac{n-2}{2}\right)$ is actually a generalization of the Cantor middle one-third set C as follows:

The set $C\left(\frac{n-2}{2}\right)$ for $n = 3$ is formed by removing an open interval $\left(\frac{1}{3}, 1 - \frac{1}{3} = \frac{2}{3}\right)$ of length $\frac{1}{3}$ from the centre of each closed interval, starting from the unit closed interval $[0, 1]$. So, in each step of this process one closed interval of length $\frac{1}{3}$ remains in each end of the interval. Moreover, the generator of $C\left(\frac{n-2}{2}\right)$ for $n = 3$ is

$$F_1 = \left[0, \frac{1}{3}\right] \cup \left[\left(1 - \frac{1}{3}\right), 1\right] = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

which is the generator of the Cantor middle one-third set C.
Also, $D_B\left(C\left(\frac{n-2}{2}\right)\right)\big]_{n=3} = \frac{\log 2}{\log 3}$ which agrees with the dimension of the Cantor middle one-third set C. This allow us to conclude that $C\left(\frac{n-2}{2}\right)$ is generalization of the Cantor middle one-third set C.

Theorem 4.2.2 : The sequence $\left\langle D_B\left(C\left(\frac{n-2}{2}\right)\right)\right\rangle_{n=3}^{\infty}$ of dimensions of the family of the Cantor set $\left\{C\left(\frac{n-2}{2}\right); n = 3, 4, 5, \cdots\right\}$ is a decreasing function of n and it asymptotically approaches zero.

Proof : We have, $D_B\left(C\left(\frac{n-2}{2}\right)\right) = \frac{\log 2}{\log n}$

$$\frac{d}{dn} D_B\left(C\left(\frac{n-2}{2}\right)\right) = -\frac{\log 2}{n(\log n)^2} < 0 \quad (\text{since } n \geq 3).$$

Also

$$\lim_{n \to \infty} D_B\left(C\left(\frac{n-2}{2}\right)\right) = \lim_{n \to \infty} \frac{\log 2}{\log n} = 0.$$

□

The following figure is drawn by plotting $D_B\left(C\left(\frac{n-2}{2}\right)\right)$ for different values of n which graphically verifies the above mentioned result.

Remark 4.2.3 : Theorem 4.2.2 shows that as the value of n increases, the dimension of $\left(C\left(\frac{n-2}{2}\right)\right)$ decreases asymptotically to 0 i.e. the dimension is inversely proportional to the number of divisions n. This is due to the increase of the length of the removed open interval. Moreover, as n increases i.e. the length of the removed middle open interval increases and the dimension approaches zero, which is the topological dimension of a point.

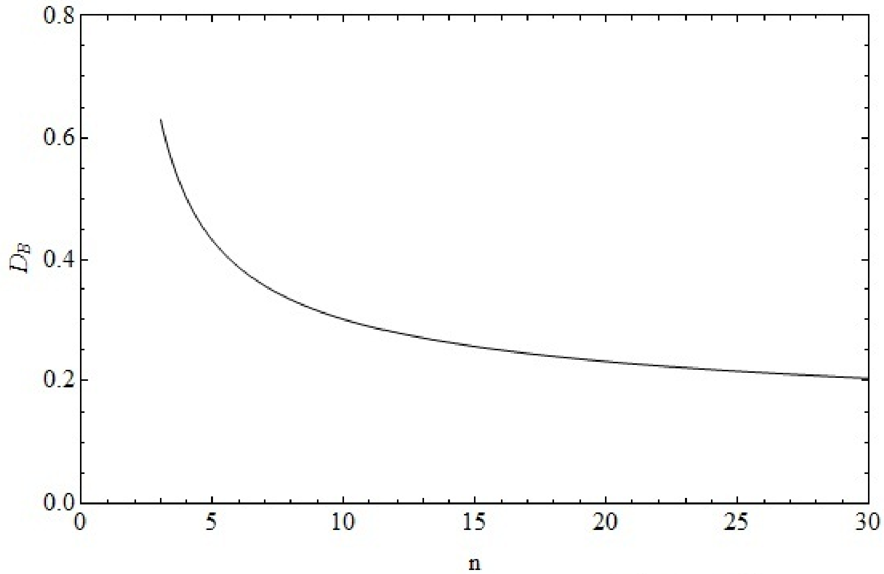

Figure 5 : Graphical representation of $D_B\left(C\left(\frac{n-2}{n}\right)\right)$ *vs* n.

4.3 Generalization III

In this generalization we divide the unit interval $F_0 = [0,1]$ into n $(n \geq 3)$ equal parts, where n is an odd number, say $n = 2m+1$ and then remove from it the alternate m open intervals, viz.

$$\left(\frac{1}{n}, \frac{2}{n}\right), \ \left(\frac{3}{n}, \frac{4}{n}\right), \cdots, \left(\frac{2m-1}{n}, \frac{2m}{n}\right)$$

i.e. we remove the set

$$\Delta_1 = \left\{x : \frac{1}{n} < x < \frac{2}{n}\right\} \cup \left\{x : \frac{3}{n} < x < \frac{4}{n}\right\} \cup \cdots \cup \left\{x : \frac{2m-1}{n} < x < \frac{2m}{n}\right\}.$$

After removing Δ_1, the remaining set F_1 turns out to be the union of $(m+1)$ closed intervals i.e.

$$F_1 = F_0 - \Delta_1 = \left[0, \frac{1}{n}\right] \cup \left[\frac{2}{n}, \frac{3}{n}\right] \cup \cdots \cup \left[\frac{2m}{n}, 1\right]$$

which is the generator of our proposed generalization III.

Next, we sub-divide each of these $(m+1)$ closed intervals into n equal parts and from each of such intervals, remove the $2^{nd}, 4^{th}, \cdots, 2m^{th}$ open intervals. Continuing this process we get a sequence $\{F_k\}$ of closed intervals : One of length 1 in the step 0,

$(m+1)$ of length n^{-1} each in the step 1, $(m+1)^2$ of length n^{-2} each in the step 2 i.e. in general, $(m+1)^k$ closed intervals of length n^{-k} each in the step k.
The limiting set obtained in this process is called Cantor $\frac{m}{n}$ th set and we denote it by $C\left(\frac{m}{n}\right)$. Therefore $C\left(\frac{m}{n}\right) = F_0 - \left\{\bigcup\limits_{i=1}^{\infty} \Delta_i\right\} = \bigcap\limits_{k=0}^{\infty} F_k$.

Geometrical representation of $C\left(\frac{3}{7}\right)$:

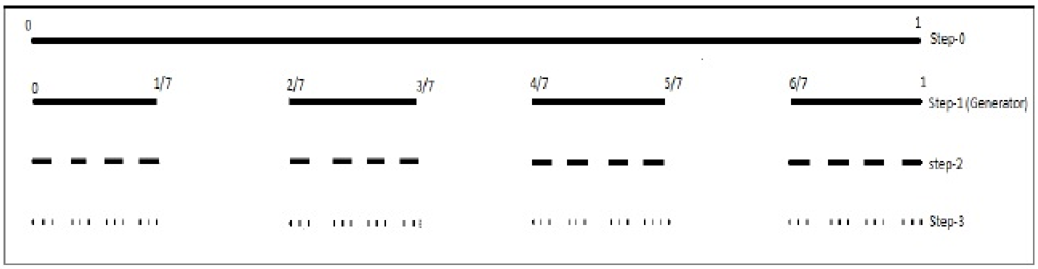

Figure – 6 :Some initial stepsin the construction of $C\left(\frac{3}{7}\right)$

Theorem 4.3.1 : $C\left(\frac{m}{n}\right)$ is compact, totally disconnected and perfect sub set of $[0,1]$.
Proof : Clearly $C\left(\frac{m}{n}\right) \subseteq [0,1]$. For any two points $x, y \in C\left(\frac{m}{n}\right)$ we have, $|x-y| \leq 1$ i.e. $C\left(\frac{m}{n}\right)$ is bounded. Also, it is closed as infinite union of closed sets is closed which implies that $C\left(\frac{m}{n}\right)$ is a compact set.
To show totally disconnectedness, we use the recurrence relation

$$m^*(F_k) = \frac{m+1}{2m+1} m^*(F_{k-1}),\text{‘} \quad k = 1, 2, 3, \cdots .$$

We have

$$m^*\left(C\left(\frac{m}{n}\right)\right) = \lim_{k\to\infty} m^*(F_k) = \lim_{k\to\infty} \left(\frac{m+1}{2m+1}\right)^k m^*(F_0) = 0.$$

Therefore, $C\left(\frac{m}{n}\right)$ does not contain any non-empty interval which implies that $C\left(\frac{m}{n}\right)$ is totally disconnected.
Lastly, we show that $C\left(\frac{m}{n}\right)$ is perfect. For this, we consider an arbitrary point x_0 of $C\left(\frac{m}{n}\right)$. Let ϵ be any positive real number. Now, we can choose a sufficiently large natural number k such that $n^{-k} < \epsilon$. The sequence $\langle x_k \rangle$ of points of the set $C\left(\frac{m}{n}\right)$ in the interval F_k are infinite in number, and all contained in the open interval $(x_0-\epsilon, x_0+\epsilon)$ which implies that

$$|x_k - x_0| < \epsilon \ \text{ as } \ k \to \infty.$$

Therefore, x_0 is a limit point of $C\left(\frac{m}{n}\right)$. Now, as x_0 is arbitrary in $C\left(\frac{m}{n}\right)$, $C\left(\frac{m}{n}\right)$ is a perfect set. $\square$

Remark 4.3.1 : In view of the definition 2.5 and theorem 4.3.1 it can be concluded that the constructed set $C\left(\frac{m}{n}\right)$ is a Cantor set.

4.3.2 Dimension of $C\left(\frac{m}{n}\right)$ and its Characteristics

To compute the dimension of the sets $C\left(\frac{m}{n}\right)$ we proceeds as follows:
By representing the number of square boxes needed to cover the set $C\left(\frac{m}{n}\right)$ by $N_{box}\left(C\left(\frac{m}{n}\right), \epsilon\right)$ we have

$$N_{box}\left(C\left(\frac{m}{n}\right), n^{-k}\right) = (m+1)^k; \quad k = 0, 1, 2, 3, \cdots .$$

Therefore,

$$\begin{aligned} D_B\left(C\left(\frac{m}{n}\right)\right) &= \lim_{k\to\infty} \frac{\log(m+1)^k}{\log n^k} = \frac{\log(1+m)}{\log n} \\ &= \frac{\log(1+m)}{\log(1+2m)}; \quad m = 1, 2, 3, \cdots . \end{aligned}$$

In the following Table-3 we have shown box counting dimension of first few cantor sets of type $C\left(\frac{m}{n}\right)$.

Table 3 : Box counting dimension of first few Cantor sets of type $C\left(\frac{m}{n}\right)$

Cantor set of type $C\left(\frac{m}{n}\right)$	Box counting dimension of $C\left(\frac{m}{2}\right)$
$C\left(\frac{1}{3}\right)$	$D_B = 0.63093$
$C\left(\frac{2}{5}\right)$	$D_B = 0.68261$
$C\left(\frac{3}{7}\right)$	$D_B = 0.71241$

Remark 4.3.2 : Here, we show that the set $C\left(\frac{m}{n}\right)$ is actually a generalization of the Cantor middle one-third set C as follows:

Since the set $C\left(\frac{m}{n}\right)$, for $m = 1$ i.e. $n = 3$ is formed by dividing first the unit interval $[0, 1]$ into three equal parts and then removing from it alternate open intervals which in this case is the central open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, so, the generator in this case is $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. This is the generator of Cantor middle one-third set C.

Also, $D_B\left(C\left(\frac{m}{n}\right)\right)\Big|_{n=3} = \frac{\log(1+1)}{\log 3} = \frac{\log 2}{\log 3}$ which agree with the dimension of the Cantor middle one-third set C. Therefore, we may conclude that $C\left(\frac{m}{n}\right)$ is generalization of Cantor middle one-third set C.

Theorem 4.3.2 : The sequence $\left\langle D_B\left(C\left(\frac{m}{n}\right)\right)\right\rangle_{n=3}^{\infty}$ of dimensions of the family of the Cantor set$\left\{C\left(\frac{m}{n}\right); n = 3, 4, 5, \cdots\right\}$ is an increasing function of n and asymptotically approaches unity.

Proof : Let us consider the function $f : [1, \infty) \to R$ defined by

$$f(x) = \frac{\log(1 + x)}{\log(1 + 2x)}.$$

Now, differentiating with respect to x, we get

$$\begin{aligned} \frac{df(x)}{dx} &= \frac{(1 + 2x)\log(1 + 2x) - 2(1 + x)\log(1 + x)}{(1 + x)(1 + 2x)[\log(1 + 2x)]^2} \\ &= \frac{h(x)}{g(x)} \quad \text{(say)}; \end{aligned}$$

where

$$\begin{aligned} g(x) &= (1 + x)(1 + 2x)[\log(1 + 2x)^2 \quad \text{for all} \quad x \in [1, \infty) \\ h(x) &= h_1(x) - h_2(x) \quad \text{for all} \quad x \in [1, \infty) \end{aligned}$$

in which

$$h_1(x) = (1 + 2x)\log(1 + 2x) \quad \text{for all} \quad x \in [1, \infty)$$

and

$$h_2(x) = 2(1 + x)\log(1 + x) \quad \text{for all} \quad x \in [1, \infty); \quad \text{respectively.}$$

Now let us consider $g : [1, \infty) \to R$ where

$$\begin{aligned} g(x) &= (1 + x)(1 + 2x)[\log(1 + 2x)]^2 \\ &= 2\{2x^2 + 3x + 1\}\log(1 + 2x) > 0 \quad \text{for all} \quad x \in [1, \infty). \end{aligned}$$

Similarly, for $h : [1, \infty) \to R$, we have

$$\begin{aligned} h_1(x) &= (1+2x)\log(1+2x) \\ &= (1+2x)\left\{2x + 2x^2 + \frac{8}{3}x^3 + \cdots, \infty\right\} \\ &= 2x + 6x^2 + \frac{20}{3}x^3 + \cdots + \infty \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} h_2(x) &= 2(1+x)\log(1+x) \\ &= 2(1+x)\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \infty\right\} \\ &= 2x + 3x^2 + \frac{5}{3}x^3 + \cdots + \infty. \end{aligned} \tag{4.2}$$

Subtracting (3.2) from (3.1), we get $h(x) = 3x^2 + 5x^3 + \cdots > 0$ for all $x \in [1, \infty)$ and therefore $\frac{df(x)}{dx} = \frac{h(x)}{g(x)} > 0$ for all $x \in [1, \infty)$. This shows that f is an increasing function of x and hence the dimension function $D_B\left(C\left(\frac{m}{n}\right)\right)$ is increasing function of n.
Finally, let us consider the limiting case of $D_B\left(C\left(\frac{m}{n}\right)\right)$

$$\lim_{n\to\infty} D_B\left(C\left(\frac{m}{n}\right)\right) = \lim_{m\to\infty}\frac{\log(1+m)}{\log(1+2m)} = 1.$$

$\square$.

Remark 4.3.3 : Thus if $\left\langle C\left(\frac{m}{n}\right)\right\rangle_{m=1}^{\infty}$, $n = 2m+1$ is a sequence of Cantor set, then the sequence of their dimensions $\left\langle D\left(C\left(\frac{m}{n}\right)\right)\right\rangle_{m=1}^{\infty}$ is an increasing function of m. Moreover, this sequence of dimensions converges to 1 which is the topological dimension of the closed interval $[0, 1]$
The following figure is drawn by plotting $D_B\left(C\left(\frac{m}{n}\right)\right)$ for different values of n which graphically verifies the above mentioned theorem.

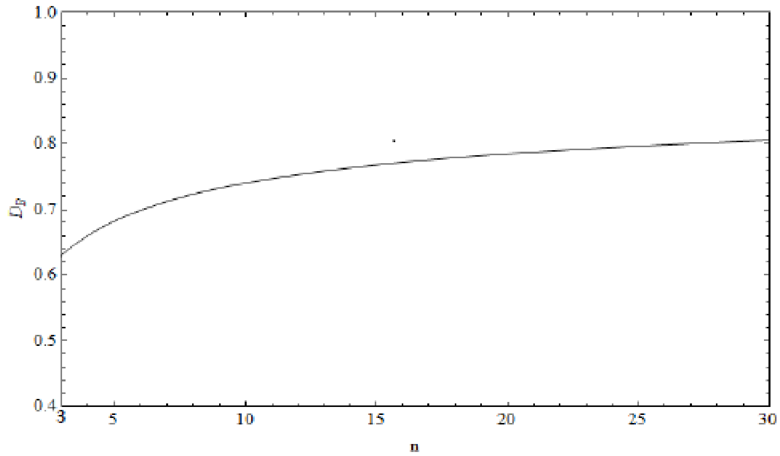

Figure 7: Graphical representation of $D_B\left(C\left(\frac{m}{n}\right)\right)$ *vs* n.

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