CSIAM Trans. Appl. Math.
doi:

Vol. **x**, No. 1, pp. 1-21
February 202x

Support Material

Yixuan Liang, Xiaoxian Tang* and Qian Zhang

School of Mathematical Sciences, Beihang University, Beijing 100191, P.R. China.
Key Laboratory of Mathematics, Informatics and Behavioral Semantics, Ministry of Education, Beijing 100191, China.
State Key Laboratory of Software Development Environment, Beihang University, 100083 Beijing, P.R. China.

Abstract. This is the supporting material for this article.

AMS subject classifications: xxxxx, xxxxx

Key words: Moving mesh method, conservative interpolation, iterative method, l^2 projection.

1 ldd

Lemma 1.1. *Given a network G (3.1) with a one-dimensional stoichiometric subspace, suppose $cap_{pos}(G) < +\infty$. Let $g(z)$ be the function defined in (**??**). For any given $\{d_i\}_{i=1}^s$, κ_1 and κ_2, x is a stable positive steady state of G if and only if the real point z satisfies $g(z) = \ln(-\frac{\lambda\kappa_2}{\kappa_1})$ and $\frac{dg}{dz}(z) < 0$, where x and z satisfy the equalities in (4.7).*

Proof. In bi-reaction network, equation (2.3) is transformed into

$$\begin{pmatrix} f_1 \\ \vdots \\ f_s \end{pmatrix} = \mathcal{N} \cdot \begin{pmatrix} \kappa_1 \prod\limits_{i=1}^{s} x_i^{\alpha_{i1}} \\ \kappa_2 \prod\limits_{i=1}^{s} x_i^{\alpha_{i2}} \end{pmatrix}.$$

By (**??**), we have

$$f_i = \gamma_i \cdot l(x_1, \cdots, x_s)\ , \text{ where } l(x_1, \cdots, x_s) := \kappa_1 \prod_{i=1}^{s} x_i^{\alpha_{i1}} + \lambda\kappa_2 \prod_{i=1}^{s} x_i^{\alpha_{i2}}. \tag{1.1}$$

Since G has a one-dimensional stoichiometric subspace, $\text{rank}(\text{Jac}_f) = 1$. So the only non-zero eigenvalue is $\rho = tr(\text{Jac}_f) = \gamma_1 \frac{\partial l}{\partial x_1} + \cdots + \gamma_s \frac{\partial l}{\partial x_s}$.

*Corresponding author. *Email address:* `xiaoxian@buaa.edu.cn` (Xiaoxian Tang)

According to definitions in section 2.2, $x=(x_1,\cdots,x_s)^T$ is a stable positive steady state if and only if

$$l(x_1,\cdots,x_s)=0, \tag{1.2}$$

$$x_i>0 \text{ for } 1\leqslant i\leqslant s, \tag{1.3}$$

$$\rho=\gamma_1\frac{\partial l}{\partial x_1}+\cdots+\gamma_s\frac{\partial l}{\partial x_s}<0. \tag{1.4}$$

Let

$$\hat{l}(z):=l(\gamma_1(z+d_1),\cdots,\gamma_s(z+d_s)). \tag{1.5}$$

By (1.1),

$$\hat{l}(z)=\kappa_1\prod_{i=1}^{s}(\gamma_i(z+d_i))^{\alpha_{i1}}+\lambda\kappa_2\prod_{i=1}^{s}(\gamma_i(z+d_i))^{\alpha_{i2}}.$$

By (3.3),

$$\hat{l}(z)=\kappa_1\prod_{i=1}^{s}(\gamma_i(z+d_i))^{\alpha_{i2}}\left(\prod_{i=1}^{s}(\gamma_i(z+d_i))^{\alpha_i}+\frac{\lambda\kappa_2}{\kappa_1}\right).$$

By (**??**),

$$\hat{l}(z)=\kappa_1\prod_{i=1}^{s}(\gamma_i(z+d_i))^{\alpha_{i2}}\left(e^{g(z)}+\frac{\lambda\kappa_2}{\kappa_1}\right). \tag{1.6}$$

Then we have

$$\begin{aligned}\frac{dl}{dz}(z)&=\gamma_1\frac{\partial l}{\partial x_1}+\cdots+\gamma_s\frac{\partial l}{\partial x_s}\bigg|_{x_i=\gamma_i z+e_i}\\ &=\kappa_1\prod_{i=1}^{s}(\gamma_i(z+d_i))^{\alpha_{i2}}\left[e^{g(z)}\frac{dg}{dz}(z)+\left(e^{g(z)}+\frac{\lambda\kappa_2}{\kappa_1}\right)\sum_{i=1}^{s}\frac{\alpha_{i2}}{z+d_i}\right].\end{aligned} \tag{1.7}$$

Then we prove the necessity and sufficiency. We assume x and z satisfy the equalities in (4.7):

$\Rightarrow$) If (1.2)–(1.4) hold, by (1.6), we have $e^{g(z)}+\frac{\lambda\kappa_2}{\kappa_1}=0$. So $g(z)=\ln\left(-\frac{\lambda\kappa_2}{\kappa_1}\right)$. Then by (1.7), we have $e^{g(z)}\frac{dg}{dz}(z)<0$. So $\frac{dg}{dz}(z)<0$.

$\Leftarrow$) If $g(z)=\ln(-\frac{\lambda\kappa_2}{\kappa_1})$ and $\frac{dg}{dz}(z)<0$, by (1.6), we have $\hat{l}(z)=0$. Then by (1.7), we have $\frac{dl}{dz}(z)<0$. By (1.2)–(1.4), it indicates that x is a stable positive steady state. □

Lemma 1.2. *Given a network G (2.1) with a one-dimensional stoichiometric subspace, suppose $cap_{pos}(G) < +\infty$. $g(z)$ is defined as in (**??**). G admits multistability if and only if there exist $\{d_i\}_{i=1}^s \subseteq \mathbb{R}$ and $K \in \mathbb{R}$ such that the equation $g(z) = K$ has at least 2 solutions z_1 and z_2 satisfying $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$.*

Proof. G admits multistability if and only if there exist $\{d_i\}_{i=1}^s$, κ_1 and κ_2 such that G has at least 2 stable positive steady state.

By Lemma (1.1), the above condition holds if and only if: $g(z) = \ln(-\frac{\lambda\kappa_2}{\kappa_1})$ has at least 2 solutions z_1 and z_2 satisfying $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$.

Let $K = \ln(\frac{\lambda\kappa_2}{\kappa_1})$ and we come to the conclusion. □

Proof of Lemma 4.1:

Proof. By Corollary 1.2, G admits multistability if and only if there exist $\{d_i\}_{i=1}^s$ and K such that the equation $g(z) = K$ has at least 2 solutions z_1 and z_2 satisfying $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$.

For $i=1,2$, let $\frac{dg}{dz}(z_i) = -t_i < 0$. According to the continuity of functions, for $\min\{\frac{t_1}{2}, \frac{t_2}{2}\}$, there exists $\delta > 0$ such that when $z \in (z_i - \delta,\, z_i + \delta)$, $\frac{dg}{dz}(z) \in (-\frac{3}{2}t_i,\, -\frac{1}{2}t_i)$ and $(z_1 - \delta,\, z_1 + \delta) \cap (z_2 - \delta,\, z_2 + \delta) = \emptyset$.

Then we consider a certain d_i. For any $n \in \mathbb{N}$, we define $g_n(z)$ is the function which replaces d_i with $d_i + \frac{1}{n}$ in $g(z)$. Then we get a function sequence $\{g_n(z)\}$. We have that $\frac{dg_n}{dz}(z)$ uniformly converges to $\frac{dg}{dz}(z)$ in $(z_1 - \delta,\, z_1 + \delta) \cup (z_2 - \delta,\, z_2 + \delta)$. So for $\varepsilon = \frac{t_1}{2}$, there exists an integer N such that for $n > N$ and $z \in (z_1 - \delta,\, z_1 + \delta) \cup (z_2 - \delta,\, z_2 + \delta)$, $\left|\frac{dg_n}{dz}(z) - \frac{dg}{dz}(z)\right| < \frac{1}{2}\min\{t_1, t_2\}$, and hence, $\frac{dg_n}{dz}(z) < 0$.

Moreover, for $i=1,2$, $g_n(z_i - \delta) \to g(z_i - \delta)$, $g_n(z_i + \delta) \to g(z_i + \delta)$ when $n \to \infty$ and $g(z_i - \delta) > 0$, $g(z_i + \delta) < 0$. So there exists $n > N$ such that $g_n(z_i - \delta) > 0$, $g_n(z_i + \delta) < 0$. Thus, $g_n(z) = K$ has a solution z_i^* in $(z_i - \delta,\, z_i + \delta)$ and $\frac{dg_n}{dz}(z_i^*) < 0$.

For any d_i, we can perform the above actions to change d_i, until d_i are distinct from each other. □

Proof of Lemma 4.2:

Proof. (i) $\Rightarrow$): By Corollary 1.2, $g(z)=K$ has at least 2 real solutions. So, $g(z)$ is not monotonic, and there must be $\widetilde{z}$ such that $\frac{dg}{dz}(\widetilde{z})>0$.

$\Leftarrow$): If there exists $\widetilde{z}$ such that $\frac{dg}{dz}(\widetilde{z})>0$, then there exists a neighborhood $(\widetilde{z}-\delta,\widetilde{z}+\delta)\subseteq (p,q)$ such that for any $z\in(\widetilde{z}-\delta,\widetilde{z}+\delta)$, $\frac{dg}{dz}(z)>0$. So, $g(z)$ is increasing in $(\widetilde{z}-\delta,\widetilde{z}+\delta)$. Let $K\in(g(\widetilde{z}-\delta),g(\widetilde{z}+\delta))$. Because $\lim\limits_{z\to q^-}g(z)=-\infty$ and $g(\widetilde{z}+\delta)>K$, the equation $g(z)=K$ has a solution in $(\widetilde{z}+\delta,q)$. Note that $\frac{dg}{dz}(z)=0$ has finitely many solutions since it is rational. $g(z)=K$ has finitely many solutions in I. So, we can define

$$z_1:=\min\{z:g(z)=K,z>\widetilde{z}+\delta\}, \tag{1.8}$$

Similarly,

$$z_2:=\max\{z:g(z)=K,z<\widetilde{z}-\delta\}.$$

If $\frac{dg}{dz}(z_1)>0$, then there exists $\delta_0>0$ such that $z_1-\delta_0\in(\widetilde{z}+\delta,z_1)$ and $g(z_1-\delta_0)<K$. So, $g(z)$=K has a solution in $(\widetilde{z}+\delta,z_1)$, which is a contradiction to (1.8). So, $\frac{dg}{dz}(z_1)\leqslant 0$.Similarly, $\frac{dg}{dz}(z_2)\leqslant 0$.

Note that for any $K\in(g(\widetilde{z}-\delta),g(\widetilde{z}+\delta))$, there exists z_1,$z_2\in I(z_1\neq z_2)$ such that $g(z_1)=g(z_2)=K$, $\frac{dg}{dz}(z_1)\leqslant 0$, and $\frac{dg}{dz}(z_2)\leqslant 0$. When $K\neq K^*$, since $g(z_1)=K$ and $g(z_1^*)=K^*$, then $z_1\neq z_1^*$. Similarly, $z_2\neq z_2^*$. If for every $K\in(g(\widetilde{z}-\delta),g(\widetilde{z}+\delta))$, $\frac{dg}{dz}(z_1)=0$ or $\frac{dg}{dz}(z_2)=0$, then $\frac{dg}{dz}(z)=0$ has infinitely many solutions, which is a contradiction to the fact that $\frac{dg}{dz}(z)$ is rational. So, there exists K such that $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. By Corollary 1.2, G admits multistability.

(ii) (1) $\Rightarrow$): By Corollary 1.2, there exist z_1, $z_2(z_1\neq z_2)$ satisfying $g(z_1)=K$, $g(z_2)=K$, $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. Assume that $z_1<z_2$. By $\frac{dg}{dz}(z_1)<0$, there exists δ such that $z_1+\delta<z_2$ and $g(z_1+\delta)<K$. By $g(z_2)=K$, there exists $\widetilde{z}_2\in(z_1+\delta,z_2)$ such that $\frac{dg}{dz}(\widetilde{z}_2)>0$. Let $\widetilde{z}_1=z_1$. Then, $\widetilde{z}_1,\widetilde{z}_2$ meet the requirements.

$\Leftarrow$): If there exist $\widetilde{z}_1$ and $\widetilde{z}_2$ such that $\frac{dg}{dz}(\widetilde{z}_1)<0$, $\frac{dg}{dz}(\widetilde{z}_2)>0$ and $\widetilde{z}_1<\widetilde{z}_2$.

Let

$$a:=\inf\{z:\frac{dg}{dz}(z^*)\geqslant 0,\text{ for any } z^*\in[z,\widetilde{z}_2]\}. \tag{1.9}$$

By $\frac{dg}{dz}(\widetilde{z}_1)<0$, a is finite and $\widetilde{z}_1<a$. If $\frac{dg}{dz}(a)>0$, then there exists a neighborhood of a such that $\frac{dg}{dz}(z)>0$ in it, which is a contradiction to (1.9). Similarly, $\frac{dg}{dz}(a)<0$ can also lead to a contradiction. So, $\frac{dg}{dz}(a)=0$.

Because $\lim_{z\to p^+}\frac{dg}{dz}(z)=+\infty$ and $\frac{dg}{dz}(\widetilde{z}_1)<0$, $\frac{dg}{dz}(z)$ has a zero point in (p,a). Note that $\frac{dg}{dz}(z)=0$ has finitely many solutions since it is rational. We can let

$$c=\max\{z: z<a, \frac{dg}{dz}(z)=0\}. \tag{1.10}$$

By (1.9), we have

$$\frac{dg}{dz}(z)<0 \text{ for any } z\in(c,a) \text{ and } \frac{dg}{dz}(z)\geqslant 0 \text{ for any } z\in(a,\widetilde{z}_2). \tag{1.11}$$

There should be a graph.

Let $k_1=g(a)$, $k_2=\min\{g(c),g(\widetilde{z}_2)\}$. By (1.11), $k_1<k_2$. Then, we let $K\in(k_1,k_2)$. Since $g(\widetilde{z}_2)>K$ and $\lim_{z\to q^-}\frac{dg}{dz}(z)=-\infty$, $g(z)=K$ has at least one solution in $(\widetilde{z}_2,q)$. So, we can let

$$z^*=\min\{z: g(z)=K, z\in(\widetilde{z}_2,q)\}. \tag{1.12}$$

If $\frac{dg}{dz}(z^*)>0$, then there exists a neighborhood $(z^*-\delta_0, z^*+\delta_0)$ to let $g(z)$ increase in it. So, $g(z)=K$ has solutions in $(\widetilde{z}_2,z^*)$ and it is a contradiction to (1.12). Thus, $\frac{dg}{dz}(z^*)\leqslant 0$. Note that $\frac{dg}{dz}(z)=0$ has finitely many solutions since it is rational. So, there exists $K\in(k_1,k_2)$ such that $\frac{dg}{dz}(z^*)<0$.

Since $g(c)>K$ and $g(a)<K$, $g(z)=K$ has a solution z_1 in (c,a). According to (1.11), $\frac{dg}{dz}(z_1)<0$. Let $z_2=z^*$. Then, z_1 and z_2 meet the requirements.

(2) By (1), there exist $\widetilde{z}_1$ and $\widetilde{z}_2$ such that $\frac{dg}{dz}(\widetilde{z}_1)<0$, $\frac{dg}{dz}(\widetilde{z}_2)>0$ and $\widetilde{z}_1<\widetilde{z}_2$. So, $\frac{dg}{dz}(z)=0$ has at least one solution in $(\widetilde{z}_1,\widetilde{z}_2)$. Note that $\frac{dg}{dz}(z)=0$ has finitely many solutions since it is rational. We can let

$$z_0:=\min\{z: \frac{dg}{dz}(z)=0, z\in(\widetilde{z}_1,\widetilde{z}_2)\}. \tag{1.13}$$

If $\frac{d^2g}{dz^2}(z_0)<0$, then there exists $\delta>0$ such that $z_0-\delta>\widetilde{z}_1$ and $\frac{dg}{dz}(z_0-\delta)>0$. So, $\frac{dg}{dz}(z)=0$ has a solution in $(\widetilde{z}_1,z_0-\delta)$, which is a contradiction to (1.13). Thus, $\frac{d^2g}{dz^2}(z_0)\geqslant 0$.

(iii) The proof is similar to the proof of (ii).

(iv) (1) By Corollary 1.2, there exist $z_1, z_2 (z_1 \neq z_2)$ satisfying $g(z_1)=K$, $g(z_2)=K$, $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$. Assume that $z_1 < z_2$. Because $g(z_1) = g(z_2)$, it is impossible that $\frac{dg}{dz}(z) \leqslant 0$ when $z \in (z_1, z_2)$. So there exists $\tilde{z}_2 \in (z_1, z_2)$ such that $\frac{dg}{dz}(\tilde{z}_2) > 0$. Let $\tilde{z}_1 = z_1$ and $\tilde{z}_3 = z_2$. Then, $\tilde{z}_1, \tilde{z}_2$ and $\tilde{z}_3$ meet the requirements.

(2) By (1), there exist $\tilde{z}_1$, $\tilde{z}_2$ and $\tilde{z}_3$ such that $\frac{dg}{dz}(\tilde{z}_1) < 0$, $\frac{dg}{dz}(\tilde{z}_2) > 0$, $\frac{dg}{dz}(\tilde{z}_3) < 0$ and $\tilde{z}_1 < \tilde{z}_2 < \tilde{z}_3$. So, there exist solutions of $\frac{dg}{dz}(z) = 0$ in $(\tilde{z}_1, \tilde{z}_2)$ and $(\tilde{z}_2, \tilde{z}_3)$ respectively. Note that $\frac{dg}{dz}(z) = 0$ has finitely many solutions since it is rational. We can let

$$\begin{aligned} a &:= \max\{z : \frac{dg}{dz}(z) = 0, z \in (\tilde{z}_1, \tilde{z}_2)\}, \\ b &:= \min\{z : \frac{dg}{dz}(z) = 0, z \in (\tilde{z}_2, \tilde{z}_3)\}. \end{aligned} \tag{1.14}$$

By (1.14), $\frac{dg}{dz}(z) = 0$ has no solution in (a, b). By $\frac{dg}{dz}(\tilde{z}_2) > 0$, we have

$$\frac{dg}{dz}(z) > 0, \text{ for any } z \in (a, b). \tag{1.15}$$

there should be a graph.

If $\frac{d^2g}{dz^2}(a) < 0$, then there exists $\delta > 0$ such that $a + \delta < \tilde{z}_2$ and $\frac{dg}{dz}(a+\delta) < 0$. So, $\frac{dg}{dz}(z) = 0$ has a solution in $(a+\delta, \tilde{z}_2)$, which is a contradiction to (1.14). Thus,

$$\frac{d^2g}{dz^2}(a) \geqslant 0. \tag{1.16}$$

Since $\frac{dg}{dz}(a) = \frac{dg}{dz}(b)$, by Rolle's theorem, $\frac{d^2g}{dz^2}(z)$ has a zero point in (a, b). Note that $\frac{d^2g}{dz^2}(z) = 0$ has finitely many solutions since it is rational. Let

$$z_0 = \min\{z : \frac{d^2g}{dz^2}(z) = 0, z \in (a, b)\}. \tag{1.17}$$

Next, we prove that $\frac{d^3g}{dz^3}(z_0) \leqslant 0$. Assume that $\frac{d^3g}{dz^3}(z_0) > 0$. Then, there exists δ_0 such that $(z_0 - \delta_0, z_0 + \delta_0) \subseteq (a, b)$ and $\frac{d^3g}{dz^3}(z) > 0$ for any $z \in (z_0 - \delta_0, z_0 + \delta_0)$. So $\frac{d^2g}{dz^2}(z_0 - \delta_0) < \frac{d^2g}{dz^2}(z_0) = 0$.

Recall $\frac{d^2g}{dz^2}(a)\geqslant 0$ in (1.16). If $\frac{d^2g}{dz^2}(a)>0$, then $\frac{d^2g}{dz^2}(z)$ has a zero point in $(a,z_0-\delta_0)$. It is a contradiction to (1.17). If $\frac{d^2g}{dz^2}(a)=0$. Since $\frac{d^2g}{dz^2}(z)$ has no zero point in $(a,z_0-\delta_0)$, we have $\frac{d^2g}{dz^2}(z)<0$ for any $z\in(a,z_0-\delta_0)$. That means $\frac{dg}{dz}(z)$ is decreasing in $(a,z_0-\delta_0)$. So $\frac{dg}{dz}(z_0-\delta_0)<\frac{dg}{dz}(a)=0$. It is a contradiction with (1.15).

(v) Let $w(z)=\frac{dg}{dz}(z)-a$. Then, the conclusion follows from (iv)(2).

□

2 zq

Lemma 2.1. *For any $\beta_1,\beta_2,e_1,e_2,x_1,x_2,x_3\in R$ satisfying*

$$\begin{aligned}&\beta_1>0,\beta_2>0,\\&x_i+e_j>0(i=1,2,3,j=1,2),\\&e_1\neq e_2.\end{aligned}\tag{2.1}$$

There exist $a,b,c\in R$ such that

$$\frac{\beta_1}{x_1+e_1}+\frac{\beta_2}{x_1+e_2}=a+\frac{c}{x_1+b},\tag{2.2}$$

$$\frac{\beta_1}{x_2+e_1}+\frac{\beta_2}{x_2+e_2}=a+\frac{c}{x_2+b},\tag{2.3}$$

$$\frac{\beta_1}{x_3+e_1}+\frac{\beta_2}{x_3+e_2}=a+\frac{c}{x_3+b},\tag{2.4}$$

where

$$a>0,b>\min\{e_1,e_2\},\textit{and}\min\{\beta_1,\beta_2\}<c<\beta_1+\beta_2.\tag{2.5}$$

Proof. Without loss of generality, assume $e_1<e_2$. Define $\gamma(y):=\frac{1}{(x_1+y)(x_2+y)(x_3+y)}$. Notice that we can eliminate b and c from (2.2)–(2.4), and we get

$$a=\frac{(e_2-e_1)^2\beta_1\beta_2\gamma(e_1)\gamma(e_2)}{\beta_1\gamma(e_1)+\beta_2\gamma(e_2)}>0.\tag{2.6}$$

Similarly, we can eliminate a and c from (2.2)–(2.4), and we get

$$b=\frac{e_1\beta_1\gamma(e_1)+e_2\beta_2\gamma(e_2)}{\beta_1\gamma(e_1)+\beta_2\gamma(e_2)}.\tag{2.7}$$

Note that

$$b-e_1=\frac{(e_2-e_1)\beta_2\gamma(e_2)}{\beta_1\gamma(e_1)+\beta_2\gamma(e_2)}>0. \tag{2.8}$$

So, we have $b>e_1$. Notice that we can eliminate a from (2.2)–(2.3), and we get

$$c=(x_1+b)(x_2+b)\Big(\frac{\beta_1}{(x_1+e_1)(x_2+e_1)}+\frac{\beta_2}{(x_1+e_2)(x_2+e_2)}\Big) \tag{2.9}$$

Then, by the fact that $b>e_1$, we have

$$\begin{aligned} c&>(x_1+e_1)(x_2+e_1)\Big(\frac{\beta_1}{(x_1+e_1)(x_2+e_1)}+\frac{\beta_2}{(x_1+e_2)(x_2+e_2)}\Big)\\ &=\beta_1+\beta_2\frac{(x_1+e_1)(x_2+e_1)}{(x_1+e_2)(x_2+e_2)}\\ &>\beta_1 \end{aligned}$$

So, $c>\min\{\beta_1,\beta_2\}$. Below we show that

$$c<\beta_1+\beta_2. \tag{2.10}$$

Let

$$\mathcal{H}:=\frac{(e_2-e_1)^2(x_1+x_2+x_3+e_2+2e_1)}{(x_1+e_1)(x_2+e_1)(x_3+e_1)(x_1+e_2)(x_2+e_2)(x_3+e_2)},$$

$$\mathcal{G}:=\frac{(e_2-e_1)^2(x_1+x_2+x_3+e_1+2e_2)}{(x_1+e_1)(x_2+e_1)(x_3+e_1)(x_1+e_2)(x_2+e_2)(x_3+e_2)}.$$

Plugging (2.7) into (2.9), we have

$$\beta_1+\beta_2-c=\frac{\beta_1^2\beta_2}{(x_1+e_1)(x_2+e_1)(x_3+e_1)}\mathcal{H}+\frac{\beta_1\beta_2^2}{(x_1+e_2)(x_2+e_2)(x_3+e_2)}\mathcal{G}. \tag{2.11}$$

Obviously, $\mathcal{H}>0$ and $\mathcal{G}>0$. So, $c<\beta_1+\beta_2$.

□

Lemma 2.2. *Let* $G(z):=\sum_{i=1}^{n}\frac{a_i}{z+d_i}$*, where* $d_i\in R$*, and* $a_i>0$*. Let* $M:=\min_{i\in\{1,\cdots,n\}} d_i$*. Then, for any three different numbers* z_1,z_2,z_3 *satisfying* $z_j>-M\,(j=1,2,3)$*, there exists* $A,D,\theta\in R$ *such that*

$$G_1(z_j)=\frac{A}{z_j+D}+\theta \text{ for } j=1,2,3, \tag{2.12}$$

where

$$\min_{i\in\{1,\cdots,n\}} a_i\le A\le\sum_{i=1}^{n}a_i,\ D\ge M,\ \theta\ge 0 \tag{2.13}$$

Proof. We prove the conclusion by induction.
For $n=1$, let $A=a_1, D=d_1, \theta=0$. Clearly, the conclusion holds.
Assume for $n=m$, the conclusion holds. Below, we prove that for $n=m+1$, the conclusion holds. Note that for $n=m+1$, we have $G(z)=\sum_{i=1}^{m}\frac{a_i}{z+d_i}+\frac{a_{m+1}}{z+d_{m+1}}$ and $M=\min_{i\in\{1,\cdots,m+1\}} d_i$. Let $G^*(z)=\sum_{i=1}^{m}\frac{a_i}{z+d_i}$ and $M^*=\min_{i\in\{1,\cdots,m\}} d_i$. Then, for any $z_j>M$, we have $z_j>M^*$. By hypothesis, there exist $A^*, D^*, \theta\in R$ such that

$$G^*(z_j)=\frac{A^*}{z_j+D^*}+\theta^* \text{ for } j=1,2,3, \tag{2.14}$$

$$\min_{i\in\{1,\cdots,m\}} a_i \leq A^* \leq \sum_{i=1}^{m} a_i, D^*>M^*, \theta^*\geq 0. \tag{2.15}$$

Since $G(z)=G^*(z)+\frac{a_{m+1}}{z+d_{m+1}}$, we have

$$G(z_j)=\frac{a_{m+1}}{z_j+d_{m+1}}+\frac{A^*}{z_j+D^*}+\theta^* \text{ for } j=1,2,3. \tag{2.16}$$

Let $\beta_1=a_{m+1}, \beta_2=A^*, e_1=d_{m+1}, e_2=D^*, x_j=z_j$ for $j=1,2,3$. Note that $\beta_1, \beta_2, e_1, e_2\, x_j$ satisfy the conditions in (2.1). By Lemma 2.1, there exist $A, D, \theta\in R$ such that

$$\frac{a_{m+1}}{z_j+d_{m+1}}+\frac{A^*}{z_j+D^*}=\theta+\frac{A}{z_j+D} \text{ for } j=1,2,3 \tag{2.17}$$

$$\theta>0, D>\min\{d_{m+1}, D^*\}, \text{and} \min\{\alpha_{m+1}, A^*\}<A<\alpha_{m+1}+A^*. \tag{2.18}$$

By (2.16) and (2.17), we have

$$G(z_j)=\frac{A}{z_j+D}+\theta+\theta^* \text{ for } j=1,2,3. \tag{2.19}$$

By (2.15) and (2.18), it is straightforward to check that $\min_{i\in\{1,\cdots,m+1\}} a_i\leq A\leq \sum_{i=1}^{m+1} a_i, D\geq M, \theta+\theta^*\geq 0$. Therefore, the conclusion holds. □

Lemma 2.3. *For any two sequences* $\{a_i\}_{i=0}^n$, $\{e_i\}_{i=0}^n$ *such that*

$$a_i>1, e_i>1, \textit{and} \tag{2.20}$$

$$a_0>\sum_{i=1}^{n} a_i. \tag{2.21}$$

the following inequalities can not hold simultaneously.

$$\frac{a_0}{e_0} \leq \sum_{i=1}^{n} \frac{a_i}{e_i} - 1 \tag{2.22}$$

$$\frac{a_0}{e_0^2} \geq \sum_{i=1}^{n} \frac{a_i}{e_i^2} - 1 \tag{2.23}$$

$$\frac{a_0}{e_0^3} \leq \sum_{i=1}^{n} \frac{a_i}{e_i^3} - 1 \tag{2.24}$$

Proof. We prove by contradiction?. Assume that (2.22)–(2.24) hold. Notice that by (2.20), $a_0 > 0$ and $e_0 > 0$. So, by (2.24), we have

$$\sum_{i \in S_3} \frac{a_i}{e_i^3} \geq 1. \tag{2.25}$$

By (2.20), for any $i \in \{1, 2, \cdots, n\}$, we have $e_i > 1$, and so,

$$\frac{a_i}{e_i{}^2} > \frac{a_i}{e_i{}^3}. \tag{2.26}$$

So, by (2.25), we have

$$\sum_{i=1}^{n} \frac{a_i}{e_i{}^2} > 1 \tag{2.27}$$

By (2.22) and (2.23), we have

$$\sum_{i=1}^{n} \frac{a_i}{e_i^2} - 1 \leq \frac{a}{e^2} = \frac{1}{a} \frac{a^2}{e^2} \leq \frac{1}{a} (\sum_{i=1}^{n} \frac{a_i}{e_i} - 1)^2 \tag{2.28}$$

Then, by (2.21) and (2.28), we have

$$\sum_{i=1}^{n} \frac{a_i}{e_i^2} - 1 \leq \frac{1}{\sum_{i=1}^{n} a_i} (\sum_{i=1}^{n} \frac{a_i}{e_i} - 1)^2. \tag{2.29}$$

We multiply the both sides of (2.29) by $\sum_{i=1}^{n} a_i$, and we get

$$\sum_{i=1}^{n} \frac{a_i}{e_i^2} \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i \leq (\sum_{i=1}^{n} \frac{a_i}{e_i})^2 - 2 \sum_{i=1}^{n} \frac{a_i}{e_i} + 1, \tag{2.30}$$

which is equivalent to

$$\sum_{i,j \in \{1,2,\cdots,n\}, i \neq j} a_i a_j (\frac{1}{e_i} - \frac{1}{e_j})^2 \leq \sum_{i=1}^{n} a_i - 2 \sum_{i=1}^{n} \frac{a_i}{e_i} + 1. \tag{2.31}$$

By (2.31) and (2.27), we have

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i a_j(\frac{1}{e_i}-\frac{1}{e_j})^2 < \sum_{i=1}^{n} a_i - 2\sum_{i=1}^{n}\frac{a_i}{e_i} + \sum_{i=1}^{n}\frac{a_i}{{e_i}^2} = \sum_{i=1}^{n} a_i(1-\frac{1}{e_i})^2. \tag{2.32}$$

By (2.22) and (2.24), we have

$$(\sum_{i=1}^{n}\frac{a_i}{e_i}-1)(\sum_{i=1}^{n}\frac{a_i}{e_i^3}-1) \geq \frac{a^2}{e^4}. \tag{2.33}$$

By (2.23) and (2.27), we have

$$\frac{a^2}{e^4} \geq (\sum_{i=1}^{n}\frac{a_i}{e_i^2}-1)^2. \tag{2.34}$$

So,

$$(\sum_{i=1}^{n}\frac{a_i}{e_i}-1)(\sum_{i=1}^{n}\frac{a_i}{e_i^3}-1) \geq (\sum_{i=1}^{n}\frac{a_i}{e_i^2}-1)^2. \tag{2.35}$$

Note that (2.35) is equivalent to

$$(\sum_{i=1}^{n}\frac{a_i}{e_i})(\sum_{i=1}^{n}\frac{a_i}{e_i^3}) - (\sum_{i=1}^{n}\frac{a_i}{e_i^2})^2 \geq \sum_{i=1}^{n}\frac{a_i}{e_i} + \sum_{i=1}^{n}\frac{a_i}{e_i^3} - 2\sum_{i=1}^{n}\frac{a_i}{e_i^2} = \sum_{i=1}^{n}\frac{a_i}{e_i}(1-\frac{1}{e_i})^2. \tag{2.36}$$

Note that

$$(\sum_{i=1}^{n}\frac{a_i}{e_i})(\sum_{i=1}^{n}\frac{a_i}{e_i^3}) - (\sum_{i=1}^{n}\frac{a_i}{e_i^2})^2 = \sum_{i,j\in\{1,2,\cdots,n\},i\neq j}\frac{a_i a_j}{e_i e_j}(\frac{1}{e_i}-\frac{1}{e_j})^2. \tag{2.37}$$

So,

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j}\frac{a_i a_j}{e_i e_j}(\frac{1}{e_i}-\frac{1}{e_j})^2 \geq \sum_{i=1}^{n}\frac{a_i}{e_i}(1-\frac{1}{e_i})^2. \tag{2.38}$$

By (2.32) and (2.38), we have

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i a_j(\frac{1}{e_i}-\frac{1}{e_j})^2\sum_{i=1}^{n}\frac{a_i}{e_i}(1-\frac{1}{e_i})^2 < \sum_{i,j\in\{1,2,\cdots,n\},i\neq j}\frac{a_i a_j}{e_i e_j}(\frac{1}{e_i}-\frac{1}{e_j})^2\sum_{i=1}^{n} a_i(1-\frac{1}{e_i})^2. \tag{2.39}$$

Let

$$E_1(x,y) = (xy-x)(1-x)^2(x-y)^2, \tag{2.40}$$
$$E_2(x,y,z) = (1-x)^2(yz-x)(y-z)^2 + (1-y)^2(xz-y)(x-z)^2 + (1-z)^2(xy-z)(x-y)^2. \tag{2.41}$$

Note that (2.39) is equivalent to

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i^2 a_j\, E_1(\frac{1}{e_i},\frac{1}{e_j})+\sum_{i,j,k\in\{1,2,\cdots,n\},i\neq j\neq k} a_i a_j a_k\, E_2(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k})>0. \tag{2.42}$$

By (2.20), we have

$$\frac{1}{e_i e_j}-\frac{1}{e_i}=\frac{1}{e_i}(\frac{1}{e_j}-1)<0. \tag{2.43}$$

So, for any $i,j\in\{1,2,\cdots,n\}, i\neq j$, we have

$$E_1(\frac{1}{e_i},\frac{1}{e_j})=(\frac{1}{e_i e_j}-\frac{1}{e_i})(1-\frac{1}{e_i})^2(\frac{1}{e_i}-\frac{1}{e_j})<0. \tag{2.44}$$

So,

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i^2 a_j\, E_1(\frac{1}{e_i},\frac{1}{e_j})<0 \tag{2.45}$$

By (2.20), we have

$$0<\frac{1}{e_i}<1\,(i=1,2,\cdots,n). \tag{2.46}$$

For any $i,j,k\in\{1,2,\cdots,n\}, i\neq j\neq k$, by Lemma 2.4, we have $E_2(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k})<0$. Therefore, for any $i,j,k\in\{1,2,\cdots,n\}, i\neq j$, we have

$$E_2(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k})<0. \tag{2.47}$$

Then, by (2.20), we have

$$\sum_{i,j,k\in\{1,2,\cdots,n\},i\neq j\neq k} a_i a_j a_k\, E_2(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k})<0 \tag{2.48}$$

By (2.45), we have

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i^2 a_j\, E_1(\frac{1}{e_i},\frac{1}{e_j})+\sum_{i,j,k\in\{1,2,\cdots,n\},i\neq j\neq k} a_i a_j a_k\, E_2(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k})<0, \tag{2.49}$$

which contradicts to (2.42). □

Lemma 2.4. *Let*

$$E(x,y,z)=(1-x)^2(yz-x)(y-z)^2+(1-y)^2(xz-y)(x-z)^2+(1-z)^2(xy-z)(x-y)^2. \tag{2.50}$$

Then, for any

$$x,y,z\in(0,1), \tag{2.51}$$

we have $E(x,y,z)<0$.

Proof. Suppose that

$$x \le y \le z. \tag{2.52}$$

Below, we discuss two cases.

(1) Assume that

$$x \ge yz. \tag{2.53}$$

By (2.51) and (2.52), we have

$$xz - y \le yz - y = y(z-1) < 0. \tag{2.54}$$

Similarly, we have

$$xy - z < 0. \tag{2.55}$$

By (2.53), we have

$$E_2(x,y,z) = (1-x)^2(yz-x)(y-z)^2 + (1-y)^2(xz-y)(x-z)^2 + (1-z)^2(xy-z)(x-y)^2 < 0. \tag{2.56}$$

(2) Assume that

$$x < yz \tag{2.57}$$

Note that

$$\begin{aligned}\frac{\partial E_2}{\partial x} =& 2(1-x)(x-yz)(y-z)^2 + 2(1-y)^2(y-xz)(z-x) + 2(1-z)^2(z-xy)(y-x)\\ &+ (1-x)(1-yz)(y-z)^2 + (y+z^2-2xz)(z-x)(1-y)^2 + (z+y^2-2xy)(y-x)(1-z)^2.\end{aligned} \tag{2.58}$$

Below we first prove when for any $y,z \in (0,1)$, $\frac{\partial E_2}{\partial x} > 0$. By (2.51), we have

$$(1-x)(1-yz)(y-z)^2 > 0. \tag{2.59}$$

In fact, since (2.51) and (2.57), we have

$$y + z^2 - 2xz > y + z^2 - 2yz^2 = y(1-z^2) + z^2(1-y) > 0, \tag{2.60}$$

$$x < yz < z. \tag{2.61}$$

So, we have

$$(y + z^2 - 2xz)(z-x)(1-y)^2 > 0. \tag{2.62}$$

Similarly, we have

$$(z+y^2-2xy)(y-x)(1-z)^2. \tag{2.63}$$

By (2.59)–(2.63), we have

$$\frac{\partial E_2}{\partial x} > 2(1-x)(x-yz)(y-z)^2+2(1-y)^2(y-xz)(z-x)+2(1-z)^2(z-xy)(y-x) \tag{2.64}$$

Recall that by (2.61), we have $x<z$. Similarly, we have $x<y$. By (2.51), we have $x+y>z(x+y)$ i.e. $y-xz>yz-x$. Similarly, $z-xy>yz-x$. So, by (2.64), we have

$$\frac{\partial E_2}{\partial x} > 2(yz-x)((1-y)(z-x)^2+(1-z)(y-x)^2-(1-x)(y-z)^2) \tag{2.65}$$

$$=2(yz-x)(y+z-2x)(1-y)(1-z). \tag{2.66}$$

Note that by (2.51) and (2.57), we have

$$y+z-2x > y+z-2yz = y(1-z)+z(1-y) > 0. \tag{2.67}$$

Then by (2.51) and (2.57), we have $\frac{\partial E_2}{\partial x}>0$. By (2.57), we have $E_2(x,y,z)<E_2(yz,y,z)$. Notice that by (2.41), we have

$$E_2(yz,y,z) = (1-y)^2 y(z^2-1)(x-z)^2+(1-z)^2 z(y^2-1)(x-y)^2. \tag{2.68}$$

By (2.51), we have $E_2(yz,y,z)<0$. So, when $x,y,z\in(0,1)$, we have $E_2(x,y,z)<0$.

□

2.0.1 When S_2, S_3 and S_4 are non-empty

According to Theorem 3.1(c)(2), we assume that S_2, S_3 and S_4 are non-empty. By (4.8), we have

$$g(z) = -\sum_{i\in S_2} a_i \ln(-z+d_i)+\sum_{i\in S_3} a_i \ln(-z+d_i)-\sum_{i\in S_4} a_i \ln(z+d_i).$$

Let

$$h(z)=-g(-z)=\sum_{i\in S_2} a_i \ln(z+d_i)-\sum_{i\in S_3} a_i \ln(z+d_i)+\sum_{i\in S_4} a_i \ln(-z+d_i).$$

We will prove by 3.1(c)(1) and Corollary 1.2. Notice that $\frac{dh}{dz}(z)=\frac{dg}{dz}(-z)$. Let $I^*=\{-z|z\in I\}$. Then there exists $z_1,z_2\in I$, for which $g(z_1)=0, g(z_2)=0, \frac{dg}{dz}(z_1)<0, \frac{dg}{dz}(z_2)<0$ if and only if there exists $z_1^*, z_2^*\in I^*$, for which $h(z_1^*)=0, h(z_2^*)=0, \frac{dh}{dz}(z_1^*)<0, \frac{dh}{dz}(z_2^*)<0$. Since we have proved Theorem 3.1(c)(1), by Corollary 1.2, there exists $z_1^*, z_2^*\in I^*$, for which $h(z_1^*)=0, h(z_2^*)=0, \frac{dh}{dz}(z_1^*)<0, \frac{dh}{dz}(z_2^*)<0$ if and only if $\sum_{i\in S_2} a_i > \min_{i\in S_3}\{a_i\}$. By Corollary 1.2, G admits multistability if and only if $\sum_{i\in S_2} a_i > \min_{i\in S_3}\{a_i\}$.

2.0.2 When S_1, S_2 and S_4 are non-empty

According to Theorem 3.1(c)(3), we assume that S_1, S_2 and S_4 are non-empty. By (4.8), we have

$$g(z)=\sum_{i\in S_1}a_i\ln(z+d_i)-\sum_{i\in S_2}a_i\ln(-z+d_i)-\sum_{i\in S_4}a_i\ln(z+d_i).$$

Let

$$h(z)=-g(-z)=-\sum_{i\in S_1}a_i\ln(-z+d_i)+\sum_{i\in S_2}a_i\ln(z+d_i)+\sum_{i\in S_4}a_i\ln(-z+d_i).$$

Similarly, by 3.1(c)(4) and Corollary 1.2, G admits multistability if and only if there exists a subset S_1^* of S_1, $\sum\limits_{i\in S_4}\{a_i\}>\sum\limits_{i\in S_1^*}a_i>\min\limits_{i\in S_4}\{a_i\}$.

3 c′

3.0.1 When S_2, S_3 and S_4 are non-empty

According to Theorem 3.1(c)(2), we assume that S_2, S_3 and S_4 are non-empty. By (4.8), we have

$$g(z)=-\sum_{i\in S_2}a_i\ln(-z+d_i)+\sum_{i\in S_3}a_i\ln(-z+d_i)-\sum_{i\in S_4}a_i\ln(z+d_i).$$

Let

$$h(z)=-g(-z)=\sum_{i\in S_2}a_i\ln(z+d_i)-\sum_{i\in S_3}a_i\ln(z+d_i)+\sum_{i\in S_4}a_i\ln(-z+d_i).$$

We will prove by 3.1(c)(1) and Corollary 1.2. Notice that $\frac{dh}{dz}(z)=\frac{dg}{dz}(-z)$. Let $I^*=\{-z|z\in I\}$. Then there exists $z_1,z_2\in I$, for which $g(z_1)=0, g(z_2)=0, \frac{dg}{dz}(z_1)<0, \frac{dg}{dz}(z_2)<0$ if and only if there exists $z_1^*,z_2^*\in I^*$, for which $h(z_1^*)=0, h(z_2^*)=0, \frac{dh}{dz}(z_1^*)<0, \frac{dh}{dz}(z_2^*)<0$. Since we have proved Theorem 3.1(c)(1), by Corollary 1.2, there exists $z_1^*,z_2^*\in I^*$, for which $h(z_1^*)=0, h(z_2^*)=0, \frac{dh}{dz}(z_1^*)<0, \frac{dh}{dz}(z_2^*)<0$ if and only if $\sum\limits_{i\in S_2}a_i>\min\limits_{i\in S_3}\{a_i\}$. By Corollary 1.2, G admits multistability if and only if $\sum\limits_{i\in S_2}a_i>\min\limits_{i\in S_3}\{a_i\}$.

3.0.2 When S_1, S_2 and S_4 are non-empty

According to Theorem 3.1(c)(3), we assume that S_1, S_2 and S_4 are non-empty. By (4.8), we have

$$g(z)=\sum_{i\in S_1}a_i\ln(z+d_i)-\sum_{i\in S_2}a_i\ln(-z+d_i)-\sum_{i\in S_4}a_i\ln(z+d_i).$$

Let

$$h(z)=-g(-z)=-\sum_{i\in S_1}a_i\ln(-z+d_i)+\sum_{i\in S_2}a_i\ln(z+d_i)+\sum_{i\in S_4}a_i\ln(-z+d_i).$$

Similarly, by 3.1(c)(4) and Corollary 1.2, G admits multistability if and only if there exists a subset S_1^* of S_1, $\sum_{i\in S_4}\{a_i\}>\sum_{i\in S_1^*}a_i>\min_{i\in S_4}\{a_i\}$.

4 b

4.0.1 When S_2 and S_3 are non-empty

According to Theorem 3.1(b)(2), we assume that S_2 and S_3 are non-empty. By (4.8), we have

$$\begin{aligned} g(z) &= -\sum_{i\in S_2}a_i\ln(-z+d_i)+\sum_{i\in S_3}a_i\ln(-z+d_i),\\ \frac{dg}{dz}(z) &= \sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}. \end{aligned}$$

Then we have

$$\frac{d^2g}{dz^2}(z)=\sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}.$$

First, we prove the sufficiency. Assume that the statement(2) in 3.1(b) holds, i.e., there is a subset S_2^* of S_2, such that

$$\sum_{i\in S_3}a_i>\sum_{i\in S_2^*}a_i>\min_{i\in S_3}a_i. \tag{4.1}$$

There exists $q\in S_3$, such that $a_q=\min_{i\in S_3}a_i$. So,

$$\sum_{i\in S_3}a_i>\sum_{i\in S_2^*}a_i \tag{4.2}$$

$$\sum_{i\in S_2^*}a_i>a_q \tag{4.3}$$

Let

$$\begin{aligned} &\text{for } i\in S_2^*, d_i=w_1>1,\\ &\text{for } i\in S_2\setminus S_2^*, d_i=w_2>1,\\ &d_q=1,\\ &\text{for } i\in S_3\setminus\{1\}, d_i=w_3>1 \end{aligned} \tag{4.4}$$

where $d_i>1(i=1,2,3)$.

Then,

$$\frac{dg}{dz}(z)=\frac{\sum\limits_{i\in S_2^*}a_i}{-z+w_1}+\frac{\sum\limits_{i\in S_2\backslash S_2^*}a_i}{-z+w_2}-\frac{a_q}{-z+1}-\frac{\sum\limits_{i\in S_3\backslash\{1\}}a_i}{-z+w_3}.$$

Note that I defined in (**??**) is $(-\infty,1)$. Let

$$h(z)=\frac{\sum\limits_{i\in S_2^*}a_i}{-z+w_1}-\frac{a_q}{-z+1}-\frac{\sum\limits_{i\in S_3\backslash\{1\}}a_i}{-z+w_3}. \tag{4.5}$$

By here need a lemmaLemma4.2(ii)(1), we only need to prove that there exist $z_1,z_2\in I$ such that $\frac{dg}{dz}(z_1)<0$, $\frac{dg}{dz}(z_2)>0$ and $z_1<z_2$. Then, we will choose appropriate number w_1,w_2 and w_3 to make it happen. We first choose w_3. Note that by (4.4):($w_1>1$)

$$h(z)<\frac{\sum\limits_{i\in S_2^*}a_i-a_q}{-z+1}-\frac{\sum\limits_{i\in S_3\backslash\{1\}}a_i}{-z+w_3}. \tag{4.6}$$

Let the RHS of (4.6) be $H(z)$. We solve d from $H(z)<0$, and we get

$$w_3<\frac{\sum\limits_{i\in S_3\backslash\{1\}}a_i+((\sum\limits_{i\in S_2^*}a_i-a_q)z)/(-z+1)}{(\sum\limits_{i\in S_2^*}a_i-a_q)/(-z+1)}. \tag{4.7}$$

Let $\mathcal{N}(z)$ and $\mathcal{D}(z)$ be the numerator and denominator of RHS of (4.7). By (4.3), we have for any $z\in(-\infty,1)$, $\mathcal{N}(z)>0$ and $\mathcal{D}(z)>0$. Note that

$$\begin{aligned}&\mathcal{N}(z)-\mathcal{D}(z)\\ =&\sum_{i\in S_3}a_i-\sum_{i\in S_2^*}a_i,\end{aligned}$$

By (4.2), we have $\sum\limits_{i\in S_3}a_i-\sum\limits_{i\in S_2^*}a_i>0$. So, there exists $z_1\in(-\infty,1)$ such that $\mathcal{N}(z_1)-\mathcal{D}(z_1)>0$. Hence,

$$\frac{\mathcal{N}(z_1)}{\mathcal{D}(z_1)}>1.$$

By (4.7), there exists $w_3>1$, such that $H(z_1)<0$. Note that for any $w_1>1$, we have (4.6). So for any $w_1>1$, we have $h(z_1)<0$.

We solve w_1 from $h(z)>0$, and we get

$$w_1<\frac{\sum\limits_{i\in S_2^*}a_i+\frac{z}{-z+1}a_{3,1}+(\sum\limits_{i\in S_3\backslash\{1\}}a_i)\frac{z}{-z+w_3}}{\frac{a_{3,1}}{-z+1}+(\sum\limits_{i\in S_3\backslash\{1\}}a_i)\frac{1}{-z+w_3}}. \tag{4.8}$$

Let $\widetilde{\mathcal{N}}(z)$ and $\widetilde{\mathcal{D}}(z)$ be the numerator and denominator of RHS of (4.8).

$$\begin{aligned}&\widetilde{\mathcal{N}}(z)-\widetilde{\mathcal{D}}(z)\\=&\sum_{i\in S_2^*}a_i-a_q+(\sum_{i\in S_3\backslash\{1\}}a_i)\frac{z-1}{-z+w_3}.\end{aligned}$$

Since $w_3>1$, we have

$$\lim_{z_2\to 1}\widetilde{\mathcal{N}}(z)-\widetilde{\mathcal{D}}(z)=\sum_{i\in S_2^*}a_i-a_q$$

By (4.3), the above limit is positive. Therefore, we can choose $z_2\in(z_1,1)$ such that

$$\frac{\widetilde{\mathcal{N}}(z_2)}{\widetilde{\mathcal{D}}(z_2)}>1,$$

i.e. the RHS of (4.8) is greater than 1. So, we can choose appropriate $w_1>1$, such that $h(z_2)>0$ and $h(z_1)<0$. By (4.3) and (4.5), $\lim\limits_{z\to-\infty}(-z)h(z)=\sum\limits_{i\in S_3}a_i-\sum\limits_{i\in S_2^*}a_i>0$ Therefore, there exists $z_0\in(-\infty,z_1)$, such that $h(z_0)>0$. We can choose

$$w_2=\max\{\frac{2\sum\limits_{i\in S_2\backslash S_2^*}a_i}{-h(z_1)}+z_1,2\}$$

Therefore,

$$\begin{aligned}\frac{dg}{dz}(z_1)=h(z_1)+\frac{\sum\limits_{i\in S_2\backslash S_2^*}a_i}{-z_1+w_2}\leq\frac{h(z_1)}{2}<0,\\\frac{dg}{dz}(z_0)>h(z_0)>0,\\\frac{dg}{dz}(z_2)>h(z_2)>0.\end{aligned}$$

here need a lemmaAccording to 4.2 (ii)(1), G admits multistability.

$\Leftarrow$)

We will prove that if there does not exist a subset S_2^* of S_2 such that

$$\sum_{i\in S_3}a_i>\sum_{i\in S_2^*}a_i>\min_{i\in S_3}a_i,\tag{4.9}$$

the network G does not admit multistability. We assume that G admits multistability, then we derive a contradiction.

Note that there does not exist a subset S_2^* of S_2 such that $\sum\limits_{i\in S_3}a_i>\sum\limits_{i\in S_2^*}a_i>\min\limits_{i\in S_3}a_i$, then we have two cases.

(Case I) $|S_3|=1$.

(Case II) Notice that $s_1=0$. Assume $a_1\le a_2\le\cdots\le a_{s_2}$. There exists $k\in\{1,2,...,s_2\}$, such that $\sum_{i=1}^{k}a_i\le\min_{i\in S_3}a_i<\sum_{i\in S_3}a_i\le a_{k+1}\le\cdots\le a_{s_2}$.

(Case I) Assume that $|S_3|=1$. Notice that $s_1=0$. Suppose $d_1\le d_2\le\cdots\le d_{s_2}$.

If $d_{s_2+1}<d_1$, notice that by (4.86), for all $i\in\{1,2,...,s_2-1\}$, $\lim_{z\to d_i^+}\frac{dg}{dz}(z)=+\infty$ and $\lim_{z\to d_{i+1}^-}\frac{dg}{dz}(z)=-\infty$. Note that $\frac{dg}{dz}(z)$ is continuous in (d_i,d_{i+1}). So, there exists $z_i\in(d_i,d_{i+1})$ such that $\frac{dg}{dz}(z_i)=0$. Hence, $\frac{dg}{dz}(z)=0$ has at least s_2-1 solutions in $(d_{s_2+1},+\infty)$. Since the numerator of $\frac{dg}{dz}(z)$ is a polynomial with degree s_2, there are no more than s_2 null points in $(-\infty,+\infty)$.
Recall that $I=(-\infty,d_{s_2+1})$ defined in (**??**). Hence, there are no more than 2 solutions in I. By Linkexin, we have $\sum_{i\in S_2}a_i>a_{s_2+1}$. So, there exists R, for any $z<R$, $\frac{dg}{dz}(z)>0$. Notice that $\lim_{z\to d_{s_2+1}^-}\frac{dg}{dz}(z)=-\infty$. According to 4.2(ii)(1),newlemma there are at least 3 solutions in I, which is a contradiction.

If $d_{s_2+1}>d_1$, notice that by (4.86), for all $i\in\{1,2,...,s_2-1\}$, we have $\lim_{z\to d_i^+}\frac{dg}{dz}(z)=+\infty$ and $\lim_{z\to d_{i+1}^-}\frac{dg}{dz}(z)=-\infty$. Note that if $d_{s_2+1}\notin(d_i,d_{i+1})$, $\frac{dg}{dz}(z)$ is continuous in (d_i,d_{i+1}). Therefore, if $d_{s_2+1}\notin(d_i,d_{i+1})$, there exists $z_i\in(d_i,d_{i+1})$ such that $\frac{dg}{dz}(z_i)=0$. Hence, $\frac{dg}{dz}(z)=0$ has at least s_2-2 solutions in $(d_1,+\infty)$. Since the numerator of $\frac{dg}{dz}(z)$ is a polynomial with degree s_2, there are no more than s_2 null points in $(-\infty,+\infty)$. Therefore there are no more than 3 null solutions in I. By (4.64), notice that $\lim_{z\to d_m^+}g(z)=-\infty$, $\lim_{z\to d_n^-}g(z)=+\infty$, $\lim_{z\to d_m^+}\frac{dg}{dz}(z)=+\infty$, and $\lim_{z\to d_n^-}\frac{dg}{dz}(z)=+\infty$. By Lemma 4.2 (iv)(1), there exist $\tilde z_1,\tilde z_2,\tilde z_3\in I$ such that $\frac{dg}{dz}(\tilde z_1)<0$, $\frac{dg}{dz}(\tilde z_2)>0$, $\frac{dg}{dz}(\tilde z_3)<0$ and $\tilde z_1<\tilde z_2<\tilde z_3$. So, there are at least 4 null points in I, which is a contradiction.
newlemma

(Case II) Assume there exists $k\in\{1,2,...,s_2\}$, such that

$$\sum_{i=1}^{k}a_i\le\min_{i\in S_3}a_i\le\sum_{i\in S_3}a_i\le a_{k+1}\le\cdots\le a_n. \tag{4.10}$$

As G admits multistability, I defined in (**??**) is not an empty set. So,

$$I=(d_m,d_n) \tag{4.11}$$

defined in (**??**), we have

$$d_m=-\infty \tag{4.12}$$
$$d_n=\min\{\min_{i\in S_2} d_i,\min_{i\in S_3} d_i\}. \tag{4.13}$$

Since G admits multistability, newlemmaby Lemma 4.2 (iv) (1), there exist $\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3\in I$, such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0, \frac{dg}{dz}(\widetilde{z}_2)>0, \frac{dg}{dz}(\widetilde{z}_3)<0 \text{ and } \widetilde{z}_1<\widetilde{z}_2<\widetilde{z}_3. \tag{4.14}$$

As G admits multistability, there exists $z_0<z_1<z_2,\ z_0,z_1,z_2\in I$, for which $\frac{dg}{dz}(z_0)<0, \frac{dg}{dz}(z_1)>0, \frac{dg}{dz}(z_2)<0$. $\frac{dg}{dz}(z)=\sum_{i\in S_2}\frac{a_{2,i}}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}$. Let $w_1=-z_0+1, a_1=\frac{\min\{-\frac{dg}{dz}(z_0),-\frac{dg}{dz}(z_2)\}}{2}$. Let $h(z)=\frac{a_1}{z+w_1}+\sum_{i\in S_2}\frac{a_{2,i}}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}$ and $I^*=(-w_1,M)$, then $z_0,z_1,z_2\in I^*$ and $h(z_0)<0, h(z_1)>0, h(z_2)<0$. According to 4.2.3, it lead to a contradiction.

4.0.2 When S_1 and S_4 are non-empty

As defined in (4.8):

$$g(z):=\sum_{i\in S_1} a_i\ln(z+d_i)-\sum_{i\in S_4} a_i\ln(z+d_i).$$

Let

$$h(z)=-g(-z)=-\sum_{i\in S_1} a_i\ln(-z+d_i)+\sum_{i\in S_4} a_i\ln(-z+d_i).$$

Notice that $\frac{dh}{dz}(z)=\frac{dg}{dz}(-z)$. Let $I^*=\{-z|z\in I\}$. Then there exists $z_1,z_2\in I$, for which $g(z_1)=0, g(z_2)=0, \frac{dg}{dz}(z_1)<0, \frac{dg}{dz}(z_2)<0$ if and only if there exists $z_1^*,z_2^*\in I^*$, for which $h(z_1^*)=0, h(z_2^*)=0, \frac{dh}{dz}(z_1^*)<0, \frac{dh}{dz}(z_2^*)<0$. According to 4.0.1, there exists $z_1^*,z_2^*\in I^*$, for which $h(z_1^*)=0, h(z_2^*)=0, \frac{dh}{dz}(z_1^*)<0, \frac{dh}{dz}(z_2^*)<0$ if and only if there exists a subset S_1^* of S_1, $\sum_{i\in S_4}\{|a_i|\}|>\sum_{i\in S_1^*}|a_i|>\min_{i\in S_4}\{|a_i|\}$. Therefore G admits multistability if and only if there exists a subset S_1^* of S_1, $\sum_{i\in S_4}\{|a_i|\}|>\sum_{i\in S_1^*}|a_i|>\min_{i\in S_4}\{|a_i|\}$.

References

[1] M. J. Berger and P. Collela, Local adaptive mesh refinement for shock hydrodynamics, J. Comput. Phys., 82 (1989), 62-84.
[2] C. de Boor, Good Approximation By Splines With Variable Knots II, in Springer Lecture Notes Series 363, Springer-Verlag, Berlin, 1973.
[3] Z. J. Tan, T. Tang and Z. R. Zhang, A simple moving mesh method for one- and two-dimensional phase-field equations, J. Comput. Appl. Math., to appear.
[4] E. F. Toro, Riemann Solvers and Numerical Methods for Fluid Dynamics, Springer-Verlag Berlin Heidelbert, 1999.