Supplementary Material

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Abstract. This is the supplementary material for the article "Multistability of Bi-reaction Networks (arXiv:2405.05103)".

1 Proof of Lemmas 4.1–4.2

1.1 Proof of Lemma 4.1

In order to prove Lemma 4.1, we first provide two lemmas.

Lemma 1.1. *Given a network G (3.1) with a one-dimensional stoichiometric subspace, suppose $cap_{pos}(G) < +\infty$. Let $g(z)$ be the function defined as in (4.11). For any given $\{d_i\}_{i=1}^s$, κ_1 and κ_2, x is a stable positive steady state of G iff the real point z satisfies $g(z) = \ln(-\frac{\lambda\kappa_2}{\kappa_1})$ and $\frac{dg}{dz}(z) < 0$, where x and z satisfy the equalities in (4.8).*

Proof. By (4.3), we have

$$f_i = (\beta_{i1} - \alpha_{i1}) \cdot \ell(x_1, \cdots, x_s) \text{ , where } \ell(x_1, \cdots, x_s) := \kappa_1 \prod_{i=1}^{s} x_i^{\alpha_{i1}} + \lambda\kappa_2 \prod_{i=1}^{s} x_i^{\alpha_{i2}}. \tag{1}$$

Since G has a one-dimensional stoichiometric subspace, rank$(\text{Jac}_f) = 1$. So, the only non-zero eigenvalue of Jac_f is $\rho := tr(\text{Jac}_f) = (\beta_{11} - \alpha_{11})\frac{\partial \ell}{\partial x_1} + \cdots + (\beta_{s1} - \alpha_{s1})\frac{\partial \ell}{\partial x_s}$. Notice that $x = (x_1, \cdots, x_s)^\top \in \mathbb{R}^s_{>0}$ is a stable steady state if and only if

$$\ell(x_1, \cdots, x_s) = 0, \text{and} \tag{2}$$

$$\rho = (\beta_{11} - \alpha_{11})\frac{\partial \ell}{\partial x_1} + \cdots + (\beta_{s1} - \alpha_{s1})\frac{\partial \ell}{\partial x_s} < 0. \tag{3}$$

Let

$$\hat{\ell}(z) := \ell((\beta_{11} - \alpha_{11})(z + \mu_1), \cdots, (\beta_{s1} - \alpha_{s1})(z + \mu_s)). \tag{4}$$

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By (1), we have

$$\begin{aligned}\hat{\ell}(z)&=\kappa_1\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i1}}+\lambda\kappa_2\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i2}}.\\ &=\kappa_1\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i2}}\left(\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i1}-\alpha_{i2}}+\frac{\lambda\kappa_2}{\kappa_1}\right).\end{aligned}$$

By (4.10), we have

$$\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i1}-\alpha_{i2}}=e^{g(z)}.$$

So,

$$\hat{\ell}(z)=\kappa_1\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i2}}\left(e^{g(z)}+\frac{\lambda\kappa_2}{\kappa_1}\right). \tag{5}$$

Similarly, we have

$$\begin{aligned}\frac{d\hat{\ell}}{dz}(z)&=(\beta_{11}-\alpha_{11})\frac{\partial\ell}{\partial x_1}+\cdots+(\beta_{s1}-\alpha_{s1})\frac{\partial\ell}{\partial x_s}\Big|_{x_i=(\beta_{i1}-\alpha_{i1})(z+\mu_i)}\\ &=\kappa_1\prod_{i=1}^{s}((\beta_{i1}-\alpha_{i1})(z+\mu_i))^{\alpha_{i2}}\left[e^{g(z)}\frac{dg}{dz}(z)+\left(e^{g(z)}+\frac{\lambda\kappa_2}{\kappa_1}\right)\sum_{i=1}^{s}\frac{\alpha_{i2}}{z+\mu_i}\right].\end{aligned} \tag{6}$$

Next, we prove the necessity and sufficiency. We assume that x and z satisfy the equalities in (4.8).

$\Rightarrow$) If (2) holds, then by (5), we have $e^{g(z)}+\frac{\lambda\kappa_2}{\kappa_1}=0$. So, $g(z)=\ln\left(-\frac{\lambda\kappa_2}{\kappa_1}\right)$. If (2)–(3) hold, then by (6), we have $e^{g(z)}\frac{dg}{dz}(z)<0$. So, $\frac{dg}{dz}(z)<0$.

$\Leftarrow$) If $g(z)=\ln(-\frac{\lambda\kappa_2}{\kappa_1})$ and $\frac{dg}{dz}(z)<0$, then by (5), we have $\hat{\ell}(z)=0$, and by (6), we have $\frac{d\hat{\ell}}{dz}(z)<0$. By (2)–(3), it indicates that x is a stable positive steady state. □

Lemma 1.2. *Given a network G (3.1) with a one-dimensional stoichiometric subspace, suppose $cap_{pos}(G)<+\infty$. Let $g(z)$ be the function defined as in (4.11). Then, G admits multistability iff there exist $\{d_i\}_{i=1}^{s}\subseteq\mathbb{R}$ and $K\in\mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in I defined in (4.12) satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$.*

Proof. Note that G admits multistability iff there exist $\{d_i\}_{i=1}^{s}$, κ_1 and κ_2 such that G has at least 2 stable positive steady states. By Lemma 1.1, the above condition holds iff $g(z)=\ln(-\frac{\lambda\kappa_2}{\kappa_1})$ has at least 2 solutions z_1 and z_2 in I defined in (4.12) satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. Let $K=\ln(-\frac{\lambda\kappa_2}{\kappa_1})$ and we come to the conclusion. □

Proof of Lemma 4.1:

Proof. By Lemma 1.2, G admits multistability iff there exist $\{d_i\}_{i=1}^{s}$ and K such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in I defined in (4.12) satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. For $i=1,2$, let $\frac{dg}{dz}(z_i)=-t_i<0$. According to the continuity of functions, for $\min\{\frac{t_1}{2}, \frac{t_2}{2}\}$, there exists $\delta>0$ such that $(z_1-\delta,\ z_1+\delta)\cap(z_2-\delta,\ z_2+\delta)=\emptyset$ and for any $z\in(z_i-\delta,\ z_i+\delta)$, we have $\frac{dg}{dz}(z)\in(-\frac{3}{2}t_i,\ -\frac{1}{2}t_i)$. For any fixed d_i, for any $n\in\mathbb{N}$, we define $g_n(z):=g(z)|_{d_i=d_i+\frac{1}{n}}$. Notice that the sequence $\{\frac{dg_n}{dz}(z)\}_{n\in\mathbb{N}}$ uniformly converges to $\frac{dg}{dz}(z)$ in $(z_1-\delta,\ z_1+\delta)\cup(z_2-\delta,\ z_2+\delta)$. So, there exists an integer N such that for any $n>N$ and for any $z\in(z_1-\delta,\ z_1+\delta)\cup(z_2-\delta,\ z_2+\delta)$, $\left|\frac{dg_n}{dz}(z)-\frac{dg}{dz}(z)\right|<\frac{1}{2}\min\{t_1,t_2\}$, and hence, $\frac{dg_n}{dz}(z)<0$. Moreover, for $i=1,2$, we have $g_n(z_i-\delta)\to g(z_i-\delta)$ and $g_n(z_i+\delta)\to g(z_i+\delta)$ when $n\to\infty$, and we have $g(z_i-\delta)>K$ and $g(z_i+\delta)<K$. So, there exists $n>N$ such that $g_n(z_i-\delta)>K$ and $g_n(z_i+\delta)<K$. Thus, $g_n(z)=K$ has a solution z_i^* in $(z_i-\delta,\ z_i+\delta)$ and $\frac{dg_n}{dz}(z_i^*)<0$. For any d_i, we can perform the above actions to change d_i until d_i's are distinct from each other. □

1.2 Proof of Lemma 4.2

Proof. (i) $\Rightarrow$) By Lemma 1.2, $g(z)=K$ has at least 2 real solutions in I defined in (4.12). So, $g(z)$ is not monotonic, and hence there exists $\widetilde{z}\in I$ such that $\frac{dg}{dz}(\widetilde{z})>0$.

$\Leftarrow$) If there exists $\widetilde{z}\in I$ such that $\frac{dg}{dz}(\widetilde{z})>0$, then there exists a neighborhood $(\widetilde{z}-\delta,\widetilde{z}+\delta)\subseteq I$ such that $g(z)$ is increasing in $(\widetilde{z}-\delta,\widetilde{z}+\delta)$. For any $K\in(g(\widetilde{z}-\delta),g(\widetilde{z}+\delta))$, since $g(\widetilde{z}+\delta)>K$ and $\lim\limits_{z\to\mathcal{R}^-}g(z)=-\infty$, the equation $g(z)=K$ has a solution in $(\widetilde{z}+\delta,\mathcal{R})$. Note that $\frac{dg}{dz}(z)=0$ has finitely many solutions since it is rational. So, $g(z)=K$ has finitely many solutions in I, and we can define

$$z_1:=\min\{z:g(z)=K, z>\widetilde{z}+\delta\}, \tag{7}$$

Similarly, we can define

$$z_2:=\max\{z:g(z)=K, z<\widetilde{z}-\delta\}.$$

If $\frac{dg}{dz}(z_1)>0$, then there exists $\delta_0>0$ such that $z_1-\delta_0\in(\widetilde{z}+\delta,z_1)$ and $g(z_1-\delta_0)<K$. So, $g(z)=K$ has a solution in $(\widetilde{z}+\delta,z_1)$, which is a contradiction to (8). So, $\frac{dg}{dz}(z_1)\leqslant0$. Similarly, we have $\frac{dg}{dz}(z_2)\leqslant0$. So, we have shown that for any $K\in(g(\widetilde{z}-\delta),g(\widetilde{z}+\delta))$, there exist

$z_1, z_2 \in I (z_1 \neq z_2)$ such that $g(z_1) = g(z_2) = K$, $\frac{dg}{dz}(z_1) \leqslant 0$, and $\frac{dg}{dz}(z_2) \leqslant 0$. Consider $K, K^* \in (g(\tilde{z}-\delta), g(\tilde{z}+\delta))$ $(K \neq K^*)$. Since $g(z_1) = K$ and $g(z_1^*) = K^*$, then $z_1 \neq z_1^*$. Similarly, $z_2 \neq z_2^*$. If for every $K \in (g(\tilde{z}-\delta), g(\tilde{z}+\delta))$, $\frac{dg}{dz}(z_1) = 0$ or $\frac{dg}{dz}(z_2) = 0$, then $\frac{dg}{dz}(z) = 0$ has infinitely many solutions, which is a contradiction to the fact that $\frac{dg}{dz}(z)$ is rational. So, there exists K such that $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$. By Lemma 1.2, G admits multistability.

(ii) $\Rightarrow$) By Lemma 1.2, there exist $z_1, z_2\, (z_1 \neq z_2)$ satisfying $g(z_1) = K$, $g(z_2) = K$, $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$. Assume that $z_1 < z_2$. Since $\frac{dg}{dz}(z_1) < 0$, there exists δ such that $z_1 + \delta < z_2$ and $g(z_1+\delta) < K$. Since $g(z_2) = K$, there exists $\tilde{z}_2 \in (z_1+\delta, z_2)$ such that $\frac{dg}{dz}(\tilde{z}_2) > 0$. Let $\tilde{z}_1 := z_1$. Then, $\tilde{z}_1$ and $\tilde{z}_2$ meet the requirements.

$\Leftarrow$) Assume that there exist $\tilde{z_1}, \tilde{z_2} \in I$ $(\tilde{z_1} < \tilde{z_2})$ such that

$$\frac{dg}{dz}(\tilde{z_1}) < 0, \text{and} \frac{dg}{dz}(\tilde{z_2}) > 0,$$

then there exists $z^* \in (\tilde{z_1}, \tilde{z_2})$, such that $g(z^*) = \min_{z \in (z_1, z_2)} g(z)$. So, there exists $\delta > 0$, such that $\frac{dg}{dz}(z) < 0$ for any $z \in (z^* - \delta, z^*)$ and $\frac{dg}{dz}(z) > 0$ for any $z \in (z^*, z^* + \delta)$. For any $K \in (g(z^*), \min\{g(z^*-\delta), g(z^*+\delta)\})$, as $\lim_{z \to \mathcal{R}^-} g(z) = -\infty$ and $g(z^*+\delta) > K$, the equation $g(z) = K$ has a solution in $(z^*+\delta, \mathcal{R})$. Since $g(z) = K$ has finitely many solutions in I, we can define

$$z_m := \min\{z : g(z) = K, z > z^* + \delta\}. \tag{8}$$

If $\frac{dg}{dz}(z_m) > 0$, then there exists $\delta_0 > 0$ such that $z_m - \delta_0 \in (z^*+\delta, z_m)$ and $g(z_m - \delta_0) < K$. So, $g(z) = K$ has a solution in $(z^*+\delta, z_m)$, which is a contradiction to (8). So, $\frac{dg}{dz}(z_m) \leqslant 0$.

Note that if $K \neq K^*$, $g(z_m) = K$ and $g(z_m^*) = K^*$, then we have $z_m \neq z_m^*$. If for any $K \in (g(z^*), \min\{(g(z^*-\delta), g(z^*+\delta))\})$, $\frac{dg}{dz}(z_m) = 0$. Then $\frac{dg}{dz}(z) = 0$ has infinitely many solutions, which is a contradiction to the fact that $\frac{dg}{dz}(z)$ is rational. So, there exists K such that $\frac{dg}{dz}(z_m) < 0$. Note that $K \in (g(z^*), \min\{(g(z^*-\delta), g(z^*+\delta))\})$, there exists $z_1 \in (z^*-\delta, z^*)$, such that

$$g(z_1) = K, \text{and} \frac{dg}{dz}(z_1) < 0\,.$$

Let z_2 be z_m. So the conclusion holds.

(iii) By Lemma 1.2, there exist z_1, $z_2\, (z_1 \neq z_2)$ satisfying $g(z_1) = K$, $g(z_2) = K$, $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$. Assume that $z_1 < z_2$. Since $g(z_1) = g(z_2)$, it is impossible that $\frac{dg}{dz}(z) \leqslant 0$

for any $z \in (z_1, z_2)$. So, there exists $\widetilde{z}_2 \in (z_1, z_2)$ such that $\frac{dg}{dz}(\widetilde{z}_2) > 0$. Let $\widetilde{z}_1 := z_1$ and $\widetilde{z}_3 := z_2$. Then, $\widetilde{z}_1, \widetilde{z}_2$ and $\widetilde{z}_3$ meet the requirements.

□

2 Proof of Lemmas 4.3–4.6

2.1 Proof of Lemma 4.3

Proof. Without loss of generality, assume that $e_1 < e_2$. Define $\gamma(y) := \frac{1}{(x_1+y)(x_2+y)(x_3+y)}$. Notice that we can eliminate b and c from (4.76) (note that there are three equalities in (4.76) since $i = 1,2,3$), and we get

$$a = \frac{(e_2 - e_1)^2 \beta_1 \beta_2 \gamma(e_1)\gamma(e_2)}{\beta_1 \gamma(e_1) + \beta_2 \gamma(e_2)} > 0. \tag{1}$$

Similarly, we can eliminate a and c from (4.76), and we get

$$b = \frac{e_1 \beta_1 \gamma(e_1) + e_2 \beta_2 \gamma(e_2)}{\beta_1 \gamma(e_1) + \beta_2 \gamma(e_2)}. \tag{2}$$

Note that

$$b - e_1 = \frac{(e_2 - e_1)\beta_2 \gamma(e_2)}{\beta_1 \gamma(e_1) + \beta_2 \gamma(e_2)} > 0. \tag{3}$$

So, we have $b > e_1$. Notice that we can eliminate a from the two equalities for $i = 1,2$ in (4.76), and we get

$$c = (x_1+b)(x_2+b)\left(\frac{\beta_1}{(x_1+e_1)(x_2+e_1)} + \frac{\beta_2}{(x_1+e_2)(x_2+e_2)}\right) \tag{4}$$

Then, by the fact that $b > e_1$, we have

$$\begin{aligned} c &> (x_1+e_1)(x_2+e_1)\left(\frac{\beta_1}{(x_1+e_1)(x_2+e_1)} + \frac{\beta_2}{(x_1+e_2)(x_2+e_2)}\right) \\ &= \beta_1 + \beta_2 \frac{(x_1+e_1)(x_2+e_1)}{(x_1+e_2)(x_2+e_2)} \\ &> \beta_1 \end{aligned}$$

So, $c > \min\{\beta_1, \beta_2\}$. Below we show that

$$c < \beta_1 + \beta_2. \tag{5}$$

Let

$$\mathcal{H} := \frac{(e_2-e_1)^2(x_1+x_2+x_3+e_2+2e_1)}{(x_1+e_1)(x_2+e_1)(x_3+e_1)(x_1+e_2)(x_2+e_2)(x_3+e_2)},$$

and

$$\mathcal{G} := \frac{(e_2-e_1)^2(x_1+x_2+x_3+e_1+2e_2)}{(x_1+e_1)(x_2+e_1)(x_3+e_1)(x_1+e_2)(x_2+e_2)(x_3+e_2)}.$$

Plugging (2) into (4), we have

$$\beta_1+\beta_2-c = \frac{\beta_1^2\beta_2}{(x_1+e_1)(x_2+e_1)(x_3+e_1)}\mathcal{H} + \frac{\beta_1\beta_2^2}{(x_1+e_2)(x_2+e_2)(x_3+e_2)}\mathcal{G}. \tag{6}$$

Obviously, $\mathcal{H} > 0$ and $\mathcal{G} > 0$. So, $c < \beta_1+\beta_2$.

□

2.2 Proof of Lemma 4.4

Proof. We prove the conclusion by induction. For $n=1$, let $A := a_1$, $D := d_1$, $\theta := 0$. Clearly, the conclusion holds. Assume for $n=m$, the conclusion holds. Note that for $n=m+1$, we have $G(z) = \sum_{i=1}^{m} \frac{a_i}{z+d_i} + \frac{a_{m+1}}{z+d_{m+1}}$ and $M = \min_{i\in\{1,\cdots,m+1\}} d_i$. Let $G^*(z) = \sum_{i=1}^{m} \frac{a_i}{z+d_i}$ and $M^* = \min_{i\in\{1,\cdots,m\}} d_i$. Note here, for any $z_j > -M$, we have $z_j > -M^*$. By the inductive hypothesis, there exist $A^*, D^*, \theta \in R$ such that

$$G^*(z_j) = \frac{A^*}{z_j+D^*} + \theta^* (j=1,2,3), \tag{7}$$

$$\min_{i\in\{1,\cdots,m\}} a_i \le A^* \le \sum_{i=1}^{m} a_i,\, D^* > M^*,\, \theta^* \ge 0. \tag{8}$$

Since $G(z) = G^*(z) + \frac{a_{m+1}}{z+d_{m+1}}$, we have

$$G(z_j) = \frac{a_{m+1}}{z_j+d_{m+1}} + \frac{A^*}{z_j+D^*} + \theta^* (j=1,2,3). \tag{9}$$

Let $\beta_1 := a_{m+1}$, $\beta_2 := A^*$, $e_1 := d_{m+1}$, $e_2 := D^*$, $x_j := z_j\,(j=1,2,3)$. Note that $\beta_1, \beta_2, e_1, e_2, x_j$ satisfy the conditions in (4.73) - (4.75). By Lemma 4.3, there exist $A, D, \theta \in R$ such that

$$\frac{a_{m+1}}{z_j+d_{m+1}} + \frac{A^*}{z_j+D^*} = \theta + \frac{A}{z_j+D} \text{ for } j=1,2,3 \tag{10}$$

$$\theta > 0,\, D > \min\{d_{m+1}, D^*\}, \text{and } \min\{\alpha_{m+1}, A^*\} < A < \alpha_{m+1}+A^*. \tag{11}$$

By (9) and (10), we have

$$G(z_j)=\frac{A}{z_j+D}+\theta+\theta^* \text{ for } j=1,2,3. \tag{12}$$

By (8) and (11), it is straightforward to check that $\min_{i\in\{1,\cdots,m+1\}} a_i \leq A \leq \sum_{i=1}^{m+1} a_i$, $D\geq M$, and $\theta+\theta^*\geq 0$. Therefore, the conclusion holds. □

2.3 Proof of Lemma 4.5

Proof. We prove the conclusion by deducing a contradiction. Assume that (4.81)–(4.83) hold simultaneously. Notice that by (4.80), we have $a_0>0$ and $e_0>0$. So, by (4.83), we have

$$\sum_{i\in S_3}\frac{a_i}{e_i^3}\geq 1. \tag{13}$$

Notice that by (4.80), for any $i\in\{1,2,\cdots,n\}$, we have $e_i>1$. So,

$$\frac{a_i}{{e_i}^2}>\frac{a_i}{{e_i}^3}. \tag{14}$$

Then, by (13), we have

$$\sum_{i=1}^{n}\frac{a_i}{{e_i}^2}>1 \tag{15}$$

Notice that by (4.81) and (4.82), we have

$$\sum_{i=1}^{n}\frac{a_i}{e_i^2}-1\leq\frac{a}{e^2}=\frac{1}{a}\frac{a^2}{e^2}\leq\frac{1}{a}\Big(\sum_{i=1}^{n}\frac{a_i}{e_i}-1\Big)^2 \tag{16}$$

Then, by (4.80) and (16), we have

$$\sum_{i=1}^{n}\frac{a_i}{e_i^2}-1\leq\frac{1}{\sum_{i=1}^{n}a_i}\Big(\sum_{i=1}^{n}\frac{a_i}{e_i}-1\Big)^2. \tag{17}$$

We multiply the both sides of (17) by $\sum_{i=1}^{n}a_i$, and we get

$$\sum_{i=1}^{n}\frac{a_i}{e_i^2}\sum_{i=1}^{n}a_i-\sum_{i=1}^{n}a_i\leq\Big(\sum_{i=1}^{n}\frac{a_i}{e_i}\Big)^2-2\sum_{i=1}^{n}\frac{a_i}{e_i}+1, \tag{18}$$

which is equivalent to

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i a_j(\frac{1}{e_i}-\frac{1}{e_j})^2 \leq \sum_{i=1}^{n} a_i - 2\sum_{i=1}^{n}\frac{a_i}{e_i}+1. \tag{19}$$

By (19) and (15), we have

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i a_j(\frac{1}{e_i}-\frac{1}{e_j})^2 < \sum_{i=1}^{n} a_i - 2\sum_{i=1}^{n}\frac{a_i}{e_i}+\sum_{i=1}^{n}\frac{a_i}{e_i{}^2} = \sum_{i=1}^{n} a_i(1-\frac{1}{e_i})^2. \tag{20}$$

On the other hand, by (4.81) and (4.83), we have

$$(\sum_{i=1}^{n}\frac{a_i}{e_i}-1)(\sum_{i=1}^{n}\frac{a_i}{e_i^3}-1) \geq \frac{a^2}{e^4}. \tag{21}$$

By (4.82) and (15), we have

$$\frac{a^2}{e^4} \geq (\sum_{i=1}^{n}\frac{a_i}{e_i^2}-1)^2. \tag{22}$$

So,

$$(\sum_{i=1}^{n}\frac{a_i}{e_i}-1)(\sum_{i=1}^{n}\frac{a_i}{e_i^3}-1) \geq (\sum_{i=1}^{n}\frac{a_i}{e_i^2}-1)^2. \tag{23}$$

Note that (23) is equivalent to

$$(\sum_{i=1}^{n}\frac{a_i}{e_i})(\sum_{i=1}^{n}\frac{a_i}{e_i^3})-(\sum_{i=1}^{n}\frac{a_i}{e_i^2})^2 \geq \sum_{i=1}^{n}\frac{a_i}{e_i}+\sum_{i=1}^{n}\frac{a_i}{e_i^3}-2\sum_{i=1}^{n}\frac{a_i}{e_i^2} = \sum_{i=1}^{n}\frac{a_i}{e_i}(1-\frac{1}{e_i})^2. \tag{24}$$

Note that

$$(\sum_{i=1}^{n}\frac{a_i}{e_i})(\sum_{i=1}^{n}\frac{a_i}{e_i^3})-(\sum_{i=1}^{n}\frac{a_i}{e_i^2})^2 = \sum_{i,j\in\{1,2,\cdots,n\},i\neq j}\frac{a_i a_j}{e_i e_j}(\frac{1}{e_i}-\frac{1}{e_j})^2. \tag{25}$$

So,

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j}\frac{a_i a_j}{e_i e_j}(\frac{1}{e_i}-\frac{1}{e_j})^2 \geq \sum_{i=1}^{n}\frac{a_i}{e_i}(1-\frac{1}{e_i})^2. \tag{26}$$

By (20) and (26), we have

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i a_j(\frac{1}{e_i}-\frac{1}{e_j})^2\sum_{i=1}^{n}\frac{a_i}{e_i}(1-\frac{1}{e_i})^2 < \sum_{i,j\in\{1,2,\cdots,n\},i\neq j}\frac{a_i a_j}{e_i e_j}(\frac{1}{e_i}-\frac{1}{e_j})^2\sum_{i=1}^{n} a_i(1-\frac{1}{e_i})^2. \tag{27}$$

Let

$$E_1(x,y)=(xy-x)(1-x)^2(x-y)^2, \tag{28}$$
$$E_2(x,y,z)=(1-x)^2(yz-x)(y-z)^2+(1-y)^2(xz-y)(x-z)^2+(1-z)^2(xy-z)(x-y)^2. \tag{29}$$

Note that (27) is equivalent to

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i^2 a_j\, E_1\Big(\frac{1}{e_i},\frac{1}{e_j}\Big)+\sum_{i,j,k\in\{1,2,\cdots,n\},i\neq j\neq k} a_i a_j a_k\, E_2\Big(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k}\Big)>0. \tag{30}$$

By (4.80), we have

$$\frac{1}{e_ie_j}-\frac{1}{e_i}=\frac{1}{e_i}\Big(\frac{1}{e_j}-1\Big)<0. \tag{31}$$

So, for any $i,j\in\{1,2,\cdots,n\}$ $(i\neq j)$, we have

$$E_1\Big(\frac{1}{e_i},\frac{1}{e_j}\Big)=\Big(\frac{1}{e_ie_j}-\frac{1}{e_i}\Big)\Big(1-\frac{1}{e_i}\Big)^2\Big(\frac{1}{e_i}-\frac{1}{e_j}\Big)<0. \tag{32}$$

So,

$$\sum_{i,j\in\{1,2,\cdots,n\},i\neq j} a_i^2 a_j\, E_1\Big(\frac{1}{e_i},\frac{1}{e_j}\Big)<0 \tag{33}$$

By (4.80), we have

$$0<\frac{1}{e_i}<1\,(i=1,2,\cdots,n). \tag{34}$$

For any $i,j,k\in\{1,2,\cdots,n\}$ $(i\neq j\neq k)$, by Lemma 4.6, we have $E_2(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k})<0$. Therefore, for any $i,j,k\in\{1,2,\cdots,n\}$ $(i\neq j)$, we have

$$E_2\Big(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k}\Big)<0. \tag{35}$$

Then, by (4.80), we have

$$\sum_{i,j,k\in\{1,2,\cdots,n\},i\neq j\neq k} a_i a_j a_k\, E_2\Big(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k}\Big)<0 \tag{36}$$

So, by (33), we have

$$\sum_{i,j\in\{1,2,\cdots,n\},(i\neq j)} a_i^2 a_j\, E_1\Big(\frac{1}{e_i},\frac{1}{e_j}\Big)+\sum_{i,j,k\in\{1,2,\cdots,n\},i\neq j\neq k} a_i a_j a_k\, E_2\Big(\frac{1}{e_i},\frac{1}{e_j},\frac{1}{e_k}\Big)<0, \tag{37}$$

which contradicts to (30). □

2.4 Proof of Lemma 4.6

Proof. Suppose that

$$x \le y \le z. \tag{38}$$

Below, we discuss two cases.

(1) Assume that

$$x \ge yz. \tag{39}$$

Note that

$$x, y, z \in (0,1). \tag{40}$$

By (38) and (40), we have

$$xz - y \le yz - y = y(z-1) < 0. \tag{41}$$

Similarly, we have

$$xy - z < 0. \tag{42}$$

By (39), we have

$$E_2(x,y,z) = (1-x)^2(yz-x)(y-z)^2 + (1-y)^2(xz-y)(x-z)^2 + (1-z)^2(xy-z)(x-y)^2 < 0. \tag{43}$$

(2) Assume that

$$x < yz. \tag{44}$$

Note that

$$\begin{aligned}\frac{\partial E_2}{\partial x} =& 2(1-x)(x-yz)(y-z)^2 + 2(1-y)^2(y-xz)(z-x) + 2(1-z)^2(z-xy)(y-x) \\ &+ (1-x)(1-yz)(y-z)^2 + (y+z^2-2xz)(z-x)(1-y)^2 + (z+y^2-2xy)(y-x)(1-z)^2.\end{aligned} \tag{45}$$

Below, we first prove that for any $y, z \in (0,1)$, $\frac{\partial E_2}{\partial x} > 0$. By (40), we have

$$(1-x)(1-yz)(y-z)^2 > 0. \tag{46}$$

In fact, by (40) and (44), we have

$$y + z^2 - 2xz > y + z^2 - 2yz^2 = y(1-z^2) + z^2(1-y) > 0, \text{ and} \tag{47}$$

$$x < yz < z. \tag{48}$$

So, we have

$$(y+z^2-2xz)(z-x)(1-y)^2>0. \tag{49}$$

Similarly, we have

$$(z+y^2-2xy)(y-x)(1-z)^2>0. \tag{50}$$

So, by (46)–(50), we have

$$\frac{\partial E_2}{\partial x}>2(1-x)(x-yz)(y-z)^2+2(1-y)^2(y-xz)(z-x)+2(1-z)^2(z-xy)(y-x). \tag{51}$$

Recall that by (48), we have $x<z$. Similarly, we have $x<y$. By (40), we have $x+y>z(x+y)$, i.e. $y-xz>yz-x$. Similarly, we have $z-xy>yz-x$. So, by (51), we have

$$\begin{aligned}\frac{\partial E_2}{\partial x}&>2(yz-x)((1-y)(z-x)^2+(1-z)(y-x)^2-(1-x)(y-z)^2) &(52)\\ &=2(yz-x)(y+z-2x)(1-y)(1-z). &(53)\end{aligned}$$

Note that by (40) and (44), we have

$$y+z-2x>y+z-2yz=y(1-z)+z(1-y)>0. \tag{54}$$

Then, we have $\frac{\partial E_2}{\partial x}>0$. By (44), we have $E_2(x,y,z)<E_2(yz,y,z)$. Notice that by (29), we have

$$E_2(yz,y,z)=(1-y)^2y(z^2-1)(x-z)^2+(1-z)^2z(y^2-1)(x-y)^2. \tag{55}$$

By (40), we have $E_2(yz,y,z)<0$. So, if $x,y,z\in(0,1)$, then we have $E_2(x,y,z)<0$.

□

3 Proof of Theorem 3.1 (c)

3.1 Proof of Theorem 3.1 (c) (1)

According to the hypothesis of Theorem 3.1 (c) (1), we assume that S_2 and S_3 are non-empty. By (4.11), we have

$$\begin{aligned}g(z)&=-\sum_{i\in S_2}a_i\ln(-z+d_i)+\sum_{i\in S_3}a_i\ln(-z+d_i), &(1)\\ \frac{dg}{dz}(z)&=\sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}. &(2)\end{aligned}$$

$\Rightarrow$) First, we prove the sufficiency. Assume that there exists a subset S_2^* of S_2 such that

$$\sum_{i\in S_3} a_i > \sum_{i\in S_2^*} a_i > \min_{i\in S_3}\{a_i\}. \tag{3}$$

The goal is to prove that G admits multistability. Assume that $a_p = \min\limits_{i\in S_3}\{a_i\}$, where $p \in S_3$. First, we let $d_p := 1$. For any $i \in S_2^*$, we let $d_i := w_1$. For any $S_2\backslash S_2^*$, we make all d_i's the same, i.e. we let $d_i := w_2$ for any $i \in S_2\backslash S_2^*$. Similarly, for any $S_3\backslash\{p\}$, we also make all d_i's the same, i.e. we let $d_i := w_3$ for any $i \in S_3\backslash\{p\}$. We assume that $w_i > 1$ $(i=1,2,3)$. Then, by (2), we have

$$\frac{dg}{dz}(z) = \frac{\sum\limits_{i\in S_2^*} a_i}{-z+w_1} + \frac{\sum\limits_{i\in S_2\backslash S_2^*} a_i}{-z+w_2} - \frac{a_p}{-z+1} - \frac{\sum\limits_{i\in S_3\backslash\{p\}} a_i}{-z+w_3}.$$

Note that the interval I defined in (4.12) is $(-\infty, 1)$. By (1) and (2), we have $\lim\limits_{z\to 1^-} g(z) = -\infty$ and $\lim\limits_{z\to 1^-} \frac{dg}{dz}(z) = -\infty$. Similar to the proof of Lemma 4.2 (ii), there exists $K \in \mathbb{R}$ such that the equation $g(z) = K$ has at least 2 solutions z_1 and z_2 in I satisfying $\frac{dg}{dz}(z_1) < 0$ and $\frac{dg}{dz}(z_2) < 0$ iff there exist $\tilde{z}_1, \tilde{z}_2 \in I$ $(\tilde{z}_1 < \tilde{z}_2)$ such that

$$\frac{dg}{dz}(\tilde{z}_1) < 0 \,\text{and}\, \frac{dg}{dz}(\tilde{z}_2) > 0.$$

Then, by Lemma 4.1, we only need to choose proper numbers w_1, w_2, and w_3 such that there exist $\tilde{z}_1, \tilde{z}_2 \in I$ $(\tilde{z}_1 < \tilde{z}_2)$ satisfying $\frac{dg}{dz}(\tilde{z}_1) < 0$ and $\frac{dg}{dz}(\tilde{z}_2) > 0$. Define

$$h(z) := \frac{\sum\limits_{i\in S_2^*} a_i}{-z+w_1} - \frac{a_p}{-z+1} - \frac{\sum\limits_{i\in S_3\backslash\{p\}} a_i}{-z+w_3}. \tag{4}$$

We complete the proof by the following three steps.

(Step 1) In this step, we prove that there exist $w_3 > 1$ and $\tilde{z}_1 \in I = (-\infty, 1)$ such that for any $w_1 > 1$, we have $h(\tilde{z}_1) < 0$. In fact, for any $w_1 > 1$, we have

$$h(z) < \frac{\sum\limits_{i\in S_2^*} a_i - a_p}{-z+1} - \frac{\sum\limits_{i\in S_3\backslash\{p\}} a_i}{-z+w_3}. \tag{5}$$

Let the RHS of (5) be $H(z)$. We solve w_3 from $H(z) < 0$, and we get

$$w_3 < \frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \tag{6}$$

where $\mathcal{N}(z) := \sum_{i\in S_3\setminus\{p\}} a_i(-z+1) + (\sum_{i\in S_2^*} a_i - a_p)z$ and $\mathcal{D}(z) := \sum_{i\in S_2^*} a_i - a_p$. Recall that $a_p = \min_{i\in S_3}\{a_i\}$. Then, by (3), we have

$$\sum_{i\in S_3} a_i > \sum_{i\in S_2^*} a_i, \text{ and} \tag{7}$$

$$\sum_{i\in S_2^*} a_i > a_p. \tag{8}$$

By (8), for any $z\in(0,1)$, we have $\mathcal{N}(z) > 0$ and $\mathcal{D}(z) > 0$. Note that

$$\begin{aligned}&\mathcal{N}(z) - \mathcal{D}(z)\\ =&(\sum_{i\in S_3} a_i - \sum_{i\in S_2^*} a_i)(-z+1).\end{aligned}$$

So, by (7), for any $z\in(0,1)$, we have $\mathcal{N}(z)-\mathcal{D}(z)>0$. Hence, let $\widetilde{z}_1 := \frac{1}{2}$. Then, we have

$$\frac{\mathcal{N}(z_1)}{\mathcal{D}(z_1)} > 1.$$

By (6), there exists $w_3 > 1$ such that $H(\widetilde{z}_1) < 0$. Recall that for any $w_1 > 1$, we have (5). So for any $w_1 > 1$, we have $h(\widetilde{z}_1) < 0$.

(Step 2) In this step, we prove that there exist $w_1 > 1$ and $\widetilde{z}_2 \in (\widetilde{z}_1, 1)$ such that $h(\widetilde{z}_2) > 0$. In fact, we can solve w_1 from $h(z) > 0$, and we get

$$w_1 < \frac{\widetilde{\mathcal{N}}(z)}{\widetilde{\mathcal{D}}(z)}, \tag{9}$$

where

$$\widetilde{\mathcal{N}}(z) := \sum_{i\in S_2^*} a_i + a_p \frac{z}{-z+1} + (\sum_{i\in S_3\setminus\{p\}} a_i)\frac{z}{-z+w_3}$$

and

$$\widetilde{\mathcal{D}}(z) := a_p \frac{1}{-z+1} + (\sum_{i\in S_3\setminus\{p\}} a_i)\frac{1}{-z+w_3}.$$

For any $z\in(0,1)$, we have $\widetilde{\mathcal{N}}(z) > 0$ and $\widetilde{\mathcal{D}}(z) > 0$. Note that

$$\begin{aligned}&\widetilde{\mathcal{N}}(z) - \widetilde{\mathcal{D}}(z)\\ =&\sum_{i\in S_2^*} a_i - a_p + (\sum_{i\in S_3\setminus\{p\}} a_i)\frac{z-1}{-z+w_3}.\end{aligned}$$

Since $w_3 > 1$, we have

$$\lim_{z\to 1}(\widetilde{\mathcal{N}}(z) - \widetilde{\mathcal{D}}(z)) = \sum_{i\in S_2^*} a_i - a_p.$$

By (8), the above limit is positive. Therefore, we can choose $\widetilde{z}_2 \in (\widetilde{z}_1, 1)$ such that

$$\frac{\widetilde{\mathcal{N}}(z_2)}{\widetilde{\mathcal{D}}(z_2)} > 1.$$

So, by (9), we can choose appropriate $w_1 > 1$ such that $h(\widetilde{z}_2) > 0$.

(Step 3) In this step, we prove that there exists $w_2 > 1$ such that $\frac{dg}{dz}(\widetilde{z}_1) < 0$ and $\frac{dg}{dz}(\widetilde{z}_2) > 0$. In fact, we let

$$w_2 = \max\{\frac{2\sum\limits_{i\in S_2\setminus S_2^*} a_i}{-h(\widetilde{z}_1)} + \widetilde{z}_1, 2\}.$$

Therefore, we have

$$\frac{dg}{dz}(\widetilde{z}_1) = h(\widetilde{z}_1) + \frac{\sum\limits_{i\in S_2\setminus S_2^*} a_i}{-\widetilde{z}_1 + w_2} \leq \frac{h(\widetilde{z}_1)}{2} < 0, \text{and} \tag{10}$$

$$\frac{dg}{dz}(\widetilde{z}_2) = h(\widetilde{z}_2) + \frac{\sum\limits_{i\in S_2\setminus S_2^*} a_i}{-\widetilde{z}_2 + w_2} > h(\widetilde{z}_2) > 0. \tag{11}$$

$\Leftarrow$) Next, we prove the necessity. Our goal is to prove that if there does not exist a subset S_2^* of S_2 such that

$$\sum_{i\in S_3} a_i > \sum_{i\in S_2^*} a_i > \min_{i\in S_3}\{a_i\}, \tag{12}$$

then the network G does not admit multistability. Below, we prove the conclusion by deducing a contradiction. Assume that the network G admits multistability. By Lemma 4.1, G admits multistability iff there exist $\{d_i\}_{i=1}^{s} \subset \mathbb{R}$ and $K \in \mathbb{R}$ such that the equation $g(z)=K$ has at least 2 solutions z_1 and z_2 in the interval $I=(\mathcal{L},\mathcal{R})$ defined in (4.12) satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$, where these d_i's are distinct from each other. Assume that $|S_i|=s_i$ (i=2,3), and assume that $S_2=\{1,\cdots,s_2\}$, and $S_3=\{s_2+1,\cdots,s_2+s_3\}$. Note that if there does not exist a subset S_2^* of S_2 such that $\sum\limits_{i\in S_3} a_i > \sum\limits_{i\in S_2^*} a_i > \min\limits_{i\in S_3}\{a_i\}$, then we have the following cases.

(Case 1) $s_3 = 1$.

(Case 2) $s_3 \geq 2$ and for any $i \in S_2$, we have $\sum\limits_{i\in S_3} a_i \leq a_i$.

(Case 3) Assume that $a_{s_1+1} \leq a_{s_1+2} \leq \cdots \leq a_{s_1+s_2}$. There exists $k \in \{1, ..., s_2\}$, such that $\sum\limits_{i=s_1+1}^{s_1+k} a_i \leq \min\limits_{i\in S_3}\{a_i\} < \sum\limits_{i\in S_3} a_i \leq a_{s_1+k+1} \leq \cdots \leq a_{s_1+s_2}$.

Below, we will prove the conclusion by discussing the three cases.

(Case 1) Assume that $s_3=1$. Then, $S_3=\{s_2+1\}$. Suppose $d_1<d_{s_1+2}<\cdots<d_{s_2}$.

(Case 1.1) If $d_{s_2+1}<d_1$, then the interval I defined in (4.12) is $(-\infty, d_{s_2+1})$. Notice that by (2), for any $i\in\{1, ..., s_2-1\}$, $\lim\limits_{z\to d_i^+}\frac{dg}{dz}(z)=-\infty$ and $\lim\limits_{z\to d_{i+1}^-}\frac{dg}{dz}(z)=+\infty$. Note also that $\frac{dg}{dz}(z)$ is continuous in (d_i, d_{i+1}). So, there exists $z_i\in(d_i, d_{i+1})$ such that $\frac{dg}{dz}(z_i)=0$. Hence, $\frac{dg}{dz}(z)=0$ has at least s_2-1 solutions in $(d_1, +\infty)$. Since the numerator of $\frac{dg}{dz}(z)$ is a polynomial with degree s_2, $\frac{dg}{dz}(z)=0$ has no more than s_2 real solutions. Hence, there is at most 1 real solution in $I=(-\infty, d_{s_2+1})$. On the other hand, by (1) and (2), we have $\lim\limits_{z\to d_{s_2+1}^-}g(z)=-\infty$ and $\lim\limits_{z\to d_{s_2+1}^-}\frac{dg}{dz}(z)=-\infty$. By Lemma 4.1, if G admits multistability, then $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. So, there exists a sufficiently small δ such that $g(z_1+\delta)<K$ and $g(z_2-\delta)>K$. By Lagrange's Mean Value Theorem , there exists $z_3\in(z_1, z_2)$ such that $\frac{dg}{dz}(z_3)>0$. Since $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$, $\frac{dg}{dz}(z)=0$ has at least 2 solutions in I, which is a contradiction.

(Case 1.2) If $d_{s_2+1}>d_1$, then the interval I defined in (4.12) is $(-\infty, d_1)$. Notice that by (2), for any $i\in\{1, ..., s_2-1\}$, we have $\lim\limits_{z\to d_i^+}\frac{dg}{dz}(z)=-\infty$ and $\lim\limits_{z\to d_{i+1}^-}\frac{dg}{dz}(z)=+\infty$. So, for any $i\in\{1, ..., s_2-1\}$ satisfying $d_{s_2+1}\notin(d_i, d_{i+1})$, there exists $z_i\in(d_i, d_{i+1})$ such that $\frac{dg}{dz}(z_i)=0$. Note that d_{s_2+1} is located in at most one of the s_2-1 intervals $(d_i, d_{i+1})\,(i\in\{1, ..., s_2-1\})$. Hence, $\frac{dg}{dz}(z)=0$ has at least s_2-2 real solutions in $(d_1, +\infty)$. Since the numerator of $\frac{dg}{dz}(z)$ is a polynomial with degree s_2, $\frac{dg}{dz}(z)=0$ has no more than s_2 real solutions in $(-\infty, +\infty)$. Hence, $\frac{dg}{dz}(z)=0$ has no more than 2 real solutions in I. On the other hand, by (1) and (2), we have $\lim\limits_{z\to d_1^-}g(z)=+\infty$, and $\lim\limits_{z\to d_1^-}\frac{dg}{dz}(z)=+\infty$. By Lemma 4.1, if G admits multistability, then $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. So, there exists a sufficiently small δ such that $g(z_1+\delta)<K$ and $g(z_2-\delta)>K$. By Lagrange's Mean Value Theorem , there exists $z_3\in(z_1, z_2)$ such that $\frac{dg}{dz}(z_3)>0$.

Note that $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. Since $\lim\limits_{z\to d_1^-}\frac{dg}{dz}(z)=+\infty$, $\frac{dg}{dz}(z)=0$ has at least 3 solutions in I, which is a contradiction.

(Case 2) Recall that the hypothesis of this case is that $s_3\geq 2$ and for any $i\in S_2$, we have $\sum\limits_{i\in S_3} a_i \leq a_i$. Notice that the interval I defined in (4.12) is

$$I=(\mathcal{L},\mathcal{R}), \tag{13}$$

where

$$\mathcal{L}=-\infty, \text{and} \tag{14}$$
$$\mathcal{R}=\min\{d_i\}_{i\in S_2\cup S_3}. \tag{15}$$

By Lemma 4.1, if G admits multistability, then $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. So, there exists a sufficiently small δ such that $g(z_1+\delta)<K$ and $g(z_2-\delta)>K$. By Lagrange's Mean Value Theorem, there exists $z_3\in(z_1,z_2)$ such that $\frac{dg}{dz}(z_3)>0$. That means there exist $\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3\in I$ ($\widetilde{z}_1<\widetilde{z}_2<\widetilde{z}_3$), such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0, \frac{dg}{dz}(\widetilde{z}_2)>0, \text{and} \frac{dg}{dz}(\widetilde{z}_3)<0\ . \tag{16}$$

Similar to the proof of Theorem 3.1 (b) (3), if the hypothesis of this case holds, then we have

$$\sum_{i\in S_2}\frac{a_i^2}{(-z_0+d_i)^2}>(\sum_{i\in S_2}\frac{a_i}{-z_0+d_i})^2, \tag{17}$$

which is a contradiction.

(Case 3) Assume there exists $k\in\{1,2,...,s_2\}$, such that

$$\sum_{i=1}^{k}a_i\leq\min_{i\in S_3}a_i\leq\sum_{i\in S_3}a_i\leq a_{k+1}\leq\cdots\leq a_n. \tag{18}$$

As G admits multistability, the interval I defined in (4.12) is not an empty set. So, we have

$$I=(-\infty,\min\{d_i\}_{i\in S_2\cup S_3}). \tag{19}$$

By Lemma 4.1, if G admits multistability, then $g(z)=K$ has at least 2 solutions z_1 and z_2 in I satisfying $\frac{dg}{dz}(z_1)<0$ and $\frac{dg}{dz}(z_2)<0$. So, there exists a sufficiently small δ such that $g(z_1+\delta)<K$ and $g(z_2-\delta)>K$. By Lagrange's Mean Value Theorem, there exists $z_3\in(z_1,z_2)$ such that $\frac{dg}{dz}(z_3)>0$. That means there exist $\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3\in I$ ($\widetilde{z}_1<\widetilde{z}_2<\widetilde{z}_3$), such that

$$\frac{dg}{dz}(\widetilde{z}_1)<0,\ \frac{dg}{dz}(\widetilde{z}_2)>0,\ \text{and}\ \frac{dg}{dz}(\widetilde{z}_3)<0\ . \tag{20}$$

Note that here, we rename z_1, z_3 and z_2 as $\widetilde{z}_1$, $\widetilde{z}_2$ and $\widetilde{z}_3$. Let

$$\mathcal{L}(z):=\frac{1}{z+d}+\sum_{i\in S_2}\frac{a_i}{-z+d_i}-\sum_{i\in S_3}\frac{a_i}{-z+d_i}, \tag{21}$$

where

$$d:=-\widetilde{z}_1+\frac{1}{\min\limits_{i=1,2,3}\{|\frac{dg}{dz}(z_i)|\}}. \tag{22}$$

Notice that by (20), we have

$$\begin{aligned}
&\mathcal{L}(\widetilde{z}_1)=\min_{i=1,2,3}\{|\frac{dg}{dz}(z_i)|\}+\frac{dg}{dz}(\widetilde{z}_1)<0,\\
&\mathcal{L}(\widetilde{z}_2)=\frac{1}{\widetilde{z}_2+d}+\frac{dg}{dz}(\widetilde{z}_2)>0,\ \text{and}\\
&\mathcal{L}(\widetilde{z}_3)=\frac{1}{\widetilde{z}_3-\widetilde{z}_1+\min\limits_{i=1,2,3}\{|\frac{dg}{dz}(z_i)|\}}+\frac{dg}{dz}(\widetilde{z}_3)<\min_{i=1,2,3}\{|\frac{dg}{dz}(z_i)|\}+\frac{dg}{dz}(\widetilde{z}_3)<0.
\end{aligned}$$

Similarly to the proof of Theorem 3.1 (b) (3), if the hypothesis of this case holds, then the three inequalities $\mathcal{L}(\widetilde{z}_1)<0$, $\mathcal{L}(\widetilde{z}_2)>0$, and $\mathcal{L}(\widetilde{z}_3)<0$ will lead to a contradiction. So, the conclusion holds.

3.2 Proof of Theorem 3.1 (c) (2)

According to Theorem 3.1 (c) (2), we assume that S_1 and S_4 are non-empty. So, by (4.11), we have

$$g(z):=\sum_{i\in S_1}a_i\ln(z+d_i)-\sum_{i\in S_4}a_i\ln(z+d_i).$$

Define

$$\widetilde{g}(z):=-g(-z)=-\sum_{i\in S_1}a_i\ln(-z+d_i)+\sum_{i\in S_4}a_i\ln(-z+d_i).$$

Notice that $\dfrac{d\widetilde{g}}{dz}(z) = \dfrac{dg}{dz}(-z)$. Let $I^* := \{-z | z \in I\}$. Then, there exist $z_1, z_2 \in I$ such that $g(z_i) = 0$ and $\dfrac{dg}{dz}(z_i) < 0$ $(i=1,2)$ if and only if there exist $z_1^*, z_2^* \in I^*$ such that $\widetilde{g}(z_i^*) = 0$ and $\dfrac{d\widetilde{g}}{dz}(z_i^*) < 0$ $(i=1,2)$. Note that by the proof of Theorem 3.1 (c) (1), there exist $z_1^*, z_2^* \in I^*$ such that $\widetilde{g}(z_i^*) = 0$ and $\dfrac{d\widetilde{g}}{dz}(z_i^*) < 0$ $(i=1,2)$ if and only if there exists a subset S_2^* of S_2 such that $\sum\limits_{i \in S_3} |a_i| > \sum\limits_{i \in S_2^*} |a_i| > \min\limits_{i \in S_3}\{|a_i|\}$. So, by Lemma 4.1, G admits multistability if and only if there exists a subset S_1^* of S_1 such that $\sum\limits_{i \in S_4} |a_i| > \sum\limits_{i \in S_1^*} |a_i| > \min\limits_{i \in S_4}\{|a_i|\}$.