Homological Methods in Commutative Algebra

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Introduction

Professor Franke started the lecture giving an idea of what the Tor and Ext functors do. Let R be a commutative ring with 1. For an exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and T another R-module, the sequence

$$M' \otimes_R T \longrightarrow M \otimes_R T \longrightarrow M'' \otimes_R T \longrightarrow 0 \tag{1}$$

is exact but usually can't be extended by 0 on the left end. The same is true for

$$0 \longrightarrow \mathrm{Hom}_R(T, M') \longrightarrow \mathrm{Hom}_R(T, M) \longrightarrow \mathrm{Hom}_R(T, M'') \tag{2}$$

and

$$0 \longrightarrow \mathrm{Hom}_R(M'', T) \longrightarrow \mathrm{Hom}_R(M, T) \longrightarrow \mathrm{Hom}_R(M', T) \ , \tag{3}$$

but again, they can't be extended by 0 on the right in general.

Example. Take $R = \mathbb{Z}$ and consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

(*a*) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (1). Then $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$ is the zero morphism, showing that injectivity on the left end fails in (1).

(*b*) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (2). We claim that surjectivity fails on the right end. Indeed, if it was surjective, then $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ would have to have a lift

$$\begin{tikzcd} & \mathbb{Z} \arrow[d] \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, dashed] \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

which it hasn't as $\mathbb{Z}$ is 2-torsion free and thus every morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be 0.

(*c*) Let $T = \mathbb{Z}$ in (3). We claim that that surjectivity fails on the right end, or more specifically, that $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ has no preimage. Indeed, if $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is a preimage of $\mathrm{id}_{\mathbb{Z}}$, i.e. the composition $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ equals $\mathrm{id}_{\mathbb{Z}}$, then f must be given by $f(n) = \frac{n}{2}$ on $2\mathbb{Z}$, but this can't be extended to all of $\mathbb{Z}$, contradiction!

To handle this deficiency, one constructs *derived functors* Tor and Ext, which give rise to long exact sequences

$$\begin{aligned}\dots \longrightarrow \operatorname{Tor}_2^R(M'',T) \longrightarrow \operatorname{Tor}_1^R(M',T) \longrightarrow \operatorname{Tor}_1^R(M,T) \longrightarrow \operatorname{Tor}_1^R(M'',T)&\\ \longrightarrow M' \otimes_R T \longrightarrow M \otimes_R T \longrightarrow M'' \otimes_R T \longrightarrow 0\,,&\end{aligned}$$

as well as

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(T,M') \longrightarrow \operatorname{Hom}_R(T,M) \longrightarrow \operatorname{Hom}_R(T,M'')&\\ \longrightarrow \operatorname{Ext}_R^1(T,M') \longrightarrow \operatorname{Ext}_R^1(T,M) \longrightarrow \operatorname{Ext}_R^1(T,M'') \longrightarrow \operatorname{Ext}_R^2(T,M') \longrightarrow \dots&\end{aligned}$$

and

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(M'',T) \longrightarrow \operatorname{Hom}_R(M,T) \longrightarrow \operatorname{Hom}_R(M',T)&\\ \longrightarrow \operatorname{Ext}_R^1(M'',T) \longrightarrow \operatorname{Ext}_R^1(M,T) \longrightarrow \operatorname{Ext}_R^1(M',T) \longrightarrow \operatorname{Ext}_R^2(M'',T) \longrightarrow \dots\,.&\end{aligned}$$

extending the open ends of (1), (2), and (3) respectively.

A highlight of this lecture will be *Serre's characterization of regularity.*

Theorem. *For a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k, the following are equivalent.*

(*a*) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$ *(i.e., R is regular).*

(*b*) *There is some vanishing bound for* $\operatorname{Tor}_*^R(-,-)$.

(*c*) *…and* $\dim R$ *is such a vanishing bound.*

(*d*) *There is some vanishing bound for* $\operatorname{Ext}_R^*(-,-)$.

(*e*) *…and* $\dim R$ *is again such a vanishing bound.*

From this, one can deduce the following

Corollary. *If R is a regular Noetherian local ring and $\mathfrak{p} \in \operatorname{Spec} R$, then $R_{\mathfrak{p}}$ is regular as well.*

We will also introduce the notion of *Cohen–Macaulay rings* and prove that they are *universally catenary* (which is quite a generalization of what we did in Algebra I, cf. [1, Theorem 10]).

Theorem. *If R is a regular Noetherian local ring or, more generally, a Cohen–Macaulay ring, then it is* **universally catenary***: If A is an R-algebra of finite type and $X \subseteq Y \subseteq Z$ are irreducible closed subsets of* $\operatorname{Spec} A$*, then*

$$\operatorname{codim}(X,Y) + \operatorname{codim}(Y,Z) = \operatorname{codim}(X,Z)\,.$$

1. The Tor and Ext functors

From now on, unless otherwise stated, our rings are commutative with 1.

1.1. Injective and projective modules

Proposition 1 (Baer)**.** *For an R-module N, the following are equivalent.*

(a) *The functor* $\operatorname{Hom}_R(-,N)$ *is exact.*

(b) *For any embedding $M' \hookrightarrow M$ of R-modules,* $\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M',N)$ *is surjective.*

(c) *Property (b) holds for $R = M$. In other words, if $I \subseteq R$ is any ideal, then any morphism $I \to N$ of R-modules extends to a morphism $R \to N$.*

Remark. (a) Since there is a bijection

$$\begin{aligned} \operatorname{Hom}_R(R,M) &\xrightarrow{\sim} M \\ (r \mapsto r\cdot m) &\longleftarrow\!\!\shortmid\, m \\ \left(R \xrightarrow{\varphi} M\right) &\longmapsto \varphi(1)\ , \end{aligned}$$

Proposition 1(c) can be reformulated as that any morphism $I \to N$ for $I \subseteq R$ an ideal has the form $i \mapsto i \cdot m$ for some $m \in M$.

(b) Note that Proposition 1(c) is trivial when $I = 0$.

(c) When $R = \mathbb{Z}$, every ideal $I \subseteq \mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$ and a morphism $n\mathbb{Z} \xrightarrow{\varphi} N$ is uniquely determined by $\varphi(n)$. Thus, an extension $\hat{\varphi}$ of φ to $\mathbb{Z}$ exists iff there is an element $\nu \in N$ such that $n \cdot \nu = \varphi(n)$ (in that case, put $\hat{\varphi}(1) = \nu$). Hence, Proposition 1(c) amounts to whether the abelian group N is *divisible*, that is, whether $N \xrightarrow{\cdot n} N$ is surjective for all $n \in \mathbb{Z}$.

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.