Homological Methods in Commutative Algebra

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Sommersemester 2018

This text consists of notes on the lecture Homological Methods in Commutative Algebra, taught at the University of Bonn by Professor Jens Franke in the summer term (Sommersemester) 2018. The dual lecture Cohomological Methods in Mmutative Algebra was given in the winter term (Wintersemester) 2017/18.

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Introduction

Professor Franke started the lecture giving an idea of what the Tor and Ext functors do. Let R be a commutative ring with 1. For an exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and T another R-module, the sequence

$$M' \otimes_R T \longrightarrow M \otimes_R T \longrightarrow M'' \otimes_R T \longrightarrow 0 \tag{1}$$

is exact but usually can't be extended by 0 on the left end. The same is true for

$$0 \longrightarrow \mathrm{Hom}_R(T, M') \longrightarrow \mathrm{Hom}_R(T, M) \longrightarrow \mathrm{Hom}_R(T, M'') \tag{2}$$

and

$$0 \longrightarrow \mathrm{Hom}_R(M'', T) \longrightarrow \mathrm{Hom}_R(M, T) \longrightarrow \mathrm{Hom}_R(M', T)\ , \tag{3}$$

but again, they can't be extended by 0 on the right in general.

Example. Take $R = \mathbb{Z}$ and consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

(a) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (1). Then $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$ is the zero morphism, showing that injectivity on the left end fails in (1).

(b) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (2). We claim that surjectivity fails on the right end. Indeed, if it was surjective, then $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ would have to have a lift

$$\begin{tikzcd} & \mathbb{Z} \arrow[d] \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, dashed] \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

which it hasn't as $\mathbb{Z}$ is 2-torsion free and thus every morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be 0.

(c) Let $T = \mathbb{Z}$ in (3). We claim that that surjectivity fails on the right end, or more specifically, that $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ has no preimage. Indeed, if $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is a preimage of $\mathrm{id}_{\mathbb{Z}}$, i.e. the composition $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ equals $\mathrm{id}_{\mathbb{Z}}$, then f must be given by $f(n) = \frac{n}{2}$ on $2\mathbb{Z}$, but this can't be extended to all of $\mathbb{Z}$, contradiction!

To handle this deficiency, one constructs *derived functors* Tor and Ext, which give rise to long exact sequences

$$\begin{aligned}\dots \longrightarrow \operatorname{Tor}_2^R(M'',T) \longrightarrow \operatorname{Tor}_1^R(M',T) \longrightarrow \operatorname{Tor}_1^R(M,T) \longrightarrow \operatorname{Tor}_1^R(M'',T)& \\ \longrightarrow M' \otimes_R T \longrightarrow M \otimes_R T \longrightarrow M'' \otimes_R T \longrightarrow 0\,,&\end{aligned}$$

as well as

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(T,M') \longrightarrow \operatorname{Hom}_R(T,M) \longrightarrow \operatorname{Hom}_R(T,M'')& \\ \longrightarrow \operatorname{Ext}_R^1(T,M') \longrightarrow \operatorname{Ext}_R^1(T,M) \longrightarrow \operatorname{Ext}_R^1(T,M'') \longrightarrow \operatorname{Ext}_R^2(T,M') \longrightarrow \dots&\end{aligned}$$

and

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(M'',T) \longrightarrow \operatorname{Hom}_R(M,T) \longrightarrow \operatorname{Hom}_R(M',T)& \\ \longrightarrow \operatorname{Ext}_R^1(M'',T) \longrightarrow \operatorname{Ext}_R^1(M,T) \longrightarrow \operatorname{Ext}_R^1(M',T) \longrightarrow \operatorname{Ext}_R^2(M'',T) \longrightarrow \dots\,.&\end{aligned}$$

extending the open ends of (1), (2), and (3) respectively.

A highlight of this lecture will be *Serre's characterization of regularity.*

Theorem. *For a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k, the following are equivalent.*

(*a*) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$ *(i.e., R is regular).*

(*b*) *There is some vanishing bound for* $\operatorname{Tor}_*^R(-,-)$.

(*c*) *... and* $\dim R$ *is such a vanishing bound.*

(*d*) *There is some vanishing bound for* $\operatorname{Ext}_R^*(-,-)$.

(*e*) *... and* $\dim R$ *is again such a vanishing bound.*

From this, one can deduce the following

Corollary. *If R is a regular Noetherian local ring and $\mathfrak{p} \in \operatorname{Spec} R$, then $R_\mathfrak{p}$ is regular as well.*

We will also introduce the notion of *Cohen–Macaulay rings* and prove that they are *universally catenary* (which is quite a generalization of what we did in Algebra I, cf. [1, Theorem 10]).

Theorem. *If R is a regular Noetherian local ring or, more generally, a Cohen–Macaulay ring, then it is* **universally catenary***: If A is an R-algebra of finite type and $X \subseteq Y \subseteq Z$ are irreducible closed subsets of* $\operatorname{Spec} A$*, then*

$$\operatorname{codim}(X,Y) + \operatorname{codim}(Y,Z) = \operatorname{codim}(X,Z)\,.$$

1. Tor and Ext of R-modules

From now on, unless otherwise stated, our rings are commutative with 1.

1.1. Injective and projective modules and properties of Ext^*_R

Proposition 1 (Baer's criterion)**.** *For an R-module N, the following are equivalent.*

(*a*) *The functor* $\mathrm{Hom}_R(-, N)$ *is exact.*

(*b*) *For any embedding $M' \hookrightarrow M$ of R-modules,* $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$ *is surjective.*

(*c*) *Property* (*b*) *holds for $R = M$. In other words, if $I \subseteq R$ is any ideal, then any morphism $I \to N$ of R-modules extends to a morphism $R \to N$.*

Remark 1. (*a*) Since there is a bijection

$$\begin{aligned} \mathrm{Hom}_R(R, M) &\xrightarrow{\sim} M \\ (r \mapsto r \cdot m) &\longleftarrow m \\ \left(R \xrightarrow{\varphi} M\right) &\longmapsto \varphi(1) \ , \end{aligned}$$

Proposition 1(*c*) can be reformulated as that any morphism $I \to N$ for $I \subseteq R$ an ideal has the form $i \mapsto i \cdot m$ for some $m \in M$.

(*b*) Note that Proposition 1(*c*) is trivial when $I = 0$.

(*c*) When $R = \mathbb{Z}$, every ideal $I \subseteq \mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$ and a morphism $n\mathbb{Z} \xrightarrow{\varphi} N$ is uniquely determined by $\varphi(n)$. Thus, an extension $\hat{\varphi}$ of φ to $\mathbb{Z}$ exists iff there is an element $\nu \in N$ such that $n \cdot \nu = \varphi(n)$ (in that case, put $\hat{\varphi}(1) = \nu$). Hence, Proposition 1(*c*) amounts to whether the abelian group N is *divisible*, that is, whether $N \xrightarrow{\cdot n} N$ is surjective for all $n \in \mathbb{Z}$ (also cf. Definition 2).

Definition 1. (*a*) An R-module is called **injective** if it satisfies the equivalent conditions from Proposition 1.

(*b*) In an arbitrary category $\mathcal{A}$, an object I is called **injective** if for every monomorphism $X \hookrightarrow Y$, the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I)$ is surjective, that is, for every morphism $X \xrightarrow{\varphi} I$ there is a (usually non-unique) lift

$$\begin{tikzcd} I & \\ X \arrow[u, "\varphi"] \arrow[r, hook] & Y \arrow[ul, dashed, "\exists\, \hat{\varphi}"'] \end{tikzcd}$$

Proof of Proposition 1. The implication $(b) \Rightarrow (c)$ is trivial. Let's prove $(c) \Rightarrow (b)$. Let $M \xrightarrow{f} N$ be a morphism of R-modules and consider

$$\mathfrak{M} = \left\{(Q, \varphi) \mid M \subseteq Q \subseteq M' \text{ and } \widetilde{\varphi} \in \mathrm{Hom}_R(Q, N) \text{ such that } \varphi|_M = f\right\} .$$

$\mathfrak{M}$ becomes a partially ordered set via $(Q_1, \varphi_1) \preceq (Q_2, \varphi_2) \Leftrightarrow Q_1 \subseteq Q_2$ and $\varphi_2|_{Q_1} = \varphi_1$. Then it's easy to see that Zorn's lemma is applicable, hence $\mathfrak{M}$ has a $\preceq$-maximal element (Q_*, φ_*). If (c) is satisfied and $Q_* \subsetneq M'$, there is an $m \in M' \setminus L_*$. Let $I = \{r \in R \mid rm \in Q_*\}$ and let $I \xrightarrow{g} N$ be given by $g(r) = \varphi_*(rm)$. By (c), there is a morphism $R \xrightarrow{\gamma} N$ extending g, i.e., a $\nu \in N$ such that $\varphi_*(rm) = r\nu$ when $r \in I$ (using Remark 1(a)). Let $\widetilde{Q} = Q_* + Rm$ and $\widetilde{\varphi}(m_* + rm) = \varphi_*(m_*) + r\nu$ for $m_* \in Q_*$ and $r \in R$, then it's easy to see that $\widetilde{\varphi}$ is well-defined and $(Q_*, \varphi_*) \prec (\widetilde{Q}, \widetilde{\varphi})$, a contradiction.

The equivalence $(a) \Leftrightarrow (b)$ is easy to see as for any short exact sequence $0 \to X \to Y \to Z \to 0$, the sequence $0 \to \mathrm{Hom}_R(Z, N) \to \mathrm{Hom}_R(Y, N) \to \mathrm{Hom}_R(X, N)$ is exact anyways and (b) implies exactness at the right end. *q.e.d.*

Definition 2. If R is a domain and M an R-module, then M is called **divisible** if $M \xrightarrow{r\cdot} M$ is surjective for all $r \in R \setminus \{0\}$

Corollary 1. (a) *When R is a domain, the property from Proposition 1(c) for principal ideals I is equivalent do divisibility of N.*

(b) *Any injective module N is divisible in the following sense: If $r \in R$ is not a zero divisor, $N \xrightarrow{r\cdot} N$ is surjective.*

(c) *In particular, if N is injective and $S \subseteq R$ a multiplicative subset not containing zero divisors, then the morphism $N \to N_S$ to the localization of N at S is surjective.*

Proof. Part (a) can be seen using the arguments from Remark 1(c). For (b), note that $R \xrightarrow{r\cdot} R$ is injective when r is no zero divisor, hence, for any $n \in N$, the morphism $\varphi \in \mathrm{Hom}_R(R, N)$ given by $\varphi(1) = n$ extends to $\hat{\varphi} \in \mathrm{Hom}_R(R, N)$ such that $\varphi = r\hat{\varphi}$. Then $\hat{\varphi}(1)$ is a preimage of n under $N \xrightarrow{r\cdot} N$. Part (c) follows from (b) and the universal property of localization. *q.e.d.*

Remark. Note that $R = \mathbb{Z}/p^2\mathbb{Z}$, for $p \in \mathbb{Z}$ a prime, is injective over itself, but $R \xrightarrow{p\cdot} R$ fails to be injective. Indeed, the only ideal of R where Baer's criterion is in question is $(p) \subseteq R$. We need to show that any R-morphism $(p) \to R$ extends to an R-morphism $R \to R$. But any $(p) \xrightarrow{\varphi} R$ maps p to the p-torsion part of R, i.e., to (p) itself, hence is given by $\varphi(p) = rp$ for some $r \in R$ and can be extended via $\hat{\varphi}$ given by $\hat{\varphi}(1) = r$. This shows that Corollary 1(b) is somewhat sharp.

Corollary 2. *A module over a principal ideal domain is injective iff it is divisible.*

Proof. Follows from Corollary 1(a). *q.e.d.*

Remark. The same holds for Dedekind domains, see Corollary 6 (which is not there yet).

Corollary 3. *When R is a principal ideal domain, then any quotient of an injective module is injective again. The category of R-modules has* **sufficiently many injective objects** *in* *the sense that for any object X there is a monomophism $X \hookrightarrow I$ with I injective. Thus, any R-module X has an* **injective resolution**, *i.e., an exact sequence*

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

with injective objects $I^0, I^1, I^2, \dots$. In fact, any R-module, for R a principal ideal domain, has an injective resolution $0 \to X \to I^0 \to I^1 \to 0$ of length 1.

Proof. The first assertion follows as the quotient of divisible modules is divisible again. Note that K/R is divisible, K being the quotient field of R, hence it is injective. If M is any R-module and $m \in M \setminus \{0\}$. We have to distinguish to cases.

Case 1. Suppose $\mathrm{Ann}_R(m)$ is non-zero, i.e., $\mathrm{Ann}_R(m) = (\alpha)$ for some $\alpha \in R \setminus \{0\}$ (remember we have a principal ideal domain). Then we have a morphism from $Rm \subseteq M$ to K/R given by $rm \mapsto \frac{r}{\alpha} \bmod R$ (note that modding out R is necessary for this to be well-defined – we couldn't just have used K). By injectivity of K/R, there is an extension $M \xrightarrow{\varphi_m} K/R$, satisfying $\varphi_m(m) \neq 0$. Let $I_m \subseteq K/R$ be the target of φ_m.

Case 2. If $\mathrm{Ann}_R(m) = 0$, we get a morphism from $Rm \subseteq M$ to K instead, sending $rm \mapsto r$ (this time, using K doesn't cause problems thanks to $\mathrm{Ann}_R(m) = 0$). By injectivity of K, this extends to a morphism $M \xrightarrow{\varphi_m} K$ such that $\varphi_m(m) \neq 0$. Let $I_m = K$ be the target of φ_m.

Now put $I = \prod_{m \in M \setminus \{0\}} I_m$. Then I is divisible (since every I_m is), hence injective, and $M \to I$, $\mu \mapsto (\varphi_m(\mu))_{m \in M \setminus \{0\}}$ is a monomorphism. As a quotient of $I^0 = I$, $I^1 = \mathrm{coker}\,(M \to I^0)$ is injective as well, hence $0 \to M \to I^0 \to I^1 \to 0$ is an injective resolution of length 1. *q.e.d.*

Proposition 2 (a.k.a. "Satz 2")**.** *For any ring R, the category of R-modules has sufficiently many injective objects.*

Proof. This will follow from Lemma 1(b) and (c) below. *q.e.d.*

Remark. This holds in vast more generality, and in particular, Proposition 2 follows immediately from the following theorem, which, however, we are not going to prove in this lecture.

Theorem (Grothendieck)**.** *Any AB5 category with a generator has sufficiently many injective objects.*

Lemma 1. *Let R be any ring.*

(a) *The forgetful functor from R-Mod to the category $\mathbb{Z}$-Mod of abelian groups has a right-adjoint functor, namely $\mathrm{Hom}_{\mathbb{Z}}(R, -)$. That is, there is a bijection*

$$\mathrm{Hom}_{\mathbb{Z}}(M, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \tag{$*$}$$

for any R-module M and any abelian group A. Here, we equip $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ with an R-module structure via $(r \cdot \varphi)(x) = \varphi(xr)$ for $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$ and $r, x \in R$.

(b) *For any injective abelian group I, $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module.*

(c) *Let M be any R-module and I and abelian group and $M \xhookrightarrow{\varphi} I$ a monomorphism of abelian groups, then the R-morphism $M \to \mathrm{Hom}_{\mathbb{Z}}(R, I)$ obtained by applying ($*$) is injective.*

Proof. Part (a). The proof given in the lecture was rather computational, so I decided to include a more elegant one. It is easy to see that $\mathrm{Hom}_{\mathbb{Z}}(R,-)$ is indeed a functor $\mathbb{Z}$-Mod $\to R$-Mod. From the well-known tensor-hom adjunction we obtain a canonical bijection

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R R, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \ .$$

But M is an R-module and so $M \otimes_R R \simeq M$ canonically, proving $(*)$.

Part (b). Since the forgetful functor R-Mod $\to \mathbb{Z}$-Mod clearly preserves injectivity of morphisms (i.e., monomorphisms), this comes down to the following more general fact about adjoint pairs of functors.

Fact 1. Let $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ be an adjoint pairs of functors. Suppose that L preserves monomorphisms. Then R preserves injective objects.

Proof of Fact 1. Let $I \in \mathrm{Ob}(\mathcal{B})$ be injective and $X \hookrightarrow Y$ be a monomorphism in $\mathcal{A}$. By assumption, $LX \hookrightarrow LY$ is a monomorphism in $\mathcal{B}$. In the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(Y, RI) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathcal{A}}(X, RI) \arrow[d, "\wr"] \\ \mathrm{Hom}_{\mathcal{B}}(LY, I) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(LX, I) \end{tikzcd}$$

the lower horizontal arrow is surjective by injectivity of I, hence so is the upper horizontal arrow. *q.e.d.*

Back to the proof of Lemma 1 and let's prove (c). Let $M \overset{\varphi}{\hookrightarrow} I$ be a monomorphism of abelian groups. The corresponding morphism $M \xrightarrow{\psi} \mathrm{Hom}_{\mathbb{Z}}(R, I)$ sends $m \in M$ to $\psi(m)\colon R \to I$ given by $\psi(m)(r) = \varphi(rm)$. If $\psi(m)$ is the zero morphism for some $m \in M$, then $0 = \psi(m)(1) = \varphi(m)$, proving $m = 0$ by injectivity of φ. Then ψ is also injective. *q.e.d.*

A. Appendix – category theory corner

A.1. Derived functors and Ext^*_R

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories (cf. [2, Definition A.1.4]). A **homological ∂-functor** $F_*\colon \mathcal{A} \to \mathcal{B}$ is a sequence $(F_n)_{n\geq 0}$ of additive functors $\mathcal{A} \xrightarrow{F_i} \mathcal{B}$ together with a natural transformation $\partial = \partial_F\colon F_{i+1}(A'') \to F_i(A')$ on the category of short exact sequences on $\mathcal{A}$, such that the sequence

$$\begin{aligned} \ldots \longrightarrow F_{i+1}(A'') \xrightarrow{\partial} F_i(A') \longrightarrow F_i(A) \longrightarrow F_i(A'') \xrightarrow{\partial} \ldots \\ \ldots \longrightarrow F_1(A'') \xrightarrow{\partial} F_0(A') \longrightarrow F_0(A) \longrightarrow F_0(A'') \longrightarrow 0\,. \end{aligned}$$

is exact whenever $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{A}$.

A **morphism** $F_* \xrightarrow{\varphi} G_*$ of homological ∂-functors is a sequence $(\varphi_n)_{n\geq 0}$ of natural transformations $F_i \xrightarrow{\varphi_i} G_i$ such that for any short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd} F_{i+1}(A'') \arrow[r, "\partial_F"] \arrow[d, "\varphi_{i+1}"'] & F_i(A') \arrow[d, "\varphi_i"] \\ G_{i+1}(A'') \arrow[r, "\partial_G"] & G_i(A') \end{tikzcd}$$

commutes.

Similarly, a **cohomological ∂-functor** $F^*\colon \mathcal{A} \to \mathcal{B}$ is a sequence $(F^n)_{n\geq 0}$ of additive functors $\mathcal{A} \xrightarrow{F^i} \mathcal{B}$ together with connecting morphism $\partial = \partial_F\colon F^i(A'') \to F^{i+1}(A')$ such that for a short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence

$$\begin{aligned} 0 \longrightarrow F^0(A') \longrightarrow F^0(A) \longrightarrow F^0(A'') \xrightarrow{\partial} F^1(A') \longrightarrow \ldots \\ \ldots \xrightarrow{\partial} F^i(A') \longrightarrow F^i(A) \longrightarrow F^i(A'') \xrightarrow{\partial} F^{i+1}(A') \longrightarrow \ldots \end{aligned}$$

is required to be exact. And the notion of a **morphism** $F_* \xrightarrow{\varphi} G_*$ of cohomological ∂-functors is defined in the obvious way.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a *right-exact* functor, i.e., for any short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence $FA' \to FA \to FA'' \to 0$ is exact. A **left-derived functor** of F is a homological functor L_*F from $\mathcal{A}$ to $\mathcal{B}$ with a natural isomorphism $L_0F \simeq F$ such that for any homological functor $\Phi_*\colon \mathcal{A} \to \mathcal{B}$, any natural transformation $\Phi_0 \to L_0F$ extends in a unique way to a morphism $\Phi_* \to L_*F$ of homological functors.

Similar, a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is *right-exact* functor if $0 \to FA' \to FA \to FA''$ is exact for any short exact sequence $0 \to A' \to A \to A'' \to 0$. A **right-derived functor** of F is a homological functor R^*F from $\mathcal{A}$ to $\mathcal{B}$ with a natural isomorphism $R^0F \simeq F$ such that for any homological functor $\Psi^*\colon \mathcal{A} \to \mathcal{B}$, any natural transformation $R^0F \to \Psi^0$ extends in a unique way to a morphism $R^*F \to \Psi^*$ of homological functors.

Remark 1. (*a*) It follows (by the usual Yoneda argument) that derived functors are unique up to unique isomorphism of (co)homological functors if they exist.

(*b*) If F is left-exact in the above sense, it preserves monomorphisms and it can be shown that $0 \to FX' \to FX \to FX''$ is exact even when only $0 \to X' \to X \to X''$ is exact (a nasty technical proof which won't appear here). Similar for right-exact functors.

(*c*) A generalized definition drops the exactness assumptions and requires $F \to L_0F$ with the universal property that any diagram

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & L_0F \\ & \Phi_0 \arrow[ur] & \end{tikzcd}$$

can be uniquely extended to a morphism $\Phi_* \to L_*F$ of homological functors. Similar for right-derived functors.

Example 1. If F is an exact functor, then left- and right-derived functors of F are given by $L_0F = R^0F = F$ and $L_iF = R^iF = 0$ for $i \geq 1$.

Definition 2. An object I in an abelian category $\mathcal{A}$ is injective iff the following equivalent conditions hold.

(*a*) When $X \xhookrightarrow{\xi} Y$ is a monomorphism, then any morphism $X \xrightarrow{\iota} I$ extends to a morphism $Y \to I$.

(*b*) Any short exact sequence $0 \to I \to X \to X'' \to 0$ splits.

Proof. To see $(a) \Rightarrow (b)$, extend id_I to $X \xrightarrow{\pi} I$, then π gives a split of the exact sequence (the argument used in the case of R-modules still works in arbitrary abelian categories).

For $(b) \Rightarrow (a)$ consider $C = \operatorname{coker}\left(X \xrightarrow{\iota\times\xi} I \oplus Y\right)$ and let $I \oplus Y \xrightarrow{p} C$ be the associated morphism. Let $i = \mathrm{id}_I \times 0$ and $j = 0 \times \mathrm{id}_Y$ be the canonical inclusions $I \to I \oplus Y$ and $Y \to I \oplus Y$. We claim that the composition

$$I \xrightarrow{i} I \oplus Y \xrightarrow{p} C$$

is a monomorphism. First note that $X \xrightarrow{\iota\times\xi} I \oplus Y$ is a monomorphism (since its composition with the projection to Y equals ξ, which was supposed to be monic), hence it's the kernel of its own cokernel as we are working in an abelian category and thus every monomorphism is an *effective monomorphism*, cf. [2, Definition A.1.3(*d*) and Definition A.1.4]. That is, $X = \ker(p)$.

Suppose now that $T \xrightarrow{\tau} I$ is a morphism satisfying $pi\tau = 0$, then $i\tau$ factors over $X = \ker(p)$. We thus have a diagram

$$\begin{tikzcd} T \arrow[rr, "\tau"] \arrow[d, dashed, "!\exists\,\vartheta"'] & & I \arrow[d, "i"] \\ X \arrow[rr, "\iota \times \xi"] & & I \oplus Y \end{tikzcd} \tag{$*$}$$

Postcomposing with the canonical projection $I \oplus Y \xrightarrow{\pi} Y$ we see that $\pi i\tau = 0 \circ \tau = 0$, hence also $\xi\vartheta = 0$ as $(*)$ commutes and $\pi \circ (\iota \times \xi) = \xi$. But ξ is a monomorphism, hence $\vartheta = 0$. By $(*)$, this implies $\tau = 0$ as i is a monomorphism. This shows that α is indeed a monomorphism.

We thus obtain a short exact sequence

$$0 \longrightarrow I \longrightarrow C \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0$$

which splits due to (b), i.e., $C \simeq I \oplus \operatorname{coker}(\alpha)$. Let $C \xrightarrow{q} I$ be the associated projection. Consider the composition

$$Y \xrightarrow{j} I \oplus Y \xrightarrow{p} C \xrightarrow{q} I\ .$$

We claim that $\upsilon = -qpj$ is a morphism $Y \xrightarrow{\upsilon} I$ extending $X \xrightarrow{\iota} I$. We have $qp \circ (\iota \times \xi) = q \circ 0 = 0$ since C is precisely the cokernel of $\iota \times \xi$. Also $qp \circ (\iota \times 0) = \iota$ as $qpi = \mathrm{id}_I$ by construction of q. Then

$$\upsilon\xi = -qpj\xi = -qp \circ (0 \times \xi) = qp \circ ((\iota \times 0) - (\iota \times \xi)) = \iota - 0 = \iota\ ,$$

hence υ has indeed the required property. *q.e.d.*

Theorem A. *Let $\mathcal{A}$ be an abelian category with sufficiently many injective objects.*

(*a*) *Any left-exact functor $\mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to any abelian category $\mathcal{B}$ has a right-derived functor.*

(*b*) *Let $\Phi^*\colon \mathcal{A} \to \mathcal{B}$ be a cohomological functor, then Φ^* is a right-derived functor of Φ^0 iff $\Phi^i I = 0$ for any injective object $I \in \mathrm{Ob}(\mathcal{A})$ and all $i \geq 1$.*

(*c*) *Let $\mathcal{F}\colon 0 \to \Phi' \to \Phi \to \Phi'' \to 0$ be a sequence of left-exact functors $\mathcal{A} \to \mathcal{B}$ and functor morphisms between them such that $0 \to \Phi' I \to \Phi I \to \Phi'' I \to 0$ is exact when I is an injective object of $\mathcal{A}$. Then there is a unique sequence of natural transformations $R^i\Phi'' \xrightarrow{d_{\mathcal{F}}} R^{i+1}\Phi'$ such that*

$$0 \longrightarrow \Phi' X \longrightarrow \Phi X \longrightarrow \Phi'' X \xrightarrow{d_{\mathcal{F}}} R^1\Phi' X \longrightarrow \dots$$
$$\dots \longrightarrow R^{i-1}\Phi'' X \xrightarrow{d_{\mathcal{F}}} R^i\Phi' X \longrightarrow R^i\Phi X \longrightarrow R^i\Phi'' X \xrightarrow{d_{\mathcal{F}}} R^{i+1}\Phi' X \longrightarrow \dots$$

is exact for arbitrary $X \in \mathrm{Ob}(\mathcal{A})$ and such that the diagram

$$\begin{tikzcd} R^i\Phi'' X'' \arrow[r, "d_{\mathcal{F}}"] \arrow[d, "\partial_{R^*\Phi''}"'] & R^{i+1}\Phi' X'' \arrow[d, "-\partial_{R^*\Phi'}"] \\ R^{i+1}\Phi'' X' \arrow[r, "d_{\mathcal{F}}"] & R^{i+2}\Phi' X' \end{tikzcd}$$

commutes. Note the minus sign on the right vertical arrow!

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.

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