Algebraic Geometry I

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This text consists of notes on the lecture Algebraic Geometry I taught at the University of Bonn by Professor Jens Franke in the winter term (Wintersemester) 2017/18.

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Introduction

The lecture will be an introduction to the language of *schemes*. The topics include but are not limited to the category of (pre-)schemes, properties of schemes, morphisms of schemes, sheaves of $\mathcal{O}_X$-modules and cohomology of quasi-coherent sheaves.

Professor Franke said the lecture requires a firm knowledge of commutative algebra and affine and projective varieties. If you are not familiar with this terms you may want to think again about visiting this lecture. If you want to brush up your knowledge about these topics the following literature is recommended:

- Matsumura, H.: *Commutative Ring Theory*,
- Hartshorne, R.: *Algebraic Geometry*,
- Mumford, D.: *The Red Book of Varieties and Schemes*,
- Schwab, N. & Wagner, F.: *Algebra I by Jens Franke* [1]. **Warning!** Somewhere in the middle of this text, the term *irreducible* is redefined as irreducible *and closed*. So don't let yourself get confused.

Let it be said that the first three recommendations are from Professor Franke while the last one is from the (not so) humble authors of these notes.

1. Varieties and Schemes

Definition 1 (sheaf and presheaf)**.** A **presheaf $\mathcal{F}$ of rings** on a topological space X associates

- to any open subset $U \subseteq X$ a ring $\mathcal{F}(U)$ called the *ring of sections* of $\mathcal{F}$ on U
- and to any inclusion of open subsets $V \subseteq U$ a ring homomorphism

  $$\cdot|_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

  such that $f|_V = f$ for all $f \in \mathcal{F}(V)$ and $(f|_V)|_W = f|_W$ for any inclusion $W \subseteq V \subseteq U$ of open subsets.

Note that while this notation (intentionally) reminds of the restriction of functions, behaves similarly and often the restriction is indeed used for this homomorphism, the elements of the rings $\mathcal{F}(U)$ are not always functions.

A so defined presheaf is furthermore a **sheaf** if additionally, the following condition, called *sheaf axiom*, holds:

For every open covering $U = \bigcup_{i \in I} U_i$ of an open subset $U \subseteq X$ the map

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \left\{ (f_i)_{i\in I} \in \prod_{i \in I} \mathcal{F}(U_i) \;\middle|\; f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for } i, j \in I \right\} \\ f &\longmapsto (f|_{U_i})_{i \in I} \end{aligned}$$

is bijective.

Remark 1. When $U = \emptyset$ one can take $I = \emptyset$ and obtains $\mathcal{F}(\emptyset) = \{0\}$.

Remark 2. Sheaves of groups, sets, etc. are defined in a similar way. A sheaf of rings $\mathcal{R}$ on X defines two sheaves of groups on X, namely $U \mapsto (\mathcal{R}(U), +)$ and $U \mapsto (\mathcal{R}(U)^\times, \cdot)$.

Remark 3. Elements of $\mathcal{R}(U)$ are called *sections*, elements of $\mathcal{R}(X)$ are called *global sections*.

Example 1. Let R be a ring. The sheaf $\mathcal{F}_X$ of R-valued functions on X associates to any open subset $U \subseteq X$ the ring of R-valued functions $f : U \to R$ with the inclusion morphism being the restriction of functions to subsets.

Remark. If $\mathcal{G}$ is any (pre)sheaf on X and $U \subseteq X$ an open subset, we get a a sheaf $\mathcal{G}|_U$ on U by $\mathcal{G}|_U(V) = \mathcal{G}(V)$ for the open subsets $V \subseteq U$ equipped with the same restriction morphisms.

Definition 2 (algebraic prevarieties)**.** Let k be an algebraically closed field. An **algebraic prevariety** over k is a pair $(X, \mathcal{O}_X)$, where X is an irreducible Noetherian topological space together with a sheaf $\mathcal{O}_X$ of rings on X such that the following property is satisfied.

Any $x \in X$ has an open neighbourhood U such that there is a homeomorphism $U \xrightarrow[\varphi]{\sim} V$ where $V \subseteq k^n$ is a Zariski-closed subset such that φ identifies $\mathcal{O}_X|_U$ with the structure sheaf $\mathcal{O}_V$ of V. That is, if $W \subseteq V$ is open then any k-valued function $f\colon W \to k$ is regular (i.e. an element of $\mathcal{O}_V(W)$) if and only if

$$\begin{aligned} g\colon \varphi^{-1}(W) &\longrightarrow k \\ x &\longmapsto f(\varphi(x)) \end{aligned}$$

is an element of $\mathcal{O}_X(\varphi^{-1}(W))$. One denotes $g = \varphi^* f$ in this case.

A **morphism of prevarieties** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continous map $X \xrightarrow{\varphi} Y$ such that for all $U \subseteq Y$ and $\lambda \in \mathcal{O}_Y(U)$ we have $\varphi^*\lambda \in \mathcal{O}(\varphi^{-1}(U))$. As above, $\varphi^*\lambda$ is defined as $(\varphi^*\lambda)(x) = \lambda(\varphi(x))$. In particular, φ^* induces a *morphism of sheaves* $\varphi^*\colon \mathcal{O}_Y \to \mathcal{O}_X$.

Remark. (*a*) In fact, the $V \subseteq k^n$ in the above definition of varieties is even *irreducible*, as V is homeomorphic to an open (and hence irreducible) subset U of the irreducible space X. In particular, V is an *affine algebraic variety* (in the sense of [1, Definition 2.2.1]) and one can think of varieties as irreducible spaces which are locally isomorphic to (or glued together from) affine varieties.

(*b*) The n in the above definition is *not* required to be constant, not even for a single $x \in X$. In fact, this wouldn't be a sensible thing to ask for, as e.g. $k \subseteq k^1$ and $k \times \{0\} \subseteq k^2$ are isomorphic affine varieties. However, the *Krull dimension* $\dim X$ (in the sense of [1, Definition 2.1.4]) is a well-defined thing and one can show that $\dim X = \dim V$ in the above situation (this is a consequence of [1, Theorem 6] and the *locality of codimension*, cf. [1, Remark 2.1.3]).

Example 2. Let $V \subseteq k^n$ be Zariski-closed, $W \subseteq V$ open. The ring $\mathcal{O}_V(W)$ of *regular functions* on W is the ring of functions $\lambda\colon W \to k$ such that for any $x \in W$ there is an open neighbourhood Ω of x and polynomials $p, q \in R = k[X_1, \dots, X_n]$ such that q does not vanish on $\Omega \cap W$ and such that we have $\lambda(y) = \frac{p(y)}{q(y)}$ for every $y \in \Omega \cap W$. (cf. [1, Definition 2.2.2]).

The sheaf $\mathcal{O}_V$ defined by $W \mapsto \mathcal{O}_V(W)$ is called the *structure sheaf* on V. If $W = V$ it can be shown that any $f \in \mathcal{O}_V(V)$ can be written as $f = p|_V$ where $p \in R$ (cf. [1, Proposition 2.2.2]).

Example 3. The *projective space* $\mathbb{P}(V)$, where V is a k-vector space, is the set of one-dimensional subspaces of V. Let $\mathbb{P}^n(k) = \mathbb{P}(k^{n+1})$. If $(x_0, \dots, x_n) \in k^{n+1} \setminus \{0\}$, let $[x_0, \dots, x_n]$ denote the subspace generated by $(x_1, \dots, x_n)$.

Recall that an ideal $I \subseteq R = k[X_0, \dots, X_n]$ is called *homogenous* if it is generated by homogenous elements (i.e. polynomials in which every monomial has the same total degree). Let I be homogenous, let $V(I) \subseteq \mathbb{P}^n(k)$ be the set of all $[x_0, \dots, x_n] \in \mathbb{P}^n(k)$ such that $f(x_0, \dots, x_n)$ vanishes for all $f \in I$. Call a subset $A \subseteq \mathbb{P}^n(k)$ *Zariski-closed* if there is a homogenous ideal I such that $A = V(I)$. This turns $\mathbb{P}^n(k)$ into an irreducible, n-dimensional, Noetherian topological space.

Let $V \subset \mathbb{P}^n(k)$ be closed, $W \subseteq V$ open and $\lambda\colon W \to k$ any function. We call λ *regular* on W, or $\lambda \in \mathcal{O}_V(W)$, if any $x \in W$ has an open neighbourhood Ω such that there are two polynomials

$p, q \in k[X_0, \dots, X_n]$ homogenous of the same degree such that $q(y_0, \dots, y_n) \neq 0$ and

$$\lambda([y_0, \dots, y_n]) = \frac{p(y_0, \dots, y_n)}{q(y_0, \dots, y_n)}$$

for all $[y_0, \dots, y_n] \in W \cap \Omega$.

The *affine space* $\mathbb{A}^n(k)$ is just good old k^n equipped with its Zariski topology. Consider the map

$$\begin{aligned} \mathbb{P}^n(k) \setminus V(X_i) &\xrightarrow{\sim} \mathbb{A}^n(k) \\ [x_0, \dots, x_n] &\longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) . \end{aligned}$$

This is a homeomorphism and identifies the structure sheaves with each other. Hence, for any irreducible closed subset $A \subseteq \mathbb{P}^n(k)$ such that $Y \coloneqq A \cap (\mathbb{P}^n(k) \setminus V(X_i)) \neq \emptyset$, $(Y, \mathcal{O}_A|_Y)$ is isomorphic to an affine algebraic variety. Thus, quasi-projective algebraic varieties (i.e. $(U, \mathcal{O}_U)$ where $U \subseteq \mathbb{P}^n(k)$ is a non-empty open subset of an irreducible closed subset) are algebraic prevarieties in the sense of Definition 2.

Example 3a. When X is prevariety in the sense of Definition 2 and $U \subseteq X$ is open and $U \neq \emptyset$, then $(U, \mathcal{O}_X|_U)$ is a prevariety. Note that any non-empty open subset of an irreducible set is necessarily irreducible as well, so irreducibility of U is not required as an extra condition.

Example 3b. Let X be a prevariety, $A \subseteq X$ irreducible and closed. Then $(A, \mathcal{O}_A)$ is a prevariety, wherein the structure sheaf $\mathcal{O}_A$ is defined as follows: If $W \subseteq A$ is open, let

$$\mathcal{O}_A(W) = \left\{ f \colon W \to k \;\middle|\; \begin{array}{c} \text{every } x \in W \text{ has an open neighbourhood } x \in U \subseteq X \\ \text{and } \varphi \in \mathcal{O}_X(U) \text{ such that } f(y) = \varphi(y)\ \forall y \in U \cap W \end{array} \right\} .$$

Then one may check that $(A, \mathcal{O}_A)$ is a prevariety. Note that $\mathcal{O}_A$ is *not* given by the restriction $\mathcal{O}_X|_A$ (which is only defined for open subsets $U \subseteq X$ anyway). If one prefers a more functorial definition of $\mathcal{O}_A$, the following will do: For each open subset $U \subseteq X$, let

$$I(U) = \{\varphi \in \mathcal{O}_X(U) \mid \varphi \text{ lies in the maximal ideal of } \mathcal{O}_{A,a}\ \forall a \in U \cap A\}$$

(the *stalk* $\mathcal{O}_{A,a}$ is defined in [1, Definition 2.2.5]). Then

$$\mathcal{O}_A(W) = \varinjlim_{U} \mathcal{O}_X(U)/I(U) ,$$

where the colimit is taken over all open subsets $U \subseteq X$ such that $U \cap A = W$.

Remark. The following conditions on a topological space X are equivalent:

- (a) If $x \neq y$ are points of X, there are open neighbourhoods $V, W \subseteq X$ seperating them, i.e. $x \in V$, $y \in W$ and $V \cap W = \emptyset$. This is the usual definition for X being *Hausdorff*.
- (b) The diagonal $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$ is a closed subset.
- (c) If a and b are continous maps $T \to X$, then their *equalizer* $K = \{t \in T \mid a(t) = b(t)\}$ is closed in T.

Definition 3 (variety)**.** Let X be a prevariety over k in the sense of Definitions 2. We call X **separated** or a **variety** over k if and only if $\{t \in T \mid a(t) = b(t)\}$ is closed in T whenever a and b are a pair of *morphisms of prevarieties* $T \overset{a}{\underset{b}{\rightrightarrows}} X$.

Example 4 (Line with two origins)**.** Let $n > 0$ and $X = (\mathbb{A}^n(k) \setminus \{0\}) \cup \{0_+\} \cup \{0_-\}$. Consider two morphisms $\iota_+, \iota_- \colon \mathbb{A}^n(k) \to X$ defined by

$$\iota_\pm(x) = \begin{cases} x & \text{if } x \neq 0 \\ 0_\pm & \text{if } x = 0 \end{cases}$$

Let $U \subseteq X$ be open iff both $\iota_+^{-1}(U)$ and $\iota_-^{-1}(U)$ are both open in $\mathbb{A}^n(k)$, and let

$$\mathcal{O}_X(U) = \left\{ f \colon U \to k \;\middle|\; \iota_\pm^* f = f\iota_\pm \in \mathcal{O}_{\mathbb{A}^n(k)}\left(\iota_\pm^{-1}(U)\right) \right\}.$$

Then $U_\pm = \iota_\pm^{-1}(\mathbb{A}^n(k))$ are both open and $\mathbb{A}^n(k) \xrightarrow{\iota_\pm} U_\pm$ is a homeomorphism identifying the respective structure sheaves. Thus, X is a prevariety, but not a variety in the sense of Definition 3 as we may take $T = \mathbb{A}^n(k) \xrightarrow{\iota_\pm} X$ in Definition 3 and obtain $K = \{x \in \mathbb{A}^n(k) \mid \iota_+(x) = \iota_-(x)\} = \mathbb{A}^n \setminus \{0\}$, which is not closed.

Example 5. (*a*) Any affine algebraic variety is a variety in the sense of Definition 3.

(*b*) In particular, $\mathbb{A}^0 = \{0\}$ is a variety, as is any one-point prevariety.

(*c*) Non-empty open and irreducible closed subsets Y of varieties X are varieties

Proof. For part (*c*) note that the inclusion $Y \xrightarrow{\iota} X$ (together with $\iota^* \colon \mathcal{O}_X \to \mathcal{O}_Y$) is a morphism of prevarieties, and for a pair $a, b \colon T \to A$ of morphisms of prevarieties we have $\{t \in T \mid a(t) = b(t)\} = \{t \in T \mid (\iota a)(t) = (\iota b)(t)\}$. The latter is closed in T because X is a variety.

For part (*a*) let $X \subseteq k^n$ be closed and irreducible, $a, b \colon T \to X$ be morphisms of prevarieties and $K = \{t \in T \mid a(t) = b(t)\}$. To show that K is closed in T, it is sufficient to show that any $t \in T$ has a neighbourhood Ω such that $K \cap \Omega$ is closed in Ω. Choosing Ω such that it is isomorphic to an affine algebraic variety, which is possible because T is a prevariety, we may assume without loss of generality that $T \subseteq k^n$ is an affine algebraic variety in (i.e. an irreducible subset of) k^n. Let

$$\begin{aligned} X_i \colon X &\longrightarrow k \\ (x_1, \ldots, x_n) &\longmapsto x_i \end{aligned}$$

denote the projection to the i^{th} coordinate. Then $X_i \in \mathcal{O}_X(X)$, hence $\alpha_i = a^* X_i$ and $\beta_i = b^* X_i$ are in $\mathcal{O}_T(T)$ and

$$K = \{t \in T \mid \alpha_i(t) = \beta_i(t)\ \forall i\} = \bigcap_{i=1}^{n} V(\alpha_i - \beta_i)\,. \tag{1}$$

But we proved in Algebra I that $V(\varphi)$ is closed in T whenever T is an affine algebraic variety and $\varphi \in \mathcal{O}_T(T)$ (cf. [1, Proposition 2.2.1]).

Part (*b*) is trivial from (*a*). *q.e.d.*

Remark. (*a*) $K \subseteq T$ is closed iff for all $t \in T$ there is an open neighbourhood Ω_t such that $\Omega_t \cap K$ is closed in Ω_t, since

$$T \setminus K = \bigcup_{t \in T} (\Omega_t \setminus (K \cap \Omega_t))$$

is open as a union of open subsets.

(*b*) It is *not* sufficient to require this just for all $t \in K$.

Proposition 1. *Let X be any prevariety such that for arbitrary $x, y \in X$ there is a common open neighbourhood U of x and y which is affine (that is, isomorphic as a prevariety to an affine variety in some k^n). Then X is a variety.*

Proof. Let $a, b \colon T \to X$ as in Definition 3 and $t \in T$ and let $U \subseteq X$ be an affine open subset of X containing both $a(t)$ and $b(t)$. Let $V = a^{-1}(U) \cap b^{-1}(U) \subseteq T$. This is an open subset of T containing t. It is easily seen that $a|_V$ and $b|_V$ are morphisms $V \to U$. By the previous example, $K \cap V = \{t \in U \mid a(t) = b(t)\}$ is closed in V. Because such a neighbourhood can be found for any $t \in T$, K is closed in T by the previous remark. *q.e.d.*

Corollary 1. *Quasi-projective and quasi-affine algebraic varieties are varieties.*

Proof. Step 1. Let $X \subseteq k^n$ be irreducible and closed. Recall that for any $f \in \mathcal{O}_X(X) \setminus \{0\}$, $X \setminus V(f)$ is affine: Let $X = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \in R = k[X_1, \ldots, X_n]$. We identify $f \in \mathcal{O}_X(X) = R/\mathfrak{p}$ with an arbitrary representative $f \in R$. Now consider the ideal $\mathfrak{q} \subseteq k[X_1, \ldots, X_n, T]$ generated by $\mathfrak{p}$ and $1 - T \cdot f$. One can show that

$$\begin{aligned} V(\mathfrak{q}) &\xrightarrow{\sim} X \setminus V(f) \\ (x, t) &\longmapsto x \\ (x, f(x)^{-1}) &\longleftarrow x \end{aligned}$$

is a homeomorphism topological spaces. Then $V(\mathfrak{q})$ is irreducible (as $X \setminus V(f)$ is), hence an affine variety, which proves that $X \setminus V(f)$ must be affine as well (cf. [1, Proposition 2.2.4]).

Let $U \subseteq X$ be open and $F \subseteq U$ be finite. Let $X \setminus U = V(I)$ with $I \subseteq R$ an ideal. If $n = 0$, $U = \{0\}$ is affine and we have nothing to prove. Let $n \geq 1$. Because k is infinite, the k-vector space I cannot be the union of its finitely many codimension one subspaces $I_x = \{p \in I \mid p(x) = 0\}$ for $x \in F$. Therefore, there is $p \in I$ such that $F \subseteq X \setminus V(p)$. By our initial remark, $X \setminus V(p)$ is affine. As $p \in I$, $X \setminus V(p) \subseteq U$ and the claim follows.

Step 2. Let $X \subseteq \mathbb{P}^n(k)$ be quasi-projective and let $F \subseteq X$ be finite. We can write

$$F = \left\{ [f_0^{(i)}, \ldots, f_n^{(i)}] \;\middle|\; 1 \leq i \leq N \right\} .$$

As k^{n+1} is larger than the union of the N codimension one subspaces

$$V_i = \left\{ (\xi_j)_{j=0}^n \;\middle|\; \sum_{j=0}^n \xi_j f_j^{(i)} = 0 \right\} \quad \text{for } i = 1, \ldots, N ,$$

there is a homogenous polynomial $p \neq 0$ of degree 1 such that $p(f_0^{(i)}, \ldots, f_n^{(i)}) \neq 0$ for all $1 \leq i \leq N$. Then $F \subseteq X \setminus V(p)$. But $\mathbb{P}^n(k) \setminus V(p)$ is isomorphic to k^n as this is the case when $p = X_0$ and $\mathrm{GL}_{n+1}(k)$ transitively acts on $k^{n+1} \setminus \{0\}$. Thus, $F \subseteq X \setminus V(p)$ and $X \setminus V(p)$ is isomorphic to a quasi-affine variety. The assertion now follows from Step 1. *q.e.d.*

Remark.

- Let X be a prevariety. If two arbitrary points have a common neighbourhood which is a variety, then X is a variety.
- We have actually seen that arbitrary finite subsets of quasi-projective algebraic varieties have open neighbourhoods which are affine. This is usefull, e.g., when forming quotients by finite groups.
- Hironaka (see Hartshorne for examples of a non-quasi-projective variety) has an example of a variety where there are two points without a common affine neighbourhood.

A. Useful stuff from category theory

A.1. Fundamental concepts

Definition 1 (category)**.** A **category** $\mathcal{A}$ is a class $\mathrm{Ob}(\mathcal{A})$ of *objects* of $\mathcal{A}$ together with:

(a) For two arbitrary $X, Y \in \mathrm{Ob}(\mathcal{A})$, a set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ of *morphisms* from X to Y in $\mathcal{A}$.

(b) For $X, Y, Z \in \mathrm{Ob}(\mathcal{A})$, a map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Z) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

called the *composition of morphisms* in $\mathcal{A}$.

The following assumptions must be satisfied:

(i) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ are morphisms in $\mathcal{A}$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) For any $A \in \mathrm{Ob}(\mathcal{A})$ there is a unique morphism $A \xrightarrow{\mathrm{id}_A} A$ in $\mathrm{Hom}_{\mathcal{A}}(A, A)$ such that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ for any $f \in \mathrm{Hom}_{\mathcal{A}}(B, A)$ and any $g \in \mathrm{Hom}_{\mathcal{A}}(A, C)$ for arbitrary $B, C \in \mathrm{Ob}(\mathcal{A})$.

Example 1.

- The category Set of sets where $\mathrm{Ob}(\mathcal{A})$ is the class of sets, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is the set of maps from X to Y and the composition of morphism is the composition of maps.
- The category Grp of groups where $\mathrm{Ob}(\mathcal{A})$ is the class of groups, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is the set of group morphisms from X to Y and the composition of morphisms is the composition of maps.
- The categories of rings Ring, commutative rings Ab and abelian groups Ab are all defined similarly.
- The topological spaces with the continous maps Top.
- The Banach spaces with bounded (continous) maps.
- The k-vector spaces with k-linear maps Vect_K or R-modules with R-linear maps R-Mod.

A category is called *small* if its class of objects is a set.

Definition 2. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We call $\mathcal{B}$

- a **subcategory** of $\mathcal{A}$ if $\mathrm{Ob}(\mathcal{B}) \subseteq \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{B}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and for arbitrary objects X, Y of $\mathcal{B}$, the identity id_X of X in $\mathcal{A}$ is a morphism in $\mathrm{Hom}_{\mathcal{B}}(X, X)$.

- a **full subcategory** of $\mathcal{A}$ if additionally $\operatorname{Hom}_{\mathcal{A}}(X,Y) = \operatorname{Hom}_{\mathcal{B}}(X,Y)$ for arbitrary objects $X, Y \in \operatorname{Ob}(\mathcal{B})$.
- an **equivalent subcategory** of $\mathcal{A}$ if it is a full subcategory and every object $X \in \operatorname{Ob}(\mathcal{A})$ is isomorphic to some $Y \in \operatorname{Ob}(\mathcal{B})$ (where a morphism $X \xrightarrow{f} Y$ is an isomorphism iff there is a (unique) $Y \xrightarrow{g} X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

A further example is the category of prevarieties (with morphisms according to Definition 1.0.2) and its full subcategory of varieties, containing all varieties as objects.

Definition 3. A (covariant) **functor** $\mathcal{A} \xrightarrow{F} \mathcal{B}$ consists of the following data:

- a map $F\colon \operatorname{Ob}(\mathcal{A}) \to \operatorname{Ob}(\mathcal{B})$,
- for $X, Y \in \operatorname{Ob}(\mathcal{A})$ a map $F\colon \operatorname{Hom}_{\mathcal{A}}(X,Y) \to \operatorname{Hom}_{\mathcal{B}}(FX, FY)$ such that $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ whenever $X \in \operatorname{Ob}(\mathcal{A})$ and $F(\psi\varphi) = F(\psi)F(\varphi)$ when $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ are morphisms in $\mathcal{A}$.

A functor is called *faithful* if $F : \operatorname{Hom}_{\mathcal{A}}(X,Y) \to \operatorname{Hom}_{\mathcal{B}}(FX, FY)$ is injective and *fully faithful* if it's bijective.

Professor Franke would like to draw a line here. We hope, this one will do: ________________.

Example. We have *forgetful* functors like $\mathrm{Vect}_k \to \mathrm{Ab}$ or $\mathrm{Ab} \to \mathrm{Set}$. These functors are faithful (at least for the two examples).

A.2. Products and equalizers

Our goal is to formulate a notion of *sheaf* that allows to define sheafs for arbitrary target categories. Recall that the sheaf axiom (this one) requires

$$\mathcal{G}(U) \longrightarrow \left\{ (g_i)_{i\in I} \in \prod_{i\in I} \mathcal{G}(U_i) \;\middle|\; g_i|_{U_{ij}} = g_j|_{U_{ij}} \ \forall i,j \in I \right\} \tag{$*$}$$

to be bijective (i.e. an isomorphism) for any open cover $U = \bigcup_{i\in I} \mathcal{G}(U_i)$, where we set $U_{ij} = U_i \cap U_j$ for convenience.

Definition 1 (products)**.** A **product** of objects $(A_i)_{i\in I}$ of $\mathcal{A}$ is an object $\prod_{i\in I} A_i$ together with morphisms $\prod_{i\in I} A_i \xrightarrow{\pi_i} A_i$ for each $i \in I$ such that the following *universal property* holds:

> If $T \in \operatorname{Ob}(\mathcal{A})$ comes with morphisms $T \xrightarrow{\tau_i} A_i$ for each $i \in I$, then there is a unique morhism $T \xrightarrow{\exists!\, f} \prod_{i\in I} A_i$ such that $\tau_i = \pi_i \circ f$.

In other words, $\prod_{i\in I} A_i = \varprojlim_{i\in I} A_i$ is the *limiting cone* over the trivial diagram consisting only of the A_i without any arrows.

Note that nobody ever guaranteed that products exist in general.

Remark. (*a*) Compare this to e.g. the universal property of localizations where the universal object is on the left – here it is on the right.

(*b*) In the case of sets, (abelian) groups, R-modules, and rings, one can take $\prod_{i\in I} A_i$ to be the set-theoretic product equipped with the respective product structure and the set-theoretic projections $\prod_{i\in I} A_i \xrightarrow{\pi_i} A_i$, $\pi_i\big((a_j)_{j\in I}\big) = a_i$.

(*c*) The above definition characterizes $\prod_{i\in I} A_i$ up to *unique isomorphism*: If the T above satisfies the same universal property, then f is an isomorphism.

(*d*) If $I = \emptyset$, the empty product is the *final object*, i.e. an object $F \in \mathrm{Ob}(\mathcal{A})$ such that for any $T \in \mathrm{Ob}(\mathcal{A})$ there is precisely one morphism $T \to F$ in $\mathcal{A}$. We have the following dual notion: I (now an object and no indexing set anymore) is called *initial object* if $\mathrm{Hom}_{\mathcal{A}}(I, T)$ has precisely one element for each $T \in \mathrm{Ob}(\mathcal{A})$.

Remark. In Set, $\emptyset$ is the only initial object and the one-point sets are the final objects. For the *abelian* categories R-Mod, the canonical morphisms from the (only) initial to the (only) final object is an isomorphism.

Let X be a topological space. A *presheaf* on X with values in $\mathcal{A}$ is a map associating

- to each open subset $U \subseteq X$ an object $\mathcal{G}(U) \in \mathrm{Ob}(\mathcal{A})$
- and to each inclusion $V \subseteq U$ a morphism $\cdot|_V^U \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(U), \mathcal{G}(V))$ which equals $\mathrm{id}_{\mathcal{G}(U)}$ if $U = V$ and such that $\cdot|_W^U = \cdot|_W^V \circ \cdot|_V^U$ whenever $W \subseteq V \subseteq U$ is an inclusion of open sets.

To formulate the sheaf axiom, it is convenient to assume that $\prod_{i\in I} \mathcal{G}(U_i)$ exists, i.e. that $\mathcal{A}$ has arbitrary products.

Recall our convention that $U_{ij} = U_i \cap U_j$. There are unique morphisms

$$\alpha, \beta\colon \prod_{i\in I} \mathcal{G}(U_i) \longrightarrow \prod_{(i,j)\in I\times I} \mathcal{G}(U_{ij})$$

characterized by $\pi_{ij} \circ \alpha = \cdot|_{U_{ij}}^{U_i} \circ \pi_i$ and $\pi_{ij} \circ \beta = \cdot|_{U_{ij}}^{U_j} \circ \pi_j$, where $\prod_{i\in I} \mathcal{G}(U_i) \xrightarrow{\pi_i} \mathcal{G}(U_i)$ and $\prod_{(i,j)\in I\times I} \mathcal{G}(U_{ij}) \xrightarrow{\pi_{ij}} \mathcal{G}(U_{ij})$ are the morphisms defining the product structure.

In the example of presheaves of sets, rings, etc. we have

$$\alpha\big((g_i)_{i\in I}\big) = \Big(g_i|_{U_{ij}}\Big)_{(i,j)\in I\times I} \quad \text{and} \quad \beta\big((g_i)_{i\in I}\big) = \Big(g_j|_{U_{ij}}\Big)_{(i,j)\in I\times I},$$

such that $(*)$ is the “largest subobject on which α and β coincide".

Definition 2 (equalizer)**.** Let $A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$ be a pair of morphisms. An **equalizer** of this pair is an object $E \in \mathrm{Ob}(\mathcal{A})$ together with a morphism $E \xrightarrow{\varepsilon} A$ such that $\alpha\varepsilon = \beta\varepsilon$ and such that the following universal property holds:

> If $T \xrightarrow{\tau} A$ is any morphism in $\mathcal{A}$ such that $\alpha\tau = \beta\tau$, then there's a unique $T \xrightarrow{t} E$ such that the following diagram commutes.

$$\begin{tikzcd}
 & T \arrow[d, dashed, "\exists!\, t"] \arrow[ddl, bend right, "\tau"'] \arrow[ddr, bend left] & \\
 & E \arrow[dl, "\varepsilon"'] \arrow[dr] & \\
A \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\beta"'] & & B
\end{tikzcd}$$

In other word, E is the *limiting cone* over the diagram $A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$.

Remark. (*a*) By the usual Yoneda argument one sees that the universal property characterizes E up to unique isomorphism.

(*b*) The sheaf axiom for presheaves with values in an arbitrary category with products now translates into the condition that $\mathcal{G}(U) \to \prod_{i\in I} \mathcal{G}(U_i)$ is, for any open cover $U = \bigcup_{i\in I} U_i$, an equalizer of the above pair of morphisms $\prod_{i\in I} \mathcal{G}(U_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{G}(U_{ij})$.

By merging the universal properties, the notion of a sheaf can be generalized to arbitrary target categories.

(*c*) For sets, abelian groups etc. the equalizer is

$$\ker\left(A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B\right) = \{a \in A \mid \alpha(a) = \beta(a)\} \ .$$

(*d*) For the *abelian* category R-Mod,

$$\ker\left(M \xrightarrow{f} N\right) = \ker\left(M \overset{f}{\underset{0}{\rightrightarrows}} N\right) \quad \text{and} \quad \ker\left(M \overset{f}{\underset{g}{\rightrightarrows}} N\right) = \ker\left(M \xrightarrow{f-g} N\right) .$$

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.