Exercises to „Algebraic geometry I“, 1

Exercise 1 (2 points). Let R be any $\mathbb{N}$-graded ring and $I \subseteq R$ be any homogenuous ideal. Show that $\sqrt{I}$ is homogenuous.

Remark 1. It is easily seen that intersections, finite products, and (possibly infinite) sums of homogenuous ideals are homogenuous as well.

Let $\mathfrak{k}$ be an algebraically closed field and $R = \mathfrak{k}[X_0, \dots, X_n]$ be equipped with the usual grading by total degree of monomials. Let $R_+ \subseteq R$ be the augmentation ideal, consisting of all polynomials vanishing at the $(n+1)$-tuple 0. Obviously, any proper homogenuous ideal of R must be contained in R_+.

Exercise 2 (Projective version of the Nullstellensatz, 3 points). Show that a homogenuous ideal $I \subseteq R$ has a zero in $\mathbb{P}^n(\mathfrak{k})$ if and only if $\sqrt{I}$ is strictly contained in R_+.

Let $V(I) \subseteq \mathbb{P}^n(\mathfrak{k})$ be the set of (projective) zeros of the homogenuous ideal R.

Exercise 3 (3 points). Show there is a topology on $\mathbb{P}^n(\mathfrak{k})$ (called the Zariski topology) for which the closed subsets are precisely the sets of the form $V(I)$, for homogenuous ideals I of R.

Exercise 4 (3 points). Show that

$$\begin{aligned} \mathbb{A}^n(\mathfrak{k}) &\to \mathbb{P}^n(\mathfrak{k}) \setminus V(X_0) \\ (x_1, \dots, x_n) &\to [1, x_1, \dots, x_n] \end{aligned}$$

is a homeomorphism.

Exercise 5 (3 points). Show that the Zariski topology on $\mathbb{P}^n(\mathfrak{k})$ is Noetherian.

Exercise 6 (Another projective form of the Nullstellensatz, 5 points). Show that we have mutually inverse bijections between the set of Zariski-closed subsets Z of $\mathbb{P}^n(\mathfrak{k})$ and the set of homogenuous ideals $I \subseteq R_+$ such that $I = \sqrt{I}$ sending Z to

$$I = \left\{ f \in R \;\middle|\; f(x_0, \dots, x_n) = 0 \text{ whenever } [x_0, \dots, x_n] \in Z \right\}$$

and I to $Z = V(I)$. Also, show that the irreducible subsets correspond to the prime ideals which are strictly contained in R_+.

Solutions should be submitted Tuesday, October 24, in the lecture.