Algebraic Geometry II

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Introduction

This lecture will develop the cohomology of (quasi)coherent sheaves of modules. Professor Franke assumes familiarity with the contents of last term's Algebraic Geometry I. In particular, this includes the category of (pre)schemes, equalizers and fibre products of preschemes as well as in arbitrary categories and quasi-coherent $\mathcal{O}_X$-modules. If you are want to brush up your knowledge about these topics, the *lecture notes from Algebraic Geometry I* [1] might be your friend.

Professor Franke started the lecture with an example of sheaf cohomology entering the game. Let X be a topological space, $\mathcal{C}_X$ the sheaf of continuous $\mathbb{C}$-valued functions on X and $\underline{\mathbb{Z}}_X$ the sheaf of locally constant (i.e., continuous) functions on X with values in $\mathbb{Z}$. Then there is a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \xrightarrow{\cdot 2\pi \mathrm{i}} \mathcal{C}_X \xrightarrow{\exp} \mathcal{C}_X^\times \longrightarrow 0$$

of sheaves of abelian groups. In general, taking global section doesn't preserve exactness but gives rise to a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{Z}}_X(X) \arrow[r] \arrow[d, "\wr"] & \mathcal{C}_X(X) \arrow[r] \arrow[d, "\wr"] & \mathcal{C}_X^\times(X) \arrow[r, "d"] \arrow[d, "\wr"] & H^1(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^1(X, \mathcal{C}_X) \arrow[r] & \dots \\
0 \arrow[r] & H^0(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^0(X, \mathcal{C}_X) \arrow[r] & H^0(X, \mathcal{C}_X^\times) \arrow[ur] & & &
\end{tikzcd}$$

in which the $H^k(X, \underline{\mathbb{Z}}_X)$, $H^k(X, \mathcal{C}_X)$, and $H^k(X, \mathcal{C}_X^\times)$ are *sheaf cohomology groups*. There is the more general notion of *derived functors* (Grothendieck, Tôhoku paper), but this won't appear in the lecture.

Background in homological algebra is not required safe for cohomology groups of cochain complexes, the long exact cohomology sequence and the following famous lemma.

Lemma (Five lemma)**.** *Given a diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

of (abelian) groups/R-modules/etc. with exact rows, in which α, β, δ, and ε are isomorphisms, then γ is an isomorphism as well.

Proof. Easy diagram chase. *q.e.d.*

1. Cohomology of quasi-coherent sheaves of modules

1.1. Recollection of basic definitions and results

Definition 1 ([1, Definition 1.5.2 and Definition 1.5.9(b)])**.** (a) A **prescheme** (Franke uses "EGA termology") is a locally ringed space $(X, \mathcal{O}_X)$ which locally has the form $\operatorname{Spec} R$ for some rings R.

(b) A prescheme X is called a **scheme**, if, for any prescheme T and any pair of morphisms $T \overset{a}{\underset{b}{\rightrightarrows}} X$, the equalizer $\operatorname{Eq}\left(T \overset{a}{\underset{b}{\rightrightarrows}} X\right)$ is a closed subprescheme of X.

Remark. Equivalently, a prescheme X is a scheme iff the diagonal $\Delta\colon X \xrightarrow{(\mathrm{id}_X, \mathrm{id}_X)} X \times X$ is a closed immersion (cf. [1, Fact 1.5.8]). In other words, schemes are *separated* preschemes

Proposition 1. *If U and V are affine open subsets of a scheme X, then their intersection $U \cap V$ is again affine (and open of course).*

Proof. This was proved in [1, Proposition 1.5.4]. *q.e.d.*

Note that open subsets of the form $\operatorname{Spec}(R_f) \simeq \operatorname{Spec} R \setminus V(f)$ form a topology base on $\operatorname{Spec} R$ and that the *saturation* of $\{1, f, f^2, \dots\}$ (i.e. the largest multiplicative subset of R which still gives the same localization) depends only on $\operatorname{Spec} R \setminus V(f)$. Hence, for any R-module M, the localization M_f depends (up to canonical isomorphism) only on $\operatorname{Spec} R \setminus V(f)$ and M. One defines a sheaf of modules $\widetilde{M}$ on $\operatorname{Spec} R$ as the sheafification of $\operatorname{Spec}(R_f) \mapsto M_f$. Then

$$\widetilde{M}(U) = \left\{ (m_\mathfrak{p})_{\mathfrak{p} \in U} \in \prod_{\mathfrak{p} \in U} M_\mathfrak{p} \;\middle|\; \begin{array}{c} \text{for every } \mathfrak{q} \in U \text{ there are } f \in R \setminus \mathfrak{q} \text{ and } \mu \in M_f \text{ such that} \\ m_\mathfrak{p} = (\text{image of } \mu \text{ under } M_f \to M_\mathfrak{p}) \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R_f) \end{array} \right\}$$

Definition 2 ([1, Definition 1.4.2])**.** A sheaf of modules $\mathcal{M}$ on $\operatorname{Spec} R$ is called **quasi-coherent** if $\mathcal{M} \simeq \widetilde{M}$ for some R-module M.

Definition 3 ([1, Definition 1.5.3])**.** A prescheme is called **quasi-compact** if the underlying topological space is quasi-compact and **quasi-separated** if the intersection of any two quasi-compact open subsets is quasi-compact.

Definition 4 ([1, Definition 1.5.4])**.** Let X be a prescheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is called **quasi-coherent** if it satisfies the following equivalent conditions.

(*a*) X may be covered by affine open subsets U such that $\mathcal{M}|_U$ is quasi-coherent in the sense of Definition 2.

(*b*) For any affine open $U \subseteq X$, $\mathcal{M}|_U$ is quasi-coherent.

(*c*) For any quasi-compact and quasi-separated open $U \subseteq X$ and $f \in \mathcal{O}_X(U)$, the canonical morphism

$$\mathcal{M}(U)_f \longrightarrow \mathcal{M}(U \setminus V(f)) \tag{1}$$

(coming from the universal property of localization) is an isomorphism.

(*d*) The morphism (1) is an isomorphism when U is quasi-compact and quasi-separated and injective when U is only quasi-compact.

(*e*) When $U \subseteq X$ is affine, the canonical morphism

$$\mathcal{M}(U)_{\mathfrak{p}_x} \longrightarrow \mathcal{M}_x \tag{2}$$

is an isomorphism for all $x \in U$, where $\mathfrak{p}_x = \{f \in \mathcal{O}_X(U) \mid x \in V(f)\}$ is the prime ideal in $\operatorname{Spec} \mathcal{O}_X(U)$ corresponding to x.

Remark. To be fair: Despite Professor Franke's usual fondness of proving definitions (or rather stating definitions in a way they need a proof), the equivalence of (*a*) to (*d*) wasn't proposed as a definition in Algebraic Geometry I.

Proof of Definition 4. The equivalence of (*a*) to (*d*) was proved in [1, Proposition 1.5.1], but property (*e*) is something we haven't seen yet. Recall the *adjunction* (cf. Definition A.1.5)

$$\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} R}}(\widetilde{M}, \mathcal{N}) \xrightarrow{\sim} \operatorname{Hom}_R(M, \mathcal{N}(\operatorname{Spec} R)) \tag{3}$$

for M an R-module and $\mathcal{N}$ a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules (cf. [1, Proposition 1.4.3]). When (2) is an isomorphism for all $x \in U \simeq \operatorname{Spec} R$, it follows that the canonical morphism $\widetilde{M} \to \mathcal{M}|_U$ (with $M = \mathcal{M}(U)$) coming from (3) is an isomorphism on stalks, hence an isomorphism. This shows (*e*) $\Rightarrow$ (*b*).

Conversely, if (*b*) holds, then $\widetilde{M} \to \mathcal{M}|_U$ (with $M = \mathcal{M}(U)$) is an isomorphism for all affine open $U \subseteq X$, hence induces isomorphisms on stalks, which shows (*b*) $\Rightarrow$ (*e*). Hence, (*e*) is equivalent to the other properties. *q.e.d.*

Let $\mathcal{A}$ be the category R-Mod and $\mathcal{B}$ be $\mathcal{O}_{\operatorname{Spec} R}$-Mod, then the functor L given by $M \mapsto \widetilde{M}$ and the functor $\mathcal{M} \mapsto \mathcal{M}(\operatorname{Spec} R)$ are an adjoint pair of functors by (3). It follows that L commutes with cokernels and coproducts. In particular, the full subcategory $\operatorname{QCoh}(X) \subseteq \mathcal{O}_X$-Mod of quasi-coherent $\mathcal{O}_X$-modules is closed under taking cokernels and direct sums for $X = \operatorname{Spec} R$, and by locality of quasi-coherentness this holds for all preschemes X.

Definition 5 ([1, Definition 2.1.1 and Definition 2.1.2])**.** A morphism $X \xrightarrow{f} Y$ of preschemes is **quasi-compact** if it satisfies the following equivalent conditions.

(*a*) For quasi-compact open $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact.

(*b*) For affine open $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact.

(*c*) One can cover Y by affine open U such that $f^{-1}(U)$ is quasi-compact.

It is called **quasi-separated** if it satisfies the following equivalent conditions.

(*d*) For an open quasi-separated $U \subseteq Y$, $f^{-1}(U)$ is quasi-separated again.

(*e*) For affine open subsets $U \subseteq Y$, $f^{-1}(U)$ is quasi-separated.

(*f*) It is possible to cover Y by affine open U such that $f^{-1}(U)$ is quasi-separated.

Proof. Equivalence was proved in [1, Fact 2.1.1] for quasi-compactness and [1, Lemma 2.1.1] for quasi-separatedness. *q.e.d.*

Proposition 2. *If* $X \xrightarrow{f} Y$ *is quasi-compact and quasi-separated morphism of preschemes and* $\mathcal{M} \in \operatorname{Ob}(\operatorname{QCoh}(X))$*, then* $f_*\mathcal{M} \in \operatorname{Ob}(\operatorname{QCoh})$.

Proof. This is [1, Proposition 1.5.2(*b*)]. *q.e.d.*

Proposition 3. (*a*) *The full subcategory* $\operatorname{QCoh}(X) \subseteq \mathcal{O}_X\text{-Mod}$ *of quasi-coherent sheaves of* $\mathcal{O}_X$*-modules on a prescheme* X *is closed under taking kernels and cokernels of morphisms and under taking (finite) direct sums.*

(*b*) *If* $\mathcal{M}$ *is a quasi-coherent* $\mathcal{O}_X$*-module and* $U \subseteq X$ *open, then* $\mathcal{M}|_U \in \operatorname{Ob}(\operatorname{QCoh}(U))$.

Proof. Part (*a*). For cokernels and finite direct sums (which are finite coproducts since $\mathcal{O}_X$-Mod is an abelian category by Proposition A.1.1), consider the case $X = \operatorname{Spec} R$ first. Then $R\text{-Mod} \xrightarrow{L} \mathcal{O}_X\text{-Mod}$, $M \mapsto \widetilde{M}$ and $\mathcal{O}_X\text{-Mod} \to R\text{-Mod}$, $\mathcal{M} \mapsto \mathcal{M}(X)$ are adjoint functors by (3). By Remark A.1.4, L preserves cokernels and coproducts. By locality of quasi-coherentness, this follows for all preschemes X.

Closedness under taking kernels was proved in [1, Fact 1.5.3]. It's worth pointing out that in fact, the proof given there shows that $M \mapsto \widetilde{M}$ preserves kernels as well.

Part (*b*) follows immediately from (e.g.) Definition 4(*b*). *q.e.d.*

Corollary 1. *Let* X *be a prescheme,* $\mathcal{M} \xrightarrow{f} \mathcal{N}$ *a morphism of quasi-coherent* $\mathcal{O}_X$*-modules and* $U \subseteq X$ *open, then*

$$\ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \ker(f)(U)\,.$$

If U *is, in addition, affine, then*

$$\operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \operatorname{coker}(f)(X)\,.$$

Proof. The first assertion holds by our explicit construction of $\ker(f)$ in Lemma A.1.2. For the second one, we may assume $X = U = \operatorname{Spec} R$. Denoting $M = \mathcal{M}(X)$, $N = \mathcal{N}(X)$, then

$$\operatorname{coker}\left(\mathcal{M} \xrightarrow{f} \mathcal{N}\right) = \operatorname{coker}\left(\widetilde{M} \xrightarrow{f} \widetilde{N}\right) \simeq \left(\operatorname{coker}\left(M \xrightarrow{f} N\right)\right)^{\sim}$$

as $M \mapsto \widetilde{M}$ preserves cokernels. *q.e.d.*

Corollary 2. *Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules on a prescheme X and $U \subseteq X$ be affine open, then*

$$0 \longrightarrow \mathcal{M}'(U) \longrightarrow \mathcal{M}(U) \longrightarrow \mathcal{M}''(U) \longrightarrow 0$$

is exact as well.

Proof. Follows from Corollary 1. *q.e.d.*

Remark. It turns out to be sufficient to assume that two of the above three sheaves $\mathcal{M}', \mathcal{M}, \mathcal{M}''$ are quasi-coherent. Indeed, we proved in Proposition 3 that kernels and cokernels of morphisms between quasi-coherent sheaves are quasi-coherent again, so the only case in question is where $\mathcal{M}$ is not required to be quasi-coherent. This case, however, will be treated by cohomological methods.

Our plan is to associate to any quasi-coherent $\mathcal{O}_X$-module on a scheme X *cohomology groups* $H^i(X, \mathcal{M})$ such that

- $H^0(X, \mathcal{M}) \simeq \mathcal{M}(X)$.
- when $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, we have a canonical long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{M}') \longrightarrow H^0(X, \mathcal{M}) \longrightarrow H^0(X, \mathcal{M}'') \\ \xrightarrow{d} H^1(X, \mathcal{M}') \longrightarrow H^1(X, \mathcal{M}) \longrightarrow H^1(X, \mathcal{M}'') \xrightarrow{d} H^2(X, \mathcal{M}') \longrightarrow \dots \end{aligned}$$

But before we do this, we to introduce the notion of *coherent* $\mathcal{O}_X$-modules.

Proposition 4. *If X is a prescheme, associating to (the isomorphism class of) a closed embedding $Y \xrightarrow{i} X$ the sheaf of ideals $\mathcal{J} = \ker\left(\mathcal{O}_X \xrightarrow{i^*} i_*\mathcal{O}_X\right)$ gives a bijection between the set of closed subpreschemes of X and the quasi-coherent sheaves of ideals in $\mathcal{O}_X$.*

Proof. This is [1, Proposition 1.5.3]. *q.e.d.*

Lemma 1. *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ on a prescheme X, the following conditions are equivalent.*

(*a*) *For any affine open $U \subseteq X$, $\mathcal{M}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.*

(*b*) *It is possible to cover X by affine open subsets $U \subseteq X$, for which $\mathcal{M}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.*

Proof. This will follow from Lemma 2 and Lemma 3 below. *q.e.d.*

Lemma 2. *Let $\mathcal{P}$ be a property of affine open subsets of a prescheme X such that*

(α) *If $U \subseteq X$ is affine and $f \in \mathcal{O}_X(U)$, then $\mathcal{P}(U)$ implies $\mathcal{P}(U \setminus V(f))$.*

(β) *If U is affine and $f_1, \dots, f_n \in \mathcal{O}_X(U)$ are such that $\bigcap_{i=1}^n V(f_i) = \emptyset$ and such that $\mathcal{P}(U \setminus V(f_i))$ holds for all $i = 1, \dots, n$, then $\mathcal{P}(U)$ holds.*

Then the following assertions about X are equivalent.

(*a*) *If* $U \subseteq X$ *is affine open,* $\mathcal{P}(U)$ *holds.*

(*b*) X *may be covered by affine open* U *for which* $\mathcal{P}(U)$ *holds.*

Proof. We proved this in [1, Lemma 2.2.2]. *q.e.d.*

Lemma 3. (*a*) *If* M *is a finitely generated* R*-module, then* M_f *is a finitely generated* R_f*-module.*

(*b*) *If* M *is an* R*-module and* $f_1, \dots, f_n \in R$ *such that* $\bigcap_{i=1}^n V(f_i) = \emptyset$ *in* $\operatorname{Spec} R$ *and such that* M_{f_i} *is finitely generated over* R_{f_i}*, then* M *is finitely generated over* R*.*

Proof. Part (*a*) is trivial, as the images of R-generators of M in M_f generate it as an R_f-module.

Now for part (*b*). As M_{f_i} is finitely generated over R_{f_i}, there are $k \in \mathbb{N}$ and $m_{i,j} \in M$, $j = 1, \dots, N_i$ such that $m_{i,j} f^{-k}$ generate M_{f_i} over R_{f_i} (as there are only finitely many generators, we can choose a common exponent k for all of them). Then also the $m_{i,j}$ generate M_{f_i} since f_i is a unit in R_{f_i}. We claim that the $\{m_{i,j} \mid i = 1, \dots, n \text{ and } j = 1, \dots, N_i\}$ generate M as an R-module. Indeed, let $m \in M$, then

$$m = \sum_{j=1}^{N_i} \frac{r_{i,j}}{f_i^\ell} m_{i,j} \quad \text{in } M_{f_i} \ ,$$

where $r_{i,j} \in R$ and $\ell \in \mathbb{N}$ (again, we can choose a common exponent ℓ). Then there is some $\ell' \in \mathbb{N}$ such that

$$f_i^{\ell+\ell'} m = \sum_{j=1}^{N_i} r_{i,j} f_i^{\ell'} m_{i,j} \quad \text{in } M \ .$$

Replacing ℓ by $\ell + \ell'$ and $r_{i,j}$ by $f_i^{\ell'} r_{i,j}$ we may assume $\ell' = 0$, i.e.

$$f_i^\ell m = \sum_{j=1}^{N_i} r_{i,j} m_{i,j} \quad \text{in } M \ .$$

We now have $\bigcap_{i=1}^n V(f_i^\ell) = \bigcap_{i=1}^n V(f_i) = \emptyset$, hence the ideal generated by the f_i^ℓ is R and we thus find $g_1, \dots, g_n \in R$ such that $\sum_{i=1}^n f_i^\ell g_i = 1$ in R. It follows that

$$m = \sum_{i=1}^{n} f_i^\ell g_i m = \sum_{i=1}^{n} \sum_{j=1}^{N_i} r_{i,j} g_i m_{i,j}$$

is an element of the submodule generated by the $m_{i,j}$. *q.e.d.*

Definition 6. (*a*) We call a quasi-coherent $\mathcal{O}_X$-module **locally finitely generated** it it satisfies the equivalent conditions from Lemma 1.

(*b*) When X is locally Noetherian (cf. [1, Definition 2.2.2]), an $\mathcal{O}_X$-module is called **coherent** if it is quasi-coherent and locally finitely generated.

Remark. There is a general definition of *coherent* sheaves of modules on arbitrary ringed spaces, which in the case of a locally Noetherian prescheme is equivalent to the above.

1.2. Čech cohomology

Let $\mathcal{U}\colon X = \bigcup_{i\in I} U_i$ be an open cover of a topological space X. In the following, we will use the convention

$$U_{i_0,\dots,i_n} = \bigcap_{k=0}^{n} U_{i_k} \ . \tag{1}$$

Definition 1. For an open cover $\mathcal{U}$ of a topological space X (e.g., a prescheme) and $\mathcal{M}$ a presheaf of abelian groups (e.g., a quasi-coherent $\mathcal{O}_X$-module) on X the **Čech complex** $\check{C}^*(\mathcal{U},\mathcal{M})$ is the cochain complex defined as follows. Let

$$\check{C}^n(\mathcal{U},\mathcal{M}) \coloneqq \prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathcal{M}(U_{i_0,\dots,i_n}) \ .$$

Let the elements of $\check{C}^n(\mathcal{U},\mathcal{M})$ be denoted $\psi = (\psi_{i_0,\dots,i_n})_{(i_0,\dots,i_n)\in I^{n+1}}$. The differentials $\check{C}^n(\mathcal{U},\mathcal{M}) \xrightarrow{\check{d}^n} \check{C}^{n+1}(\mathcal{U},\mathcal{M})$ are defined by

$$(\check{d}^n\psi)_{i_0,\dots,i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j \psi_{i_0,\dots,\hat{i}_j,\dots,i_{n+1}}|_{U_{i_0,\dots,i_{n+1}}}$$

where $\hat{i}_j$ denotes the omission of the index i_j. For instance,

$$(\check{d}^0\psi)_{i,j} = \psi_j|_{U_{i,j}} - \psi_i|_{U_{i,j}} \quad \text{and} \quad (\check{d}^1\psi)_{i,j,k} = \psi_{j,k}|_{U_{i,j,k}} - \psi_{i,k}|_{U_{i,j,k}} + \psi_{i,j}|_{U_{i,j,k}} \ .$$

The **Čech cohomology** $\check{H}^*(\mathcal{U},\mathcal{M})$ is defined as the cohomology of the Čech complex, i.e.,

$$\check{H}^i(\mathcal{U},\mathcal{M}) = H^i\left(\check{C}^*(\mathcal{U},\mathcal{M})\right) \ .$$

To see that $\check{C}^*(\mathcal{U},\mathcal{M})$ is indeed a cochain complex, we need to prove $\check{d}^2 = 0$ – and we won't do this in a remark!

Proof of Definition 1. For $\ell = 0,\dots,n+1$ let $\check{C}^n(\mathcal{U},\mathcal{M}) \xrightarrow{d_\ell} \check{C}^{n+1}(\mathcal{U},\mathcal{M})$ be given by

$$(d_\ell\psi)_{i_0,\dots,i_{n+1}} = \psi_{i_0,\dots,\hat{i}_\ell,\dots,i_{n+1}}|_{U_{i_0,\dots,i_{n+1}}} \ .$$

Again, $\hat{i}_\ell$ denotes the omission of the index i_ℓ.

Step 1. We prove that

$$d_m d_\ell = d_{\ell+1} d_m \quad \text{when } \ell \geq m \ . \tag{2}$$

Indeed, we have $(d_m d_\ell \psi)_{\mathbf{i}} = \psi_{\mathbf{j}}|_{U_{\mathbf{i}}}$, where $\mathbf{j}$ is obtained from $\mathbf{i}$ by omitting the indices i_ℓ and i_m when $\ell < m$ and the indices $i_{\ell+1}$ and i_m when $\ell \geq m$. The assertion follows.

Step 2. We prove the following. Let C^* is any family of abelian groups (or objects of an abelian category) and $C^n \xrightarrow{d_\ell} C^{n+1}$ morphisms for $\ell = 0,\dots,n+1$. Suppose that $C^n = 0$ for $n < 0$ and that (2) holds. Then

$$d = \sum_{i=0}^{n+1} (-1)^i d_i$$

satisfies $d^2 = 0$. Indeed,

$$\begin{aligned} d^2 &= \sum_{m=0}^{n+2}\sum_{\ell=0}^{n+1}(-1)^{\ell+m}d_m d_\ell = \sum_{m=0}^{n+2}\sum_{\ell=0}^{m-1}(-1)^{\ell+m}d_m d_\ell + \sum_{m=0}^{n+2}\sum_{\ell=m}^{n+1}(-1)^{\ell+m}d_m d_\ell \\ &= \sum_{m=0}^{n+2}\sum_{\ell=0}^{m-1}(-1)^{\ell+m}d_m d_\ell + \sum_{m=0}^{n+1}\sum_{\ell=m}^{n+1}(-1)^{\ell+m}d_{\ell+1} d_m \\ &= \sum_{i>j}(-1)^{i+j}d_i d_j + \sum_{i>j}(-1)^{i+j-1}d_i d_j = 0\ , \end{aligned}$$

as required. *q.e.d.*

Remark. Our program is to show that $\check{H}^*(\mathcal{U},\mathcal{M})$ is independent of $\mathcal{U}$ and has the desired properties, when X is a scheme, $\mathcal{M} \in \mathrm{Ob}(\mathrm{QCoh}(X))$ and $\mathcal{U}$ is an affine open cover.

Remark. For instance, the cohomology of $\mathbb{P}^1_R = \mathrm{Proj}(R[X_0, X_1])$ can be calculated using the affine open cover

$$U_i = \mathbb{P}^1_R \setminus V(X_i) \simeq \mathrm{Spec}\left(R[X_0,X_1]_{X_i}\right)_0 \simeq \mathrm{Spec}\, R[t_i] \quad \text{where } t_i = \begin{cases} X_1\cdot X_0^{-1} & \text{if } i=0 \\ X_0\cdot X_1^{-1} & \text{if } i=1 \end{cases}.$$

Unfortunately, calculations become complicated by the fact that there are infinitely many non-zero terms in $\check{C}^*(\mathcal{U},\mathcal{M})$.

Remark 1. Let $\check{C}^n_{\mathrm{alt}}(\mathcal{U},\mathcal{M}) \subseteq \check{C}^n(\mathcal{U},\mathcal{M})$ be the subgroup containing all $\psi \in \check{C}^n(\mathcal{U},\mathcal{M})$ such that

$$\psi_{i_{\pi(0)},,\dots,i_{\pi(n)}} = \mathrm{sgn}(\pi)\psi_{i_0,\dots,i_n} \in \mathcal{M}(U_{i_0,\dots,i_n}) \quad \text{and} \quad \psi_{i_0,\dots,i_{n-1},i_{n-1}} = 0 \in \mathcal{M}(U_{i_0,\dots,i_{n-1}})$$

for all permutations $\pi \in \mathfrak{S}_n$. Note that $U_{i_0,\dots,i_n} = U_{i_{\pi(0)},\dots,i_{\pi(n)}}$ as permuting indices doesn't change intersections, so the first property makes sense. Also note that both properties together imply that $\psi_{i_0,\dots,i_n} = 0$ whenever $(i_0,\dots,i_n)$ contains a repeated index.

Claim. $\check{C}^*_{\mathrm{alt}}(\mathcal{U},\mathcal{M}) \subseteq \check{C}^*(\mathcal{U},\mathcal{M})$ is a subcomplex, called the **alternating Čech complex**.

Proof. Define *codegeneracy maps*

$$\check{C}^n(\mathcal{U},\mathcal{M}) \xrightarrow{s_\ell} \check{C}^{n-1}(\mathcal{U},\mathcal{M})\ , \quad (s_\ell\psi)_{i_0,\dots,i_{n-1}} = \psi_{i_0,\dots,i_\ell,i_\ell,\dots,i_{n-1}} \quad \text{for } \ell = 0,\dots,n-1$$

(i.e., s_ℓ repeats the ℓ^{th} index) as well as *transposition maps*

$$\check{C}^n(\mathcal{U},\mathcal{M}) \xrightarrow{t_\ell} \check{C}^n(\mathcal{U},\mathcal{M})\ , \quad (t_\ell\psi)_{i_0,\dots,i_n} = \psi_{i_0,\dots,i_{\ell-1},i_{\ell+1},i_\ell,i_{\ell+2},\dots,i_{n-1}} \quad \text{for } \ell = 0,\dots,n-1$$

(i.e., t_ℓ swaps the ℓ^{th} and $(\ell+1)^{\text{st}}$ index). As any permutation may be expressed as a composition of elementary transpositions, $\check{C}^n_{\mathrm{alt}}(\mathcal{U},\mathcal{M}) \subseteq \check{C}^n(\mathcal{U},\mathcal{M})$ is given by the relations

$$s_\ell\psi = 0 \quad \text{and} \quad t_\ell\psi = -\psi \quad \text{for } \ell = 0,\dots,n-1$$

So what we need to check to confirm that $\check{C}^*_{\mathrm{alt}}(\mathcal{U},\mathcal{M})$ is indeed a subcomplex of $\check{C}^*(\mathcal{U},\mathcal{M})$ is that the above relations are preserved by the differential $\check{d}$.

One may easily check the relations

$$s_\ell d_i = \begin{cases} d_i s_{\ell-1} & \text{if } i < \ell \\ \mathrm{id} & \text{if } i = \ell \text{ or } i = \ell+1 \\ d_{i-1} s_\ell & \text{if } i > \ell+1 \end{cases} \tag{3}$$

and

$$t_\ell d_j = \begin{cases} d_j t_\ell & \text{if } \ell < j-1 \\ d_\ell & \text{if } \ell = j-1 \\ d_{\ell+1} & \text{if } \ell = j \\ d_j t_{\ell-1} & \text{if } \ell > j \end{cases} . \tag{4}$$

Now let $\psi \in \check{C}^n(\mathcal{U}, \mathcal{M})$ such that $t_j\psi = -\psi$ for all $j = 0, \dots, n-1$. Using (4), we get

$$\begin{aligned} t_\ell \check{d}\psi &= \sum_{j=0}^{\ell-1} (-1)^j t_\ell d_j \psi + (-1)^\ell t_\ell d_\ell \psi + (-1)^{\ell+1} t_\ell d_{\ell+1}\psi + \sum_{j=\ell+2}^{n} (-1)^j t_\ell d_j \psi \\ &= \sum_{j=0}^{\ell-1} (-1)^j d_j t_{\ell-1} \psi + (-1)^\ell d_{\ell+1} \psi + (-1)^{\ell+1} d_\ell \psi + \sum_{j=\ell+2}^{n} (-1)^j d_j t_\ell \psi \\ &= -\sum_{j=0}^{\ell-1} (-1)^j d_j \psi - (-1)^\ell d_\ell \psi - (-1)^{\ell+1} d_{\ell+1} \psi - \sum_{j=\ell+2}^{n} (-1)^j d_j \psi \\ &= -\check{d}\psi \, . \end{aligned}$$

Similarly, one can check that $s_\ell \check{d}\psi = 0$ when $s_j\psi = 0$ for all $j = 0, 1, \dots, n-1$. This shows that $\check{d}$ restricts to a differential on $\check{C}^n_{\mathrm{alt}}(\mathcal{U}, \mathcal{M})$, as required. *q.e.d.*

It will eventually turn out that the cohomology groups $\check{H}^i_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) = H^i\left(\check{C}^*_{\mathrm{alt}}(\mathcal{U}, \mathcal{M})\right)$ obtained from the alternating Čech complex are the same as the regular Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{M})$.

Remark 2. A *cosimplicial object* of a category $\mathcal{A}$ is a sequence of objects $(X^n)_{n\geq 0}$ with morphisms $d_j \colon X^n \to X^{n+1}$ for $j = 0, \dots, n+1$ satisfying (2) and $s_j \colon X^n \to X^{n-1}$ for $j = 0, \dots, n$ satisfying a version of (2) together with (3). In other words, a cosimplicial object is a covariant functor from the *simplex category* Δ to $\mathcal{A}$.

There is a *Dold–Puppe correspondence* between cochain complexes concentrated in nonnegative degrees and cosimplicial objects of an abelian category.

Example 1. (*a*) By the sheaf axiom,

$$\begin{aligned} \mathcal{M}(X) &\simeq \left\{ (m_i)_{i\in I} \in \prod_{i\in I} \mathcal{M}(U_i) \;\middle|\; m_i|_{U_{i,j}} = m_j|_{U_{i,j}} \right\} \\ &= \ker\left(\check{C}^0(\mathcal{U}, \mathcal{M}) \xrightarrow{\check{d}^0} \check{C}^1(\mathcal{U}, \mathcal{M})\right) \simeq \check{H}^0(\mathcal{U}, \mathcal{M}) \\ &= \ker\left(\check{C}^0_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \xrightarrow{\check{d}^0} \check{C}^1_{\mathrm{alt}}(\mathcal{U}, \mathcal{M})\right) \simeq \check{H}^0_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \end{aligned}$$

(*b*) For the trivial cover $\mathcal{U}_0\colon X = X$, the Čech complex $\check{C}^*(\mathcal{U}_0, \mathcal{M})$ has the form

$$\mathcal{M}(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

and the alternating Čech complex $\check{C}^*_{\mathrm{alt}}(\mathcal{U}_0, \mathcal{M})$ looks like

$$\mathcal{M}(X) \xrightarrow{0} \mathcal{M}(X) \xrightarrow{\mathrm{id}} \mathcal{M}(X) \xrightarrow{0} \dots ,$$

so

$$\check{H}^n(\mathcal{U}_0, \mathcal{M}) = \check{H}^n_{\mathrm{alt}}(\mathcal{U}_0, \mathcal{M}) = \begin{cases} \mathcal{M}(X) & \text{if } n = 0 \\ 0 & \text{else} \end{cases} .$$

(*c*) If $\mathcal{M}$ and $\mathcal{N}$ are presheaves of modules on X and $\mathcal{U}$ is an open cover of X, then

$$\check{C}^*(\mathcal{U}, \mathcal{M} \oplus \mathcal{N}) = \check{C}^*(\mathcal{U}, \mathcal{M}) \oplus \check{C}^*(\mathcal{U}, \mathcal{N})$$

and, more general,

$$\check{C}^*\left(\mathcal{U}, \prod_{i \in I} \mathcal{M}_i\right) = \prod_{i \in I} \check{C}^*(\mathcal{U}, \mathcal{M}_i) .$$

The same holds for $\check{C}^*_{\mathrm{alt}}(\mathcal{U}, -)$.

If $\mathcal{U}\colon X = \bigcup_{i \in I} U_i$ is a cover of a scheme X by affine open subsets U_i, then all intersections $U_{i_0,\dots,i_n}$ are affine again by Proposition 1.1.1. If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is a short exact sequence in $\mathrm{QCoh}(X)$, Corollary 1.1.2 provides short exact sequences

$$0 \longrightarrow \check{C}^*(\mathcal{U}, \mathcal{M}') \longrightarrow \check{C}^*(\mathcal{U}, \mathcal{M}) \longrightarrow \check{C}^*(\mathcal{U}, \mathcal{M}'') \longrightarrow 0$$

and

$$0 \longrightarrow \check{C}^*_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}') \longrightarrow \check{C}^*_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \longrightarrow \check{C}^*_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}'') \longrightarrow 0$$

of chain complexes. For $\check{C}^*(\mathcal{U}, -)$ this is immediate from Definition 1 and from the fact that products of short exact sequences are short exact again. To see this for $\check{C}^*_{\mathrm{alt}}(\mathcal{U}, -)$, choose any linear ordering of I and note that

$$\check{C}^n_{\mathrm{alt}}(\mathcal{U}, \mathcal{M}) \simeq \prod_{i_0 < \dots < i_n \in I} \mathcal{M}(U_{i_0,\dots,i_n}) ,$$

then the same argument may be applied.

Taking long exact cohomology sequences we just proved

Proposition 1. *If $\mathcal{U}$ is an affine open cover of a scheme X and $\mathcal{M}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then there is a long exact cohomology sequence*

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(\mathcal{U}, \mathcal{M}') \arrow[r] \arrow[d, "\wr"] & \check{H}^0(\mathcal{U}, \mathcal{M}) \arrow[r] \arrow[d, "\wr"] & \check{H}^0(\mathcal{U}, \mathcal{M}'') \arrow[r, "d"] \arrow[d, "\wr"] & \check{H}^1(\mathcal{U}, \mathcal{M}') \arrow[r] & \check{H}^1(\mathcal{U}, \mathcal{M}) \arrow[r] & \dots \\ 0 \arrow[r] & \mathcal{M}'(X) \arrow[r] & \mathcal{M}(X) \arrow[r] & \mathcal{M}''(X) & & & \end{tikzcd}$$

and similar for $\check{H}^*_{\mathrm{alt}}(\mathcal{U}, -)$.

Remark 3. For arbitrary preschemes, the situation is more difficult (cf. Thomason, *The Grothendieck Festschrift*).

Definition 2. An open cover $\mathcal{V}\colon X = \bigcup_{j\in J} V_j$ is a **refinement** of $\mathcal{U}\colon X = \bigcup_{i\in I} U_i$ if there is a map $v\colon J \to I$ such that $V_j \subseteq U_{v(j)}$ for all $j \in J$. Such a v is called a **refinement map** for the pair $(\mathcal{V},\mathcal{U})$.

Note that Professor Franke isn't sure whether *refinement map* is the usual term. Assuming the axiom of choice, the existence of v is equivalent to every V_j being contained in some U_i.

A refinement map v induces a morphism

$$\check{C}^*(\mathcal{U},\mathcal{M}) \xrightarrow{v^*} \check{C}^*(\mathcal{V},\mathcal{M})\ , \quad (v^n\psi)_{j_0,\dots,j_n} = \psi_{v(j_0),\dots,v(j_n)}|_{V_{j_0,\dots,j_n}} \quad \text{for } \psi \in \check{C}^n(\mathcal{U},\mathcal{M})\ .$$

Clearly, v^* commutes with the d_j, s_j, and t_j, hence restricts to a morphism of chain complexes $\check{C}^*_{\mathrm{alt}}(\mathcal{U},\mathcal{M}) \xrightarrow{v^*} \check{C}^*_{\mathrm{alt}}(\mathcal{V},\mathcal{M})$.

Lemma 1. (*a*) *A refinement $\mathcal{W}$ of a refinement $\mathcal{V}$ of $\mathcal{U}$ is a refinement of $\mathcal{U}$ and if v and w are associated refinement maps for $(\mathcal{V},\mathcal{U})$ and $(\mathcal{W},\mathcal{U})$, then vw is a refinement map for $(\mathcal{W},\mathcal{U})$. Moreover, $(vw)^* = w^*v^*$ and the identity id_I is a refinement map for $(\mathcal{U},\mathcal{U})$ and $\mathrm{id}_I^* = \mathrm{id}_{\check{C}^*(\mathcal{U},-)}$.*

(*b*) *Two arbitrary open covers have a common refinement. When X is a prescheme, this common refinement can be chosen affine.*

(*c*) *If $v_1, v_2\colon J \to I$ are two refinement maps for $(\mathcal{V},\mathcal{U})$, then v_1^* and v_2^* induce the same morphism on Čech cohomology.*

Proof. Part (*a*) is obvious. For (*b*), let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. Then $X = \bigcup_{(i,j)\in I\times J} U_i \cap V_j$ is a common refinement. When X is a prescheme, we may cover each $U_i \cap V_j$ by affine open subsets and thus obtain a common affine refinement of $\mathcal{U}$ and $\mathcal{V}$.

Now for part (*c*). Define maps $h^n\colon \check{C}^n(\mathcal{U},\mathcal{M}) \to \check{C}^{n-1}(\mathcal{V},\mathcal{M})$ as follows: We put

$$h^n = \sum_{\ell=0}^{n-1} (-1)^\ell h_\ell\ ,$$

where $h_\ell\colon \check{C}^n(\mathcal{U},\mathcal{M}) \to \check{C}^{n-1}(\mathcal{V},\mathcal{M})$ is given by

$$(h_\ell\psi)_{j_0,\dots,j_{n-1}} = \psi_{v_1(j_0),\dots,v_1(j_\ell),v_2(j_\ell),\dots,v_2(j_{n-1})}|_{V_{j_0,\dots,j_{n-1}}} \quad \text{for } \psi \in \check{C}^n(\mathcal{U},\mathcal{M})\ .$$

Then it's a straightforward but tedious check that the following relations hold:

$$h_\ell d_k = \begin{cases} d_k h_{\ell-1} & \text{if } 0 \le k < \ell \\ h_{\ell-1} d_k & \text{if } 0 < k = \ell \\ v_2^n & \text{if } 0 = k = \ell \\ h_{\ell+1} d_k & \text{if } k = \ell+1 < n \\ v_1^n & \text{if } k = \ell+1 = n \\ d_{k-1} h_\ell & \text{if } k > \ell+1 \end{cases}\ . \tag{5}$$

(I tried my best to get the indices right and I claim my hit ratio is way higher than Franke's). Our goal is to show

$$\check{d}^{n-1}h^n + h^{n+1}\check{d}^n = v_2^n - v_1^n \ ,$$

for then h^* is a cochain homotopy between v_1^* and v_2^* and it's a well-known fact from homological algebra that cochain homotopic maps induce the same morphisms on cohomology. Indeed, using (5) we get

$$\begin{aligned}
h^{n+1}\check{d}^n &= \sum_{\ell=0}^{n}\sum_{k=0}^{n+1}(-1)^{\ell+k}h_\ell d_k \\
&= \sum_{\ell=0}^{n}\sum_{k=0}^{\ell-1}(-1)^{\ell+k}h_\ell d_k + (-1)^0 h_0 d_0 + \sum_{\ell=1}^{n}(-1)^{2\ell}h_\ell d_\ell \\
&\qquad + \sum_{\ell=0}^{n-1}(-1)^{2\ell+1}h_\ell d_{\ell+1} + (-1)^{2n+1}h_n d_{n+1} + \sum_{\ell=0}^{n}\sum_{k=\ell+2}^{n+1}(-1)^{\ell+k}h_\ell d_k \\
&= \sum_{\ell=0}^{n}\sum_{k=0}^{\ell-1}(-1)^{\ell+k}d_k h_{\ell-1} + v_2^n + \sum_{\ell=1}^{n}h_{\ell-1}d_\ell \\
&\qquad - \sum_{\ell=0}^{n-1}h_\ell d_{\ell+1} - v_1^n + \sum_{\ell=0}^{n}\sum_{k=\ell+2}^{n+1}(-1)^{\ell+k}d_{k-1}h_\ell \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{\ell}(-1)^{\ell+1+k}d_k h_\ell + \sum_{\ell=0}^{n}\sum_{k=\ell+1}^{n}(-1)^{\ell+k+1}d_k h_\ell + v_2^n - v_1^n \\
&= -\sum_{k=0}^{n}\sum_{\ell=0}^{n-1}(-1)^{\ell+k}d_k h_\ell + v_2^n - v_1^n \\
&= -\check{d}^{n-1}h^n + v_2^n - v_1^n \ ,
\end{aligned}$$

as required. *q.e.d.*

A. Appendix – category theory corner

A.1. Towards abelian categories

Definition 1. (*a*) A **pointed** category is a category with initial and final objects, such that the canonical (unique) morphism from the initial to the final object is an isomorphism.

(*b*) An **additive** category $\mathcal{A}$ is a pointed category which has a product $X \times Y$ (i.e., a fibre product over the final object $*$) and coproduct $X \amalg Y$ (i.e., a dual fibre product with respect to the initial object $*$) such that the canonical morphism $X \amalg Y \to X \times Y$ is an isomorphism for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ and such that the resulting addition law on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ defines a group structure for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

Remark. (*a*) When $\mathcal{A}$ is a pointed category and $X, Y \in \mathrm{Ob}(\mathcal{A})$, let the *zero morphism* (which we denote 0) $X \xrightarrow{0} Y$ be defined by $X \to * \to Y$, where $*$ is the both initial and final object.

(*b*) We will construct the canonical morphism $X \amalg Y \xrightarrow{c} X \times Y$ from Definition 1(*b*). The product $X \times Y$ comes with canonical projections $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ such that given morphisms $T \xrightarrow{\xi} X$ and $T \xrightarrow{\upsilon} Y$ there is a unique $T \xrightarrow{\xi\times\upsilon} X \times Y$ such that

$$\begin{tikzcd}
X & & \\
& X \times Y \arrow[ul, "p_1"] \arrow[dl, "p_2"'] & T \arrow[l, dashed, "\exists!\ \xi \times \upsilon"'] \arrow[ull, bend right, "\xi"'] \arrow[dll, bend left, "\upsilon"] \\
Y & &
\end{tikzcd}$$

commutes.

Similarly, the coproduct $X \amalg Y$ has morphisms $X \xrightarrow{i_1} X \amalg Y \xleftarrow{i_2} Y$ such that given morphisms $X \xrightarrow{\xi} T$ and $Y \xrightarrow{\upsilon} T$ there is a unique morphism $X \amalg Y \xrightarrow{\xi \amalg \upsilon} T$ such that

$$\begin{tikzcd}
X \arrow[dr, "i_1"] \arrow[drr, bend left, "\xi"] & & \\
& X \amalg Y \arrow[r, dashed, "\exists!\ \xi \amalg \upsilon"] & T \\
Y \arrow[ur, "i_2"'] \arrow[urr, bend right, "\upsilon"'] & &
\end{tikzcd}$$

commutes.

Using the universal property of $X \times Y$, we get a unique morphism $X \xrightarrow{\alpha} X \times Y$ such that $p_1\alpha = \mathrm{id}_X$, $p_2\alpha = 0$ and a unique morphism $Y \xrightarrow{\beta} X \times Y$ such that $p_1\beta = 0$ and $p_2\beta = \mathrm{id}_Y$. Then

$$c\colon X \amalg Y \xrightarrow{\alpha \amalg \beta} X \times Y$$

is the morphism we are looking for. It is unique with the property that $p_1ci_1 = \mathrm{id}_X$, $p_1ci_2 = 0$, $p_2ci_1 = 0$, and $p_2ci_2 = \mathrm{id}_Y$.

(*c*) For abelian groups and modules over a ring, both $X \amalg Y$ and $X \times Y$ are given by $\{(x, y) \mid x \in X,\ y \in Y\}$ with component-wise operations and $p_1(x, y) = x$, $p_2(x, y) = y$, $i_1(x) = (x, 0)$, and $i_2(y) = (0, y)$.

(*d*) For an additive category $\mathcal{A}$, it follows that finite products $\prod_{i=1}^n X_i$ and coproducts $\coprod_{i=1}^n X_i$ (of some objects $X_1, \ldots, X_n \in \mathrm{Ob}(\mathcal{A})$) exist and are canonically isomorphic. We typically denote both by $\bigoplus_{i=1}^n X_i$ in that case.

(*e*) We would like to describe the addition on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. For a pair of morphisms $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ we denote the composition

$$X \xrightarrow{\mathrm{id}_X \times \mathrm{id}_X} X \oplus X \xrightarrow{a \amalg b} Y$$

by $a + b$. Then 0 is a neutral element and associativity holds, but the existence of inverse elements needs to be imposed to obtain indeed a group structure.

(*f*) It is, however, automatically abelian. What we need to show is $(a \amalg b) \circ \Delta = (b \amalg a) \circ \Delta$ with $\Delta = \mathrm{id}_X \times \mathrm{id}_X$. The universal property of coproducts gives a unique $X \oplus X \xrightarrow{\sigma} X \oplus X$ such that

$$\begin{tikzcd}
& X \arrow[dl, "i_2"'] \arrow[d, "i_1"] \arrow[drr, bend left, "a"] & & \\
X \oplus X \arrow[r, dashed, "\exists!\, \sigma"] & X \oplus X \arrow[rr, "a \amalg b"] & & Y \\
& X \arrow[ul, "i_1"] \arrow[u, "i_2"'] \arrow[urr, bend right, "b"'] & &
\end{tikzcd}$$

commutes. Then σ is easily seen to be an isomorphism and $b \amalg a = (a \amalg b) \circ \sigma$ by the uniqueness of $b \amalg a$. It thus suffices to show $\sigma\Delta = \Delta$. By the uniqueness of Δ, this is equivalent to $p_1\sigma\Delta = \mathrm{id}_X$ and $p_2\sigma\Delta = \mathrm{id}_X$. We claim that $p_1\sigma = p_2$ and vice versa, which would finish the proof. To see this, note that $p_1\sigma = p_2$ is equivalent to $p_1\sigma i_1 = p_2 i_1 = 0$ and $p_1\sigma i_2 = p_2 i_2 = \mathrm{id}_X$ by the universal property of the coproduct $X \oplus X$. This follows from $\sigma i_1 = i_2$ and $\sigma i_2 = i_1$ by definition of σ.

Example. The following are additive categories.

(*a*) Modules over a given ring R (in particular, abelian groups).

(*b*) Sheaves of modules.

(*c*) Banach spaces with bounded linear maps as morphisms. The common initial and final object is the zero space and $A \oplus B = \{(a,b) \mid a \in A,\ b \in B\}$ with $\max\{\|a\|, \|b\|\}$ or $\|a\| + \|b\|$ as norm (this category will turn out not to be abelian).

(*d*) Free or projective modules over a ring R.

Definition 2. Let $A \xrightarrow{\alpha} B$ be a morphism in an additive category $\mathcal{A}$. The **kernel** $\ker(A \xrightarrow{\alpha} B)$ of α (if it exists) comes with a morphism $\ker(\alpha) \xrightarrow{\iota} A$ satisfying the universal property

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}\left(T, \ker\left(A \xrightarrow{\alpha} B\right)\right) &\xrightarrow{\sim} \{f \in \mathrm{Hom}_{\mathcal{A}} \mid \alpha f = 0\} \\ \left(T \xrightarrow{\tau} A\right) &\longmapsto f = \iota\tau \end{aligned}$$

for any test object $T \in \mathrm{Ob}(\mathcal{A})$.

Definition 2a. Similarly, the **cokernel** of α (if existent) comes with a morphism $B \xrightarrow{\pi} \mathrm{coker}(\alpha)$ and satisfies

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}\left(\mathrm{coker}\left(A \xrightarrow{\alpha} B\right), T\right) &\xrightarrow{\sim} \{g \in \mathrm{Hom}_{\mathcal{A}}(B,T) \mid g\alpha = 0\} \\ \left(\mathrm{coker}(\alpha) \xrightarrow{\tau} T\right) &\longmapsto g = \tau\pi \end{aligned}$$

for any test object $T \in \mathrm{Ob}(\mathcal{A})$.

Remark 1. Kernels and cokernels in an additive category $\mathcal{A}$ are special cases of *equalizers* and *coequalizers* (cf. [1, Definition A.3.2 and Definition A.3.4]), respectively. Indeed, we have

$$\ker\left(A \xrightarrow{\alpha} B\right) = \mathrm{Eq}\left(A \underset{0}{\overset{\alpha}{\rightrightarrows}} B\right) \quad \text{and} \quad \mathrm{coker}\left(A \xrightarrow{\alpha} B\right) = \mathrm{Coeq}\left(A \underset{0}{\overset{\alpha}{\rightrightarrows}} B\right).$$

But we can reconstruct equalizers and coequalizers from kernels and cokernels via

$$\mathrm{Eq}\left(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\right) = \ker\left(A \xrightarrow{\alpha-\beta} B\right) \quad \text{and} \quad \mathrm{Coeq}\left(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\right) = \mathrm{coker}\left(A \xrightarrow{\alpha-\beta} B\right)$$

(the minus here is the one obtained from additivity of $\mathcal{A}$).

Definition 3. A morphism $A \xrightarrow{i} B$ is an **effective monomorphism**, if the following equivalent conditions hold.

(*a*) (In any category) We have a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(T,A) &\xrightarrow{\sim} \left\{ f \in \mathrm{Hom}_{\mathcal{A}}(T,B) \;\middle|\; \begin{array}{c} \alpha f = \beta f \text{ if } B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} S \text{ is any pair of} \\ \text{morphisms such that } \alpha i = \beta i \end{array} \right\} \\ t \in \mathrm{Hom}_{\mathcal{A}}(T,A) &\longmapsto f = it\,. \end{aligned}$$

(*b*) (If the category has finite colimits) i is an equalizer of something.

(*c*) (In additive categories with kernels and cokernels) i is the kernel of an appropriate morphism.

(*d*) (In additive categories with kernels and cokernels) i is the kernel of its cokernel.

Definition 3a. Dually, $A \xrightarrow{p} B$ is an **effective epimorphism** if the following equivalent conditions hold.

(*a*) (In any category) We have a bijection

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}(B,T) &\xrightarrow{\sim} \left\{ f \in \operatorname{Hom}_{\mathcal{A}}(A,T) \;\middle|\; \begin{array}{c} f\alpha = f\beta \text{ if } S \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \text{ is any pair of} \\ \text{morphisms such that } p\alpha = p\beta \end{array} \right\} \\ t \in \operatorname{Hom}_{\mathcal{A}}(B,T) &\longmapsto f = tp\ . \end{aligned}$$

(*b*) (If the category has finite limits) p is a coequalizer of something.

(*c*) (In additive categories with kernels and cokernels) p is the cokernel of an appropriate morphism.

(*d*) (In additive categories with kernels and cokernels) p is the cokernel of its kernel.

(*e*) $B^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} A^{\mathrm{op}}$ is an effective monomorphism in the dual category $\mathcal{A}^{\mathrm{op}}$.

In any category, a morphism which is mono and effectively epi (or epi and effectively mono) is an isomorphism, but there are examples of morphisms which are simultaneously mono and epi but not an isomorphism (e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category of rings). This needs to be ruled out by a definition, and that's what is happening now!

Definition 4. A category $\mathcal{A}$ is **abelian**, if it is additive, has kernels and cokernels and such that every monomorphism is effectively mono, every epimorphism is effectively epi, and (thus) any morphism which is both a mono- and an epimorphism is an isomorphism.

Remark. (*a*) The three conditions on mono- and epimorphisms are not independent. The last condition, i.e. that every morphism which is both a mono- and an epimorphism is an isomorphism, follows from either of the former two.

(*b*) Since equalizers and coequalizers in an abelian category $\mathcal{A}$ can be constructed from kernels and cokernels (cf. Remark 1) and we already have finite products and coproducts from additivity of $\mathcal{A}$, we deduce that $\mathcal{A}$ has arbitrary finite limits and colimits. Indeed, we proved on exercise sheet #7 from Algebraic Geometry I that finite limits can be constructed from equalizers and finite products, and, given coequalizers and finite coproducts instead, it's just the same for finite colimits.

Conversely, the existence of finite limits and colimits guarantees that $\mathcal{A}$ has equalizers, coequalizers, finite products, and finite products, all of them being special cases of finite limits and colimits.

The category of modules (over a ring R) or sheaves of modules are abelian categories (as we are going to prove in a moment), but not Banach spaces or projective modules over most rings.

Proposition 1. *The category $\mathcal{R}$-*Mod *of sheaves of modules (over a sheaf of rings $\mathcal{R}$ on some topological space X) is abelian.*

For clarity (and to better distinguish between the proof and Professor Franke's remarks about it), we will chop the proof into some lemmas.

Lemma 1. *The category* $\mathcal{R}$-Mod *is additive.*

Proof. First note that the zero sheaf 0 is a common initial and final object. A direct sum of $\mathcal{M}, \mathcal{N} \in \mathrm{Ob}(\mathcal{R}\text{-Mod})$ is given by

$$(\mathcal{M} \oplus \mathcal{N})(U) = \{(m,n) \mid m \in \mathcal{M}(U), n \in \mathcal{N}(U)\} \quad \text{for all } U \subseteq X \text{ open}$$

(it's clear that this is a presheaf and it inherits the sheaf axiom from $\mathcal{A}$ and $\mathcal{N}$) with componentwise module operations and with $\mathcal{M} \xleftarrow{p} \mathcal{M} \oplus \mathcal{N} \xrightarrow{q} \mathcal{N}$ and $\mathcal{M} \xrightarrow{i} \mathcal{M} \oplus \mathcal{N} \xleftarrow{j} \mathcal{N}$ given by $p(m,n) = m$, $q(m,n) = n$, $i(m) = (m,0)$, and $j(n) = (0,n)$ on open subsets $U \subseteq X$ and $m \in \mathcal{M}(U)$, $n \in \mathcal{N}(U)$.

If $\mathcal{M} \xrightarrow{\mu} \mathcal{T} \xleftarrow{\nu} \mathcal{N}$ are given, $\mathcal{M} \oplus \mathcal{N} \xrightarrow{\mu \amalg \nu} \mathcal{T}$ sending $(m,n) \in (\mathcal{M} \oplus \mathcal{N})(U)$ to $\mu(m) + \nu(n)$ verifies the universal property of the coproduct for $\mathcal{M} \oplus \mathcal{N}$. Similarly, $\mathcal{T} \xrightarrow{\mu \times \nu} \mathcal{M} \oplus \mathcal{N}$ given by $(\mu \times \nu)(t) = (\mu(t), \nu(t))$ for $t \in \mathcal{T}(U)$ confirms the universal property of the product for $\mathcal{M} \oplus \mathcal{N}$. Also, $c = \mathrm{id}_{\mathcal{M} \oplus \mathcal{N}}$ is the unique endomorphism c of that object such that $pci = \mathrm{id}_{\mathcal{M}}$, $qcj = \mathrm{id}_{\mathcal{N}}$, $pcj = 0$, and $qci = 0$. Thus, $\mathcal{R}$-Mod is additive (the group structure on Hom sets being easily verified). *q.e.d.*

Lemma 2. *The category* $\mathcal{R}$-Mod *has kernels.*

Proof. Let $\mathcal{M} \xrightarrow{f} \mathcal{N}$ be a morphism of sheaves of $\mathcal{R}$-modules and $\mathcal{K}$ be the sheaf given by

$$\mathcal{K}(U) = \ker\left(\mathcal{M} \xrightarrow{f} \mathcal{N}\right)(U) := \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right)$$

(you should convince yourself that this indeed satisfies the sheaf axiom). Then the inclusion $\mathcal{K} \xrightarrow{\kappa} \mathcal{M}$ is a monomorphism as $\mathcal{K}(U) \hookrightarrow \mathcal{M}(U)$ is injective for every open subset $U \subseteq X$.

If $\mathcal{T} \xrightarrow{\tau} \mathcal{M}$ is a morphism of $\mathcal{R}$-modules such that $f\tau = 0$, then, for every $t \in \mathcal{T}(U)$, we have $f(\tau(t)) = 0$, hence $\widetilde{\tau}(t) := \tau(t) \in \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) = \mathcal{K}(U)$ and τ factors over

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\tau"] \arrow[dr, dashed, "\exists!\, \widetilde{\tau}"'] & & \mathcal{M} \\ & \mathcal{K} \arrow[ur, hook, "\kappa"'] & \end{tikzcd}$$

This proves that $\mathcal{K}$ is indeed a kernel of f in the category $\mathcal{R}$-Mod. *q.e.d.*

Remark 2. (a) It is a consequence of the exactness of the $\varinjlim$ functor (for filtered systems of abelian groups; exactness of $\varinjlim$ does *not* hold in general, not even for filtered colimits in abelian categories), that

$$\mathcal{K}_x = \varinjlim_{U \ni x} \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \ker\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right).$$

This isomorphism can also be seen in a straightforward way.

(b) One may check that in any additive category (with kernels), a morphism i is a monomorphism iff $\ker(i) = 0$. Thus, in our example we have the equivalent conditions

(α) $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is a monomorphism.

(β) $\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)$ is injective for all open subsets $U \subseteq X$.

(γ) $\ker(f) = 0$ (the zero sheaf).

(δ) $\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x$ is injective for all $x \in X$.

The construction of cokernels won't be that straightforward (duh!), related to the fact that epimorphisms in categories of sheaves aren't as simple as you might think. If $\mathcal{G}$ and $\mathcal{H}$ are sheaves on some topological space X and f is a morphism between them such that $\mathcal{G}(U) \xrightarrow{f} \mathcal{H}(U)$ is surjective for all open U, then f is an epimorphism, but there are epimorphisms f for which this fails.

However, it follows from the fact that a sheaf $\mathcal{G}$ is canonically isomorphic to its sheafification $\mathcal{G}^{\mathrm{sh}}$ (cf. [1, Proposition 1.2.1(d)]) that a morphism between sheaves (of sets, groups, ...) is uniquely determined by the maps it induces on stalks. Thus, $\mathcal{G} \to \mathcal{H}$ is an epimorphism if $\mathcal{G}_x \to \mathcal{H}_x$ is an epimorphism in the respective target category for all $x \in X$.

Lemma 3. *The category* $\mathcal{R}$-Mod *has cokernels.*

Proof. For a morphism $\mathcal{M} \xrightarrow{f} \mathcal{N}$ of sheaves of $\mathcal{R}$-modules, the map

$$U \mapsto \operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) = \mathcal{M}(U)/\mathcal{N}(U) \quad \text{for } U \subseteq X \text{ open}$$

defines a presheaf $\mathcal{F}$ of $\mathcal{R}$-modules, but in general, $\mathcal{F}$ will fail to be a sheaf. We put $\mathcal{C} = \mathcal{F}^{\mathrm{sh}}$ (the *sheafification* of $\mathcal{F}$, cf. [1, Definition 1.2.3]) and claim that $\mathcal{N} \to \mathcal{C}$ is a cokernel of f.

Our first goal is to show that

$$\mathcal{C}_x \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right) . \tag{$*$}$$

In the lecture, we did a direct proof, which was somewhat ugly and (in my opinion) lacking the essential step. From [1, Proposition 1.2.1(a)], we get that $\mathcal{C}_x \simeq \mathcal{F}_x$ (which is basically what we proved in the lecture for this particular special case), so we need to show that

$$\mathcal{F}_x = \varinjlim_{U \ni x} \operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right) .$$

Since $\mathcal{M}_x = \varinjlim_{U \ni x} \mathcal{M}(U)$ and similar for $\mathcal{N}_x$, this amounts to showing that cokernels and certain colimits commute. But by Remark 1, cokernels are just a special case of colimits, so what we are actually going to show is that colimits commute with colimits – in the following sense.

Lemma 4. *Let $(X_{i,j})_{i\in I, j\in J}$ be objects of a category $\mathcal{A}$. For each $i_1, i_2 \in I$ let there be an indexing set I_{i_1,i_2} and for each $\alpha \in I_{i_1,i_2}$ and $j \in J$ a morphism*

$$f_\alpha^j \colon X_{i_1,j} \longrightarrow X_{i_2,j} \ .$$

Similarly, for each $j_1, j_2 \in J$ let there be an indexing set J_{j_1,j_2} and for each $\beta \in J_{j_1,j_2}$ and $i \in I$ a morphism

$$g_\beta^i \colon X_{i,j_1} \longrightarrow X_{i,j_2} \ .$$

Moreover, suppose that for each $i_1, i_2 \in I$ and $j_1, j_2 \in J$ and $\alpha \in I_{i_1,i_2}$ and $\beta \in J_{j_1,j_2}$ the diagram

$$\begin{tikzcd}
X_{i_1,j_1} \arrow[r, "f_\alpha^{j_1}"] \arrow[d, "g_\beta^{i_1}"'] & X_{i_2,j_1} \arrow[d, "g_\beta^{i_2}"] \\
X_{i_1,j_2} \arrow[r, "f_\alpha^{j_2}"] & X_{i_2,j_2}
\end{tikzcd} \tag{\#}$$

commutes. Then there is an isomorphism

$$\varinjlim_{i\in I} \varinjlim_{j\in J} X_{i,j} \simeq \varinjlim_{j\in J} \varinjlim_{i\in I} X_{i,j} \simeq \varinjlim_{(i,j)\in I\times J} X_{i,j} \ .$$

Proof of Lemma 4. Clearly, it is enough to show the rightmost isomorphism. What we need to show is that $L \coloneqq \varinjlim_j \varinjlim_i X_{i,j}$ satisfies the universal property of $L' \coloneqq \varinjlim_{(i,j)} X_{i,j}$.

Let T be an object of $\mathcal{A}$ and $\big(X_{i,j} \xrightarrow{\tau_{i,j}} T\big)_{i\in I, j\in J}$ be a cocone below the diagram $(X_{i,j})_{i,j}$. That is, for every $\alpha \in I_{i_1,i_2}$ and $j \in J$ the diagram

$$\begin{tikzcd}
X_{i_1,j} \arrow[rr, "f_\alpha^j"] \arrow[dr, "\tau_{i_1,j}"'] & & X_{i_2,j} \arrow[dl, "\tau_{i_2,j}"] \\
& T &
\end{tikzcd}$$

commutes. By the universal properties of the $L_j \coloneqq \varinjlim_i X_{i,j}$, the $\tau_{i,j}$ factor over some maps $L_j \xrightarrow{\tau_j} T$. Moreover, for each $j_1, j_2 \in J$ and $\beta \in J_{j_1,j_2}$, the compositions

$$X_{i,j_1} \xrightarrow{g_\beta^i} X_{i,j_2} \longrightarrow L_{j_2}$$

induce a map $L_{j_1} \xrightarrow{g_\beta} L_{j_2}$ by the universal property of L_{j_1} (here, we silently used the commutativity of (#), otherwise the above compositions wouldn't be a cocone below $(X_{i,j_1})_{i\in I}$). We thus get a diagram

$$\begin{tikzcd}
X_{i,j_1} \arrow[rrrr, "g_\beta^i"] \arrow[dr] \arrow[ddrr, bend right, "\tau_{i,j_1}"'] & & & & X_{i,j_2} \arrow[dl] \arrow[ddll, bend left, "\tau_{i,j_2}"] \\
& L_{j_1} \arrow[rr, dashed, "g_\beta"] \arrow[dr, "\tau_{j_1}"'] & & L_{j_2} \arrow[dl, "\tau_{j_2}"] & \\
& & T & &
\end{tikzcd}$$

in which everything but the bottom-middle triangle commutes. We show that this triangle commutes as well. Indeed, by the universal property of L_{j_1}, τ_{j_1} is the unique morphism $L_{j_1} \to T$ making each

$$\begin{tikzcd} X_{i,j_1} \arrow[rr] \arrow[dr, "\tau_{i,j_1}"'] & & L_{j_1} \arrow[dl] \\ & T & \end{tikzcd}$$

commute. But apparently, $\tau_{j_2} g_\beta$ has this property as well, proving $\tau_{j_1} = \tau_{j_2} g_\beta$. Then the morphisms $\left(L_j \xrightarrow{\tau_j} T\right)_{j \in J}$ form a cocone below the diagram $(L_j)_{j\in J}$, hence factor uniquely over some $L \xrightarrow{\tau} T$ by the universal property of L.

It remains to prove uniqueness of τ. If $L \xrightarrow{\tau} T$ is a morphism over which each $X_{i,j} \xrightarrow{\tau_{i,j}} T$ factors, then the composition $L_j \to L \xrightarrow{\tau} T$ must equal τ_j since τ_j is uniquely determined by the universal property of L_j. But τ is uniquely determined by the τ_j, proving uniqueness. *q.e.d.*

Having thus proved $(*)$, we now proceed with the proof of Lemma 3. We have a morphism $\mathcal{N} \to \mathcal{C}$ sending $n \in \mathcal{N}(U)$ to

$$\left(\text{image of } n \text{ under } \mathcal{N}(U) \longrightarrow \mathcal{N}_x \longrightarrow \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)\right)_{x\in U}.$$

Since $\mathcal{C}_x \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)$, this morphism $\mathcal{N} \to \mathcal{C}$ induces surjections on stalks, hence is an epimorphism of sheaves. We show that the morphism $\mathcal{N} \to \mathcal{C}$ satisfies the universal property of the cokernel.

Let $\mathcal{N} \xrightarrow{\tau} \mathcal{T}$ be a morphism of sheaves of $\mathcal{R}$-modules such that $\tau f = 0$. Let $U \subseteq X$ be open. For

$$\nu = (\nu_x)_{x\in U} \in \mathcal{C}(U) \subseteq \prod_{x\in U} \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)$$

we define $\tau_1(\nu) \in \prod_{x\in U} \mathcal{T}_x$ by selecting $n \in \mathcal{N}_x$ whose image in $\operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)$ equals ν_x, then put $\tau_1(\nu)_x = \tau(n)_x$ which is independent of the choice of n as $\tau f = 0$. It follows from the coherence condition for $\mathcal{C}$ that $\tau_1(\nu)$ satisfies the coherence condition for $\mathcal{T}^{\mathrm{sh}}$, i.e. $\tau_1(\nu) \in \mathcal{T}^{\mathrm{sh}}(U) \subseteq \prod_{x\in U} \mathcal{T}_x$. Hence there is $\mathcal{C} \xrightarrow{\tau_2} \mathcal{T}$ such that $\tau_1 = \left(\mathcal{T} \xrightarrow{\sim} \mathcal{T}^{\mathrm{sh}}\right) \circ \tau_2$ and τ_2 makes

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "\tau"] \arrow[dr, dashed, "\exists!\ \tau_2"'] & & \mathcal{T} \\ & \mathcal{C} \arrow[ur] & \end{tikzcd}$$

commutative. Uniqueness of τ_2 is easy to see stalk-wise. It follows that $\mathcal{N} \to \mathcal{C}$ is ineed a cokernel of f. *q.e.d.*

Remark 3. One may check that in any additive category (with cokernels) a morphism f is an epimorphism if $\operatorname{coker}(f) = 0$. By our previous construction of cokernels and the description of stalks, we have equivalent conditions

(*a*) $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is an epimorphism of sheaves of $\mathcal{R}$-modules.

(*b*) $\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x$ is surjective for all $x \in X$.

(*c*) For every open $U \subseteq X$ and $n \in \mathcal{N}(U)$ there are an open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $m_\lambda \in \mathcal{M}(U_\lambda)$ such that $n|_{U_\lambda} = f(m_\lambda)$

... but (*c*) does *not* imply the surjectivity of $\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)$, unless, e.g., f is also a monomorphism.

Proof of Proposition 1. We verify the rest of the abelianness conditions. First, let $\mathcal{M} \xrightarrow{f} \mathcal{N}$ be a mono- and epimorphism. Then it induces isomorphisms on stalks (by Remark 2(*b*) and Remark 3), hence is an isomorphism itself.

Let $\mathcal{M} \xrightarrow{i} \mathcal{N}$ be a monomorphism and $\mathcal{N} \to \mathcal{C}$ be its cokernel. Then

$$\ker\left(\mathcal{N} \longrightarrow \mathcal{C}\right)_x = \ker\left(\mathcal{N}_x \longrightarrow \mathcal{C}_x\right) = \ker\left(\mathcal{N}_x \longrightarrow \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{i} \mathcal{N}_x\right)\right) \simeq \mathcal{M}_x$$

as $\mathcal{M}_x \xrightarrow{i} \mathcal{N}_x$ is injective. Hence $\mathcal{M} \to \ker(\mathcal{N} \to \mathcal{C})$ induces isomorphisms on stalks and thus is an isomorphism itself. It follows by Definition 3(*d*) that any monomorphism is an effective monomorphism.

Similar arguments apply to epimorphisms. *q.e.d.*

Recall the definition of an adjoint pair of functors.

Definition 5 ([1, Definition A.2.3])**.** Let $\mathcal{A}, \mathcal{B}$ be categories. A pair $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ of (covariant) functors is called **adjoint**, if there is a canonical bijection

$$\operatorname{Hom}_{\mathcal{A}}(X, RY) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{B}}(LX, Y)$$

which is functorial in both $X \in \operatorname{Ob}(\mathcal{A})$ and $Y \in \operatorname{Ob}(\mathcal{B})$.

Remark 4. It can be easily seen that L preserves colimits (in particular, coproducts, and in particular again, initial objects) and R preserves limits (in particular, products, and in particular again, final objects). When $\mathcal{A}$ and $\mathcal{B}$ are additive, it follows that both L and R map 0 to 0 and are compatible with finite direct sums. Moreover, L preserves cokernels and R preserves kernels since these are special cases of colimits and limits, respectively (in particular, I have no idea what the purpose of Franke's extra calculation was).

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebraic Geometry I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgGeoI.