Algebra I

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1. The Hilbert Basis- and Nullstellensatz

1.1. Noetherian Rings

Definition 1. Let R be a ring, and $f_1, \ldots, f_n \in R$, then the *ideal generated by the f_i* is

$$(f_1, \ldots, f_n)_R = \left\{ \sum \lambda_i f_i \,\middle|\, \lambda_i \in R \right\} = \bigcap_{f_1, \ldots, f_n \in I \text{ ideal}} I \ .$$

The f_i are called a *basis* or *generators* of I.

Remark 1. If I is not necessarily finite,

$$(f_i \mid i \in I)_R = \left\{ \sum_{i \in I} \lambda_i f_i \,\middle|\, \lambda_i = 0 \text{ for all but finitely many } i \right\} = \bigcap_{(f_i)_{i \in I} \subseteq I} I \ .$$

Definition 2. Let k be a field, $I \subseteq k[X_1, \ldots, X_n]$ an ideal, ℓ a field extension of k. Call $x \in \ell^n$ a *zero* of I iff $f(x_1, \ldots, x_n) = 0$ for all $f \in I$.

Remark 2. An element x is a common zero of the $f_i \in k[X_1, \ldots, X_n]$ iff it is a zero of the ideal generated by the f_i.

Proposition 1. *For a ring R the following conditions are equivalent:*

(i) *Every ideal has a finite set of generators (i.e. is finitely generated).*

(ii) *Every ascending chain $I_0 \subseteq I_1 \subseteq \ldots$ of ideals in R terminates after finitely many steps, i.e. there is some $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n \geq N$.*

(iii) *Every non-empty set $\mathfrak{M}$ of ideals in R has an $\subseteq$-maximal element I.*

Definition 3. A ring with these properties is called *Noetherian.*

Example 1. Fields and principal ideal domains are Noetherian.

Theorem 1 (Hilbert's Basissatz)**.** *If R is Noetherian, so is $R[X_1, \ldots, X_n]$.*

Corollary 1 (of the Basissatz)**.** *Every polynomial system of equations in finitely many variables over a field has finite subsystem with the same set of solutions.*

Theorem 2 (Hilbert's Nullstellensatz)**.** *Let k be a algebraically closed field and I be a proper ideal of $k[X_1, \ldots, X_n]$. Then I has a zero $x \in k^n$.*

Both Hilbert's Nullstellensatz and Hilbert's Basissatz will be proved later on.

1.2. Modules over rings

Definition 1. An R-Module (where R is a ring) is an abelian group $(M, +)$ with an operation

$$\cdot : R \times M \longrightarrow M \;, \quad (r, m) \longmapsto r \cdot m$$

such that for all $r, s \in R$ and $m, n \in M$

$$\begin{aligned} r \cdot (s \cdot m) &= (r \cdot s) \cdot m & \qquad (r + s) \cdot m &= r \cdot m + s \cdot m \\ 1 \cdot m &= m & r \cdot (m + n) &= r \cdot m + r \cdot n \;. \end{aligned}$$

A *morphism* of R-Modules is a map $M \xrightarrow{f} N$ which is a homomorphism of abelian groups compatible with $\cdot$. A *submodule* of M is a subgroup $X \subseteq M$ of $(M, +)$ such that $R \cdot X \subseteq X$.

Example 1. The R-submodules of R are the ideals in R.

Proposition 1. *If $N \subseteq M$ is a R-submodule of the R-module M the quotient group M/N has a unique structure of an R-submodule such that the projection $M \xrightarrow{\pi} M/N$ is a morphism of R-modules, and for arbitrary R-modules T the map*

$$\begin{aligned} \operatorname{Hom}_R(M/N, T) &\longrightarrow \{\tau \in \operatorname{Hom}_R(M, T) \mid \tau|_N = 0\} \\ t &\longmapsto \tau = t \circ \pi \end{aligned}$$

is bijective, where t is surjective iff τ is and t is injective iff $\ker(\tau)$ equals N.

Corollary 1. *Let $N, L \subseteq M$ be submodules of some R-Module M.*

(i) *There is a unique isomorphism $L/(N \cap L) \xrightarrow{\sim} (N + L)/N$ such that the following diagram commutes:*

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, "\pi_{L/(N\cap L)}"'] & N + L \arrow[d, "\pi_{(N+L)/N}"] \\ L/(N \cap L) \arrow[r, dashed, "\sim"] & (N + L)/N \end{tikzcd}$$

(ii) *If further $L \subseteq N$, there is a unique isomorphism $M/N \xrightarrow{\sim} (M/L)/(N/L)$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\pi_{M/L}"] \arrow[d, "\pi_{M/N}"'] & M/L \arrow[d, "\pi_{(M/L)/(N/L)}"] \\ M/N \arrow[r, dashed, "\sim"] & (M/L)/(N/L) \end{tikzcd}$$

Definition 2. If M and N are R-modules, $M \oplus N = M \times N$ equipped with component-by-component addition and scalar multiplication. This can be generalized to finitely many summands.

Example 2. $R^n = \{(r_i)_{i=1}^n \mid r_i \in R\}$ is an R-module.

Definition 3. If M is an R-module and $m_1, \ldots, m_k \in M$, then the *submodule generated by* $\{m_1, \ldots, m_k\}$ *is*

$$\langle m_1, \ldots, m_k \rangle_R = Rm_1 + \ldots + Rm_k = \left\{ \sum r_i \cdot m_i \;\middle|\; r_i \in R \right\} = \bigcap_{m_1, \ldots, m_k \in X \text{ submodule}} X$$

As was the case for Definition 1.1.1, this can be generalized to infinitely many generators. M is *finitely generated* iff there are $m_1, \ldots, m_k \in M$ such that the submodules of M generated by the m_i equals M.

Proposition 2. *Consider an exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

of R-modules.

(i) *If M is finitely generated, then so is L.*

(ii) *If N and L are finitely generated, then so is M.*

Corollary 2. *$M \oplus N$ is finitely generated iff M and N are.*

Proposition 3. *Let M be an R-module. The following properties are equivalent:*

(a) *Every submodule $N \subseteq M$ of M is finitely generated.*

(b) *Every ascending sequence $N_0 \subseteq N_1 \subseteq \ldots$ of submodules of N terminates.*

(c) *Every non-empty set $\mathfrak{M}$ of R-submodules of M has a $\subseteq$-maximal element.*

Proof. **(a) → (b)** Let $N_\infty = \bigcup_{i=0}^\infty N_i$, then this is a submodule, hence finitely generated by a). Let $n_1, \ldots, n_k$ generate N_∞. Choose ℓ_i such that $n_i \in N_{\ell_i}$ and let $\ell = \max_{i \le k} \ell_i$, then $N_\ell = N_\infty$.

(b) → (c) From b) we conclude, that in the $\subseteq$-ordered set $\mathfrak{M}$ every ascending chain has an upper bound in $\mathfrak{M}$, namely the ideal, that terminates the chain. Therefore by Zorn's Lemma there is $\subseteq$-maximal element in $\mathfrak{M}$.

(c) → (a) Let $\mathfrak{M}$ be the set of finitely generated submodules of N. Since $\{0\} \subseteq N$ is a module, this set is not empty. Therefore there is a $\subseteq$-maximal submodule P in $\mathfrak{M}$ generated by $p_1, \ldots, p_n$. Therefore there is no $f \in N \setminus P$ such that $\langle p_1, \ldots, p_n, f \rangle_R$ is a submodule of N since this would be a superset of P. Hence we have $N = P$ is finitely generated.

□

Definition 4. A module over a ring R is *Noetherian* iff the equivalent conditions above are fulfilled.

Remark 1. Sub- and quotient modules of Noetherian rings are Noetherian. If N is a submodule of M and if N and M/N are Noetherian, then M is Noetherian.

Proof. The first assertion follows easily from Proposition 2 and the characterization of *Noetherian modules* by Proposition 3(a). For the second assertion let N and M/N be Noetherian and $X \subseteq M$ be a submodule. Since both $(X \cap N) \subseteq N$ and $X/(X \cap N) \simeq (X + N)/N \subseteq M/N$ are finitely generated as submodules of N, M/N respectively, we obtain the exact sequence

$$0 \longrightarrow X \cap N \longrightarrow X \longrightarrow X/(X \cap N) \longrightarrow 0 \, ,$$

proving that X is finitely generated by Proposition 2. □

Remark 2. Any Noetherian module is finitely generated.

Proposition 4. *Let R be a Noetherian ring. Then any finitely generated R-module is Noetherian.*

Proof. We proceed by induction on the number of generators of M. The case of only one generator is immediate. Now let $M = Rm_1 + \ldots + Rm_k$ and any Ry -module with less than k generators be Noetherian. In particular, $N = Rm_1 + \ldots + Rm_{k-1}$ is Noetherian. The map $R \to M/N$ sending $r \in R$ to $rm_k + N$ is surjective, hence M/N is isomorphic to some quotient of R and thus Noetherian by Remark 1. Then, again by Remark 1, M is Noetherian. □

Definition 5. For a module M over a ring R, define

$$\mathrm{Ann}(M) = \{r \in R \mid r \cdot M = \{0\}\} = \{r \in R \mid r \cdot m = 0 \; \forall m \in M\} \, .$$

It is called the *annihilator* or *annulator* of M.

Proposition 5. *A module M over a ring R is Noetherian iff it is finitely generated and $R/\operatorname{Ann}(M)$ is a Noetherian ring.*

1.3. Proof of the Hilbert basis theorem

Proof. Let R be a Noetherian ring and $I \subseteq R[T]$ be an ideal. Let $R[T]_{\leq n}$ be the set of polynomials over R of degree smaller or equal to n. This is isomorphic to R^{n+1} $(1, \ldots, T^n$ being free generators) as R-modules, thus Noetherian (Proposition 1.2.4) which implies that $I_{\leq n} = I \cap R[T]_{\leq n}$ is a finitely generated R-module. Let I_n be the set of all $a_n \in R$, such that $a_0 + a_1 T + \ldots + a_n T^n \in I$ for some $a_0, \ldots, a_{n-1} \in R\}$. This is an ideal ($R$-submodule) of R, being the image of $I_{\leq n} \to R$ sending $a_0 + a_+ \ldots + a_n T^n \in I_{\leq n}$ to a_n. We have $I_n \subseteq I_{n+1}$ as $T \cdot I_{\leq n} \subseteq I_{\leq n+1}$. As R is Noetherian, this chain terminates at some $N \in \mathbb{N}$ with $I_n = I_N$ for $n \geq N$. Let $f_1, \ldots, f_k$ be generators of $I_{\leq N}$ as an R-module. We claim that they generate I as an $R[T]$-module. Since they generate $I_{\leq N}$ as an R-module, their N-th coefficients $f_N^{(i)}$, where $i \leq k$, generate $I_n = I_N$, for $n \geq N$, as an R-module.

We show by induction on n, that any $g \in I_{\leq n}$ belongs to $(f_1, \ldots, f_k)_{R[T]}$, thus establishing $I = (f_1, \ldots, f_k)_{R[T]}$. For $n \leq k$ we have $g \in I_{\leq N}$ and the assertion is obvious. Let $n > N$ let the assertion be valid for all $h \in I_{\leq n-1}$. Let $g = \sum_{i=1}^n g_i T^i$, $g_n = \sum_{i=1}^k \gamma_i f_N^{(i)}$ and $h = g - \sum_{i=1}^k \gamma_i T^{n-N} f_i$, then $h \in I_{\leq n-1}$ as the coefficient of T^n cancels. Thus, $h = \sum_{i=1}^k \rho_i f_i$ with $\rho_i \in R[T]$ by the induction assumption and

$$g = \sum_{i=1}^k (\gamma_i T^{n-k} + \rho_i) f_i \in (f_1, \ldots, f_k)_{R[T]}$$

as claimed. This shows that I is finitely $R[T]$-generated, hence $R[T]$ is Noetherian. □

Corollary 1. *If R is a Noetherian ring, so is $R[X_1, \ldots, X_n]$ for all $n \in \mathbb{N}$.*

1.4. Finiteness properties of R-algebras

Definition 1. Let R be a ring. An R-*algebra* is a ring A (commutative, with 1) together with a ring homomorphism $R \xrightarrow{\alpha} A$. Then A becomes an R-module via $r \cdot a := \alpha(r) \cdot a$. We call A *finite over* R (or *finite as an* R-*algebra*) if it is finitely generated as an R-module. We call A of *finite type over* R if it is finitely generated as an R-algebra in the sense that there are $f_1, \ldots, f_k \in A$, $k \in \mathbb{N}$, such that any R-subalgebra $B \subseteq A$ (i.e. any subring $B \subseteq A$ which is also a R-submodule, or, equivalently, a subring containing the image of α) containing the f_i must equal A.

Remark 1. If A is an R-algebra and $f_1, \ldots, f_k \in A$, the following subsets of A coincide:

- $\left\{\sum r_\alpha f_1^{\alpha_1} \cdot \ldots \cdot f_k^{\alpha_k} \;\middle|\; r_\alpha \in R, r_\alpha \neq 0 \text{ only for finitely many } \alpha\right\}$
- The image of the ring homomorphism $R[X_1, \ldots, X_k] \to A$ sending $p \in R[X_1, \ldots, X_k]$ to $p(f_1, \ldots, f_k)$.
- The intersection of all R-subalgebras of A containing the f_i.

Thus, an R-algebra A is of finite type iff it is isomorphic to a quotient of $R[X_1, \ldots, X_k]$ by some ideal I for finite k.

Remark 2. a) Obviously, if $f_1, \ldots, f_i \in A$ generate A as an R-module, they generate it as an R-algebra. Thus any finite R-algebra is of finite type. On the other side, when $R \neq \{0\}$ and and $n > 0$, $R[X_1, \ldots, X_n]$ is an R-algebra of finite type that is not finitely generated as an R-module.

b) Obviously, if L/K is a field extension then L is a finite K-algebra iff the field extension is finite. The fact that this still holds if L is a K-algebra of finite type turns out to be essentially equivalent to the Nullstellensatz.

Proposition 1. *Let R be a ring, A an R-algebra. Any A-algebra B becomes an R-algebra via the composition $R \to A \to B$.*

(i) *If A is finite over R, it is of finite type over R.*

(ii) *(transitivity of finiteness) If B is finite over A and A finite over R, then B is finite over R.*

(iii) *If B over A and A over R are of finite type, then B is of finite type over R.*

(iv) *An algebra of finite type over a Noetherian ring is a Noetherian ring.*

Proof. (i) Trivial.

(ii) If $b_1, \ldots, b_m$ generate B as an A-module and $a_1, \ldots, a_n$ generate A as an R-module, the $\beta_{i,j} = a_j \cdot b_i$ generate B as an R-module: Indeed, let $b \in B$, then $b = \sum_{i=1}^m \alpha_i b_i$ (with $\alpha_i \in A$) and each α_i can be written as $\alpha_i = \sum_{j=1}^n r_{i,j} a_j$. Then $b = \sum_{i=1}^m \sum_{j=1}^n r_{i,j} \beta_{i,j}$.

(iii) By Remark 1, we obtain surjective homomorphisms $A[Y_1, \ldots, Y_m] \xrightarrow{\beta} B$ (as A-algebras, hence also as R-algebras) and $R[X_1, \ldots, X_n] \xrightarrow{\alpha} A$ (as R-algebras). Lifting the latter to a surjective homomorphism $R[X_1,\dots,X_n,Y_1,\dots,Y_m] \to A[Y_1,\dots,Y_m]$ and composing them provides us with a surjective homomorphism

$$R[X_1,\dots,X_n,Y_1,\dots,Y_m] \longrightarrow B\ ,$$

proving that B is of finite type over R. In particular, if $b_1,\dots,b_m$ generate B as an A-algebra and $a_1,\dots,a_n$ generate A as an R-algebra, then B is generated by $a_1,\dots,a_n,b_1,\dots,b_m$ as an R-algebra.

(iv) Note that the quotient of a Noetherian ring by an ideal stays Noetherian: The preimage of an infinitely ascending chain of ideals of the quotient ring would be an infinitely ascending chain of ideals of the original ring. Now if $a_1,\dots,a_m \in A$ generate A as an R-algebra, then

$$\begin{aligned} R[X_1,\dots,X_m] &\longrightarrow A \\ p &\longmapsto p(a_1,\dots,a_m) \end{aligned}$$

is surjective and A is isomorphic to a quotient of $R[X_1,\dots,X_m]$, which by the Basissatz is Noetherian if R is.

$\square$

Proposition 2 (Artin-Tate)**.** *Let R be a Noetherian ring, A an R-algebra of finite type and $B \subseteq A$ an R-subalgebra such that A is finite over B. Then B is an R-algebra of finite type.*

Proof. ¡¡¡¡¡¡¡ HEAD Let $a_1,\dots,a_m$ generate A as an R-algebra and let $\alpha_1,\dots,\alpha_n$ generate it as a B-module. We have expressions

$$a_i = \sum_{j=1}^{n} b_{i,j}\alpha_j \quad \text{and} \quad \alpha_k \cdot \alpha_k = \sum_{j=1}^{n} \beta_{j,k,l}\alpha_j. \tag{$*$}$$

Let $\mathfrak{B} \subseteq B$ be the R-algebra generated by the $b_{i,j}$ and the $\beta_{j,k,l}$. It is of finite type over R thus Noetherian. Let $\mathfrak{A} \subseteq A$ be the $\mathfrak{B}$-submodule generated by $\alpha_1,\dots,\alpha_n$. It is a subring containing the a_i by (2) and is an R-algebra because $\mathfrak{B}$ is. Then $\mathfrak{A} = A$ and A is finite over $\mathfrak{B}$. Since $\mathfrak{B}$ is Noetherian, $B \subseteq A$ is a $\mathfrak{B}$-subalgebra, and B is finitely generated as $\mathfrak{B}$-module ($\mathfrak{B}$ being Noetherian), B is of finite type over $\mathfrak{B}$ (Proposition 1(i)) and thus also over R (Proposition 1(iii)). ======= Let $(a_i)_{i=1}^n$ generate A as an R-algebra and let $(\alpha_j)_{j=1}^n$ generate it as a B-module. We have expressions

$$a_i = \sum_{j=1}^{n} b_{i,j}\alpha_j \tag{1}$$

$$\alpha_k \cdot \alpha_k = \sum_{j=1}^{n} \beta_{j,k,l}\alpha_j. \tag{2}$$

Let $\tilde{B} \subseteq B$ be the R-algebra generated by the $b_{i,j}$ and the $\beta_{j,k,l}$. It is of finite type over R thus Noetherian. Let $\tilde{A} \subseteq A$ be the $\tilde{B}$-submodule generated by the $(\alpha_k)_{k=1}^n$. It is a subring by (2) and contains the a_i by (1) and is an R-algebra because $\tilde{B}$ is. Then $\tilde{A} = A$ and A is finite over $\tilde{B}$. Since $\tilde{B}$ is Noetherian and $B \subseteq A$ is a $\tilde{B}$-subalgebra and B is finitely generated as $\tilde{B}$-module ($\tilde{B}$ being Noetherian), hence B is of finite type over $\tilde{B}$ (Proposition 1.4.1a), hence B is of finite type over R (Proposition 1.4.1c) ¿¿¿¿¿¿¿ parent of c0a011f... lecture from May 4th 2017 $\square$

Proposition 3 (Eakin-Nagata)**.** *Let A be a Noetherian ring and $B \subseteq A$ be a subring such that A is finite over B. Then B is Noetherian.*

Remark 3. See Matsumura, CRT, for Eakin-Nagata.

1.5. The notion of integrity and the Noether Normalization Theorem

Remark of the author: It's called integrity not entireness...

Definition 1. Let $A \subseteq B$ be a ring extension. We call $b \in B$ *integral* over A if it satisfies an equation

$$b^n + a_{n-1}b^{n-1} + \ldots + a_1 b + a_0 = 0$$

with $a_0, \ldots, a_{n-1} \in A$. We call B over A *integral*, if every element of B is integral.

Remark 1. It is not really necessary to assume $A \to B$ to be injective.

Proposition 1. *¡¡¡¡¡¡¡ HEAD*

(i) *An element $b \in B$ is integral over A iff there is an intermediate ring $A \subseteq C \subseteq B$ containing b which is finite over A. If $b_1, \ldots, b_n$ are finitely many integral elements of B, there is an A-subalgebra $A \subseteq C \subseteq B$ containing all b_i which is finite over A.*

(ii) *The elements of B which are integral over A form a subring of B, the* integral closure *of A in B.*

(iii) *If C/B and B/A are integral, so is C/A.*

(iv) *Let B/A be integral (where it is essential that A is a subring of B). If B is a field, then so is A.*

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a) $b \in B$ is integral over A iff there is an intermediate ring $A \subseteq C \subseteq B$ containing b which is finite over A. If $b_1, \ldots, b_n$ are finitely many elements of B which are integral over A, the there is an A-subalgebra $A \subseteq C \subseteq B$ which is finite over A and containing all b_i.

b) The elements of B which are integral over A form a subring of B, the integral closure of A in B.

c) If C/B and B/A are integral, C/A is integral.

d) Let B/A be integral (where it is essential that A is a subring of B). If B is a field, then A is a field.

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Proof. (i) Let $b_1, \ldots, b_n$ be integral over A. Each b_i satisfies an equation

$$b_j^{d_i} = \sum_{i=0}^{d_i - 1} a_{i,j} b_j^i \quad \text{where } a_{i,j} \in A\ .$$

Then the subring $C = A[b_1, \ldots, b_n]$ is generated by all $b_1^{k_1} \cdots b_n^{k_n}$ where $0 \leq k_i < d_i$, hence it is finite over A. The first assertion of (i) follows as a special case.

For the other direction let $C \subseteq B$ be an A-subalgebra which is finitely generated as an A-module, say, by $\gamma_1, \ldots, \gamma_n$. Let $b \in C$ and choose $m_{i,j} \in A$ such that

$$b\gamma_j = \sum_{i=1}^{n} m_{i,j}\gamma_j$$

The matrix $M = (m_{i,j})_{i,j=1}^{n}$ satisfies its own characteristic equation by Cayley-Hamilton theorem: $M^n = p_0 + p_1 M + \ldots + p_{n-1}M^{n-1}$ for suitable $p_0, \ldots, p_{n-1} \in A$. Since b^j in C can be expressed by M^j (in the sense that

$$\begin{tikzcd}
(a_1, \ldots, a_n) \arrow[d, mapsto] & A^n \arrow[r, "M^j \cdot"] \arrow[d, two heads, "\gamma"'] & A^n \arrow[d, two heads, "\gamma"] & (a_1, \ldots, a_n) \arrow[d, mapsto] \\
\sum a_i\gamma_i & C \arrow[r, "\cdot b^j"] & C & \sum a_i\gamma_i
\end{tikzcd}$$

commutes) it follows, that $b^n \cdot c = p_0 c + p_1 bc + \ldots + p_{n-1}b^{n-1}c$ (first for $c = \gamma_i$, then all $c \in C$). Taking $c = 1$ shows that b is indeed integral over A.

(ii) If C is as in A and contains b_1, b_2, then it contains $b_1 \pm b_2$ and $b_1 \cdot b_2$, showing that these are integral over A. ¡¡¡¡¡¡¡ HEAD

(iii) Let, more generally, B/A be integral and $c \in C$ integral over B. It satisfies an equation $c^d = \beta_0 + \beta_1 c + \ldots + \beta_{d-1}c^{d-1}$ with $\beta_i \in B$. By (i), there is an A-subalgebra $\mathfrak{B} \subseteq B$ which is finite over A and contains the β_i. Then c is integral over $\mathfrak{B}$, hence by (i) there is a $\mathfrak{B}$-subalgebra $\mathfrak{C} \subseteq C$ containing c and finite over $\mathfrak{B}$. Now $\mathfrak{C}/A$ is finite by Proposition 1.4.1(ii), hence c is integral over A by (i).

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Let, more generally, B/A be integral and $c \in C$ integral over B. It satisfies an equation $c^d = \sum_{i=0}^{d-1} \beta_i c^i$ with $\beta_i \in B$. By a), there is an A-subalgebra $\tilde{B} \subseteq B$ which is finite over A and contains the β_i. Then c is integral over $\tilde{B}$, hence by a) there is a $\tilde{B}$-subalgebra $\tilde{C} \subseteq C$ containing c and finite over $\tilde{B}$. Now $\tilde{C}/A$ is finite by Proposition 1.4.1b), hence c is integral over A by a).

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