Algebra II

Nicholas Schwab & Ferdinand Wagner

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This text consists of notes of the lecture Algebra II taught at the University of Bonn by Professor Jens Franke in the winter term (Wintersemester) 2017/18.

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Contents

Introduction

After a slight delay due to the Professor being confused by the large attendance to his lecture, Franke briefly recaps the contents of his lecture course Algebra I. Our notes to this lecture can be found here[1]. He mentions specifically

- Hilbert's Basissatz and Nullstellensatz,
- the Noether Normalization Theorem,
- the Zariski-topology on k^n,
- irreducible topological spaces and their correspondence to the prime ideals of $k[X_1, \ldots, X_n]$,
- Noetherian topological spaces and their unique decomposition into irreducible subsets,
- the dimension of topological spaces and codimension of their irreducible subsets,
- catenary topological spaces,
- the fact that k^n is catenary and $\dim(k^n) = n$,
- quasi-affine varieties,
- structure sheaves,
- the fact that quasi-affine varieties X are catenary and $\dim(X) = \deg\operatorname{tr}(K(X)/k)$, where $K(X)$ is the quotient field of $\mathcal{O}(X)$. By the way, there is a nice alternative characterization as a direct limit (or colimit)

$$K(X) = \varinjlim_{\substack{\emptyset \neq U \subseteq X \\ U \text{ open}}} \mathcal{O}(U) \; .$$

- going up and going down for integral ring extensions,

- localizations.

Exercises will be held on Wednesday from 16 to 18 and Friday from 12 to 14 in Room 0.008. It is necessary to have achieved at least half the points on the exercise sheets in order to attend the exams.

Introduction to Krull dimension and all that

Professor Franke recapitulated on some topics of his previous lecture, Algebra I (of which detailed lecture notes may be found in [1]). Note that although the numbering of theorems in the following might seem messy, it is *intentionally* messy at least.

Definition 1 ([1, Definition 2.1.2])**.** A topological space X is called **quasi-compact** if every open cover $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ admits a finite subcover.

X is **Noetherian** if it satisifies the following equivalent conditions:

(a) Every open subset is quasi-compact.

(b) There is no infinite properly descending chain of closed subsets.

(c) Every set of closed subsets of X has a $\subseteq$-minimal element.

Definition 2 ([1, Definition 2.1.3])**.** A topological space $X \neq \emptyset$ is **irreducible** if it satisifies the following equivalent conditions:

(a) If $X = X_1 \cup X_2$ where X_1 and X_2 are closed subsets, then $X = X_1$ or $X = X_2$. Also, $X \neq \emptyset$.

(b) Any two non-empty open subsets of X have non-empty intersection.

(c) Every non-empty open subset of X is dense.

Condition (a) implies, by induction, the following more general property: For any finite cover $X = \bigcup_{i=1}^n X_i$ by closed subsets, there is $1 \leq i \leq n$ such that $X = X_i$.

Proposition 1. (a) *Any subset of a Noetherian topological space is Noetherian with it's induced subspace topology* (cf. [1, Remark 2.2.1]).

(b) *If X is Noetherian, there is a unique (that is, up to permutation of the X_i) decomposition $X = \bigcup_{i=1}^n X_i$ into irreducible closed subsets $X_i \subseteq X$ such that $X_i \nsubseteq X_j$ for $i \neq j$* (cf. [1, Proposition 2.1.1]).

Definition 3 ([1, Definition 2.1.4]). Let X be a topological space, $Z \subseteq X$ irreducible. We put

$$\begin{aligned}
\operatorname{codim}(Z, X) &= \sup \left\{ \ell \;\middle|\; \begin{array}{c} \text{there is a strictly descending chain} \\ Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_\ell \subseteq X \text{ of irreducible } Z_i \subseteq X \end{array} \right\} \\
\dim(X) &= \sup \{\operatorname{codim}(Z, X) \mid Z \subseteq X \text{ irreducible}\}
\end{aligned}$$

Example 1 ([1, Section 1.7 and 2.1]). Let $k = \overline{k}$ be an algebraically closed field. For an ideal $I \subseteq R = k[X_1, \ldots, X_n]$ let

$$V(I) = \{x \in k^n \mid f(x) = 0 \ \forall f \in I\}$$

be the set of zeroes of I. By the Hilbert Nullstellensatz, $V(I) \neq \emptyset$ when $I \subsetneq R$. Moreover

$$\begin{aligned}
V(I) &= V\left(\sqrt{I}\right) \\
V(I \cdot J) &= V(I) \cup V(J) \\
V\Big(\sum_{\lambda \in \Lambda} I_\lambda\Big) &= \bigcap_{\lambda \in \Lambda} V(I_\lambda)\ .
\end{aligned}$$

It follows that there is a topology (called the *Zariski topology*) on k^n containing precisely the subsets of the form $V(I)$ as closed subsets. A version of the Nullstellensatz ([1, Proposition 1.7.1]) says

$$\{f \in R \mid f(x) = 0 \ \forall f \in I\} = \{f \in R \mid V(f) \supseteq V(I)\} = \sqrt{I}\ .$$

This means that there is strictly antimonotonic bijective correspondence between the ideals I of R with $I = \sqrt{I}$ and the Zariski-closed subsets $A \subseteq k^n$ via

$$\begin{aligned}
\left\{\text{ideals } I \subseteq R \text{ such that } I = \sqrt{I}\right\} &\xrightarrow{\sim} \{\text{Zariski-closed subsets } A \subseteq k^n\} \\
\{f \in R \mid V(f) \supseteq A\} &\longleftarrow A \\
I &\longmapsto V(I)\ .
\end{aligned}$$

(cf. [1, Remark 2.1.1]). As R is Noetherian, any strictly ascending chain of ideals in R terminates, implying that k^n is a Noetherian topological space. Under the above correspondence prime ideals correspond to irreducible subsets and vice versa (cf. [1, Proposition 2.1.2]).

Remark 1 ([1, Remark 2.1.3]). In general for $A \subseteq B \subseteq C \subseteq X$

$$\operatorname{codim}(A, B) + \operatorname{codim}(B, C) \leq \operatorname{codim}(A, C) \tag{1}$$

$$\operatorname{codim}(A, X) + \dim A \leq \dim X. \tag{2}$$

may be strict. A Noetherian topological space is called *catenary* if (1) is an equality whenever A, B and C are irreducible.

Theorem 5 ([1, Theorem 5])**.** *The space $X = k^n$ is catenary and in this case equality always occurs in (2).*

Example 2. For $n = 1$, the closed subsets of k are k itself and the finite subsets. Since k is infinite, the points and k are the irreducible subsets, implying $\dim(k) = 1$ and the other assertions for $n = 1$.

Example 3. The irreducible subsets of k^2 are k^2 itself, single points, and $V(f)$ where $f \in k[X, Y]$ is a prime element.

Definition 4 (transcendence degree)**.** Let $K \subseteq L$ be a field extension. A set $S \subseteq L$ is called *algebraically independent* over K if for all polynomials $P \in K[X_1, \ldots, X_n]$ and pairwise different $s_1, \ldots, s_n \in S$,

$$P(s_1, \ldots, s_n) = 0 \quad \text{implies} \quad P = 0 \; .$$

A *transcendence base* of L/K is a subset $S \subseteq L$ which is algebraically independent over K and such that $L/K(s_1, \ldots, s_n)$ is algebraic. The **transcendence degree** $\deg \operatorname{tr} L/K$ of L/K is the cardinality of any transcendence base.

Example. The empty set is a transcendence base of K/K.

Definition 5 (regular functions, [1, Definition 2.2.2])**.** Let $X \subseteq k^n$ be closed, $U \subseteq X$ open. A function $f \colon U \to k$ is called *regular* at $x \in U$ if x has a neighbourhood $\Omega \subseteq k^n$ for which there are polynomials $p, q \in k[X_1, \ldots, X_n]$ such that $V(q) \cap \Omega = \emptyset$ and

$$f(y) = \frac{p(y)}{q(y)} \quad \text{for all } y \in U \cap \Omega$$

The ring $\mathcal{O}(U)$ of **regular functions** on U consists of all functions $U \xrightarrow{f} k$ which are regular at every $x \in U$.

Proposition 2. *If $X \subseteq k^n$ is closed then $R = k[X_1, \ldots, X_n] \to \mathcal{O}(X)$ is surjective.*

In [1, Proposition 2.2.2], we actually proved a stronger result: If $X \subseteq k^n$ is irreducible closed, i.e. $X = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subseteq R$, then $\mathcal{O}(X) \simeq R/\mathfrak{p}$. Proposition 2 immediately follows from this, as any closed subset decomposes into irreducible closed subsets according to Proposition 1 (it is crucial that each X_i occuring in such a contains a non-empty open subset of X, cf. [1, Proposition 2.1.1]).

Remark 2. When $X \subseteq k^n$ is an irreducible open-closed subset (that is, an open subset of an irreducible closed subset – a.k.a. a *quasi-affine variety*, cf. [1, Definition 2.2.1]) then $\mathcal{O}(X)$ is a domain.

Remark 3. Let T be any topological space, $A \subseteq T$ such that every $t \in T$ has an open neighbourhood $U \subseteq T$ such that $A \cap U$ is closed in U, then A is closed in T (we suspect that this is mentioned only because Professor Franke ~~mistook this class for Algebraic Geometry I~~ recently used this in Algebraic Geometry I). If the condition is required only for $t \in A$, then A is called *locally closed*.

If X is irreducible, let $K(X)$ be the quotient field of $\mathcal{O}(X)$. This is called the *field of rational functions* on X.

Theorem 6 ([1, Theorem 6])**.** *If $X \subseteq k^n$ is locally closed and irreducible, then*

$$\dim(X) = \deg\operatorname{tr}(K(X)/k)\ .$$

Moreover, X is catenary and equality always holds in (2), i.e. $\dim Y + \operatorname{codim}(Y, X) = \dim X$ *whenever $Y \subseteq X$ is closed, irreducible.*

One may check that locally closed sets are precisely the open subsets of closed sets. In particular, X from the above theorem is a quasi-affine variety, as we used to call it in Algebra I.

Remark 4. It is easy to see that $\dim k^n \geq n$ since we have the chain

$$\{0\}^n \subsetneq k \times \{0\}^{n-1} \subsetneq \ldots \subsetneq k^{n-1} \times \{0\} \subsetneq k^n$$

of irreducible closed subsets. To prove $\dim(k^n) \leq n$, and $\dim(X) \leq \deg\operatorname{tr}(K(X)/k)$, one proves $\deg\operatorname{tr}(\mathfrak{K}(\mathfrak{p})/k) > \deg\operatorname{tr}(\mathfrak{K}(\mathfrak{q})/k)$ whenever A/k is an algebra of finite type over k, $\mathfrak{q} \supsetneq \mathfrak{p}$ are prime ideals and $\mathfrak{K}(\mathfrak{p})$ denotes the quotient field of $A/\mathfrak{p}$.

For general affine X one uses the Noether Normalization theorem to get a finite morphism $X \xrightarrow{(f_1,\ldots,f_d)} \mathbb{A}^d(k) = k^d$ (i.e., $\mathcal{O}(X)$ is integral over $k[f_1, \ldots, f_d]$) and $f_1, \ldots, f_d$ are k-algebraically independent). One then uses the going-up (going-down) for (certain) integral ring extensions to lift chains of irreducible subsets of $\mathbb{A}^d(k) = d^d$ to chains of irreducible subsets of X (all of this may be found in much more detail in [1, Section 2.4-2.6]).

Professor Franke at this point recommends the books *Algebraic Geometry* by R. Hartshorne, *The Red Book of Varieties and Schemes* by D. Mumford, *Commutative Ring Theory* by H. Matsumura and *Introduction to Commutative Algebra* by M. Atiyah and I. MacDonald. The oh-so-humble authors of these notes want to use this opportunity to recommend *Algebra I by Jens Franke (lecture notes)* by N. Schwab and F. Wagner [1] as well.

Localization of rings

Definition 6 (Multiplicative Subsets)**.** Let R be any ring (commutative, with 1). A subset $S \subseteq R$ is called a **multiplicative subset** of R if it is closed under finite products (in particular $\prod_{s\in\emptyset} s = 1 \in S$).

Definition 7 (Localization of a ring)**.** A **localization** R_S of R with respect to S is a ring R_S with a ring morphism $R \xrightarrow{\psi_S} R_S$ such that $\psi_S(S) \subseteq R_S^\times$ (the group of units of R_S) and such that ψ_S has the universal property (on the left) for such ring morphisms:

> If $R \xrightarrow{\alpha} A$ is any ring morphism such that $\alpha(S) \subseteq A^\times$ then there is a unique ring morphism $R_S \xrightarrow{\mu} A$ such that the diagram
>
> $$\begin{tikzcd} R \arrow[rr, "\alpha"] \arrow[dr, "\psi_S"'] & & A \\ & R_S \arrow[ur, dashed, "\exists!\ \mu"'] & \end{tikzcd}$$
>
> commutes.

It turns out (by a Yoneda-style argument) that this universal property characterizes R_S uniquely up to unique isomorphism. One constructs R_S (and thereby proves its existence) by $R_S = (R \times S)/_\sim$ where $(r, s) \sim (\rho, \sigma)$ iff there is $t \in S$ such that $t \cdot r \cdot \sigma = t \cdot \rho \cdot s$ (note that since R is not necessarily a domain the factor t on both sides cannot be omitted). One thinks of $(r, s)/_\sim$ as $\frac{r}{s}$ and introduces the ring operations in an obvious way.

If $I \subseteq R$ is any ideal then $I_S = I \cdot R_S = \left\{ \frac{i}{s} \;\middle|\; i \in I, s \in S \right\}$ is an ideal in R_S, and any ideal in R_S can be obtained in this way: $J = (J \sqcap R) \cdot R_S$ for any ideal $J \subseteq R_S$ where $J \sqcap R$ denotes the preimage of J in R under ψ_S. It follows then R_S is Noetherian when R is. For prime ideals one obtains a bijection (cf. [1, Corollary 2.3.1(e)])

$$\begin{aligned} \operatorname{Spec} R_S &\xrightarrow{\sim} \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \cap S = \emptyset\} \\ \mathfrak{p} &\longmapsto \mathfrak{p} \sqcap R \\ \mathfrak{q} \cdot R_S &\longleftarrow\!\shortmid \mathfrak{q}\ . \end{aligned}$$

We have an equivalence of categories between the category of R_S-modules and the category of R-modules M on which $M \xrightarrow{s\cdot} M$ acts bijectively for every $s \in S$. For every R-module M there is an R-module M_S belonging to the right hand side together with a morphism of R-modules $M \to M_S$, which has the universal property (on the left) for all morphisms from M to some R_S-module. It can be constructed as $\left\{ \frac{m}{s} \mid m \in Ma, s \in S \right\}/_\sim$ with $\frac{m}{s} \sim \frac{\mu}{\sigma}$ iff $m \cdot \sigma \cdot t = \mu \cdot s \cdot t$ for some $t \in S$. $M = I$ is an ideal in R, one can take $M_S = I_S = I \cdot R_S$. As for rings, we call M_S the *localization* of M (cf. [1, Proposition 2.3.2]).

"Advanced" Galois Theory: Trace and Norm

Let L/K be a finite field extension, $\overline{L}$ an algebraic closure of L.

Definition 8 (Characteristic Polynomial, Trace and Norm)**.** Let $x \in L$, $\mathrm{Min}_{x/K} = T^d + \sum_{i=0}^{d-1} m_i T^i$ the unique normed generator of $\{P \in K[T] \mid P(x) = 0\}$. Recall that $d = [K(x) : K]$ is called the *degree* and $\mathrm{Min}_{x/K}$ the *minimal polynomial* of x. The **characteristic polynomial** of x with respect to L is $P_{x,L} = \det(T \cdot \mathrm{id} - x) = T^D + \sum_{i=0}^{D-1} p_i T^i$. The Trace and Norm $\mathrm{Tr}_{L/K}(x)$ and $N_{L/K}(x)$ are $-p_{D-1}$ and $(-1)^D p_0$, respectively. In other words, there are the trace and norm of the endomorphism $L \xrightarrow{x\cdot} L$ of the K-vector space L.

Theorem 7. *(a) If V is any finite dimensional L-vector space and $f \in \mathrm{End}_L(V)$, then $\det_K(f) = N_{L/K}(\det_L(f))$ and $\mathrm{Tr}_K(f) = \mathrm{Tr}_{L/K}(\mathrm{Tr}_L(f))$ where $\mathrm{Tr}_M(f)$ and $\det_M(f)$ are trace and determinant of the endomorphism of the M-vector space V.*

(b) If M/L is a finite field extension and $m \in M$ the $\mathrm{Tr}_{M/K}(m) = \mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(m))$ and $N_{M/K}(m) = N_{L/K}(N_{M/L}(m))$.

Let $x \in L$ and let $x = x_1, \ldots, x_e$ be the pairwise different images of x under the K-linear embeddings $L \longrightarrow \overline{L}$. Also, let $d = \deg(x/K)$ and $D = [L : K]$ as before.

(c) If $e = 1$ there is a positive integer a which is a power of the characteristic of K such that $x^a \in K$. In particular $x \in K$ for $\mathrm{char}\, K = 0$.

(d) $\mathrm{Min}_{x/K} = \prod_{i=1}^{e}(T - x_i)^{d/e}$ and $P_{x,L/K} = \prod_{i=1}^{e}(T - x_i)^{D/e} = \prod_{\chi}(T - \chi(x))^{D/E}$ where χ runs over the different embeddings $L \longrightarrow \overline{L}$ and E is their number.

(e) We have $P_{x,L/K} = \mathrm{Min}_{x/K}^{[L/K(x)]}$. More general, if $K \subseteq F \subseteq L$ and $x \in F$ then $P_{x,L/K} = P_{x,L/E}^{[L:E]}$.

Proof. (e) Let $m_1, \ldots, m_k$ be a base of L/E and $e_1, \ldots, e_\ell$ a base of E/K and M be the matrix of $E \xrightarrow{x\cdot} E$ for thar base. Then it is known from basic Galois theory that $\{e_i \cdot m_j \mid 1 \leq i \leq \ell, 1 \leq j \leq k\}$ is a base of L/K. The representation of $L \xrightarrow{x\cdot} L$ in that base is a block matrix with k times the block M on the diagonal. This shows that $P_{x,L/K} = P_{x,E/K}^k$ as stated. If $E = K(x)$ then $P_{x,E/K} = \mathrm{Min}_{x/K}$ since x is zero of the left hand side by Cayley-Hamilton and $\deg P_{x,E/K} = [E : K] = [K(x) : K] = \deg(x/K) = \deg \mathrm{Min}_{x/K}$ and both polynomials are normed.

(a) If $C = (\ell_1, \ldots, \ell_k)$ is a base of L/K and $B = (v_1, \ldots, v_m)$ a base of V as an L-vector space and $\mathrm{Mat}_B(f) = \{f_{i,j}\}i, j = 1^m$ the matrix representing f to the base of B. Then $\widetilde{B} = \{\ell_i \cdot v_j \mid 1 \leq i \leq k, 1 \leq j \leq m\}$ is a base of V and

Matrix under construction, please stand by

Since the trace of a matrix is the sum of its diagonal, the assertion about traces follows. The assertion about determinants also follows in the case $f_{i,j} = 0$ when $i > j$ because $\det X = \prod_{i=1}^{m} \det(\mathrm{Mat}_C(f_{i,i})) = \prod_{i=1}^{m} N_{L/K}(f_{i,i}) = N_{L/K}(\prod_{i=1}^{m} f_{i,i}) = N_{l/K}(\det \mathrm{Mat}_B(f)) = N_{L/K} \det_L f$. Moreover, the assertion is trivial when $\mathrm{Mat}_B(f)$ is a permutation matrix, and that it holds forr $f \times g$ when it holds for both f and g. By Gaussian elimination, f can be turned into upper triangular form by permutation of rows and addtion of multiples of a row vector tor another row, corresponding to left multiplication by endomorphisms of a type dealt with before. Therefore (a) and (b) follow.

(c) Induction on $d = \deg(x/K)$. When $d = 1$, $e = 1$, all assertions are trivial. Let $d > 1$ and let the assertion be proven for elements of L of degree strictly less then d. If $d = e$ we have $\mathrm{Min}_{x/K} = \prod_{i=1}^{e}(T - x_i)$ by a comparison of degree and highest coefficient as all x_i must be zeroes of the left hand side. Assertion c is also trivial in this case. It is known from basic Galois theory that all zeroes of the $\mathrm{Min}_{x,K}$ are simple unless $\operatorname{char} K = p > 0$ and $\mathrm{Min}_{x/K} = Q(T^p)$ for some $Q \in K[T]$. In the first case it is known from basic Galois theory that every zero of $\mathrm{Min}_{x/K}$ is $\psi(x)$ where $\psi\colon K(x) \longrightarrow \overline{L}$ is some K-linear embedding and that all such ψ extend to some $L \longrightarrow \overline{L}$ (K-linear embedding of fields). Thus $e = d$ in the first case. Let thus be $p > 0$ and $\mathrm{Min}_{x/K} = Q(T^p)$. Thus $d = p \cdot \delta$. It follows that Q is the minimal polynomial of x^p. Moreover, $x_1^p, \ldots, x_e^p$ are the pairwise different images of x under the K-linear $L \longrightarrow \overline{L}$ (since $x_i^p - x_j^p = (x_i - x_j)^p \neq 0$ when $i \neq j$). By the induction assumption, $Q = \mathrm{Min}_{x^p/K} = \prod_{i=1}^{e}(T - x_i^p)^{d/(pe)}$ and $\mathrm{Min}_{x/K} = Q(T^p) = \prod_{i=1}^{e}(T^p - x^p)^{d/(pe)} = \prod_{i=1}^{e}(T - x_i)^{d/e}$. Also, when $e = 1$ the induction assumption implies $(x^p)^\alpha \in K$ where $\alpha \in p^{\mathbb{N}}$. Thus, $x^a \in K$ with $a = p \cdot \alpha \in p^{\mathbb{N}}$. Part (d), $P_{x,L/K} = \prod_{i=1}^{e}(T - x_i)^{D/e}$ follows from (e) and the assertion just shown.

(d) Let $\psi_1, \ldots, \psi_e$ be the different K-linear embeddings of $K(x) \longrightarrow \overline{L}$. It is easy to see, that each of the ψ_i has the same number n of extensions to a K-linear embedding $L \longrightarrow \overline{L}$. Then by the previous step the left hand side is $\prod_{i=1}^{e}(T - x_i)^{D/e} = \prod_{\chi}(T - \chi(x))^{D/(ne)} = \prod_{\chi}(T - \chi(x))^{D/E}$.

q.e.d.

1. Krull's Principal Ideal Theorm

1.1. Formulation

Theorem 11. *Let R be Noetherian, $f \in R$, $\mathfrak{p} \in \operatorname{Spec} R$ minimal among all prime ideals containing f. Then* $\operatorname{ht}(\mathfrak{p}) \leq 1$*. In other words,* $\mathfrak{p}$ *is a minimal prime ideal (if* $\operatorname{ht}(\mathfrak{p}) = 0$*) or all prime ideals strictly contained in* $\mathfrak{p}$ *are minimal.*

Remark. (*a*) The *height* of a prime ideal is defined as

$$\operatorname{ht}(\mathfrak{p}) = \sup \left\{ \ell \;\middle|\; \begin{matrix} \text{there is a strictly descending chain} \\ \mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \ldots \supsetneq \mathfrak{p}_\ell \text{ of prime ideals } \mathfrak{p}_i \in \operatorname{Spec} R \end{matrix} \right\} .$$

(*b*) Recall the *Zariski topology* on $\operatorname{Spec} R$: For any ideal $I \subseteq R$, let

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\} .$$

We have

$$\begin{aligned} V(I) &= V\left(\sqrt{I}\right) \\ V(I \cdot J) &= V(I) \cup V(J) \\ V\left(\sum_{\lambda \in \Lambda} I_\lambda\right) &= \bigcap_{\lambda \in \Lambda} V(I_\lambda). \end{aligned}$$

This implies (together with $V(0) = \operatorname{Spec} R$ and $V(R) = \emptyset$) that $\operatorname{Spec} R$ can be equipped with a topology in which the closed subsets are precisely the subsets of them for $V(I)$ where I is some ideal in R. This topology is Noetherian when R is, hence any closed subset can be decomposed into irreducible components. For $V(f) = V(f \cdot R)$, they are precisely those $V(\mathfrak{p})$ for which $\mathfrak{p}$ is minimal among all prime ideals containing f. Theorem 11 thus states that all irreducible components of $V(f)$ have codimension smaller or equal to 1 in $\operatorname{Spec} R$.

Corollary 1. *If $X \subseteq k^n$ is quasi-affine in k^n (with k algebraically closed) and $f \in \mathcal{O}(X) \setminus \{0\}$ then every irreducible component of $V(f)$ has codimension 1 in X.*

Remark. (*a*) It is possible, without losing generality, to assume that $X \subseteq k^n$ is Zariski-closed (thus affine, thus isomorphic to an affine variety in some k^n). This is so by (b) and (d) below. It is thus possible, while proving the corollary, to assume that X is isomorphic to some affine algebraic variety in some k^n.

(*b*) Let $U \subseteq X$ be open, then there is a bijective correspondence between the irreducible closed subsets of U and the irreducible closed subsets $A \subseteq X$ not disjoint from U:

$$\begin{aligned} A \cap U &\longleftarrow A \\ B &\longmapsto \overline{B} \end{aligned}$$

This shows that $\operatorname{codim}(A \cap U, U) = \operatorname{codim}(A, X)$ whenever $A \subseteq X$ is irreducible, closed and $U \subseteq X$ open and not disjoint from A.

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.