Algebraic Geometry I

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This text consists of notes on the lecture Algebraic Geometry I, taught at the University of Bonn by Professor Jens Franke in the winter term (Wintersemester) 2017/18.

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Contents

Introduction

The lecture will be an introduction to the language of *schemes*. The topics include but are not limited to the category of (pre-)schemes, properties of schemes, morphisms of schemes, sheaves of $\mathcal{O}_X$-modules and cohomology of quasi-coherent sheaves.

Professor Franke said the lecture requires a firm knowledge of commutative algebra and affine and projective varieties. If you are not familiar with this terms you may want to think again about visiting this lecture. If you want to brush up your knowledge about these topics the following literature is recommended:

- Matsumura, H.: *Commutative Ring Theory*,
- Hartshorne, R.: *Algebraic Geometry*,
- Mumford, D.: *The Red Book of Varieties and Schemes*,
- Schwab, N. & Wagner, F.: *Algebra I by Jens Franke* [1]. **Warning!** Somewhere in the middle of this text, the term *irreducible* is redefined as irreducible *and closed*. So don't let yourself get confused.

Let it be said that the first three recommendations are from Professor Franke while the last one is from the (not so) humble authors of these notes.

1. Varieties and Schemes

1.1. Introductory definitions

Definition 1 (Sheaf and presheaf)**.** A **presheaf $\mathcal{F}$ of rings** on a topological space X associates

- to any open subset $U \subseteq X$ a ring $\mathcal{F}(U)$ called the *ring of sections* of $\mathcal{F}$ on U
- and to any inclusion of open subsets $V \subseteq U$ a ring homomorphism

  $$(-)|_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

  such that $f|_V = f$ for all $f \in \mathcal{F}(V)$ and $(f|_V)|_W = f|_W$ for any inclusion $W \subseteq V \subseteq U$ of open subsets.

Note that while this notation (intentionally) reminds of the restriction of functions, behaves similarly and often the restriction is indeed used for this homomorphism, the elements of the rings $\mathcal{F}(U)$ are not always functions.

A so defined presheaf is furthermore a **sheaf** if additionally, the following condition, called *sheaf axiom*, holds:

For every open covering $U = \bigcup_{i \in I} U_i$ of an open subset $U \subseteq X$ the map

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \left\{ (f_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(U_i) \;\middle|\; f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for } i, j \in I \right\} \\ f &\longmapsto (f|_{U_i})_{i \in I} \end{aligned}$$

is bijective.

Remark 1. When $U = \emptyset$ one can take $I = \emptyset$ and obtains $\mathcal{F}(\emptyset) = \{0\}$.

Remark 2. Sheaves of groups, sets, etc. are defined in a similar way. A sheaf of rings $\mathcal{R}$ on X defines two sheaves of groups on X, namely $U \mapsto (\mathcal{R}(U), +)$ and $U \mapsto (\mathcal{R}(U)^\times, \cdot)$.

Remark 3. Elements of $\mathcal{R}(U)$ are called *sections*, elements of $\mathcal{R}(X)$ are called *global sections*.

Example 1. Let R be a ring. The sheaf $\mathcal{F}_X$ of R-valued functions on X associates to any open subset $U \subseteq X$ the ring of R-valued functions $f : U \to R$ with the inclusion morphism being the restriction of functions to subsets.

Remark. If $\mathcal{G}$ is any (pre)sheaf on X and $U \subseteq X$ an open subset, we get a a sheaf $\mathcal{G}|_U$ on U by $\mathcal{G}|_U(V) = \mathcal{G}(V)$ for the open subsets $V \subseteq U$ equipped with the same restriction morphisms.

Definition 2 (Algebraic prevarieties)**.** Let k be an algebraically closed field. An **algebraic prevariety** over k is a pair $(X, \mathcal{O}_X)$, where X is an irreducible Noetherian topological space together with a sheaf $\mathcal{O}_X$ of rings on X such that the following property is satisfied.

> Any $x \in X$ has an open neighbourhood U such that there is a homeomorphism $U \xrightarrow[\varphi]{\sim} V$ where $V \subseteq k^n$ is a Zariski-closed subset such that φ identifies $\mathcal{O}_X|_U$ with the structure sheaf $\mathcal{O}_V$ of V. That is, if $W \subseteq V$ is open then any k-valued function $f\colon W \to k$ is regular (i.e. an element of $\mathcal{O}_V(W)$) if and only if
>
> $$\begin{aligned} g\colon \varphi^{-1}(W) &\longrightarrow k \\ x &\longmapsto f(\varphi(x)) \end{aligned}$$
>
> is an element of $\mathcal{O}_X(\varphi^{-1}(W))$. One denotes $g = \varphi^* f$ in this case.

A **morphism of prevarieties** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $X \xrightarrow{\varphi} Y$ such that for all $U \subseteq Y$ and $\lambda \in \mathcal{O}_Y(U)$ we have $\varphi^*\lambda \in \mathcal{O}_X(\varphi^{-1}(U))$. As above, $\varphi^*\lambda$ is defined as $(\varphi^*\lambda)(x) = \lambda(\varphi(x))$. In particular, φ^* induces a *morphism of sheaves* $\varphi^*\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ (cf. Definition 1.2.2a and Definition 1.2.4).

Remark. (*a*) In fact, the $V \subseteq k^n$ in the above definition of varieties is even *irreducible*, as V is homeomorphic to an open (and hence irreducible) subset U of the irreducible space X. In particular, V is an *affine algebraic variety* (in the sense of [1, Definition 2.2.1]) and one can think of varieties as irreducible spaces which are locally isomorphic to (or glued together from) affine varieties.

(*b*) The n in the above definition is *not* required to be constant, not even for a single $x \in X$. In fact, this wouldn't be a sensible thing to ask for, as e.g. $k \subseteq k^1$ and $k \times \{0\} \subseteq k^2$ are isomorphic affine varieties. However, the *Krull dimension* $\dim X$ (in the sense of [1, Definition 2.1.4]) is a well-defined thing and one can show that $\dim X = \dim V$ in the above situation (this is a consequence of [1, Theorem 6] and the *locality of codimension*, cf. [1, Remark 2.1.3]).

Example 2. Let $V \subseteq k^n$ be Zariski-closed, $W \subseteq V$ open. The ring $\mathcal{O}_V(W)$ of *regular functions* on W is the ring of functions $\lambda\colon W \to k$ such that for any $x \in W$ there is an open neighbourhood Ω of x and polynomials $p, q \in R = k[X_1, \dots, X_n]$ such that q does not vanish on $\Omega \cap W$ and such that we have $\lambda(y) = \frac{p(y)}{q(y)}$ for every $y \in \Omega \cap W$. (cf. [1, Definition 2.2.2]).

The sheaf $\mathcal{O}_V$ defined by $W \mapsto \mathcal{O}_V(W)$ is called the *structure sheaf* on V. If $W = V$ it can be shown that any $f \in \mathcal{O}_V(V)$ can be written as $f = p|_V$ where $p \in R$ (cf. [1, Proposition 2.2.2]).

Example 3. The *projective space* $\mathbb{P}(V)$, where V is a k-vector space, is the set of one-dimensional subspaces of V. Let $\mathbb{P}^n(k) = \mathbb{P}(k^{n+1})$. If $(x_0, \dots, x_n) \in k^{n+1} \setminus \{0\}$, let $[x_0, \dots, x_n]$ denote the subspace generated by $(x_1, \dots, x_n)$.

Recall that an ideal $I \subseteq R = k[X_0, \dots, X_n]$ is called *homogenous* if it is generated by homogenous elements (i.e. polynomials in which every monomial has the same total degree). Let I be homogenous, let $V(I) \subseteq \mathbb{P}^n(k)$ be the set of all $[x_0, \dots, x_n] \in \mathbb{P}^n(k)$ such that $f(x_0, \dots, x_n)$ vanishes for all $f \in I$. Call a subset $A \subseteq \mathbb{P}^n(k)$ *Zariski-closed* if there is a homogenous ideal I such that $A = V(I)$. This turns $\mathbb{P}^n(k)$ into an irreducible, n-dimensional, Noetherian topological space.

Let $V \subset \mathbb{P}^n(k)$ be closed, $W \subseteq V$ open and $\lambda\colon W \to k$ any function. We call λ *regular* on W, or $\lambda \in \mathcal{O}_V(W)$, if any $x \in W$ has an open neighbourhood Ω such that there are two polynomials $p, q \in k[X_0, \dots, X_n]$ homogenous of the same degree such that $q(y_0, \dots, y_n) \neq 0$ and

$$\lambda([y_0, \dots, y_n]) = \frac{p(y_0, \dots, y_n)}{q(y_0, \dots, y_n)}$$

for all $[y_0, \dots, y_n] \in W \cap \Omega$.

The *affine space* $\mathbb{A}^n(k)$ is just good old k^n equipped with its Zariski topology. Consider the map

$$\begin{aligned} \mathbb{P}^n(k) \setminus V(X_i) &\xrightarrow{\sim} \mathbb{A}^n(k) \\ [x_0, \dots, x_n] &\longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) . \end{aligned}$$

This is a homeomorphism and identifies the structure sheaves with each other. Hence, for any irreducible closed subset $A \subseteq \mathbb{P}^n(k)$ such that $Y \coloneqq A \cap (\mathbb{P}^n(k) \setminus V(X_i)) \neq \emptyset$, $(Y, \mathcal{O}_A|_Y)$ is isomorphic to an affine algebraic variety. Thus, quasi-projective algebraic varieties (i.e. $(U, \mathcal{O}_U)$ where $U \subseteq \mathbb{P}^n(k)$ is a non-empty open subset of an irreducible closed subset) are algebraic prevarieties in the sense of Definition 2.

Example 3a. When X is prevariety in the sense of Definition 2 and $U \subseteq X$ is open and $U \neq \emptyset$, then $(U, \mathcal{O}_X|_U)$ is a prevariety. Note that any non-empty open subset of an irreducible set is necessarily irreducible as well, so irreducibility of U is not required as an extra condition.

Example 3b. Let X be a prevariety, $A \subseteq X$ irreducible and closed. Then $(A, \mathcal{O}_A)$ is a prevariety, wherein the structure sheaf $\mathcal{O}_A$ is defined as follows: If $W \subseteq A$ is open, let

$$\mathcal{O}_A(W) = \left\{ f\colon W \to k \;\middle|\; \begin{array}{c} \text{every } x \in W \text{ has an open neighbourhood } x \in U \subseteq X \\ \text{and } \varphi \in \mathcal{O}_X(U) \text{ such that } f(y) = \varphi(y)\ \forall y \in U \cap W \end{array} \right\} .$$

Then one may check that $(A, \mathcal{O}_A)$ is a prevariety. Note that $\mathcal{O}_A$ is *not* given by the restriction $\mathcal{O}_X|_A$ (which is only defined for open subsets $U \subseteq X$ anyway). If one prefers a more functorial definition of $\mathcal{O}_A$, the following will do: For each open subset $U \subseteq X$, let

$$I(U) = \{\varphi \in \mathcal{O}_X(U) \mid \varphi \text{ lies in the maximal ideal of } \mathcal{O}_{A,a}\ \forall a \in U \cap A\}$$

(the *stalk* $\mathcal{O}_{A,a}$ is defined in [1, Definition 2.2.5] or, for arbitrary presheaves, in Definition 1.2.1). Then

$$\mathcal{O}_A(W) = \varinjlim_U \mathcal{O}_X(U)/I(U) ,$$

where the colimit is taken over all open subsets $U \subseteq X$ such that $U \cap A = W$.

Remark. The following conditions on a topological space X are equivalent:

- (a) If $x \neq y$ are points of X, there are open neighbourhoods $V, W \subseteq X$ seperating them, i.e. $x \in V$, $y \in W$ and $V \cap W = \emptyset$. This is the usual definition for X being *Hausdorff*.
- (b) The diagonal $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$ is a closed subset.

(*c*) If a and b are continuous maps $T \to X$, then their *equalizer* $K = \{t \in T \mid a(t) = b(t)\}$ is closed in T.

Definition 3 (Variety)**.** Let X be a prevariety over k in the sense of Definitions 2. We call X **separated** or a **variety** over k if and only if $\{t \in T \mid a(t) = b(t)\}$ is closed in T whenever a and b are a pair of *morphisms of prevarieties* $T \underset{b}{\overset{a}{\rightrightarrows}} X$.

Example 4 (Line with two origins)**.** Let $n > 0$ and $X = (\mathbb{A}^n(k) \setminus \{0\}) \cup \{0_+\} \cup \{0_-\}$. Consider two morphisms $\iota_+, \iota_- \colon \mathbb{A}^n(k) \to X$ defined by

$$\iota_\pm(x) = \begin{cases} x & \text{if } x \neq 0 \\ 0_\pm & \text{if } x = 0 \end{cases}$$

Let $U \subseteq X$ be open iff both $\iota_+^{-1}(U)$ and $\iota_-^{-1}(U)$ are both open in $\mathbb{A}^n(k)$, and let

$$\mathcal{O}_X(U) = \left\{ f \colon U \to k \;\middle|\; \iota_\pm^* f = f\iota_\pm \in \mathcal{O}_{\mathbb{A}^n(k)}\left(\iota_\pm^{-1}(U)\right)\right\}.$$

Then $U_\pm = \iota_\pm^{-1}(\mathbb{A}^n(k))$ are both open and $\mathbb{A}^n(k) \xrightarrow{\iota_\pm} U_\pm$ is a homeomorphism identifying the respective structure sheaves. Thus, X is a prevariety, but not a variety in the sense of Definition 3 as we may take $T = \mathbb{A}^n(k) \xrightarrow{\iota_\pm} X$ in Definition 3 and obtain $K = \{x \in \mathbb{A}^n(k) \mid \iota_+(x) = \iota_-(x)\} = \mathbb{A}^n \setminus \{0\}$, which is not closed.

Example 5. (*a*) Any affine algebraic variety is a variety in the sense of Definition 3.

(*b*) In particular, $\mathbb{A}^0 = \{0\}$ is a variety, as is any one-point prevariety.

(*c*) Non-empty open and irreducible closed subsets Y of varieties X are varieties

Proof. For part (*c*) note that the inclusion $Y \xrightarrow{\iota} X$ (together with $\iota^* \colon \mathcal{O}_X \to \mathcal{O}_Y$) is a morphism of prevarieties, and for a pair $a, b \colon T \to A$ of morphisms of prevarieties we have $\{t \in T \mid a(t) = b(t)\} = \{t \in T \mid (\iota a)(t) = (\iota b)(t)\}$. The latter is closed in T because X is a variety.

For part (*a*) let $X \subseteq k^n$ be closed and irreducible, $a, b \colon T \to X$ be morphisms of prevarieties and $K = \{t \in T \mid a(t) = b(t)\}$. To show that K is closed in T, it is sufficient to show that any $t \in T$ has a neighbourhood Ω such that $K \cap \Omega$ is closed in Ω. Choosing Ω such that it is isomorphic to an affine algebraic variety, which is possible because T is a prevariety, we may assume without loss of generality that $T \subseteq k^n$ is an affine algebraic variety in (i.e. an irreducible subset of) k^n. Let

$$\begin{aligned} X_i \colon X &\longrightarrow k \\ (x_1, \dots, x_n) &\longmapsto x_i \end{aligned}$$

denote the projection to the i^{th} coordinate. Then $X_i \in \mathcal{O}_X(X)$, hence $\alpha_i = a^* X_i$ and $\beta_i = b^* X_i$ are in $\mathcal{O}_T(T)$ and

$$K = \{t \in T \mid \alpha_i(t) = \beta_i(t)\ \forall i\} = \bigcap_{i=1}^n V(\alpha_i - \beta_i)\,. \tag{1}$$

But we proved in Algebra I that $V(\varphi)$ is closed in T whenever T is an affine algebraic variety and $\varphi \in \mathcal{O}_T(T)$ (cf. [1, Proposition 2.2.1]).

Part (b) is trivial from (a). *q.e.d.*

Remark. (a) $K \subseteq T$ is closed iff for all $t \in T$ there is an open neighbourhood Ω_t such that $\Omega_t \cap K$ is closed in Ω_t, since

$$T \setminus K = \bigcup_{t \in T} (\Omega_t \setminus (K \cap \Omega_t))$$

is open as a union of open subsets.

(b) It is *not* sufficient to require this just for all $t \in K$.

Proposition 1. *Let X be any prevariety such that for arbitrary $x, y \in X$ there is a common open neighbourhood U of x and y which is affine (that is, isomorphic as a prevariety to an affine variety in some k^n). Then X is a variety.*

Proof. Let $a, b \colon T \to X$ as in Definition 3 and $t \in T$ and let $U \subseteq X$ be an affine open subset of X containing both $a(t)$ and $b(t)$. Let $V = a^{-1}(U) \cap b^{-1}(U) \subseteq T$. This is an open subset of T containing t. It is easily seen that $a|_V$ and $b|_V$ are morphisms $V \to U$. By the previous example, $K \cap V = \{t \in U \mid a(t) = b(t)\}$ is closed in V. Because such a neighbourhood can be found for any $t \in T$, K is closed in T by the previous remark. *q.e.d.*

Corollary 1. *Quasi-projective and quasi-affine algebraic varieties are varieties.*

Proof. Step 1. Let $X \subseteq k^n$ be irreducible and closed. Recall that for any $f \in \mathcal{O}_X(X) \setminus \{0\}$, $X \setminus V(f)$ is affine: Let $X = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \in R = k[X_1, \dots, X_n]$. We identify $f \in \mathcal{O}_X(X) = R/\mathfrak{p}$ with an arbitrary representative $f \in R$. Now consider the ideal $\mathfrak{q} \subseteq k[X_1, \dots, X_n, T]$ generated by $\mathfrak{p}$ and $1 - T \cdot f$. One can show that

$$\begin{aligned} V(\mathfrak{q}) &\xrightarrow{\sim} X \setminus V(f) \\ (x, t) &\longmapsto x \\ (x, f(x)^{-1}) &\longleftarrow\!\shortmid\ x \end{aligned}$$

is a homeomorphism topological spaces. Then $V(\mathfrak{q})$ is irreducible (as $X \setminus V(f)$ is), hence an affine variety, which proves that $X \setminus V(f)$ must be affine as well (cf. [1, Proposition 2.2.4]).

Let $U \subseteq X$ be open and $F \subseteq U$ be finite. Let $X \setminus U = V(I)$ with $I \subseteq R$ an ideal. If $n = 0$, $U = \{0\}$ is affine and we have nothing to prove. Let $n \geq 1$. Because k is infinite, the k-vector space I cannot be the union of its finitely many codimension one subspaces $I_x = \{p \in I \mid p(x) = 0\}$ for $x \in F$. Therefore, there is $p \in I$ such that $F \subseteq X \setminus V(p)$. By our initial remark, $X \setminus V(p)$ is affine. As $p \in I$, $X \setminus V(p) \subseteq U$ and the claim follows.

Step 2. Let $X \subseteq \mathbb{P}^n(k)$ be quasi-projective and let $F \subseteq X$ be finite. We can write

$$F = \left\{ [f_0^{(i)}, \dots, f_n^{(i)}] \;\middle|\; 1 \leq i \leq N \right\} .$$

As k^{n+1} is larger than the union of the N codimension one subspaces

$$V_i = \left\{ (\xi_j)_{j=0}^n \;\middle|\; \sum_{j=0}^n \xi_j f_j^{(i)} = 0 \right\} \quad \text{for } i = 1, \ldots, N \;,$$

there is a homogenous polynomial $p \neq 0$ of degree 1 such that $p(f_0^{(i)}, \ldots, f_n^{(i)}) \neq 0$ for all $1 \leq i \leq N$. Then $F \subseteq X \setminus V(p)$. But $\mathbb{P}^n(k) \setminus V(p)$ is isomorphic to k^n as this is the case when $p = X_0$ and $\mathrm{GL}_{n+1}(k)$ transitively acts on $k^{n+1} \setminus \{0\}$. Thus, $F \subseteq X \setminus V(p)$ and $X \setminus V(p)$ is isomorphic to a quasi-affine variety. The assertion now follows from Step 1. *q.e.d.*

Remark.
- Let X be a prevariety. If two arbitrary points have a common neighbourhood which is a variety, then X is a variety.
- We have actually seen that arbitrary finite subsets of quasi-projective algebraic varieties have open neighbourhoods which are affine. This is usefull, e.g., when forming quotients by finite groups.
- Hironaka (see Hartshorne for examples of a non-quasi-projective variety) has an example of a variety where there are two points without a common affine neighbourhood.

1.2. General properties and sheaf constructions

Definition 1 (Stalk)**.** Let $\mathcal{G}$ be a presheaf on the topological space X and $x \in X$. Let $\mathcal{G}_x = \varinjlim_U \mathcal{G}(U)$, denote the **stalk** of $\mathcal{G}$ at x, where $\varinjlim$ is taken over the open neighbourhoods U of x.

Remark. In general, $\varinjlim$ is the colimit in the target category. For the target categories of abelian groups, sets or rings we have

$$\mathcal{G}_x = \{(U, g) \mid x \in U \text{ open }, g \in \mathcal{G}(U)\} /_{\sim} \;,$$

where $(U, g) \sim (V, h)$ iff there is an open neighbourhood $W \subseteq U \cap V$ of x such that $g|_W = h|_W$.

Example 1. The stalk at x of the structure sheaf of a prevariety X is called the local ring $\mathcal{O}_{X,x}$ of X at x.

Remark. A set $\mathcal{B}$ of open subsets of a topological space X is called a *base* of the topology if every open subset of X may be written as a union of elements of $\mathcal{B}$. A set of subsets of some set X is the base of a (uniquely determined) topology on X iff the intersection of two arbitrary elements of $\mathcal{B}$ can be written as the union of elements of $\mathcal{B}$ and $\mathcal{B}$ covers X.

Definition 2 (Presheaves defined on a topological base)**.** Let $\mathcal{B}$ be a base of the topology on X. A **presheaf** on $\mathcal{B}$ associates

- a set (group, ring, ...) $\mathcal{G}(U)$ to any $U \in \mathcal{B}$
- and to each inclusion of elements of $\mathcal{B}$ a *restriction morphism* $(-)|_V \colon \mathcal{G}(U) \to \mathcal{G}(V)$ such that $\mathcal{G}(U) \xrightarrow{(-)|_U} \mathcal{G}(U)$ equals $\mathrm{id}_{\mathcal{G}(U)}$ and $f|_W = (f|_V)|_W$ for all $f \in \mathcal{G}(U)$ if $W \subseteq V \subseteq U$ is a inclusion of elements of $\mathcal{B}$.

If $\mathcal{G}$ is a presheaf on $\mathcal{B}$ and $x \in X$ we put $\mathcal{G}_x = \varinjlim_U \mathcal{G}(U)$ where the direct limit (*direct limit* means the same as *colimit*) is taken over alle $U \in \mathcal{B}$ containing x.

Remark. Equip $\mathcal{B}$ with partial order $\preceq$ where $V \preceq U$ iff $V \supseteq U$. Any partially ordered set $(\mathfrak{X}, \preceq)$ defines a category in which there is a morphism $X \to Y$ precisely if $X \preceq Y$. Then, a presheaf on $\mathcal{B}$ is a functor from (the category made out of) $\mathcal{B}$ and the target category.

Definition 2a. A morphism $\mathcal{G} \xrightarrow{f} \mathcal{H}$ of presheaves on $\mathcal{B}$ is a collection $\mathcal{G}(U) \xrightarrow{f_U} \mathcal{H}(U)$ in the target category such that

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, "(-)|_V"] \arrow[d, "f_U"'] & \mathcal{G}(V) \arrow[d, "f_V"] \\ \mathcal{H}(U) \arrow[r, "(-)|_V"] & \mathcal{H}(V) \end{tikzcd}$$

commutes. We will often write $\mathcal{G}(U) \xrightarrow{f} \mathcal{H}(U)$ instead of $\mathcal{G}(U) \xrightarrow{f_U} \mathcal{H}(U)$.

Fact 1. If $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ is a smaller base of the same topology there is an obvious restriction $\widetilde{\mathcal{G}} = \mathcal{G}|_{\widetilde{\mathcal{B}}}$. We have a canonical isomorphism

$$\widetilde{\mathcal{G}}_x \xrightarrow{\sim} \mathcal{G}_x$$

sending $(U, g)/_\sim \in \widetilde{\mathcal{G}}_x$ (where $U \in \widetilde{\mathcal{B}}$, $g \in \mathcal{G}(U)$, $x \in U$) to $(U, g)/_\sim \in \mathcal{G}_x$. It is easy to see that this is well-defined and an isomorphism.

Definition 3 (Sheafification)**.** Let $\mathcal{B}$ be a topology base on X, $\mathcal{G}$ a presheaf on $\mathcal{B}$. We define the **sheafification** (German: **Garbifizierung**) $\mathcal{G}^{\mathrm{sh}}$ (Professor Franke uses the notation $\mathrm{Sheaf}(\mathcal{G})$) of $\mathcal{G}$ as follows:

$$\mathcal{G}^{\mathrm{sh}}(U) = \left\{ (g_x) \in \prod_{x \in U} \mathcal{G}_x \;\middle|\; \begin{array}{c} \text{for all } y \in U \text{ there are an open neighbourhood } V \in \mathcal{B} \\ \text{and } \gamma \in \mathcal{G}(V) \text{ such that } g_x = (V, \gamma)/_\sim \ \forall x \in U \cap V \end{array} \right\} \qquad (*)$$

Moreover, we define the restriction morphisms via $(g_x)_{x \in U}|_V = (g_x)_{x \in V}$.

Remark. (*a*) The definition $(*)$ obviously does not change if the coherence condition on its right hand side is replaced by

> for every $y \in U$, there are an open neighbourhood V of y in X and $\gamma \in \mathcal{G}(U)$ such that $V \subseteq U$ and $V \in \mathcal{B}$ and such that $g_x = (V, \gamma)/_\sim$ for all $x \in V$.

(*b*) It is easy to see that $\mathcal{H} = \mathcal{G}^{\mathrm{sh}}$ satisfies the sheaf axiom: If $U = \bigcup_{i \in I} U_i$ is an open cover and $g^{(i)} = (g_x^{(i)})_{x \in U_i} \in \mathcal{H}(U_i)$ as on the right hand side of the sheaf axiom, define $g \in \mathcal{H}(U)$ by $g_x = g_x^{(i)}$ where i is chosen such that $x \in U_i$. This depends on x only. It is easy to see that indeed $g \in \mathcal{H}(U)$ and that g this is the only element of $\mathcal{H}(U)$ with $g|_{U_i} = g_i$ for all $i \in I$.

(c) It is easy to see that for every $x \in X$, $\mathcal{G} \to \mathcal{G}_x$ is a functor from the category of presheaves to the respective target category, provided that the colimits occuring in the definition of $\mathcal{G}_x$ exist in that category. Moreover $(-)^{\mathrm{sh}}$ is a functor from the category of presheaves with values in sets, groups, or rings to the respective category of sheaves.

We define a *canonical morphism* (cf. Definition A.2.2)

$$\begin{aligned} \Gamma_{\mathcal{G}} \colon \mathcal{G} &\longrightarrow \mathcal{G}^{\mathrm{sh}}|_{\mathcal{B}} \\ g \in \mathcal{G}(U) &\longmapsto (g_x)_{x \in U} \end{aligned}$$

of presheaves on $\mathcal{B}$.

Proposition 1. *We consider (pre-)sheaves of sets, (abelian) groups or rings.*

(a) *The morphism $\mathcal{G}_x \to \mathcal{G}_x^{\mathrm{sh}}$ induced by $\Gamma_{\mathcal{G}}$ is an isomorphism, for arbitrary $x \in X$.*

(b) *If $\mathcal{G}$ and $\mathcal{H}$ are presheaves on $\mathcal{B}$ and $\mathcal{G} \xrightarrow{\varphi} \mathcal{H}$ is a morphism between them such that the induced morphism $\mathcal{G}_x \xrightarrow{\varphi_x} \mathcal{H}_x$ is injective (respectively bijective) for arbitrary $x \in X$, then*

$$\mathcal{G}^{\mathrm{sh}}(U) \xrightarrow{\varphi_U^{\mathrm{sh}}} \mathcal{H}^{\mathrm{sh}}(U)$$

is injective (respectively bijective) for every open subset U of X.

(c) *If $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ is a topology base then the canonical morphism*

$$(\mathcal{G}|_{\widetilde{\mathcal{B}}})^{\mathrm{sh}} \longrightarrow \mathcal{G}^{\mathrm{sh}}$$

(defined using $(\mathcal{G}|_{\widetilde{\mathcal{B}}})_x \xrightarrow{\sim} \mathcal{G}_x$) is an isomorphism.

(d) *If $\mathcal{B}$ is closed under intersections and $\mathcal{G}$ satisfies the sheaf axiom for coverings of elements of $\mathcal{B}$ by elements of $\mathcal{B}$, then*

$$\Gamma_{\mathcal{G}} \colon \mathcal{G} \xrightarrow{\sim} \mathcal{G}^{\mathrm{sh}}|_{\mathcal{B}} \,.$$

Proof. (c) If you think about it for a while, this becomes rather obvious from the definitions.

For part (a), we prove surjectivity first: Let $g \in \mathcal{G}^{\mathrm{sh}}(U)$ and $x \in U$, we need to find an open neighbourhood $V \in \mathcal{B}$ of x and $\gamma \in \mathcal{G}(V)$ such that the images of $\Gamma_{\mathcal{G}}(\gamma) \in \mathcal{G}^{\mathrm{sh}}(V)$ and of g in $\mathcal{G}_x^{\mathrm{sh}}$ coincide. We chose V and γ as on the right hand of $(*)$. As $\mathcal{B}$ is a topology base there is $W \in \mathcal{B}$ sucht that $x \in W \subseteq U \cap V$. It follows from the definition of $\Gamma_{\mathcal{G}}$ and the coherence condition that $g|_W = \Gamma_{\mathcal{G}}(\gamma)|_W$. Consequently, their images in the stalk at x coincide, as asserted.

To show injectivity define

$$\begin{aligned} \iota_x \colon \mathcal{G}_x^{\mathrm{sh}} &\longrightarrow \mathcal{G}_x \\ \begin{pmatrix} \text{image of} \\ (g_y)_{y \in U} \in \mathcal{G}^{\mathrm{sh}}(U) \text{ in } \mathcal{G}_x^{\mathrm{sh}} \end{pmatrix} &\longmapsto g_x \,. \end{aligned}$$

It is easy to see that this is well-defined and that it is a left inverse to the map $\mathcal{G}_x \xrightarrow{(\Gamma_{\mathcal{G}})_x} \mathcal{G}_x^{\mathrm{sh}}$ studied. Therefore, $(\Gamma_{\mathcal{G}})_x$ is also injective.

Let's prove (b). The assertion about injectivity is trivial. Let us assume that $\mathcal{G}_x \xrightarrow{\varphi_x} \mathcal{H}_x$ is an isomorphism for arbitrary $x \in X$ and let $h = (h_x)_{x\in U} \in \mathcal{H}^{\mathrm{sh}}(U)$. We put $g = (g_x)_{x\in U} \in \prod_{x\in U} \mathcal{G}_x$ where $g_x = \varphi_x^{-1}(h_x)$. If $g \in \mathcal{G}^{\mathrm{sh}}(U)$ then it is obvious from the definitions that $\varphi_U^{\mathrm{sh}}(g) = h$, proving the bijectivity of $\mathcal{G}^{\mathrm{sh}}(U) \xrightarrow{\varphi_U^{\mathrm{sh}}} \mathcal{H}^{\mathrm{sh}}(U)$.

To verify the coherence condition for g, fix $x \in U$. As h satisfies the coherence condition there are an open neighbourhood $V \in \mathcal{B}$ of x and $\eta \in \mathcal{H}(V)$ such that h_y equals the image of η in $\mathcal{H}_y$ for all $y \in U \cap V$. As the image of η in $\mathcal{H}_x$ is in the image of $\mathcal{G}_x \xrightarrow{\varphi_x} \mathcal{H}_x$, there are a neighbourhood $W \in \mathcal{B}$ of x and $\gamma \in \mathcal{G}(W)$ such that φ_x maps the image $(W,\gamma)/_\sim$ of γ in $\mathcal{G}_x$ to the image $(V,\eta)/_\sim$ of η in $\mathcal{H}_x$. By definition of stalks and φ_x, this means that there is an open neighbourhood $\Omega \in \mathcal{B}$ of x such that $\Omega \subseteq V \cap W$ and $\varphi_W(\gamma)|_\Omega = \eta|_\Omega$. Replacing V by Ω and γ and η by their restrictions to V (which equals Ω now), we may assume that there is $\gamma \in \mathcal{G}(V)$ such that $\varphi(\gamma) = \eta$. For $y \in V$,

$$\varphi_y(g_y) = h_y = (V,\eta)/_\sim = \big(V,\varphi(\gamma)\big)/_\sim = \varphi_y\big((V,\gamma)/_\sim\big) \ .$$

Since φ_y is an isomorphism, this implies $g_y = (V,\gamma)/_\sim$. Thus $g \in \mathcal{G}^{\mathrm{sh}}(U)$.

And finally part (d). We first show injectivity of $\mathcal{G}(U) \xrightarrow{\Gamma_\mathcal{G}} \mathcal{G}^{\mathrm{sh}}(U)$, for $U \in \mathcal{B}$. Let $g, g' \in \mathcal{G}(U)$ have the same image in $\mathcal{G}^{\mathrm{sh}}(U)$, i.e. for every $x \in U$ their images g_x and g'_x in $\mathcal{G}_x$ coincide. By definition of $\mathcal{G}_x$, this means that for every $x \in U$ there is an open neighbourhood $V_x \in \mathcal{B}$ of x, such that $g|_{V_x} = g'|_{V_x}$. As the V_x cover U, this implies $g = g'$ by the sheaf axiom.

For surjectivity, let $g = (g_x)_{x\in U} \in \mathcal{G}^{\mathrm{sh}}(U)$. For every $x \in U$ there are, by the coherence condition on the right hand side of $(*)$, an open neighbourhood $V_x \in \mathcal{B}$ and $\gamma^{(x)} \in \mathcal{G}(V_x)$ such that g_y equals the image of $\gamma^{(x)}$ in $\mathcal{G}_y$ for all $y \in U \cap V_x$. Replacing V_x by $U \cap V_x$ and $\gamma^{(x)}$ by its restriction we may assume $V_x \subseteq U$. If $x, \xi \in U$ and $y \in V_x \cap V_\xi$, then the images of $\gamma^{(x)}$ and $\gamma^{(\xi)}$ in $\mathcal{G}_y$ coincide (they are g_y). Thus, the images of $\gamma^{(x)}|_{V_x\cap V_\xi}$ and $\gamma^{(\xi)}|_{V_x\cap V_\xi}$ under $\mathcal{G}(V_x \cap V_\xi) \xrightarrow{\Gamma_\mathcal{G}} \mathcal{G}^{\mathrm{sh}}(V_x \cap V_\xi)$ coincide. Consequently, $\gamma^{(x)}|_{V_x\cap V_\xi} = \gamma^{(\xi)}|_{V_x\cap V_\xi}$, as injectivity has already been shown. By the sheaf axiom for $\mathcal{G}$, there is $\gamma \in \mathcal{G}(U)$ such that $\gamma^{(x)} = \gamma|_{V_x}$ simultaneously for all $x \in U$ (note that in all of this we silently used that $\mathcal{B}$ is closed under intersections). Then

$$g_x = (V_x,\gamma^{(x)})/_\sim = (V_x,\gamma|_{V_x})/_\sim = (U,\gamma)/_\sim \quad \text{for all } x \in U \ .$$

Thus, $\Gamma_\mathcal{G}(\gamma) = g$. *q.e.d.*

Corollary 1. *The following assertions hold in the categories of sets, (abelian) groups or rings.*

(a) *If $\mathcal{F}$ is a sheaf of the given type, then any $f \in \mathcal{F}(U)$ is uniquely determined by the family of its images in $\mathcal{F}_x$ for $x \in U$, that is,*

$$\mathcal{F}(U) \longrightarrow \prod_{x\in U} \mathcal{F}_x \tag{1}$$

is injective. The image of $\mathcal{F}(U)$ under this map is the set of all $(f_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ such that for every $x \in U$ there are an open neighbourhood V of x and $\varphi \in \mathcal{F}(V)$ such that f_y equals the image φ_y of φ under $\mathcal{F}(U) \to \mathcal{F}_y$ for all $y \in U \cap V$.

(*b*) *A morphism of sheaves inducing isomorphisms (respectively injective maps) on stalks defines isomorphisms (respectively injective maps) on sections of open subsets.*

(*c*) *If two morphism* $\mathcal{F} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathcal{G}$ *define identical maps* $\mathcal{F}_x \underset{\beta_x}{\overset{\alpha_x}{\rightrightarrows}} \mathcal{G}_x$ *for each* $x \in X$*, then* $\alpha = \beta$*.*

(*d*) *If* $\mathcal{G} \xrightarrow{\alpha} \mathcal{H}$ *is a morphism of sheaves such that* $\mathcal{G}_x \xrightarrow{\alpha_x} \mathcal{H}_x$ *is injective (respectively surjective) for all* x*, then* α *is a monomorphism (respectively an epimorphism).*

(*e*) *Let* $\mathcal{R}$ *be a sheaf of rings on* x *and* $\rho \in \mathcal{R}(U)$*, then* $\rho \in \mathcal{R}(U)^\times$ *iff for each* $x \in U$ *the image* ρ_x *of* ρ *in* $\mathcal{R}_x$ *is in* $\mathcal{R}_x^\times$*.*

Proof. (*a*) is an immediate consequence of Proposition 1(*d*).

Part (*b*) follows from Proposition 1(*b*) as

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\alpha"] \arrow[d, "\Gamma_{\mathcal{F}}"', "\wr"] & \mathcal{G}(U) \arrow[d, "\Gamma_{\mathcal{G}}"', "\wr"] \\
\mathcal{F}^{\mathrm{sh}}(U) \arrow[r, "\alpha"] & \mathcal{G}^{\mathrm{sh}}(U)
\end{tikzcd}$$

commutes. Here the vertical arrows are isomorphisms by Proposition 1(*d*).

Part (*c*) immediately follows from (*a*).

Part (*d*) is quite obvious. Let α_x be injective for arbitrary $x \in X$, and let $\mathcal{T} \underset{\theta}{\overset{\tau}{\rightrightarrows}} \mathcal{F}$ be morphisms such that $\alpha\tau = \alpha\theta$. Then $\alpha_x\tau_x = \alpha_x\theta_x$ for all $x \in X$. As α_x is an injective map this implies $\tau_x = \theta_x$. By (*c*) we have $\tau = \theta$. Hence α is a monomorphism.

Let α_x be surjective for arbitrary $x \in X$, and let $\mathcal{F} \underset{\theta}{\overset{\tau}{\rightrightarrows}} \mathcal{T}$ be morphisms such that $\tau\alpha = \theta\alpha$. Then $\tau_x\alpha_x = \theta_x\alpha_x$ for all $x \in X$. As α_x is a surjective map this implies $\tau_x = \theta_x$. By (*c*) we have $\tau = \theta$. Hence α is a epimorphism.

Proving the *only if* part of (*e*) is quite trivial since $\mathcal{R}(U) \to \mathcal{R}_x$ is a ring homomorphism, thus maps units to units. Conversely, let ρ_x be a unit for every $x \in U$. We consider the morphism $\mathcal{R}|_U \xrightarrow{\rho\cdot} \mathcal{R}|_U$ (of sheaves of *sets*, multipliying by ρ is no ring homomorphism!). On stalks it defines the map $\mathcal{R}_x \xrightarrow{\rho_x\cdot} \mathcal{R}_x$, which is bijective. By (*b*) it is an isomorphism, hence there is $\sigma \in \mathcal{R}(U)$ such that $\rho\sigma = 1$ and $\rho \in \mathcal{R}(U)^\times$. *q.e.d.*

Fact 2. If $\mathcal{F}$ is a presheaf on $\mathcal{B}$ and $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ is another topology base, then $\left(\mathcal{F}|_{\widetilde{\mathcal{B}}}\right)^{\mathrm{sh}} \simeq \mathcal{F}^{\mathrm{sh}}$.

Proof. This is Proposition 1(*c*) and we proved it there. Ok, actually this doesn't count as a proof, so let's do it now.

By Fact 1, $\left(\mathcal{F}|_{\widetilde{\mathcal{B}}}\right)_x \xrightarrow{\sim} \mathcal{F}_x$ is an isomorphism for all $x \in X$. It is clear that

$$f = (f_x) \in \prod_{x \in U} \mathcal{F}_x \simeq \prod_{x \in U} (\mathcal{F}|_{\widetilde{\mathcal{B}}})_x$$

satisfying the coherence condition for $(\mathcal{F}|_{\widetilde{\mathcal{B}}})^{\mathrm{sh}}$ also satisfies it for $\mathcal{F}^{\mathrm{sh}}$. Conversely, let f satisfy the coherence condition for $\mathcal{F}^{\mathrm{sh}}$. For every $x \in U$, this means that there is $V \in \mathcal{B}$ and $\varphi \in \mathcal{F}(V)$

such that $\varphi_y = f_y$ for $y \in U \cap V$. As $\widetilde{\mathcal{B}}$ is a topology base there is $W \in \widetilde{\mathcal{B}}$ such that $x \in W \subseteq V$. Let $\widetilde{\varphi} = \varphi|_W$, then $\widetilde{\varphi}_y = f_y$ when $y \in U \cap W$. Hence $f \in (\mathcal{F}|_{\widetilde{\mathcal{B}}})^{\mathrm{sh}}(U)$. *q.e.d.*

Proposition 2. *The following holds for sheaves of sets, rings and (abelian) groups. Let $\mathcal{B}$ be a topology base and $\mathcal{F}$ a presheaf on $\mathcal{B}$. Any morphism $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}|_{\mathcal{B}}$, where $\mathcal{G}$ is a sheaf on X, has the form*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\alpha"] \arrow[dr, "\Gamma_{\mathcal{F}}"'] & & \mathcal{G}|_{\mathcal{B}} \\ & \mathcal{F}^{\mathrm{sh}}|_{\mathcal{B}} \arrow[ur, dashed, "\exists!\ \beta|_{\mathcal{B}}"'] & \end{tikzcd}$$

in which $\mathcal{F}^{\mathrm{sh}} \xrightarrow{\beta} \mathcal{G}$ is a uniquely determined morphism of sheaves.

Proof. If β is as above then α_x equals

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\alpha_x"] \arrow[dr, "(\Gamma_{\mathcal{F}})_x"'] & & (\mathcal{G}|_{\mathcal{B}})_x \arrow[r, "\sim"] & \mathcal{G}_x \\ & (\mathcal{F}^{\mathrm{sh}}|_{\mathcal{B}})_x \arrow[ur, "(\beta|_{\mathcal{B}})_x"'] \arrow[r, "\sim"] & \mathcal{F}^{\mathrm{sh}}_x \arrow[ur, "\beta_x"'] & \end{tikzcd}$$

in which also $(\Gamma_{\mathcal{F}})_x$ is an isomorphism by Proposition 1. It follows that β_x is uniquely determined by α_x, hence by α. By Corollary 1(c) this means that β is uniquely determined by α.

To show existence of β we construct it as

$$\mathcal{F}^{\mathrm{sh}} \xrightarrow{\alpha^{\mathrm{sh}}} (\mathcal{G}|_{\mathcal{B}})^{\mathrm{sh}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{sh}} \simeq \mathcal{G}$$

(the first isomorphism follows from Fact 2 and the latter from Proposition 1(d)). It is easily verified that this has the required properties. *q.e.d.*

Remark 1. By the usual Yoneda-type argument, it is clear that the above universal property characterizes $\mathcal{F}^{\mathrm{sh}}$ uniquely up to unique isomorphism. For other target categories the above universal property should be taken as the definition of $\mathcal{F}^{\mathrm{sh}}$. It may or may not exist, but is unique if it does.

Remark 2. Proposition 2 essentially says, that, for $\mathcal{F}$ a presheaf and $\mathcal{G}$ a sheaf of sets, (commutative) rings or (abelian) groups, we have a bijection

$$\mathrm{Hom}_{\mathrm{Shv}(X)}(\mathcal{F}^{\mathrm{sh}}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{PreShv}(\mathcal{B})}(\mathcal{F}, \mathcal{G}|_{\mathcal{B}})\ .$$

In other words, $L = (-)^{\mathrm{sh}}\colon\ \mathrm{PreShv}(\mathcal{B}) \to \mathrm{Shv}(X)$ and $R = (-)|_{\mathcal{B}}\colon\ \mathrm{Shv}(X) \to \mathrm{PreShv}(\mathcal{B})$ is an adjoint pair of functors.

Definition 4 (Direct image)**.** Let $X \xrightarrow{f} Y$ be a continuous map and $\mathcal{G}$ a presheaf on X. By $f_*\mathcal{G}$ we denote the presheaf on Y defined by

$$f_*\mathcal{G}(W) = \mathcal{G}(f^{-1}(W))$$

for open subsets $W \subseteq Y$, where the restriction $(-)|_V$ in $f_*\mathcal{G}$ is defined as $(-)|_{f^{-1}(V)}$ in $\mathcal{G}$. If $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is a morphism of presheaves on X, we define $f_*\alpha\colon f_*\mathcal{F} \to f_*\mathcal{G}$ by

$$f_*\mathcal{F}(W) = \mathcal{F}(f^{-1}(W)) \xrightarrow{\alpha} \mathcal{G}(f^{-1}(W)) = f_*\mathcal{G}(W)\ .$$

Fact 3. It is easy to see that the axioms of a presheaf hold for $f_*\mathcal{G}$ that $f_*\alpha$ is indeed a morphism of presheaves, that f_* is a functor and $g_*f_* = (gf)_*$. Also, id_* is the identity functor.

Remark. If $\mathcal{B}_Y$ is a topology base for Y and $\mathcal{B}_X \supseteq f^{-1}(\mathcal{B}_Y) = \{f^{-1}(U) \mid U \in \mathcal{B}\}$ (we possibly need to extend $f^{-1}(\mathcal{B}_Y)$ to obtain a topology base of X), then we have $\mathrm{PreShv}(\mathcal{B}_X) \xrightarrow{f_*} \mathrm{PreShv}(\mathcal{B}_X)$ defined in the same way and with similar properties.

Remark 3. Obviously we have a morphism

$$\begin{gathered} f_x^*\colon (f_*\mathcal{F})_{f(x)} \longrightarrow \mathcal{F}_x \\ \begin{pmatrix} \text{image of } (W,\varphi)\text{, where} \\ \varphi \in f_*\mathcal{F}(W)\text{, in } (f_*\mathcal{F})_{f(x)} \end{pmatrix} \longmapsto \begin{pmatrix} \text{image of} \\ (f^{-1}(W),\varphi) \text{ in } \mathcal{F}_x \end{pmatrix} \end{gathered} \tag{2}$$

defined by mapping the image of $(W, \varphi(f_*\mathcal{F})(W))$ in $(f_*\mathcal{F})(W)$ to the image of $(f^{-1}(W),\varphi)$ in $\mathcal{F}_x$. This is a natural transformation, since

$$\begin{tikzcd} (f_*\mathcal{F})_{f(x)} \arrow[r, "(f_*\alpha)_{f(x)}"] \arrow[d, "f_x^*"'] & (f_*\mathcal{G})_{f(x)} \arrow[d, "f_x^*"] \\ \mathcal{F}_x \arrow[r, "\alpha_x"] & \mathcal{G}_x \end{tikzcd}$$

commutes for any morphism $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ of presheaves. Also, the diagram

$$\begin{tikzcd} (g_*f_*\mathcal{F})_{g(f(x))} \arrow[r, equal] \arrow[d, "g_{f(x)}^*"'] & ((gf)_*\mathcal{F})_{(gf)(x)} \arrow[d, "(gf)_x^*"] \\ (f_*\mathcal{F})_{f(x)} \arrow[r, "f_x^*"] & \mathcal{F}_x \end{tikzcd}$$

commutes. And $(f_*\mathcal{F})_{f(x)} \xrightarrow{f^*} \mathcal{F}_x$ equals $\mathrm{id}_{\mathcal{F}_x}$ when $f = \mathrm{id}_X$.

Remark (about $f^\flat/f^\sharp$)**.** If $X \xrightarrow{f} Y$ is continuous and $\mathcal{G}$ a presheaf (of sets, (abelian) groups, (commutative) rings) on Y one defines

$$f^\flat\mathcal{G}(U) = \varinjlim_{W \supseteq f(U)} \mathcal{G}(W) = \left\{ (W,\gamma)/_\sim \;\middle|\; \begin{array}{c} W \subseteq Y \text{ open, } W \supseteq f(U),\ \gamma \in \mathcal{G}(W) \\ \text{and } (W,\gamma) \sim (V,\eta) \text{ iff there's an open set } \Omega \\ \text{such that } f(U) \subseteq \Omega \subseteq V \cap W \text{ and } \gamma|_\Omega = \eta|_\Omega \end{array} \right\}$$

Restriction to $V \subseteq U$ maps $(W,\gamma)/_\sim$ to $(W,\gamma)/_\approx$ where $\approx$ is the equivalence relation for the definition of $f^\flat\mathcal{G}(V)$.

Let $f^\sharp\mathcal{G} = (f^\flat\mathcal{G})^{\mathrm{sh}}$. If turns out that $\mathrm{PreShv}(Y) \xrightarrow{f^\flat} \mathrm{PreShv}(X)$ is left-adjoint to $\mathrm{PreShv}(X) \xrightarrow{f_*} \mathrm{PreShv}(Y)$ and $\mathrm{Shv}(Y) \xrightarrow{f^\sharp = f^\flat(-)^{\mathrm{sh}}} \mathrm{Shv}(X)$ is a left-adjoint functor to $\mathrm{Shv}(X) \xrightarrow{f_*} \mathrm{Shv}(Y)$.

Example 2. (*a*) If $f(U)$ is open in Y, then $(f^\flat\mathcal{G})(U) = \mathcal{G}(f(U))$.

(*b*) If $f(U)$ is open for any open subset U (i.e. if f is an open map) then (*a*) is a complete description of $f^\flat$.

(*c*) If $f\colon X \to \{*\}$ is the projection to the one-point space and the category $\mathrm{Shv}(\{*\})$ of sheaves on $\{*\}$ with values in the category $\mathcal{A}$ of sets, abelian groups, rings, etc, is identified with $\mathcal{A}$ and $A \in \mathrm{Ob}(\mathcal{A})$, then

$$(f^\flat(A))(U) = \begin{cases} A & \text{if } U \neq \emptyset \\ \{0\} & \text{if } U = \emptyset \end{cases}$$
$$(f^\sharp(A))(U) = \{f\colon U \to A \mid f \text{ is locally constant}\}\ .$$

(*d*) If $V \xrightarrow{f} Y$ is the embedding of an open subset then $f^\flat\mathcal{G}(U) = \mathcal{G}(U)$ for open $U \subseteq V$. In other words, $f^\flat\mathcal{G} = \mathcal{G}|_V$ and the sheaf property is preserved. Thus, $f^\sharp\mathcal{G} = \mathcal{G}|_V$ if $\mathcal{G}$ is a sheaf.

(*e*) If $\{x\} \xrightarrow{\iota} X$ is the inclusion of a point then

$$\iota^\flat\mathcal{G}(U) = \begin{cases} \mathcal{G}(\emptyset) & \text{if } U = \emptyset \\ \mathcal{G}_x & \text{if } U = \{x\} \end{cases}$$

which is a sheaf if $\mathcal{G}$ is. Thus, $\iota^\sharp\mathcal{G}$ can be identified, under the identification of $\mathrm{Shv}(\{x\})$ with $\mathcal{A}$ made in (*c*), with $\mathcal{G}_x$.

1.3. Locally ringed spaces

Definition 1 (Ringed space)**.** A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a *structure sheaf* which is a sheaf of rings. A **locally ringed space** is ringed space $(X, \mathcal{O}_X)$ such that for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring, called the *local ring of X at x*. Its maximal ideal is denoted $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$, or sometimes just $\mathfrak{m}_x$. Finally, let $\mathfrak{K}(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field.

Remark 1. Obviously, prevarieties (in the sense of Definition 1.1.2) are locally ringed spaces, $\mathcal{O}_{X,x}$ is the ring of germs of regular functions at x and $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$.

Definition 2 (Morphism of ringed spaces)**.** A **morphism of ringed spaces** $X \xrightarrow{f} Y$ is a pair (f_{Top}, f^*) where $X \xrightarrow{f_{\mathrm{Top}}} Y$ is a continuous map and $\mathcal{O}_Y \xrightarrow{f^*} (f_{\mathrm{Top}})_*\mathcal{O}_X$ is a morphism of sheaves of rings (the index $_{\mathrm{Top}}$ will be dropped very soon so that $(f_{\mathrm{Top}})_*$ becomes only f_*). We call f_{Top} the *topological* and f^* the *algebraic component* of f.

The identity on $(X, \mathcal{O}_X)$ is $\mathrm{id}_{(X,\mathcal{O}_X)} = (\mathrm{id}_X, \mathrm{id}_{\mathcal{O}_X})$. If $Y \xrightarrow{g} Z$ is another morphism of ringed spaces, the composition gf is given by $(gf)_{\mathrm{Top}} = g_{\mathrm{Top}}f_{\mathrm{Top}}$ and $(gf)^*$ is the composition

$$\mathcal{O}_Z \xrightarrow{g^*} g_*\mathcal{O}_Y \xrightarrow{g_*(f^*)} g_*f_*\mathcal{O}_X = (gf)_*\mathcal{O}_X$$

It is easy to see that this defines a *category of ringed spaces*.

Remark. Locally ringed spaces will be a subcategory of it, but not a *full* subcategory.

Remark. As $f^\sharp$ is left-adjoint to f_* (to be precise, we're actually considering $(f_{\mathrm{Top}})_*$ and $(f_{\mathrm{Top}})^\sharp$), giving $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is equivalent to giving a morphism $f^\sharp\mathcal{O}_Y \to \mathcal{O}_X$ which is also denoted by f^*. Using this type of f^* we would still arrive at the same category, up to unique isomorphism of the sets of morphisms. In particular, we have

$$\mathcal{O}_{Y,f_{\mathrm{Top}}(x)} \simeq \iota^\sharp_{\{f_{\mathrm{Top}}(x)\}}\mathcal{O}_Y = \iota^\sharp_{\{x\}}(f^\sharp\mathcal{O}_Y) \xrightarrow{\iota^\sharp_{\{x\}}f^*} \iota^\sharp_{\{x\}}\mathcal{O}_X \simeq \mathcal{O}_{X,x}\,.$$

This yields a morphism $\mathcal{O}_{Y,f_{\mathrm{Top}}(x)} \xrightarrow{f^*} \mathcal{O}_{X,x}$. Herein, $\iota_{\{x\}}$ and $\iota_{\{f_{\mathrm{Top}}(x)\}}$ denote the inclusions $\{x\} \hookrightarrow X$ respectively $\{f_{\mathrm{Top}}(x)\} \hookrightarrow Y$. The equality $\iota^\sharp_{\{f_{\mathrm{Top}}(x)\}}\mathcal{O}_Y = \iota^\sharp_{\{x\}}(f^\sharp\mathcal{O}_Y)$ follows from the fact that $\mathcal{O}_Y$ and $f^\sharp\mathcal{O}_Y$ have the same stalk at $f_{\mathrm{Top}}(x)$, namely $\mathcal{O}_{Y,f_{\mathrm{Top}}(x)}$ (quite easy to check). The arrow $\iota^\sharp_{\{x\}}f^*$ should be read as the functor $\iota^\sharp_{\{x\}}\colon \mathrm{Shv}(X) \to \mathrm{Shv}(\{x\})$ applied to the morphism $f^*\colon f^\sharp\mathcal{O}_Y \to \mathcal{O}_X$.

This may be described directly in terms of structures occurring in Definition 1:

Definition 2a. For $f\colon X \to Y$ a morphism of ringed spaces, we define $\mathcal{O}_{Y,f_{\mathrm{Top}}(x)} \xrightarrow{f^*} \mathcal{O}_{X,x}$ as the unique morphism such that

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^*"] \arrow[d] & f_*\mathcal{O}_X(U)\ (= \mathcal{O}_X(f^{-1}U)) \arrow[d] \\
\mathcal{O}_{Y,f_{\mathrm{Top}}(x)} \arrow[r, "f^*"] & \mathcal{O}_{X,x}
\end{tikzcd}$$

commutes. We can characterize $\mathcal{O}_{Y,f_{\mathrm{Top}}(x)} \xrightarrow{f^*} \mathcal{O}_{X,x}$ explicitly as the composition

$$\mathcal{O}_{Y,f_{\mathrm{Top}}(x)} \xrightarrow{(f^*)_{f_{\mathrm{Top}}(x)}} (f_*\mathcal{O}_X)_{f_{\mathrm{Top}}(x)} \xrightarrow{(f_{\mathrm{Top}})^*_x} \mathcal{O}_{X,x}\,.$$

As you might have noticed, the symbol $*$ is *massively* overloaded and deserves a careful explanation: The left arrow in the above composition means the stalk at $f_{\mathrm{Top}}(x)$ of the *algebraic component* f^* of f. The right arrow is the kind of morphism of stalks we defined in Remark 1.2.3, (1.2.2).

The symbol $*$ in the index, however, always means the *direct image* from Definition 1.2.4.

Definition 3 (Vanishing set)**.** Let X be a locally ringed space, $f \in \mathcal{O}_X(U)$, then we define its **vanishing set** by

$$V(f) = \{x \in U \mid \text{image of } f \text{ under } \mathcal{O}_X(U) \to \mathcal{O}_{X,x} \text{ is in } \mathfrak{m}_x\}\,.$$

Remark. (*a*) In view of what was recalled for prevarieties, this extends our previous definition of $V(f)$ in that case.

(*b*) $V(f) \cap V(g) \subseteq V(f+g)$ (as $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is an ideal) and $V(fg) = V(f) \cup V(g)$ (as $\mathfrak{m}_x$ is a prime ideal) and $V(f|_W) = V(f) \cap W$ if $W \subseteq U$ is open (pretty obvious).

Fact 1. For $f \in \mathcal{O}_X(U)$, $V(f) \subseteq U$ is closed. When f is nilpotent, $V(f) = U$.

Proof. Let $x \in U \setminus V(f)$. The image of φ of f in $\mathcal{O}_{X,x}$ has an inverse ψ which is the image of $g \in \mathcal{O}_X(W)$ where W is an open neighbourhood of x. Replacing W by $W \cap U$ and g by $g|_{W\cap U}$ we may assume $W \subseteq U$, and shrinking W further (using the definition of stalk) we may assume $f|_W \cdot g = 1$ in $\mathcal{O}_X(W)$. For $y \in W$ the image of f in $\mathcal{O}_{X,y}$ has an inverse given by g, hence it is not in $\mathfrak{m}_y$. Thus, $W \cap V(f) = \emptyset$. Since such W may be found for any $x \in U \setminus V(f)$, $U \setminus V(f)$ is open.

If f is nilpotent, say, $f^n = 0$, then $U = V(0) = V(f^n) = \bigcup_{i=1}^n V(f) = V(f)$ by (b) of the previous remark. *q.e.d.*

Definition 4. (a) Let $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ be local rings with their respective maximal ideals. A ring homomorphism $R \xrightarrow{\varphi} S$ is **local** iff $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.

(b) A **morphism of locally ringed spaces** is a morphism $X \xrightarrow{f} Y$ of ringed spaces such that X and Y are locally ringed spaces and such that the following equivalent conditions hold

(α) For any open subset $U \subseteq Y$ and $\lambda \in \mathcal{O}_Y(U)$ we have $V(f^*\lambda) = f^{-1}(V(\lambda))$.

(β) For arbitrary $x \in X$, the ring morphism $\mathcal{O}_{Y,f(x)} \xrightarrow{f^*} \mathcal{O}_{X,x}$ is local in the sense of (a).

Remark. (a) A composition of local ring morphisms is local.

(b) Note that for a ring morphism $R \xrightarrow{\varphi} S$ where R and S are local, we automatically have that $\varphi^{-1}(\mathfrak{m}_S) \subseteq \mathfrak{m}_R$ as φ maps $R^\times = R \setminus \mathfrak{m}_R$ to $S^\times = S \setminus \mathfrak{m}_S$. Thus, to verify that φ is local it is only necessary to show $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

(c) It is obvious that a composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. Also, id_X is a morphism of locally ringed spaces. Thus, locally ringed spaces form a subcategory (but not a full subcategory) of the category of ringed spaces.

Proof of Definition 4. We first prove $(\beta) \to (\alpha)$. Let (β) hold for given $X \xrightarrow{f} Y$ of ringed spaces, with X and Y locally ringed, then in the situation of (α) we have

$$\begin{aligned} V(f^*\lambda) &= \left\{x \in f^{-1}(U) \;\middle|\; (\text{image of } f^*\lambda \text{ in } \mathcal{O}_{X,x}) \in \mathfrak{m}_{X,x}\right\} \\ &\overset{\text{Def. 2a}}{=} \left\{x \in f^{-1}(U) \;\middle|\; f^*(\text{image of } \lambda \text{ in } \mathcal{O}_{X,x}) \in \mathfrak{m}_{X,x}\right\} \\ &\overset{(\beta)}{=} \left\{x \in f^{-1}(U) \;\middle|\; (\text{image of } \lambda \text{ in } \mathcal{O}_{Y,f(x)}) \in \mathfrak{m}_{Y,f(x)}\right\} \\ &= \left\{x \in f^{-1}(U) \;\middle|\; f(x) \in V(\lambda)\right\} \\ &= f^{-1}(V(\lambda))\ . \end{aligned}$$

To prove $(\alpha) \to (\beta)$, let $x \in X$, $y = f(x)$, $\ell \in \mathfrak{m}_{Y,y}$. We need to show $f^*\ell \in \mathfrak{m}_{X,x}$. There are an open neighbourhood U of y and $\lambda \in \mathcal{O}_Y(U)$ such that ℓ is the image of λ in $\mathcal{O}_{Y,y}$. Then $y \in V(\lambda)$, hence $x \in f^{-1}(V(\lambda)) = V(f^*\lambda)$. Thus $f^*\ell$ is in $\mathfrak{m}_{X,x}$ as it is the image of $f^*\lambda$ in $\mathcal{O}_{X,x}$. *q.e.d.*

Fact 2. (*a*) We have $f^{-1}(V(\lambda)) = V(f^*\lambda)$ when f is a morphism of locally ringed spaces.

(*b*) Let $f\colon X \to Y$ be a morphism in LRS (the category of locally ringed spaces), $x \in X$, $y = f(x)$. Since $f^*(\mathfrak{m}_{Y,y}) \subseteq \mathfrak{m}_{X,x}$, we have a ring morphism of the residue fields

$$\mathfrak{K}(y) \xrightarrow{f^*} \mathfrak{K}(x)$$

induced by $\mathcal{O}_{Y,y} \xrightarrow{f^*} \mathcal{O}_{X,x}$. If $Y \xrightarrow{g} Z$ is another morphism in LRS and $z = g(y)$, then

$$\begin{tikzcd} \mathfrak{K}(z) \arrow[rr, "g^*"] \arrow[dr, "(gf)^*"'] & & \mathfrak{K}(y) \arrow[dl, "f^*"] \\ & \mathfrak{K}(x) & \end{tikzcd}$$

commutes.

(*c*) Let R, S, T be local rings with maximal ideals $\mathfrak{m}_R, \mathfrak{m}_S, \mathfrak{m}_T$ and $R \xrightarrow{\rho} S \xrightarrow{\sigma} T$ (not necessarily local) morphism of rings.

(i) If ρ and σ are local, then so is $\sigma\rho$.

(ii) If $\sigma\rho$ is local, then so is ρ. Note that in the lecture we made the additional but *unnecessary* assumption that σ must be local as well.

(iii) If $\sigma\rho$ is local and ρ is surjective, then σ is local.

(*d*) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in the category of ringed spaces, where X, Y, Z are locally ringed.

(i) If f and g are morphisms of locally ringed spaces, then so is gf.

(ii) If gf is a morphism of locally ringed spaces and $X \xrightarrow{f_{\mathrm{Top}}} Y$ is surjective, then g is a morphism of locally ringed spaces. As in (*c*), in the lecture we made the unnecessary assumption that f is a morphism of locally ringed spaces as well.

(iii) If gf is a morphism of locally ringed spaces and $g^*\colon \mathcal{O}_{Z,z} \to \mathcal{O}_{Y,y}$ is surjective for all $y \in Y$ and $z = g(y)$, then f is a morphism of locally ringed spaces.

Proof. (*a*) is just (α) from Definition 4(*b*). We got it, ok?

Part (*b*) is immediately follows from the fact that $\mathcal{O}_{Z,z} \xrightarrow{g^*} \mathcal{O}_{Y,y} \xrightarrow{f^*} \mathcal{O}_{X,x}$ equals $\mathcal{O}_{Z,z} \xrightarrow{(gf)^*} \mathcal{O}_{X,x}$ by Remark 1.2.3.

Now about part (*c*). Point (i) is trivial. For (ii), note that $\sigma^{-1}(\mathfrak{m}_T) \subseteq \mathfrak{m}_S$ by this remark, hence

$$\rho^{-1}(\mathfrak{m}_S) \supseteq \rho^{-1}(\sigma^{-1}(\mathfrak{m}_T)) = (\sigma\rho)^{-1}(\mathfrak{m}_T) = \mathfrak{m}_R$$

since $\sigma\rho$ is local. On the other hand, $\rho^{-1}(\mathfrak{m}_S) \subseteq \mathfrak{m}_R$ (again by this remark), proving $\rho^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, that is, ρ is local.

For point (iii), we may assume that $S = R/I$ for some proper ideal $I \subseteq R$. Then $I \subseteq \mathfrak{m}_R$, hence $\mathfrak{m}_R \to \mathfrak{m}_S = \mathfrak{m}_R/I$ is surjective. We thus obtain

$$\sigma(\mathfrak{m}_S) = \sigma\big(\rho(\mathfrak{m}_R)\big) \subseteq \mathfrak{m}_T \ ,$$

the inclusion due to the fact that $\sigma\rho$ is local. This shows that σ is local as well.

Part (e) now reduces to (d) by considering $\mathcal{O}_{Z,z} \xrightarrow{g^*} \mathcal{O}_{Y,y} \xrightarrow{f^*} \mathcal{O}_{X,x}$, where $y = f(x)$, $z = g(y)$. Note that for (ii) we do need the extra condition that $X \xrightarrow{f_{\mathrm{Top}}} Y$ is surjective, otherwise we wouldn't know anything about the $y \notin f_{\mathrm{Top}}(X)$. *q.e.d.*

Fact 3. If $R \xrightarrow{\varphi} S$ is a surjective ring homomorphism where $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ are local rings, then φ is local.

Proof. Since φ is surjective, it induces an isomorphism $\overline{\varphi}\colon R/\ker\varphi \xrightarrow{\sim} S$. As local rings are forbidden to be the zero ring, $\ker\varphi$ must be a proper ideal of R and hence contained in $\mathfrak{m}_R$. Then φ maps $\mathfrak{m}_R$ to $\mathfrak{m}_R/\ker\varphi = \mathfrak{m}_S$. *q.e.d.*

Example 1. Let $U \subseteq X$ be an open subset, let $\iota\colon U \to X$ its embedding, where $(X, \mathcal{O}_X)$ is a locally ringed space, and let $\mathcal{O}_U = \mathcal{O}_X|_U$. Let $\iota^*\colon \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) = \mathcal{O}_X(\iota^{-1}(V))$ be given by $(-)|_{U\cap V}\colon \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$. It follows that $\mathcal{O}_{X,x} \xrightarrow{\iota^*} \mathcal{O}_{U,x}$, for $x \in U$, is an isomorphism (the canonical isomorphism $(\mathcal{O}_X|_U)_x \xrightarrow{\sim} \mathcal{O}_{X,x}$), hence local. Hence ι is a morphism in LRS, and

$$\begin{aligned} \mathrm{Hom}_{\mathrm{LRS}}(T, U) &\xrightarrow{\sim} \{g \in \mathrm{Hom}_{\mathrm{LRS}}(T, X) \mid g(T) \subseteq U\} \\ f &\longmapsto g = \iota f \end{aligned} \tag{Fact 5}$$

Fact 4. If $X \xrightarrow{\iota} Y$ is a morphism in LRS such that ι_{Top} is an injective map and such that for any $x \in X$, $\iota_x^*\colon \mathcal{O}_{Y,\iota(x)} \to \mathcal{O}_{X,x}$ is surjective, then ι is a monomorphism in LRS.

Proof. It suffices to note that ι_{Top} is a monomorphism of topological spaces and ι^* an epimorphism of sheaves on Y (since it stalk-wise an epimorphism). *q.e.d.*

Proposition 1. *Let* $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ *be morphisms in* LRS*. Then an equalizer of these morphisms is given as follows*

- *Let*

$$K = \left\{ x \in X \;\middle|\; \begin{array}{c} a(x) = b(x) \textit{ and } \mathfrak{K}(y) \overset{a^*}{\underset{b^*}{\rightrightarrows}} \mathfrak{K}(x) \textit{ (the morphisms induced} \\ \textit{on the residue fields) coincide, where } y := a(x) = b(x) \end{array} \right\},$$

  equipped with the subspace topology induced from X*.*

- *Let* $\mathcal{O}_{K,[k]} = \mathrm{Coeq}\left(\mathcal{O}_{Y,y} \overset{a^*}{\underset{b^*}{\rightrightarrows}} \mathcal{O}_{X,k}\right)$ *where* $y = a(k) = b(k)$*. The square brackets are to indicate that we do not yet know if the pretender stalk* $\mathcal{O}_{K,[k]}$ *is really the stalk at* k *of the yet-to-define structure sheaf* $\mathcal{O}_K$ *(spoiler: it is).*

- *Let for open* $U \subseteq K$

$$\mathcal{O}_K(U) = \left\{ (f_k) \in \prod_{k\in U} \mathcal{O}_{K,[k]} \;\middle|\; \begin{array}{c} \textit{for every } \ell \in U, \textit{ there are an open neighbourhood} \\ \ell \in V \subseteq X \textit{ and } \varphi \in \mathcal{O}_X(V) \textit{ such that for every} \\ k \in U \cap V, f_k \textit{ is equal to the image of } \varphi \textit{ under} \\ \mathcal{O}_X(V) \to \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{array} \right\}.$$

The ring operations $* \in \{+, \cdot\}$ on $\mathcal{O}_K(U)$ are defined component-wise via

$$(f_x)_{x \in U} * (g_x)_{x \in U} = (f_x * g_x)_{x \in U} \ .$$

- *The morphism $K \xrightarrow{\kappa} X$ is given by the inclusion $K \subseteq X$ and $\kappa^* \colon \mathcal{O}_X \to \kappa_* \mathcal{O}_K$ is defined as follows: For $\varphi \in \mathcal{O}_X(U)$, define*

$$\kappa^*(\varphi) = \left(\begin{array}{c} \textit{image of } \varphi \textit{ under the composition} \\ \mathcal{O}_X(U) \to \prod_{x \in U \cap K} \mathcal{O}_{X,x} \to \prod_{x \in (U \cap K)} \mathcal{O}_{K,[x]} \end{array} \right) \in \mathcal{O}_K(K \cap U) = \kappa_* \mathcal{O}_K(U) \ .$$

Moreover, we have an isomorphism

$$\begin{aligned} \mathcal{O}_{K,k} &\xrightarrow{\sim} \mathcal{O}_{K,[k]} \\ (U, (f_\ell)_{\ell \in U})/\sim &\longmapsto f_k \end{aligned}$$

Proof. The following steps are rather trivial:

- K is a topological space, κ is continuous, $\mathcal{O}_{K,[k]}$ is a local ring (by Fact A.3.1 about coequalizers of local ring morphisms, also see exercise sheet #5), and $\mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]}$ is local (it is, in fact, a quotient projection, i.e. surjective).
- $\mathcal{O}_K$ is a sheaf of rings, $\mathcal{O}_X \xrightarrow{\kappa^*} \kappa_* \mathcal{O}_K$ is a morphism of sheaves of rings.

Step 1. We start proving that $\mathcal{O}_{K,k} \xrightarrow{\sim} \mathcal{O}_{K,[k]}$ is indeed an isomorphism.

To show injectivity of $\mathcal{O}_{K,k} \to \mathcal{O}_{K,[k]}$, let $U \subseteq K$ be an open neighbourhood of k and let $f = (f_\ell)_{\ell \in U} \in \mathcal{O}_K(U) \subseteq \prod_{\ell \in U} \mathcal{O}_{K,[\ell]}$ be such that f_k vanishes. We have to show that the image if f in $\mathcal{O}_{K,k}$ vanishes. By the coherence condition for sections of $\mathcal{O}_K$, there are an open neighbourhood V of k in X and $\varphi \in \mathcal{O}_X(V)$ such that

$$f_\ell = \left(\begin{array}{c} \text{image of } \varphi \text{ under} \\ \mathcal{O}_X(V) \to \mathcal{O}_{X,\ell} \to \mathcal{O}_{K,[\ell]} \end{array} \right) \quad \text{for } \ell \in U \cap V \ .$$

Shrinking U if necessary, we may assume $V \cap K \supseteq U$. Let φ_k be the image of φ in $\mathcal{O}_{X,k}$ and $y = a(k) = b(k)$. By Fact A.3.1 we have $\mathcal{O}_{K,[k]} = \mathcal{O}_{X,k}/I$, where I is the ideal $I = (a^*(v) - b^*(v) \mid v \in \mathcal{O}_{Y,y}) \subseteq \mathcal{O}_{X,k}$. As we assume that f_k vanishes,

$$0 = f_k = (\varphi_k \mod I) \in \mathcal{O}_{K,[k]} = \mathcal{O}_{X,k}/I \ ,$$

hence we have $\rho_1, \dots, \rho_N \in \mathcal{O}_{X,k}$ and $v_1, \dots, v_N \in \mathcal{O}_{Y,y}$ such that

$$\varphi_k = \sum_{i=1}^N \rho_i \big(a^*(v_i) - b^*(v_i)\big) \ .$$

By the definitions of $\mathcal{O}_{X,k}$ and $\mathcal{O}_{Y,y}$ there are open neighbourhoods W and $\Omega_1, \dots, \Omega_N$ of k in X and $r_i \in \mathcal{O}_X(\Omega_i)$ such that $\rho_i = (r_i)_k$. Moreover, there are open neighbourhoods $\Theta_1, \dots, \Theta_N$ of y in Y and $y_i \in \mathcal{O}_Y(\Theta_i)$ satisfying $(y_i)_y = v_i$, such that

$$W \subseteq V \cap \bigcap_{i=1}^N \Omega_i \cap \bigcap_{i=1}^N \left(a^{-1}(\Theta_i) \cap b^{-1}(\Theta_i)\right)$$

and finally

$$\varphi|_W = \sum_{i=1}^{N} r_i|_W \cdot (a^*(y_i)|_W - b^*(y_i)|_W)\,.$$

For $\ell \in W \cap K$, φ_ℓ is the image of the above expression in $\mathcal{O}_{X,\ell}$, hence

$$\varphi_\ell \in \Big(a^*(v) - b^*(v) \;\Big|\; v \in \mathcal{O}_{Y,a(\ell)} = \mathcal{O}_{Y,b(\ell)}\Big)_{\mathcal{O}_{X,\ell}}$$

and $f_\ell = 0$ as $\mathcal{O}_{K,[\ell]}$ is the quotient of $\mathcal{O}_{X,\ell}$ by the ideal on the right hand side. Thus, $f|_{U\cap W} = 0$, proving the injectivity claim.

Surjectivity is somewhat easier. Let $\gamma \in \mathcal{O}_{K,[k]}$, and choose a pre-image $\gamma' \in \mathcal{O}_{X,k}$ of γ under the quotient projection $\mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} = \mathcal{O}_{X,k}/I$. There are an open neighbourhood $k \in V \subseteq X$ and $\varphi \in \mathcal{O}_X(V)$ such that γ' is the image of φ in $\mathcal{O}_{X,k}$. Let $U = V \cap K$ and $f = \kappa^*\varphi \in \kappa_*\mathcal{O}_K(V) = \mathcal{O}_K(V \cap K)\mathcal{O}_K(U)$, then the component f_k of f at k equals

$$f_k = \begin{pmatrix} \text{image of } \varphi \text{ under} \\ \mathcal{O}_X(V) \to \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix} = \begin{pmatrix} \text{image of } \gamma' \text{ under} \\ \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix} = \gamma\,,$$

proving surjectivity.

Hence, $(K, \mathcal{O}_K)$ is a locally ringed space and κ is a morphism of locally ringed spaces. Indeed, κ is a morphism of ringed spaces and $\mathcal{O}_{X,k} \xrightarrow{\kappa^*} \mathcal{O}_{K,k} \simeq \mathcal{O}_{X,k}/I$ is surjective and thus local by Fact 3.

Step 2. We check that $K \xrightarrow{\kappa} X$ equalizes $X \overset{a}{\underset{b}{\rightrightarrows}} Y$.

The fact that $a\kappa = b\kappa$ is a trivial consequence of the definition of K as far as only the topological component is concernced. For the algebraic component, let $V \subseteq Y$ be open, $U = a^{-1}(V) \cap K = b^{-1}(V) \cap K$ and let $\lambda \in \mathcal{O}_K(V)$. We need to show that $\kappa^*a^*\lambda = \kappa^*b^*\lambda$. Indeed, let $y = a(k) = b(k)$. Using that

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "a^*"] \arrow[d] & \mathcal{O}_X(a^{-1}V) = a_*\mathcal{O}_X(V) \arrow[d] \\
\mathcal{O}_{Y,y} \arrow[r, "a^*"] & \mathcal{O}_{X,k}
\end{tikzcd}$$

commutes (analogously for b) and the fact that $\mathcal{O}_{Y,y} \overset{a^*}{\underset{b^*}{\rightrightarrows}} \mathcal{O}_{X,k}$ is coequalized by $\mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]}$ we get

$$\begin{aligned}
\kappa^*a^*\lambda &= \begin{pmatrix} \text{image of } a^*\lambda \text{ under} \\ \mathcal{O}_X(a^{-1}V) \to \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix}_{k\in U} = \begin{pmatrix} \text{image of } \lambda \text{ under} \\ \mathcal{O}_Y(V) \to \mathcal{O}_{Y,y} \xrightarrow{a^*} \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix}_{k\in U} \\
&= \begin{pmatrix} \text{image of } \lambda \text{ under} \\ \mathcal{O}_Y(V) \to \mathcal{O}_{Y,y} \xrightarrow{b^*} \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix}_{k\in U} = \begin{pmatrix} \text{image of } b^*\lambda \text{ under} \\ \mathcal{O}_X(b^{-1}V) \to \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix}_{k\in U} \\
&= \kappa^*b^*\lambda\,.
\end{aligned}$$

Step 3. The last thing to check is the universal property.

Let $T \xrightarrow{\tau} X$ be a morphism in the category of locally ringed spaces such that $a\tau = b\tau$. For $t \in T$ put $k = \tau(t)$ and we have $a(k) = b(k) =: y$. Moreover, the compositions of pull-backs on residue fields

$$\mathfrak{K}(y) \underset{b^*}{\overset{a^*}{\rightrightarrows}} \mathfrak{K}(k) \xrightarrow{\tau^*} \mathfrak{K}(t)$$

must coincide with

$$\mathfrak{K}(y) \xrightarrow{(a\tau)^* = (b\tau)^*} \mathfrak{K}(t) \ .$$

As $\mathfrak{K}(k) \xrightarrow{\tau^*} \mathfrak{K}(t)$ is injective (any ring morphism between fields is injective), it follows that $\mathfrak{K}(y) \underset{b^*}{\overset{a^*}{\rightrightarrows}} \mathfrak{K}(k)$ coincide. Thus $k \in K$ by definition of K and we obtain a map $T \xrightarrow{\vartheta} K$ such that $\kappa(\vartheta(t)) = \tau(t)$ and $a(\vartheta(t)) = b(\vartheta(t))$ for $t \in T$. This map is continuous because K carries the induced topology with respect to X and for open subsets $V \subseteq X$, $\theta^{-1}(V \cap K) = \tau^{-1}(V)$ is open in T.

Let $U \subseteq K$ be an open subset and $\lambda = (\lambda_k)_{k \in U} \in \mathcal{O}_K(U) \subseteq \prod_{k \in U} \mathcal{O}_{K,[k]}$. For $t \in T$ and $y = a(\vartheta(t)) = b(\vartheta(t))$, the morphism $\mathcal{O}_{X,\vartheta(t)} \xrightarrow{\tau^*} \mathcal{O}_{T,t}$ coequalizes the pair $\mathcal{O}_{Y,y} \underset{b^*}{\overset{a^*}{\rightrightarrows}} \mathcal{O}_{X,\vartheta(t)}$, hence factors over

$$\mathcal{O}_{X,\vartheta(t)} \longrightarrow \mathcal{O}_{K,[\vartheta(t)]} \xrightarrow{\widetilde{\tau}_t} \mathcal{O}_{T,t} \ .$$

We claim that

$$\widetilde{\tau}^*\lambda = \left(\widetilde{\tau}_t\left(\lambda_{\vartheta(t)}\right)\right)_{t \in \vartheta^{-1}U} \in \prod_{t \in \vartheta^{-1}U} \mathcal{O}_{T,t} \tag{$*$}$$

belongs to the image of $\mathcal{O}_T(\vartheta^{-1}U)$ in $\prod_{t \in \vartheta^{-1}U} \mathcal{O}_{T,t}$. Believing this for the moment, we finish the rest of the proof. From $(*)$ we obtain a morphism $\vartheta^* \colon \mathcal{O}_K(U) \to \mathcal{O}_T(\vartheta^{-1}U)$ such that the image of $\vartheta^*\lambda$ in $\prod_{t \in \vartheta^{-1}U} \mathcal{O}_{T,t}$ equals $\widetilde{\tau}^*\lambda$. It is rather easy to show that $\vartheta^* \colon \mathcal{O}_K \to \vartheta_*\mathcal{O}_T$ is a morphism of sheaves of rings, thus turning ϑ into a morphism of ringed spaces satisfying $\tau = \kappa\vartheta$ (which may be easily checked on stalks). Since $\mathcal{O}_{X,k} \xrightarrow{\kappa^*} \mathcal{O}_{K,k} \simeq \mathcal{O}_{X,k}/I$ is surjective, Fact 2(*d*)(iii) tells us that ϑ is indeed a morphism of locally ringed spaces.

We need to show that ϑ is unique. As before, the topological component is easy: As $K \xrightarrow{\kappa_{\mathrm{Top}}} X$ is injective, it follows that ϑ_{Top} is uniquely determined by $\tau = \kappa\vartheta$. Now for the algebraic component. As $\mathcal{O}_{X,\kappa(\vartheta(t))} \xrightarrow{\kappa^*} \mathcal{O}_{K,\vartheta(t)} = \mathcal{O}_{K,\tau(t)}$ is surjective, it follows that $\vartheta^* \colon \mathcal{O}_{K,\vartheta(t)} \to \mathcal{O}_{T,t}$ is uniquely determined by τ and the commutativity of

$$\begin{tikzcd}
\mathcal{O}_{X,\kappa(\vartheta(t))} \arrow[rr, "\kappa^*"] \arrow[dr, "\tau^*"'] & & \mathcal{O}_{K,\tau(t)} \arrow[dl, "\vartheta^*"] \\
& \mathcal{O}_{T,t} &
\end{tikzcd}$$

which again follows from $\tau = \kappa\vartheta$. As

$$\begin{tikzcd}
\mathcal{O}_K(U) \arrow[r, "\vartheta^*"] \arrow[d] & \mathcal{O}_T(\vartheta^{-1}U) \arrow[d] \\
\prod\limits_{k \in U} \mathcal{O}_{K,k} \arrow[r, "\vartheta^*"] & \prod\limits_{t \in \vartheta^{-1}U} \mathcal{O}_{T,t}
\end{tikzcd}$$

commutes, it follows that $\vartheta^*\colon \mathcal{O}_K(U) \to \mathcal{O}_T(\vartheta^{-1}U)$ is also uniquely determined by this condition. Therefore, the algebraic component $\vartheta^*\colon \mathcal{O}_K \to \vartheta_*\mathcal{O}_T$ is also uniquely determined, which proves the universal property.

Step 4. It remains to verify $(*)$, i.e. that the coherence condition characterizing the image of $\mathcal{O}_T(\vartheta^{-1}U)$ in $\prod_{t\in\vartheta^{-1}U}\mathcal{O}_{T,t}$ is satisfied. For this purpose let $z \in \vartheta^{-1}U$. By the coherence condition for $\lambda \in \mathcal{O}_K(U)$, there are an open neighbourhood V of $\kappa(\vartheta(z)) = \tau(z)$ in X and $\ell \in \mathcal{O}_X(V)$ such that

$$\lambda_k = \begin{pmatrix} \text{image of } \ell \text{ under} \\ \mathcal{O}_X(V) \to \mathcal{O}_{X,k} \to \mathcal{O}_{K,[k]} \end{pmatrix} \quad \text{for all } k \in U \cap V\,.$$

When $t \in \vartheta^{-1}U \cap \tau^{-1}V =: W$ (which is an open neighbourhood of z in T), this may be applied to $k = \vartheta(t) = \tau(t)$ and gives

$$\widetilde{\tau}_t(\lambda_k) = \begin{pmatrix} \text{image of } \tau^*\ell \text{ under} \\ \mathcal{O}_T(\tau^{-1}V) \to \mathcal{O}_T(W) \to \mathcal{O}_{T,t} \end{pmatrix}$$

by the commutativity of

$$\begin{tikzcd}
\mathcal{O}_X(V) \arrow[rr, "\tau^*"] \arrow[d] & & \mathcal{O}_T(\tau^{-1}V) = \tau_*\mathcal{O}_T(V) \arrow[d] \\
\mathcal{O}_{X,k} \arrow[rr, "\tau^*"] \arrow[dr, "\kappa^*"'] & & \mathcal{O}_{T,t} \\
& \mathcal{O}_{K,k} \arrow[ur, "\widetilde{\tau}_k"'] &
\end{tikzcd}$$

It follows that

$$\widetilde{\tau}^*\lambda = \begin{pmatrix} \text{image of } \tau^*\ell \text{ under} \\ \mathcal{O}_T(\tau^{-1}V) \to \mathcal{O}_T(W) \to \prod_{t\in W}\mathcal{O}_{T,t} \end{pmatrix}\,.$$

Since such a neighbourhood W may be found for any $z \in \vartheta^{-1}U$, $\widetilde{\tau}^*\lambda$ desires the satisfied coherence condition. *q.e.d.*

Corollary 1. *Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be a pair of morphisms of locally ringed spaces and $K \xrightarrow{\kappa} X$ be an equalizer of that pair:*

(a) *κ is a homeomorphism of K onto a subset of X which is equipped with the induced topology.*

(b) *$\kappa^*\colon \mathcal{O}_{X,\kappa(k)} \to \mathcal{O}_{K,k}$ is surjective for all $k \in K$.*

(c) *If $U \subseteq X$ is open, then $\kappa^{-1}(U) \xrightarrow{\kappa|_U} U$ is an equalizer of $U \underset{g|_U}{\overset{f|_U}{\rightrightarrows}} Y$.*

Proof. All of this is rather trivial. *q.e.d.*

Remark. If $Y \xrightarrow{j} Z$ is a monomorphism, then the equalizers of jf with jg and of f with g are canonically isomorphic. This holds in any category as the equalizer represents

$$\{\tau \in \operatorname{Hom}(T,X) \mid f\tau = g\tau\} = \{\tau \in \operatorname{Hom}(T,X) \mid jf\tau = jg\tau\}\,.$$

Fibre products. Our goal now is to characterize fibre products of locally ringed spaces. Let $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$ be morphisms of locally ringed spaces. For $U \subseteq X$, $V \subseteq Y$, $W \subseteq S$ open subsets in their respective spaces such that $\xi(U) \cup \upsilon(V) \subseteq W$ we put

$$\mathcal{R}(U,V;W) \coloneqq \mathcal{O}_X(U) \otimes_{\mathcal{O}_S(W)} \mathcal{O}_Y(V) \quad \text{and also} \quad \mathcal{R}(U,V) \coloneqq \mathcal{R}(U,V;S)\ .$$

For open subsets $\widetilde{U} \subseteq U$, $\widetilde{V} \subseteq V$, and $\widetilde{W} \subseteq W$ we have a ring homomorphism

$$\begin{aligned}\mathcal{R}(U,V;W) = \mathcal{O}_X \otimes_{\mathcal{O}_S(W)} \mathcal{O}_Y(V) &\longrightarrow \mathcal{O}_X(\widetilde{U}) \otimes_{\mathcal{O}_S(\widetilde{W})} \mathcal{O}_Y(\widetilde{V}) = \mathcal{R}(\widetilde{U},\widetilde{V};\widetilde{W}) \\ f \otimes g &\longmapsto (f|_{\widetilde{U}}) \otimes (g|_{\widetilde{V}})\end{aligned}$$

denoted $(-)|_{\widetilde{U},\widetilde{V};\widetilde{W}}$ or $(-)|_{\widetilde{U},\widetilde{V}}$.

Remark. If $U = \widetilde{U}$, $V = \widetilde{V}$ then $(-)|_{U,V;\widetilde{W}} \colon \mathcal{R}(U,V;W) \to \mathcal{R}(U,V;\widetilde{W})$ is surjective and $\varinjlim \mathcal{R}(U,V;W) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ where we take the colimit over all triples $(U,V;W)$ such that $x \in U$, $y \in V$ and $s \in W$.

We're now prepared to describe the fibre product $F = X \times_S Y$:

- *Underlying set.* Set-theoretically, $F = X \times_S Y$ is given by

$$F = X \times_S Y = \left\{ (x,y,\mathfrak{p}) \;\middle|\; \begin{array}{c} x \in X,\ y \in Y \text{ such that } \xi(x) = \upsilon(y) =: s \text{ and the} \\ \text{prime ideal } \mathfrak{p} \in \operatorname{Spec}\left(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}\right) \text{ satisfies} \\ \alpha^{-1}(\mathfrak{p}) = \mathfrak{m}_{X,x} \text{ and } \beta^{-1}(\mathfrak{p}) = \mathfrak{m}_{Y,y} \end{array} \right\}$$

  Herein, as usual $\mathfrak{m}_{X,x}$ and $\mathfrak{m}_{Y,y}$ denote the maximal ideals of $\mathcal{O}_{X,x}$ respectively $\mathcal{O}_{Y,y}$ and the morphisms α, β are given by $\alpha(\lambda) = \lambda \otimes 1$ for $\lambda \in \mathcal{O}_{X,x}$ and $\beta(\vartheta) = 1 \otimes \vartheta$ for $\vartheta \in \mathcal{O}_{Y,y}$.

- *Topology.* The sets of the form

$$\Omega(U,V,\theta) \coloneqq \left\{ (x,y,\mathfrak{p}) \;\middle|\; \begin{array}{c} x \in U,\ y \in V \text{ and } \mathfrak{p} \text{ doesn't contain the} \\ \text{image of } \theta \text{ under } \mathcal{R}(U,V) \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \end{array} \right\}$$

  form a topology base of F (in view of the above-mentioned surjectivity we only need to consider $\mathcal{R}(U,V)$ rather than all $\mathcal{R}(U,V;W)$).

- *Structure sheaf.* Let $\mathcal{O}_{F,[x,y,\mathfrak{p}]} = (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_{\mathfrak{p}}$. This *pretender stalk* eventually will turn out to be the actual stalk of $\mathcal{O}_F$ at $(x,y,\mathfrak{p})$. For $\Omega \subseteq F$ open the section of $\mathcal{O}_F$ on Ω is

$$\mathcal{O}_F(\Omega) = \left\{ \lambda = \left(\lambda_{(x,y,\mathfrak{p})}\right)_{(x,y,\mathfrak{p}) \in \Omega} \in \prod_{(x,y,\mathfrak{p}) \in \Omega} \mathcal{O}_{F,[x,y,\mathfrak{p}]} \;\middle|\; \begin{array}{c} \lambda \text{ fulfills the} \\ \textit{coherence condition} \end{array} \right\}.$$

  The *coherence condition* says that every $\omega \in \Omega$ has an open neighbourhood $\Omega(U,V,\theta)$ together with an $\ell \in \mathcal{R}(U,V)_\theta$ such that for $(x,y,\mathfrak{p}) \in \Omega \cap \Omega(U,V,\theta)$ we have

$$\lambda_{(x,y,\mathfrak{p})} = \left(\begin{array}{c} \text{image of } \ell \text{ under } \mathcal{R}(U,V,\theta) = \left(\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_Y(V)\right)_\theta \\ \to \left(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}\right)_\theta \to \left(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}\right)_{\mathfrak{p}} \end{array} \right).$$

- *Projections.* We obtain projections $X \xleftarrow{p} F \xrightarrow{p} Y$ by $p(x,y,\mathfrak{p}) = x$ and $q(x,y,\mathfrak{p}) = y$. They are continuous, e.g. $p^{-1}(U) = \Omega(U, Y, 1)$. For $f \in \mathcal{O}_X(U)$, $g \in \mathcal{O}_Y(V)$ we define the pullbacks p^*, q^* via

$$p^* f = \left(\begin{array}{c} \text{image of } f \text{ under} \\ \mathcal{O}_X(U) \to \mathcal{O}_{X,x} \xrightarrow{\alpha} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \to (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_{\mathfrak{p}} \end{array} \right)_{(x,y,\mathfrak{p}) \in p^{-1}(U)}$$

$$q^* g = \left(\begin{array}{c} \text{image of } g \text{ under} \\ \mathcal{O}_Y(V) \to \mathcal{O}_{Y,y} \xrightarrow{\beta} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \to (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_{\mathfrak{p}} \end{array} \right)_{(x,y,\mathfrak{p}) \in q^{-1}(V)}$$

Proposition 2. *The above is the correct description of a fibre product $F = X \times_S Y$ of $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$. Moreover, the morphism*

$$\begin{array}{rcl} & \mathcal{O}_{F,(x,y,\mathfrak{p})} \longrightarrow \mathcal{O}_{F,[x,y,\mathfrak{p}]} & \\ \textit{image of } (\lambda_\omega) \in \displaystyle\prod_{\omega \in \Omega} \mathcal{O}_{F,[\omega]} & \longmapsto & \lambda_{(x,y,\mathfrak{p})} \in \mathcal{O}_{F,[x,y,\mathfrak{p}]} \end{array}$$

is an isomorphism.

Proof. Step 1. First we show that the $\Omega(U, V, \theta)$ form a base of a topology. We have $F = \Omega(X, Y, 1)$ and $\emptyset = \Omega(X, Y, 0)$ and $\Omega(U, V, \theta) \cap \Omega(\widetilde{U}, \widetilde{V}, \widetilde{\theta}) = \Omega(U \cap \widetilde{U}, V \cap \widetilde{V}, (\theta|_\Gamma) \cdot (\widetilde{\theta}_\Gamma))$ where $\Gamma = (U \cap \widetilde{U}, V \cap \widetilde{V})$ for short. This suffices.

Step 2. Furthermore, $\mathcal{O}_F$ is a sheaf of rings. Let $\lambda = (\lambda_\omega)$ and $\widetilde{\lambda} = (\widetilde{\lambda}_\omega)$ be sections in $\mathcal{O}_F(\Omega)$. We have to show that $\lambda * \widetilde{\lambda} \in \mathcal{O}_F(\Omega)$ for $* \in \{+, \cdot\}$. If $\ell \in \mathcal{R}(U, V)_\theta$ and $\widetilde{\ell} \in \mathcal{R}(\widetilde{U}, \widetilde{V})_{\widetilde{\theta}}$ verify the coherence condition for λ respectively $\widetilde{\lambda}$ at $\omega \in \Omega$, then

$$\left(\begin{array}{c} \text{image of } \ell|_\Gamma \text{ under} \\ \mathcal{R}(\Gamma)_{\theta|_\Gamma} \to \mathcal{R}(\Gamma)_{\theta|_\Gamma \cdot \widetilde{\theta}|_\Gamma} \end{array} \right) * \left(\begin{array}{c} \text{image of } \widetilde{\ell}|_\Gamma \text{ under} \\ \mathcal{R}(\Gamma)_{\theta|_\Gamma} \to \mathcal{R}(\Gamma)_{\theta|_\Gamma \cdot \widetilde{\theta}|_\Gamma} \end{array} \right)$$

verifies the coherence condition for $\lambda * \widetilde{\lambda}$ at ω. Again, Γ denotes $(U \cap \widetilde{U}, V \cap \widetilde{V})$.

Step 3. We prove that the pretender stalks are isomorphic to the actual ones. This time, surjectivity first. Let $\lambda \in \mathcal{O}_{F,[x,y,\mathfrak{p}]} = (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_{\mathfrak{p}}$ be given as $\frac{\varphi}{\sigma}$ where φ and σ are elements of this tensor product and $\sigma \notin \mathfrak{p}$. Then there exist open neighbourhoods U of x and V of y as well as elements $f \in \mathcal{R}(U, V)$ and $t \in \mathcal{R}(U, V)$ such that the images of f and t in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ equal φ and σ. Then $\frac{f}{t}$ defines an element of $\mathcal{O}_F(\Omega(U, V, t))$ whose image in $\mathcal{O}_{F,(x,y,\mathfrak{p})}$ is mapped to $\lambda = \frac{\varphi}{\sigma} \in \mathcal{O}_{F,[x,y,\mathfrak{p}]}$.

For injectivity, let $\lambda = (\lambda_\omega) \in \mathcal{O}_F(\Omega) \subseteq \prod_{\omega \in \Omega} \mathcal{O}_{F,[\omega]}$ and let $(x, y, \mathfrak{p}) \in \Omega$ such that $\lambda_{x,y,\mathfrak{p}}$ vanishes in $\mathcal{O}_{F,[x,y,\mathfrak{p}]}$. We must show that λ vanishes in some neighbourhood of $(x, y, \mathfrak{p})$. By the coherence condition there is an open neighbourhood $\Omega(U, V, \theta)$ of $(x, y, \mathfrak{p})$ together with $\ell \in \mathcal{R}(U, V)_\theta$ such that the image of ℓ under $\mathcal{R}(U, V)_\theta \to \mathcal{O}_F(\Omega(U, V, \theta)) \to \mathcal{O}_F(\Omega \cap \Omega(U, V, \theta))$ equals the restriction $\lambda|_{\Omega \cap \Omega(U,V,\theta)}$ of λ to $\Omega \cap \Omega(U, V, \theta)$. Replacing Ω by $\Omega \cap \Omega(U, V, \theta)$ and λ by its restriction, we achieve that lambda itself equals this image. Let $\ell = \frac{n}{\theta}$ with $n \in \mathcal{R}(U, V)$ (we may w.l.o.g. assume the denominator is θ because localizing at an arbitrary power of θ still gives the same localization). Now $0 = \lambda_{(x,y,\mathfrak{p})}$ equals the image of ℓ under

$$\mathcal{R}(U, V)_\theta = (\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_Y(V)) \longrightarrow (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_{\mathfrak{p}} \ .$$

Since $\lambda_{(x,y,\mathfrak{p})} = 0$, there is $h \in (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}) \setminus \mathfrak{p}$ such that $n \cdot h = 0$ in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \simeq \varinjlim \mathcal{O}_X(\widetilde{U}) \otimes_{\mathcal{O}_S(W)} \mathcal{O}_Y(\widetilde{V})$. We may assume that h is the image of $\eta \in \mathcal{R}(\widetilde{U}, \widetilde{V})$ in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ and that $n \cdot \eta = 0$ in $\mathcal{R}(\widetilde{U}, \widetilde{V}; W)$. Replacing both η and θ by $\eta \cdot \theta$ as well as U by $\widetilde{U}$ and V by $\widetilde{V}$, we may assume $U = \widetilde{U}$, $V = \widetilde{V}$ and $\eta = \theta$. Then the image of $\ell = \frac{n}{\theta}$ under $\mathcal{R}(U,V)_\theta \to \mathcal{R}(U,V;W)_\theta$ vanishes. But the map $\mathcal{R}(U,V)_\theta \to \mathcal{O}_F(\Omega(U,V,\theta))$ factors over $\mathcal{R}(U,V;W)_\theta$ as this is the case for the maps

$$\begin{tikzcd}
\Big(\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_Y(V)\Big)_\theta \arrow[r] \arrow[dr] & \Big(\mathcal{O}_{X,x'} \otimes_{\mathcal{O}_{S,s'}} \mathcal{O}_{Y,y'}\Big)_\theta \arrow[r] & \mathcal{O}_{F,[x',y',\mathfrak{p}']} \\
& \Big(\mathcal{O}_X(U) \otimes_{\mathcal{O}_S(W)} \mathcal{O}_Y(V)\Big)_{\theta|_{U,V;W}} \arrow[u] &
\end{tikzcd}$$

(where $s' = \xi(x') = \upsilon(y')$) from which it is made up. It follows that the image of ℓ under this map vanishes and λ als vanishes.

Step 4. We have to show that p and q are morphisms of locally ringed spaces. Only the locality of $\mathcal{O}_{X,x} \xrightarrow{p^*} \mathcal{O}_{F,(x,y,p)}$ (and q^* likewise) is questionable. By the previous step, p^* can be identified with

$$\mathcal{O}_{X,x} \xrightarrow{\alpha} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{F,[x,y,p]} = (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_\mathfrak{p}$$

and this is local since the pre-image of the maximal ideal $\mathfrak{p}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y})_\mathfrak{p}$ in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ equals $\mathfrak{p}$ and that of $\mathfrak{p}$ under α equals $\alpha^{-1}(\mathfrak{p}) = \mathfrak{m}_{X,x}$ by our definition of F as a set. For q^* it's just the same.

Step 5. We show $\xi p = \upsilon q$. On the level of points, $\xi(p(x,y,\mathfrak{p})) = \xi(x) = s = \upsilon(y) = \upsilon(q(x,y,\mathfrak{p}))$, hence the topological components of the involved morphisms have the desired property.

Before we continue, recall the following

Fact. If X is a locally ringed space, $U \subseteq X$ and $f \in \mathcal{O}_X(U)$, then there is a unique dotted arrow making the following diagram commute:

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r] \arrow[dr] & \mathcal{O}_X(U \setminus V(f)) \\
& \mathcal{O}_X(U)_f \arrow[u, dashed, "\exists!"']
\end{tikzcd}$$

Sketch of a proof. This basically comes down to proving that f is invertible in $\mathcal{O}_X(U \setminus V(f))$. For any $x \in U \setminus V(f)$, the image of x in $\mathcal{O}_{X,x}$ is not in $\mathfrak{m}_{X,x}$, hence invertible. Consequently, there is an open neighbourhood U_x of x (of which we may assume $U_x \subseteq U \setminus V(f)$) and some $g_x \in \mathcal{O}_X(U_x)$ such that $f|_{U_x} \cdot g_x = 1$. Denote $U_{xy} = U_x \cap U_y$, then $g_x|_{U_{xy}} = g_y|_{U_{xy}}$ since $f|_{U_{xy}}$ has a unique inverse in $\mathcal{O}_X(U_{xy})$ if it has any. Now the sheaf axiom does the rest. *q.e.d.*

Now we show that $\xi p = \upsilon q$ holds for the algebraic components: If $W \subseteq S$ is open and $\sigma \in \mathcal{O}_S(W)$, then

$$p^*\xi^*\sigma = \left(\begin{array}{c} \text{image of } \xi^*\sigma \text{ under} \\ \mathcal{O}_X(\xi^{-1}W) \to \mathcal{O}_{X,x} \xrightarrow{\alpha} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \to \mathcal{O}_{F,[x,y,\mathfrak{p}]} \end{array} \right)_{(x,y,\mathfrak{p})\in F \text{ s.t. } x\in\xi^{-1}(W)}$$

$$\begin{aligned}
&= \left(\begin{array}{c} \text{image of } \sigma \text{ under} \\ \mathcal{O}_S(W) \to \mathcal{O}_{S,s} \xrightarrow{\xi^*} \mathcal{O}_{X,x} \xrightarrow{\alpha} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \to \mathcal{O}_{F,[x,y,\mathfrak{p}]} \end{array} \right)_{(x,y,\mathfrak{p})\in(\xi p)^{-1}W} \\
&= \left(\begin{array}{c} \text{image of } \sigma \text{ under} \\ \mathcal{O}_S(W) \to \mathcal{O}_{S,s} \xrightarrow{\upsilon^*} \mathcal{O}_{Y,y} \xrightarrow{\beta} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \to \mathcal{O}_{F,[x,y,\mathfrak{p}]} \end{array} \right)_{(x,y,\mathfrak{p})\in(\upsilon q)^{-1}W} \\
&= \left(\begin{array}{c} \text{image of } \upsilon^*\sigma \text{ under} \\ \mathcal{O}_Y(\upsilon^{-1}W) \to \mathcal{O}_{Y,y} \xrightarrow{\beta} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \to \mathcal{O}_{F,[x,y,\mathfrak{p}]} \end{array} \right)_{(x,y,\mathfrak{p})\in(\upsilon q)^{-1}W} \\
&= q^*\upsilon^*\sigma \ .
\end{aligned}$$

Herein, we used that for $t \in \mathcal{O}_{S,s}$, $\alpha(\xi^* t) = \xi^*(t) \otimes 1 = 1 \otimes \upsilon^*(t) = \beta(\upsilon^* t)$ by the $\mathcal{O}_{S,s}$-bilinearity of $- \otimes_{\mathcal{O}_{S,s}} -$.

Step 6. We show the universal property. Let $X \xleftarrow{\tau_X} T \xrightarrow{\tau_Y} Y$ be morphisms of locally ringed spaces such that $\xi\tau_X = \upsilon\tau_Y$. We define a map $T \xrightarrow{\tau} F$ as follows: For $t \in T$, we put $x = \tau_X(t), y = \tau_Y(t), s = \xi(x) = \xi(\tau_X(t)) = \upsilon(\tau_Y(t)) = \upsilon(y)$. By the universal property of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ as a cofibre product in the category of rings, there is a unique ring homomorphism τ^* from this ring to $\mathcal{O}_{T,t}$ such that

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "\alpha"] \arrow[d, "\tau_X^*"'] & \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \arrow[dl, dashed, "\tau^*"] \\
\mathcal{O}_{T,t} & \mathcal{O}_{Y,y} \arrow[l, "\tau_Y^*"] \arrow[u, "\beta"']
\end{tikzcd} \tag{$*$}$$

commutes. It is given by $\tau^*(f \otimes g) = \tau_X^*(f) \cdot \tau_Y^*(g)$. Let $\mathfrak{p} = (\tau^*)^{-1}(\mathfrak{m}_{T,t})$. By the above diagram,

$$\alpha^{-1}(\mathfrak{p}) = \alpha^{-1}(\tau^*)^{-1}(\mathfrak{m}_{T,t}) = \tau_X^*(\mathfrak{m}_{T,t}) = \mathfrak{m}_{X,x} \ ,$$

as τ_X is a morphism of locally ringed spaces. Similarly, $\beta^{-1}(\mathfrak{p}) = (\mathfrak{m}_{Y,y})$ Thus

$$\tau(t) \coloneqq (x, y, \mathfrak{p}) \in F \ .$$

Clearly, we have $p\tau = \tau_X$ and $q\tau = \tau_Y$ as maps of underlying sets topological spaces. It remains to prove that τ is continuous and a morphism of locally ringed spaces

Continuity of τ. Let $U \subseteq X, V \subseteq Y$ be open and $f \in \mathcal{R}(U, V)$. As the last ring is a cofibre product $\mathcal{O}_X(V) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_Y(V)$, there is a unique ring homomorphism $\tau^\sharp$ from that ring to $\mathcal{O}_T(\Theta)$, where $\Theta = \tau_X^{-1}(U) \cap \tau_Y^{-1}(Y)$, such that the diagram

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r, "f \mapsto f\otimes 1"] \arrow[d, "\tau_X^*"'] & \mathcal{R}(U,V) \arrow[dl, dashed, "\tau^\sharp"] \\
\mathcal{O}_T(\Theta) & \mathcal{O}_Y(V) \arrow[l, "\tau_Y^*"] \arrow[u, "g \mapsto 1\otimes g"']
\end{tikzcd}$$

commutes, and $\tau^\sharp(f\otimes g) = \tau_X^*(f)\cdot\tau_Y^*(g)$. Then, for $\lambda \in \mathcal{R}(U,V)$,

$$\begin{aligned}
\tau^{-1}\Omega(U,V,\lambda) &= \left\{ t \in T \;\middle|\; \begin{array}{c} \tau_X(t) \in U,\ \tau_Y(t) \text{ and the image of } \lambda \text{ in } \mathcal{O}_{X,x}\otimes_{\mathcal{O}_{S,s}}\mathcal{O}_{Y,y} \\ (\text{where } s = \xi(\tau_X(t)) = \upsilon(\tau_Y(t))) \text{ is } \notin (\tau^*)^{-1}(\mathfrak{m}_{T,t}) \end{array} \right\} \\
&= \Big\{ t \in \Theta \;\Big|\; \Big(\text{image of } \lambda \text{ in } \mathcal{O}_{X,x}\otimes_{\mathcal{O}_{S,s}}\mathcal{O}_{Y,y}\Big) \notin (\tau^*)^{-1}(\mathfrak{m}_{T,t}) \Big\} \\
&= \Big\{ t \in \Theta \;\Big|\; \Big(\text{image of } \tau^\sharp\lambda \text{ in } \mathcal{O}_T(\Theta) \longrightarrow \mathcal{O}_{T,t}\Big) \notin \mathfrak{m}_{T,t} \Big\} \\
&= \Theta \setminus V(\tau^\sharp\lambda)\,, \qquad (\#)
\end{aligned}$$

which is open. The transition from the second line to the third line uses the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{R}(U,V) \arrow[r, "\tau^\sharp"] \arrow[d] & \mathcal{O}_T(\Theta) \arrow[d] \\
\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{S,s}}\mathcal{O}_{Y,y} \arrow[r, "\tau^*"] & \mathcal{O}_{T,t}
\end{tikzcd}$$

This can be seen as follows: The element $f\otimes g$ in the upper-left corner $\mathcal{R}(U,V) = \mathcal{O}_X(V)\otimes_{\mathcal{O}_S(S)}\mathcal{O}_Y(V)$ is mapped to via the upper-right corner to

$$\begin{aligned}
\left(\begin{array}{c} \text{image of } \tau^\sharp(f\otimes g) \text{ under} \\ \mathcal{O}_T(\theta) \to \mathcal{O}_{T,t} \end{array} \right) &= \left(\begin{array}{c} \text{image of } \tau_X^*(f)\cdot\tau_Y^*(g) \text{ under} \\ \mathcal{O}_T(\theta) \to \mathcal{O}_{T,t} \end{array} \right) \\
&= \big(\text{image of } \tau_X^*(f) \text{ in } \mathcal{O}_{T,t}\big)\cdot\big(\text{ image of } \tau_Y^*(g) \text{ in } \mathcal{O}_{T,t}\big) \\
&= \tau_X^*\big(\text{image of } f \text{ in } \mathcal{O}_{X,x}\big)\cdot\tau_Y^*\big(\text{image of } g \text{ in } \mathcal{O}_{Y,y}\big) \\
&= \tau^*\big(\,(\text{image of } f \text{ in } \mathcal{O}_{X,x})\otimes(\text{image of } g \text{ in } \mathcal{O}_{Y,y})\,\big),
\end{aligned}$$

which is the image of $f\otimes g$ via the lower-left corner. As these pure tensor products generate $\mathcal{R}(U,V)$ as an abelian group, the stated commutativity of the diagram follows.

Definition of the algebraic component τ^* of τ: Fix an open subset $\Omega \subseteq F$. We define $\tau(\varphi)$, where $\varphi = (\varphi_\omega)_{\omega\in\Omega} \in \prod_{\omega\in\Omega}\mathcal{O}_{F,[\omega]}$ must satisfy the coherence condition for sections of $\mathcal{O}_F$, to be the preimage of $(\tau^\flat\varphi_{\tau(t)})_{t\in\tau^{-1}(\Omega)}$ under $\mathcal{O}_T(\tau^{-1}\Omega) \to \prod_{t\in\tau^{-1}(\Omega)}\mathcal{O}_{T,t}$. Here, $\tau^\flat$ is the unique ring homomorphism $\mathcal{O}_{F,[\tau(t)]} \to \mathcal{O}_{T,t}$ making the diagram

$$\begin{tikzcd}
\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{S,s}}\mathcal{O}_{Y,y} \arrow[r] \arrow[dr, "\tau^*"'] & \Big(\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{S,s}}\mathcal{O}_{Y,y}\Big)_{\mathfrak{p}} = \mathcal{O}_{F,[x,y,\mathfrak{p}]} \arrow[d, dashed, "\exists!\,\tau^\flat"] \\
& \mathcal{O}_{T,t}
\end{tikzcd}$$

commute. It exists by the universal property of localizations, as $\mathfrak{p} = (\tau^*)^{-1}(\mathfrak{m}_{T,t})$. To confirm that $(\tau^\flat \varphi_{\tau(t)})_{t \in \tau^{-1}(\Omega)}$ has the coherence condition characterizing the image of $\mathcal{O}_T(\tau^{-1}\Omega)$, let $\widetilde{\Omega} = \Omega(U, V, \lambda)$ be an open neighbourhood in F of $\tau(t)$ such that on $\widetilde{\Omega} \cap \Omega$, φ is given by $f \in R(U,V)_\lambda$. By (#), $\tau^{-1}(\widetilde{\Omega}) = \left(\tau_X^{-1}(V) \cap \tau_Y^{-1}(V)\right) \setminus V(\tau^\sharp(\lambda))$ is an open neighbourhood of t on which $\tau^\sharp(g) \cdot \tau^\sharp(\lambda)^{-k}$ defines a section of $\mathcal{O}_T$, where $g \in R(U,V)$ and $k \geq 0$ such that $f = g \cdot \lambda^{-k}$. Then

$$\tau^\flat\left(\varphi_{\tau(t)}\right) = \frac{\tau^*(g_{\tau(t)})}{\tau^*(\lambda_{\tau(t)})^k} = \left(\text{image of } \frac{\tau^\sharp(g)}{\tau^\sharp(\lambda)^k} \text{ in } \mathcal{O}_{T,t}\right)$$

for $t, \in \tau^{-1}(\Omega) \cap \tau^{-1}(\widetilde{\Omega})$, using the same equality as in the proof of (#). This completes the description of the algebraic component τ^* of τ. It is clear that it makes τ a morphism of ringed spaces, that $\tau^* p^* = \tau_X^*$ and that $\tau^* q^* = \tau_Y^*$ hold (this follows from the commutativity of the triangles in in $(*)$ and the fact that the map τ^* defined there coincides with the map on stalks induced by the new τ^*).

Locality of $\tau^*\colon \mathcal{O}_{F,\tau(t)} \to \mathcal{O}_{T,t}$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{F,\tau(t)} \arrow[rr] \arrow[d, phantom, "\wr|"] & & \mathcal{O}_{T,t} \\
\mathcal{O}_{F,[\tau(t)]} = \left(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}\right)_{\mathfrak{p}} \arrow[urr, "\tau^\flat"] & & \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} =: \mathcal{L} \arrow[ll] \arrow[u, "\tau^*"']
\end{tikzcd}$$

and $\tau^\flat$ maps $\mathfrak{p}\mathcal{L}_\mathfrak{p}$ to $\mathfrak{m}_{T,t}$ as $\tau^*(\mathfrak{p}) \subseteq \mathfrak{m}_{T,t}$ as $\mathfrak{p} = (\tau^*)^{-1}\mathfrak{m}_{T,t}$ by the definition of T.

Due to the lecture being over, we do without proving the uniqueness part of the universal property.

This follows now. Let $X \xleftarrow{p} F \xrightarrow{q} Y$ be the claimed fibre product of $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$ constructed before, T a locally ringed space, $X \xleftarrow{\tau_X} T \xrightarrow{\tau_Y} Y$ morphisms of locally ringed spaces such that $\xi\tau_X = \upsilon\tau_Y$. We already have shown that there exists $T \xrightarrow{\tau} F$ such that $\tau_X = p\tau$ and $\tau_Y = q\tau$ and still have to proof that it is unique with this property. Let $t \in T$, $\tau(t) = (x, y, \mathfrak{p})$, then obviously $x = \tau_X(t)$ and $y = \tau_Y(t)$. In the above diagram, the dashed arrow $\tau^\flat$ is uniquely determined by the co-cartesianess of the lower right corner. Because τ is a morphism of locally ringed spaces, the dotted arrow τ^* must be a local ring morphism, hence

$$(\tau^*)^{-1}(m_{T,t}) = \mathfrak{p}\mathcal{O}_{F,[x,y,\mathfrak{p}]} = \mathfrak{p}\left(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}\right)_{\mathfrak{p}} .$$

Since the preimage of this ideal in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$ equals $\mathfrak{p}$ and the diagram commutes, $\mathfrak{p} = (\tau^\flat)^{-1}(\mathfrak{m}_{T,t})$ is uniquely determined. By the universal property of localization, the dotted arrow is also unique. Thus τ^* is unique at stalks, hence unique. *q.e.d.*

Corollary 2. *(a) Let X_{Top} be the topological space underlying X, then the map*

$$(X \times_S Y)_{\mathrm{Top}} \longrightarrow X_{\mathrm{Top}} \times_{S_{\mathrm{Top}}} Y_{\mathrm{Top}}$$

(determined by the diagram on the right) is surjective.

(*b*) *If $U \subseteq X$, $V \subseteq Y$ and $W \subseteq S$ are open subsets then $p^{-1}(U) \cap q^{-1}(V) \subseteq F$ together with the restrictions of p and q is a fibre product of U and V over W.*

Proof. To prove (a) we use that for all $x \in X_{\mathrm{Top}}$ and $y \in Y_{\mathrm{Top}}$ such that $s = \xi(x) = \upsilon(y)$ there is an appropriate $\mathfrak{p}$. We must thus show that there is $\mathfrak{p} \in \operatorname{Spec}\left(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}\right)$ such that $\alpha^{-1}(\mathfrak{p}) = \mathfrak{m}_{X,x}$, $\beta^{-1}(\mathfrak{p}) = \mathfrak{m}_{Y,y}$. Consider the ring homomorphism

$$\begin{aligned} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} &\xrightarrow{\varphi} \mathfrak{K}(x) \otimes_{\mathfrak{K}(s)} \mathfrak{K}(y) = A \\ f \otimes g &\longmapsto (f \mod \mathfrak{m}_{X,x}) \otimes (g \mod \mathfrak{m}_{Y,y}) \ . \end{aligned}$$

If $\mathfrak{m}$ is any maximal ideal of A, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$ has the required property. For instance,

$$(f \otimes 1) \in \mathfrak{p} \Leftrightarrow \varphi(f \otimes 1) \in \mathfrak{m} \Leftrightarrow ((f \mod \mathfrak{m}_{X,x}) \otimes 1) \in \mathfrak{m} \ .$$

When $f \in \mathfrak{m}_{X,x}$ the right hand side vanishes, otherwise f and the right hand side are units.

The part (b) is a straightforward consequence of our construction of $\mathcal{R}(U,V) \to \mathcal{R}(U,V,W)$. *q.e.d.*

Remark 2. The proof would work with infinitely many factors as well, with some modifications. When S and X_λ are preschemes, the fibre product may fail to be a prescheme unless all but finitely many of the morphisms $X_\lambda \to S$ are affine.

Definition 5 (Immersive morphisms)**.** A morphism $Y \xrightarrow{\upsilon} S$ of locally ringed spaces is called **immersive** if, topologically, υ is a homeomorphism to its image (equipped with the induced topology by S), and for every $y \in Y$ the map $\mathcal{O}_{S,\upsilon(y)} \xrightarrow{\upsilon^*} \mathcal{O}_{Y,y}$ is surjective. If $\upsilon(Y) \subseteq S$ is a closed subset, υ is called a closed immersion.

Corollary 3. *The base change $F \xrightarrow{\widetilde{\upsilon}} X$ of an immersive morphism is immersive, and its image is $\xi^{-1}(\upsilon(Y))$. In particular, the base change of a closed immersion is itself closed.*

Proof. For $(x, y, \mathfrak{p})$ and in the description of F, let $I_S \subseteq \mathcal{O}_{S,s}$ be the kernel of $\mathcal{O}_{S,s} \xrightarrow{\upsilon^*} \mathcal{O}_{Y,y}$. Then $\mathcal{O}_{Y,y} \simeq \mathcal{O}_{S,s}/I_s$, hence $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x}/(\xi^*(I_s) \cdot \mathcal{O}_{X,x}) =: \mathcal{O}_{X,x}/(I_s \cdot \mathcal{O}_{X,x}$. Since $\alpha^{-1}(\mathfrak{p})$ must be $\mathfrak{m}_{X,x}$, $\mathfrak{p} \supseteq \mathfrak{m}_{X,x}/(I_s \cdot \mathcal{O}_{X,x}$ which must be an equality as the right hand side is the maximal ideal of local ring $\mathcal{O}_{X,x}/(I_s \cdot \mathcal{O}_{X,x})$. Thus, $\mathfrak{p}$ and y are uniquely determined by x, $\mathcal{O}_{F,[x,y,\mathfrak{p}]} \simeq \mathcal{O}_{X,x}/(I_s \cdot \mathcal{O}_{X,x})$ is a quotient of $\mathcal{O}_{X,x}$ implying surjectivity of ξ^* on stalks. It remains to confirm that the topology is the induced topology. Note that

$$\Omega(U,V,f) = \bigcup_{\lambda \in \Lambda} \Omega(U, V_\lambda, f_{U,V_\lambda}) \tag{1}$$

when $V = \bigcup_{\lambda \in \Lambda} V_\lambda$. Let $f \in \mathcal{R}(U,V)$, $f = \sum_{i=1}^n g_i \otimes h_i$, $g_i \in \mathcal{O}_X(U)$, $h_i \in \mathcal{O}_Y(V)$. As Y carries the induced topology and υ^* is surjective on stalks, V may be covered by open subsets of the form $V_\lambda = Y \cap W_\lambda$, $W_\lambda \subseteq S$ open, such that there are $\eta_{i,\lambda} \in \mathcal{O}_S(W_\lambda)$ such that $h_i|_{V_\lambda} = \upsilon^*(\eta_{i,\lambda})$. Then

$$f_{U,V_\lambda} = \sum_{i=1}^n g_i \otimes \upsilon^*(\eta_{i,\lambda}) = \sum_{i=1}^n (\xi^*(\eta_i) \cdot g_i) \otimes 1 = \varphi \otimes 1$$

in $\mathcal{R}(\xi^{-1}(W_\lambda) \cap U, V_\lambda, W_\lambda)$. Thus $\Omega(U, V_\lambda, f|_{U,V_\lambda}) = ((U \cap \xi^{-1}(W_\lambda) \setminus V(\varphi)) \cap F$ is open in the induced topology. By (1), $\Omega(U,V,f)$ is open in that topology. *q.e.d.*

Definition 6 (Sheaf of modules)**.** Let $\mathcal{R}$ be a sheaf of rings on X. A **sheaf of modules** over $\mathcal{R}$, or simply an $\mathcal{R}$-module, is a sheaf $\mathcal{M}$ of abelian groups on X with multiplications

$$\mathcal{R}(U) \times \mathcal{M}(U) \longrightarrow \mathcal{M}(U)$$

giving $\mathcal{M}(U)$ the structure of an $\mathcal{R}(U)$-module and such that $(\rho \cdot \mu)|_V = (\rho|_V) \cdot (\mu|_V)$ when $V \subseteq U$ is open. In the case where $\mathcal{M}(U) \subseteq \mathcal{R}(U)$ is an ideal in that ring, with module operations and $|_V$ are the restrictions of the ring operations in $\mathcal{R}(U)$ and of $|_V$ for $\mathcal{R}$ to $\mathcal{M}(U)$, we call $\mathcal{M}$ a **sheaf of ideals** in $\mathcal{R}$.

Proposition 3. *If $(I_{[x]})_{x \in X}$ is a family of ideals in the stalks $\mathcal{R}_x$, then the following conditions are equivalent.*

(*a*) *There is a sheaf of ideals I on X such that $I_{[x]}$ equals the (isomorphic) image I_x in $\mathcal{R}_x$.*

(*b*) *For every $x \in X$, every $\iota \in I_{[x]}$ is the image of some $i \in \mathcal{R}(U)$ for some open neighbourhood U of x, such that $i_y \in I_{[y]}$ for all $y \in U$.*

In this case I is determined uniquely.

Corollary 4. $U = \left\{ x \in X \;\middle|\; I_{[x]} = \mathcal{R}_x \right\}$ *is open in X.*

Proposition 4. *For a locally ringed space X, let two immersions $K_1 \xrightarrow{\kappa_1} X \xleftarrow{\kappa_2} K_2$ be called equivalent if there is an isomorphism $K_1 \xrightarrow{\iota} K_2$ of locally ringed spaces such that $\kappa_2 \iota = \kappa_1$. Then there is a bijection from the closed immersions up to equivalence and the sheaves of ideals $I \subseteq \mathcal{O}_X$ by*

$$(K \xrightarrow{\kappa} X)/_\sim \longmapsto I = \ker(\mathcal{O}_X \xrightarrow{\kappa^*} \kappa_* \mathcal{O}_K) \ .$$

Remark. When X is a prescheme, K will fail to be a prescheme unless I is quasi-coherent.

A. Useful stuff from category theory

A.1. Fundamental concepts

Definition 1 (Category)**.** A **category** $\mathcal{A}$ is a class $\mathrm{Ob}(\mathcal{A})$ of *objects* of $\mathcal{A}$ together with:

(a) For two arbitrary $X, Y \in \mathrm{Ob}(\mathcal{A})$, a set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ of *morphisms* from X to Y in $\mathcal{A}$.

(b) For $X, Y, Z \in \mathrm{Ob}(\mathcal{A})$, a map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Z) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

called the *composition of morphisms* in $\mathcal{A}$.

The following assumptions must be satisfied:

(i) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ are morphisms in $\mathcal{A}$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) For any $A \in \mathrm{Ob}(\mathcal{A})$ there is a unique morphism $A \xrightarrow{\mathrm{id}_A} A$ in $\mathrm{Hom}_{\mathcal{A}}(A, A)$ such that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ for any $f \in \mathrm{Hom}_{\mathcal{A}}(B, A)$ and any $g \in \mathrm{Hom}_{\mathcal{A}}(A, C)$ for arbitrary $B, C \in \mathrm{Ob}(\mathcal{A})$.

Example 1.

- The category Set of sets where $\mathrm{Ob}(\mathcal{A})$ is the class of sets, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is the set of maps from X to Y and the composition of morphism is the composition of maps.
- The category Grp of groups where $\mathrm{Ob}(\mathcal{A})$ is the class of groups, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is the set of group morphisms from X to Y and the composition of morphisms is the composition of maps.
- The categories of rings Ring, commutative rings Ab and abelian groups Ab are all defined similarly.
- The topological spaces with the continuous maps Top.
- The Banach spaces with bounded (continuous) maps.
- The k-vector spaces with k-linear maps Vect_K or R-modules with R-linear maps R-Mod.

A category is called *small* if its class of objects is a set.

Definition 2. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We call $\mathcal{B}$

- a **subcategory** of $\mathcal{A}$ if $\mathrm{Ob}(\mathcal{B}) \subseteq \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{B}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and for arbitrary objects X, Y of $\mathcal{B}$, the identity id_X of X in $\mathcal{A}$ is a morphism in $\mathrm{Hom}_{\mathcal{B}}(X, X)$.

- a **full subcategory** of $\mathcal{A}$ if additionally $\mathrm{Hom}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{B}}(X,Y)$ for arbitrary objects $X, Y \in \mathrm{Ob}(\mathcal{B})$.
- an **equivalent subcategory** of $\mathcal{A}$ if it is a full subcategory and every object $X \in \mathrm{Ob}(\mathcal{A})$ is isomorphic to some $Y \in \mathrm{Ob}(\mathcal{B})$ (where a morphism $X \xrightarrow{f} Y$ is an isomorphism iff there is a (unique) $Y \xrightarrow{g} X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

A further example is the category of prevarieties (with morphisms according to Definition 1.1.2) and its full subcategory of varieties, containing all varieties as objects.

A.2. Functors, functor morphisms, and the Yoneda lemma

Definition 1. A (covariant) **functor** $\mathcal{A} \xrightarrow{F} \mathcal{B}$ consists of the following data:

- a map $F\colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$,
- for $X, Y \in \mathrm{Ob}(\mathcal{A})$ a map $F\colon \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ such that $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ whenever $X \in \mathrm{Ob}(\mathcal{A})$ and $F(\psi\varphi) = F(\psi)F(\varphi)$ when $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ are morphisms in $\mathcal{A}$.

A functor is called *faithful* if $F : \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is injective and *fully faithful* if it's bijective.

Professor Franke would like to draw a line here. We hope, this one will do: ____________.

Example. We have *forgetful* functors like $\mathrm{Vect}_k \to \mathrm{Ab}$ or $\mathrm{Ab} \to \mathrm{Set}$. These functors are faithful (at least for the two examples).

Remark.

- If $\mathcal{A} \xrightarrow{F} \mathrm{Set}$ is a faithful functor, then any morphism f such that Ff is injective (respectively surjective) then f is a monomorphism (respectively epismorphism).
- In the categories of sets or R-modules, f is a monomorphism (epimorphism) iff f is injective (surjective).
- In the category of Banach spaces and bounded linear maps, f is a epimorphism iff its image is dense.
- The map from a domain to its field of quotients is an epimorphism in the category of rings.

Definition 2 (Canonical morphism)**.** Let $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ be functors. A **functor morphism** or **canonical morphism** from F to G is a collection of morphisms $F(A) \xrightarrow{\varphi_A} G(A)$, for all $A \in \mathrm{Ob}\,\mathcal{A}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(\alpha)"'] & G(A) \arrow[d, "G(\alpha)"] \\ F(B) \arrow[r, "\varphi_B"] & G(B) \end{tikzcd}$$

commutes for arbitrary morphisms $A \xrightarrow{\alpha} \mathcal{B}$ in $\mathcal{A}$. The functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ as objects with canonical morphisms as morphisms thus form a category, the *functor category* $\mathrm{Funct}(\mathcal{A}, \mathcal{B})$. Also cf. [1, page 26].

Definition 3 (Adjoint functors)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An **adjoint pair of functors**

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$$

is a pair of functors of the specified type together with an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(LX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X, RY)$$

which in functorial in X and Y. We may say that L is left-adjoined to R and that R is right adjoined to L. By Yoneda-style arguments, RY is uniquely determined (up to unique morphism) by L (and Y) if it exists, and LX is uniquely determined by R (and X) if it exists.

Definition 4. (*a*) If $\mathcal{A}$ is a category, its **opposite** or **dual category** $\mathcal{A}^{\mathrm{op}}$ is defined by $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, X)$.

(*b*) A **contravariant functor** $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. In other words, it consists of maps $\mathrm{Ob}(\mathcal{A}) \xrightarrow{F} \mathrm{Ob}(\mathcal{B})$ and $\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(FY, FX)$ such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(\beta\alpha) = F(\alpha)F(\beta)$ where $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ are composable morphisms in $\mathcal{A}$.

Remark.

- Equivalently, a contravariant functor $\mathcal{A} \to \mathcal{B}$ is a covariant functor $\mathcal{A} \to \mathcal{B}^{\mathrm{op}}$.
- The map $V \to V^*$ (to the dual vector space) on the category of vector spaces over some fixed field k is an example.

Lemma 1 (Yoneda lemma for covariant functors)**.** *For any* $M \in \mathrm{Ob}\,\mathcal{A}$*, the map* $\mathrm{Cov}_M(X) = \mathrm{Hom}_{\mathcal{A}}(M, X)$ *defines a covariant functor* $\mathrm{Cov}_M = \mathrm{Hom}_{\mathcal{A}}(M, -)\colon \mathcal{A} \to \mathrm{Set}$ *and*

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \xrightarrow{\mathcal{Y}} \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A},\mathrm{Set})}(\mathrm{Cov}_N, \mathrm{Cov}_M),$$

is bijective. The image $\mathcal{Y}(M \xrightarrow{\mu} N)$ *of* $\mu \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ *under the Yoneda map* $\mathcal{Y}$ *maps* $N \xrightarrow{\xi} X$ *to* $M \xrightarrow{\mu} N \xrightarrow{\xi} X$*.*

Lemma 2 (Yoneda lemma, contravariant case)**.** *For any* $M \in \mathrm{Ob}\,\mathcal{A}$*,* $\mathrm{Con}_M(X) = \mathrm{Hom}_{\mathcal{A}}(X, M)$ *defines a contravariant functor* $\mathrm{Con}_M = \mathrm{Hom}_{\mathcal{A}}(-, M)\colon \mathcal{A} \to \mathrm{Set}$ *and*

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \xrightarrow{\mathcal{Y}} \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A}^{\mathrm{op}},\mathrm{Set})}(\mathrm{Con}_M, \mathrm{Con}_N)$$

is bijective. The image $\mathcal{Y}(M \xrightarrow{\mu} N)$ *of* $\mu \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ *under the Yoneda map* $\mathcal{Y}$ *maps* $X \xrightarrow{\xi} M \in \mathrm{Con}_M(X)$ *to* $X \xrightarrow{\xi} M \xrightarrow{\mu} N \in \mathrm{Con}_N(X)$*.*

Definition 5. (*a*) We say that a functor $\mathcal{A} \xrightarrow{F} \mathrm{Set}$ is **representable** if there is a functor-isomorphism $F \simeq \mathrm{Cov}_M$, i.e. $F(X) \simeq \mathrm{Hom}_{\mathcal{A}}(M, X)$ for all $X \in \mathrm{Ob}\,\mathcal{A}$. In this case, we say that M *represents* F. The Yoneda-Lemma shows that M is unique up to unique isomorphism.

(*b*) We say that a contravariant functor $\mathcal{A} \xrightarrow{F} \mathrm{Set}$ is **representable** iff $F \simeq \mathrm{Con}_M$ for some $M \in \mathrm{Ob}\,\mathcal{A}$ in which case we say that M *represents* F.

Remark. (*a*) For instance, LX exists in Definition 3 iff the covariant functor

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X, R(-)) \colon \mathcal{B} &\longrightarrow \mathrm{Set} \\ Y &\longmapsto \mathrm{Hom}_{\mathcal{A}}(X, RY) \end{aligned}$$

is representable.

(*b*) In the above Definition 3, RY exists iff

$$\begin{aligned} \mathrm{Hom}_{\mathcal{B}}(L(-), Y) \colon \mathcal{A}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \mathrm{Hom}_{\mathcal{B}}(LX, Y) \end{aligned}$$

is representable.

(*c*) Many important constructions in algebraic topology (classifying spaces) and algebraic geometry (Moduli space, Picard/Hilbert schemes) amount to representing certain functors.

Lemma 3 (Yoneda Lemma, 3rd try or something)**.** *Let* $\mathrm{Hom}_{\mathrm{Funct}(\mathcal{A},\mathcal{B})}(F, G)$ *denote the "class" of functor-morphisms between covariant functors* $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$.

(*a*) *The Yoneda map*

$$\begin{aligned} \mathcal{Y} \colon \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A},\mathrm{Set})}(\mathrm{Cov}_M, F) &\longrightarrow F(M) \\ \alpha &\longmapsto \alpha_M(\mathrm{id}_M) \\ F(-)(\mu) = \alpha &\longleftarrow\!\shortmid \mu \end{aligned}$$

is bijective. Herein, $\alpha = F(-)(\mu)$ *should be read as follows: A morphism* $(M \xrightarrow{\xi} X) \in \mathrm{Cov}_M(X)$ *induces a morphism* $F(M) \xrightarrow{F(\xi)} F(X)$*. Evaluating at* $\mu \in F(M)$*, we get an element* $F(\xi)(\mu) \in F(X)$*. Now,* $\mathrm{Cov}_M \xrightarrow{\alpha} F$ *is given by*

$$\begin{aligned} \alpha_X \colon \mathrm{Cov}_M(X) &\longrightarrow F(X) \\ \xi &\longmapsto F(\xi)(\mu)\ . \end{aligned}$$

(*b*) *The Yoneda map*

$$\begin{aligned} \mathcal{Y} \colon \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A}^{\mathrm{op}},\mathrm{Set})}(\mathrm{Con}_M, F) &\longrightarrow F(M) \\ \alpha &\longmapsto \alpha_M(\mathrm{id}_M) \\ F(-)(\mu) = \alpha &\longleftarrow\!\shortmid \mu \end{aligned}$$

is bijective. Herein, $\alpha = F(-)(\mu)$ *should be read as follows: A morphism* $(X \xrightarrow{\xi} M) \in \mathrm{Con}_M(X)$ *induces a morphism* $F(M) \xrightarrow{F(\xi)} F(X)$ *(remember that this time* F *is contravariant). Evaluating at* $\mu \in F(M)$*, we get an element* $F(\xi)(\mu) \in F(X)$*. Now,* $\mathrm{Con}_M \xrightarrow{\alpha} F$ *is given by*

$$\begin{aligned} \alpha_X \colon \mathrm{Con}_M(X) &\longrightarrow F(X) \\ \xi &\longmapsto F(\xi)(\mu) \, . \end{aligned}$$

Proof. We prove only (a), (b) being the dual assertion. The well-definedness of $\mapsto$ is obvious. For $\leftarrow\!\!\shortmid$ is is necessary to show that α is a natural transformation. For this, let $X \xrightarrow{\nu} Y$ be a morphism in $\mathcal{A}$ and $\xi \in \mathrm{Cov}_M(X)$, we have to show that

$$\begin{tikzcd} \mathrm{Cov}_M(X) \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{Cov}_M(\nu)"'] & F(X) \arrow[d, "F(\nu)"] \\ \mathrm{Cov}_M(Y) \arrow[r, "\alpha_Y"] & F(Y) \end{tikzcd}$$

commutes. Let $M \xrightarrow{\xi} X$. As F is a functor and $M \xrightarrow{\xi} X \xrightarrow{\nu} Y$, we have $F(\nu)F(\xi) = F(\nu\xi)$. Also, $\mathrm{Cov}_M(\nu)(\xi) = \nu\xi \in \mathrm{Cov}_M(Y)$. Hence,

$$F(\nu)\big(\alpha_X(\xi)\big) = F(\nu)F(\xi)(\mu) = F(\nu\xi)(\mu) = F\big(\mathrm{Cov}_M(\nu)(\xi)\big)(\mu) = \alpha_Y\big(\mathrm{Cov}_M(\nu)(\xi)\big)$$

and that's exactly what we wanted to show.

To show that the maps are inverse to each other first look at $\mu \in F(M)$ and its image $\alpha \in \mathrm{Hom}_{\mathrm{Funct}(\mathcal{A},\mathrm{Set})}(\mathrm{Cov}_M, F)$. Then we have $\alpha_M(\mathrm{id}_M) = F(\mathrm{id}_M)(\mu) = \mathrm{id}_{F(M)}(\mu) = \mu$. Conversely, let $\beta \colon \mathrm{Cov}_M \to F$ be a natural transformation and $\mu = \beta_M(\mathrm{id}_M)$ and let α be defined by μ as above. Let $M \xrightarrow{\xi} X$. As β is a natural transformation, the diagram

$$\begin{tikzcd} \mathrm{Cov}_M(M) \arrow[r, "\beta_M"] \arrow[d, "\mathrm{Cov}_M(\xi)"'] & F(X) \arrow[d, "F(\xi)"] \\ \mathrm{Cov}_M(X) \arrow[r, "\beta_X"] & F(Y) \end{tikzcd}$$

commutes, i.e. $F(\xi)\beta_M = \beta_X \,\mathrm{Cov}_M(\xi)$. Hence,

$$\alpha_X(\xi) = F(\xi)(\mu) = F(\xi)\big(\beta_M(\mathrm{id}_M)\big) = \beta_X\big(\mathrm{Cov}_M(\xi)(\mathrm{id}_M)\big) = \beta_X(\xi) \, ,$$

and thus $\alpha = \beta$, as we wanted to show. *q.e.d.*

Putting $F = \mathrm{Cov}_N$ respectively $F = \mathrm{Con}_N$ we obtain the above special cases.

A.3. Products and (co-)equalizers

Our goal is to formulate a notion of *sheaf* that allows to define sheafs for arbitrary target categories. Recall that the sheaf axiom (this one) requires

$$\mathcal{G}(U) \longrightarrow \left\{ (g_i)_{i\in I} \in \prod_{i\in I} \mathcal{G}(U_i) \;\middle|\; g_i|_{U_{ij}} = g_j|_{U_{ij}} \ \forall i,j \in I \right\} \tag{$*$}$$

to be bijective (i.e. an isomorphism) for any open cover $U = \bigcup_{i\in I} \mathcal{G}(U_i)$, where we set $U_{ij} = U_i \cap U_j$ for convenience.

Definition 1 (Products)**.** A **product** of objects $(A_i)_{i\in I}$ of $\mathcal{A}$ is an object $\prod_{i\in I} A_i$ together with morphisms $\prod_{i\in I} A_i \xrightarrow{\pi_i} A_i$ for each $i \in I$ such that the following *universal property* holds:

> If $T \in \mathrm{Ob}(\mathcal{A})$ comes with morphisms $T \xrightarrow{\tau_i} A_i$ for each $i \in I$, then there is a unique morhism $T \xrightarrow{\exists!\ f} \prod_{i\in I} A_i$ such that $\tau_i = \pi_i \circ f$.

In other words, $\prod_{i\in I} A_i = \varprojlim_{i\in I} A_i$ is the *limiting cone* over the trivial diagram consisting only of the A_i without any arrows.

Note that nobody ever guaranteed that products exist in general.

Remark. (*a*) Compare this to e.g. the universal property of localizations where the universal object is on the left – here it is on the right.

(*b*) In the case of sets, (abelian) groups, R-modules, and rings, one can take $\prod_{i\in I} A_i$ to be the set-theoretic product equipped with the respective product structure and the set-theoretic projections $\prod_{i\in I} A_i \xrightarrow{\pi_i} A_i$, $\pi_i\big((a_j)_{j\in I}\big) = a_i$.

(*c*) The above definition characterizes $\prod_{i\in I} A_i$ up to *unique isomorphism*: If the T above satisfies the same universal property, then f is an isomorphism.

(*d*) If $I = \emptyset$, the empty product is the *final object*, i.e. an object $F \in \mathrm{Ob}(\mathcal{A})$ such that for any $T \in \mathrm{Ob}(\mathcal{A})$ there is precisely one morphism $T \to F$ in $\mathcal{A}$. We have the following dual notion: I (now an object and no indexing set anymore) is called *initial object* if $\mathrm{Hom}_{\mathcal{A}}(I, T)$ has precisely one element for each $T \in \mathrm{Ob}(\mathcal{A})$.

Remark. In Set, $\emptyset$ is the only initial object and the one-point sets are the final objects. For the *abelian* categories R-Mod, the canonical morphisms from the (only) initial to the (only) final object is an isomorphism.

Let X be a topological space. A *presheaf* on X with values in $\mathcal{A}$ is a map associating

- to each open subset $U \subseteq X$ an object $\mathcal{G}(U) \in \mathrm{Ob}(\mathcal{A})$
- and to each inclusion $V \subseteq U$ a morphism $(-)|_V^U \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(U), \mathcal{G}(V))$ which equals $\mathrm{id}_{\mathcal{G}(U)}$ if $U = V$ and such that $(-)|_W^U = (-)|_W^V \circ (-)|_V^U$ whenever $W \subseteq V \subseteq U$ is an inclusion of open sets.

To formulate the sheaf axiom, it is convenient to assume that $\prod_{i\in I}\mathcal{G}(U_i)$ exists, i.e. that $\mathcal{A}$ has arbitrary products.

Recall our convention that $U_{ij} = U_i \cap U_j$. There are unique morphisms

$$\alpha, \beta\colon \prod_{i\in I}\mathcal{G}(U_i) \longrightarrow \prod_{(i,j)\in I\times I}\mathcal{G}(U_{ij})$$

characterized by $\pi_{ij}\circ\alpha = (-)|^{U_i}_{U_{ij}}\circ\pi_i$ and $\pi_{ij}\circ\beta = (-)|^{U_j}_{U_{ij}}\circ\pi_j$, where $\prod_{i\in I}\mathcal{G}(U_i)\xrightarrow{\pi_i}\mathcal{G}(U_i)$ and $\prod_{(i,j)\in I\times I}\mathcal{G}(U_{ij})\xrightarrow{\pi_{ij}}\mathcal{G}(U_{ij})$ are the morphisms defining the product structure.

In the example of presheaves of sets, rings, etc. we have

$$\alpha((g_i)_{i\in I}) = \left(g_i|_{U_{ij}}\right)_{(i,j)\in I\times I} \quad\text{and}\quad \beta((g_i)_{i\in I}) = \left(g_j|_{U_{ij}}\right)_{(i,j)\in I\times I}\ ,$$

such that $(*)$ is the "largest subobject on which α and β coincide".

Definition 2 (Equalizer)**.** Let $A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$ be a pair of morphisms. An **equalizer** of this pair is an object $E\in\mathrm{Ob}(\mathcal{A})$ together with a morphism $E\xrightarrow{\varepsilon}A$ such that $\alpha\varepsilon = \beta\varepsilon$ and such that the following universal property holds:

> If $T\xrightarrow{\tau}A$ is any morphism in $\mathcal{A}$ such that $\alpha\tau = \beta\tau$, then there's a unique $T\xrightarrow{t}E$ such that the following diagram commutes.
>
> $$\begin{tikzcd}
 & T \arrow[d, dashed, "\exists!\, t"] \arrow[ddl, bend right, "\tau"'] \arrow[ddr, bend left] & \\
 & E \arrow[dl, "\varepsilon"'] \arrow[dr] & \\
A \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\beta"'] & & B
\end{tikzcd}$$
>
> In other words, E is the *limit* over the diagram $A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$.

Remark. (*a*) By the usual Yoneda argument one sees that the universal property characterizes E up to unique isomorphism.

(*b*) The sheaf axiom for presheaves with values in an arbitrary category with products now translates into the condition that $\mathcal{G}(U)\to\prod_{i\in I}\mathcal{G}(U_i)$ is, for any open cover $U = \bigcup_{i\in I}U_i$, an equalizer of the above pair of morphisms $\prod_{i\in I}\mathcal{G}(U_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{(i,j)\in I\times I}\mathcal{G}(U_{ij})$.

By merging the universal properties, the notion of a sheaf can be generalized to arbitrary target categories.

(*c*) For sets, abelian groups etc. the equalizer is

$$\ker\left(A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B\right) = \{a\in A \mid \alpha(a) = \beta(a)\}\ .$$

(*d*) For the *abelian* category R-Mod,

$$\ker\left(M \xrightarrow{f} N\right) = \ker\left(M \underset{0}{\overset{f}{\rightrightarrows}} N\right) \quad \text{and} \quad \ker\left(M \underset{g}{\overset{f}{\rightrightarrows}} N\right) = \ker\left(M \xrightarrow{f-g} N\right).$$

Definition 3 (Mono- and epimorphism)**.** Let $\mathcal{A}$ be a category, $X \xrightarrow{f} Y$ a morphism in $\mathcal{A}$. We say that f is **monomorphism** (**epimorphism**) if for any object T and any pair $T \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$ (respectively $Y \underset{\beta}{\overset{\alpha}{\rightrightarrows}} T$) with $f\alpha = f\beta$ (respectively $\alpha f = \beta f$) we get $\alpha = \beta$.

Definition 4 (Coequalizer)**.** Moreover, $Y \xrightarrow{c} C$ is a **coequalizer** in $\mathcal{A}$ (of $X \underset{b}{\overset{a}{\rightrightarrows}} Y$) iff $C^{\mathrm{op}} \xrightarrow{c^{\mathrm{op}}} X^{\mathrm{op}}$ is an equalizer of $Y^{\mathrm{op}} \underset{b^{\mathrm{op}}}{\overset{a^{\mathrm{op}}}{\rightrightarrows}} X^{\mathrm{op}}$ in $\mathcal{A}^{\mathrm{op}}$. In other words,

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}(C,T) &\xrightarrow{\sim} \{g \in \operatorname{Hom}_{\mathcal{A}}(Y,T) \mid ga = gb\} \\ f &\longmapsto fc \end{aligned}$$

must be bijective. In even more other words C is a coequalizer iff it fulfills the following universal property:

> If $Y \xrightarrow{\tau} T$ is any morphism in $\mathcal{A}$ such that $\tau a = \tau b$, then there's a unique $C \xrightarrow{t} T$ such that the following diagram commutes.
>
> $$\begin{tikzcd} X \arrow[rr, bend left, "a"] \arrow[rr, bend right, "b"'] \arrow[dr] \arrow[ddr, bend right] & & Y \arrow[dl, "c"] \arrow[ddl, bend left, "\tau"] \\ & C \arrow[d, dashed, "\exists!\ t"] & \\ & T & \end{tikzcd}$$
>
> In yet another other words, C is the *colimit* below the diagram $X \underset{b}{\overset{a}{\rightrightarrows}} Y$.

Fact 1. Let $A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$ be a ring morphism. Then a coequalizer is given by B/I where I is the ideal generated by $\{\alpha(a) - \beta(a) \mid a \in A\}$. When A, B are local rings and α, β local ring morphisms, this is a local ring iff α and β induce the same morphism on the residue fields, and $\{0\}$ otherwise.

Let us denote this by $\operatorname{Coeq}\left(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\right)$.

Proof. Let $C = B/I \overset{\pi}{\twoheadleftarrow} B$. Obviously, $\pi\alpha = \pi\beta$. Let $B \xrightarrow{t} T$ be a ring homomorphism such that $t\alpha = t\beta$. Then $t(\alpha(a) - \beta(a)) = 0$ hence $\{\alpha(a) - \beta(a) \mid a \in A\} \subseteq \ker(t)$ hence $I \subseteq \ker(t)$ as $\ker(t)$ is an ideal, and t has a unique representation as $t = \tau\pi$ for $C \xrightarrow{\tau} T$, by the universal property of the quotient ring.

The second assertion is also easy: Denote the maximal ideals by $\mathfrak{m}_A$ and $\mathfrak{m}_B$, then I is a proper ideal iff $\alpha(a) - \beta(a) \in \mathfrak{m}_B$ for all $a \in A$, i.e. if the induced morphisms $\mathfrak{K}(\mathfrak{m}_A) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathfrak{K}(\mathfrak{m}_B)$ coincide. In this case, B/I is local with maximal ideal $\mathfrak{m}_B/I$, and $\{0\}$ otherwise. *q.e.d.*

A.4. Fibre products

Definition 1 (Fibre product)**.** Given morphisms $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$ in some category $\mathcal{A}$, a **fibre product** of X with Y over S is an object $P \in \mathrm{Ob}(\mathcal{A})$ with morphisms $X \xleftarrow{p} P \xrightarrow{q} Y$ satisfying $\xi p = \upsilon q$ and enjoying the following universal property:

> Whenever $T \in \mathrm{Ob}(\mathcal{A})$ comes with morphisms $X \xleftarrow{\tau_X} T \xrightarrow{\tau_Y} Y$ such that $\xi\tau_X = \upsilon\tau_Y$, then there is a unique morphism $T \xrightarrow{t} P$ such that the below diagram commutes.
>
> $$\begin{tikzcd} T \arrow[dr, dashed, "\exists!\, t"] \arrow[drr, bend left, "\tau_Y"] \arrow[ddr, bend right, "\tau_X"'] & & \\ & P \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\upsilon"] \\ & X \arrow[r, "\xi"] & S \end{tikzcd}$$
>
> That is, in terms of limits, P is the *limit* over the diagram $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$.

Often we will write $P = X \times_S Y$. By the typical Yoneda-style argument the fibre product is unique up to unique isomorphism.

Remark. (*a*) In other words, P must represent the functor

$$T \longmapsto \{(\tau_X, \tau_Y) \in \mathrm{Hom}_{\mathcal{A}}(T, X) \times \mathrm{Hom}_{\mathcal{A}}(T, Y) \mid \xi\tau_X = \upsilon\tau_Y\} \tag{1}$$

(*b*) If $S = F$ is a final object, then the equality $\xi\tau_X = \upsilon\tau_Y$ is automatic and we obtain $X \times_F Y = X \times Y$.

(*c*) If S is arbitrary and $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ is a product of X and Y in $\mathcal{A}$ then

$$\mathrm{Eq}\left(X \times Y \underset{\upsilon\pi_Y}{\overset{\xi\pi_X}{\rightrightarrows}} S\right) \simeq X \times_S Y.$$

(*d*) Let the *comma category* $\mathcal{A} \to S$ (or $\mathcal{A}/S$) be the category whose objects are pairs (X, ξ) with $X \in \mathrm{Ob}(\mathcal{A})$ an object and $X \xrightarrow{\xi} S$ a morphism in $\mathcal{A}$, and where a morphism $(X, \xi) \xrightarrow{f} (Y, \upsilon)$ is a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ such that we have $\upsilon f = \xi$, i.e.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\xi"'] & & Y \arrow[dl, "\upsilon"] \\ & S & \end{tikzcd}$$

commutes. Then $X \times_S Y$ is the product of (X, ξ) and (Y, υ) in $\mathcal{A} \to S$.

(*e*) Fibre products with any (possibly cardinal) number of factors can be defined in the same way or as products in $A \to S$.

(*f*) The smallest square in the diagram above illustrating the universal property of fibre products is called *cartesian* if P is a fibre product. If is also often useful to think of p as morphism such that the *fibre* $p^{-1}(x)$ is isomorphic to $\upsilon^{-1}(\xi(x))$. Then one prefers a terminology

$$\begin{tikzcd} P \arrow[r, "\xi'"] \arrow[d, "\upsilon'"'] & Y \arrow[d, "\upsilon"] \\ X \arrow[r, "\xi"] & S \end{tikzcd}$$

and calls υ' a *base change* of υ with respect to ξ and likewise ξ' a base change of ξ with respect to υ.

(*g*) In a number of easy cases (sets, (abelian) groups, (commutative) rings, topological spaces) fibre products may be constructed as

$$X \times_S Y = \{(x, y) \in X \times Y \mid \xi(x) = \upsilon(y)\},$$

equipped with component-wise operations $(x, y) \circ (\widetilde{x}, \widetilde{y}) = (x \circ \widetilde{x}, y \circ \widetilde{y})$ and (possibly) the induced topology from $X \times Y$.

Definition 2 (Cofibre product)**.** A **dual fibre product** or **cofibre product** or also **push-out** of morphisms $X \leftarrow B \to Y$ in some category $\mathcal{A}$ is a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "k"] \\ X \arrow[r, "j"] & P \end{tikzcd}$$

enjoying the universal property for such diagrams, that is:

Whenever $T \in \mathrm{Ob}(\mathcal{A})$ comes with morphisms $X \to T \leftarrow Y$ such that the left of the below diagrams commutes, there is a unique (dashed) morphism $P \to T$ in the right diagram making it commute.

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[dd, bend left] \\ X \arrow[dr, bend right] & \\ & T \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "k"] \arrow[dd, bend left] \\ X \arrow[r, "j"] \arrow[dr, bend right] & P \arrow[d, dashed, "\exists!"] \\ & T \end{tikzcd}$$

In colimited language, P is the *colimit* below the diagram $X \leftarrow B \to Y$.

Example 1. Let R be a ring, A and B be R-algebras. Then their tensor product $A \otimes_R B$ over R has a unique ring structure such that $(a \otimes b) \cdot (\alpha \otimes \beta) = (a \cdot \alpha) \otimes (b \cdot \beta)$. Then $A \to A \otimes_R \leftarrow B$ sending $a \in A$ to $a \otimes 1$ and $b \in B$ to $1 \otimes b$ are ring morphisms such that

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & B \arrow[d, "b \mapsto 1 \otimes b"] \\
A \arrow[r, "a \mapsto a \otimes 1"] & A \otimes_R B
\end{tikzcd}$$

is *cocartesian* in the category of rings, i.e. a cofibre product of A and B over R.

Bibliography

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