Algebraic Geometry II

Ferdinand Wagner

Sommersemester 2018

This text consists of notes on the lecture Algebraic Geometry II, taught at the University of Bonn by Professor Jens Franke in the summer term (Sommersemester) 2018.

Please report bugs, typos etc. through the *Issues* feature of github.

Contents

Introduction

This lecture will develop the cohomology of (quasi)coherent sheaves of modules. Professor Franke assumes familiarity with the contents of last term's Algebraic Geometry I. In particular, this includes the category of (pre)schemes, equalizers and fibre products of preschemes as well as in arbitrary categories and quasi-coherent $\mathcal{O}_X$-modules. If you are want to brush up your knowledge about these topics, the *lecture notes from Algebraic Geometry I* [1] might be your friend.

Professor Franke started the lecture with an example of sheaf cohomology entering the game. Let X be a topological space, $\mathcal{C}_X$ the sheaf of continuous $\mathbb{C}$-valued functions on X and $\underline{\mathbb{Z}}_X$ the sheaf of locally constant (i.e., continuous) functions on X with values in $\mathbb{Z}$. Then there is a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \xrightarrow{\cdot 2\pi \mathrm{i}} \mathcal{C}_X \xrightarrow{\exp} \mathcal{C}_X^\times \longrightarrow 0$$

of sheaves of abelian groups. In general, taking global section doesn't preserve exactness but gives rise to a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{Z}}_X(X) \arrow[r] \arrow[d, "\wr"] & \mathcal{C}_X(X) \arrow[r] \arrow[d, "\wr"] & \mathcal{C}_X^\times(X) \arrow[r, "d"] \arrow[d, "\wr"] & H^1(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^1(X, \mathcal{C}_X) \arrow[r] & \dots \\
0 \arrow[r] & H^0(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^0(X, \mathcal{C}_X) \arrow[r] & H^0(X, \mathcal{C}_X^\times) \arrow[ur] & & &
\end{tikzcd}$$

in which the $H^k(X, \underline{\mathbb{Z}}_X)$, $H^k(X, \mathcal{C}_X)$, and $H^k(X, \mathcal{C}_X^\times)$ are *sheaf cohomology groups.* There is the more general notion of *derived functors* (Grothendieck, Tôhoku paper), but this won't appear in the lecture.

Background in homological algebra is not required safe for cohomology groups of cochain complexes, the long exact cohomology sequence and the following famous lemma.

Lemma (Five lemma)**.** *Given a diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

of (abelian) groups/R-modules/etc. with exact rows, in which α, β, δ, and ε are isomorphisms, then γ is an isomorphism as well.

Proof. Easy diagram chase. *q.e.d.*

1. Cohomology of quasi-coherent sheaves of modules

1.1. Recollection of basic definitions and results

Definition 1 ([1, Definition 1.5.2 and Definition 1.5.9(b)])**.** (a) A **prescheme** (Franke uses "EGA termology") is a locally ringed space $(X, \mathcal{O}_X)$ which locally has the form $\operatorname{Spec} R$ for some rings R.

(b) A prescheme X is called a **scheme**, if, for any prescheme T and any pair of morphisms $T \overset{a}{\underset{b}{\rightrightarrows}} X$, the equalizer $\operatorname{Eq}\left(T \overset{a}{\underset{b}{\rightrightarrows}} X\right)$ is a closed subprescheme of X.

Remark. Equivalently, a prescheme X is a scheme iff the diagonal $\Delta\colon X \xrightarrow{(\mathrm{id}_X, \mathrm{id}_X)} X \times X$ is a closed immersion (cf. [1, Fact 1.5.8]). In other words, schemes are *separated* preschemes

Proposition 1. *If U and V are affine open subsets of a scheme X, then their intersection $U \cap V$ is again affine (and open of course).*

Proof. This was proved in [1, Proposition 1.5.4]. *q.e.d.*

A. Appendix – category theory corner

A.1. Towards abelian categories

Definition 1. (*a*) A **pointed** category is a category with initial and final objects, such that the canonical (unique) morphism from the initial to the final object is an isomorphism.

(*b*) An **additive** category $\mathcal{A}$ is a pointed category which has a product $X \times Y$ (i.e., a fibre product over the final object $*$) and coproduct $X \amalg Y$ (i.e., a dual fibre product with respect to the initial object $*$) such that the canonical morphism $X \amalg Y \to X \times Y$ is an isomorphism for all objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ and such that the resulting addition law on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ defines a group structure for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

Remark. (*a*) When $\mathcal{A}$ is a pointed category and $X, Y \in \mathrm{Ob}(\mathcal{A})$, let the *zero morphism* (which we denote 0) $X \xrightarrow{0} Y$ be defined by $X \to * \to Y$, where $*$ is the both initial and final object.

(*b*) We will construct the canonical morphism $X \amalg Y \xrightarrow{c} X \times Y$ from Definition 1(*b*). The product $X \times Y$ comes with canonical projections $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ such that given morphisms $T \xrightarrow{\xi} X$ and $T \xrightarrow{\upsilon} Y$ there is a unique $T \xrightarrow{\xi \times \upsilon} X \times Y$ such that

$$\begin{tikzcd}
X & & \\
& X \times Y \arrow[ul, "p_1"'] \arrow[dl, "p_2"] & T \arrow[l, dashed, "\exists!\ \xi \times \upsilon"'] \arrow[ull, bend right, "\xi"'] \arrow[dll, bend left, "\upsilon"] \\
Y & &
\end{tikzcd}$$

commutes.

Similarly, the coproduct $X \amalg Y$ has morphisms $X \xrightarrow{i_1} X \amalg Y \xleftarrow{i_2} Y$ such that given morphisms $X \xrightarrow{\xi} T$ and $Y \xrightarrow{\upsilon} T$ there is a unique morphism $X \amalg Y \xrightarrow{\xi \amalg \upsilon} T$ such that

$$\begin{tikzcd}
X \arrow[dr, "i_1"] \arrow[drr, bend left, "\xi"] & & \\
& X \amalg Y \arrow[r, dashed, "\exists!\ \xi \amalg \upsilon"] & T \\
Y \arrow[ur, "i_2"'] \arrow[urr, bend right, "\upsilon"'] & &
\end{tikzcd}$$

commutes.

Using the universal property of $X \times Y$, we get a unique morphism $X \xrightarrow{\alpha} X \times Y$ such that $p_1\alpha = \mathrm{id}_X$, $p_2\alpha = 0$ and a unique morphism $Y \xrightarrow{\beta} X \times Y$ such that $p_1\beta = 0$ and $p_2\beta = \mathrm{id}_Y$. Then

$$c\colon X \amalg Y \xrightarrow{\alpha\amalg\beta} X \times Y$$

is the morphism we are looking for. It is unique with the property that $p_1ci_1 = \mathrm{id}_X$, $p_1ci_2 = 0$, $p_2ci_1 = 0$, and $p_2ci_2 = \mathrm{id}_Y$.

(*c*) For abelian groups and modules over a ring, both $X \amalg Y$ and $X \times Y$ are given by $\{(x, y) \mid x \in X,\ y \in Y\}$ with component-wise operations and $p_1(x, y) = x$, $p_2(x, y) = y$, $i_1(x) = (x, 0)$, and $i_2(y) = (0, y)$.

(*d*) For an additive category $\mathcal{A}$, it follows that finite products $\prod_{i=1}^n X_i$ and coproducts $\coprod_{i=1}^n X_i$ (of some objects $X_1, \ldots, X_n \in \mathrm{Ob}(\mathcal{A})$) exist and are canonically isomorphic. We typically denote both by $\bigoplus_{i=1}^n X_i$ in that case.

(*e*) We would like to describe the addition on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. For a pair of morphisms $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ we denote the composition

$$X \xrightarrow{\mathrm{id}_X \times \mathrm{id}_X} X \oplus X \xrightarrow{a\amalg b} Y$$

by $a + b$. Then 0 is a neutral element and associativity holds, but the existence of inverse elements needs to be imposed to obtain indeed a group structure.

(*f*) It is, however, automatically abelian. What we need to show is $(a\amalg b)\circ\Delta = (b\amalg a)\circ\Delta$ with $\Delta = \mathrm{id}_X \times \mathrm{id}_X$. The universal property of coproducts gives a unique $X \oplus X \xrightarrow{\sigma} X \oplus X$ such that

$$\begin{tikzcd}
 & X \arrow[dl, "i_2"'] \arrow[d, "i_1"] \arrow[drr, bend left, "a"] & & \\
X \oplus X \arrow[r, dashed, "\exists!\,\sigma"] & X \oplus X \arrow[rr, "a \amalg b"] & & Y \\
 & X \arrow[ul, "i_1"] \arrow[u, "i_2"'] \arrow[urr, bend right, "b"'] & &
\end{tikzcd}$$

commutes. Then σ is easily seen to be an isomorphism and $b \amalg a = (a \amalg b) \circ \sigma$ by the uniqueness of $b \amalg a$. It thus suffices to show $\sigma\Delta = \Delta$. By the uniqueness of Δ, this is equivalent to $p_1\sigma\Delta = \mathrm{id}_X$ and $p_2\sigma\Delta = \mathrm{id}_X$. We claim that $p_1\sigma = p_2$ and vice versa, which would finish the proof. To see this, note that $p_1\sigma = p_2$ is equivalent to $p_1\sigma i_1 = p_2 i_1 = 0$ and $p_1\sigma i_2 = p_2 i_2 = \mathrm{id}_X$ by the universal property of the coproduct $X \oplus X$. This follows from $\sigma i_1 = i_2$ and $\sigma i_2 = i_1$ by definition of σ.

Example. The following are additive categories.

(*a*) Modules over a given ring R (in particular, abelian groups).

(*b*) Sheaves of modules.

(*c*) Banach spaces with bounded linear maps as morphisms. The common initial and final object is the zero space and $A \oplus B = \{(a,b) \mid a \in A,\ b \in B\}$ with $\max\{\|a\|, \|b\|\}$ or $\|a\| + \|b\|$ as norm (this category will turn out not to be abelian).

(*d*) Free or projective modules over a ring R.

Remark 1. For kernels and cokernels in an additive category $\mathcal{A}$, the following universal properties are imposed. The kernel $\ker(A \xrightarrow{\alpha} B)$ must come with a morphism $\ker(\alpha) \xrightarrow{\iota} A$ and satisfy

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}\Big(T, \ker\Big(A \xrightarrow{\alpha} B\Big)\Big) &\xrightarrow{\sim} \{f \in \mathrm{Hom}_{\mathcal{A}} \mid \alpha f = 0\} \\ \Big(T \xrightarrow{\tau} A\Big) &\longmapsto f = \iota\tau \end{aligned}$$

for any test object $T \in \mathrm{Ob}(\mathcal{A})$. Similarly, cokernels come with a morphism $B \xrightarrow{\pi} \mathrm{coker}(\alpha)$ and satisfy

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}\Big(\mathrm{coker}\Big(A \xrightarrow{\alpha} B\Big), T\Big) &\xrightarrow{\sim} \{g \in \mathrm{Hom}_{\mathcal{A}}(B,T) \mid g\alpha = 0\} \\ \Big(\mathrm{coker}(\alpha) \xrightarrow{\tau} T\Big) &\longmapsto g = \tau\pi \end{aligned}$$

for any test object $T \in \mathrm{Ob}(\mathcal{A})$.

Thus,

$$\ker\Big(A \xrightarrow{\alpha} B\Big) = \mathrm{Eq}\Big(A \underset{0}{\overset{\alpha}{\rightrightarrows}} B\Big) \quad \text{and} \quad \mathrm{coker}\Big(A \xrightarrow{\alpha} B\Big) = \mathrm{Coeq}\Big(A \underset{0}{\overset{\alpha}{\rightrightarrows}} B\Big).$$

For abelian categories, the existence of kernels and cokernels is required, in addition to additivity. Equivalent conditions are the existence of equalizers and coequalizers, fibre products and dual fibre products, or the existence of finite limits and colimits as

$$\mathrm{Eq}\Big(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\Big) = \ker\Big(A \xrightarrow{\alpha-\beta} B\Big) \quad \text{and} \quad \mathrm{Coeq}\Big(A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\Big) = \mathrm{coker}\Big(A \xrightarrow{\alpha-\beta} B\Big)$$

(the minus here is the one obtained from additivity of $\mathcal{A}$).

Definition 2. A morphism $A \xrightarrow{i} B$ is an *effective monomorphism*, if the following equivalent conditions hold.

(*a*) (In any category) We have a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(T,A) &\xrightarrow{\sim} \left\{ f \in \mathrm{Hom}_{\mathcal{A}}(T,B) \;\middle|\; \begin{array}{c} \alpha f = \beta f \text{ if } B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} S \text{ is any pair of} \\ \text{morphisms such that } \alpha i = \beta i \end{array} \right\} \\ t \in \mathrm{Hom}_{\mathcal{A}}(T,A) &\longmapsto f = it\,. \end{aligned}$$

(*b*) (If the category has finite colimits) i is an equalizer of something.

(*c*) (In additive categories with kernels and cokernels) i is the kernel of an appropriate morphism.

(*d*) (In additive categories with kernels and cokernels) i is the kernel of its cokernel.

Definition 2a. Dually, $A \xrightarrow{p} B$ is an *effective epimorphism* if the following equivalent conditions hold.

(*a*) (In any category) We have a bijection

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}(B,T) &\xrightarrow{\sim} \left\{ f \in \operatorname{Hom}_{\mathcal{A}}(A,T) \;\middle|\; \begin{array}{c} f\alpha = f\beta \text{ if } S \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \text{ is any pair of} \\ \text{morphisms such that } p\alpha = p\beta \end{array} \right\} \\ t \in \operatorname{Hom}_{\mathcal{A}}(B,T) &\longmapsto f = tp\ . \end{aligned}$$

(*b*) (If the category has finite limits) p is a coequalizer of something.

(*c*) (In additive categories with kernels and cokernels) p is the cokernel of an appropriate morphism.

(*d*) (In additive categories with kernels and cokernels) p is the cokernel of its kernel.

(*e*) $B^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} A^{\mathrm{op}}$ is an effective monomorphism in the dual category $\mathcal{A}^{\mathrm{op}}$.

In any category, a morphism which is mono and effectively epi (or epi and effectively mono) is an isomorphism, but there are examples of morphisms which are simultaneously mono and epi but not an isomorphism (e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category of rings). This needs to be ruled out by a definition, and that's what is happening now!

Definition 3. A category $\mathcal{A}$ is **abelian**, if it is additive, has kernels and cokernels and such that every monomorphism is effectively mono, every epimorphism is effectively epi, and (thus) any morphism which is both a mono- and an epimorphism is an isomorphism.

The category of modules (over a ring R) or sheaves of modules are abelian categories, but not Banach spaces or projective modules over most rings.

Proposition 1. *The category* $\mathcal{R}$-Mod *of sheaves of modules (over a sheaf of rings* $\mathcal{R}$ *on some topological space* X*) is abelian.*

For clarity (and to better distinguish between the proof and Professor Franke's remarks about it), we will chop the proof into some lemmas.

Lemma 1. *The category* $\mathcal{R}$-Mod *is additive.*

Proof. First note that the zero sheaf 0 is a common initial and final object. A direct sum of $\mathcal{M}, \mathcal{N} \in \mathrm{Ob}(\mathcal{R}\text{-Mod})$ is given by

$$(\mathcal{M} \oplus \mathcal{N})(U) = \{(m,n) \mid m \in \mathcal{M}(U), n \in \mathcal{N}(U)\} \quad \text{for all } U \subseteq X \text{ open}$$

(it's clear that this is a presheaf and it inherits the sheaf axiom from $\mathcal{A}$ and $\mathcal{N}$) with componentwise module operations and with $\mathcal{M} \xleftarrow{p} \mathcal{M} \oplus \mathcal{N} \xrightarrow{q} \mathcal{N}$ and $\mathcal{M} \xrightarrow{i} \mathcal{M} \oplus \mathcal{N} \xleftarrow{j} \mathcal{N}$ given by $p(m,n) = m$, $q(m,n) = n$, $i(m) = (m,0)$, and $j(n) = (0,n)$ on open subsets $U \subseteq X$ and $m \in \mathcal{M}(U)$, $n \in \mathcal{N}(U)$.

If $\mathcal{M} \xrightarrow{\mu} \mathcal{T} \xleftarrow{\nu} \mathcal{N}$ are given, $\mathcal{M} \oplus \mathcal{N} \xrightarrow{\mu \amalg \nu} \mathcal{T}$ sending $(m, n) \in (\mathcal{M} \oplus \mathcal{N})(U)$ to $\mu(m) + \nu(n)$ verifies the universal property of the coproduct for $\mathcal{M} \oplus \mathcal{N}$. Similarly, $\mathcal{T} \xrightarrow{\mu \times \nu} \mathcal{M} \oplus \mathcal{N}$ given by $(\mu \times \nu)(t) = (\mu(t), \nu(t))$ for $t \in \mathcal{T}(U)$ confirms the universal property of the product for $\mathcal{M} \oplus \mathcal{N}$. Also, $c = \mathrm{id}_{\mathcal{M} \oplus \mathcal{N}}$ is the unique endomorphism c of that object such that $pci = \mathrm{id}_{\mathcal{M}}$, $qcj = \mathrm{id}_{\mathcal{N}}$, $pcj = 0$, and $qci = 0$. Thus, $\mathcal{R}$-Mod is additive (the group structure on Hom sets being easily verified). *q.e.d.*

Lemma 2. *The category* $\mathcal{R}$-Mod *has kernels.*

Proof. Let $\mathcal{M} \xrightarrow{f} \mathcal{N}$ be a morphism of sheaves of $\mathcal{R}$-modules and $\mathcal{K}$ be the sheaf given by

$$\mathcal{K}(U) = \ker\left(\mathcal{M} \xrightarrow{f} \mathcal{N}\right)(U) := \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right)$$

(you should convince yourself that this indeed satisfies the sheaf axiom). Then the inclusion $\mathcal{K} \xrightarrow{\kappa} \mathcal{M}$ is a monomorphism as $\mathcal{K}(U) \hookrightarrow \mathcal{M}(U)$ is injective for every open subset $U \subseteq X$.

If $\mathcal{T} \xrightarrow{\tau} \mathcal{M}$ is a morphism of $\mathcal{R}$-modules such that $f\tau = 0$, then, for every $t \in \mathcal{T}(U)$, we have $f(\tau(t)) = 0$, hence $\widetilde{\tau}(t) := \tau(t) \in \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) = \mathcal{K}(U)$ and τ factors over

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, "\tau"] \arrow[dr, dashed, "\exists!\, \widetilde{\tau}"'] & & \mathcal{M} \\
& \mathcal{K} \arrow[ur, hook, "\kappa"'] &
\end{tikzcd}$$

This proves that $\mathcal{K}$ is indeed a kernel of f in the category $\mathcal{R}$-Mod. *q.e.d.*

Remark 2. (a) It is a consequence of the exactness of the $\varinjlim$ functor (for filtered systems of abelian groups; exactness of $\varinjlim$ does *not* hold in general, not even for filtered colimits in abelian categories), that

$$\mathcal{K}_x = \varinjlim_{U \ni x} \ker\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \ker\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right).$$

This isomorphism can also be seen in a straightforward way.

(b) One may check that in any additive category (with kernels), a morphism i is a monomorphism iff $\ker(i) = 0$. Thus, in our example we have the equivalent conditions

(α) $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is a monomorphism.

(β) $\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)$ is injective for all open subsets $U \subseteq X$.

(γ) $\ker(f) = 0$ (the zero sheaf).

(δ) $\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x$ is injective for all $x \in X$.

The construction of cokernels won't be that straightforward (duh!), related to the fact that epimorphisms in categories of sheaves aren't as simple as you might think. If $\mathcal{G}$ and $\mathcal{H}$ are sheaves on some topological space X and f is a morphism between them such that $\mathcal{G}(U) \xrightarrow{f} \mathcal{H}(U)$ is surjective for all open U, then f is an epimorphism, but there are epimorphisms f for which this fails.

However, it follows from the fact that a sheaf $\mathcal{G}$ is canonically isomorphic to its sheafification $\mathcal{G}^{\mathrm{sh}}$ (cf. [1, Proposition 1.2.1(d)]) that a morphism between sheaves (of sets, groups, ...) is uniquely determined by the maps it induces on stalks. Thus, $\mathcal{G} \to \mathcal{H}$ is an epimorphism if $\mathcal{G}_x \to \mathcal{H}_x$ is an epimorphism in the respective target category for all $x \in X$.

Lemma 3. *The category* $\mathcal{R}$-Mod *has cokernels.*

Proof. For a morphism $\mathcal{M} \xrightarrow{f} \mathcal{N}$ of sheaves of $\mathcal{R}$-modules, the map

$$U \mapsto \operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) = \mathcal{M}(U)/\mathcal{N}(U) \quad \text{for } U \subseteq X \text{ open}$$

defines a presheaf $\mathcal{F}$ of $\mathcal{R}$-modules, but in general, $\mathcal{F}$ will fail to be a sheaf. We put $\mathcal{C} = \mathcal{F}^{\mathrm{sh}}$ (the *sheafification* of $\mathcal{F}$, cf. [1, Definition 1.2.3]) and claim that $\mathcal{N} \to \mathcal{C}$ is a cokernel of f.

Our first goal is to show that

$$\mathcal{C}_x \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right). \tag{$*$}$$

In the lecture, we did a direct proof, which was somewhat ugly and (in my opinion) lacking the essential step. From [1, Proposition 1.2.1(a)], we get that $\mathcal{C}_x \simeq \mathcal{F}_x$ (which is basically what we proved in the lecture for this particular special case), so we need to show that

$$\mathcal{F}_x = \varinjlim_{U \ni x} \operatorname{coker}\left(\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\right) \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right).$$

Since $\mathcal{M}_x = \varinjlim_{U \ni x} \mathcal{M}(U)$ and similar for $\mathcal{N}_x$, this amounts to showing that cokernels and certain colimits commute. But by Remark 1, cokernels are just a special case of colimits, so what we are actually going to show is that colimits commute with colimits – in the following sense.

Lemma 4. *Let $(X_{i,j})_{i \in I, j \in J}$ be objects of a category $\mathcal{A}$. For each $i_1, i_2 \in I$ let there be an indexing set I_{i_1,i_2} and for each $\alpha \in I_{i_1,i_2}$ and $j \in J$ a morphism*

$$f_\alpha^j \colon X_{i_1,j} \longrightarrow X_{i_2,j}.$$

Similarly, for each $j_1, j_2 \in J$ let there be an indexing set J_{j_1,j_2} and for each $\beta \in J_{j_1,j_2}$ and $i \in I$ a morphism

$$g_\beta^i \colon X_{i,j_1} \longrightarrow X_{i,j_2}.$$

Moreover, suppose that for each $i_1, i_2 \in I$ and $j_1, j_2 \in J$ and $\alpha \in I_{i_1,i_2}$ and $\beta \in J_{j_1,j_2}$ the diagram

$$\begin{tikzcd} X_{i_1,j_1} \arrow[r, "f_\alpha^{j_1}"] \arrow[d, "g_\beta^{i_1}"'] & X_{i_2,j_1} \arrow[d, "g_\beta^{i_2}"] \\ X_{i_1,j_2} \arrow[r, "f_\alpha^{j_2}"] & X_{i_2,j_2} \end{tikzcd} \tag{$\#$}$$

commutes. Then there is an isomorphism

$$\varinjlim_{i\in I}\varinjlim_{j\in J} X_{i,j} \simeq \varinjlim_{j\in J}\varinjlim_{i\in I} X_{i,j} \simeq \varinjlim_{(i,j)\in I\times J} X_{i,j} \,.$$

Proof of Lemma 4. Clearly, it is enough to show the rightmost isomorphism. What we need to show is that $L \coloneqq \varinjlim_j \varinjlim_i X_{i,j}$ satisfies the universal property of $L' \coloneqq \varinjlim_{(i,j)} X_{i,j}$.

Let T be an object of $\mathcal{A}$ and $\left(X_{i,j} \xrightarrow{\tau_{i,j}} T\right)_{i\in I, j\in J}$ be a cocone below the diagram $(X_{i,j})_{i,j}$. That is, for every $\alpha \in I_{i_1,i_2}$ and $j \in J$ the diagram

$$\begin{tikzcd} X_{i_1,j} \arrow[rr, "f^j_\alpha"] \arrow[dr, "\tau_{i_1,j}"'] & & X_{i_2,j} \arrow[dl, "\tau_{i_2,j}"] \\ & T & \end{tikzcd}$$

commutes. By the universal properties of the $L_j \coloneqq \varinjlim_i X_{i,j}$, the $\tau_{i,j}$ factor over some maps $L_j \xrightarrow{\tau_j} T$. Moreover, for each $j_1, j_2 \in J$ and $\beta \in J_{j_1,j_2}$, the compositions

$$X_{i,j_1} \xrightarrow{g^i_\beta} X_{i,j_2} \longrightarrow L_{j_2}$$

induce a map $L_{j_1} \xrightarrow{g_\beta} L_{j_2}$ by the universal property of L_{j_1} (here, we silently used the commutativity of (#), otherwise the above compositions wouldn't be a cocone below $(X_{i,j_1})_{i\in I}$). We thus get a diagram

$$\begin{tikzcd} X_{i,j_1} \arrow[rr, "g^i_\beta"] \arrow[d] \arrow[dr, bend right=60, "\tau_{i,j_1}"'] & & X_{i,j_2} \arrow[d] \arrow[dl, bend left=60, "\tau_{i,j_2}"] \\ L_{j_1} \arrow[rr, dashed, "g_\beta"] \arrow[r, "\tau_{j_1}"'] & T & L_{j_2} \arrow[l, "\tau_{j_2}"] \end{tikzcd}$$

in which everything but the bottom-middle triangle commutes. We show that this triangle commutes as well. Indeed, by the universal property of L_{j_1}, τ_{j_1} is the unique morphism $L_{j_1} \to T$ making each

$$\begin{tikzcd} X_{i,j_1} \arrow[rr] \arrow[dr, "\tau_{i,j_1}"'] & & L_{j_1} \arrow[dl] \\ & T & \end{tikzcd}$$

commute. But apparently, $\tau_{j_2} g_\beta$ has this property as well, proving $\tau_{j_1} = \tau_{j_2} g_\beta$. Then the morphisms $\left(L_j \xrightarrow{\tau_j} T\right)_{j\in J}$ form a cocone below the diagram $(L_j)_{j\in J}$, hence factor uniquely over some $L \xrightarrow{\tau} T$ by the universal property of L.

It remains to prove uniqueness of τ. If $L \xrightarrow{\tau} T$ is a morphism over which each $X_{i,j} \xrightarrow{\tau_{i,j}} T$ factors, then the composition $L_j \to L \xrightarrow{\tau} T$ must equal τ_j since τ_j is uniquely determined by the universal property of L_j. But τ is uniquely determined by the τ_j, proving uniqueness. *q.e.d.*

Having thus proved $(*)$, we now proceed with the proof of Lemma 3. We have a morphism $\mathcal{N} \to \mathcal{C}$ sending $n \in \mathcal{N}(U)$ to

$$\left(\text{image of } n \text{ under } \mathcal{N}(U) \longrightarrow \mathcal{N}_x \longrightarrow \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)\right)_{x\in U}.$$

Since $\mathcal{C}_x \simeq \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)$, this morphism $\mathcal{N} \to \mathcal{C}$ induces surjections on stalks, hence is an epimorphism of sheaves. We show that the morphism $\mathcal{N} \to \mathcal{C}$ satisfies the universal property of the cokernel.

Let $\mathcal{N} \xrightarrow{\tau} \mathcal{T}$ be a morphism of sheaves of $\mathcal{R}$-modules such that $\tau f = 0$. Let $U \subseteq X$ be open. For

$$\nu = (\nu_x)_{x\in U} \in \mathcal{C}(U) \subseteq \prod_{x\in U} \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)$$

we define $\tau_1(\nu) \in \prod_{x\in U} \mathcal{T}_x$ by selecting $n \in \mathcal{N}_x$ whose image in $\operatorname{coker}\left(\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\right)$ equals ν_x, then put $\tau_1(\nu)_x = \tau(n)_x$ which is independent of the choice of n as $\tau f = 0$. It follows from the coherence condition for $\mathcal{C}$ that $\tau_1(\nu)$ satisfies the coherence condition for $\mathcal{T}^{\mathrm{sh}}$, i.e. $\tau_1(\nu) \in \mathcal{T}^{\mathrm{sh}}(U) \subseteq \prod_{x\in U} \mathcal{T}_x$. Hence there is $\mathcal{C} \xrightarrow{\tau_2} \mathcal{T}$ such that $\tau_1 = \left(\mathcal{T} \xrightarrow{\sim} \mathcal{T}^{\mathrm{sh}}\right) \circ \tau_2$ and τ_2 makes

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "\tau"] \arrow[dr, dashed, "\exists!\,\tau_2"'] & & \mathcal{T} \\
& \mathcal{C} \arrow[ur] &
\end{tikzcd}$$

commutative. Uniqueness of τ_2 is easy to see stalk-wise. It follows that $\mathcal{N} \to \mathcal{C}$ is ineed a cokernel of f. *q.e.d.*

Remark 3. One may check that in any additive category (with cokernels) a morphism f is an epimorphism if $\operatorname{coker}(f) = 0$. By our previous construction of cokernels and the description of stalks, we have equivalent conditions

(*a*) $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is an epimorphism of sheaves of $\mathcal{R}$-modules.

(*b*) $\mathcal{M}_x \xrightarrow{f} \mathcal{N}_x$ is surjective for all $x \in X$.

(*c*) For every open $U \subseteq X$ and $n \in \mathcal{N}(U)$ there are an open covering $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ and $m_\lambda \in \mathcal{M}(U_\lambda)$ such that $n|_{U_\lambda} = f(m_\lambda)$

...but (*c*) does *not* imply the surjectivity of $\mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)$, unless, e.g., f is also a monomorphism.

Proof of Proposition 1. We verify the rest of the abelianness conditions. First, let $\mathcal{M} \xrightarrow{f} \mathcal{N}$ be a mono- and epimorphism. Then it induces isomorphisms on stalks (by Remark 2(b) and Remark 3), hence is an isomorphism itself.

Let $\mathcal{M} \xrightarrow{i} \mathcal{N}$ be a monomorphism and $\mathcal{N} \to \mathcal{C}$ be its cokernel. Then

$$\ker\left(\mathcal{N} \longrightarrow \mathcal{C}\right)_x = \ker\left(\mathcal{N}_x \longrightarrow \mathcal{C}_x\right) = \ker\left(\mathcal{N}_x \longrightarrow \operatorname{coker}\left(\mathcal{M}_x \xrightarrow{i} \mathcal{N}_x\right)\right) \simeq \mathcal{M}_x$$

as $\mathcal{M}_x \xrightarrow{i} \mathcal{N}_x$ is injective. Hence $\mathcal{M} \to \ker\left(\mathcal{N} \to \mathcal{C}\right)$ induces isomorphisms on stalks and thus is an isomorphism itself. It follows by Definition 2(d) that any monomorphism is an effective monomorphism.

Similar arguments apply to epimorphisms. *q.e.d.*

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebraic Geometry I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgGeoI.