Homological Methods in Commutative Algebra

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This text consists of notes on the lecture Homological Methods in Commutative Algebra, taught at the University of Bonn by Professor Jens Franke in the summer term (Sommersemester) 2018.

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Introduction

Professor Franke started the lecture giving an idea of what the Tor and Ext functors do. Let R be a commutative ring with 1. For an exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and T another R-module, the sequence

$$M' \otimes_R T \longrightarrow M \otimes_R T \longrightarrow M'' \otimes_R T \longrightarrow 0 \tag{1}$$

is exact but usually can't be extended by 0 on the left end. The same is true for

$$0 \longrightarrow \mathrm{Hom}_R(T, M') \longrightarrow \mathrm{Hom}_R(T, M) \longrightarrow \mathrm{Hom}_R(T, M'') \tag{2}$$

and

$$0 \longrightarrow \mathrm{Hom}_R(M'', T) \longrightarrow \mathrm{Hom}_R(M, T) \longrightarrow \mathrm{Hom}_R(M', T)\ , \tag{3}$$

but again, they can't be extended by 0 on the right in general.

Example. Take $R = \mathbb{Z}$ and consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

(a) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (1). Then $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z}$ is the zero morphism, showing that injectivity on the left end fails in (1).

(b) Let $T = \mathbb{Z}/2\mathbb{Z}$ in (2). We claim that surjectivity fails on the right end. Indeed, if it was surjective, then $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ would have to have a lift

$$\begin{tikzcd} & \mathbb{Z} \arrow[d] \\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, dashed] \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

which it hasn't as $\mathbb{Z}$ is 2-torsion free and thus every morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be 0.

(c) Let $T = \mathbb{Z}$ in (3). We claim that that surjectivity fails on the right end, or more specifically, that $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ has no preimage. Indeed, if $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is a preimage of $\mathrm{id}_{\mathbb{Z}}$, i.e. the composition $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ equals $\mathrm{id}_{\mathbb{Z}}$, then f must be given by $f(n) = \frac{n}{2}$ on $2\mathbb{Z}$, but this can't be extended to all of $\mathbb{Z}$, contradiction!

To handle this deficiency, one constructs *derived functors* Tor and Ext, which give rise to long exact sequences

$$\begin{aligned}\dots \longrightarrow \operatorname{Tor}_2^R(M'',T) \longrightarrow \operatorname{Tor}_1^R(M',T) \longrightarrow \operatorname{Tor}_1^R(M,T) \longrightarrow \operatorname{Tor}_1^R(M'',T)\\ \longrightarrow M'\otimes_R T \longrightarrow M\otimes_R T \longrightarrow M''\otimes_R T \longrightarrow 0\ ,\end{aligned}$$

as well as

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(T,M') \longrightarrow \operatorname{Hom}_R(T,M) \longrightarrow \operatorname{Hom}_R(T,M'')\\ \longrightarrow \operatorname{Ext}_R^1(T,M') \longrightarrow \operatorname{Ext}_R^1(T,M) \longrightarrow \operatorname{Ext}_R^1(T,M'') \longrightarrow \operatorname{Ext}_R^2(T,M') \longrightarrow \dots\end{aligned}$$

and

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_R(M'',T) \longrightarrow \operatorname{Hom}_R(M,T) \longrightarrow \operatorname{Hom}_R(M',T)\\ \longrightarrow \operatorname{Ext}_R^1(M'',T) \longrightarrow \operatorname{Ext}_R^1(M,T) \longrightarrow \operatorname{Ext}_R^1(M',T) \longrightarrow \operatorname{Ext}_R^2(M'',T) \longrightarrow \dots\ .\end{aligned}$$

extending the open ends of (1), (2), and (3) respectively.

A highlight of this lecture will be *Serre's characterization of regularity.*

Theorem. *For a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k, the following are equivalent.*

(*a*) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$ *(i.e., R is regular).*

(*b*) *There is some vanishing bound for* $\operatorname{Tor}_*^R(-,-)$.

(*c*) *…and* $\dim R$ *is such a vanishing bound.*

(*d*) *There is some vanishing bound for* $\operatorname{Ext}_R^*(-,-)$.

(*e*) *…and* $\dim R$ *is again such a vanishing bound.*

From this, one can deduce the following

Corollary. *If R is a regular Noetherian local ring and $\mathfrak{p} \in \operatorname{Spec} R$, then $R_{\mathfrak{p}}$ is regular as well.*

We will also introduce the notion of *Cohen–Macaulay rings* and prove that they are *universally catenary* (which is quite a generalization of what we did in Algebra I, cf. [1, Theorem 10]).

Theorem. *If R is a regular Noetherian local ring or, more generally, a Cohen–Macaulay ring, then it is* **universally catenary***: If A is an R-algebra of finite type and $X \subseteq Y \subseteq Z$ are irreducible closed subsets of* $\operatorname{Spec} A$*, then*

$$\operatorname{codim}(X,Y) + \operatorname{codim}(Y,Z) = \operatorname{codim}(X,Z)\ .$$

1. Tor and Ext of R-modules

From now on, unless otherwise stated, our rings are commutative with 1.

1.1. Injective and projective modules and properties of Ext_R^*

Proposition 1 (Baer's criterion)**.** *For an R-module N, the following are equivalent.*

(a) *The functor $\mathrm{Hom}_R(-,N)$ is exact.*

(b) *For any embedding $M' \hookrightarrow M$ of R-modules, $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M',N)$ is surjective.*

(c) *Property (b) holds for $R = M$. In other words, if $I \subseteq R$ is any ideal, then any morphism $I \to N$ of R-modules extends to a morphism $R \to N$.*

Remark 1. (a) Since there is a bijection

$$\begin{aligned} \mathrm{Hom}_R(R,M) &\xrightarrow{\sim} M \\ (r \mapsto r\cdot m) &\longleftarrow m \\ \left(R \xrightarrow{\varphi} M\right) &\longmapsto \varphi(1)\ , \end{aligned}$$

Proposition 1(c) can be reformulated as that any morphism $I \to N$ for $I \subseteq R$ an ideal has the form $i \mapsto i \cdot m$ for some $m \in M$.

(b) Note that Proposition 1(c) is trivial when $I = 0$.

(c) When $R = \mathbb{Z}$, every ideal $I \subseteq \mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$ and a morphism $n\mathbb{Z} \xrightarrow{\varphi} N$ is uniquely determined by $\varphi(n)$. Thus, an extension $\hat{\varphi}$ of φ to $\mathbb{Z}$ exists iff there is an element $\nu \in N$ such that $n \cdot \nu = \varphi(n)$ (in that case, put $\hat{\varphi}(1) = \nu$). Hence, Proposition 1(c) amounts to whether the abelian group N is *divisible*, that is, whether $N \xrightarrow{\cdot n} N$ is surjective for all $n \in \mathbb{Z}$ (also cf. Definition 2).

Definition 1. An R-module satisfying the equivalent conditions from Proposition 1 is called **injective**.

Proof of Proposition 1. The implication (b) $\Rightarrow$ (c) is trivial. Let's prove (c) $\Rightarrow$ (b). Let $M \xrightarrow{f} N$ be a morphism of R-modules and consider

$$\mathfrak{M} = \left\{(Q,\varphi) \mid M \subseteq Q \subseteq M' \text{ and } \widetilde{\varphi} \in \mathrm{Hom}_R(Q,N) \text{ such that } \varphi|_M = f\right\}\ .$$

$\mathfrak{M}$ becomes a partially ordered set via $(Q_1,\varphi_1) \preceq (Q_2,\varphi_2) \Leftrightarrow Q_1 \subseteq Q_2$ and $\varphi_2|_{Q_1} = \varphi_1$. Then it's easy to see that Zorn's lemma is applicable, hence $\mathfrak{M}$ has a $\preceq$-maximal element (Q_*,φ_*). If (c) is satisfied and $Q_* \subsetneq M'$, there is an $m \in M' \setminus L_*$. Let $I = \{r \in R \mid rm \in Q_*\}$ and let

$I \xrightarrow{g} N$ be given by $g(r) = \varphi_*(rm)$. By (c), there is a morphism $R \xrightarrow{\gamma} N$ extending g, i.e., a $\nu \in N$ such that $\varphi_*(rm) = r\nu$ when $r \in I$ (using Remark 1(a)). Let $\widetilde{Q} = Q_* + Rm$ and $\widetilde{\varphi}(m_* + rm) = \varphi_*(m_*) + r\nu$ for $m_* \in Q_*$ and $r \in R$, then it's easy to see that $\widetilde{\varphi}$ is well-defined and $(Q_*, \varphi_*) \prec (\widetilde{Q}, \widetilde{\varphi})$, a contradiction.

The equivalence $(a) \Leftrightarrow (b)$ is easy to see as for any short exact sequence $0 \to X \to Y \to Z \to 0$, the sequence $0 \to \mathrm{Hom}_R(Z, N) \to \mathrm{Hom}_R(Y, N) \to \mathrm{Hom}_R(X, N)$ is exact anyways and (b) implies exactness at the right end. *q.e.d.*

Definition 2. If R is a domain and M an R-module, then M is called **divisible** if $M \xrightarrow{r\cdot} M$ is surjective for all $r \in R \setminus \{0\}$

Corollary 1. (a) *When R is a domain, the property from Proposition 1(c) for principal ideals I is equivalent do divisibility of N.*

(b) *Any injective module N is divisible in the following sense: If $r \in R$ is not a zero divisor, $N \xrightarrow{r\cdot} N$ is surjective.*

(c) *In particular, if N is injective and $S \subseteq R$ a multiplicative subset not containing zero divisors, then the morphism $N \to N_S$ to the localization of N at S is surjective.*

Proof. Part (a) can be seen using the arguments from Remark 1(c). For (b), note that $R \xrightarrow{r\cdot} R$ is injective when r is no zero divisor, hence, for any $n \in N$, the morphism $\varphi \in \mathrm{Hom}_R(R, N)$ given by $\varphi(1) = n$ extends to $\hat{\varphi} \in \mathrm{Hom}_R(R, N)$ such that $\varphi = r\hat{\varphi}$. Then $\hat{\varphi}(1)$ is a preimage of n under $N \xrightarrow{r\cdot} N$. Part (c) follows from (b) and the universal property of localization. *q.e.d.*

Remark. Note that $R = \mathbb{Z}/p^2\mathbb{Z}$, for $p \in \mathbb{Z}$ a prime, is injective over itself, but $R \xrightarrow{p\cdot} R$ fails to be injective. Indeed, the only ideal of R where Baer's criterion is in question is $(p) \subseteq R$. We need to show that any R-morphism $(p) \to R$ extends to an R-morphism $R \to R$. But any $(p) \xrightarrow{\varphi} R$ maps p to the p-torsion part of R, i.e., to (p) itself, hence is given by $\varphi(p) = rp$ for some $r \in R$ and can be extended via $\hat{\varphi}$ given by $\hat{\varphi}(1) = r$. This shows that Corollary 1(b) is somewhat sharp.

Corollary 2. *A module over a principal ideal domain is injective iff it is divisible.*

Proof. Follows from Corollary 1(a). *q.e.d.*

Remark. The same holds for Dedekind domains, see Corollary 6 (which is not there yet).

Corollary 3. *When R is a principal ideal domain, then any quotient of an injective module is injective again. The category of R-modules has* **sufficiently many injective objects** *in the sense that for any object X there is a monomophism $X \hookrightarrow I$ with I injective. Thus, any R-module X has an* **injective resolution***, i.e., an exact sequence*

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

with injective objects $I^0, I^1, I^2, \dots$. In fact, any R-module, for R a principal ideal domain, has an injective resolution $0 \to X \to I^0 \to I^1 \to 0$ of length 1.

Proof. The first assertion follows as the quotient of divisible modules is divisible again. Note that K/R is divisible, K being the quotient field of R, hence it is injective. If M is any R-module and $m \in M \setminus \{0\}$. We have to distinguish to cases.

Case 1. Suppose $\mathrm{Ann}_R(m)$ is non-zero, i.e., $\mathrm{Ann}_R(m) = (\alpha)$ for some $\alpha \in R \setminus \{0\}$ (remember we have a principal ideal domain). Then we have a morphism from $Rm \subseteq M$ to K/R given by $rm \mapsto \frac{r}{\alpha} \bmod R$ (note that modding out R is necessary for this to be well-defined – we couldn't just have used K). By injectivity of K/R, there is an extension $M \xrightarrow{\varphi_m} K/R$, satisfying $\varphi_m(m) \neq 0$. Let $I_m \subseteq K/R$ be the target of φ_m.

Case 2. If $\mathrm{Ann}_R(m) = 0$, we get a morphism from $Rm \subseteq M$ to K instead, sending $rm \mapsto r$ (this time, using K doesn't cause problems thanks to $\mathrm{Ann}_R(m) = 0$). By injectivity of K, this extends to a morphism $M \xrightarrow{\varphi_m} K$ such that $\varphi_m(m) \neq 0$. Let $I_m = K$ be the target of φ_m.

Now put $I = \prod_{m \in M\setminus\{0\}} I_m$. Then I is divisible (since every I_m is), hence injective, and $M \to I$, $\mu \mapsto (\varphi_m(\mu))_{m \in M\setminus\{0\}}$ is a monomorphism. As a quotient of $I^0 = I$, $I^1 = \mathrm{coker}\,(M \to I^0)$ is injective as well, hence $0 \to M \to I^0 \to I^1 \to 0$ is an injective resolution of length 1. *q.e.d.*

Proposition 2 (a.k.a. "Satz 2")**.** *For any ring R, the category of R-modules has sufficiently many injective objects.*

Proof. This will follow from Lemma 1(b) and (c) below. *q.e.d.*

Remark. This holds in vast more generality, and in particular, Proposition 2 follows immediately from the following theorem, which, however, we are not going to prove in this lecture.

Theorem (Grothendieck)**.** *Any AB5 category with a generator has sufficiently many injective objects.*

Lemma 1. *Let R be any ring.*

(a) *The forgetful functor from R-Mod to the category of abelian groups has a right-adjoint functor, namely $\mathrm{Hom}_{\mathbb{Z}}(R, -)$. That is, there is a bijection*

$$\mathrm{Hom}_{\mathbb{Z}}(M, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, A)) \qquad (*)$$

for any R-module M and any abelian group A. Here, we equip $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ with an R-module structure via $(r \cdot \varphi)(x) = \varphi(rx)$ for $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$ and $r, x \in R$.

(b) *For any injective abelian group I, $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module.*

(c) *Let M be any R-module and I and abelian group and $M \overset{\iota}{\hookrightarrow} I$ a monomorphism of abelian groups, then the R-morphism $M \to \mathrm{Hom}_{\mathbb{Z}}(R, A)$ obtained by applying $(*)$ is injective.*

Bibliography

[1] Nicholas Schwab; Ferdinand Wagner. *Algebra I by Jens Franke (lecture notes)*. GitHub: https://github.com/Nicholas42/AlgebraFranke/tree/master/AlgebraI.