Assorted Examples of Finite Dimensional Lie Superalgebras

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This final project for Math 222 is organized as follows, in Section I we give basic definitions related to Lie superalgebras, in Section II we consider the problem of classification and introduce the supertrace, in Section II.1 we examine $\mathbf{A}(m|n)$, in Section II.2 we examine $\mathbf{B}(m|n), \mathbf{C}(n), \mathbf{D}(m|n)$, in Section II.3 we look at the remaining contragredient Lie superalgebras $\mathbf{F}(4), \mathbf{G}(3), \mathbf{D}(2,1;\alpha)$ as well as the strange Lie superalgebras $\mathbf{P}(n)$ and $\mathbf{Q}(n)$, in Section III we look at the example $\mathfrak{gl}(1|1)$, in Section IV we consider the example $\mathfrak{sl}(1|2)$, and in Section V we provide further directions, including a roadmap connecting Lie superalgebras to quantum invariants by sketching how the Alexander polynomial can be related to the R-matrix of a deformation of the universal enveloping quantum superalgebra $U_q\mathfrak{gl}(1|1)$.

I. PRELIMINARIES

In the interest of extending what we are learning about Lie groups in Math 222, we endeavor to explain superalgebras and Lie superalgebras in the hopes of building a roadmap of how to use these concepts to explore quantum invariants and/or knot invariants such as the Alexander polynomial. It is also worth noting that physicists employ Lie superalgebras in the study of supersymmetry (the idea that there is an involution between bosonic and fermionic Fock spaces, see p. 579 in [1]).

We follow the introduction to Lie superalgebras given by Victor Kac in his seminal paper of the same name [2].

Definition 1.1 A $\mathbb{Z}_n$**-graded algebra G** is an algebra that can be decomposed into a direct sum of n finite-dimensional subspaces $G = \bigoplus_{i\in\mathbb{Z}_n} G_i$ such that $G_iG_j \subseteq G_{(i+j) \mod n}$.

Definition 1.2 A **superalgebra H** is a $\mathbb{Z}_2$-graded algebra $H = H_0 \oplus H_1$ satisfying the associativity relation that for $a \in H_\alpha,\ b \in H_\beta$ with $\alpha,\ \beta \in \mathbb{Z}_2 = \{0,1\}$, then $ab \in H_{\alpha+\beta}$.

Definition 1.3 A **Lie superalgebra L** is a superalgebra $L = L_0 \oplus L_1$ satisfying the bracketing relations,

1. graded commutator, called the supercommutator: $[a,b] = -(-1)^{\alpha\beta}[b,a]$
2. graded Jacobi identity: $[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$.

for all $a \in L_\alpha,\ b \in L_\beta$.

II. CLASSIFICATION

Similar to the Cartan classification of Lie algebras, we can classify the classical Lie superalgebras. Recall that being classical is the requirement to be both simple and completely reducible. Consider the $\mathbb{Z}_2$-graded space $V = \bigoplus_{i\in\mathbb{Z}_2} V_i$. The associative algebra End V then inherits the $\mathbb{Z}_2$-grading as End $V = \bigoplus_{i\in\mathbb{Z}_2}$ End V_i, where

$$\mathrm{End}_i\, V = \{a \in \mathrm{End}\, V \mid aV_s \subseteq V_{i+s}\}, \quad i, s \in \mathbb{Z}_2, \tag{2.0.1}$$

making it an associative superalgebra End $V = \mathrm{End}_0 V \oplus \mathrm{End}_1 V$.

Imposing the additional condition that End V must satisfy the braket relation $[a,b] = ab - (-1)^{(\deg a)(\deg b)}ba$ makes End V the Lie superalgebra $\ell(V)$. Another way to denote the Lie superalgebra $\ell(V)$ is as $\ell(m|n)$ since dim $V_0 = m$ and dim $V_1 = n$. Note that dealing with matrix algebras, we are looking at block matrices of the form:

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$$(m+n)\times(m+n) \text{ matrices of the form } \left[\begin{array}{ccc|ccc} w_{11} & \dots & w_{1m} & x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ w_{m1} & \dots & w_{mm} & x_{m1} & \dots & x_{mn} \\ \hline y_{11} & \dots & y_{1m} & z_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{n1} & \dots & y_{nm} & z_{n1} & \dots & z_{nn} \end{array}\right] = \left[\begin{array}{c|c} W_{m\times m} & X_{m\times n} \\ \hline Y_{n\times m} & Z_{n\times n} \end{array}\right] \in \mathfrak{gl}(m|n) \quad (2.0.2)$$

Note that in examining $\mathbb{Z}_2$-graded spaces, for matrices of the form (2.0.2),

$$\left[\begin{array}{c|c} W_{m\times m} & 0 \\ \hline 0 & Z_{n\times n} \end{array}\right] \in \mathfrak{gl}_{\bar{0}}(m|n) \qquad \text{is called the } \underline{\text{even part}} \tag{2.0.3}$$

$$\left[\begin{array}{c|c} 0 & X_{m\times n} \\ \hline Y_{n\times m} & 0 \end{array}\right] \in \mathfrak{gl}_{\bar{1}}(m|n) \qquad \text{is called the } \underline{\text{odd part}}. \tag{2.0.4}$$

Definition 2.0.5 For $a, b \in \ell(V)$, we define the **supertrace str** as

$$\mathrm{str} : \ell \to k \tag{2.0.5}$$
$$[a, b] \mapsto 0 \tag{2.0.6}$$

The supertrace is consistent, supersymmetric, and invariant (Prop 1.1.2 in [2]). We define each of these properties as follows,

Definition 2.0.6 A bilinear form f on a $\mathbb{Z}_2$ graded space $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is considered **consistent** when $f(a, b) = 0$ for $a \in G_{\bar{0}}$ and $b \in G_{\bar{1}}$.

Definition 2.0.7 The bilinear form f is considered **supersymmetric** if $f(a, b) = (-1)^{(\deg a)(\deg b)} f(b, a)$.

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Definition 2.0.8 The bilinear form f is considered **invariant** when $f([a, b], c) = f(a, [b, c])$ for $a, b, c \in L$, where L is a Lie superalgebra as defined in Def. 1.3.

II.1 CLASSIFYING LIE SUPERALGEBRAS AS A$(m|n)$

Consider the subspace of $\ell(m|n)$ given as,

$$\mathfrak{sl}(m|n) = \{a \in \ell(m, n) \mid \mathrm{str}\, a = 0\} \tag{2.1.1}$$

The condition $\mathrm{str}\, a = 0$ means $\mathrm{Tr}(W) - \mathrm{Tr}(Z) = 0$. We can see that this makes $\mathfrak{sl}(m|n)$ a subalgebra of $\ell(m|n)$ since the bracket relations in Def. 1.3 are satisfied and we see that $\mathfrak{sl}$ is an ideal of $\ell(m|n)$:

$$\mathrm{str}([\ell(m|n), \mathfrak{sl}(m|n)]) = 0 \implies [\ell(m|n), \mathfrak{sl}(m|n)] \subseteq \mathfrak{sl}(m|n). \tag{2.1.2}$$

To make this more concrete and explicit, let us consider the example $\mathfrak{sl}(1|1)$, which is of the form,

$$\left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}\right] \text{ s.t. } \alpha - \delta = 0 \text{ where } \alpha, \beta, \gamma, \delta \in \mathbb{K} \tag{2.1.3}$$

We consider the decomposition of $\mathfrak{sl}(1|1)$,

$$\left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \alpha \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline \gamma & 0 \end{array}\right] \oplus \left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & \alpha \end{array}\right] \oplus \left[\begin{array}{c|c} 0 & \beta \\ \hline 0 & 0 \end{array}\right] \in \mathfrak{sl}_{-1}(1|1) \oplus \mathfrak{sl}_0(1|1) \oplus \mathfrak{sl}_1(1|1) = \mathfrak{sl}_{\bar{0}}(1|1) \oplus \mathfrak{sl}_{\bar{1}}(1|1) \tag{2.1.4}$$

$$\text{The even part is then } \left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & \delta \end{array}\right] = \left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & \alpha \end{array}\right] \in \mathfrak{sl}_{\bar{0}}(1|1). \tag{2.1.5}$$

$$\text{The odd part is then } \left[\begin{array}{c|c} 0 & 0 \\ \hline \gamma & 0 \end{array}\right] \in \mathfrak{sl}_{-1}(1|1) \text{ and } \left[\begin{array}{c|c} 0 & \beta \\ \hline 0 & 0 \end{array}\right] \in \mathfrak{sl}_{1}(1|1), \text{ where } \mathfrak{sl}_{-1}(1|1) \oplus \mathfrak{sl}_{1}(1|1) = \mathfrak{sl}_{\bar{1}}(1|1). \tag{2.1.6}$$

We see that the free variables are $\alpha, \beta,$ and γ, so $\mathfrak{sl}(1|1)$ is dimension 3.
This leads to the classification,

$$\mathbf{A}(m|n) = \mathfrak{sl}(m+1|n+1) \quad \text{for } m \neq n, \quad m, n \geq 0, \tag{2.1.7}$$

$$\mathbf{A}(n|n) = \mathfrak{sl}(n+1|n+1)/\langle \mathbb{1}_{2n+2} \rangle, \quad n > 0 \tag{2.1.8}$$

II.2 CLASSIFYING LIE SUPERALGEBRAS AS $\mathbf{B}(m|n)$, $\mathbf{C}(n)$, or $\mathbf{D}(m|n)$

Consider the subspace of $\ell(m|n)$ given as,

$$\mathfrak{osp}(m|n)_s = \{a \in \ell(m|n)_s \text{ s.t. } F(a(x), y) = -(-1)^{s(\deg x)} F(x, a(y))\}, \qquad s \in \mathbb{Z}_2 \tag{2.2.1}$$

This is the **orthogonal-symplectic superalgebra** $\mathfrak{osp}(m|n)$. Examine the condition $F(a(x), y) = -(-1)^{s(\deg x)} F(x, a(y))$, which effectively is a form of skew-adjointness $M^T F = -FM$ where $M \in \mathfrak{osp}(m|n)$ and F is the orthogonal-symplectic transformation matrix. The F matrix has the form,

$$\left[\begin{array}{ccc|cc} 0 & 1_l & 0 & & \\ 1_l & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & 1_r \\ & & & -1_r & 0 \end{array}\right] \text{ for } m = 2q+1 \qquad \text{OR} \qquad \left[\begin{array}{cc|cc} 0 & 1_l & & \\ 1_l & 0 & & \\ \hline & & 0 & 1_r \\ & & -1_r & 0 \end{array}\right] \text{ for } m = 2q. \tag{2.2.2}$$

The M matrix has the form,

$$\left[\begin{array}{ccc|cc} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ \hline y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{array}\right] \in \mathfrak{osp}(2q+1|n) \qquad \text{OR} \qquad \left[\begin{array}{cc|cc} a & b & x & x_1 \\ c & -a^T & y & y_1 \\ \hline y_1^T & x_1^T & d & e \\ -y^T & -x^T & f & -d^T \end{array}\right] \in \mathfrak{osp}(2q|n) \tag{2.2.3}$$

When $n = 0$, we have the orthogonal Lie algebra,

$$\begin{bmatrix} a & b & u \\ c & -a^T & v \\ -v^T & -u^T & 0 \end{bmatrix} \in \mathfrak{osp}(2q+1|0) \cong \mathfrak{o}(2q+1) \qquad \text{OR} \qquad \begin{bmatrix} a & b \\ c & -a^T \end{bmatrix} \in \mathfrak{osp}(2q|0) \cong \mathfrak{o}(2q) \tag{2.2.4}$$

When $m = 0$, we have the symplectic Lie algebra,

$$\begin{bmatrix} d & e \\ f & -d^T \end{bmatrix} \in \mathfrak{osp}(0|n) \cong \mathfrak{sp}(n). \tag{2.2.5}$$

Kac then classifies some finite dimensional Lie superalgebras as being of the form,

$$\mathbf{B}(m|n) = \mathfrak{osp}(2m+1, 2n), \quad m \geq 0, \ n > 0 \tag{2.2.6}$$
$$\mathbf{D}(m|n) = \mathfrak{osp}(2m, 2n), \quad m \geq 2, \ n > 0 \tag{2.2.7}$$
$$\mathbf{C}(n) = \mathfrak{osp}(2, 2n-2), \quad n \geq 2. \tag{2.2.8}$$

II.3 CLASSIFYING LIE SUPERALGEBRAS AS $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{D}(2,1;\alpha), \mathbf{P}(n)$, or $\mathbf{Q}(n)$

Kac's classification of the finite dimensional Lie superalgebras (stated as Theorem 2 in [2]) also specifies that Lie superalgebras can be of the following forms:

Aside from $\mathbf{A}(m|n), \mathbf{B}(m|n), \mathbf{C}(n), \mathbf{D}(m|n)$, the other simple, contragredient (meaning they can be specified using a Cartan matrix) Lie superalgebras are (see Prop. 2.1.1 in [2]),

$$\mathbf{F}(4) \tag{2.3.1}$$
$$\text{with Lie algebra } F(4)_{\bar{0}} \text{ of type } B_3 \oplus A_1 \tag{2.3.2}$$
$$\text{and representation on } F(4)_{\bar{1}} \text{ being } spin_7 \otimes sl_2 \tag{2.3.3}$$

$$\mathbf{G}(3) \tag{2.3.4}$$
$$\text{with Lie algebra } G(3)_{\bar{0}} \text{ of type } G_2 \oplus A_1 \tag{2.3.5}$$
$$\text{and representation on } G(3)_{\bar{1}} \text{ being } G_2 \otimes sl_2 \tag{2.3.6}$$

$$\mathbf{D}(2,1;\alpha) \qquad \text{where } \alpha \in \mathbb{K}^* \setminus \{0,-1\} \tag{2.3.7}$$
$$\text{with Lie algebra } \mathbf{D}(2,1;\alpha)_{\bar{0}} \text{ of type } A_1 \oplus A_1 \oplus A_1 \tag{2.3.8}$$
$$\text{and representation on } \mathbf{D}(2,1;\alpha)_{\bar{1}} \text{ being } sl_2 \otimes sl_2 \otimes sl_2 \tag{2.3.9}$$

And the simple strange Lie superalgebras (see 2.1.3 and 2.1.4 in [2]) are, for $2 \leq n$,

$$\mathbf{P}(n) \ni \left[\begin{array}{c|c} a & b \\ \hline c & -a^T \end{array}\right] \qquad \text{where } \mathrm{tr}(a) = 0, \text{ b is a symmetric matrix, and c is a skew-symmetric matrix} \tag{2.3.10}$$

$$\mathbf{Q}(n) = \widetilde{Q}(n) / \langle 1_{2n+2} \rangle \tag{2.3.11}$$
$$\text{where } \widetilde{Q}(n) \text{ is of the form } \left[\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right] \in \mathfrak{sl}(n+1, n+1) \text{ s.t. } \mathrm{tr}(b) = 0 \tag{2.3.12}$$
$$\text{and we have quotiented out by the center } C(\widetilde{Q}(n)) = \langle 1_{2n+2} \rangle . \tag{2.3.13}$$

III. THE LIE SUPERALGEBRA $\mathfrak{gl}(1|1)$

Recall 2.0.2,

$$\left[\begin{array}{c|c} W_{m\times m} & X_{m\times n} \\ \hline Y_{n\times m} & Z_{n\times n} \end{array}\right] \in \mathfrak{gl}(m|n) \tag{3.1.1}$$

Now suppose that we want to consider the Lie superalgebra $\mathfrak{gl}(1|1)$,

$$\left[\begin{array}{c|c} W_{1\times 1} & X_{1\times 1} \\ \hline Y_{1\times 1} & Z_{1\times 1} \end{array}\right] \in \mathfrak{gl}(1|1) \tag{3.1.2}$$

As on p. 59 of [3], we take the generators of the Lie superalgebra $\mathfrak{gl}(1|1)$ to be,

$$E = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right], \qquad G = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right], \qquad Y = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right], \qquad X = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \tag{3.1.3}$$

The even part is E and G, while the odd part is Y and X. We want to verify that these generators satisfy the,

1. graded commutator, called the supercommutator: $[a,b] = -(-1)^{\alpha\beta}[b,a]$
2. graded Jacobi identity: $[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$.

It is sufficient to show,

$$\{X,Y\} = XY + YX = [X,Y] = XY - (-1)YX = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \tag{3.1.4}$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3.1.5}$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] = E \tag{3.1.6}$$

$$X^2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \tag{3.1.7}$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3.1.8}$$

$$= 0 \tag{3.1.9}$$

$$Y^2 = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] \tag{3.1.10}$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3.1.11}$$

$$= 0 \tag{3.1.12}$$

$$[G,X] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{3.1.13}$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{3.1.14}$$

$$= X \tag{3.1.15}$$

$$[G,Y] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{3.1.16}$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] \tag{3.1.17}$$

$$= -Y \tag{3.1.18}$$

$$[E,G] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] - \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{3.1.19}$$

$$= 0 \tag{3.1.20}$$

$$[E,X] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{3.1.21}$$

$$= 0 \tag{3.1.22}$$

$$[E,Y] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{3.1.23}$$

$$= 0 \tag{3.1.24}$$

These relations holding is sufficient to show that the 1) supercommutator and the 2) graded Jacobi identity hold for the generators E, G, X, and Y because we have directly shown that the supercommutator holds and then if we consider the pairings,

$$
\begin{aligned}
&[E,[G,X]] && (3.1.25)\\
&[G,[X,Y]] && (3.1.26)\\
&[X,[G,Y]] && (3.1.27)\\
&[X,[Y,G]] && (3.1.28)
\end{aligned}
$$

We evaluate the nested commutator as,

$$
\begin{aligned}
&[G,X]=0 && (3.1.29)\\
&[X,Y]=E && (3.1.30)\\
&[G,Y]=-Y && (3.1.31)\\
&[Y,G]=Y && (3.1.32)
\end{aligned}
$$

And then evaluating the overall commutator as,

$$
\begin{aligned}
&[E,0]=0 && (3.1.33)\\
&[G,E]=0 && (3.1.34)\\
&[X,-Y]=-E && (3.1.35)\\
&[X,Y]=E && (3.1.36)
\end{aligned}
$$

And from this it is straightforward computation to see that the graded Jacobi identity is satisfied.

Consider the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}(1|1)$ and its dual $\mathfrak{h}^*$. A basis for $\mathfrak{h}$ is $\{E, G\}$ and a dual basis, which is for $\mathfrak{h}^*$, is $\{X, Y\}$. Then we have the dual weight lattice $\mathbb{Z}E \oplus \mathbb{Z}G \subset \mathfrak{h}$ and the weight lattice $\mathbb{Z}X \oplus \mathbb{Z}Y \subset \mathfrak{h}^*$ (compare to bottom of p. 4 in [4]).

IV. THE LIE SUPERALGEBRA $\mathfrak{sl}(1|2)$

Consider $\mathfrak{sl}(1|2)$. This is $A(m|n) = \mathfrak{sl}(m+1|n+1)$ where $m = 0,\ n = 1$.

This means that we are looking at matrices of the form,

$$A_{(0|1)} = \left[\begin{array}{c|cc} w & x_1 & x_2 \\ \hline y_1 & z_{11} & z_{12} \\ y_2 & z_{21} & z_{22} \end{array}\right] \qquad \text{s.t. } \operatorname{str}(A_{(0|1)}) = \operatorname{tr}(W) - \operatorname{tr}(Z) = 0 \tag{4.1.1}$$

From the supertrace condition, we have that $w = z_{11} + z_{22}$. This means that we have a 2 dimensional Cartan subalgebra, which we can form using the generators,

$$h_1 = \left[\begin{array}{c|cc} w=1 & & \\ \hline & z_{11}=1 & \\ & & z_{22}=0 \end{array}\right] \qquad h_2 = \left[\begin{array}{c|cc} w=0 & & \\ \hline & z_{11}=1 & \\ & & z_{22}=-1 \end{array}\right] \tag{4.1.2}$$

And we can check that the supertrace for both of these is zero as follows,

$$
\begin{aligned}
&\operatorname{str}(h_1) = 1-(1+0) = 0 && (4.1.3)\\
&\operatorname{str}(h_2) = 0-(1+(-1)) = 0 && (4.1.4)
\end{aligned}
$$

Now we can consider the even part containing,

$$E = \left[\begin{array}{c|cc} & & \\ \hline & & z_{12}=1 \\ & & \end{array}\right] \qquad F = \left[\begin{array}{c|cc} & & \\ \hline & & \\ & z_{21}=1 & \end{array}\right] \tag{4.1.5}$$

We can check a case of the

1. graded commutator, i.e. the supercommutator: $[a,b] = -(-1)^{\alpha\beta}[b,a]$

being satisfied since we can compute that,

$$2E = [h_2, E] = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right] \tag{4.1.6}$$

$$= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right] \tag{4.1.7}$$

$$= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right] = 2E \tag{4.1.8}$$

and if we had reversed the order of the commutator, we have,

$$-2E = [E, h_2] = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \tag{4.1.9}$$

$$= - \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right] = -2E \tag{4.1.10}$$

which makes sense because we wished to show that

$$[h_2, E] = -(-1)^0 [E, h_2] \tag{4.1.11}$$

and we did because

$$[h_2, E] = 2E = -(-2E) = -[E, h_2]. \tag{4.1.12}$$

We then have that the odd part of $\mathfrak{sl}(1|2)$ contains,

$$X^1 = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad X^2 = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad Y^1 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad Y^2 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \tag{4.1.1}$$

We then have that $\mathfrak{sl}(1|2)$ is 9 free variables - supertrace condition = 8 dimensional, generated by,

$$\{h_1, h_2, E, F, X^1, X^2, Y^1, Y^2\}. \tag{4.1.2}$$

V. FURTHER READING

The Chevalley-Serre Relations, commonly referred to as the "Serre relations," along with a specified Cartan matrix uniquely determine a semisimple complex Lie algebra. A complete treatment of an analogue of these Serre relations for finite dimensional simple contragredient Lie superalgebras is developed by Zhang in [5]. The associated Dynkin diagrams are also developed and a (slightly) novel positive and negative labeling is adopted to distinguish between some Dynkin diagrams that were previously indistinguishable in the literature [5]. It should be noted that Kac's original account of Lie superalgebras and his original classification in [2] includes associated Dynkin diagrams; for examples see Table V on p. 54 in [2]; for the general cases see Table VI on p. 56. For a refresher on how to construct Dynkin diagrams for (classical) semisimple Lie algebras, we refer to Tables I, II, and III on pp. 26-27 in [2], as well as to pp. 319 – 327 in [6]. We would be remiss if we did not mention a textbook on the subject of Lie superalgebras and so we refer the interested reader to [7].

An extremely interesting result using Lie superalgebras is the computation of the Alexander polynomial. An expository account of this can be found in [4]. A key ingredient in the construction in [4] is the R-matrix of deformation of the quantum Lie superalgebra $U_q\mathfrak{gl}\,(1|1)$ by a parameter h. Note that the quantum group is formed as a deformation of the universal enveloping algebra $U\mathfrak{gl}(1|1)$, which is done by taking the lie superalgebra $gl(1|1)$ that we developed in Section III, and keeping the same defining relations, but replacing the resulting evaluation of the supercommutator $[X, Y]$ as,

$$\frac{q^E - q^{-E}}{q - q^{-1}} \tag{4.1.3}$$

instead of just E in 3.1.6 (see p. 59 in [3] for more details). Note that here $q = e^h$, where h was the aforementioned parameter. Background on the R-matrix can be found on p. 18 of [8], or p. 60 of [3], or p. 8 of [4], and a definition of a quantum group $U_q(\mathfrak{g})$ can be found on p. 19 in [8].

A few references providing definitions and background on the Alexander polynomial are [9], [10], and [11]. In [9], two approaches are discussed: 1) a geometric way of defining the Alexander polynomial using the cell complex structure of a knot complement, and 2) an algebraic approach that uses Fox calculus. In both sections, the corresponding calculation is shown for computing the Alexander polynomial of the trefoil.

In the case of the trefoil 3_1, the Alexander Polynomial is

$$t^2 - t + 1 \text{ [1]} \qquad \text{or} \qquad x + x^{-1} - 1 \text{ [12] [13]}, \tag{4.1.4}$$

depending on notation convention. The way to convert between these two is the fact that the Alexander polynomial is unique up to a unit, so multiplying $t^2 - t + 1$ by t^{-1} results in $t - 1 + t^{-1}$, which is equivalent to the notation used by $x + x^{-1} - 1$.

The Alexander polynomial for prime knots up to 7_7 are listed in Appendix 2 of [1]. Examples of how to compute the Alexander polynomial from Seifert matrices are on pp. 248 − 249 of the same source, [1].

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