# Understanding Liu's Functional Integral Construction of TQFT

Alex Leonardi[1, 2, *]

[1] Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA
[2] Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138, USA

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We develop Liu's framework for $(n + 1)$ unitary alterfold TQFT and explain Liu's examples of $0 + 1$ and $1 + 1$ TQFT in the framework. This is a final project for Physics 216: Mathematics of Modern Physics, and as stressed in the course, we place particular emphasis on the idea of reflection positivity. The appendices include material on the Wightman axioms, the Osterwalder-Scrader Axioms, and Frobenius algebras.

## I. PRELIMINARIES

Topological quantum field theories (TQFTs) have been a rich field of research, drawing the attention of physicists and mathematicians alike. Witten constructed a 2+1 TQFT through Chern-Simons theory, a 3+1 TQFT, and an n-dimensional TQFT with Dijkgraaf, though higher dimensional cases have been difficult to fully categorize and study [1]. Recently, a functional integral approach for constructing n + 1 TQFTs has been developed by Liu and called an alterfold TQFT [1] [2] [3] [4]. Liu's framework includes these existing TQFTs that are known, in addition to paving the way to new TQFTs.

TQFTs can be formulated using the set of axioms for TQFT that Atiyah specified, where the data of the theory are assumed to satisfy certain conditions [5]. We outline Atiyah's approach as follows,

1. The data:
   (a) ground ring $\Lambda$
      i. For our purposes, we take $\Lambda$ as a field $\mathbb{K}$
   (b) $d$-dimensional, oriented, closed, smooth manifold $W$
      i. associated $\Lambda$-module $Z(W)$ that is finitely generated
   (c) $(d + 1)$-dimensional, oriented, smooth manifold $Q$
      i. associated element $Z(Q) \in Z(\partial Q)$

Note that we consider $\partial Q = W$ and that $Z(W) \in \mathbb{K}$. In effect, we are describing the partition function $Z$ that requires a quantization to define a measure on a manifold to make sense of a path-integral formulation. This partition function is subjected to,

1. The conditions:
   (a) $Z$ is a functor
      i. For a diffeomorphism $\phi : W \to W'$ preserving orientation, $Z(\phi) : Z(W) \to Z(W')$ is an isomorphism
   (b) "$Z$ is involutory"
      i. This means $Z(W^*) = Z(W)^*$. Note $W^*$ has the opposite orientation of $W$. Recall that we are working with $\Lambda$ as the field $\mathbb{K}$, so $Z(W)^*$ is the dual module which is effectively a dual vector space (at least it is a dual vector space when $\mathbb{K}$ is $\mathbb{C}, \mathbb{R}$.
   (c) "$Z$ is multiplicative"
      i. Then $Z(W) = Z(W_1) \otimes Z(W_2)$ when $W = W_1 \sqcup W_2$.

In effect, we can sum up Atiyah's axioms by describing an $(n + 1)$ TQFT as arising from a symmetric monoidal functor $Z : \mathrm{Bord}_{n+1,n} \to \mathrm{Vect}_{\mathbb{K}}$ where $\mathrm{Bord}_{n+1,n}$ is the category of with $n$-manifolds as objects and $(n+1)$ cobordisms, and we also have that $\mathrm{Vect}_{\mathbb{K}}$ is the category of vector spaces over a field $\mathbb{K}$ [6].

A simple introduction to Atiyah's axioms for a TQFT is given in [7], though it should be noted that the author (Freed) states the caveat that many assumptions are made and certain specifics are omitted for the sake of clarity and

* alexleonardi@college.harvard.edu

building up the ideas from first principles. Nevertheless, it is a great resource for gaining a working understanding of TQFT and Freed gives references to material for further, more detailed descriptions, much of which includes his own research.

Freed remarks (see p. 3 in [7]) that his presentation of Atiyah's axioms for TQFT is more akin to Steenrod's axioms, which apply to homology theory, rather than being more akin to the Wightman axioms for a relativistic QFT. Our discussion in this paper focuses on Liu's functional integral construction of TQFT found in [1], and our approach will stress aspects of both Atiyah's axioms and the Wightman axioms, in addition to the Osterwalder-Schrader (OS) axioms for a unitary Euclidean QFT. We include a discussion of the OS axioms since they have a correspondance with the Wightman axioms, and also because the OS axioms are unique in that they were the first to present the idea of reflection positivity; the idea of reflection positivity is central to Liu's construction.

After presenting Liu's construction and showing how reflection positivity, among other axioms, are formulated in Liu's approach, we then turn to some simple examples using Liu's framework in 2D.

For reference, we provided a detailed account of the Wightman Axioms in Appendix B and Appendix C, the Osterwalder-Schrader Axioms in Appendix D, and functional topological field theory, including Frobenius algebras for 2D TQFT, in Appendix E.

## II. LIU'S FUNCTIONAL INTEGRAL CONSTRUCTION FOR TQFT

In section 6.4 of [1], in a way that resembles the definition of an $(n+1)$ cobordism, Liu defines an alterfold as an ordered triple,

$$(M^{n+1}, B^{n+1}, M^n) \tag{2.1.1}$$

where $M^{n+1}$ is an oriented, compact (n+1) manifold that is partitioned by a trivial and a bulk coloring, A and B respectively, where the intersection of the colored manifolds $A^{n+1} \cap B^{n+1}$ is the boundary of the partition, $M^n$.

Visually, we can depict this as:

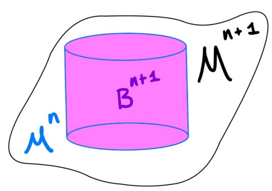



FIG. 1: The black manifold is $M^{n+1}$. The blue manifold is $M^n$ and it is the boundary of the purple manifold $B^{n+1}$.

In considering an n-dimensional lattice Hamiltonian H, Liu constructs the partition function $Z$, the operator algebra, the associated representations and categories, the (n+1) TQFT, and then the (n+1) unitary alterfold TQFT [8]. The lattice Hamiltonian motivates the labeled regular stratified piecewise linear manifold in the following manner:

- labeling gives the spin on the lattice,
- regularity gives the non-trivial data (neighborhood at any point factoring through),
- stratification allows us to deal with a manifold with all possible codimensions,
- and the piecewise linear component gives the shape of the lattice. [9]

In Theorem 6.42 found in [1], Liu performs this construction for an (n+1) unitary alterfold TQFT by enforcing the following conditions:

1. (RP) reflection positivity
2. (HI) homeomorphic invariance
3. (CF) complete finiteness

Here we see that homeomorphic invariance allows Liu to follow Atiyah's first condition that $Z$ be a functor, as the diffeomorphism $\phi$ is traded in for a homeomorphism $\phi$ between regular piecewise linear stratified manifolds. If we turn to discuss reflection positivity, the case for a general field $\mathbb{K}$ necessitates exchanging the reflection positivity condition with the requirement of strong semisimplicity [1]. The idea behind this is trying to enforce having a finite dimensional C$^*$ algebra. We have finite dimensionality in the TQFT, as this is the condition 3) (CF) complete finiteness. Then, when we consider a field $\mathbb{C}$ or $\mathbb{R}$, condition 1) (RP) reflection positivity, yields a C$^*$ algebra. This is precisely what allows us to have a Hilbert space describing the system. We note that a finite dimensional C$^*$ algebra is known to be semisimple, which is why when we allow the general field $\mathbb{K}$, we seek to enforce the condition of (strong) semisimplicity.

To construct the partition function $Z$ that is eventually defined on the alterfold in 1.1.1, we begin by specifying the vectors and vector spaces involved, that will give rise to the TQFT under the conditions RP, HI, and CF.

Consider the following notation that Liu develops in 2.8 [1],

$$\text{closed unit disk } D^n = [-1,1]^n \tag{2.1.2}$$
$$\text{boundary unit sphere } S^{n-1} = \partial D^n \tag{2.1.3}$$
$$\text{positive half n-disk } D^n_+ = D^{n-1} \times [0,1] \tag{2.1.4}$$
$$\text{negative half n-disk } D^n_- = D^{n-1} \times [-1,0] \tag{2.1.5}$$

We also specify, as Liu does in Definition 2.36 [1],

$$\text{set of piecewise linear stratified manifolds } LS_k \tag{2.1.6}$$
$$\text{elements } \mathcal{S} \in LS_k \tag{2.1.7}$$
$$\text{support } S^{n-k-1} \qquad 0 \le k \le n \tag{2.1.8}$$

With this in mind, let us now develop the Hilbert space in the following manner. Consider an $(n+1)$ hyper-disk algebra $V$ formed by

$$\text{closed unit disk } D^{n+1} = [-1,1]^{n+1} \tag{2.1.9}$$
$$= D^n \times [-1,1] \tag{2.1.10}$$

$$\text{boundary unit sphere } S^n = \partial D^{n+1} \tag{2.1.11}$$
$$= D^n_+ \cup D^n_- \tag{2.1.12}$$

Visually, $S^n$ can be shown as,

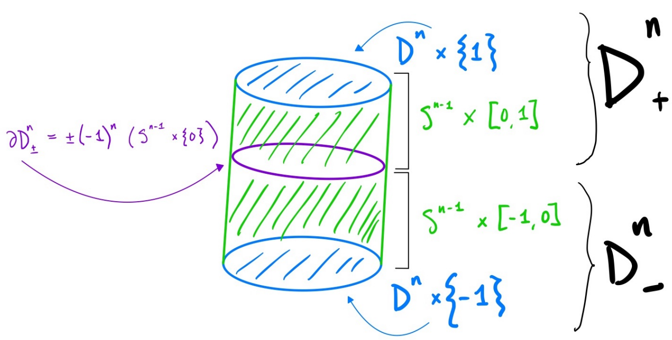



FIG. 2

$$\text{where } D^n_+ = D^n \times \{1\} \cup S^{n-1} \times [0,1] \tag{2.1.13}$$
$$D^n_- = D^n \times \{-1\} \cup S^{n-1} \times [-1,0] \tag{2.1.14}$$

Then the L-labeled stratified manifold $\mathcal{S}$ with support $D^n_\pm$ spans the vector space $V_{\mathcal{S},\pm}$.

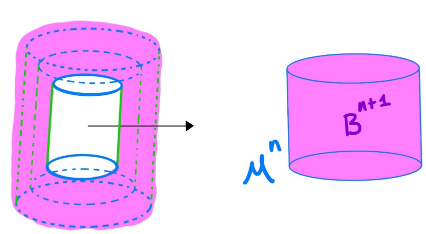



FIG. 3: (On left:) $S^n$ is the support of $V_{\mathcal{S},\pm}$. (On right:) Turning $V_{\mathcal{S},\pm}$ into a Hilbert space allows us to form $B^{n+1}$.

The operations of disjoint union and gluing extend linearly to labeled stratified manifolds. This lends itself to the conditions in Atiyah's axioms for a TQFT, as we can follow definition 3.5 in [1] and define the $S^n$ functional $Z$ as a bilinear form on $V_{\mathcal{S},-} \times V_{\mathcal{S},+}$,

$$Z(\ell_- \times \ell_+) := Z(\ell_- \cup \ell_+). \tag{2.1.15}$$

Using the partition function to specify null vectors, along with the fact that the $V_{\mathcal{S},\pm}$ vector spaces are pre-Hilbert spaces, in a fashion similar to the Wightman reconstruction theorem, Liu quotients out by the kernel

$$K_{\mathcal{S},\pm} = \{\ell_n \in V_{\mathcal{S},\pm} \mid Z(\ell_n \cup v) = 0 \; \forall v \in V_{\mathcal{S},\mp}\} \tag{2.1.16}$$

to construct the Hilbert space

$$\widetilde{V}_{\mathcal{S},\pm} := V_{\mathcal{S},\pm}/K_{\mathcal{S},\pm}. \tag{2.1.17}$$

In definition 3.10 [1], Liu defines the 180° rotation along $S^1$ of the last two coordinates as $\rho$, which induces a linear map,

$$\rho : V_{\mathcal{S},\pm} \to V_{\rho(\mathcal{S}),\mp} \tag{2.1.18}$$

Liu defines reflection positivity in the following manner:
On a $\mathrm{D}^n$ algebra $V$ (which is a covariant representation $\pi$ of the operad $\mathcal{O}^n$, see definition 4.9) equipped with a partition function Z, we define a reflection $\theta$ as,

$$\theta : V_{\mathcal{S},\pm} \to V_{\mathcal{S},\mp} \qquad \text{such that} \qquad \theta\mathcal{T}\left(\bigotimes_{i\in I} v_i\right) = \mathcal{T}\left(\bigotimes_{i\in I} \theta(v_i)\right) \tag{4.29}$$

where $\mathcal{T}\left(\bigotimes_{i\in I} v_i\right)$ is a tangle $\mathcal{T} : \bigotimes_{i=1}^m V_{\mathcal{S}_i} \to V_{\mathcal{S}_0}$ on a stratified manifold $\mathcal{S}$ (see definition 4.29 in [1]).
In 4.31 [1], Liu defines a $S^n$ functional $Z$ to be Hermitian with respect to reflection if it satisfies,

$$Z(v_- \cup v_+)^* = Z(\theta(v_+) \cup \theta(v_-)) \qquad \forall \; v_\pm \in V_{\mathcal{S},\pm}. \tag{2.1.19}$$

We see that this is very similar to the first Wightman axiom, the hermiticity axiom W1, that is given in Appendix C as Eqn. C.1.6. Note that in order to equal the dual representation of the LHS, the argument of the partition function has the reflection $\theta$ applied and the order of the argument is inverted.

In 4.33 [1], Liu defines a hermitian $S^n$ functional $Z$ to be reflection positive over $\mathbb{K}$ when,

$$0 \le Z(v \cup \theta(v)) \qquad \forall \; v \in V_{\mathcal{S},\pm}. \tag{2.1.20}$$

where $\theta(v) \in V_{\mathcal{S},-}$ when $v \in V_{\mathcal{S},+}$, or we have that $\theta(v) \in V_{\mathcal{S},+}$ when $v \in V_{\mathcal{S},-}$. We can compare this to the reflection positivity condition (see Eqn. D.1.7) presented in Appendix D on the Osterwalder-Schrader axioms. In the OS axiom on RP, we note that the inner product uses a vector (the ket) and then takes the conjugate reflected representation of the ket (the bra) and stipulates that their inner product be positive semi-definite.

We turn our attention now to the underlying algebra. In particular, in section 5.1 Liu demonstrates for $|\mathcal{D}^{n-1}| = D^{n-1}$, we have an associative algebra $\widetilde{V}_{\partial(\mathcal{D}^{n-1}\times D^1)}$ and that when the functional $Z$ is reflection positive, then it is not only an associative algebra, but also a $C^*$ algebra. The identity element in this algebra is $\mathcal{D}^{n-1}$, which can be seen from how the trace is defined in the 1D case as,

$$\mathrm{Tr}(x) := Z(x_- \cup \rho(D^1_-)) \tag{2.1.21}$$

where $\rho$ is a 180° rotation in the last coordinate (in the 1D case it is a change in sign). We see that when we replace the identity $D^1_-$ with a vector $y_-$, the trace behaves in the expected way,

$$\mathrm{Tr}(xy) = Z(x_- \cup \rho(y_-)) \tag{2.1.22}$$
$$= Z(\rho(\rho(y_-)) \cup \rho(x_-)) \qquad \text{via hermiticity} \tag{2.1.23}$$
$$= Z(y_- \cup \rho(x_-)) \qquad \text{simplifying the representation} \tag{2.1.24}$$
$$= Tr(yx) \qquad \text{by definition of the trace} \tag{2.1.25}$$

Liu provides a this proof by isotopy and shows the corresponding visualization in section 5.1 on Idempotent Completion [1].

The algebra for $\widetilde{V}_{\partial(\mathcal{D}^{n-1}\times D^1)}$ can also be made commutative (not just associative). We denote this commutative algebra as $A(\mathcal{D}^{n-1} \times D^1)$, which identifies it with its identity element. The proof of commutatitivity of elements in this algebra $A(\mathcal{D}^{n-1} \times D^1)$ follows by isotopy,

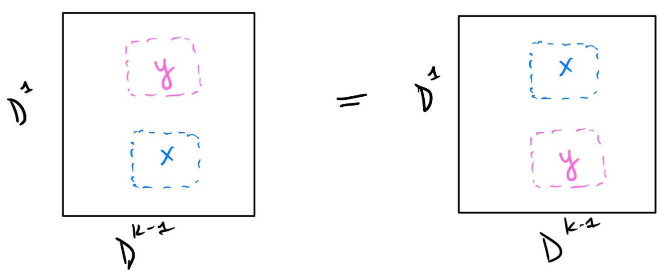



FIG. 4

In a manner similar to the conjugate reflected representations in the OS axioms (see Appendix D), we place an emphasis on $\theta$ as a reflection and $\rho$ as a $180°$ rotation, as we have seen earlier in this paper. Then we have that these operations obey $\rho\theta = \theta\rho$ on $V_{\mathcal{S},\pm}$, as visually we have that,

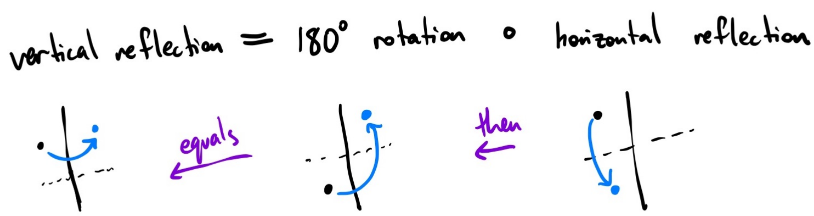



FIG. 5

Then in 5.15 and 5.16 [1], Liu defines the adjoint

$$x^* := \rho\theta(x) \qquad \text{where } x \in A(\mathcal{D}^{n-k} \times D^k) \tag{2.1.26}$$

which then gives us that $\mathrm{Tr}(x^*) = \mathrm{Tr}(x)^*$. Thus, the adjoint makes $A(\mathcal{D}^{n-k} \times D^k)$ into a *-algebra and when $A(\mathcal{D}^{n-k} \times D^k)$ is finite dimensional, along with Z being reflection positive, then $A(\mathcal{D}^{n-k} \times D^k)$ is a $C^*$-algebra.

In Section 6, Liu builds up the theorems and lemmas that lead to proving Theorem 6.42 [1], which is exactly that an $(n + 1)$ unitary alterfold TQFT results from an $S^n$ linear functional $Z$ over the field $\mathbb{C}$ being reflection positive, homeomorphic invariant, and complete finite. There is an added condition that the $\mathcal{S}^{k-1}$ relation defined in 6.4 be nondegenerate.

## III. 0+1 TQFT EXAMPLE IN LIU'S FRAMEWORK

In 0+1 TQFT, we have that $n = 0$ and so we work with a $D^0$ algebra. This $D^0$ algebra gives us the vector space $V$, which has finite dimension dim $V$. We may work with $m$ such objects and then have the tensor product of their vector spaces as $V^{\otimes m}$.

Recall from Eqn. 2.1.3 that $S^n = \partial D^{n+1}$. For $n = 0$, we have $S^0 = \partial D^{0+1}$. So then $S^0 = \{\pm 1\}$ is the boundary of the interval $D^1 = [-1, 1]$.

The partition function is then the $S^0$ functional $Z : V \otimes V \to \mathbb{K}$ which is a non-degenerate inner product.

For a reflection positive partition function $Z$, $V$ becomes a Hilbert space. The associated invariant is $\dim(V)^{\# S^1}$ because this counts the number of disjoint circles whose disjoint union form the 1-manifold. Liu discusses this example at the beginning of section 7 in [1].

## IV. 1+1 TQFT EXAMPLE IN LIU'S FRAMEWORK

In section 5.10 [1], Liu defines a minimal idempotent $\alpha$ of the algebra $A(\mathcal{D}^k \times D^{n-k})$ as an indecomposible k-morphism of type $|\mathcal{D}^k|$ and in 5.12 defines its quantum dimension as $\mathrm{Tr}(\alpha)$. This trace is nondegenerate when the algebra is semisimple. A proof of this (Prop. 5.13) is as follows: Assume that $\mathrm{Tr}(\alpha) = 0$. Then $\alpha x \alpha = c\alpha$ for some $x \in A$ and $c \in \mathbb{K}$. Taking the trace of this yields, $\mathrm{Tr}(x\alpha) = \mathrm{Tr}(\alpha x \alpha) = \mathrm{Tr}(c\alpha) = \mathrm{c\ Tr}\ (\alpha) = (\mathrm{c})(0) = 0$, which implies that $\alpha$ is null. But semisimplicity means that the indecomposible k-morphism $\alpha$ is non-trivial, so we arrive at a contradiction.

In 1+1 TQFT, we have $n = 1$ and we work with the semisimple algebra $D^1 = \bigoplus_i M_{n_i}(\mathbb{K})$. The $S^1$ functional $Z$ is the trace and when it is reflection positive, we have that $V$ is a $C^*$ algebra.

We can now consider a minimal idempotent $p_j$ of the algebra. It has a trace $\mathrm{Tr}(p_j) = d_j$, which is non-degenerate because the algebra is semisimple. The $\mathcal{S}^{k-1}$ relations (given as 6.7 in [1]) in $1 + 1$ TQFT, as we have that $0 \leq k \leq n + 1 = 2$, are given in the diagrams,

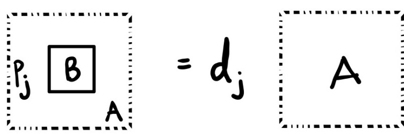



FIG. 6: (On left:) Minimal idempotent $p_j$. (On right:) Contracting the idempotent evaluates to $\mathrm{Tr}(p_j) = d_j$.
Overall, Figure 6 demonstrates the idea of a counit $\epsilon$ (see Appendix E on Frobenius algebras).

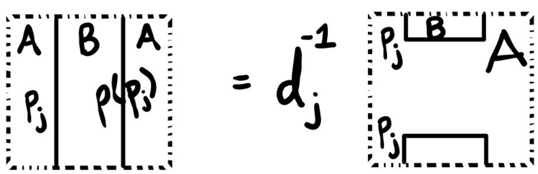



FIG. 7: (On left:) There is one face (the middle strip). The left edge is labelled with the minimal idempotent $p_j$ and the right edge is labeled with the 180° rotated $\rho(p_j)$. (On right:) There are two faces, both have edges labeled by the minimal idempotent $p_j$. A normalization factor $d_j^{-1}$ appears.

Depending on whether the equality is read from left to right or vice versa, Figure 7 demonstrates the idea of a comultiplication $\Delta$ and multiplication $\mu$ (see Appendix F on Frobenius algebras). Figure 7 is the diagram as Liu presents it. To check it more explicitly, we can insert a minimal idempotent $p_j$ and glue sides with opposite orientation:

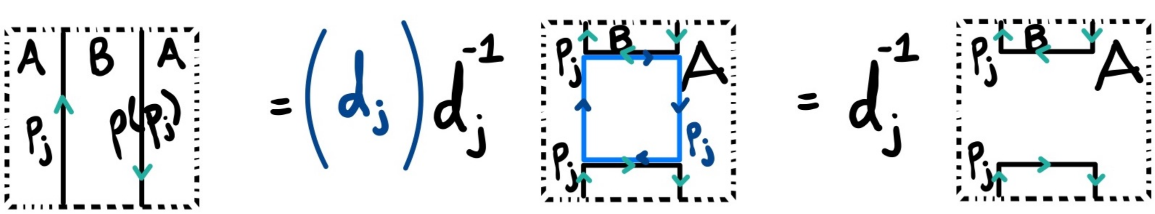



FIG. 8: (Middle:) Depending on the direction of the equality, inserting $p_j$ can be seen as using the unit or counit.

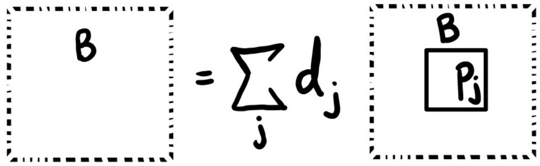



FIG. 9: Adding in the minimal idempotent requires multiplying by its trace Tr($p_j$)= $d_j$. This demonstrates the idea of a unit in a Frobenius algebra.

We arrive at the invariant,

$$Z(S) = \sum_j d_j^{n_2 - n_1 + n_0} = \sum_j d_j^E(S) \tag{4.1.1}$$

with $E(S)$ as the Euler number of the triangulated oriented surface $S$, and $n_k$ is the number of $k$-simplices in the triangulation. Let's examine this for each of the three relations,

$$\text{Figure 6 (Counit): 2 faces (A and B) - 1 edge } (p_j) + 0 \text{ vertices} = 1 \text{ face (just A) - 0 edges} + 0 \text{ vertices} \tag{4.1.2}$$

$$\text{Figure 7 (Comultiplication/Multiplication): 3 faces (A,B, and A) - 2 edges}(p_j \text{ and } \rho(p_j)) + 4 \text{ vertices} \tag{4.1.3}$$

$$= 3 \text{ faces (A,B, and B) - 2 edges}(p_j \text{ and } p_j) + 4 \text{ vertices} \tag{4.1.4}$$

$$\text{Figure 8 (Unit): 1 face (just B) - 0 edges} + 0 \text{ vertices} = 2 \text{ faces (B and interior) - 1 edge } (p_j) + 0 \text{ vertices} \tag{4.1.5}$$

And thus, we see that it is indeed invariant. It is also worth mentioning that in Section 7, Prof. Liu notes that we can consider a finite group algebra with a trace and then as a consequence of the Peter-Weyl theorem arrive at a partition function that is Mednykh's formula,

$$Z(S) = \#\mathrm{Hom}(\pi_1(S), G)|G|^{E(S)-1} = \sum_j \mathrm{dim} V_j^{E(S)}. \tag{4.1.6}$$

In exploring the literature on Dijkgraaf-Witten theory, the author came across [10], which contains a nice proof of Mednykh's formula in Section 2.3.

Further parts of section 7 of [1] discuss the cases $n = 2$ (TV TQFT, RT TQFT), $n = 3$ (Witten's 3+1 TQFT and Donaldson's invariant of 4-manifolds), and $n = n$ (Dijkgraaf-Witten TQFT).

We provide a brief summary of Liu's remarks on the $n = 2$ case as follows. Liu discusses Turaev-Viro invariants of 3-manifolds as being the invariants of the B-colored manifolds, along with mentioning Witten-Reschetikhin-Turaev TQFT, Turaev-Viro TQFT, Jones' work on $D^2$ planar algebras, and subfactor planar algebras that result from reflection positive and complete finite partition functions $Z$. In particular, Liu notes work on spherical fusion categories that uses a blow-up procedure for embedding TV TQFT and RT TQFT of the Drinfield center of the spherical fusion category into 2+1 alterfold TQFT.

## V. ENTANGLEMENT ENTROPY AND TENSOR NETWORK REPRESENTATION

In [11] Liu and Zhao find a tensor network representation for the toric code using reflection positivity. Reflection positivity holds in general for string-net Hamiltonians, as proved in [12], leading Liu and Zhao to suggest that a tensor network representation of the perturbation of string-net ground states can be found. This is one of many directions of current and future work that Prof. Liu's group is pursuing.

## VI. CONCLUSION

Alterfold theory proves to be a very useful tool in examining TQFT, as Liu demonstrates through his functional integral construction of TQFT. The key theorem that Liu builds in [1] is that a partition function satisfying the conditions of reflection positivity, homeomorphic invariance, and complete finiteness result in a unitary alterfold TQFT. In particular, this construction works in $n + 1$ dimensions. In our paper, we have shown the basics of how to build up Liu's framework and then applied it to two examples, 0+1 and 1+1 TQFT. For 2+1 TQFT, which we did not discuss in this paper, we refer the interested reader to section 7 of [1].

## VII. ACKNOWLEDGMENTS

We would like to thank Prof. Arthur Jaffe, Prof. Zhengwei Liu, and Shi Chen for many enlightening discussions related to QFT, Frobenius algebras, reflection positivity, and alterfold theory. Without their guidance, support, and encouragement, this final project would not have been possible. Along with my fellow classmates in Physics 216, they are what made Physics 216 such a wonderful learning experience!

## VIII. APPENDIX A: BACKGROUND ON WIGHTMAN AXIOMS AND OSTERWALDER-SCHRADER AXIOMS

In [13], Osterwalder and Schrader present a great background to relativistic QFT, the Wightman axioms, and the need for their Osterwalder-Schrader axioms for a unitary Euclidean QFT. We now summarize their background as containing the following main points:

1. Dyson introduced the idea of using imaginary time $it$ or imaginary energy $iE$ instead of time $t$ or energy $E$ as a technique to map to Euclidean space under this imaginary transformation. This circumvented problems caused by the indefinite Minkowski metric.

2. Schwinger investigated how the Euclidean group acted on time ordered Green's functions' analytic continuation to imaginary times.

   (a) Schwinger found these Euclidean Green's functions were solutions to specific differential equations. Euclidean Green's functions are commonly refered to as Schwinger functions.

3. Wightman, Hall, and Jost uncovered that the boundary values of Wightman functions are Green's functions, or Wightman distributions, and that they can be analytic. Then Wightman functions restricted to points with imaginary time and real space coordinates define Euclidean Green's (Schwinger) functions.

4. Symanzik preferred a Euclidean approach that sought to construct Euclidean Green's (Schwinger) functions, as constructing Wightman distributions (Green functions) could prove more difficult. This deferred the issue of continuing back to real time.

5. Nelson provided a generalization of Symanzik's approach by treating Euclidean fields as random variables, Green's functions as the expectations of the products of random variables, and Euclidean Markoff fields subject to conditions (including regularity) as a relativistic quantum field theory satisfying the Wightman axioms.

We note that by reading p. 46 in [14], we were made aware of an error in the orginal Osterwalder-Schrader axioms paper [13]. The exact error Simon refers to is unclear without a more thorough reading. Nevertheless, a formulation of the Wightman axioms (W1 - W5) can be found in [14] p. 50 - 51, while the Osterwalder-Schrader axioms (OS1 - OS5) appear in the same source, on p. 64-65, with a set of proofs following on p. 66-67.

We adopt the approach presented in [15], as it formulates W1-W5 and OS1-OS5 in a clear way, by referencing both the source material by Wightman and by Osterwalder and Schrader, in addition to the formulation of both set of axioms presented in [14].

The Osterwalder-Schrader axioms [13] for a unitary Euclidean quantum field theory are,

OS1) Temperedness
OS2) Euclidean covariance
OS3) Positivity
OS4) Symmetry
OS5) Cluster property

These conditions ensure that the Euclidean Green's (Schwinger) functions can be analytically continued such that the boundary values uniquely determine a collection of Wightman distributions [13]. Also please note that in the original paper [13], the numbering of the axioms appears as E0-E4, and this numbering corresponds to OS1-OS5, respectively. We adopt the latter numbering to make it more explicit when we reference the Osterwalder-Schrader axioms.

## IX. APPENDIX B: WIGHTMAN AXIOMS, AS FORMULATED IN 216 LECTURE

We begin by stating the Wightman Axioms, presented during Physics 216 lecture, for a relativistic quantum field theory:

1. A quantum field theory acts on a Hilbert space $\mathcal{H}$.
2. A unitary representation of the Poincaré group (Lorentz group and added translations) acts on $\mathcal{H}$.
   (a) The Lorentz group $\Lambda$ can be represented as $\Lambda = BR$, where B are boosts and R is for rotations.
   (b) We also consider spacetime translation $a = (t, \vec{x})$ with its unitary operator being expressed as $U_a = U_{(t,\vec{x})} = e^{itH - i\vec{x}\vec{p}}$.
   (c) The Poincaré group $\{a, \Lambda\}x = \Lambda x + a$ is then the semi-direct product of spacetime translations $a$ and the Lorentz group $\Lambda$: $\mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3)$.
   (d) The Poincaré group obeys the multiplication law $\{a_1, \Lambda_1\}\{a_2, \Lambda_2\} = \{\Lambda_1 a_2 + a_1, \Lambda_1\Lambda_2\}$.
   (e) The Poincaré group contains the inverse element $\{a, \Lambda\}^{-1} = \{-\Lambda^{-1}a, \Lambda^{-1}\}$.
   (f) The Poincaré group contains the identity element $\{a = 0, \Lambda = 1\}$.
3. The representation of the Poincaré group acts naturally on the field.
   (a) This can be expressed as $U_{\{a,\Lambda\}}\varphi(x)U^*_{\{a,\Lambda\}} \underbrace{=}_{\text{scalar}} \varphi(\{a, \Lambda\}x)$.
4. Fields commute or anti-commute at space-like separation.
5. We assume that the vacuum vector is cyclic. This entails $\{\varphi(f_1)\dots\varphi(f_n)\Omega\} = \mathcal{H}$.

## X. APPENDIX C: WIGHTMAN AXIOMS, AS FORMULATED IN [15]

As in Section 2.1 in [15], we present the Wightman axioms in the following manner:
Consider the local operators $\varphi_i(x), x \in \mathbb{R}^{1,d-1}$ and the $n$-point correlators $\langle\varphi_1(x_1)\dots\varphi_n(x_n)\rangle$ in a unitary QFT in Minkowski space. These $n$-point correlators are Wightman functions and they must be tempered distributions, meaning that they can be paired with associated Schwartz test functions $f(x_1, \dots, x_n)$. For a brief reminder of what Schwartz test functions are, consider the following definitions given in [16]:

> The space of *rapidly decreasing functions*, aka test functions or Schwartz functions, is expressed as,
>
> $$f \in \mathcal{S}(\mathbb{R}) \text{ s.t. } \sup_x |x^n(\partial^{(k)}f)(x)| < \infty \text{ where } 0 \le k, n \tag{1.7}$$
>
> A tempered distribution $\Phi$ is also called a *generalized function*, and it is a complex linear map $\Phi : \mathcal{S} \to \mathbb{C}$. Now consider the definition,
>
> $$\int_{\mathbb{R}} dx\ f(x)\ \Phi(x) := \Phi(f) \tag{1.9}$$
>
> where $\Phi(x)$ on its own contains no meaning (unless it is a function) since a quantum field cannot be given at a point. Smearing, or integrating against the test function, contextualizes this value.

The Wightman functions are invariant under translation and $\mathrm{SO}(1, d-1)$. We do not consider fermionic operators and spinor representations, but rather focus on the case of bosonic operators for clarity. There is a choice of basis for local operators $\mathcal{O}_i$ that transform under irreducible representations of $\mathrm{SO}(1, d-1)$ that are denoted $\rho_i$. The invariance of the Wightman functions is then expressed,

$$\mathcal{O}_i^p(x) \to \rho_i(h)^p_q \mathcal{O}_i^q(h^{-1}x), \qquad \text{where } h \in \mathrm{SO}(1, d-1) \text{ and } p, q \in \{1, \dots, \dim(\rho_i)\} \tag{C.1.1}$$

We also need to define the complex vector space $\mathcal{C}$ as having elements that are finite linear combinations of components of local operators $\mathcal{O}_i$. Now we can state the correspondance between the Wightman axioms as they appear in [15] with W1-W5 in [14] as follows,

1. Local commutativity/microcausality (W5)

   (a) When operators are spacelike separated, i.e. $0 < (x_j - x_{j+1})^2$, then we have that the operators commute, $\langle \varphi_1(x_1)\ldots\varphi_j(x_j)\varphi_{j+1}(x_{j+1})\ldots\varphi_n(x_n)\rangle = \langle \varphi_1(x_1)\ldots\varphi_{j+1}(x_{j+1})\varphi_j(x_j)\ldots\varphi_n(x_n)\rangle$.

2. Clustering (W2)

   (a) For spacelike vectors $a$, as $\lambda \to \infty$, spacelike separated correlators factorize according to,

$$\langle \varphi_1(x_1)\ldots\varphi_j(x_j)\varphi_{j+1}(x_{j+1}+\lambda a)\ldots\varphi_n(x_n)\rangle \to \langle \varphi_1(x_1)\ldots\varphi_j(x_j)\rangle \;\; \langle \varphi_{j+1}(x_{j+1}+\lambda a)\ldots\varphi_n(x_n)\rangle \tag{C.1.2}$$

3. Spectral condition (W4)

   (a) From the fact that Wightman functions are translation invariant, we can express the Wightman functions as tempered distributions with one less dimension,

$$\langle \varphi_1(x_1)\ldots\varphi_n(x_n)\rangle = W(\xi_1,\ldots,\xi_{n-1}), \qquad \text{where } \xi_r = x_r - x_{r+1} \tag{C.1.3}$$

   (b) The fourier transform of $W$ is well defined and results in the tempered distribution,

$$\widehat{W}(k_1,\ldots,k_{n-1}) = \int W(\xi_1,\ldots,\xi_{n-1})e^{i\sum_{r=1}^{n-1}k_r\cdot\xi_r}d\xi_1\ldots d\xi_n \tag{C.1.4}$$

$$\text{with } k_r = (E_r,\vec{k}_r),\; \xi_r = (t_r,\xi_r),\; \text{and } k_r\cdot\xi_r = -E_r t_r + \vec{k}_r\cdot\xi_r \tag{C.1.5}$$

   (c) The spectral condition states that in the region $\vec{k}_r \leq E_k$, where $k \in \{1,2,3,\ldots,n-1\}$, formed by the product of closed forward light cones, this fourier transform $\widehat{W}$ is supported.

4. Hermiticity (W1)

   (a) Operators $\varphi_j$ must have conjugates $\varphi_j^\dagger$ such that conjugation is an involution $\dagger\dagger = \text{id}$, meaning $(\varphi_j^\dagger)^\dagger = \varphi_j$. This conjugation is a map $\dagger : \mathcal{C} \to \mathcal{C}$ such that $(\mathcal{O}_i^p) \mapsto (\mathcal{O}_i^p)^\dagger$.

   (b) The map $\dagger$ is anti-linear: $(c_1\mathcal{O}_1 + c_2\mathcal{O}_2)^\dagger = c_1^*\mathcal{O}_1^\dagger + c_2^*\mathcal{O}_2^\dagger$.

   (c) Note that we have not made the assumption that $\varphi^\dagger$ is an adjoint operator in a Hilbert space. We have not made this assumption, because we have not assumed that we are working with Hilbert spaces.

   (d) The hermitian condition is to invert the order of the conjugate operators as,

$$\overline{\langle \varphi_1(x_1)\ldots\varphi_n(x_n)\rangle} = \langle \varphi_n^\dagger(x_n)\ldots\varphi_1^\dagger(x_1)\rangle \tag{C.1.6}$$

5. Positivity (W3)

   (a) Associate local operators $\varphi_1,\ldots,\varphi_n \in V$ and a complex Schwartz function $f$ with the ket state $|\psi(f,\varphi_1,\ldots,\varphi_n)\rangle$.

   (b) Define an inner product on the ket states:

$$\langle \psi(h,\chi_1,...,\chi_n)|\psi(f,\varphi_1,...,\varphi_n)\rangle := \int \overline{h(x_1,...,x_m)f(y_1,...,y_n)}\,\langle \chi_m^\dagger(x_m)...\chi_1^\dagger(x_1)\varphi_1(y_1)...\varphi_n(y_n)\rangle\, dydx \tag{C.1.7}$$

   (c) The positivity condition is then,

$$0 \leq \langle \Psi|\Psi\rangle\,, \qquad \text{where } |\Psi\rangle \in \mathcal{H}_0 \tag{C.1.8}$$

The Wightman reconstruction theorem then constructs the Hilbert space $\mathcal{H}$ for the QFT by taking $\mathcal{H}_0$ and modding out the vectors of zero norm that it contains. Remark 2.2 in [15] points out that hermiticity (W3) can be derived as a consequence of the positivity property (W5).

## XI. APPENDIX D: OSTERWALDER-SCHRADER AXIOMS, AS FORMULATED IN [15]

In [15] the OS axioms are formulated in the following manner:
Consider the n-point correlators known as Schwinger functions, that are invariant under SO($d$),

$$\langle\varphi_1(x_1)\dots\varphi_n(x_n)\rangle, \qquad \text{where } \varphi_i \in \mathcal{C},\ x_i \in \mathbb{R}^d,\ \text{and } x_i \neq x_j\ \forall\ i,j \tag{D.1.1}$$

Unlike the Schwartz test functions (that are paired with Wightman functions), the Schwinger functions must vanish at coincident points, hence we imposed $x_i \neq x_j\ \forall\ i,j$. We also operate under the assumption that the Schwinger functions are real-analytic and decay faster than some power (see 2.15 in [15] for the precise law). Also, in considering the bosonic case, we have commutativity of local operators.

Consider the action of the irreducible representations of SO($d$), we note that these irreps $\rho$ are tensors $T^{\mu_1\cdots\mu_l}$. We have that the action of the irreps on these tensors is,

$$T^{\mu_1\cdots\mu_l} \to (\rho_i(h)T)^{\mu_1\cdots\mu_l} = h^{\mu_1}_{\nu_1}\dots h^{\mu_l}_{\nu_l}T^{\nu_1\dots\nu_l}. \tag{D.1.2}$$

Then the conjugate irrep $\overline{\rho}$ is the complex conjugate of $\rho(h)^p_q$ and compactness of the group SO($d$) identifies the conjugate representation $\overline{\rho}$ with the dual representation $\rho^*$.

Additionally, we require $\rho^R$, the reflected representation. It is expressed as matrices of the form

$$\rho^R(h) = \rho(h^R) \text{ such that } h^R = \Theta h \Theta \tag{D.1.3}$$

$$\text{where } h \in\ \mathrm{SO}(d) \text{ and } \Theta = \mathrm{diag}(-1,1,\dots,1). \tag{D.1.4}$$

Now composing both conjugation and reflection, we can identify $\overline{\rho}^R$ with the standard representation $\rho$. This is true in all cases except $d \equiv 0 \mod 4$, where (anti-)chiral tensors must be interchanged (see 2.17 in [15]).

We now work with the reflected positions,

$$x^\theta := \Theta x \tag{D.1.5}$$

and construct the involutive and anti-linear conjugation $\dagger : \mathcal{C} \to \mathcal{C}$. The conjugate local operators $\mathcal{O}_i^\dagger$ that it defines act as the conjugate reflected irreps $\overline{\rho_i}^R$. We can then formulate the hermiticity axiom as,

$$\overline{\langle\varphi_1(x_1)\dots\varphi_n(x_n)\rangle} = \langle\varphi_n^\dagger(x_n^\theta)\dots\varphi_1^\dagger(x_1^\theta)\rangle\,. \tag{D.1.6}$$

This Osterwalder-Schrader hermiticity axiom is analogous to W1, which we stated as Equation C.1.6.

We provide the following setup necessary for stating the Osterwalder-Schrader positivity axiom:
Consider the following,

1. local operators $\varphi_1,\dots,\varphi_n \in \mathcal{C}$,
2. a complex compactly supported Schwartz test function $f(x_1,\dots,x_n)$,
3. that $f$ is nonzero only when $x_n^0 < \cdots < x_2^0 < x_1^0 < 0$,
4. and ket states $|\psi(f,\varphi_1,\dots,\varphi_n)\rangle$.

Define an inner product on the ket states as follows,

$$\langle\psi(h,\chi_1,\dots,\chi_m)|\psi(f,\varphi_1,\dots,\varphi_n)\rangle := \int \overline{h(y_1^\theta,\dots,y_m^\theta)}f(x_1,\dots,x_n) \tag{D.1.7}$$

$$\times\, \langle\chi_m^\dagger(y_m)\dots\chi_1^\dagger(y_1)\varphi_1(x_1)\dots\varphi_n(x_n)\rangle\, dydx \tag{D.1.8}$$

Then we construct the vector space $\mathcal{H}_0^{\mathrm{OS}}$ by linearity and anti-linearity, as finite linear combinations of the ket states $|\Psi\rangle$.

The Osterwalder-Schrader reflection positivity axiom is then the requirement that the inner product in D.1.7 is positive semi-definite,

$$0 \leq \langle\Psi|\Psi\rangle\,, \qquad \forall\ |\Psi\rangle \in \mathcal{H}_0^{\mathrm{OS}} \tag{D.1.9}$$

The key difference between C.1.7 (in W3) and D.1.7 (in OS3) is that D.1.7 contains reflections in the argument of h, with the reflections being as defined in D.1.5. Similar to the Wightman reconstruction, by modding out the zero vectors in $\mathcal{H}_0^{\mathrm{OS}}$, we can form the Hilbert space $\mathcal{H}^{\mathrm{OS}}$ for a unitary Euclidean QFT.

[15] also asserts that hermiticity need not be an axiom, as it is a condition that follows directly from Osterwalder-Schrader reflection positivity, analogous to the case for the Wightman axioms.

As our paper focuses on reflection positivity, we omit the remaining OS axiom (clustering) and instead direct the interested reader to 2.24 in [15] for a more complete treatment.

## XII. APPENDIX E: FUNCTIONAL TOPOLOGICAL FIELD THEORY

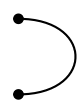
Most of the content in this section, II. Functional Topological Field Theory, is thanks to Shi Chen, who taught it during discussion section for Physics 216. Recall from (2.34) we can define a functional topological field theory as,

$$F : \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}} \tag{2.34}$$

where $F$ is a symmetric monoidal functor, $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ is the bordism category where objects are $(n+1)$ dimensional manifolds with morphisms that are n-dimensional cobordisms, $\mathcal{C}$ is a symmetric monoidal category, and $\mathrm{Vect}_{\mathbb{C}}$ is a vector space that can be decomposed into a tensor product of complex spaces.

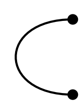
Let us first consider the 1 dimensional case, described by $\mathrm{Bord}_{\langle 1,0\rangle}$. Here we have several building blocks,

$$\emptyset \mapsto \mathbb{C} \tag{E.1.1}$$
$$\bullet \mapsto V \tag{E.1.2}$$

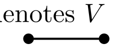


g denotes $V \times V \to \mathbb{C}$

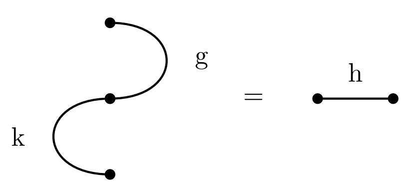
k denotes $\mathbb{C} \to V \times V$



h denotes $V \to V$



We can then examine the following braiding patterns,

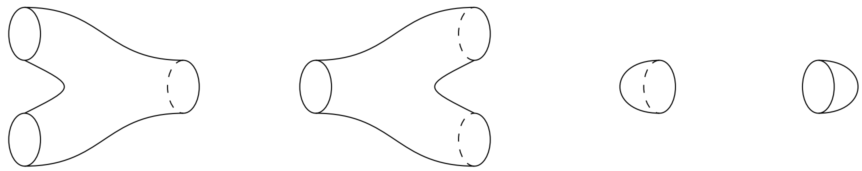



which corresponds to,

$$(g \otimes \mathbb{1}) \circ (\mathbb{1} \otimes k) = h = \mathbb{1}_V. \tag{E.1.3}$$

Note that this 1 dimensional case is general and that applying specific rules to it can result in the Temporley-Lieb algebra.

Let us now consider the 2 dimensional case. Here we have the 1 dimensional compact manifold $S^1$, equivalently referred to as the circle group $\mathbb{T}$. The partition function $Z$ applied to $S^1$ yields $Z(S^1) = V$, which is the vector space that physicists commonly attribute as a Hilbert space. Additionally, we have the 2 dimensional manifolds that are classified by genus and are known to be decomposable by handle-body decomposition in the following manner [17],



$\mu$ multiplication $Z(\mu) : V \otimes V \to V$ | $\Delta$ comultiplication $Z(\Delta) : V \to V \otimes V$ | $\eta$ unit $Z(\eta) : \mathbb{C} \to V$ | $\epsilon$ counit $Z(\epsilon) : V \to \mathbb{C}$

These five conditions: the partition function in the context of $S^1$, and having multiplication, comultiplication, a unit, and a counit, specify a 2 dimensional topological quantum field theory since we have the following,

1. $(A, \mu, \eta)$ is an algebra, because it satisfies the associativity condition and unit relation,

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes\mathbb{1}"] \arrow[d, "\mathbb{1}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \cong A \otimes \mathbb{C} \cong \mathbb{C} \otimes A \arrow[r, "\mathbb{1}\otimes\eta"] \arrow[d, "\mathbb{1}\otimes\eta"'] \arrow[dr, "\mathbb{1}"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

depicting the associativity condition, $\mu \circ (\mu \otimes \mathbb{1}) = \mu \circ (\mathbb{1} \otimes \mu)$.

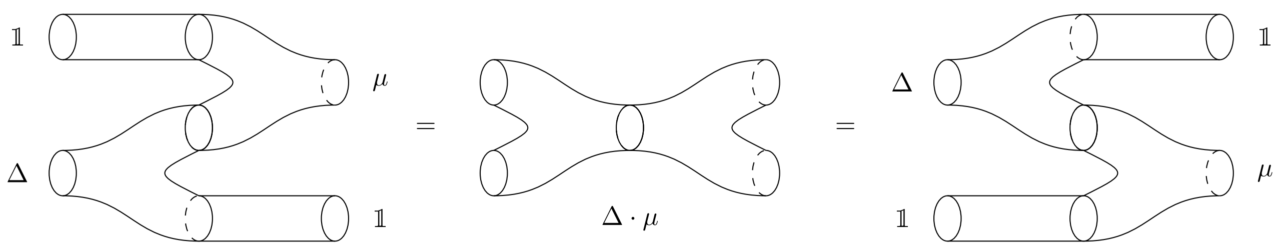
which shows the unit relation, $a \cdot e = e \cdot a = a$ for all $a \in A$.

2. $(A, \Delta, \epsilon)$ is a coalgebra, because it satisfies an analogous co-associativity condition and counit relation,

3. The algebra and coalgebra satisfy the Frobenius relation,

$$(\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \Delta) = \Delta \cdot \mu = (\mathbb{1} \otimes \mu) \circ (\Delta \otimes \mathbb{1}) : V \otimes V \to V \otimes V \tag{E.1.4}$$

By gluing the handlebodies, we can check that the algebra and coalgebra satisfy the Frobenius relation E.1.4 as follows,




$\Delta \cdot \mu$

More on these relations can be found in [18] on p.171. A more condensed version of [18] is found in [19]. A rigorous connection between frobenius algebras and TQFT is also established in [20].

Let us now consider the 3 dimensional case. Here, there is no straightforward classification of 3-manifolds. The best tool we can make use of is Kirby calculus, which gives us equivalence classes of 3-manifolds, but these equivalence classes contain too many different types of 3-manifolds that we do not know how to further classify. Another issue with higher dimensional TQFT we discussed in section and can be found in [7] is that, for the path-integral formulation (applies to the partition functional integral construction) of a TQFT, we require a notion of integration and we may encounter difficulty in assigning a suitable measure. A key technique in constructing such a measure is assigning a Haar measure at the group element, then using a measure on the conjugacy class of the element, and then integrating over the entire group by considering a volume form determined by the Jacobian, though this construction assumes that subspaces factor through the group in order to find these conjugacy classes of group elements (perhaps taking the limit of these conjugacy classes to find a notion of a maximal subspace/torus).

We do note that the theory for 2D TQFT does extend to higher dimensions through the higher-dimensional Frobenius algebra specified by the following $\mathbb{CP}^{N-1}$ model:

Consider the vector space

$$Z(S^1) = H^*(\mathbb{CP}^{N-1}) \cong \mathbb{C}[x]/(x^N = 0) \tag{E.1.5}$$

A basis for this space is $1, x, x^2, \ldots, x^{N-1}$ with nilpotency described by $x^N = 0$.

$$\epsilon : Z(S^1) \to \mathbb{C}x^k \mapsto \begin{cases} 1 & k = N-1 \\ 0 & otherwise \end{cases} \tag{E.1.6}$$

$$\Delta : Z(S^1) \to Z(S^1) \otimes Z(S^1)x^k \mapsto \left\{ \sum_{0 \leq j \leq N-1-k} x^{k+j} \otimes x^{N-1-j} \right. \tag{E.1.7}$$

Multiplication $\mu : V \otimes V \to V$ modulo $x^N = 0$:

$$\mu(x^i \otimes x^j) = \begin{cases} x^{i+j}, & \text{if } i+j < N, \\ 0, & \text{if } i+j \geq N. \end{cases} \tag{E.1.8}$$

Unit $\eta : \mathbb{C} \to V$ is given by $\eta(1) = 1 \in V$.

And the analogous comultiplication, counit, and frobenius relation can also be formulated for this model. Note that Proposition 5.8 in [1] constructs a commutative associate algebra $\tilde{V}_{\mathcal{S},-}$ that is denoted $A(\mathcal{D}^{n-k} \times D^k)$. It has unit $\mathcal{D}^{n-k} \times D^k$. This is what Liu uses to construct the Hilbert space in the alterfold TQFT.

## XIII. APPENDIX G: RECOMMENDED READING

We direct the interested reader to [21] for an account of how to prove the Wightman functions obey a positivity property that is analogous to the euclidean functions satisfying reflection positivity. The key ingredient is wedge reflection emulating euclidean time reflection.

For reflection positivity in a manifold setting that includes a curvature term, as well as for reflection positivity in the context of Dirac operators, we refer the reader to [22].

Recent work by Liu et al has placed alterfold TQFT in the context of modular fusion categories (see section 2.3 in [4]). Time permitting, we would like to explore this connection. A good reference for the category theory approach, as well as work on Drinfield centers by Drinfield himself, is [23] and [24], which provides the hexagon axioms on p. 17 (useful for comparision with the Pachner moves explored in [1]) and a description of quantum groups on p.19. Other references include [17], which builds up topological field theories.

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