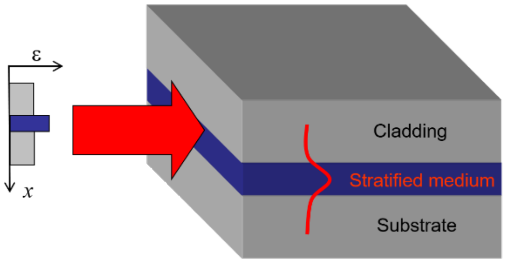
Computational Photonics

Seminar 04, 04.05.2024

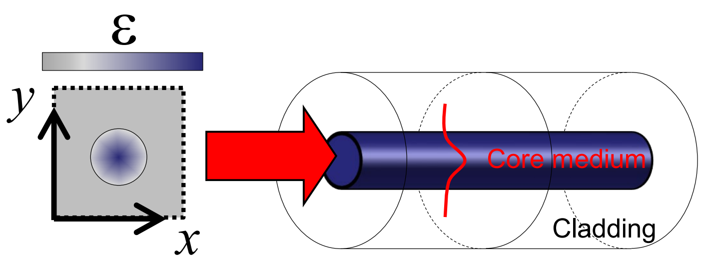
Implementation of a Finite-Difference Mode Solver

(2nd order finite difference schemes in matrix notation)

- Calculation of the guided modes in a **slab waveguide** system (1+1=2D)



- Calculation of the guided modes in a **strip waveguide** system (2+1=3D)



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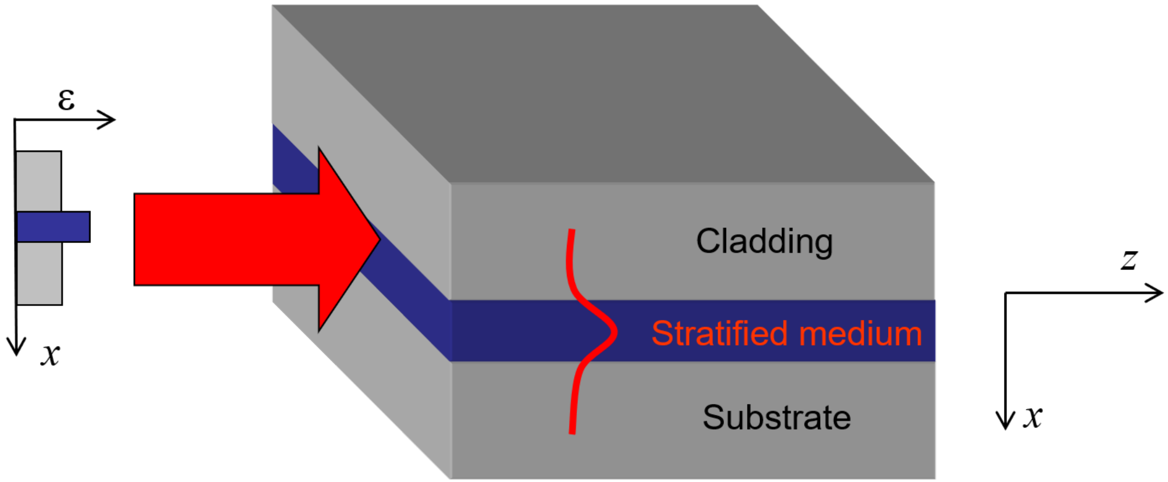
Guided modes in stratified media – recap matrix method



- no y-dependence
- phase evolution in the z-direction

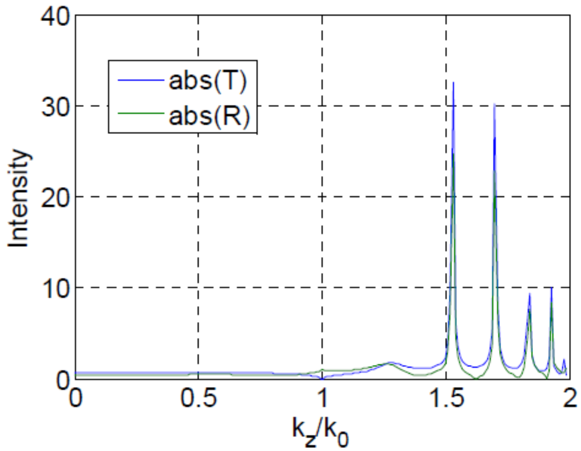
➔ looking for beams without diffraction in x-direction

Initially motivated in lecture using matrix method

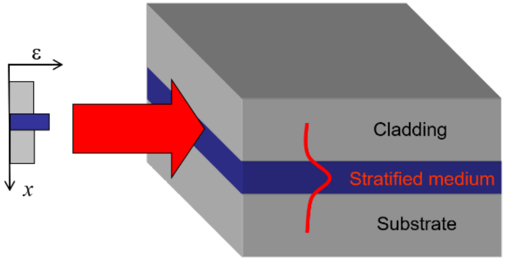
- looking for solutions which are evanescent in cladding/substrate but oscillating in "core"
- exist as resonant modes of the system independent of external excitation

  → roots of transmission/reflection problem

$$R = \frac{F_{\mathbf{R}}}{F_{\mathbf{I}}} = \frac{\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} - \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) - \mathbf{i}\left(M_{21} + \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right)}{\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} + \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) + \mathbf{i}\left(M_{21} - \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right)}$$



Guided modes in 1+1=2D systems (TE modes)



Assumptions:

- no y- and z-dependence
- weak guiding $\varepsilon_0\varepsilon(\omega)\,\mathrm{div}\,\mathbf{E}(\mathbf{r},\omega)\approx 0$

Helmholtz equation for inhomogeneous media:

$$\Delta\mathbf{E}(\mathbf{r},\omega)+\frac{\omega^2}{c^2}\varepsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)=0$$

Ansatz for stationary states (modes) with $\varepsilon(\mathbf{r},\omega)=\varepsilon(x,y,\omega)$

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_0(x,y)e^{i\beta z},\ \ \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}_0(x,y)e^{i\beta z}$$

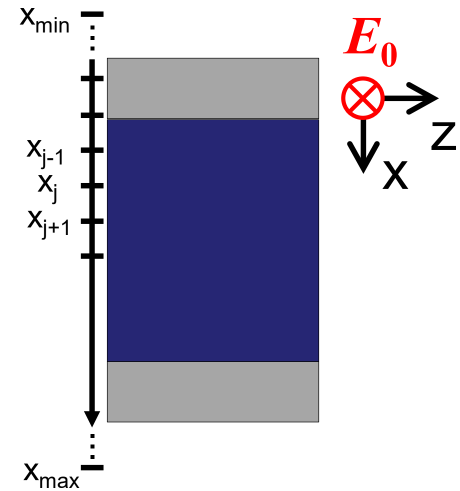
<u>For a 2D geometry & TE:</u>

$$\mathbf{E}_0(x)=\begin{pmatrix}0\\E_0\\0\end{pmatrix} \Rightarrow \boxed{\left(\frac{1}{k_0^2}\frac{\partial^2}{\partial_x^2}+\varepsilon(x,\omega)\right)E_0(x) = \varepsilon_{eff}E_0(x)}$$

With: $\varepsilon_{\text{eff}}=\left(\frac{\beta}{k_0}\right)^2$

Guided modes in 1+1D systems (TE modes)

Numerical solution by discretizing functions and operators:



1. Assuming perfectly electric conducting (PEC) boundaries

$$E_0(x_{\min}) = E_0(x_{\max}) = 0$$

2. Discretizing transverse space:

$$x_j = x_{\min}, x_{\min} + h, \ldots, x_{\max}$$

3. Discretizing the E-fields and $\boldsymbol{\varepsilon}$

$$E_j(x) = E_0\,(x_j),\ \varepsilon_j(x,\omega) = \varepsilon(x_j,\omega)$$

4. Discretizing 2[nd] order derivative:

$$\left(\frac{1}{k_0^2}\boxed{\frac{\partial^2}{\partial_x^2}} + \varepsilon(x,\omega)\right) E_0(x) = \varepsilon_{eff} E_0(x)$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_j} \approx \frac{f(x_{j+1}) - 2f(x_j) + f(x_{j-1})}{h^2}$$

Guided modes in 1+1D systems (matrix form)

Analytic equation of continuous variable x

$$\left(\frac{1}{k_0^2}\frac{\partial^2}{\partial_x^2}+\ \varepsilon(x,\omega)\right)E_0(x)=\ \varepsilon_{eff}E_0(x)$$

matrix diagonal: $-2/h^2+k_0^2\varepsilon$
adjacent diagonals: $1/h^2$

Algebraic equation of discrete variable x_i

$$\frac{1}{k_0^2}\begin{pmatrix} -\frac{2}{h^2}+k_0^2\varepsilon_1 & 1/h^2 & 0 & 0 & 0 & 0 & \cdots \\ 1/h^2 & -\frac{2}{h^2}+k_0^2\varepsilon_2 & 1/h^2 & 0 & 0 & 0 & \cdots \\ 0 & 1/h^2 & -\frac{2}{h^2}+k_0^2\varepsilon_3 & 1/h^2 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & \cdots \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}=\varepsilon_{\mathrm{eff}}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

`numpy.diag` and `numpy.linalg.eig` might be useful

Task I: Function header

```python
def guided_modes_1DTE(prm, k0, h):
    """Computes the effective permittivity of a TE polarized guided eigenmode.
    All dimensions are in µm.
    Note that modes are filtered to match the requirement that
    their effective permittivity is larger than the substrate (cladding).

    Parameters
    ----------
    prm : 1d-array
        Dielectric permittivity in the x-direction
    k0 : float
        Free space wavenumber
    h : float
        Spatial discretization

    Returns
    -------
    eff_eps : 1d-array
        Effective permittivity vector of calculated modes
    guided : 2d-array
        Field distributions of the guided eigenmodes
    """
    pass
```

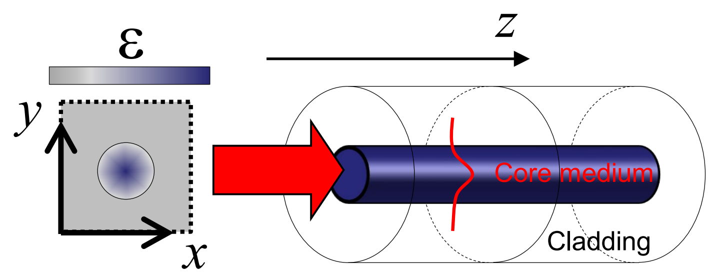
Task I: Assignment description

- Use the input variables: $\varepsilon(x,\omega), k_0, h$
- Assume a Gaussian waveguide profile: $\varepsilon(x,\omega) = \varepsilon_{\text{Substrate}} + \Delta\varepsilon e^{-(x/W)^2}$
- Select guided modes according to their eigenvalue: $\varepsilon_{\text{Substrate}} < \varepsilon_{\text{eff}} < \varepsilon(x)_{\max}$
- Use the given parameters (see separate files) for testing

Subtasks:

1. Calculate the **effective permittivity** and **field distribution** of the guided eigenmodes in TE-polarization and for PEC boundary conditions!
2. Show and discuss the dependence of your numerical solution on the discretization size *h*! (Hint: Consider the trade-off between convergence and computation time and motivate a reasonable cut-off value. `time.time` and `numpy.logspace` might be useful.)

Guided modes in 2+1 (=3D) systems (strip waveguide)
in scalar approximation



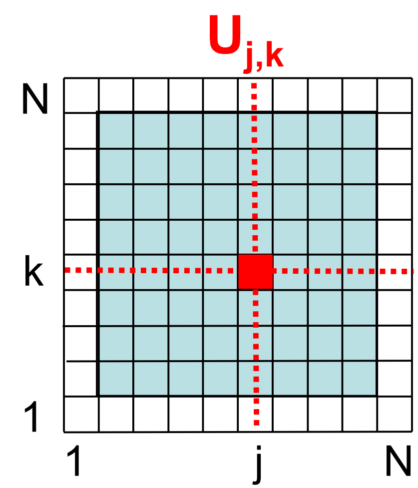
Assumptions:
- no z-dependence
- weak guiding: $\varepsilon_0 \varepsilon(\omega) \operatorname{div} \mathbf{E}(\mathbf{r},\omega) \approx 0$

Ansatz:
- scalar field and stationary states (modes)

$$v(\mathrm{r}) = u(x,y)\exp\left(\mathbf{i}\beta z\right)$$

- leads to scalar Helmholtz equation (eigenvalue problem for scalar field “intensity”)

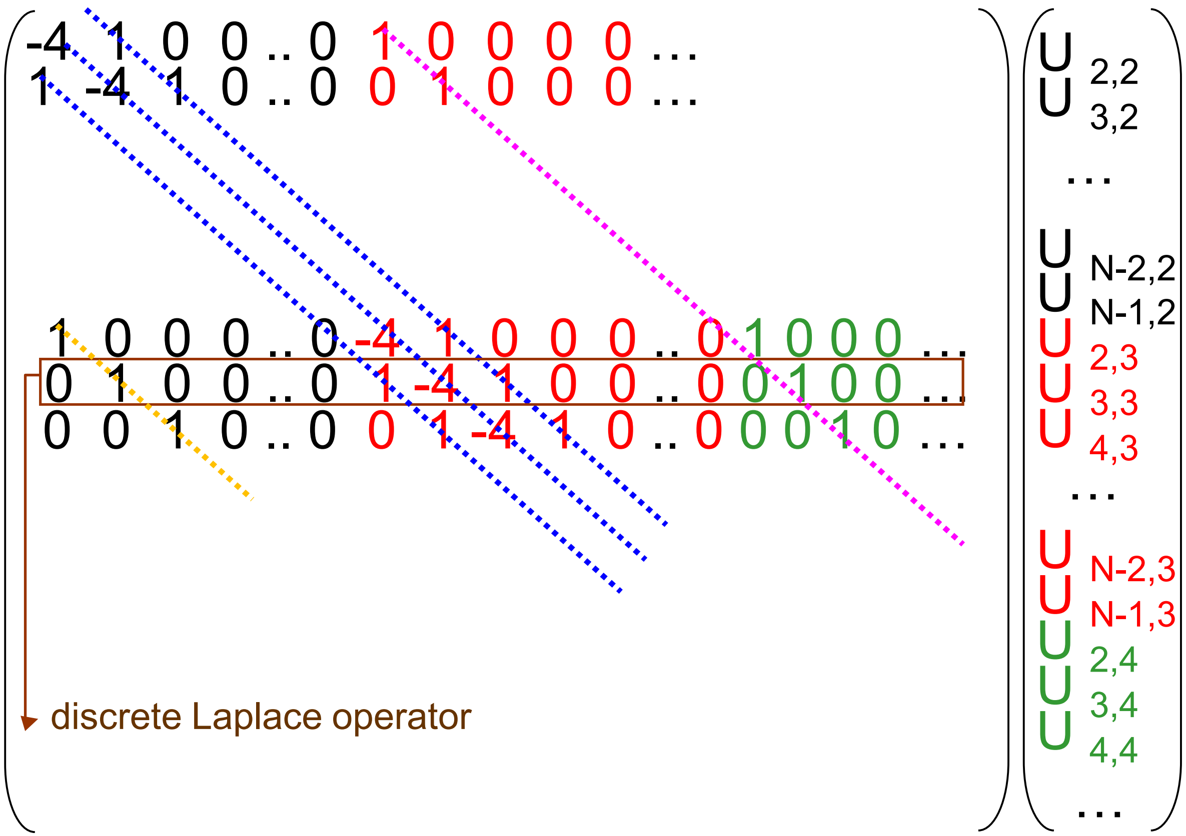
$$\left[\boxed{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}+k_0^2\varepsilon(x,y,\omega)\right]u(x,y)=\beta^2(\omega)u(x,y)$$

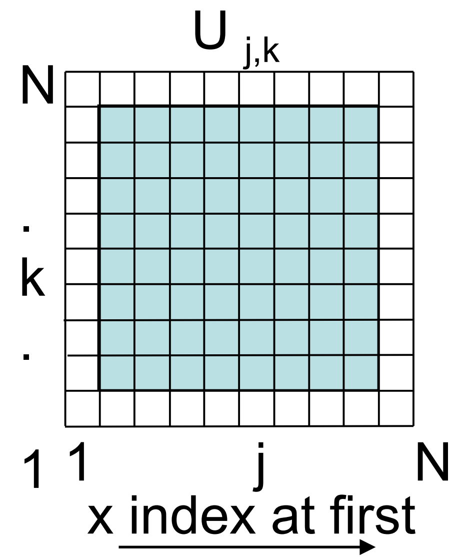


Discretization in 2D:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_j,y_k}+\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_j,y_k} \approx \frac{f\left(x_{j+1},y_k\right)+f\left(x_{j-1},y_k\right)+f\left(x_j,y_{k+1}\right)+f\left(x_j,y_{k-1}\right)-4f\left(x_j,y_k\right)}{h^2}$$

Numerical implementation of the 2D Laplace operator





Note: this only resembles the Laplace operator of the Helmholtz eq.

Differentiation with respect to x (same as 1D)

Differentiation with respect to y

Use the concept of sparse matrices: `scipy.sparse.linalg.eigs` and `scipy.sparse.spdiags` might be useful.

Task II: Function header

```
def guided_modes_2D(prm, k0, h, numb):
    """Computes the effective permittivity of a quasi-TE polarized guided
    eigenmode. All dimensions are in µm.

    Parameters
    ----------
    prm  : 2d-array
        Dielectric permittivity in the xy-plane
    k0 : float
        Free space wavenumber
    h : float
        Spatial discretization
    numb : int
        Number of eigenmodes to be calculated

    Returns
    -------
    eff_eps : 1d-array
        Effective permittivity vector of calculated eigenmodes
    guided : 3d-array
        Field distributions of the guided eigenmodes
    """
    pass
```

Task II: Assignment description

- Use the input variables: $\varepsilon(x,\omega)$, k_0, h, $numb$
- Assume a Gaussian waveguide profile: $\varepsilon(x,y,\omega) = \varepsilon_{\text{Cladding}} + \Delta\varepsilon \mathrm{e}^{-\frac{x^2+y^2}{W^2}}$
- Select guided modes according to their eigenvalue: $\varepsilon_{\text{Cladding}} < \varepsilon_{\text{eff}} < \max(\varepsilon(x,y))$
- Use the given parameters (see separate files) for testing

Subtasks:

1. Calculate the **effective permittivity** and **field distribution** of the scalar (quasi-TE polarized) guided eigenmodes for PEC boundary conditions!
2. Show and discuss the dependence of the numerical solution on the discretization size *h*! (Hint: Consider the trade-off between convergence and computation time and motivate a reasonable cut-off value. `time.time` and `numpy.logspace` might be useful.)

Summary Homework 1

- Solve tasks I & II.

- For each task we require that each group implements a program that solves the problem and documents the code and its result (e.g. with an iPython Notebook). This includes a graphical display of the result!

- Submission via email to: teaching-nanooptics@uni-jena.de

- **Due: 16.05.2024, 03:00 sharp**

- The subject line of the email should have the following format:
  - CPho24 - solution to the homework 1: group [Number]; [family_name1, family_name2, family_name3]:
  - gather all your files in a single zip archive
    (no rar, tar, 7z, gz or any other compression format)