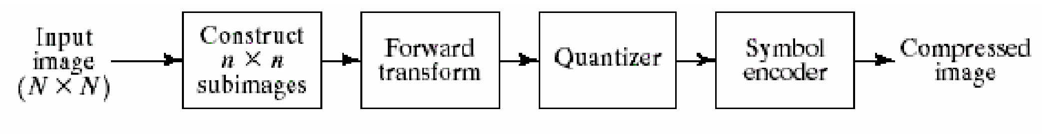
Solutions to Week 2 Lecture Notes
Exercise 2: KL Transform

(a)

The block diagram of a transform-based image compression scheme is given as follows:



Main components:

Image preparation:
Partition an $N \times N$ image into numerous $n \times n$ subimages (blocks) for transformation

Transformer (forward transform):
- Perform one-to-one transformation on each subimage.
- The transformed output is in a representation which is more suitable for efficient compression than the raw image data.
- Some transformation such as discrete cosine transform (DCT) packs the signal energy into a small number of coefficients.

Quantizer:
- Generate a limited number of symbols from the transformed coefficients
- An irreversible process, resulting in information loss

Symbol encoder:
- Assign a codeword to each symbol at the output of the quantizer
- Usually employ variable length coding (VLC) or entropy coding to assign shorter codewords to more probable symbols. This minimizes the average length of the codewords used to represent the symbols.
- Eg: Huffman coder

(b) The covariance matrix $\boldsymbol{C}$ has four eigenvectors: $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$

Therefore, it satisfies the following equation:

$$\boldsymbol{C}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \quad i = 1,2,3,4$$

when i=1:

$$\boldsymbol{C}\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1$$

$$\Rightarrow \begin{bmatrix} 5 & 5.5 & 4.5 & 7.5 \\ 5.5 & 7.25 & 8.25 & 10.5 \\ 4.5 & 8.25 & 14.25 & 15 \\ 7.5 & 10.5 & 15 & 21 \end{bmatrix} \begin{bmatrix} 0.6454 \\ -0.6916 \\ 0.3071 \\ -0.1043 \end{bmatrix} = \lambda_1 \begin{bmatrix} 0.6454 \\ -0.6916 \\ 0.3071 \\ -0.1043 \end{bmatrix}$$

根据一行即可算出λ1

$$\Rightarrow \lambda_1 = 0.036$$

when i=2:

$$\boldsymbol{C}\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2$$

$$\Rightarrow \begin{bmatrix} 5 & 5.5 & 4.5 & 7.5 \\ 5.5 & 7.25 & 8.25 & 10.5 \\ 4.5 & 8.25 & 14.25 & 15 \\ 7.5 & 10.5 & 15 & 21 \end{bmatrix} \begin{bmatrix} 0.1704 \\ 0.4808 \\ 0.4828 \\ -0.7118 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0.1704 \\ 0.4808 \\ 0.4828 \\ -0.7118 \end{bmatrix}$$

$$\Rightarrow \lambda_2 = 1.94$$

when i=3:

$$\boldsymbol{C}\boldsymbol{v}_3 = \lambda_3 \boldsymbol{v}_3$$

$$\Rightarrow \begin{bmatrix} 5 & 5.5 & 4.5 & 7.5 \\ 5.5 & 7.25 & 8.25 & 10.5 \\ 4.5 & 8.25 & 14.25 & 15 \\ 7.5 & 10.5 & 15 & 21 \end{bmatrix} \begin{bmatrix} 0.6946 \\ 0.3752 \\ -0.6138 \\ 0.0034 \end{bmatrix} = \lambda_3 \begin{bmatrix} 0.6946 \\ 0.3752 \\ -0.6138 \\ 0.0034 \end{bmatrix}$$

$$\Rightarrow \lambda_3 = 4.03$$

when i=4:

$$\boldsymbol{C}\boldsymbol{v}_4 = \lambda_4 \boldsymbol{v}_4$$

$$\Rightarrow \begin{bmatrix} 5 & 5.5 & 4.5 & 7.5 \\ 5.5 & 7.25 & 8.25 & 10.5 \\ 4.5 & 8.25 & 14.25 & 15 \\ 7.5 & 10.5 & 15 & 21 \end{bmatrix} \begin{bmatrix} 0.2681 \\ 0.3871 \\ 0.5439 \\ 0.6946 \end{bmatrix} = \lambda_4 \begin{bmatrix} 0.2681 \\ 0.3871 \\ 0.5439 \\ 0.6946 \end{bmatrix}$$

$$\Rightarrow \lambda_4 = 41.5$$

Therefore, the corresponding eigenvalues for $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ are $\lambda_1 = 0.036$, $\lambda_2 = 1.94$, $\lambda_3 = 4.03$, and $\lambda_4 = 41.5$, respectively.

(c)

As $\lambda_4 = 41.5, \lambda_3 = 4.03, \lambda_2 = 1.94$ are much greater than $\lambda_1 = 0.036$, a suitable number of principal components used to represent the image will be 3. This corresponds to 3 dominant eigenvectors on basis images $\underset{\sim}{v}_4, \underset{\sim}{v}_3$ and $\underset{\sim}{v}_2$.

To calculate the representation error

- First, express the subimage $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ as column vector $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

- To represent $\underset{\sim}{b}$ using 3 dominant principal components

$$\hat{\underset{\sim}{b}} = \sum_{i=2}^{4} c_i \underset{\sim}{v}_i$$

new basic function/image

(Note that $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ and $\underset{\sim}{v}_4$ form an orthonormal set)

$$= \sum_{i=2}^{4} (\underset{\sim}{b}^T \underset{\sim}{v}_i)\underset{\sim}{v}_i$$

$$\hat{\underset{\sim}{b}} = \sum_{i=2}^{4} (\underset{\sim}{b}^T \underset{\sim}{v}_i)\underset{\sim}{v}_i$$

$$= \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 0.1704 \\ 0.4808 \\ 0.4828 \\ -0.7118 \end{pmatrix}\right] \underset{\sim}{v}_2 + \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 0.6946 \\ 0.3752 \\ -0.6138 \\ 0.0034 \end{pmatrix}\right] \underset{\sim}{v}_3 + \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 0.2681 \\ 0.3871 \\ 0.5439 \\ 0.6946 \end{pmatrix}\right] \underset{\sim}{v}_4$$

$$= 0.6512 \begin{pmatrix} 0.1704 \\ 0.4808 \\ 0.4828 \\ -0.7118 \end{pmatrix} + 1.0698 \begin{pmatrix} 0.6946 \\ 0.3752 \\ -0.6138 \\ 0.0034 \end{pmatrix} + 0.6552 \begin{pmatrix} 0.2681 \\ 0.3871 \\ 0.5439 \\ 0.6946 \end{pmatrix}$$

$$= \begin{pmatrix} 1.030 \\ 0.968 \\ 0.014 \\ -0.005 \end{pmatrix}$$

Square error between $\underset{\sim}{b}$ and $\hat{\underset{\sim}{b}}$ is given as

$$= \left\| \underset{\sim}{b} - \hat{\underset{\sim}{b}} \right\|^2$$

$$= 0.0021$$

(d)

Limitations of KLT in practical image compression schemes:

- it is image-dependent,
- it requires estimation of image covariance matrix,
- it is computationally intensive

Example on eigen-decomposition

Determine eigenvalues & eigenvectors for $R = \begin{bmatrix} 1 & 0.9 & 0.2 \\ 0.9 & 1 & 0.1 \\ 0.2 & 0.1 & 1 \end{bmatrix}$

Eigen-decomposition:

$\mathbf{Rv} = \lambda\mathbf{v}$ -------- (1A)

$(\mathbf{R} - \lambda\mathbf{I})\mathbf{v} = 0$ --- (1B)

$\det(\mathbf{R} - \lambda\mathbf{I}) = 0$ (Characteristic equation)

$$\Rightarrow \det\left\{\begin{bmatrix} 1-\lambda & 0.9 & 0.2 \\ 0.9 & 1-\lambda & 0.1 \\ 0.2 & 0.1 & 1-\lambda \end{bmatrix}\right\} = 0$$

Solve for λ

$\lambda = 1.948, 0.958, 0.094$

To find eigenvectors:

When $\lambda = 1.948$, let $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Substitute into (1B)

$$\Rightarrow \begin{bmatrix} -0.948 & 0.9 & 0.2 \\ 0.9 & -0.948 & 0.1 \\ 0.2 & 0.1 & -0.948 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$\Rightarrow -0.948v_1 + 0.9v_2 + 0.2v_3 = 0$ ------- ①

$0.9v_1 - 0.948v_2 + 0.1v_3 = 0$ ------- ②

$0.2v_1 + 0.1v_2 - 0.948v_3 = 0$ ------- ③

②×2: $1.8v_1 - 1.896v_2 + 0.2v_3 = 0$ ------- ④

④–①: $2.748v_1 - 2.796v_2 = 0$

$v_2 = 0.9828v_1$ -------- ⑤

⑤→①: $-0.948v_1 + 0.8845v_1 + 0.2v_3 = 0$

$v_3 = 0.3175v_1$ -------- ⑥

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.9828 \\ 0.3175 \end{bmatrix} v_1$$

Normalized eigenvector $= \dfrac{1}{\sqrt{1 + 0.9828^2 + 0.3175^2}}\begin{bmatrix} 1 \\ 0.9828 \\ 0.3175 \end{bmatrix} = \begin{bmatrix} 0.696 \\ 0.684 \\ 0.221 \end{bmatrix}$