# Review of Single-phase AC Circuits

The voltage and current in an AC circuit are in the form of sinusoidal.

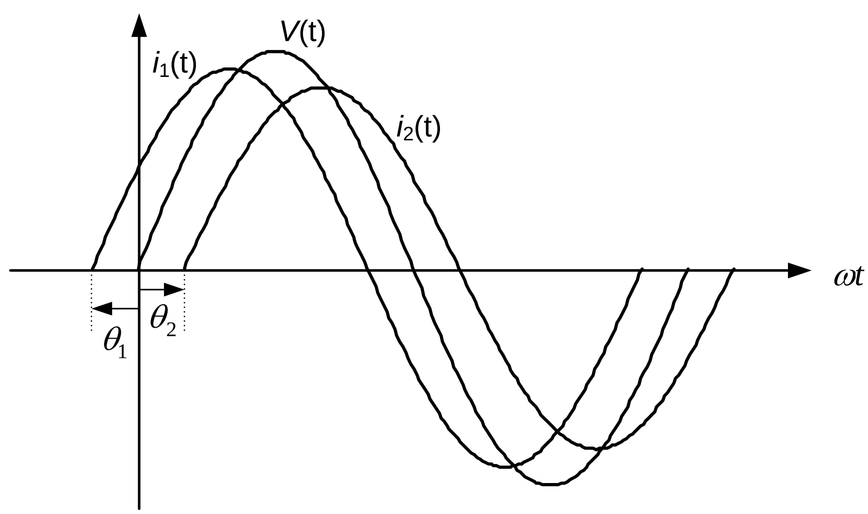
**In time domain**: a sinusoidal voltage or current can be expressed as

$$v(t) = V_m \sin(\omega t + 0^0) = \sqrt{2}\, V \sin(\omega t + 0^0) \text{ V}$$

$$i_1(t) = I_{m1} \sin(\omega t + \theta_1) = \sqrt{2}\, I_1 \sin(\omega t + \theta_1) \text{ A}$$

$$i_2(t) = I_{m2} \sin(\omega t - \theta_2) = \sqrt{2}\, I_2 \sin(\omega t - \theta_2) \text{ A}$$

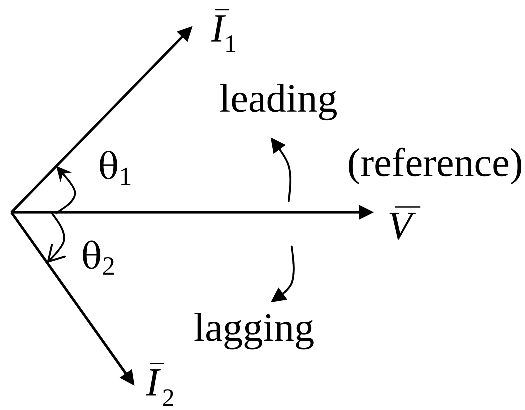
| | |
|---|---|
| $v$(t), $i_1$(t), $i_2$(t): | instantaneous values |
| $V_m$, $I_{m1}$, $I_{m2}$: | peak or maximum values |
| $V$, $I_1$, $I_2$: | root mean square (rms) values |
| $\theta_1$, $\theta_2$: | phase angles (in degree) |
| $\omega = 2\pi f$: | angular velocity in radian/sec |
| $f$: | frequency in Hz |



For a sinusoidal waveform

peak or maximum value = $\sqrt{2}\times$rms value $\Rightarrow V_m = \sqrt{2}\,V$

**In phasor domain**: a sinusoidal voltage or current (of same frequency) can be expressed in terms of its rms value and phase angle




$$\bar{V} = V\angle 0^0 \text{ V}$$
$$\bar{I}_1 = I_1\angle\theta_1 \text{ A}$$
$$\bar{I}_2 = I_2\angle -\theta_2 \text{ A}$$

| | |
|---|---|
| $\bar{V}$, $\bar{I}_1$, $\bar{I}_2$ : | phasors or complex numbers |
| $V$, $I_1$, $I_2$: | root mean square (rms) values |
| $\theta_1$, $\theta_2$: | phase angles (in degree) |

In time domain expressions

- Phase angle of $v$(t) is $0^0$ $\Rightarrow$ $v$(t) is reference
- Phase angle of $i_1$(t) is $+\theta_1$ $\Rightarrow$ $i_1$(t) leads $v$(t) by $\theta_1$ or $v$(t) lags $i_1$(t) by $\theta_1$
- Phase angle of $i_2$(t) is $-\theta_2$ $\Rightarrow$ $i_2$(t) lags $v$(t) by $\theta_2$ or $v$(t) leads $i_2$(t) by $\theta_2$

Similarly, in phasor expressions

- $\bar{V}$ has a phase angle of $0^0$ $\Rightarrow$ $\bar{V}$ is reference
- $\bar{I}_1$ has a phase angle of $\theta_1$ $\Rightarrow$ $\bar{I}_1$ leads $\bar{V}$ by $\theta_1$
- $\bar{I}_2$ has a phase angle of $-\theta_2$ $\Rightarrow$ $\bar{I}_2$ lags $\bar{V}$ by $\theta_2$

## In this subject

Mainly phasor representation will be used and that requires only the rms value and phase angle.

**Unless specified otherwise**, AC voltages and currents are always specified in terms of rms values

- Service voltage in Singapore is 230 V $\Rightarrow$ rms voltage is 230 V (NOT the maximum nor instantaneous value)
- A load taking a current of 10 A $\Rightarrow$ rms current is 10 A

## Notations Used

- Phasor (or complex number) is represented by adding a bar '–' at the top ($\bar{V}$, $\bar{I}$, etc.)
- Magnitude of a phasor (or rms value) is a scalar number and it is represented by an ordinary letter (*V*, *I*, etc.)

## A phasor (or complex number) has two forms:

**(a) Polar Form**: In polar form, a phasor is represented in terms of its magnitude (rms value) and phase angle

$$\bar{V} = V\angle\theta_1, \quad \bar{I} = I\angle\theta_2, \quad \bar{V} = 100\angle 30^0, \quad \bar{I} = 15\angle -25^0$$

Magnitude is always positive

Phase angle can be positive or negative (usually it is kept in between -$180^0$ and +$180^0$)

**(b) Rectangular Form**: In rectangular form, a phasor is represented in terms of its real part and imaginary part

$$\bar{V} = (V_r + jV_m), \quad \bar{I} = (I_r + jI_m)$$

$$\bar{V} = (86.6 + j50), \quad \bar{I} = (13.59 - j6.34)$$

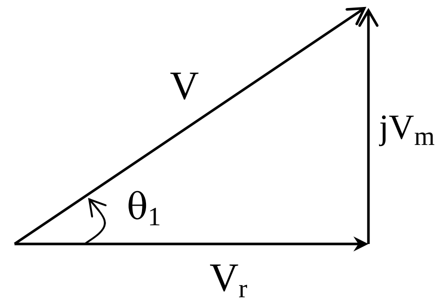
*j* is an operator: $j = \sqrt{-1} \quad or \quad j^2 = -1$

## Relationship between polar and rectangular forms

Consider a voltage phasor as shown in the following figure

In rectangular form: $\bar{V} = (V_r + jV_m)$

In polar form: $\bar{V} = V\angle\theta_1$




Conversion from polar form to rectangular form or vice versa is given in the following and it follows the properties of a right angle triangle

Polar $\Rightarrow$ Rectangular: $V_r = V\cos\theta_1$ ; $V_m = V\sin\theta_1$

Rectangular $\Rightarrow$ Polar: $V = \sqrt{V_r^2 + V_m^2}$ ; $\theta_1 = \tan^{-1}\left(\frac{V_m}{V_r}\right)$

When calculating $\theta_1$ from the above expression, care must be taken to ensure that the result belong to the correct quadrant

The above expression of $\theta_1$ is valid if $V_r \geq 0$

If $V_r < 0$, $\theta_1 = \tan^{-1}\left(\frac{V_m}{V_r}\right) \pm 180^0$ is to be used

## Complex Conjugate

Conjugate of a complex number is obtained by changing the sign of its phase angle (in polar form) or by changing the sign of its imaginary part (in rectangular form).

Conjugate is represented by a superscript '*'

$$\bar{V} = 100\angle 30^0\ V \quad \Rightarrow \quad \bar{V}^* = 100\angle -30^0\ V$$

$$\bar{I} = (15 - j12)\ A \quad \Rightarrow \quad \bar{I}^* = (15 + j12)\ A$$

## Phase Difference

It is the difference between phase angles of two phasors.

Consider $\overline{V} = 100\angle 30^0\ V \quad and \quad \bar{I} = 5\angle 10^0\ A$

Phase difference between $\overline{V}$ and $\bar{I}$ is ($30^0$ -$10^0$) = $20^0$

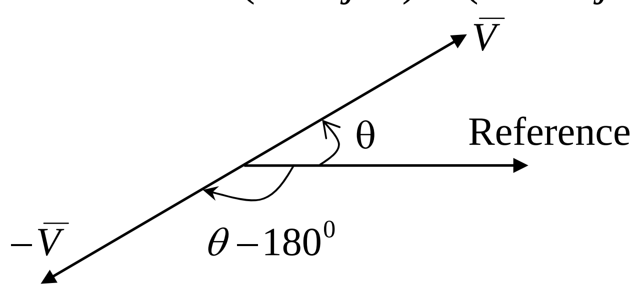
## Polarity Inversion

Inverting polarity of a phasor is equivalent to phase shift of $180^0$ (simply in opposite direction)

$$\overline{V} = 100\angle 30^0 \quad \Rightarrow \quad -\overline{V} = 100\angle(30^0 \pm 180^0) = 100\angle -150^0$$

$$\overline{V} = 100\angle -60^0 \quad \Rightarrow \quad -\overline{V} = 100\angle(-60^0 \pm 180^0) = 100\angle 120^0$$

$$\bar{I} = (15 - j12) \quad \Rightarrow \quad -\bar{I} = -(15 - j12) = (-15 + j12)$$




In general, the phase angle is kept in between -$180^0$ and +$180^0$

- Phase angle of $210^0$ is equivalent to ($210^0$ – $360^0$) = -$150^0$
- Phase angle of -$240^0$ is equivalent to (-$240^0$ + $360^0$) = $120^0$

## Phasor Rule 1:

Multiplying (or dividing) a phasor by a positive constant *k* is equivalent to multiplying (or dividing) its magnitude by the same constant *k*. The phase angle remains the same. Consider *k* = 5

$$\overline{V} = 100\angle 30^0 \quad \Rightarrow \quad k\overline{V} = 500\angle 30^0 \ \ and \ \ \frac{\overline{V}}{k} = 20\angle 30^0$$

$$\bar{I} = (15 - j12) \quad \Rightarrow \quad k\bar{I} = (75 - j60) \ \ and \ \ \frac{\bar{I}}{k} = (3 - j2.4)$$

## Phasor Rule 2:

Adding (or subtracting) two phasors is equivalent to adding (or subtracting) their real parts and imaginary parts separately

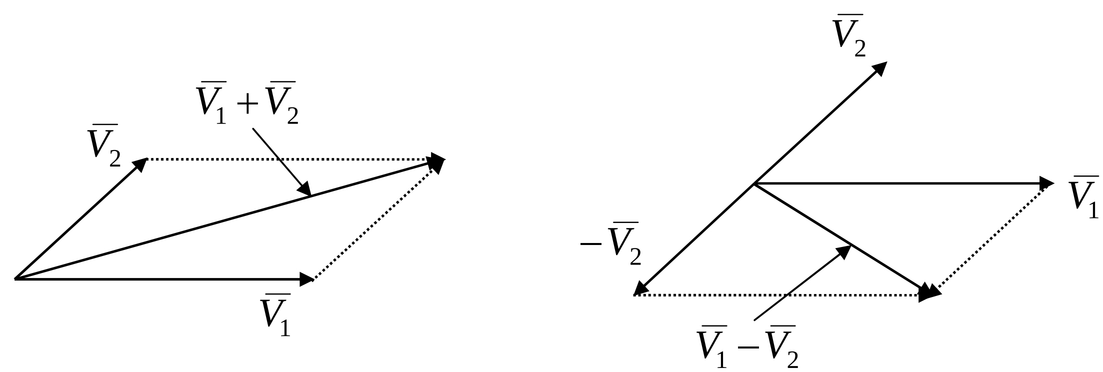
Let $\overline{V_1} = (80 + j60)$ *and* $\overline{V_2} = (50 + j20)$

$$\therefore \quad \overline{V_1} + \overline{V_2} = (80 + 50) + j(60 + 20) = (130 + j80)$$

$$\overline{V_1} - \overline{V_2} = (80 - 50) + j(60 - 20) = (30 + j40)$$

In phasor diagram, $\overline{V_1} - \overline{V_2}$ is a phasor from $\overline{V_2}$ to $\overline{V_1}$

For addition (or subtraction), it is convenient to represent the phasors in rectangular form (instead of polar form)




## Phasor Rule 3: (multiplication of phasors)

When two phasors are multiplied:

- The resultant magnitude is the product of the magnitudes
- The resultant phase angle is the sum of the phase angles

Let $\overline{X}_1 = 50\angle 30^0,\ \overline{X}_2 = 10\angle 20^0$ *and* $\overline{X}_3 = 5\angle -40^0$

$$\therefore \overline{X}_1 \times \overline{X}_2 = (50 \times 10)\angle(30^0 + 20^0) = 500\angle 50^0$$

$$\overline{X}_1 \times \overline{X}_3 = (50 \times 5)\angle(30^0 - 40^0) = 250\angle -10^0$$

**Phasor Rule 4:** (ratio of phasors)

When a phasor is divided by another phasor

- The resultant magnitude is the ratio of the magnitudes
- The resultant phase angle is the difference between two phase angles (angle of numerator minus angle of denominator)

Let $\overline{X}_1 = 50\angle 30^0$, $\overline{X}_2 = 10\angle 20^0$ *and* $\overline{X}_3 = 5\angle -40^0$

$$\therefore \frac{\overline{X}_1}{\overline{X}_2} = \frac{50\angle 30^0}{10\angle 20^0} = \frac{50}{10}\angle(30^0 - 20^0) = 5\angle 10^0$$

$$\frac{\overline{X}_1\overline{X}_2}{\overline{X}_3} = \frac{50\angle 30^0 \times 10\angle 20^0}{5\angle -40^0} = \frac{500\angle 50^0}{5\angle -40^0} = 100\angle 90^0$$

$$\frac{1}{\overline{X}_1} = \frac{1}{50\angle 30^0} = \frac{1}{50}\angle -30^0 = 0.02\angle -30^0$$

For multiplication/division, it is convenient to represent the phasors in polar form (instead of rectangular form).

## Impedance/Admittance

Impedance is a complex number and is defined as

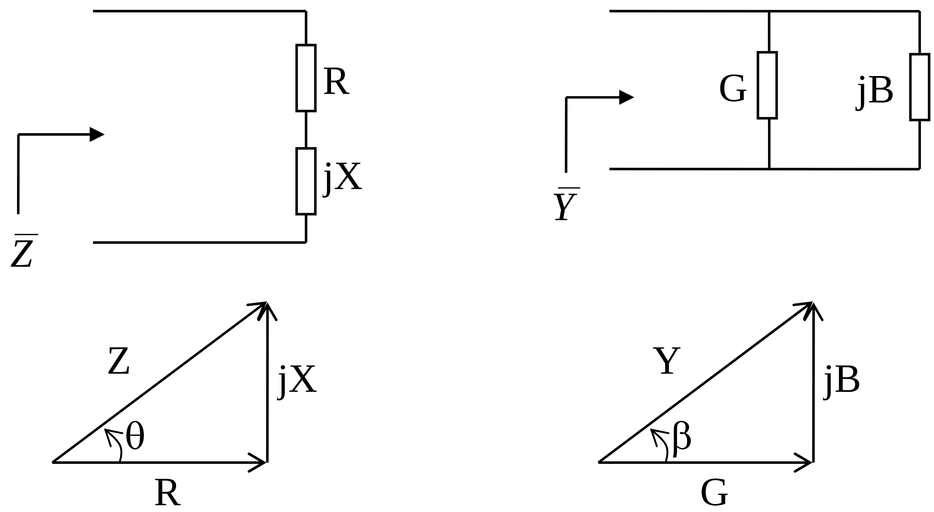
$$\overline{Z} = Z\angle\theta = (R + jX)\ \Omega$$

- Real part of $\overline{Z}$ is called resistance (R) and the imaginary part of $\overline{Z}$ is called reactance (X)
- The unit of R, X and Z is ohms (Ω)

Admittance is reciprocal of impedance and is represented by $\overline{Y}$

$$\overline{Y} = \frac{1}{\overline{Z}} = (G + jB)\ \text{S}$$

- Real part of $\overline{Y}$ is called conductance (G) and the imaginary part of $\overline{Y}$ is called susceptance (B)
- The unit of G, B and Y is siemens (S)




Let $\overline{Z} = 5\angle 30^0\,\Omega = (4.33 + j2.5)\,\Omega$

Admittance $\overline{Y} = \dfrac{1}{\overline{Z}} = \dfrac{1}{5\angle 30^0} = 0.2\angle -30^0 = (0.173 - j0.10)\ S$

$\therefore$ R = 4.33 Ω, X = 2.5 Ω and Z = 5 Ω

G = 0.173 S, B = -0.10 S and Y = 0.2 S

Active power or average power P is associated with

- Resistance R or real part of $\overline{Z}$
- Conductance G or real part of $\overline{Y}$

Reactive power Q is associated with

- Reactance X or imaginary part of $\overline{Z}$
- Susceptance B or imaginary part of $\overline{Y}$

**Note:** A scalar number may be considered as a special complex number with zero imaginary part or zero phase angle

For example: $R = 5\Omega \Rightarrow \overline{Z}_R = (5+j0)\Omega$ or $\overline{Z}_R = 5\angle 0^0\,\Omega$

## Three Basic Circuit Elements

### (a) Resistance R

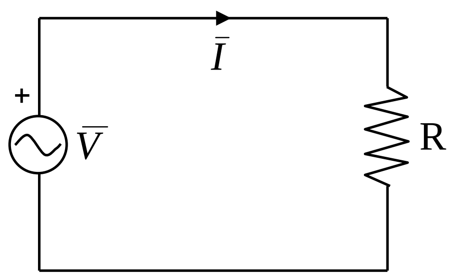
Impedance of a resistance is $\bar{Z}_R = (R+j0) = R\angle 0^0\ \Omega$

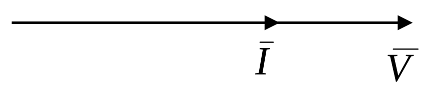
Consider $\bar{V}$ as reference, i.e. $\bar{V} = V\angle 0^0$ V

$$\therefore \text{ Current } \bar{I} = \frac{\bar{V}}{\bar{Z}_R} = \frac{V\angle 0^0}{R\angle 0^0} = \frac{V}{R}\angle 0^0 \quad \triangleq \quad I\angle 0^0 \text{ A}$$

The voltage across R and the current through R are in phase

$\therefore$ Phase difference between $\bar{V}$ and $\bar{I}$ is zero $\Rightarrow \varphi = 0$




Active power absorbed by the resistance is

$$P = VI\cos\varphi = VI = \frac{V^2}{R} = I^2R \text{ W}$$

Reactive power absorbed by the resistance is zero

$$Q = VI\sin\varphi = 0 \text{ VAr}$$

Resistance does not absorb reactive power

### (b) Inductance L

Reactance of an inductance is $X_L = \omega L = 2\pi f L\ \Omega$

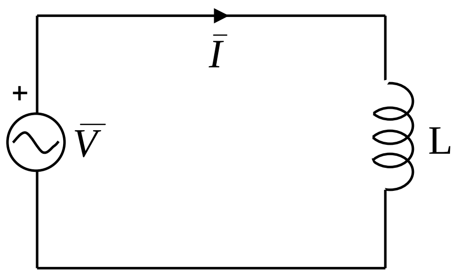
Impedance of the inductance is $\bar{Z}_L = (0 + jX_L) = X_L\angle 90^0\ \Omega$

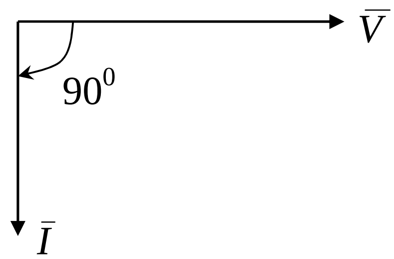
Consider $\bar{V}$ as reference, i.e. $\bar{V} = V\angle 0^0$ V

$\therefore$ Current $\bar{I} = \dfrac{\bar{V}}{\bar{Z}_L} = \dfrac{V\angle 0^0}{X_L\angle 90^0} = \dfrac{V}{X_L}\angle -90^0 \quad \underline{\underline{\Delta}} \quad I\angle -90^0$ A

Current lags voltage by $90^0$ or voltage leads current by $90^0$

$\therefore$ Phase difference between $\bar{V}$ and $\bar{I}$ is $90^0 \Rightarrow \varphi = 90^0$




Active power absorbed by the inductor is

P = $VI$ cosφ = 0 W

A pure inductor does not absorb active power

Reactive power absorbed by the inductor is

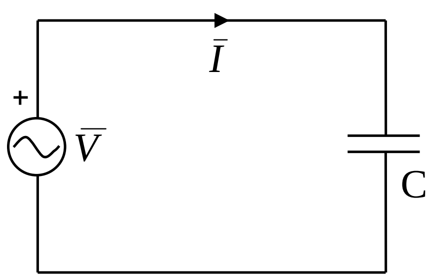
$$Q = VI\sin\varphi = VI = \frac{V^2}{X_L} = I^2 X_L \text{ VAr}$$

## (c) Capacitance C

Reactance of a capacitance is $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}\ \Omega$

Impedance of the capacitance is $\bar{Z}_C = (0 - jX_C) = X_C\angle -90^0\ \Omega$

Consider $\bar{V}$ as reference, i.e. $\bar{V} = V\angle 0^0$ V




$\therefore$ Current $\bar{I} = \dfrac{\bar{V}}{\bar{Z}_C} = \dfrac{V\angle 0^0}{X_C\angle -90^0} = \dfrac{V}{X_c}\angle 90^0 \quad \underline{\underline{\Delta}} \quad I\angle 90^0$ A

Current leads voltage by $90^0$ or voltage lags current by $90^0$

$\therefore$ Phase difference between $\bar{V}$ and $\bar{I}$ is $-90^0 \Rightarrow \varphi = -90^0$

Active power absorbed by the capacitor is

$$P = VI\cos\varphi = 0 \text{ W}$$

A pure capacitor does not absorb active power

Reactive power absorbed by the capacitor is

$$Q = VI\sin\varphi = -VI = -\frac{V^2}{X_C} = -I^2 X_C \text{ VAr}$$

**Note:** The reactive power absorbed by a capacitor is considered as negative whereas the reactive power absorbed by an inductor is considered as positive.

## Complex Power

Complex power $\bar{S}$ is composed of active power $P$ and reactive power $Q$ i.e. $\bar{S} = (P + jQ)$ VA

Complex power $\bar{S}$ can be expressed in terms of complex voltage and complex current

Let $\bar{V} = V\angle\theta_1\ V \quad and \quad \bar{I} = I\angle\theta_2\ A$

$\therefore$ The complex power $\bar{S}$ is given by

$$\bar{S} = \bar{V} \times \bar{I}^* = V\angle\theta_1 \times I\angle -\theta_2$$

$$= VI\angle(\theta_1 - \theta_2)\ \underline{\underline{\Delta}}\ S\angle\varphi = (P + jQ)\,VA$$

where $S = VI$ (VA)

$\varphi = (\theta_1 - \theta_2)$ = phase difference between $\bar{V}$ and $\bar{I}$
(also called power factor angle)

$P = S\cos\varphi = VI\cos\varphi$ (W)

$Q = S\sin\varphi = VI\sin\varphi$ (VAr)

pf = cosφ

**Example**

Consider $\bar{V} = 100\angle 30^0\ V$ and $\bar{I} = 5\angle 10^0\ A$

∴ Complex power associated with the above voltage & current

$$\bar{S} = \bar{V}\ \bar{I}^* = 100\angle 30^0 \times 5\angle -10^0 \text{ VA}$$

$$= 500\angle 20^0 = (469.85 + j171.0) \text{ VA}$$

Apparent power: $S$ = 500 VA

Active power: $P$ = 469.85 W

Reactive power: $Q$ = 171 VAr

Power factor: pf = $\cos 20^0$ = 0.9397 (lagging)
(because current lags voltage)

## Definition of Various Quantities

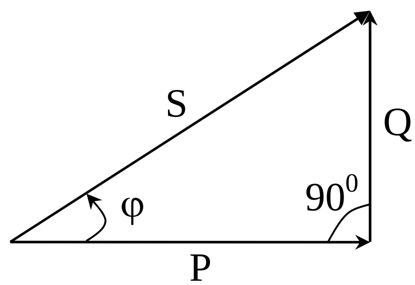
| Symbol | Definition | Unit |
|---|---|---|
| S | Apparent power | VA, kVA, MVA |
| P | Active power, average power or real power | W, kW, MW |
| Q | Reactive power | VAr, kVAr, MVAr |
| φ | Power factor angle | degree |

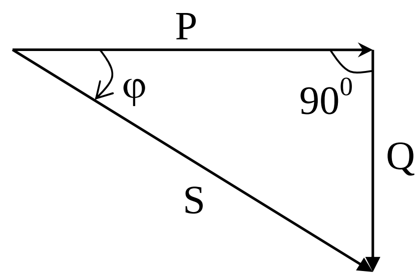
Note that there are three different powers in AC circuits (apparent power, active power and reactive power) and their units are different.

In many cases, the type of power (active, reactive or apparent) may not be specified but it can be identified from the unit.

## Power Triangle

*P, Q, S* and φ are related through a right angle triangle and is called power triangle as shown in the following figure:




By knowing any two quantities, the rest two quantities can be determined from the properties of a right angle triangle.

### Consider a load absorbs 10 kVA at 0.8 pf (lagging)

Note that kVA is the unit of *S* (not *P* nor *Q*)

$\therefore$ $S$ = 10 kVA, and φ = $\cos^{-1}(0.8) = 36.87^0$

From power triangle

$P = S\cos\varphi$ = 8 kW;   $Q = S\sin\varphi$ = +6 kVAr

(Q is positive because pf is lagging)

Complex power can directly be written as

$$\bar{S} = S\angle\cos^{-1}(pf) = 10\angle 36.87^0\ kVA$$

**Consider a load absorbs 10 kW at 0.9 pf (leading)**

Note that kW is the unit of $P$ (not $S$ nor $Q$)

$\therefore P = 10$ kW, and $\varphi = \cos^{-1}(0.9) = 25.84^0$

From power triangle

Q = -P tan(φ) = -4.84 kVAr; $S = \sqrt{P^2 + Q^2} = 11.11\,kVA$

(Q is negative because pf is leading)

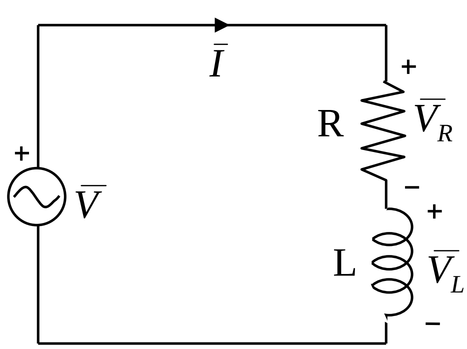
Complex power can directly be written as

$$\bar{S} = \frac{P}{pf}\angle -\cos^{-1}(pf) = 11.11\angle -25.84^0\ kVA$$

## R-L Series Circuit

The impedance of a R-L series circuit is

$$\bar{Z} = (R + j\omega L) = (R + jX_L) = Z\angle\varphi\ \Omega$$




Consider $\bar{V}$ as reference, i.e. $\bar{V} = V\angle 0^0$ V

$$\therefore \text{Current } \bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{V\angle 0^0}{Z\angle\varphi} = \frac{V}{Z}\angle -\varphi \quad \triangleq \quad I\angle -\varphi \text{ A}$$

Phase difference between $\bar{V}$ and $\bar{I}$ is φ

Complex power $\bar{S} = \bar{V}\ \bar{I}^* = VI\angle\varphi = VI(\cos\varphi + j\sin\varphi)$ VA

Apparent power: $S = VI = I^2 Z = \dfrac{V^2}{Z}$ VA

Active power: $P = VI \cos\varphi = S \cos\varphi$ W

Reactive power: $Q = VI \sin\varphi = S \sin\varphi$ VAr

Voltage across the resistance: $\bar{V}_R = R\bar{I} = RI\angle -\varphi$ V

Voltage across the inductor: $\bar{V}_L = jX_L\bar{I} = X_L I\angle(90^0 - \varphi)$ V

Also $P = V_R I = I^2 R = \dfrac{V_R^2}{R}$ W

$$Q = V_L I = I^2 X_L = \frac{V_L^2}{X_L} \text{ VAr}$$

$$S = VI = I^2 Z = \frac{V^2}{Z} \text{ VA}$$

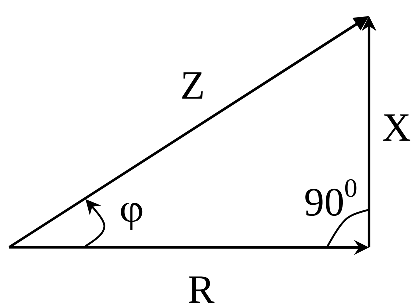
## Observations

Angle of $\bar{S}$ = angle of $\bar{Z}$

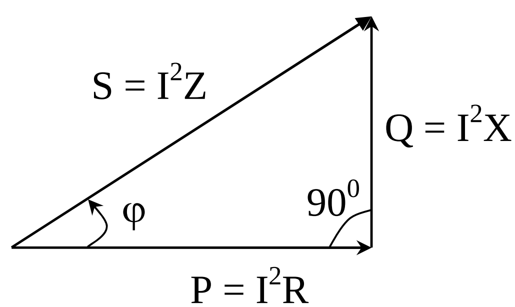
Angle of $\bar{I}$ = – angle of $\bar{Z}$ (when $\bar{V}$ is reference)

## Impedance triangle and power triangle

When all sides of an impedance triangle are multiplied by $I^2$, we can get the power triangle




Impedance triangle

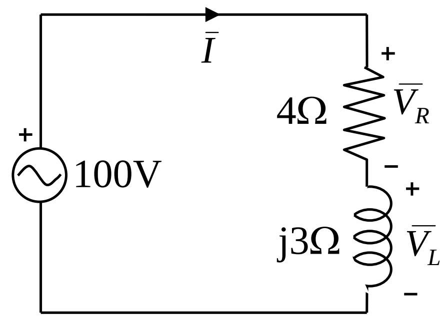



Power triangle

## Example – 1

In the following circuit, determine the apparent power, active power and reactive power supplied by the source. Also find the voltage across the resistance and the inductance.

**Solution**




Consider $\bar{V}$ as reference,

i.e. $\bar{V} = 100\angle 0^0$ V

Current $\bar{I} = \dfrac{\bar{V}}{\bar{Z}} = \dfrac{100\angle 0^0}{(4 + j3)}$

$= 20\angle -36.87^0\ A$

Complex power supplied by the source is

$$\bar{S} = \bar{V}\ \bar{I}^* = 100\angle 0^0 \times 20\angle 36.87^0\ VA$$

$$= 2000\angle 36.87^0 = (1600 + j1200)\ VA$$

$\therefore$ $S$ = 2000 VA, $P$ = 1600 W and $Q$ = 1200 VAr

Voltage across the resistance is

$$\bar{V}_R = R\bar{I} = 4 \times 20\angle -36.87^0 = 80\angle -36.87^0\ V$$

Voltage across the inductor is

$$\bar{V}_L = jX_L\bar{I} = 3\angle 90^0 \times 20\angle -36.87^0 = 60\angle 53.13^0\ V$$

**Alternative ways of finding P, Q and S**

$$P = V_R I = I^2 R = \frac{V_R^2}{R} = 1600\, W$$

Note that

$$P \neq VI \neq \frac{V^2}{R} \text{ (because } V \text{ is not across the resistance R)}$$

Similarly, $Q = V_L I = I^2 X_L = \dfrac{V_L^2}{X_L} = 1200\, VAr$

Again $Q \neq VI \neq \dfrac{V^2}{X_L}$ (because $V$ is not across the inductance L)

$$S = VI = I^2 Z = \frac{V^2}{Z} = 2000 \text{ VA}$$

Also $S = \sqrt{P^2 + Q^2}$ VA

## R-C Series Circuit

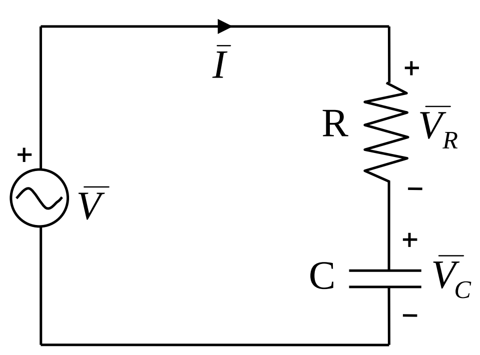
The impedance of a R-C series circuit is

$$\overline{Z} = (R - j\frac{1}{\omega C}) = (R - jX_C) = Z\angle -\varphi \ \Omega$$

Consider $\overline{V}$ as reference, i.e. $\overline{V} = V\angle 0^0$ V

$\therefore$ Current $\overline{I} = \dfrac{\overline{V}}{\overline{Z}} = \dfrac{V\angle 0^0}{Z\angle -\varphi} = \dfrac{V}{Z}\angle\varphi \quad \triangleq \quad I\angle\varphi$ A

Phase difference between $\overline{V}$ and $\overline{I}$ is –φ



Complex power $\bar{S} = \bar{V}\ \bar{I}^* = VI\angle -\varphi = VI(\cos\varphi - j\sin\varphi)$ VA

Apparent power: $S = VI = I^2 Z = \dfrac{V^2}{Z}$ VA

Active power: $P = VI\cos(-\varphi) = S\cos\varphi$ W

Reactive power: $Q = VI\sin(-\varphi) = -S\sin\varphi$ VAr

Voltage across the resistance is $\bar{V}_R = R\bar{I} = RI\angle\varphi$ V

Voltage across the capacitor is $\bar{V}_C = -jX_C\bar{I} = X_C I\angle(\varphi - 90^0)$ V

Also $P = V_R I = I^2 R = \dfrac{V_R^2}{R}$ W

$$Q = -V_C I = -I^2 X_C = -\frac{V_L^2}{X_C}\ \text{VAr}$$

$$S = \sqrt{P^2 + Q^2}\ \text{VA}$$

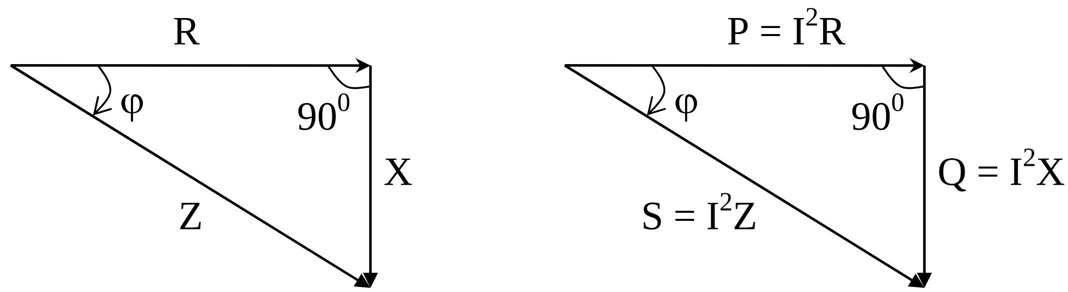
## Observations

Angle of $\bar{S}$ = angle of $\bar{Z}$

Angle of $\bar{I}$ = – angle of $\bar{Z}$ (when $\bar{V}$ is reference)

## Impedance triangle and power triangle

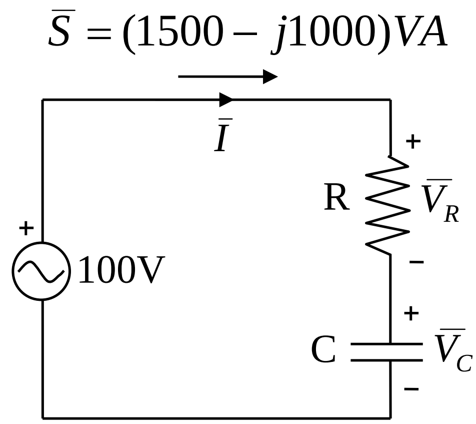
When all sides of the impedance triangle are multiplied by $I^2$, we can get the power triangle



## Example – 2

A 100 V, 50 Hz source supplies a complex power of (1500 – j1000) VA to an R-C series circuit. Determine the values of R and C. Also find the voltage across R and C.

**Solution**




Consider $\bar{V}$ as reference,

i.e. $\bar{V} = 100\angle 0^0$ V

$\bar{S} = (1500 - j1000)$

$= 1802.8\angle -33.69^0\ VA$

But $\bar{S} = \bar{V}\ \bar{I}^*$

$$\therefore\ \bar{I}^* = \frac{\bar{S}}{\bar{V}} = \frac{1802.8\angle -33.69^0}{100\angle 0^0} = 18.028\angle -33.69^0$$

$$\Rightarrow\ \bar{I} = 18.028\angle 33.69^0\ A$$

The impedance

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{100\angle 0^0}{18.028\angle 33.69^0}$$

$$= 5.547\angle -33.69^0 = (4.615 - j3.077)\ \Omega$$

$\therefore$ R = 4.615 $\Omega$ and $X_C$ = 3.077 $\Omega$

But $X_c = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$ $\Rightarrow$ C =1,034.5 $\mu$F

Voltage across the resistance is

$$\bar{V}_R = R\bar{I} = 4.615 \times 18.028\angle 33.69^0 = 83.2\angle 33.69^0\ V$$

Voltage across the capacitor is

$$\bar{V}_C = -jX_C\bar{I} = 3.077\angle -90^0 \times 18.028\angle 33.69^0 = 55.47\angle -56.31^0\ V$$

### Verification

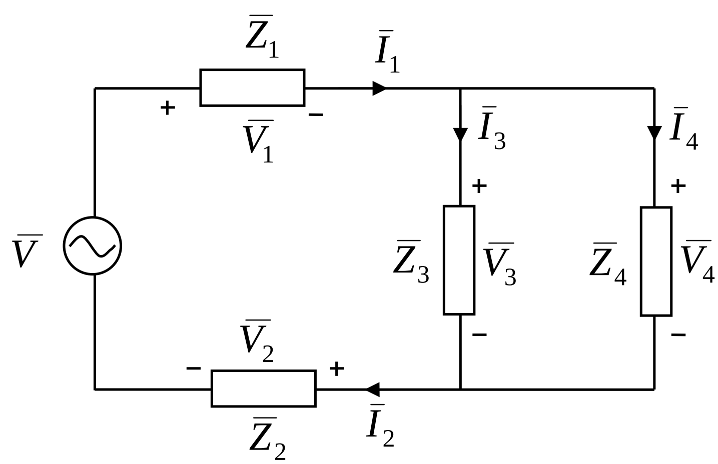
$$P = V_R I = I^2 R = \frac{V_R^2}{R} = 1500\,W$$

$$Q = -V_C I = -I^2 X_C = -\frac{V_L^2}{X_C} = -1000\,VAr$$

$$S = VI = I^2 Z = \frac{V^2}{Z} = 1802.8\ \text{VA}$$

## Example – 3

In the following circuit, $\bar{Z}_1$ = (2+j4) Ω, $\bar{Z}_2$ = (1+j2) Ω, $\bar{Z}_3$ = (10+j8) Ω, $\bar{Z}_4$ = (8-j8) Ω and $\bar{V}_4 = 100\angle 0^0\ V$. Determine the (a) current through each impedance (b) voltage across each impedance, (c) complex power absorbed by each impedance and (d) complex power supplied by the source.




### Solution

Here $\bar{Z}_3$ and $\bar{Z}_4$ are in parallel, therefore $\bar{V}_3 = \bar{V}_4 = 100\angle 0^0\ V$

$$\bar{I}_3 = \frac{\bar{V}_3}{\bar{Z}_3} = \frac{100\angle 0^0}{(10 + j8)} = 7.808\angle -38.66^0\ A$$

$$\bar{I}_4 = \frac{\bar{V}_4}{\bar{Z}_4} = \frac{100\angle 0^0}{(8 - j8)} = 8.84\angle 45^0 \ A$$

$$\bar{I}_1 = \bar{I}_2 = (\bar{I}_3 + \bar{I}_4) = 12.42\angle 6.34^0 \ A$$

(b) $\bar{V}_1 = \bar{Z}_1 \bar{I}_1 = 55.56\angle 69.78^0 \ V$

$$\bar{V}_2 = \bar{Z}_2 \bar{I}_2 = 27.78\angle 69.77^0 \ V$$

$$\bar{V}_3 = \bar{V}_4 = 100\angle 0^0 \ V$$

(c) $\bar{S}_1 = \bar{V}_1 \ \bar{I}_1^* = 690.22\angle 63.44^0 \ VA$

$$\bar{S}_2 = \bar{V}_2 \ \bar{I}_2^* = 345.11\angle 63.43^0 \ VA$$

$$\bar{S}_3 = \bar{V}_3 \ \bar{I}_3^* = 780.80\angle 38.66^0 \ VA$$

$$\bar{S}_4 = \bar{V}_4 \ \bar{I}_4^* = 883.9\angle -45.0^0 \ VA$$

(d) Source voltage is

$$\bar{V} = \left(\bar{V}_1 + \bar{V}_3 + \bar{V}_2\right) = 150.69\angle 31.26^0 \ V$$

Complex power supplied by the source is

$$\bar{S}_S = \bar{V} \ \bar{I}_1^* = 1872.0\angle 24.92^0 \ VA$$

**Verification**

Total complex power absorbed by all impedances is

$$\bar{S}_T = (\bar{S}_1 + \bar{S}_2 + \bar{S}_3 + \bar{S}_4) = 1872.0\angle 24.92^0 \ VA$$

This is exactly the same as the complex power supplied by the source.

## Equivalent Impedance

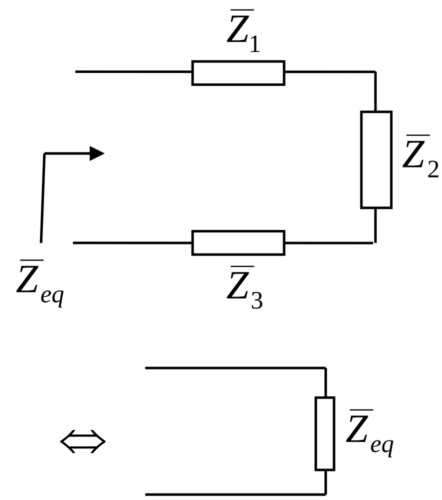
Equivalent impedance of a series or parallel circuit can be determined as follows:

## (a) **Series Circuit**:

When two or more impedances are connected in series, the equivalent impedance is the sum of impedances

$$\overline{Z}_{eq} = \overline{Z}_1 + \overline{Z}_2 + \overline{Z}_3$$

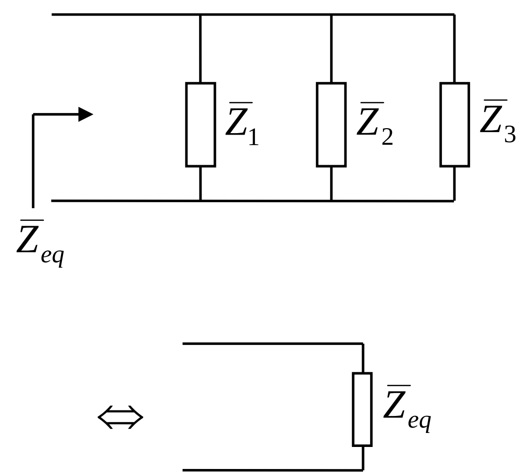
Series elements have the same current.




## (b) **Parallel Circuit**:

When two or more impedances are connected in parallel, the reciprocal of the equivalent impedance is the sum of reciprocal of the impedances

$$\frac{1}{\overline{Z}_{eq}} = \frac{1}{\overline{Z}_1} + \frac{1}{\overline{Z}_2} + \frac{1}{\overline{Z}_3}$$




That is, the equivalent admittance is the sum of admittances

$$\Rightarrow \quad \overline{Y}_{eq} = \overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3$$

Shunt elements have the same voltage.

When only **two impedances** ($\overline{Z}_1$ and $\overline{Z}_2$) are connected in parallel, the equivalent impedance is

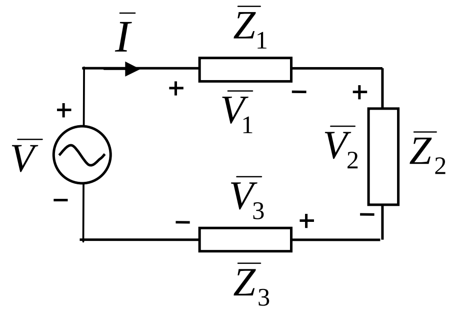
$$\overline{Z}_{eq} = \frac{\overline{Z}_1 \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}$$

When ***n*** **equal impedances** are connected in parallel, the equivalent impedance is 1/*n* of the original impedance.

# Complex Power in a Composite Circuit

## Series circuit:

Using KVL: $\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3$




Multiply both sides of the above equation by $\bar{I}^*$. Note that series elements have the same current

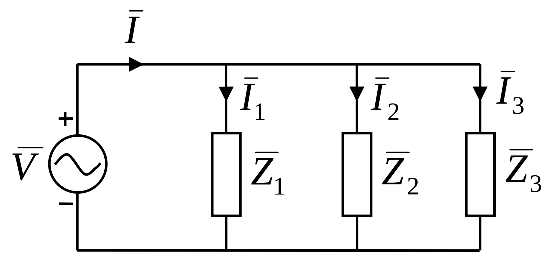
$$\bar{V}\bar{I}^* = \bar{V}_1\bar{I}^* + \bar{V}_2\bar{I}^* + \bar{V}_3\bar{I}^*$$

$$\Rightarrow \bar{S} = \bar{S}_1 + \bar{S}_2 + \bar{S}_3$$

The source complex power is the sum of complex power absorbed by each element.

## Parallel circuit:

Using KCL: $\bar{I} = \bar{I}_1 + \bar{I}_2 + \bar{I}_3$




$$\Rightarrow \bar{I}^* = \bar{I}_1^* + \bar{I}_2^* + \bar{I}_3^*$$

Multiply both sides of the above equation by $\bar{V}$. Note that parallel elements have the same voltage

$$\Rightarrow \bar{V}\,\bar{I}^* = \bar{V}\,\bar{I}_1^* + \bar{V}\,\bar{I}_2^* + \bar{V}\,\bar{I}_3^*$$

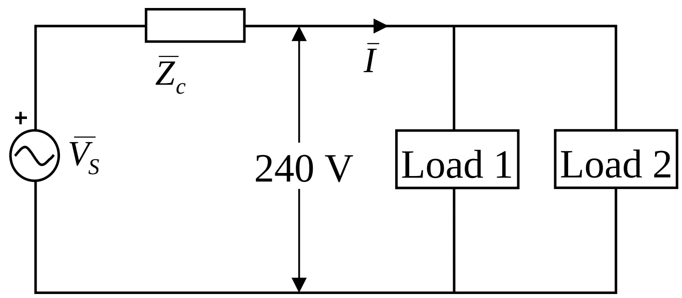
$$\Rightarrow \bar{S} = \bar{S}_1 + \bar{S}_2 + \bar{S}_3$$

The source complex power is the sum of complex power absorbed by each element.

**In summary**: complex power supplied by the source is the sum of complex power absorbed by various elements and it does not matter whether the elements are connected in series or parallel (see Example – 3).

## Example – 4

Two loads are connected in parallel. Load 1 absorbs 20 kW at 0.8 lagging pf and load 2 absorbs 15 kVA at 0.95 leading pf. Determine the complex power supplied by the source if the cable connecting the source to load has an impedance of $\bar{Z}_c = (0.1 + j0.2)\,\Omega$.




**<u>Solution:</u>**

Complex power of loads 1 and 2 are:

$$\bar{S}_1 = \frac{20}{0.8}\angle\cos^{-1}(0.8) = 25\angle 36.87^0 = (20 + j15)\ \text{kVA}$$

$$\bar{S}_2 = 15\angle -\cos^{-1}(0.95) = 15\angle -18.19^0 = (14.25 - j4.68)\ \text{kVA}$$

Total load is

$$\bar{S}_{12} = \bar{S}_1 + \bar{S}_2 = (34.25 + j10.32) = 35.77\angle 16.76^0\ \text{kVA}$$

But $\bar{S}_{12} = \bar{V}\,\bar{I}^* \quad \Rightarrow \quad \bar{I} = \left(\dfrac{\bar{S}_{12}}{\bar{V}}\right)^* = 149\angle -16.76^0\ \text{A}$

[Load voltage is considered as reference, $\bar{V} = 240\angle 0^0\,V$ ]

Cable impedance $\bar{Z}_c = (0.1 + j0.2)\,\Omega$

Losses in the cable are:

$P_{Loss} = I^2R_c = 149^2\times 0.1 = 2.22$ kW

$Q_{Loss} = I^2X_c = 149^2\times 0.2 = 4.44$ kVA

$\bar{S}_{Loss} = (2.22 + j4.44)\ \text{kVA}$

Source complex power is

$$\bar{S}_S = \bar{S}_1 + \bar{S}_2 + \bar{S}_{Loss} = 39.33\angle 22^0 = (36.46 + j14.75)\ \text{kVA}$$

**Alternative way**

Supply voltage

$$\bar{V}_S = \bar{V} + \bar{I}\,\bar{Z}_c = 240\angle 0^0 + 149\angle -16.67^0 (0.1 + j).2)$$
$$= 263.98\angle 5.27^0\ V$$

Complex power supplied by the source is

$$\bar{S}_S = \bar{V}_S \bar{I}^* = 263.98\angle 5.27 \times 149\angle 16.76^0$$
$$= 39.33\angle 22^0 = (36.52 + j14.76)\ kVA$$

**Note:** $\bar{S}_S = \bar{S}_1 + \bar{S}_2 + \bar{S}_3$ is okay

$S_S = S_1 + S_2 + S_3$ **is not okay**

$P_S = P_1 + P_2 + P_3$ is okay

$Q_S = Q_1 + Q_2 + Q_3$ is okay

## R-L-C Series Circuit

The impedance of a R-L-C series circuit is

$$\bar{Z} = (R + j\omega L - j\frac{1}{\omega C}) = [R + j(X_L - X_C)] = Z\angle\varphi$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad and \quad \varphi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

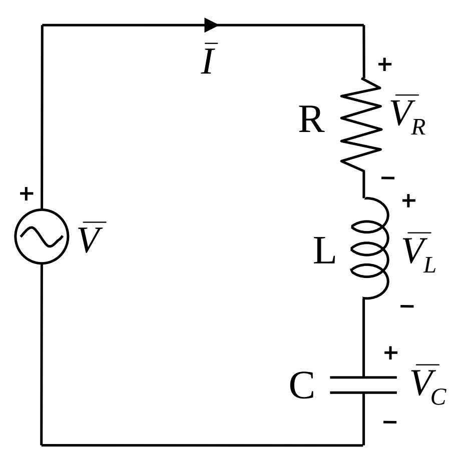
**When $X_L > X_C$**

The circuit is inductive.

It has a lagging power factor (current lags voltage).

It is similar to a R-L circuit.

Overall reactive power is positive.



**When $X_L < X_C$**

The circuit is capacitive.

It has a leading power factor (current leads voltage).

It is similar to a R-C circuit.

Overall reactive power is negative.

**When $X_L = X_C$**

The circuit is purely resistive.

It has a unity power factor (current and voltage are in phase).

It is similar to a pure resistive circuit.

## Example – 5

A 240 V, 50 Hz, small industrial plant has the following loads:

Load 1: absorbs 10 kVA at 0.8 lagging power factor
Load 2: absorbs 5 kW at 0.6 lagging power factor
Load 3: takes 25 A at 0.9 lagging power factor

Determine (a) the impedance of each load, (b) the equivalent impedance and admittance of the plant, and (c) the overall current and complex power of the plant.

A variable capacitor is now added to the plant. Determine the value of the capacitor for which the plant will have minimum current. What is the value of the corresponding current?

**<u>Solution</u>**

(a) Consider the voltage as reference $\Rightarrow \bar{V} = 240\angle 0^0\ V$

Load 1:

$$\bar{S}_1 = 10\angle\cos^{-1}(0.8) = 10\angle 36.87^0\ kVA$$

$$\bar{S}_1 = \bar{V}\ \bar{I}_1^* \quad \Rightarrow \quad \bar{I}_1 = (\bar{S}_1 / \bar{V})^* = 41.677\angle -36.87^0\ A$$

$$\bar{Z}_1 = \bar{V} / \bar{I}_1 = 5.76\angle 36.87^0\ \Omega$$

Load 2:

$$\bar{S}_2 = \frac{5}{0.6}\angle\cos^{-1}(0.6) = 8.33\angle 53.13^0\ kVA$$

$$\bar{S}_2 = \bar{V}\ \bar{I}_2^* \quad \Rightarrow \quad \bar{I}_2 = (\bar{S}_2/\bar{V})^* = 34.72\angle -53.13^0\ A$$

$$\bar{Z}_2 = \bar{V}/\bar{I}_2 = 6.91\angle 53.13^0\ \Omega$$

Load 3:

$$\bar{I}_3 = 25\angle -\cos^{-1}(0.9) = 25\angle -25.84^0\ A$$

$$\bar{Z}_3 = \bar{V}/\bar{I}_3 = 9.6\angle 25.84^0\ \Omega$$

(b) Equivalent impedance/admittance:

$$\frac{1}{\bar{Z}_{eq}} = \frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2} + \frac{1}{\bar{Z}_3} = 0.4153\angle -39.71^0\ S = \bar{Y}_{eq}$$

$$\bar{Z}_{eq} = \frac{1}{\bar{Y}_{eq}} = 2.408\angle 39.71^0\ \Omega$$

(c) $$\bar{I} = \frac{\bar{V}}{\bar{Z}_{eq}} = 99.67\angle -39.71^0 = (76.67 - j63.68)\ A$$

$$\bar{S} = \bar{V}\ \bar{I}^* = 23.92\angle 39.71^0 = (18.4 + j15.28)\ kVA$$

Also $\bar{S} = \bar{S}_1 + \bar{S}_2 + \bar{S}_3 = 23.92\angle 39.71^0\ kVA$

**When the capacitor is added**

Let the capacitor current be $\bar{I}_c = (0 + jI_c)\ A$

Total current $\bar{I}_T$ is:

$$\bar{I}_T = \bar{I} + \bar{I}_c = (76.67 - j63.68) + (0 + jI_c)$$

$$= 76.67 + j(I_c - 63.68)\ A$$

The current will have a minimum value when

$(I_c – 63.68) = 0 \quad \Rightarrow I_c = 63.68$ A

But $I_c = V/X_c \ \Rightarrow\ 63.68 = 240/X_c \ \Rightarrow X_c = 3.769\ \Omega$

But $X_c = 1/\omega C \ \Rightarrow C = 844\ \mu F$

The minimum current is (76.67 + j0) A = 76.67 A

# 3-phase Systems

## Introduction

- Almost all electric power generation and transmission in the world today is in the form of 3-phase
- Most of the large industrial loads are also 3-phase
- Small loads (such as residential loads) are 1-phase, but this is simply a tap-off from a 3-phase system
- A 3-phase system is an integration of three 1-phase systems
- A 3-phase system consists of 3-phase generators, 3-phase transmission lines and 3-phase loads
- A 3-phase power system is preferred over a 1-phase power system for the following reasons:

  3-phase generators, motors and transformers are simpler, cheaper and more efficient

  3-phase transmission lines deliver more power for a given cost or for a given weight of conductor

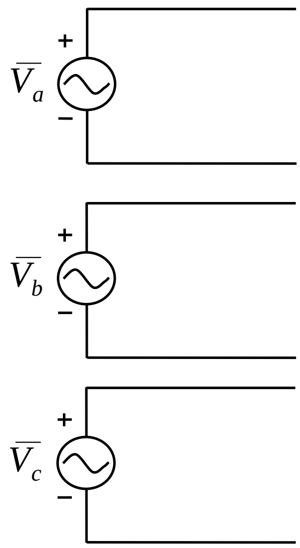
  Voltage regulation of a 3-phase system is inherently better

## Generation of 3-phase Voltages and Currents

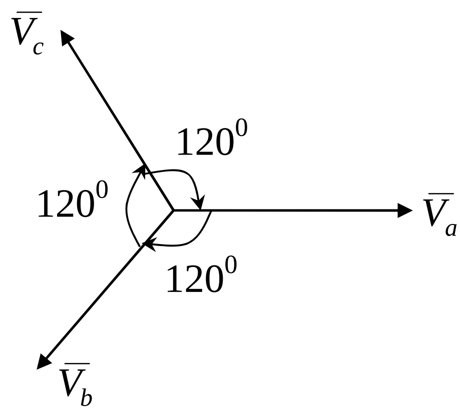
- A 3-phase generator consists of three 1-phase generators
- All three voltages are equal in magnitude but differing in phase angle from others by $120^0$
- Quantities of three phases (of a 3-phase system) are represented by using subscripts '*a*', '*b*' and '*c*'. They are called phase-*a*, phase-*b* and phase-*c*

# Expression of Three Phase Voltages

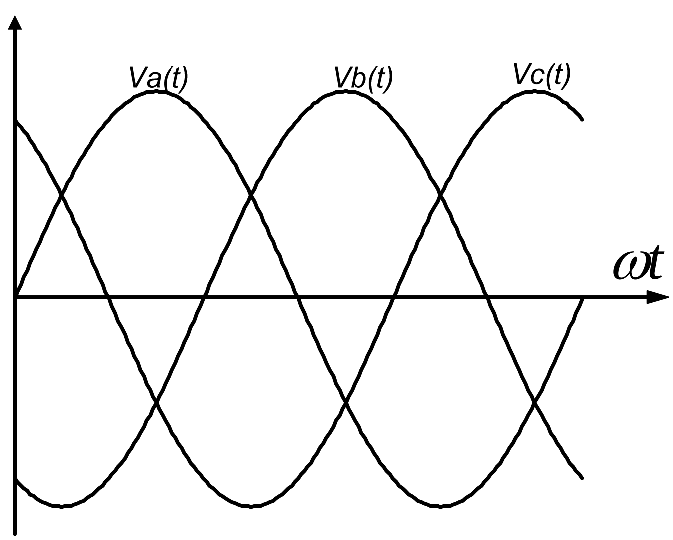
| **Time domain** | | **Phasor** |
|---|---|---|
| $v_a(t) = \sqrt{2}V\sin(\omega t)\,V$ | $\Leftrightarrow$ | $\overline{V}_a = V\angle 0^0\ V$ |
| $v_b(t) = \sqrt{2}V\sin(\omega t - 120^0)\,V$ | $\Leftrightarrow$ | $\overline{V}_b = V\angle -120^0\ V$ |
| $v_c(t) = \sqrt{2}V\sin(\omega t + 120^0)\,V$ | $\Leftrightarrow$ | $\overline{V}_c = V\angle 120^0\ V$ |



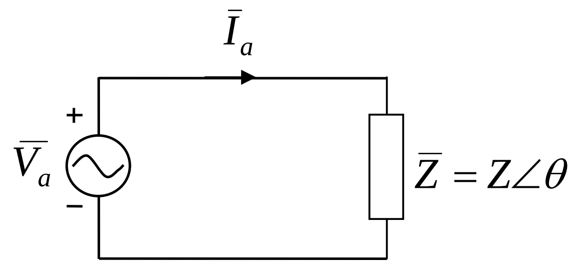

Three 1-phase generators



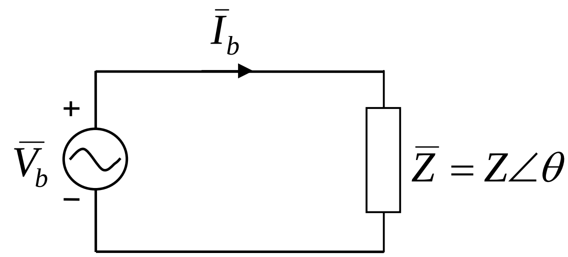

Phasor diagram



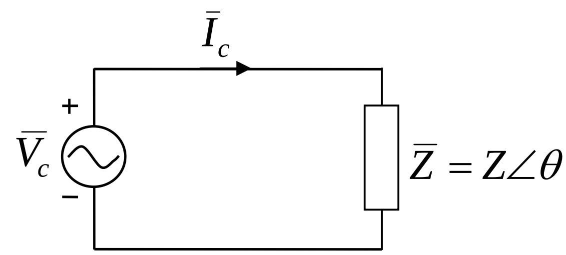
Each of these three generators can be connected to one of the three identical loads by a pair of wires.




The load impedance is considered as $\bar{Z} = Z\angle\theta\,\Omega$.




Such a system is basically three 1-phase circuits which happen to differ in phase angle by $120^0$.




Currents in the above circuits are:

$$\bar{I}_a = \frac{\bar{V}_a}{\bar{Z}} = \frac{V\angle 0^0}{Z\angle\theta} = \frac{V}{Z}\angle -\theta \triangleq I\angle -\theta\ A$$

$$\bar{I}_b = \frac{\bar{V}_b}{\bar{Z}} = \frac{V\angle -120^0}{Z\angle\theta} = \frac{V}{Z}\angle(-120^0 - \theta) \triangleq I\angle(-\theta - 120^0)\ A$$

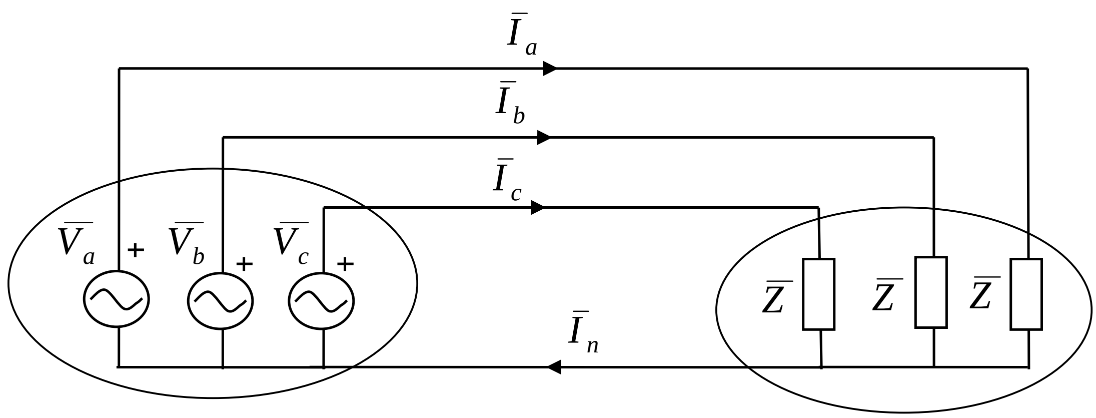
$$\bar{I}_c = \frac{\bar{V}_c}{\bar{Z}} = \frac{V\angle 120^0}{Z\angle\theta} = \frac{V}{Z}\angle(120^0 - \theta) \triangleq I\angle(-\theta + 120^0)\ A$$

All currents have the same magnitude $I$ (= $V/Z$) but a phase shift of $120^0$ from others.

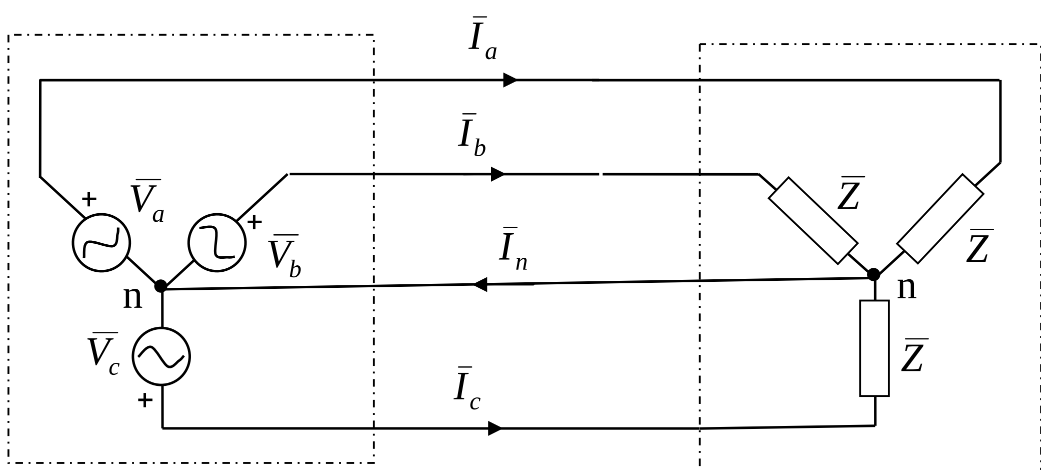
## Integration of Three Generators

- The above three 1-phase systems require six wires (two for each system)
- In fact, three of the six wires are not necessary for the generators to supply power to the loads

- Consider that the three negative ends of the generators are joined together. Similarly, the three lower ends of the loads are also joined together
- Three return wires can now be replaced by a single wire called neutral wire. Current can still return from the loads to the generators as shown in the following figure
- Such a system is called **3-phase, 4-wire** system. It consists of three phase wires and a neutral wire




Generator Load




With a neutral line

A 3-phase, 4-wire system

- Current in the neutral wire can be written as (using KCL)

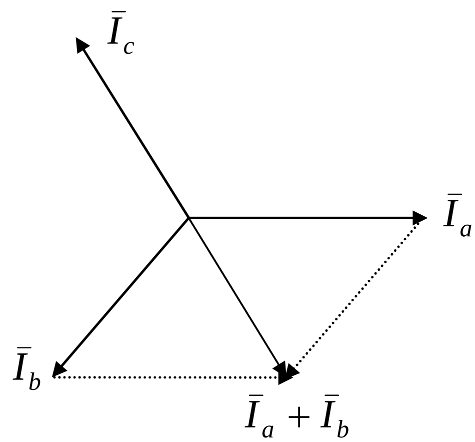
$$\bar{I}_n = \bar{I}_a + \bar{I}_b + \bar{I}_c$$

$$= I\angle -\theta + I\angle(-\theta - 120^0) + I\angle(-\theta + 120^0) = 0$$

When 3 phasors of equal magnitudes are shifted by $120^0$ from others, the sum of the phasors is always **zero.**

As you can see, $\bar{I}_a + \bar{I}_b$ is the same as $-\bar{I}_c$

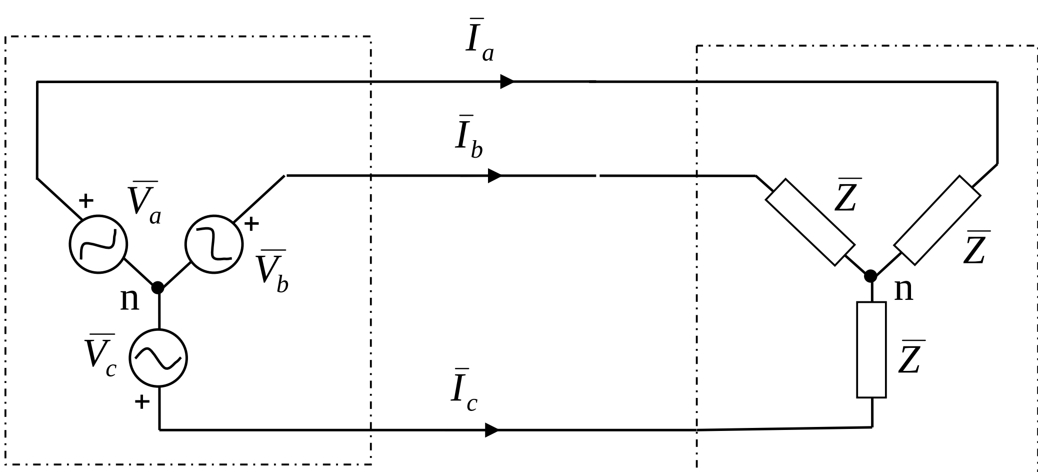
$$\Rightarrow \quad \bar{I}_a + \bar{I}_b + \bar{I}_c = 0$$




As long as the three loads are identical, the current in the neutral line is always zero.

Thus, the neutral line can be eliminated (as shown in the following figure) and power can still be transferred from the generators to the loads.

Such a system is called **3-phase, 3-wire** system.




Without a neutral line

A 3-phase, 3-wire system

## Balanced 3-phase Systems

- A 3-phase source is called balanced if it consists of three voltages of equal magnitude with phase shifts of exactly $120^0$ from others

  Example: $$\overline{V}_a = V\angle\alpha\, V$$
  $$\overline{V}_b = V\angle(\alpha - 120^0)\, V$$
  $$\overline{V}_c = V\angle(\alpha + 120^0)\, V$$

- A 3-phase source is called unbalanced if the voltage magnitudes are not equal and/or the phase shifts are not exactly $120^0$
- A 3-phase load is called balanced if all three impedances are identical. Identical means all impedances have the same magnitude and same angle
- A 3-phase transmission system is called balanced if the impedances of all three lines are identical
- A 3-phase system is called balanced if the sources, transmission systems and loads are balanced

In this subject, we will mainly concentrate on balanced 3-phase systems.

In practice

- Sources and transmission systems are usually balanced
- Loads are slightly unbalanced (because of 1-phase loads)

## Phase Sequence

There are two phase sequences in a 3-phase system:

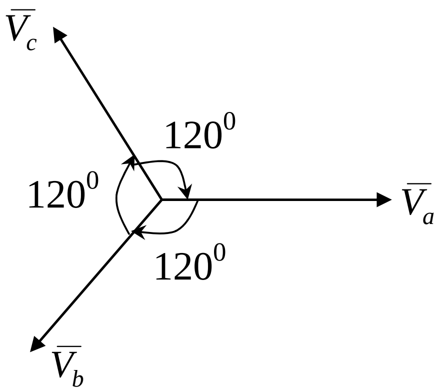
### (a) Phase sequence a-b-c

In this case, $\overline{V}_a$ leads $\overline{V}_b$, $\overline{V}_b$ leads $\overline{V}_c$ and $\overline{V}_c$ leads $\overline{V}_a$ (in the sequence of a-b-c).

$$\overline{V}_a = V\angle 0^0\ V$$

$$\overline{V}_b = V\angle -120^0\ V$$

$$\overline{V}_c = V\angle 120^0\ V$$



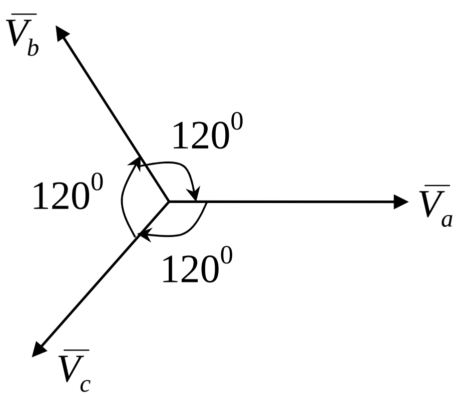

### (b) Phase sequence a-c-b

In this case, $\overline{V}_a$ leads $\overline{V}_c$, $\overline{V}_c$ leads $\overline{V}_b$ and $\overline{V}_b$ leads $\overline{V}_a$ (in the sequence of a-c-b).

$$\overline{V}_a = V\angle 0^0\ V$$

$$\overline{V}_b = V\angle 120^0\ V$$

$$\overline{V}_c = V\angle -120^0\ V$$



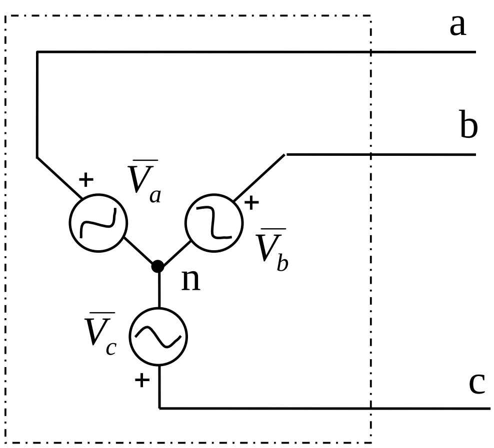

The direction of rotation of some 3-phase motors depends on phase sequence.

## Connections of 3-phase Generators and Loads

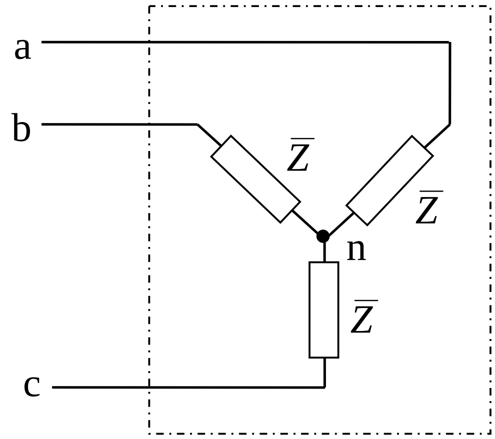
A 3-phase generator or load may be connected in two ways:

### (a) Star or Y (Wye) connection

In star or Y-connection, one end of each source (or load) is connected to a common point called neutral point. In a balanced system, the voltage at the neutral point is zero and sometimes it is connected to the ground.



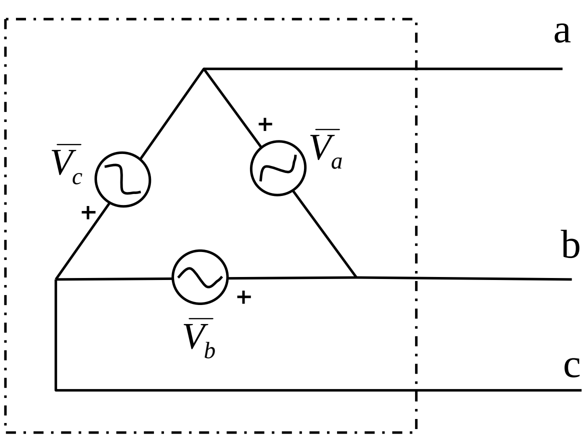

Y-connected generator



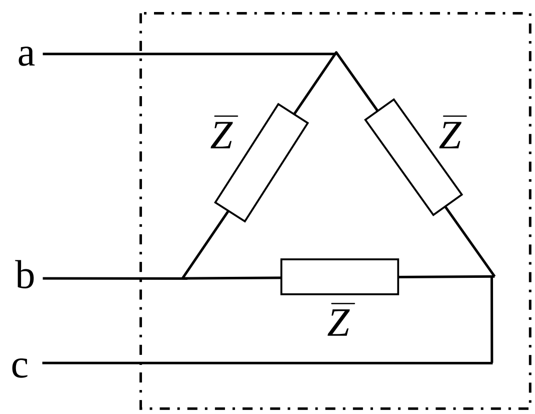

Y-connected load

## (b) Mesh or Δ (Delta) connection

In mesh or Δ-connection, three sources (or loads) are connected in Δ (or closed form) as shown below. There is no neutral point in Δ-connection.




Δ-connected generator




Δ-connected load

Any number of Y and Δ connected generators and loads may be mixed up in a power system.

## General Representation of 3-phase Generators and Loads


3-phase, 3-wire system

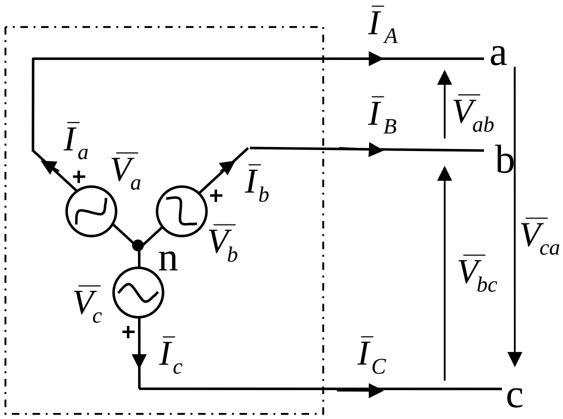

3-phase, 4-wire system

In Δ-connection, there is no neutral point and thus it is always 3-phase, 3-wire. However, a Y-connected system can be 3-phase, 3-wire or 3-phase, 4-wire.

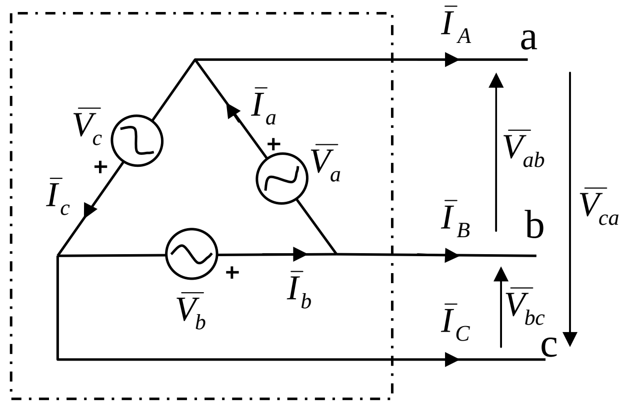
## Phase and Line Quantities

### In both Y- and Δ-connection

- Voltage of each source or across each load impedance (inside the dotted box) is called phase voltage
- Current of each source or through each load impedance (inside the dotted box) is called phase current
- Voltage between any two lines (outside the dotted box) is called line voltage. Exclude the neutral wire (if any)
- Current through any line (outside the dotted box) is called line current. Exclude the neutral wire (if any)




Y-connected generator




Δ-connected generator

## In the Above Diagrams

Phase voltages are: $\bar{V}_a$ $\bar{V}_b$ and $\bar{V}_c$

Line voltages are: $\bar{V}_{ab}$ $\bar{V}_{bc}$ and $\bar{V}_{ca}$

Phase currents are: $\bar{I}_a$ $\bar{I}_b$ and $\bar{I}_c$

Line currents are: $\bar{I}_A$ $\bar{I}_B$ and $\bar{I}_C$

Phase quantities are inside the dotted box (which may not be accessible) and line quantities are outside the dotted box (which are always available).

## By Inspection

For Y-connection

Line current = phase current

$$\bar{I}_A = \bar{I}_a, \quad \bar{I}_B = \bar{I}_b, \quad \bar{I}_C = \bar{I}_c$$

Line voltage ≠ phase voltage

For Δ-connection

Line voltage = phase voltage

$$\bar{V}_{ab} = \bar{V}_a, \quad \bar{V}_{bc} = \bar{V}_b, \quad \bar{V}_{ca} = \bar{V}_c$$

Line current ≠ phase current

## Relationship between $V_L$ and $V_p$ of Y-connection

Let the phase voltages be (for phase sequence a-b-c)

$$\bar{V}_a = V\angle 0^0\,\text{V}, \quad \bar{V}_b = V\angle -120^0\,\text{V} \quad \text{and} \quad \bar{V}_c = V\angle 120^0\,\text{V}$$

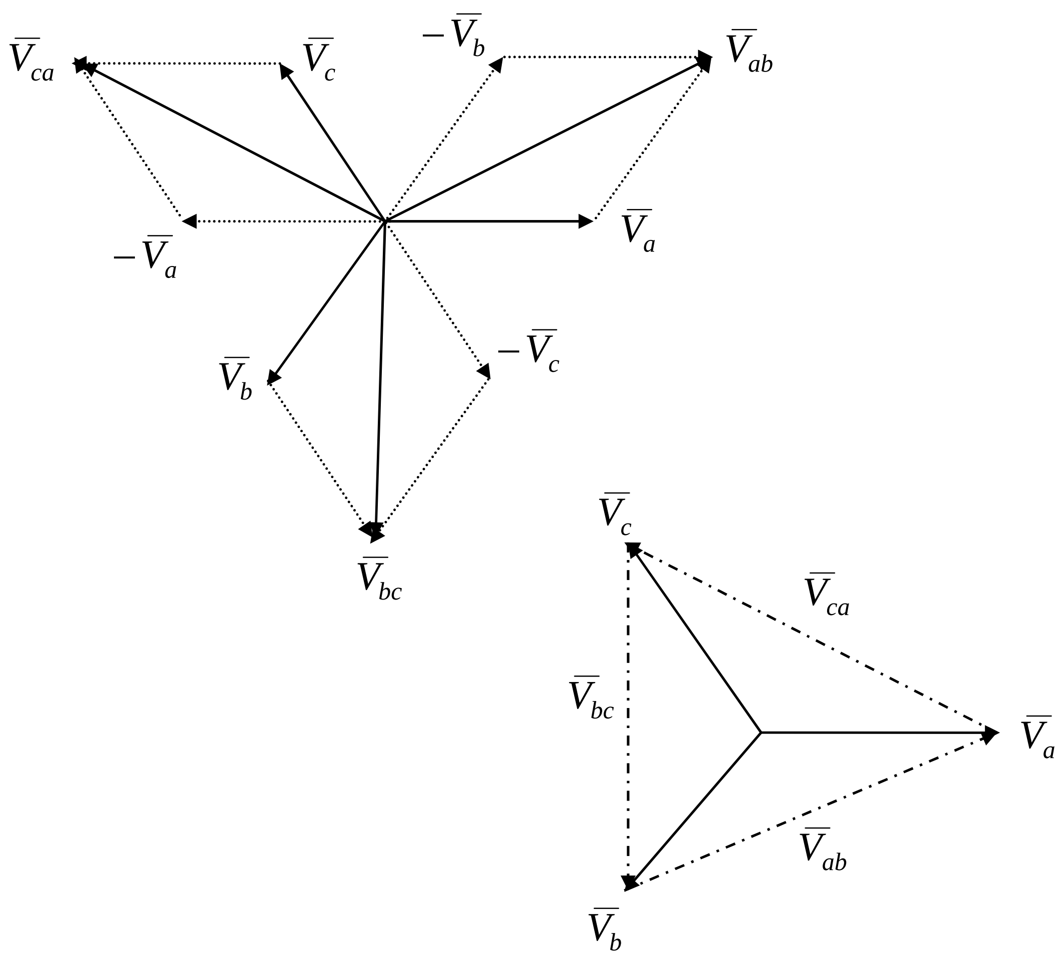
$\therefore$ Line voltages are:

$$\bar{V}_{ab} = \bar{V}_a - \bar{V}_b \quad = V\angle 0^0 - V\angle -120^0 \qquad = \sqrt{3}V\angle 30^0\,\text{V}$$

$$\bar{V}_{bc} = \bar{V}_b - \bar{V}_c \quad = V\angle -120^0 - V\angle 120^0 \qquad = \sqrt{3}V\angle -90^0\,\text{V}$$

$$\bar{V}_{ca} = \bar{V}_c - \bar{V}_a \quad = V\angle 120^0 - V\angle 0^0 \qquad = \sqrt{3}V\angle 150^0\,\text{V}$$

The above voltages are shown in the following phasor diagrams:



- Magnitude of line voltage = $\sqrt{3}\times$magnitude of phase voltage
- Line voltage leads the respective phase voltage by $30^0$ (for phase sequence a-b-c)
- Line voltage lags the respective phase voltage by $30^0$ (for phase sequence a-c-b)

## Relationship between $I_L$ and $I_p$ of $\Delta$-connection

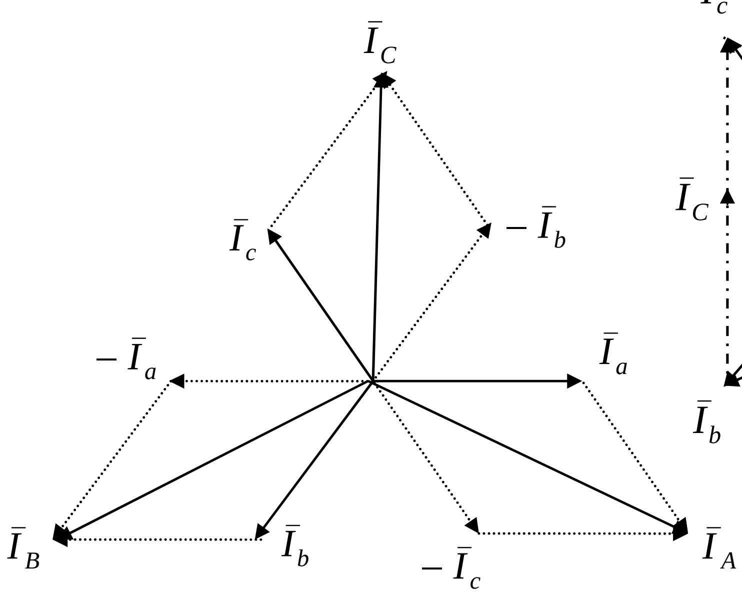
Let the phase currents be (for phase sequence a-b-c)

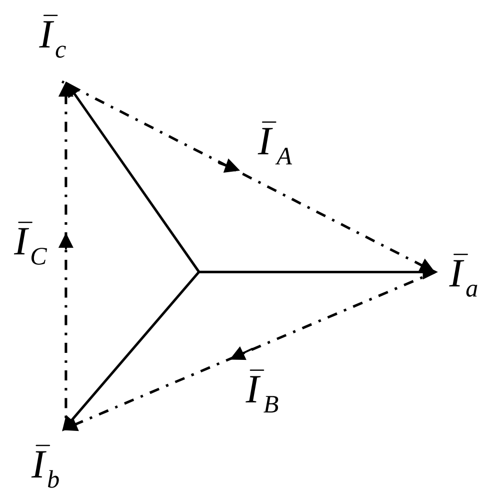
$$\bar{I}_a = I\angle 0^0\,\text{A},\quad \bar{I}_b = I\angle -120^0\,\text{A}\quad \text{and } \bar{I}_c = I\angle 120^0\,\text{A}$$

$\therefore$ Line currents are (by using KCL):

$$\bar{I}_A = \bar{I}_a - \bar{I}_c = I\angle 0^0 - I\angle 120^0 \quad = \sqrt{3}\,I\angle -30^0\,\text{A}$$
$$\bar{I}_B = \bar{I}_b - \bar{I}_a = I\angle -120^0 - I\angle 0^0 \quad = \sqrt{3}\,I\angle -150^0\,\text{A}$$
$$\bar{I}_C = \bar{I}_c - \bar{I}_b = I\angle 120^0 - I\angle -120^0 \quad = \sqrt{3}\,I\angle 90^0\,\text{A}$$

Above currents are shown in the following phasor diagrams:



- Magnitude of line current = $\sqrt{3}\times$magnitude of phase current
- Line current lags the respective phase current by $30^0$ (for phase sequence a-b-c)
- Line current leads the respective phase current by $30^0$ (for phase sequence a-c-b)

## Summary

| | Y-connection | Δ-connection |
|---|---|---|
| Voltage | $V_L = \sqrt{3}V_p$ | $\bar{V}_L = \bar{V}_p$ |
| Current | $\bar{I}_L = \bar{I}_p$ | $I_L = \sqrt{3}\, I_p$ |
| Phase shift (for phase sequence abc) | $\bar{V}_L$ leads $\bar{V}_p$ by $30^0$ | $\bar{I}_L$ lags $\bar{I}_p$ by $30^0$ |

Subscripts '*L*' and '*p*' are used to represent the line and phase quantities, respectively.

## 3-phase Power

Total complex power of a 3-phase system is the sum of complex power of all three phases.

$$\therefore \bar{S}_T = \bar{S}_a + \bar{S}_b + \bar{S}_c \text{ VA}$$

Under balanced condition, all three phases have the same complex power.

That is, $\bar{S}_a = \bar{S}_b = \bar{S}_c$

$$\therefore \bar{S}_T = \bar{S}_a + \bar{S}_b + \bar{S}_c = 3\bar{S}_a = 3\bar{V}_a \bar{I}_a^* \text{ VA}$$

$$\Rightarrow \quad \bar{S}_T = 3V_p I_P \angle \varphi = P_T + jQ_T \text{ VA,}$$

$$P_T = 3V_p I_P \cos\varphi \text{ W} \quad \text{and} \quad Q_T = 3V_p I_P \sin\varphi \text{ VAr}$$

Here φ is the angle between the phase voltage and the phase current.

- Phase voltages and phase currents are inside the generators or loads (within the dotted box) and are usually not accessible from outside terminals
- Line voltages and line currents are always available at generator or load terminals (outside the dotted box)
- Thus, it is more appropriate to express the above powers in terms of line quantities (line voltage and line current)

**For Y-connection**

$$3V_p I_p \angle\varphi = \sqrt{3}\,(\sqrt{3}V_p)I_P \angle\varphi = \sqrt{3}V_L I_L \angle\varphi$$

**For Δ-connection**

$$3V_p I_p \angle\varphi = \sqrt{3}\,V_p(\sqrt{3}I_P)\angle\varphi = \sqrt{3}V_L I_L \angle\varphi$$

$\therefore$ For both Y- and Δ-connections, $3V_p I_p \angle\varphi = \sqrt{3}V_L I_L \angle\varphi$

## Total Power of a Balance 3-phase System

| Power | In terms of phase quantities | In terms of line quantities | Unit |
|---|---|---|---|
| $\bar{S}$ | $3V_p I_p \angle\varphi$ | $\sqrt{3}V_L I_L \angle\varphi$ | VA |
| $P$ | $3V_p I_p \cos\varphi$ | $\sqrt{3}V_L I_L \cos\varphi$ | W |
| $Q$ | $3V_p I_p \sin\varphi$ | $\sqrt{3}V_L I_L \sin\varphi$ | VAr |

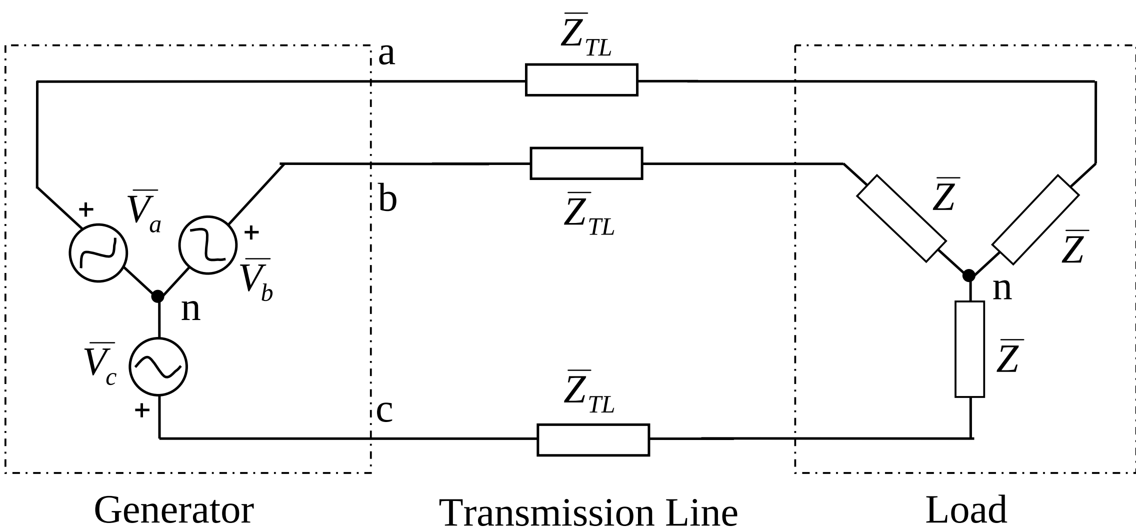
The above equations are valid for both Y- and Δ-connections but under balanced condition.

- Note that φ is the angle between phase voltage and phase current (but **NOT** between the line voltage and line current)
- Similar to a 1-phase circuit, φ is also called the power factor angle and is the same as impedance angle
- Similar to a 1-phase circuit, reactive power $Q$ is positive for a lagging power factor load (inductive circuit) and negative for a leading power factor load (capacitive circuit)

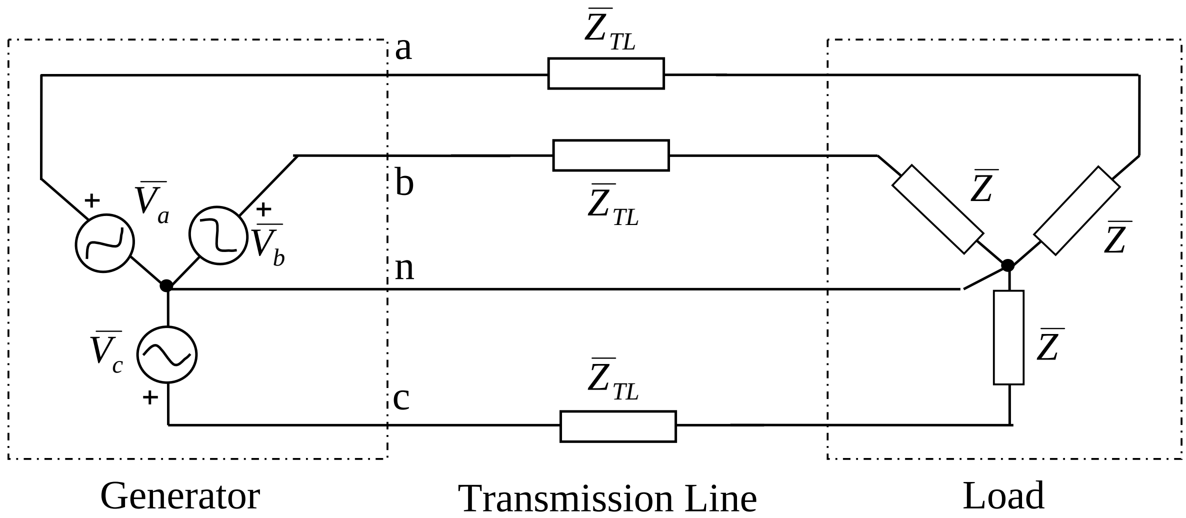
## Analysis of a Balanced 3-phase Y-connected System

For a balanced 3-phase system, it is possible to determine the voltages, currents, and powers at various points in a 3-phase circuit from its single-phase equivalent circuit.

Consider a 3-phase Y-connected generator supplying power to a 3-phase Y-connected load through a 3-phase transmission line as shown in the following figure.

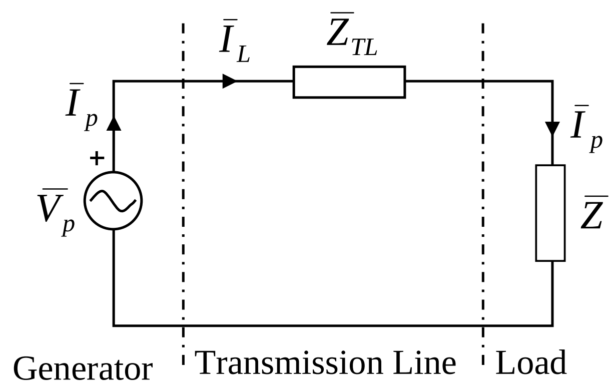



Add a fictitious neutral line between the generator neutral point and the load neutral point as shown in the following. Note that addition of the neutral line has no effect because it does not carry any current (under balanced condition).




All three phases are identical except for a shift in phase angle of $120^0$.

Thus, the 3-phase circuit can be analyzed from its single-phase equivalent circuit consisting of one of the three phases and the neutral line as shown here.

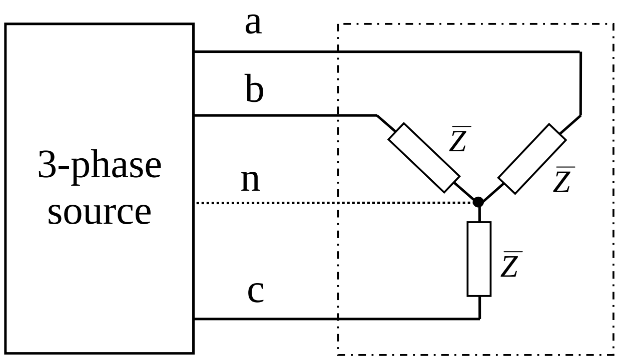


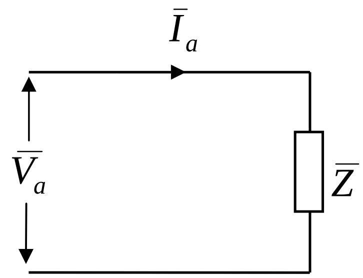

Results of the above 1-phase circuit are also valid for the other two phases when the phase shift of $120^0$ is included.

## Example – 6

A 415 V, 3-phase source supplies power to a balanced 3-phase Y-connected load having an impedance of (16+j12) Ω per phase. Determine the current and complex power absorbed by the load.

**Solution**




1-phase equivalent circuit

Here $V_L = 415$ V $\Rightarrow \quad V_p = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6$ V/phase

Consider $\bar{V}_a$ as reference $\Rightarrow \bar{V}_a = 239.6\angle 0^0$ V

$$\therefore \quad \bar{I}_a = \frac{\bar{V}_a}{\bar{Z}} = \frac{239.6\angle 0^0}{(16 + j12)} = 11.98\angle -36.87^0 \text{ A}$$

$$\bar{S}_a = \bar{V}_a \, \bar{I}_a^* = 239.6\angle 0^0 \times 11.98\angle 36.87^0$$

$$= 2870.4\angle 36.87^0 = (2296.3 + j1722.2)\,VA$$

Thus

$I_p = 11.98$ A $\Leftrightarrow$ $I_L = I_p = 11.98$ A

$\bar{S}_p = 2870.4\angle 36.87^0$ VA $\Leftrightarrow \bar{S}_T = 3\bar{S}_p$ VA

or $\bar{S}_T = 8611.2\angle 36.87^0 \; VA = (6888.9 + j5166.7)VA$

$P_T = 3P_p = 6888.9$ W and $Q_T = 3Q_p = 5166.7$ VAr

Currents and complex powers of all phases are:

$\bar{I}_a = 11.98\angle -36.87^0$ means

$\bar{I}_b = 11.98\angle -156.87^0$ A and $\bar{I}_c = 11.98\angle 83.13^0$ A

and $\bar{S}_a = \bar{S}_b = \bar{S}_c = 2870.4\angle 36.87^0$ VA

# Convention

### Unless specified otherwise

- Voltage of a 3-phase system is always specified as line voltage (only magnitude or rms value)
- Current of a 3-phase system is always specified as line current (only magnitude or rms value)
- Power (P, Q or S) of a 3-phase system is always specified as total power
- However, the impedance of a 3-phase system is always specified as **phase impedance**

# In the Above Example:

| | | |
|---|---|---|
| A 3-phase 415 V source | $\Leftrightarrow$ | Line voltage is 415 V (rms value) |
| Find the load current | $\Leftrightarrow$ | Find the line current of the load (rms value) |
| Find the load power | $\Leftrightarrow$ | Find the total load power |

# In the 1-phase Equivalent Circuit:

All quantities (voltage, current, power, impedance, etc.) are phase quantities.

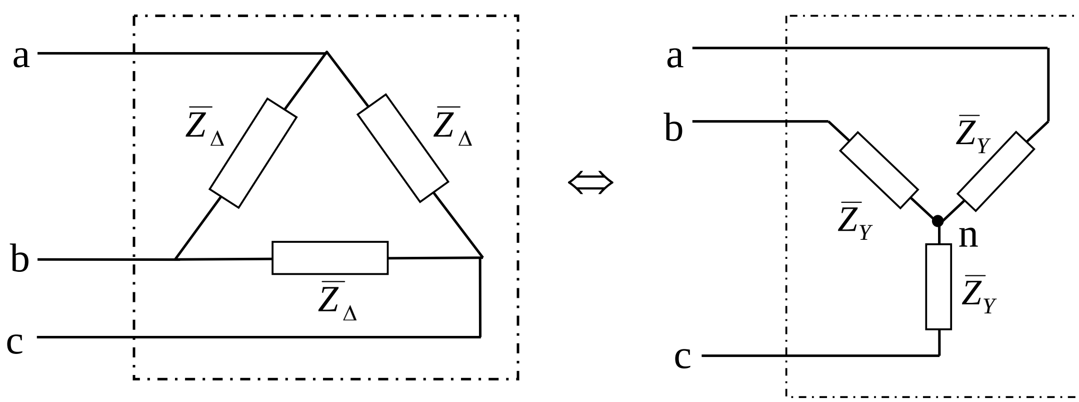
- At the beginning, it is required to find the phase quantities in order to solve the 1-phase equivalent circuit
- At the end, it is necessary to find the line quantities and total power from the results of 1-phase equivalent circuit (using standard relationships)

## Analysis of a Balanced 3-phase Δ System

- In the above 1-phase equivalent circuit, it is required to have the neutral point (at least conceptually) to provide return path for the current. This is fine with Y-connected sources and Y-connected loads. However, a Δ-connected source or load does not have a neutral point
- The standard procedure of solving the problem of a Δ-connected system is by transforming it to an equivalent Y-connected system

## Δ-Y Transformation of Load

For a balanced 3-phase load, the following equations can be used to transform a Δ-connected load into an equivalent Y-connected load or vice versa.




From Δ to Y: $\overline{Z}_Y = \dfrac{\overline{Z}_\Delta}{3}$

From Y to Δ: $\overline{Z}_\Delta = 3\overline{Z}_Y$

## Example – 7

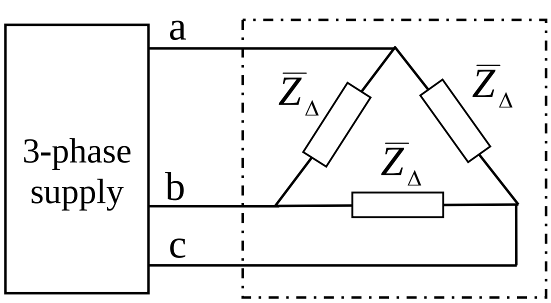
A 3-phase 415 V source supplies power to a balanced 3-phase Δ-connected load having an impedance of (16+j12) Ω per phase. Find the current and complex power of the load.

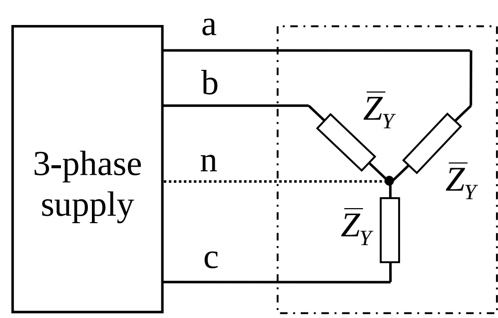
## Solution

Here $\overline{Z}_{\Delta} = (16 + j12)\,\Omega$

Transform the $\Delta$ load into an equivalent Y load

$$\therefore \quad \overline{Z}_Y = \overline{Z}_{\Delta}/3 = (5.333 + j4)\,\Omega$$



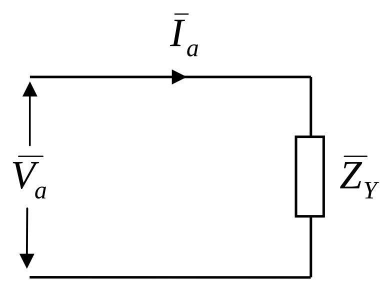




In the equivalent Y system

$V_p = V_L/\sqrt{3} = 415/\sqrt{3} = 239.6$ V

Consider $\overline{V}_a$ as reference

$$\Rightarrow \overline{V}_a = 239.6\angle 0^0 \text{ V}$$




1-phase equivalent circuit

Thus $$\overline{I}_a = \frac{\overline{V}_a}{\overline{Z}_Y} = \frac{239.6\angle 0^0}{(5.333 + j4)} = 35.94\angle -36.87^0 \; A$$

$$\overline{S}_a = \overline{V}_a \overline{I}_a^* = 239.6\angle 0^0 \times 35.94\angle 36.87^0$$

$$= 8611\angle 36.86^0 = (6889 + j5166.7)\,VA$$

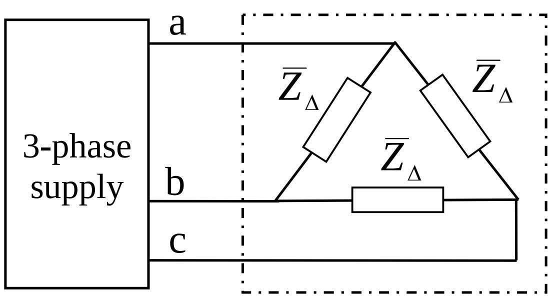
Therefore

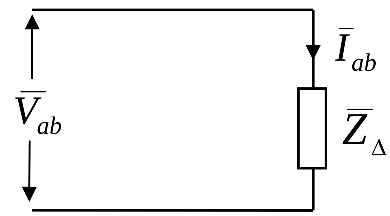
$I_p = 35.94$ A $\Leftrightarrow$ $I_L = I_p = 35.94$ A

$\overline{S}_p = 8611\angle 36.87^0$ VA $\Leftrightarrow$ $\overline{S}_T = 3\overline{S}_p = 25833\angle 36.87^0$ VA

The problem of a simple Δ-connected load can directly be solved without transforming the load from Δ to Y.

**Consider the problem of Example - 7**



1-phase equivalent circuit

In example – 7, $\bar{V}_a$ was considered as reference

$\therefore \bar{V}_p = \bar{V}_{ab} = 415\angle 30^0$ V

$$\bar{I}_p = \bar{I}_{ab} = \frac{\bar{V}_{ab}}{\bar{Z}_\Delta} = \frac{415\angle 30^0}{(16 + j12)} = 20.75\angle -6.87^0 \ A$$

$$\bar{S}_p = \bar{S}_{ab} = \bar{V}_{ab}\bar{I}_{ab}^* = 415\angle 30^0 \times 20.75\angle 6.87^0$$

$$= 8611\angle 36.86^0 = (6889 + j5166.7)\, VA$$

Thus

$I_p = 20.75$ A ⇔ $I_L = \sqrt{3}\, I_p = 35.94$ A

$\bar{S}_p = 8611\angle 36.87^0$ VA ⇔ $\bar{S}_T = 3\bar{S}_p = 25833\angle 36.87^0$ VA

Note that the above method (without converting Δ into Y) may not be applied if the cables/lines connecting the source to the load have some impedance.

## Analysis of a Balanced 3-phase System

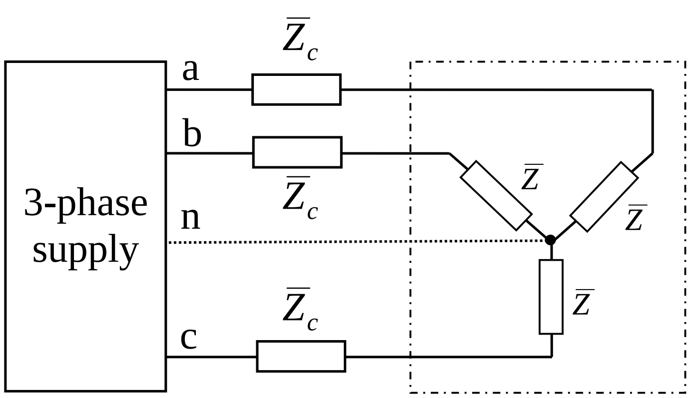
- The problem of a balanced 3-phase system can be analyzed from its single-phase equivalent circuit
- This require to get the phase quantities to analyze the 1-phase equivalent circuit
- At the end, results of the 3-phase system can be obtained from the results of the 1-phase equivalent circuit using the standard relationships

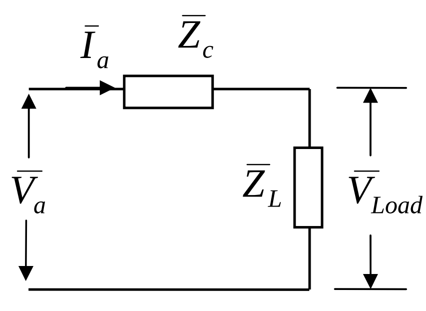
Problem of a 3-$\phi$ system → Draw the 1-$\phi$ equi. circuit → Solve the 1-$\phi$ equi. circuit → Get the 3-$\phi$ results

## Example – 8

A 400 V, 3-phase source supplies power to a 3-phase Y-connected load having an impedance of (18+j12) Ω per phase through an underground cable of impedance (0.4+j0.9) Ω per phase. Determine the load voltage, load current and load active power. Also determine the active and reactive power losses in the cable.

**Solution**




1-phase equivalent circuit

Here $\bar{Z}_c = (0.4 + j0.9)\ \Omega$ and $\bar{Z}_L = (18 + j12)\ \Omega$

$V_p = V_L/\sqrt{3} = 400/\sqrt{3} = 230.9$ V

Consider $\bar{V}_a$ as reference $\Rightarrow$ $\bar{V}_a = 230.9\angle 0^0$ V

$$\bar{I}_a = \frac{\bar{V}_a}{\bar{Z}_c + \bar{Z}_L} = \frac{230.9\angle 0^0}{(0.4 + j0.9) + (18 + j12)} = 10.28\angle -35^0\ A$$

$$\bar{V}_{Load} = \bar{I}_a \bar{Z}_L = 10.28\angle -35^0 \times (18 + j12) = 222.4\angle -1.31^0\ V/\phi$$

Load active power = $I_a^2 R_L$ = $10.28^2 \times 18$ = 1.9 kW/phase

Active power loss = $I_a^2 R_c$ = $10.28^2 \times 0.4$ = 42.3 W/phase

Reactive power loss = $I_a^2 X_c$ = $10.28^2 \times 0.9$ = 95.1 VAr/phase

| | |
|---|---|
| Load voltage | = $\sqrt{3} \times 222.4$ = 385.2 V (line) |
| Load current | = 10.28 A (line) |
| Load active power | = $3 \times 1.9$ = 5.7 kW (total) |
| Active power loss | = $3 \times 42.3$ = 126.9 W (total) |
| Reactive power loss | = $3 \times 95.1$ = 285.3 VAr (total) |

## Example – 9

A 400 V, 3-phase source supplies power to a Δ-connected load having an impedance of (18+j12) Ω/phase through a cable of impedance (0.5+j1.5) Ω/phase. Determine the (a) phase and line current of the load (b) active power of the load and source (c) power factor at the load side and source side.

> **Note:** When the cable impedances are included, it is difficult to solve the problem without converting the load from Δ to Y because the line voltages at source side do not appear across the Δ-connected load. Moreover, the cable impedance and Δ-connected load impedance are not in

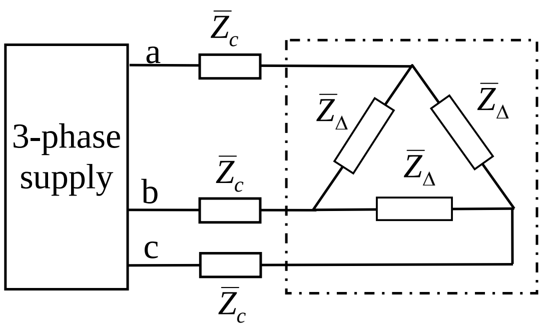
series and thus one of the phases cannot be separated out. The general way of solving this problem is by transforming the Δ load into an equivalent Y load.

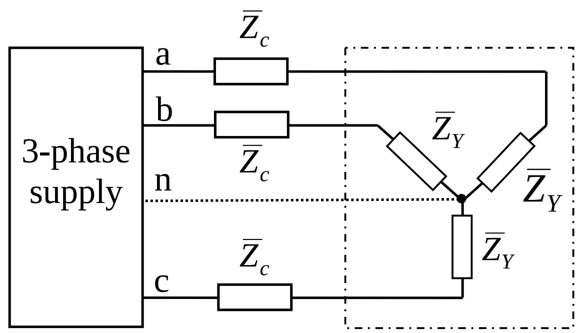
**Solution**

Here $\bar{Z}_c = (0.5 + j1.5)\,\Omega$ and $\bar{Z}_\Delta = (18 + j12)\,\Omega$

Convert the Δ load into Y

$$\Rightarrow \bar{Z}_Y = \bar{Z}_\Delta / 3 = (6 + j4)\,\Omega = 7.211\angle 33.69^0\ \Omega$$

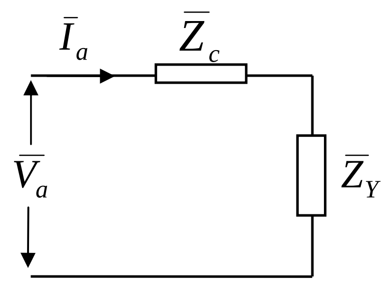



$V_p = V_L/\sqrt{3} = 400/\sqrt{3} = 230.9$ V

Consider $\bar{V}_a$ as reference

$\Rightarrow \bar{V}_a = 230.9\angle 0^0$ V




$$\bar{I}_a = \frac{\bar{V}_a}{\bar{Z}_c + \bar{Z}_Y} = \frac{230.9\angle 0^0}{(0.5 + j1.5) + (6 + j4)} = 27.12\angle -40.24^0\ A$$

(a) Line current $I_L = I_p$ (in the equivalent Y) = 27.12 A

Phase current (in Δ-connection) = $I_L/\sqrt{3}$

= $27.12/\sqrt{3} = 15.66$ A

(b) $P_{Load} = 3I_a^2R_Y = 3\times 27.12^2\times 6 = 13.24$ kW

$P_{Source} = 3V_aI_a\cos\varphi = 3\times230.9\times27.12\times\cos(40.24^0)$

$= 14.34$ kW

$P_{Loss} = 3I_a^2R_c = 3\times27.12^2\times0.5 = 1.10$ kW

Also, $P_{Source} = P_{Load} + P_{Loss}$

$= (13.24 + 1.10) = 14.34$ kW (checked)

(c) Load pf angle = angle $\overline{Z}_Y = 33.69^0$

Load power factor = $\cos(33.69^0) = 0.832$ (lagging)

Source pf angle = phase difference between $\overline{V}_a$ and $\overline{I}_a$

$= 40.24^0$

Source pf = $\cos(40.24^0) = 0.7633$ (lagging)

Also source pf angle = angle of $(\overline{Z}_c + \overline{Z}_Y)$

$$(\overline{Z}_c + \overline{Z}_Y) = (0.5 + j1.5) + (6 + j4) = 8.51\angle 40.24^0\ \Omega$$

$\therefore$ source pf angle is $40.24^0$