$$A = \begin{bmatrix} \sqrt{2} & 0.5 & 0.5 \\ \sqrt{2} & -0.5 & -0.5 \end{bmatrix}$$

To find u :

$$AA^T = \begin{bmatrix} \sqrt{2} & \frac{1}{2} & \frac{1}{2} \\ \sqrt{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

perform eigen-decomposition :

$$AA^T \underset{\sim}{u} = \lambda \underset{\sim}{u}$$

$$\begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \left(\text{let } \underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right)$$

$$\begin{bmatrix} \frac{5}{2}-\lambda & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2}-\lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underset{\sim}{0} \quad (*)$$

Char eqn :

$$\det \begin{pmatrix} \frac{5}{2}-\lambda & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2}-\lambda \end{pmatrix} = 0$$

$$\left(\frac{5}{2}-\lambda\right)^2 - \frac{9}{4} = 0$$

solving $\quad \lambda^2 - 5\lambda + 4 = 0$

$$(\lambda-4)(\lambda-1) = 0$$

$\lambda = 4, 1$ eigenvalue

$\boxed{\sigma = 2, 1} = \sqrt{\lambda}$

$$\boxed{\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \quad 2 \times 3$$

When $\lambda = 4$, sub into $(*)$.

$$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad u_1 = u_2.$$

$$\therefore \quad \underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \left[u_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^2 = 1$$

normalized $\Rightarrow \quad \underset{\sim}{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

When $\lambda = 1$, sub into $(*)$.

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad u_2 = -u_1.$$

$$\therefore \quad \underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

normalized $\Rightarrow \quad \underset{\sim}{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2×2

formed by eigenvector $\quad \therefore \boxed{U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \Leftarrow$

To find V :

$$A^TA = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{2} & \frac{1}{2} \\ \sqrt{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Perform eigen-decomposition :

$$A^TA\underset{\sim}{v} = \lambda \underset{\sim}{v}$$

$$(A^TA - \lambda I)\underset{\sim}{v} = 0$$

$$\begin{pmatrix} 4-\lambda & 0 & 0 \\ 0 & \frac{1}{2}-\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2}-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \quad (*) \qquad \text{Let } \underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Characteristic eqn :

$$\det \begin{pmatrix} 4-\lambda & 0 & 0 \\ 0 & \frac{1}{2}-\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2}-\lambda \end{pmatrix} = 0$$

(i) either from previous part, $\lambda = 4, 1$

(ii) or solving it

$$(4-\lambda)\left[(\tfrac{1}{2}-\lambda)^2 - (\tfrac{1}{2})^2\right] = 0$$

$$(4-\lambda)(1-\lambda)(-\lambda) = 0$$

$$\lambda(\lambda-4)(\lambda-1) = 0$$

$$\lambda = 4, 1, 0 :$$

When $\lambda = 4$, sub into $(*)$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{7}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow \quad -\frac{7}{2}v_2 + \frac{1}{2}v_3 = 0 \Rightarrow v_3 = 7v_2 \quad (1)$

$\quad \frac{1}{2}v_2 - \frac{7}{2}v_3 = 0 \quad (2)$

$(1) \to (2)$: $\quad \frac{1}{2}v_2 - \frac{49}{2}v_2 = 0 \Rightarrow v_2 = 0$

$\quad v_3 = 0$

$$\therefore \underset{\sim}{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \underset{\sim}{v_a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 1$, sub into (*)

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$\Rightarrow$ $v_1 = 0$

$v_2 = v_3$

$\therefore$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} v_3 \Rightarrow$ normalized $\underset{\sim}{v_b} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

To find v_c, there are 2 alternatives (i) or (ii):

(i) When $\lambda = 0$, sub into (*)

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underset{\sim}{0}$$

$\Rightarrow$ $v_1 = 0$

$v_2 = -v_3$

$\therefore$ $v_c = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} v_3 \Rightarrow$ normalized $\hat{v_c} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

OR

(ii) As $\underset{\sim}{V}$ is orthogonal

$\Rightarrow$ $\underset{\sim}{v_c}^T \underset{\sim}{v_a} = 0$ & $\hat{v_c}^T \underset{\sim}{v_b} = 0$

by observation or calculation: it can be seen that: $v_c = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\therefore \quad V = [\hat{v_a}\ \hat{v_b}\ \hat{v_c}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$