(a) To find the singular values : using eigen-decomposition

$$AA^T \underset{\sim}{u} = \lambda \underset{\sim}{u}$$

$$AA^T = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 22 \\ 22 & 36 \end{bmatrix}$$

Characteristic eqn :

$$\det\{AA^T - \lambda I\} = 0$$

$$\det\left\{\begin{bmatrix} 14-\lambda & 22 \\ 22 & 36-\lambda \end{bmatrix}\right\} = 0$$

$$(14-\lambda)(36-\lambda) - 22^2 = 0$$

$$\lambda^2 - 50\lambda + 504 - 484 = 0$$

$$\lambda^2 - 50\lambda + 20 = 0$$

use this method

Solving quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{50 \pm \sqrt{50^2 - 4(1)(20)}}{2} = \frac{50 \pm \sqrt{2420}}{2} = 49.5967, \; 0.40325$$

$$\lambda_1 = 49.5967 \; , \quad \lambda_2 = 0.40325$$

$$\sigma_1 = \sqrt{\lambda_1} = 7.0425 \; , \quad \sigma_2 = \sqrt{\lambda_2} = 0.6350$$

(b). To find u:

$$AA^T = \begin{bmatrix} 14 & 22 \\ 22 & 36 \end{bmatrix}$$

$$AA^T \underset{\sim}{u} = \lambda \underset{\sim}{u}$$

$$\Rightarrow (AA^T - \lambda I)\underset{\sim}{u} = 0$$

$$\begin{bmatrix} 14-\lambda & 22 \\ 22 & 36-\lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underset{\sim}{0} \quad (1) \quad \left(\text{let } \underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

When $\lambda_1 = 49.5967$

Substitute into (1)

$$\Rightarrow \begin{bmatrix} -35.5967 & 22 \\ 22 & -13.5967 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underset{\sim}{0}$$

$$-35.5967\, u_1 + 22 u_2 = 0$$

$$u_2 = 1.618\, u_1$$

$$\therefore \underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}$$

normalized

$$\underset{\sim}{u} = \frac{1}{\sqrt{1+1.618^2}}\begin{bmatrix} 1 \\ 1.618 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5257 \\ 0.8507 \end{bmatrix}$$

When $\lambda_2 = 0.40325$

Substitute into (1)

$$\Rightarrow \begin{bmatrix} 13.59675 & 22 \\ 22 & 35.59675 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underset{\sim}{0}$$

$$13.59675 u_1 + 22 u_2 = 0$$

$$u_2 = -0.6180 u_1$$

$$\therefore \quad \underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ -0.6180 \end{bmatrix}$$

normalized $\underset{\sim}{u}$ $= \dfrac{1}{\sqrt{1+0.618^2}} \begin{bmatrix} 1 \\ -0.6180 \end{bmatrix}$

$$= \begin{bmatrix} 0.8507 \\ -0.5257 \end{bmatrix}$$

$$\therefore \quad U = \begin{bmatrix} 0.5257 & 0.8507 \\ 0.8507 & -0.5257 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 7.0425 & 0 & 0 \\ 0 & 0.6350 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.6325 & -0.6325 & 0.4472 \\ 0.3162 & -0.3162 & -0.8944 \\ 0.7071 & 0.7071 & 0 \end{bmatrix}$$

To verify that $A = U\Sigma V^T$

RHS $= U\Sigma V^T$

$$= \begin{bmatrix} 0.5257 & 0.8507 \\ 0.8507 & -0.5257 \end{bmatrix} \begin{bmatrix} 7.0425 & 0 & 0 \\ 0 & 0.6350 & 0 \end{bmatrix} \begin{bmatrix} 0.6325 & 0.3162 & 0.7071 \\ -0.6325 & -0.3162 & 0.7071 \\ 0.4472 & -0.8944 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 4 \end{bmatrix}$$

$=$ LHS (verified).

(c) As $\sigma_1 = 7.0425 \gg \sigma_2 = 0.635$, a suitable rank for this low-rank representation is 1

Low-rank representation,

$$\hat{A} = \sigma_1 \underset{\sim}{u}_1 \underset{\sim}{v}_1^T$$

$$= 7.0425 \begin{bmatrix} 0.5257 \\ 0.8507 \end{bmatrix} \begin{bmatrix} 0.6325 & 0.3162 & 0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} 2.342 & 1.171 & 2.618 \\ 3.789 & 1.894 & 4.236 \end{bmatrix}$$

$$\text{error} = A - \hat{A}$$

$$= \begin{bmatrix} -0.342 & -0.171 & 0.382 \\ 0.211 & 0.106 & -0.236 \end{bmatrix}$$

square all value and add them together

$\therefore$ total squared error $= \| A - \hat{A} \|^2 = 0.404$