The Dynamics Model of VTOL

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Basic Dynamic Function

We define our state as :

$$\mathbf{x} = [x_r, y_r, z_r, \phi, \theta, \psi, v_x, v_y, v_z, \omega_x, \omega_y, \omega_z]^T$$

We define our input as:

$$\mathbf{u} = [\delta_{ar}, \delta_{al}, \delta_c, u_{pr}, u_{pl}]$$

(a)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} \mathrm{v_x} \\ \mathrm{v_y} \\ \mathrm{v_z} \end{pmatrix}$$

(b)

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

(c)

$$\mathbf{f}^b \triangleq (f_x, f_y, f_z)^{\mathrm{T}}$$

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = \begin{pmatrix} \omega_z v_y - \omega_y v_z \\ \omega_x v_z - \omega_z v_x \\ \omega_y v_x - \omega_x v_y \end{pmatrix} + \frac{1}{\mathrm{m}} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

(d)

$$\mathbf{m}^b \triangleq (l, \mathrm{m}, n)^{\mathrm{T}}$$

$$\dot{\omega} = \mathbf{J}^{-1}[-\omega \times (\mathbf{J}\omega) + \mathrm{m}^b]$$

$$\begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \begin{pmatrix} \Gamma_1 \omega_x \omega_y - \Gamma_2 \omega_y \omega_z \\ \Gamma_5 \omega_x \omega_z - \Gamma_6 (\omega_x^2 - \omega_z^2) \\ \Gamma_7 \omega_x \omega_y - \Gamma_1 \omega_y \omega_z \end{pmatrix} + \begin{pmatrix} \Gamma_3 l + \Gamma_4 n \\ \frac{1}{J_y} \mathrm{m} \\ \Gamma_4 l + \Gamma_8 n \end{pmatrix}$$

$$\Gamma \triangleq J_x J_z - J_{xz}^2$$

where

$$\begin{aligned}
\Gamma_1 &= \frac{J_{xz}\left(J_x - J_y + J_z\right)}{\Gamma} \\
\Gamma_2 &= \frac{J_z\left(J_z - J_y\right) + J_{xz}^2}{\Gamma} \\
\Gamma_3 &= \frac{J_z}{\Gamma} \\
\Gamma_4 &= \frac{J_{xz}}{\Gamma} \\
\Gamma_5 &= \frac{J_z - J_x}{J_y} \\
\Gamma_6 &= \frac{J_{xz}}{J_y} \\
\Gamma_7 &= \frac{\left(J_x - J_y\right)J_x + J_{xz}^2}{\Gamma} \\
\Gamma_8 &= \frac{J_x}{\Gamma}.
\end{aligned}$$

Force and Moment The total force and moment can be expressed as:

$$\begin{aligned}
\mathbf{f} &= \mathbf{f}_g + \mathbf{f}_a + \mathbf{f}_p \\
\mathbf{m} &= \mathbf{m}_a + \mathbf{m}_p
\end{aligned}$$

(a) Gravitational Forces

$$\mathbf{f}_g^b = R_g^b \begin{pmatrix} 0 \\ 0 \\ \mathrm{m}g \end{pmatrix} = \begin{pmatrix} -\mathrm{m}g\sin\theta \\ \mathrm{m}g\cos\theta\sin\phi \\ \mathrm{m}g\cos\theta\cos\phi \end{pmatrix}$$

(b) Aerodynamics Forces

The lift, drag, and moment are commonly expressed as:

$$\begin{aligned}
F_{\text{lift}} &= \frac{1}{2}\rho V_a^2 S C_L \\
F_{\text{drag}} &= -\frac{1}{2}\rho V_a^2 S C_D
\end{aligned}$$

To model forces on the aerodynamic surfaces, we use

$$\begin{aligned}
\mathbf{f}_{Ls_i} &= f_{L,s_i}\mathbf{e}_{z_s} = \frac{1}{2}C_L\rho\left|\mathbf{v}_{s_i}\right|^2 S_i\mathbf{e}_{z_s} \\
\mathbf{f}_{Ds_i} &= f_{D,s_i}\mathbf{e}_{x_s} = \frac{1}{2}C_D\rho\left|\mathbf{v}_{s_i}\right|^2 S_i\mathbf{e}_{x_s}
\end{aligned}$$

where the Cl and Cd:

$$C_L = C_{L0} + C_{L\alpha}\alpha + C_{L\delta}\delta$$
$$C_D = C_{D0} + C_{D\alpha}\alpha + C_{D\delta}\delta$$

(c) Propulsion Forces

Maybe we can get the mapping relationship between $\mathbf{f}_p$ and pwm δ_t by using tautness meter

$$\mathbf{f}_p = \mathbf{f}_p(\delta_t)$$

(d) Aerodynamics Moments

$$\mathbf{m}_a = \sum_i \left(\mathbf{l}_{s_i} \times \mathbf{R}_{s_i}^b \mathbf{f}_{s_i}\right)$$
$$\mathbf{l}_{s_i} = \mathbf{r}_{h_i} + \mathbf{R}_{s_i}^b \mathbf{r}_{s_i}$$

(e) Propulsion Forces

$$\mathbf{m}_p = \sum_i \left(\mathbf{l}_{p_i} \times \mathbf{R}_{p_i}^b \mathbf{f}_{p_i}\right)$$

The total forces above:

$$\mathbf{f} = \sum_i \mathbf{R}_{s_i}^b \mathbf{f}_{s_i} + \mathbf{R}_b^{g\,T} g \mathbf{e}_{z_r} + \sum_i \mathbf{R}_t^b \mathbf{f}_t$$