EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet **Discussion 3A**

1. Complex Algebra

(a) Express the following values in polar forms: -1, j, $-j$, $\sqrt{j}$, and $\sqrt{-j}$. Recall $j = \sqrt{-1}$.

Answer:

The answers are also found in the table:

$$
\begin{aligned}
-1 &= j^2 = e^{j\pi} = e^{-j\pi} \\
j &= e^{j\frac{\pi}{2}} = \sqrt{-1} \\
-j &= -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \\
\sqrt{j} &= (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}} \\
\sqrt{-j} &= (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}}
\end{aligned}
$$

(b) Represent $\sin\theta$ and $\cos\theta$ using complex exponentials. (*Hint:* Use Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$.)

Answer:

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

(c) For complex number $z = x + jy$ show that $|z| = \sqrt{z\bar{z}}$, where $\bar{z}$ is the complex conjugate of z.

Answer:

We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\bar{z}} = \sqrt{(x+jy)(x-jy)} = \sqrt{x^2+y^2} = |z|$$

For the next two parts, consider two complex numbers $A = 1 - j\sqrt{3}$ and $B = \sqrt{3} + j$.

(d) Express A and B in polar form.

Answer:

We can draw out A and B in the complex plane. Then,

$$|A| = 2, \ \ |B| = 2, \ \ \theta_A = -\frac{\pi}{3}, \ \ \theta_B = \frac{\pi}{6}.$$

Hence,

$$A = 2e^{-j\frac{\pi}{3}} \quad B = 2e^{j\frac{\pi}{6}}.$$

(e) Find AB, $A\bar{B}$, $\frac{A}{B}$, and $\sqrt{B}$.

Answer:

$$AB = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$
$$A\bar{B} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$
$$\frac{A}{B} = e^{-j\frac{\pi}{2}} = -j$$
$$\sqrt{B} = \sqrt{2} \cdot e^{j\frac{\pi}{12}}$$

2. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations

$$\frac{d}{dt}x_1(t) = \lambda x_1(t) \tag{1}$$
$$\frac{d}{dt}x_2(t) = \overline{\lambda} x_2(t) \tag{2}$$

with initial conditions $x_1(0) = c_0$ and $x_2(0) = \overline{c}_0$, where λ and c_0 are complex numbers and $\overline{\lambda}$ and $\overline{c}_0$ are their complex conjugates, respectively.

Suppose now that we have the following different variables related to the original ones:

$$y_1(t) = ax_1(t) + \overline{a}x_2(t) \tag{3}$$
$$y_2(t) = bx_1(t) + \overline{b}x_2(t) \tag{4}$$

where a and b are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written:

$$a = a_r + ja_i,$$
$$\overline{a} = a_r - ja_i,$$
$$b = b_r + jb_i,$$
$$\overline{b} = b_r - jb_i,$$

where a_r, a_i, b_r, b_i are all real numbers.

(a) First, assume that $\lambda = j = \sqrt{-1}$ in the equations for $x_1(t)$ and $x_2(t)$ above. **Solve $x_1(t)$ and $x_2(t)$.**

Answer:

$$x_1(t) = x_1(0)e^{\lambda t} = c_0 e^{jt}$$
$$x_2(t) = x_2(0)e^{\lambda t} = \bar{c}_0 e^{-jt}$$

(b) **How do the initial conditions for $\vec{x}(t)$ translate into the initial conditions for $\vec{y}(t)$?**

Answer:

$$y_1(0) = ac_0 + \overline{ac}_0$$
$$y_2(0) = bc_0 + \overline{b}\overline{c}_0$$

These numbers are purely real, as you can see if you expand them or by directly noticing that these are the sum of a complex number with its conjugate.

(c) **Write out a system of differential equations using $\frac{d}{dt}y_i(t)$ and $y_i(t)$.**

Answer:

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}\vec{x}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$V = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix}$$

$$V^{-1} = \frac{1}{a\overline{b} - \overline{a}b}\begin{bmatrix} \overline{b} & -\overline{a} \\ -b & a \end{bmatrix}$$

Expanding this out into the $a = a_r + ja_i$ form gives:

Answer:

$$a\overline{b} - \overline{a}b = -j(2a_rb_i - 2a_ib_r).$$

The coefficient coming from the determinant part of the inverse in front of the matrix in V^{-1} is purely imaginary — this can be seen directly from the fact that it is a complex number minus its conjugate. (The real parts cancel away.)

$$\begin{tikzcd}
x_i \text{ coordinates} & \bigcirc \arrow[rr, "{A = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}}", "\text{differentiation}"'] & & \bigcirc \arrow[d, "V"] \\
y_i \text{ coordinates} & \bigcirc \arrow[u, "V^{-1}"] \arrow[rr, dashed, "\breve{A} = VAV^{-1}"] & & \bigcirc
\end{tikzcd}$$

So:

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = VAV^{-1}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\begin{aligned} VAV^{-1} &= \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix}\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}\frac{1}{a\overline{b} - \overline{a}b}\begin{bmatrix} \overline{b} & -\overline{a} \\ -b & a \end{bmatrix} \\ &= \frac{1}{a\overline{b} - \overline{a}b}\begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix}\begin{bmatrix} +j\overline{b} & -j\overline{a} \\ +jb & -ja \end{bmatrix} \\ &= \frac{1}{a\overline{b} - \overline{a}b}\begin{bmatrix} +j(a\overline{b} + \overline{a}b) & -j2a\overline{a} \\ j2b\overline{b} & -j(a\overline{b} + \overline{a}b) \end{bmatrix} \end{aligned} \tag{5}$$

Expanding this out into the $a = a_r + ja_i$ form gives:

$$\begin{aligned} a\bar{b}+\bar{a}b &= (a_r+ja_i)(b_r-jb_i)+(a_r-ja_i)(b_r+jb_i) \\ &= (a_rb_r+a_ib_i-jb_ia_r+ja_ib_r)+(a_rb_r+a_ib_i-ja_ib_r+ja_rb_i) \\ &= 2(a_rb_r+a_ib_i) \end{aligned} \tag{6}$$

$$\begin{aligned} a\bar{a} &= (a_r+ja_i)(a_r-ja_i) \\ &= (a_r^2-ja_ra_i+ja_ra_i+a_i^2) \end{aligned} \tag{7}$$

$$\begin{aligned} a\bar{b}+\bar{a}b &= 2(a_rb_r+a_ib_i) \\ a\bar{a} &= a_r^2+a_i^2 = |a|^2 \\ b\bar{b} &= b_r^2+b_i^2 = |b|^2 \end{aligned} \tag{8}$$

All of these numbers are purely real since they are clearly either magnitude-squareds or the sum of a complex number with its conjugate.

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \frac{1}{-j(2a_rb_i-2a_ib_r)}\begin{bmatrix} j(a\bar{b}+\bar{a}b) & -j2a\bar{a} \\ j2b\bar{b} & -j(a\bar{b}+\bar{a}b) \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \frac{1}{(2a_rb_i-2a_ib_r)}\begin{bmatrix} 2(a_rb_r+a_ib_i) & -2|a|^2 \\ 2|b|^2 & -2(a_rb_r+a_ib_i) \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \end{aligned} \tag{9}$$

We notice that our differential equation matrix $\breve{A}$ is purely real.

(d) Suppose we know $x_1(t)$ and $x_2(t)$ are complex conjugates of each other. **What will this say about $y_1(t)$ and $y_2(t)$?**

Answer: Note that

$$y_1(t) = ax_1(t)+\bar{a}x_2(t) = ax_1(t)+\bar{a}\bar{x}_1(t).$$

Hence, $y_1(t)$ is purely real. Similarly,

$$y_2(t) = bx_1(t)+\bar{b}x_2(t) = bx_1(t)+\bar{b}\bar{x}_1(t),$$

and we see that $y_2(t)$ is purely real.

(e) **Find the eigenvalues λ_1, λ_2 and associated eigenspaces for the differential equation matrix for $\vec{y}(t)$ above.**

Answer: The eigenvalues of

$$\begin{bmatrix} 2(a_rb_r+a_ib_i) & -2|a|^2 \\ 2|b|^2 & -2(a_rb_r+a_ib_i) \end{bmatrix}$$

are given by:

$$\begin{aligned} (2(a_rb_r+a_ib_i)-\lambda)(-2(a_rb_r+a_ib_i)-\lambda)+4|a|^2|b|^2 &= 0 \\ (\lambda^2-4(a_rb_r+a_ib_i)^2)+4(a_r^2+a_i^2)(b_r^2+b_i^2) &= 0 \end{aligned}$$

$$(\lambda^2 - 4(a_r^2 b_r^2 + 2a_r b_r a_i b_i + a_i^2 b_i^2)) + 4(a_r^2 b_r^2 + a_r^2 b_i^2 + a_i^2 b_r^2 + a_i^2 b_i^2) = 0$$
$$(\lambda^2 + 4(a_r^2 b_i^2 - 2a_r b_r a_i b_i + a_i^2 b_r^2) = 0$$
$$\lambda^2 + 4(a_r b_i - a_i b_r)^2 = 0$$
$$\lambda = \pm j \cdot 2(a_r b_i - a_i b_r)$$

The eigenvalues of

$$\breve{A} = \frac{1}{(2a_r b_i - 2a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) \end{bmatrix}$$

are therefore given by:

$$\lambda = \pm j \cdot 2(a_r b_i - a_i b_r) \cdot \frac{1}{(2a_r b_i - 2a_i b_r)} = \pm j$$

(f) **Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t),\ z_{\lambda_2}(t)$.** (These variables should be in eigenbasis-aligned coordinates.)

Answer:

$$\begin{bmatrix} \frac{d}{dt} z_{\lambda_1}(t) \\ \frac{d}{dt} z_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} z_{\lambda_1}(t) \\ z_{\lambda_2}(t) \end{bmatrix}$$

(g) **Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.**

Answer:

$$z_{\lambda_1}(t) = K_1 e^{jt}$$
$$z_{\lambda_2}(t) = K_2 e^{-jt}$$
$$z_{\lambda_1}(0) = K_1 e^{jt} = c_0$$
$$z_{\lambda_2}(0) = K_2 e^{-jt} = \bar{c}_0$$

$$z_{\lambda_1}(t) = c_0 e^{jt}$$
$$z_{\lambda_2}(t) = \bar{c}_0 e^{-jt}$$

(h) **Convert your solution back into the $\vec{y}(t)$ coordinates to find $\vec{y}(t)$.**

Answer:

The eigenspace associated with $\lambda_1 = j$ is given by:

$$\frac{1}{(2a_r b_i - 2a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) - j & -2|a|^2 \\ 2|b|^2 & -2(a_r b_r + a_i b_i) - j \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} a \\ b \end{bmatrix}$$

The eigenspace associated with $\lambda_2 = -j$ is given by:

$$\frac{1}{(2a_rb_i - 2a_ib_r)}\begin{bmatrix} 2(a_rb_r + a_ib_i) + j & -2|a|^2 \\ 2|b|^2 & -2(a_rb_r + a_ib_i) + j \end{bmatrix}\vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_{\lambda_2} = \beta\begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix}$$

$$\begin{bmatrix} \breve{x}_1(t) \\ \breve{x}_2(t) \end{bmatrix} = V\begin{bmatrix} z_{\lambda_1}(t) \\ z_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix}\begin{bmatrix} z_{\lambda_1}(t) \\ z_{\lambda_2}(t) \end{bmatrix}$$

$$\breve{x}_1(t) = ac_0e^{jt} + \overline{ac}_0e^{-jt}$$

$$\breve{x}_2(t) = bc_0e^{jt} + \overline{b}\overline{c}_0e^{-jt}$$

Notice that this is in classic phasor form — something plus its complex conjugate.

(i) **Repeat the above for general complex λ.**

Answer:

In General
λ is complex

$$\frac{1}{a\bar{b} - \bar{a}b}\begin{bmatrix} a & \bar{a} \\ b & \bar{b} \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}\begin{bmatrix} \bar{b} & -\bar{a} \\ -b & a \end{bmatrix}$$

$$\frac{1}{a\bar{b} - \bar{a}b}\begin{bmatrix} a & \bar{a} \\ b & \bar{b} \end{bmatrix}\begin{bmatrix} \lambda\bar{b} & -\lambda\bar{a} \\ -b\bar{\lambda} & +a\bar{\lambda} \end{bmatrix}$$

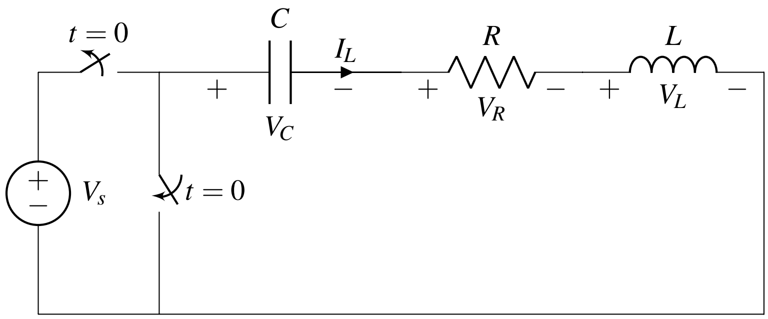
$$= \frac{1}{a\bar{b} - \bar{a}b}\begin{bmatrix} \lambda a\bar{b} - \bar{\lambda}\bar{a}b & -\lambda|a|^2 + \bar{\lambda}|a|^2 \\ \lambda|b|^2 - \bar{\lambda}|b|^2 & -\lambda\bar{a}b + \bar{\lambda}a\bar{b} \end{bmatrix}$$

$$= \frac{1}{\underbrace{a\bar{b} - \bar{a}b}_{\text{pure ims}}}\begin{bmatrix} \overbrace{\lambda ab - \bar{\lambda}\bar{a}b}^{\text{pure ims}} & +|a|^2\overbrace{(\bar{\lambda} - \lambda)}^{\text{pure ims}} \\ \underbrace{|b|^2(\lambda - \bar{\lambda})}_{\text{pure ims}} & \underbrace{-\lambda\bar{a}b + \bar{\lambda}a\bar{b}}_{\text{pure ims}} \end{bmatrix}$$

Get a real matrix. Has eigenvalues $\lambda, \bar{\lambda}$.
with eigenvectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}$ ...

3. RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

(a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s, L, R, and C.**

Answer: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{d}{dt}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{d}{dt}V_C(t) = I_L(t) \tag{10}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t). \tag{11}$$

If we use the state variable names, we can write this as

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t), \tag{12}$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have

$$V_C(t) + V_R(t) + V_L(t) = 0. \tag{13}$$

Using Ohm's Law and the inductor equation $V_L = L\frac{d}{dt}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{d}{dt}I_L(t) = 0, \tag{14}$$

which we can rewrite as

$$\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t). \tag{15}$$

If we use the state variable names, this becomes

$$\frac{d}{dt}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t), \tag{16}$$

and we have a second differential equation.

To summarize the final system is

$$\frac{d}{dt}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{17}$$

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t). \tag{18}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.**

This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Answer: By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{19}$$

which is in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{20}$$

(c) **Find the eigenvalues of the A matrix symbolically. (Hint: the quadratic formula will be involved.)**

Answer: To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{21}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{22}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{23}$$

The Quadratic Formula gives

$$\lambda = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{(\frac{R}{L})^2 - \frac{4}{LC}}. \tag{24}$$

(d) **Under what condition on the circuit parameters R, L, C are there going to be a pair of distinct real eigenvalues of A?**

Answer: For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{25}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{26}$$

(e) **Under what condition on the circuit parameters R, L, C are there going to be a pair of purely imaginary eigenvalues of A?**

Answer: The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm j\sqrt{\frac{1}{LC}}. \tag{27}$$

(f) **Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, find eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ corresponding to them.**

(HINT: Rather than trying to find the relevant nullspaces, etc., you might just want to try to find eigenvectors of the form $\begin{bmatrix} 1 \\ ? \end{bmatrix}$ *where we just want to find the missing entry. Can you see from the structure of the A matrix why we might want to try that guess?)*

Answer:

The easy way is just to remember what an eigenvector is. We want $A\vec{v}_{\lambda_i} = \lambda_1 \vec{v}_{\lambda_i}$. So, we can try to follow the hint:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ ? \end{bmatrix} = \begin{bmatrix} * \\ \frac{1}{C} \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i ? \end{bmatrix} \tag{28}$$

We want the second entry to be $?\lambda_i$, but instead we have a $\frac{1}{C}$, and so the $? = \frac{1}{C\lambda_i}$. This immediately gives us

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}$$

Since these by construction obey $A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i}$ in the second position and we know that we have the flexibility to scale eigenvectors, they must be eigenvectors. i.e. We always have the flexibility to scale a single nonzero position to be 1. Putting a zero in the first position does not result in an eigenvector by inspection of the first row since we would get $-\frac{1}{L}?$ unless $? = 0$ and $\vec{0}$ is never a valid eigenvector. (This is how being able to argue rigorously lets you avoid some algebra.)

Of course, you can grind out the minor algebra to just verify from the expressions for the eigenvalues in (24) that they indeed do satisfy $A\vec{v}_{\lambda_i} = \lambda_1 \vec{v}_{\lambda_i}$ in the first position as well since $-\frac{R}{L} - \frac{1}{\lambda_i CL}$ indeed equals λ_i for the solutions given in (24).

Alternatively, you can try to use the standard approach of finding the nullspace of $A - \lambda_i I$. Since the matrix A has a zero in the second row, and we know

$$\begin{bmatrix} -\frac{R}{L} - \lambda_1 & -\frac{1}{L} \\ \frac{1}{C} & -\lambda_1 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{R}{L} - \lambda_2 & -\frac{1}{L} \\ \frac{1}{C} & -\lambda_2 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then it is clear that the nullspace can be found as anything along the following two vectors:

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}$$

These are clearly linearly independent since $\lambda_1 \neq \lambda_2$.
Either way works.

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = [\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}]$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **What is the $\widetilde{A}$ so that $\frac{d}{dt}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$?** It is fine to have your answer expressed symbolically using λ_1, λ_2.

Answer: V is given by:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

We know that V transforms from the $\tilde{x}$ coordinate frame to the x coordinate frame, V^{-1} transforms back, and A takes gives the relationship from x to $\frac{d}{dt}x$.
Therefore to go from $\widetilde{x}$ to $\frac{d}{dt}\widetilde{x}$:

$$\widetilde{A} = V^{-1}AV = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

$$\widetilde{A} = V^{-1}AV = \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2 C} & -1 \\ -\frac{1}{\lambda_1 C} & 1 \end{bmatrix} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

$$\widetilde{A} = V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

You didn't have to multiply things out explicitly. You could have just noticed that the A matrix times the V matrix would give columns that were λ_i times $\vec{v}_i$ each, and then multiplying that by V^{-1} would just pick out the λ_i on the diagonals and zeros on the off-diagonals since $V^{-1}V = I$.

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