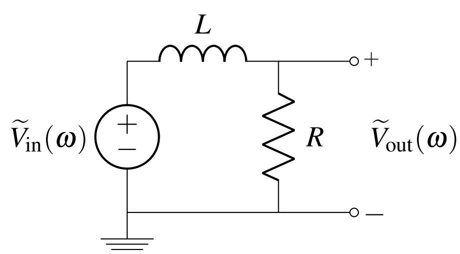
EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet Discussion 4B

1. Transfer function warmup

Now that you've seen how to derive a transfer function for a simple circuit, you'll be deriving some transfer functions on your own. To start off with, determine $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ for the following circuits, and use the transfer function you find to describe how the circuit responds to low and high frequencies.

(a) Determine $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$. How does this circuit respond as $\omega \to 0$ (low frequencies)? as $\omega \to \infty$ (high frequencies)?



Answer: The strategy is to use the voltage divider formula, *i.e.*,

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}}.$$

$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\frac{L}{R}}. \tag{1}$$

At low frequencies, we have

$$\lim_{\omega \to 0} H(\omega) = 1, \tag{2}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} H(\omega) = 0. \tag{3}$$

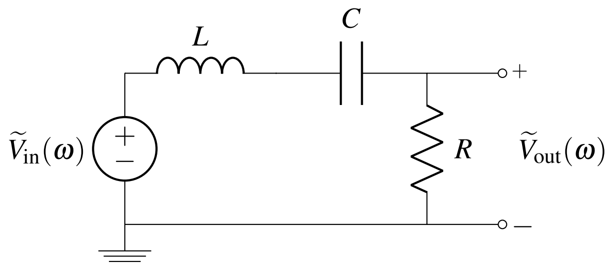
So this circuit is a *low-pass filter*.

(b) What happens if you replace the inductor above with a capacitor?

Answer: FILL THIS IN WITH CAP high-pass info

2. Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$, where $X(\omega)$ is a real valued function of ω.**

Answer: Since the capacitor, resistor and inductor are in series, the equivalent impedance is given by,

$$Z_{RLC}(\omega) = R + Z_L(\omega) + Z_C(\omega)$$
$$\implies Z_{RLC}(\omega) = R + j\omega L + \frac{1}{j\omega C}$$

Since,

$$\frac{1}{j} = -j$$
$$Z_{RLC}(\omega) = R + j(\omega L - \frac{1}{\omega C})$$

Hence,

$$A(\omega) = R$$

and

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

(b) **Write down the transfer function $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ for this circuit.**

Answer: We could solve this problem using the same voltage divider rule we've used in the past. But we've already calculated the impedance of the RLC circuit, so let's use that:

$$V_{out} = I_{in}R$$
$$I_{in} = \frac{V_{in}}{Z_{RLC}}$$
$$V_{out} = \frac{V_{in}R}{Z_{RLC}}$$
$$H(\omega) = \frac{R}{Z_{RLC}(\omega)}$$
$$H(\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})}$$

(c) **At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other.)

What happens to the relative magnitude of the impedances of the capacitor and inductor as ω moves above and below ω_n? What is the value of the transfer function at this frequency?

Answer:

$$X(\omega_n) = 0 = \omega_n L - \frac{1}{\omega_n C}$$

Multiplying both sides by ω_n:

$$0 = \omega_n^2 L - \frac{1}{C}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

As the frequency gets higher above ω_n the impedance of the inductor gets higher while the capacitor gets lower. Since the two components are in series, the impedance of the inductor will dominate. As the frequency gets below ω_n the impedance of the capacitor gets higher while the inductor gets lower. In this case the impedance of the capacitor will dominate.

At this frequency, $Z_{RLC} = R$, since the imaginary components cancel out perfectly. As a result

$$H(\omega_n) = 1$$

(d) In most filters, we are interested in the cutoff frequency, since that helps define the frequency range over which the filter operates. Remember that this is the frequency at which the magnitude of the transfer function drops by a factor of $\sqrt{2}$ from its maximum value. Notice that the real part of the impedance Z_{RLC} is not changing with frequency and stays at R. What we care about is the frequencies where the imaginary part of the impedance equals $-jR$ to $+jR$.

To do this, we want to see what happens in the neighborhood of ω_n and so express the combined effect of the capacitor and inductor in terms of $\omega = \omega_n + \delta$, where δ is (presumably) a relatively small number compared to ω_n.

Write an expression for $X(\omega_n + \delta)$, where δ is a variable shift from ω_n. Find the values of δ which give $X(\omega_n + \delta_1) = -R$, and $X(\omega_n + \delta_2) = +R$.. You may use the approximation that $\frac{1}{1+x} \approx 1 - x$ if $x \ll 1$.

Answer:

FIX THIS NOTATION

From the first part we know that,

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

So we get,

$$X(\omega_n + \delta\omega) = (\omega_n + \delta\omega)L - \frac{1}{\omega_n + \delta\omega)C}$$

$$\implies X(\omega_n + \delta\omega) = \omega_n L + \delta\omega L - \frac{1}{\omega_n C(1 + \frac{\delta\omega}{\omega_n})}$$

Using, $\frac{1}{1+x} \approx 1 - x$ for $x << 1$ we get,

$$X(\omega_n + \delta\omega) \approx \omega_n L + \delta\omega L - \frac{1}{\omega_n C}(1 - \frac{\delta\omega}{\omega_n})$$

Using, $\omega_n L = \frac{1}{\omega_n C}$ we get,

$$X(\omega_n + \delta\omega) \approx \delta\omega L + \frac{\delta\omega}{\omega_n^2 C}$$

Using $\frac{1}{\omega_n^2} = LC$ we get,

$$X(\omega_n + \delta\omega) \approx 2\delta\omega L$$

Hence, if $X(\omega_n + \delta_2\omega) = R$, then we get,

$$\delta_2\omega = \frac{R}{2L}$$

Similarly, when $X(\omega_n + \delta_1\omega) = -R$, we get,

$$\delta_1\omega = \frac{-R}{2L}$$

(e) **Simplify $X(\omega)$ in two cases, when $\omega \to \infty$ and when $\omega \to 0$. Plug this simplified $X(\omega)$ into your previously solved expressions to find the transfer function at high and low frequencies.**

Answer: At low frequencies,

$$X(\omega) \approx -\frac{1}{\omega C}$$

. Plugging this back in to the original transfer function we get a CR high-pass filter.

$$H(\omega) \approx \frac{R}{R - j\frac{1}{\omega C}}$$

At high frequencies,

$$X(\omega) \approx \omega L$$

. Plugging this back in to the original transfer function we get a LR low-pass filter.

$$H(\omega) \approx \frac{R}{R + j\omega L}$$

(f) **Draw a bode plot for a CR high pass filter, a LR low pass filter, and this LCR band-pass filter. Assume the $L = 9nH$, $R = .18\Omega$, and $C = 18.7pF$ values are the same in all three filters. Compare the locations of the cutoff frequencies, and the behavior of the filters at very high and low frequencies.**

Answer: As shown in the previous question, our band-pass filter looks like an LR low-pass filter at high frequencies and a CR high-pass filter at low frequencies. Note that in this case, the cutoff frequencies for the LR and CR filters are not the same as LCR cutoff frequencies or the resonance frequency.

FIX THIS. ADD THE PLOT!!!!!

Contributors:

- Alex Devonport.
- Nathan Lambert.
- Pavan Bhargava.
- Sanjit Batra.
- Kris Pister.
- Anant Sahai.