Hint for Problem 3 on Problem Set 8. People seem to struggle with this problem, so I decided to provide some hints. The point of this problem is for you to practice using properties of *determinants* and *adjugates*.

By the way, for this problem, feel free to assume that $\det(A)$ and $\det(B)$ are nonzero. I apologize for not writing this clearly.

The problem says that A and B are two square matrices of the same dimension:

$$A, B \in \mathbb{R}^{n\times n}.$$

However, one tricky thing about this problem is that n is not even given to you!

Instead, you are given three complicated equations:

$$\det(\mathrm{adj}(A\cdot B)) = 4^{18}, \tag{1}$$

$$\det(\mathrm{adj}(\mathrm{adj}(A))\cdot B^2) = 4^{47}, \tag{2}$$

$$\det(\mathrm{adj}(\mathrm{adj}(B)) = 4^{9}. \tag{3}$$

But how do we use them?

Recall the following properties we learned about determinants and adjugates:

Property 1. For any $X \in \mathbb{R}^{n\times n}$, we have $\mathrm{adj}(X)\cdot X = \det(X)\cdot I_n$.

Property 2. For any $X \in \mathbb{R}^{n\times n}$ and for any $\lambda \in \mathbb{R}$, we have $\det(\lambda \cdot X) = \lambda^n \cdot \det(X)$.

Property 3. For any square matrices X and Y, we have $\det(X\cdot Y) = \det(X)\cdot\det(Y)$.

Now, can we use these three properties to unwrap the three equations? In particular, can we write the three equations in terms of $\det(A)$, $\det(B)$, and n, so we can solve for their values?

Here is an example. If we use Property 1 with $X = B$, we find

$$\mathrm{adj}(B)\cdot B = \det(B)\cdot I_n.$$

Now let us take the determinant on both sides. We find

$$\det(\mathrm{adj}(B)\cdot B) = \det(\det(B)\cdot I_n).$$

By Property 3, the left-hand side is

$$\det(\mathrm{adj}(B)\cdot B) = \det(\mathrm{adj}(B))\cdot\det(B).$$

By Property 2, the right-hand side is

$$\det(\det(B)\cdot I_n) = \det(B)^n\cdot\det(I_n) = \det(B)^n.$$

Hence, we obtain

$$\det(\mathrm{adj}(B))\cdot\det(B) = \det(B)^n.$$

Assuming $\det(B) \neq 0$ so we can divide: we find

$$\det(\operatorname{adj}(B)) = \det(B)^{n-1}. \tag{4}$$

Now let us use Property 1 with $X = \operatorname{adj}(B)$ instead. We find

$$\operatorname{adj}(\operatorname{adj}(B)) \cdot \operatorname{adj}(B) = \det(\operatorname{adj}(B)) \cdot I_n.$$

Once again, by taking the determinant on both sides and using Properties 2 and 3, we find

$$\det(\operatorname{adj}(\operatorname{adj}(B)) = \det(\operatorname{adj}(B))^{n-1}.$$

Combining this with Equation (4), we obtain

$$\det(\operatorname{adj}(\operatorname{adj}(B)) = \det(B)^{(n-1)^2}.$$

Hence, Equation (3) tells us that

$$\det(B)^{(n-1)^2} = 4^9.$$

Can you do something similar with Equations (1) and (2)? Remember the goal is to solve for $\det(A)$, $\det(B)$, and n.