The simplest unsolved computational geometry problem: efficiently folding a polyhedron

Linus Hamilton* Yevhenii Diomidov

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Abstract

What is the smallest square of paper that can wrap around the surface of a given polyhedron? We develop a method to find upper bounds for this problem, and use it to find new record foldings for the tetrahedron, octahedron, dodecahedron, and icosahedron. We present evidence that the tetrahedron folding is optimal.

We released the code of our algorithm: `https://github.com/6849-2020/efficient-polyhedra-folds/blob/main/foldingPolyhedra.py`.

1 Introduction

Many computational geometry problems present themselves so simply that they ought to be solved by now. For example, Moser's worm problem [2] (what is the smallest shape that contains any length-1 curve?), or the moving sofa problem [3] (what is the smallest shape that fits through a 90° turn in a hallway?). It speaks to a fundamental gap in the field of computational geometry that we have not been able to answer these questions.

In this paper we study a similarly simple question: what is the smallest square of paper that wraps completely around the surface of a given polyhedron? We pay special attention to the simplest case, that of the tetrahedron, for which this may be a contender for the "easiest unsolved computational geometry problem." Unlike the sofa and worm conundrums, folding a square around a tetrahedron seems, at least intuitively, to have finitely many degrees of freedom. Yet, even though we present evidence that our new tetrahedron fold is optimal, the answer is still not known for sure.

In this paper, we present a method to search for efficient foldings of arbitrary polyhedra. We demonstrate the power of our method by improving the best known records for wrapping a square around a tetrahedron, octahedron, dodecahedron, and icosahedron. (The answer for a cube was already known to be optimal.) This is similar to the research in [1], which studies wrapping a

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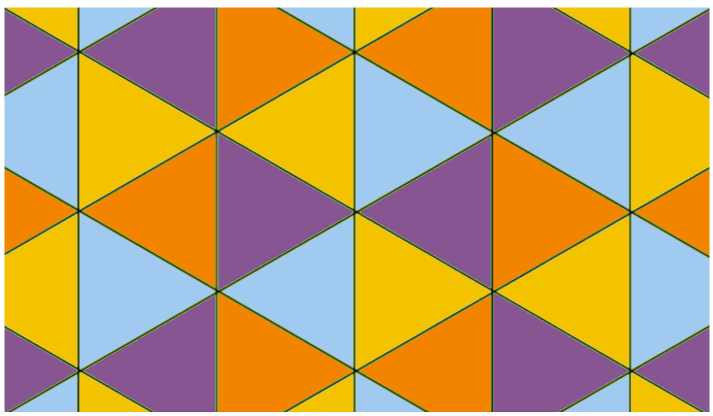
cube or a sphere using the smallest possible rectangle with a given length/width ratio. We also explore a technique, novel to this author, to obtain a restricted lower bound on tetrahedron wrapping. The technique may be useful on other computational geometry problems.

2 The tetrahedron

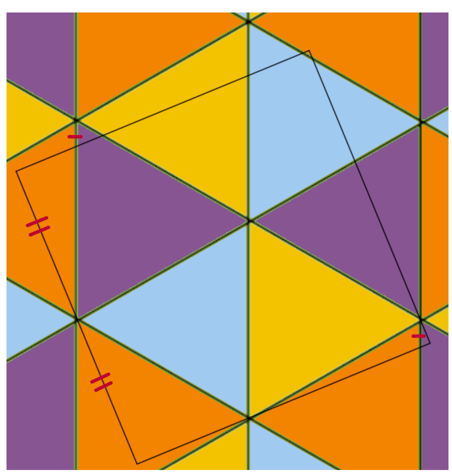
For the purpose of exposition, we will first demonstrate our method on the tetrahedron.

Consider this 4-colored tiling of unit equilateral triangles:



Surprisingly, if the paper is allowed to intersect itself, this tiling folds perfectly onto the surface of a unit tetrahedron, with each color corresponding to one face. This is because every vertex of a tetrahedron has total face degree $\pi = \tau/2$, so every vertex of the above tiling double-covers a tetrahedron vertex.

Now examine the following square. Two pairs of equal-length segments are marked.



The area inside this square covers every part of every color of triangle. (The coverings of the purple and orange faces are each split among three different copies of the triangle.) Therefore, by restricting the full tiling fold to just this square, we obtain a complete wrapping of this square around a tetrahedron. It can be checked manually that this restriction does *not* require the paper to intersect itself.

A brute-force computer search suggests that this is the smallest square with this property. Its side length (we state without proof) is

$$3\sqrt{\frac{7+2\sqrt{3}}{37}} \approx 1.5954065\ldots$$

I conjecture that this is the most efficient way to fold a tetrahedron from a square.

3 Other polyhedra

Unfortunately, the entire plane does not perfectly wrap around other polyhedra. Instead we introduce the notion of a *redundant net:*

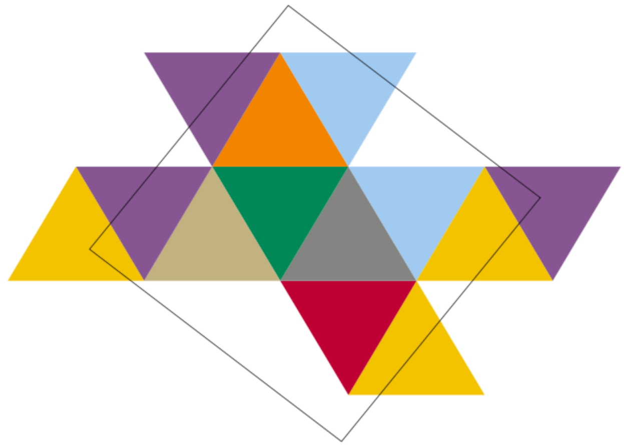
Definition 1. A *redundant net* of a polyhedron P is a partial tiling of the plane by labeled faces of P, such that a fold can map the labeled faces onto the corresponding faces of P, possibly with the paper intersecting itself.

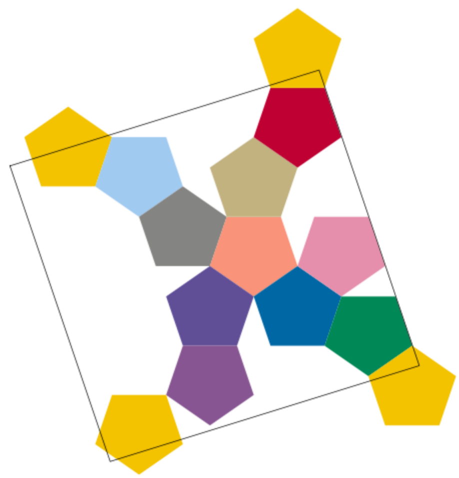
A polyhedron can have many redundant nets. This suggests a brute force method:

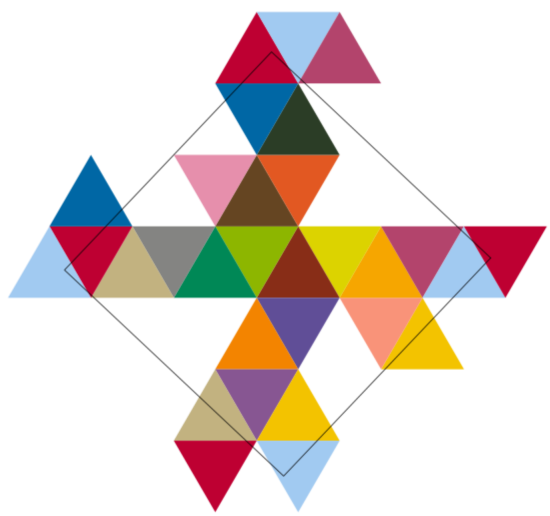
Algorithm: searching for efficient foldings of a polyhedron P
Repeat:
1. Generate a random redundant net of P. (Be careful – since folding always increases distances, some otherwise valid nets are impossible to actually fold. Our code checks that every pair of vertices is at least as far apart on the square as on the surface of the polyhedron. The diagrams below were then manually checked to be foldable, in case this isn't sufficient.)
2. Via random search followed by gradient descent, find a small square that covers every part of every face of P in the redundant net.

The random search in step 2 is fairly naive. In our implementation, first several dozen squares are randomly placed on the redundant net. Via binary search, we find for each square the smallest scaling of it that covers the polyhedron. Finally, we use the third-party module scipy to perform a final gradient descent. It is plausible that this search could be vastly sped up by a superior algorithm.

Here are the results of running this algorithm for several hours on the octahedron, dodecahedron, and icosahedron. Unlike for the tetrahedron, I do not conjecture that any of these are optimal. The dodecahedron fold in particular might even be locally improvable by "hinging" the bottommost yellow pentagon.







The side lengths of these squares are at most 2.35557, 6.00181, and 3.63663, respectively.

Python code at`https://github.com/6849-2020/efficient-polyhedra-folds/blob/main/foldingPolyhedra.py`allows you to run this algorithm on any polyhedron of your choice, as well try paper shapes other than square.

4 A conditional lower bound for the tetrahedron

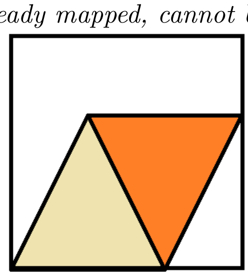
One classical lower bound on folding a tetrahedron is known:

Theorem 2. *One cannot fold a unit tetrahedron from a square of side length less than* $\sqrt{2}$.

Proof. No matter which point x on the tetrahedron the center of the square maps to, there is some point distance ≥ 1 away on the surface of the tetrahedron. Therefore, to cover this point, the square must have a point distance ≥ 1 from its center. It follows that the side length of the square is $\geq \sqrt{2}$. □

We have not improved this bound unconditionally. However, we can show a conditional bound for folding patterns containing what we call a "diamond" configuration. For the purpose of exposition, let us do an example:

Fact 3. *The partial folding shown below, with two adjacent tetrahedron faces already mapped, cannot be extended to cover the whole tetrahedron.*



Proof. (Informal) We have already chosen points on the square p_A, p_B, p_C, p_D mapping to the four vertices A, B, C, D of the tetrahedron. Folding cannot increase distances. So for any point T on the tetrahedron, it must be covered by a point p_T on the square satisfying $d(p_T, p_A) \geq d_{\text{tetrahedron}}(T, A)$ and analogously for B, C, D.

But (I claim here without proof) there exists a point x on the tetrahedron such that these bounds rule out the entire square. So this point x cannot be covered. □

We shall use this technique to prove the following restricted lower bound:

Theorem 4. *Suppose a folding of a tetrahedron from a square contains a "diamond" configuration as shown above. Then the side length of the square is more than 1.55.*

Using a proof of the form above, we can rule out a single possible placement of a diamond in the side-1.55 square. In fact, we can do a little more. If the proof has any slack in the inequalities, then we can rule out a small neighborhood of possible diamond locations. The following lemma makes this explicit.

Lemma 5. *Parameterize the space of possible diamond positions by x, y, θ, where (x, y) is the center of the diamond and θ is its angle. Say a diamond with center (x, y) and angle θ, having vertices p_A, p_B, p_C, p_D, is placed overlapping a square S.*

(*a*) *Suppose for some constant δ that some vertex of the diamond – one of p_A, p_B, p_C, p_D – lies outside the square S by distance more than $(\sqrt{2} + \frac{\sqrt{3}}{2})\delta$. Then that vertex still lies outside the square even if the parameters x, y, θ are first each adjusted by as much as δ.*

(*b*) *Suppose for some constant δ there exists a tetrahedron point T such that* no *point $p_T \in S$ satisfies all four of these bounds: $d(p_T, p_A) > d_{tetrahedron}(T, A) + (\sqrt{2} + \frac{\sqrt{3}}{2})\delta$ and analogously for B, C, D. Then the diamond folding cannot be extended to cover point T, even if the parameters x, y, θ of the diamond are first each adjusted by as much as δ.*

Proof. If the x, y, θ parameters of the diamond change by at most δ each, then every vertex of the diamond p_A, p_B, p_C, p_D moves by at most $\sqrt{2}\delta$ (from changing x and y), plus $\frac{\sqrt{3}}{2}\delta$ (from the rotation by θ applied to the vertices $\frac{\sqrt{3}}{2}$ from the center of the diamond). This proves (a). This also shows that moving the diamond weakens the inequalities $d(p_T, p_A) > d_{\text{tetrahedron}}(T, A) + (\sqrt{2} + \frac{\sqrt{3}}{2})\delta$ in (b) to merely $d(p_T, p_A) > d_{\text{tetrahedron}}(T, A)$, which is still enough to apply the proof of the previous fact. This proves (b). $\square$

To prove our main lower bound, we use the lemma to incrementally rule out all possible locations of the diamond. The configuration space for the x, y, θ parameters of the diamond is a box $[0, 1.55] \times [0, 1.55] \times [0, \tau/2]$. Now run the following algorithm:

Algorithm: ruling out a box $[x_{min}, x_{max}] \times [y_{min}, y_{max}] \times [\theta_{min}, \theta_{max}]$

1. Let x, y, θ be the center of this box.
2. Via randomly probing tetrahedron points T, search for a proof using the lemma showing that T cannot be covered whenever the diamond pattern lies in the box.
3. If no proof is found quickly, then cut the box along each axis to obtain 8 smaller boxes, and recursively run this algorithm on each smaller box.

If the algorithm halts on the original box $[0, 1.55] \times [0, 1.55] \times [0, \tau/2]$, then the theorem is proved. In short: it does. We omit the execution for brevity, as the program checks approximately ten thousand sub-boxes. See the github for code. It is plausible that running the code for longer could prove a stronger bound than 1.55. It would be interesting to take this technique all the way to the conjectured upper bound $1.5954065\ldots$.

5 Future Work

A simple next step: run my code to search for the smallest rectangle with a given side length ratio that covers a cube. Compare the results to those in [1].

Though the configuration-space-search proof in the previous section is unwieldy and requires computer time, it may open the door to lower bounds for other simple computational geometry problems.

Also, a natural question to ask: is the problem "What is the smallest square that folds to cover the polyhedron P" *computable*? It does not obviously fall to Tarski's proof that the theory of the reals is decidable.

References

[1] Alex Cole, Erik D Demaine, and Eli Fox-Epstein. On wrapping spheres and cubes with rectangular paper. In *Japanese Conference on Discrete and Computational Geometry and Graphs*, pages 31–43. Springer, 2013.

[2] Rick Norwood, George Poole, and Michael Laidacker. The worm problem of leo moser. *Discrete & Computational Geometry*, 7(2):153–162, 1992.

[3] Neal R Wagner. The sofa problem. *Amer. Math. Monthly*, 83:188–189, 1976.

6 Appendix: TODOs if this gets published

As an addendum, there are some rough spots in this paper to fix if we want to publish this result.

1. Finally now that I've updated my code to handle paper shapes other than a square, try to improve the wrapping-a-rectangle-around-a-cube bounds from [1]. Or try to wrap a triangle around a cube, etc.

2. Give a quick proof in an appendix that the tetrahedron folding achieves side length $3\sqrt{\frac{7+2\sqrt{3}}{37}}$.

3. Is there a procedural way to check that the final results for the octahedron/dodecahedron/icosahedron can be folded without stretching or overlapping paper? I manually checked them here, but I'm not sure how to turn that into a proof.

4. Convert the conditional 1.55 lower bound proof to some text file format so that it can be quickly computer checked for correctness. Also write the "check for correctness" code in a good style so that it's clear that if it outputs True, then the proof is correct.

5. Make the Platonic solid folds colorblind accessible, e.g. label each face with a number.