Counting Rationals Final Report (Lab 7)

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1 Introduction

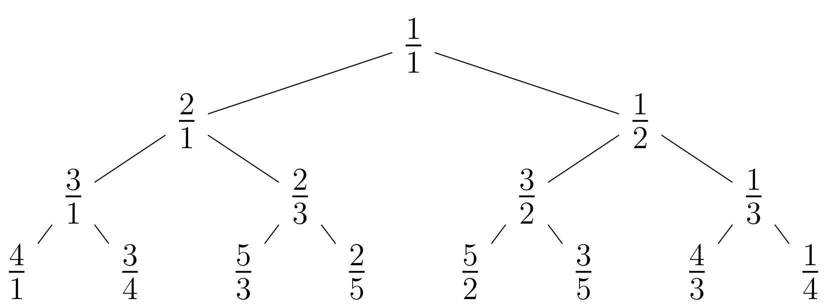
Let $f : \mathbb{N} \to \mathbb{Q}^+$ be defined by

$$f(1) = \frac{1}{1} \text{ and } f(n+1) = \frac{1}{[f(n)] + 1 - \{f(n)\}} (n \geq 1).$$

Where $[f(n)]$ is the integral part of $f(n)$ and $\{f(n)\}$ is the fractional part. We begin by computing the first sixteen values of f:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{1}$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\frac{3}{4}$	$\frac{4}{1}$	$\frac{1}{5}$

Now define T to be the infinite binary tree with entries in $\mathbb{Q}^+$ where the root is $\frac{1}{1}$ and for each node $\frac{p}{q}$, the left child is $\frac{p+q}{q}$ and the right child is $\frac{p}{q+p}$. The first four rows of T are shown below.



Moving through this tree from right to left ($\frac{1}{1} \to \frac{1}{2} \to \frac{2}{1} \to \frac{1}{3} \to \ldots$), we noticed that $f(n)$ appeared to be the nth entry in T.

2 Properties of T

2.1 T Contains All Positive Rationals

Proposition 2.1. *Any positive rational $\frac{p}{q} \in \mathbb{Q}^+$ is a node in T.*

Proof. Assume not. Let $\frac{p}{q}$ where $p, q \in \mathbb{N}$ and $\mathrm{GCD}(p, q) = 1$ be the counterexample whose sum of its numerator and denominator $(p + q)$ is the smallest possible.

If $p = q$, then $\frac{p}{q} = \frac{1}{1}$ which is in T.

If $p > q$, then $\frac{p-q}{q}$ is a positive rational with a smaller numerator-denominator sum, so $\frac{p-q}{q}$ is in the tree. However, the left child of $\frac{p-q}{q}$ is $\frac{(p-q)+q}{q} = \frac{p}{q}$, so we have a contradiction.

Similarly, if $p < q$, then $\frac{p}{q-p}$ is a positive rational in the tree, and $\frac{p}{q}$ is its right child. In all cases, $\frac{p}{q}$ turns out to be actually in the tree, so there exist no counterexamples. □

This proof uses the idea that we can get the parent node of any rational $\frac{p}{q}$. In fact, we can also determine whether $\frac{p}{q}$ is the left or right child of its parent. We turn this idea into an algorithm for calculating $f(n)$ in section 4.

3 Properties of f

3.1 f is Injective

We show that $f(1) = f(m) \iff m = 1$ and $f(n) = f(m) \iff f(n-1) = f(m-1)$. Together these imply that $f(n) = f(m) \iff n = m$, i.e. f is injective.

Lemma 3.1. *Let $n \in \mathbb{N}$. Then $f(n) = 1$ if and only in $n = 1$.*

Proof. The reverse direction of the implication follows from the definition of f. To prove the forward direction, we suppose by way of contradiction that $n \neq 1$. Then

$$f(n) = \frac{1}{[f(n-1)] + 1 - \{f(n-1)\}}$$

. Rearranging, we have

$$[f(n-1)] = \{f(n-1)\}$$

The RHS is a value in the range $[0, 1)$, and the LHS is an integer. Therefore both sides are equal to zero. However, this implies that $f(n-1) = 0$, which is not possible since 0 is not in the range of f. This is a contradiction, so our assumption is incorrect. $f(n) = 1 \implies n = 1$. Both directions of the lemma are proven. □

Proposition 3.2. *Let $m, n \in \mathbb{N}$. If $f(m) = f(n)$ then $m = n$.*

Proof. Let $f(m) = f(n)$ for $m, n \in \mathbb{N}$ and let $m \leq n$. Expanding our recursion,

$$f(m) = f(n)$$
$$\Rightarrow \frac{1}{[f(m-1)] + 1 - \{f(m-1)\}} = \frac{1}{[f(n-1)] + 1 - \{f(n-1)\}}$$
$$\Rightarrow [f(m-1)] + 1 - \{f(m-1\} = [f(n-1)] + 1 - \{f(n-1)\}$$
$$\Rightarrow [f(m-1)] - [f(n-1)] = \{f(m-1)\} - \{f(n-1)\}.$$

Since, by definition, $0 \leq \{f(m-1)\}, \{f(n-1)\} < 1$, we have $-1 < \{f(m-1)\} - \{f(n-1)\} < 1$, and hence $\{f(m-1)\} - \{f(n-1)\} = 0$. Thus,

$$\{f(m-1)\} = \{f(n-1)\} \text{ and } [f(m-1)] = [f(n-1)]$$
$$\Rightarrow f(m-1) = f(n-1).$$

Repeating this process, because $n \leq m$, $1 = f(1) = f(n-m+1)$. By Lemma 2.1, this implies that $n - m + 1 = 1$ and thus $n = m$. □

3.2 f Alternates Between < 1 and > 1

Additionally, consider the following lemma about the magnitude of the outputs of f. This lemma in instrumental in section 3.3

Lemma 3.3. *For all $n \in \mathbb{N}$,*

$$\begin{aligned} f(2n) &< 1 \\ f(2n+1) &> 1. \end{aligned}$$

Proof. We prove the lemma by induction. Note that $f(2(1)) = \frac{1}{2} < 1$ and $f(2(1)+1) = 2 > 1$ for the base case.

For the inductive step assume that $f(2n) < 1, f(2n+1) > 1$. So $\lfloor f(2n+1) \rfloor \geq 1$ and $\{f(2n+1)\} < 1$. Then,

$$\begin{aligned} f(2(n+1)) &= \frac{1}{\lfloor f(2n+1) \rfloor + 1 - \{f(2n+1)\}} \\ &\leq \frac{1}{2 - \{f(2n+1)\}} \\ &< \frac{1}{1} = 1. \end{aligned}$$

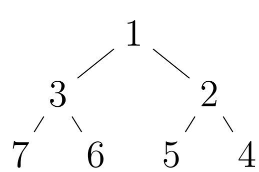
Similarly, as $\lfloor f(2n+2) \rfloor = 0$ and $0 < \{f(2n+2)\} < 1$ since $f(2n+2) \neq 0$,

$$\begin{aligned} f(2(n+1)+1) &= \frac{1}{\lfloor f(2n+2) \rfloor + 1 - \{f(2n+2)\}} \\ &= \frac{1}{1 - \{f(2n+1)\}} \\ &> \frac{1}{1} = 1. \end{aligned}$$

Thus the lemma is proven. $\square$

3.3 Relation Between f and Binary Tree of Rationals

A useful property of binary trees is that if we number the nodes of the tree with the natural numbers, going from top to bottom and right to left, then the right child of the nth node is the $(2n)$th node and the left child is the $(2n+1)$th node.



We conjectured that the function $f(n)$ enumerates a right-to-left breadth-first traversal of T. Because of the tree numbering above, this means we need to show that going from a node to its right child is the same as going from $f(n)$ to $f(2n)$, and that going from a node to its left child is the same as going from $f(n)$ to $f(2n+1)$.

An equivalent way of defining the binary tree of rationals T is as follows: Letting the value of a certain node be A, the left child is $1 + A$ and the right child is $\frac{A}{A+1}$. Substituting $f(n)$ into this alternate definition, we see that in order to prove our connection between T and f, we need to show that f satisfies the following recursive relationships.

Proposition 3.4. *For all $n \in \mathbb{N}$,*

$$f(2n+1) = f(n) + 1$$
$$f(2n) = \frac{f(n)}{f(n)+1}.$$

Proof. We proceed by induction. For the base case, note that $f(2(1)+1) = f(3) = 2$ and $f(1)+1 = 2$ thus $f(2(1)+1) = f(1)+1$. Similarly $f(2) = \frac{1}{2} = \frac{1}{1+1} = \frac{f(1)}{f(1)+1}$.

For the inductive step, assume that $f(2n+1) = f(n)+1$. Using the Lemma 2.3, note that $\lfloor f(2n+2) \rfloor = 0$ so $\{f(2n+2)\} = f(2n+2)$. Then,

$$\begin{aligned}
f(2(n+1)+1) &= \frac{1}{\lfloor f(2n+2) \rfloor + 1 - \{f(2n+2)\}} \\
&= \frac{1}{1 - f(2n+2)} \\
&= \frac{1}{1 - \frac{1}{\lfloor f(2n+1) \rfloor + 1 - \{f(2n+1)\}}} \\
&= \frac{\lfloor f(2n+1) \rfloor + 1 - \{f(2n+1)\}}{\lfloor f(2n+1) \rfloor - \{f(2n+1)\}} \\
&= \frac{\lfloor f(n) \rfloor + 2 - \{f(n)\}}{\lfloor f(n) \rfloor + 1 - \{f(n)\}} \\
&= 1 + \frac{1}{\lfloor f(n) \rfloor + 1 - \{f(n)\}} \\
&= 1 + f(n+1).
\end{aligned}$$

Using this result,

$$\begin{aligned}
f(2(n+1)) &= \frac{1}{\lfloor f(2n+1) \rfloor + 1 - \{f(2n+1)\}} \\
&= \frac{1}{\lfloor f(n) \rfloor + 2 - \{f(n)\}} \\
&= \frac{1}{\frac{1}{f(n+1)} + 1} \\
&= \frac{f(n+1)}{f(n+1)+1}
\end{aligned}$$

completing the inductive step. Thus the proposition is proven. $\square$

3.4 f is Bijective

Proposition 3.5. *f is a bijection.*

Proof. As shown in section 3.3, every value in T is a value of $f(n)$. By proposition 2.1, any $\frac{p}{q} \in \mathbb{Q}^+$ is in T. Therefore, for any $\frac{p}{q} \in \mathbb{Q}^+$, there exists a $n \in \mathbb{N}$ such that $f(n) = \frac{p}{q}$, so f is surjective. f is also injective by proposition 3.2, so f is a bijection. $\square$

4 Non-Recursive Description of f

4.1 Algorithm for Closed Form of f

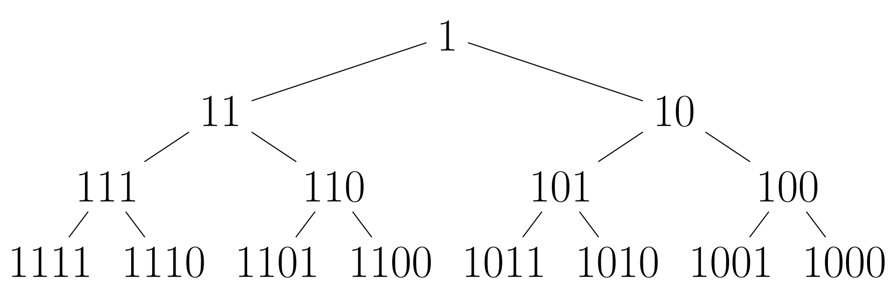
The following is a non-recursive algorithmic formulation of f:

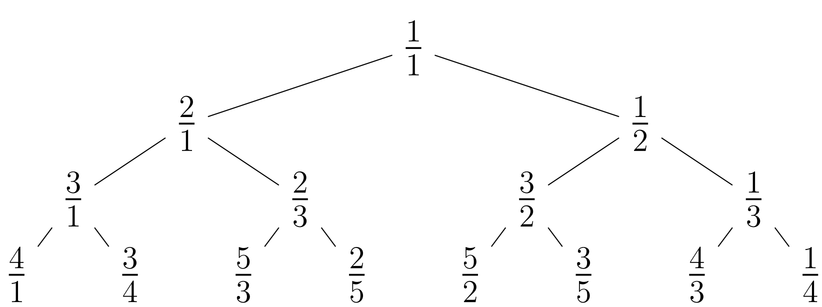
1. To calculate $f(n)$ start by converting n to binary.
2. Create a list of numbers: the number of 1's at the end of n's binary representation (which could be 0), then the number of consecutive 0's in a row before the 1's, then the number of consecutive 1's before those 0's, and so on until you get to the beginning of the binary representation of n.
3. If the second digit of the binary representation of n is a 0, treat the first digit as a 0 when generating the list.
4. This list is the continued fraction of of the value of $f(n)$.

Example 4.1. Consider $n = 35$, where $(35)_2 = 100011$; the generated list is $2, 4$ because it ends with two 1's, and since the second number is a 0, the first 1 is treated as a 0, so there are 4 zeros after the 1s hence $f(100011) = [2, 4]$

Example 4.2. Now let $n = 52$. Then, $(52)_{10} = 110100_2$; $f(52) = [0, 2, 1, 1, 2]$ as it ends with 0 ones, then 2 zeros, 1 one, then 1 zero, and finally 2 ones.

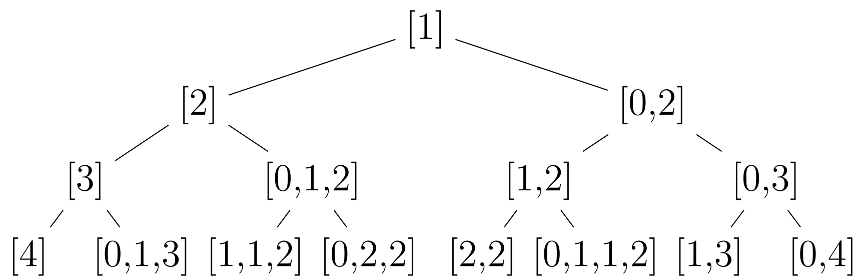
Now consider the following trees of representing, the binary representation of n, $f(n)$ as a fraction, and the continued fraction of $f(n)$ respectively.





Further examples referencing the above trees,

$$f(11_2) = \frac{2}{1} = [2]$$
$$f(110_2) = \frac{2}{3} = [0, 1, 2].$$



4.2 Motivation and Proof of Algorithm

To motivate such construction of the nodes of the tree, consider the following recursive definitions.

Note that when traversing the tree, from parent to child nodes, there are two options. One option is the right node, giving $n \to 2n$. The other option is the left node giving $n \to 2n+1$.

Going from $n \to 2n$ can be represented by adding a 0 at the end of the binary representation of n. Similarly going from $n \to 2n+1$ is equivalent to adding a 1 to the end of the binary representation.

On the tree of continued fractions, we claim going left is the same as adding 1 to the first integer in the continued fraction. This follows as a left move is equivalent to $f(n) \to f(2n+1)$ and $f(2n+1) = f(n)+1$. Further increasing a continued fraction by 1 is the same as adding to to the first value, so if $f(x) = [a_1, a_2, \ldots, a_k]$,

$$\begin{aligned} f(x) \to f(2x+1) &= f(x) + 1 \\ &= [a_1 + 1, a_2, \ldots, a_k]. \end{aligned}$$

E.g $f(2) \to f(5)$ is $[0,2] \to [1,2]$.

Now consider going right on the tree.

If $f(x) > 1$ then the continued fraction for $f(x)$ does not start with a 0. Therefore, going right will increase the second term in the continued fraction by 1. This is because going right takes $f(x) \to f(2x)$ hence,

$$\begin{aligned} f(2x) &= \frac{f(x)}{1+f(x)} \\ &= \frac{1}{1+\frac{1}{(f(x))}} \\ &= 0 + \frac{1}{1+\frac{1}{[a_1,a_2,\ldots,a_k]}} \\ &= [0, 1, a_1, a_2, \ldots, a_k] \end{aligned}$$

So going right from f(x) where $f(x) > 1$ appends 0, 1 to the beginning of the continued fraction of $f(x)$

Now consider going right from $f(x)$ where $f(x) < 1$ (i.e the first digit of $f(x)$'s continued fraction is 0) if the continued fraction of $f(x)$ is $f(x) = [0, a_1, ..., a_k]$,

$$\begin{aligned} f(x) \to f(2x) &= \frac{1}{1 + \frac{1}{f(x)}} \\ &= \frac{1}{1 + \frac{1}{[0, a_1, ..., a_k]}} \\ &= \frac{1}{1 + [a_1, ..., a_k]} \\ &= \frac{1}{[a_1 + 1, ..., a_k]} \\ &= [0, a_1 + 1, ..., a_k]. \end{aligned}$$

So going right from $f(x)$ where $f(x) < 1$ increases the second number of the continued fraction, the one right after the first zero, by 1 Example: going right from $[0, 1, 2]$ gives $\to [0, 2, 2]$

5 Ternary Tree of All Relatively Prime Ordered Triples

We want to create a ternary tree that has all relatively prime ordered triples.

We generalize our findings for rational numbers by switching to relatively prime ordered tuples. If the tree for rationals is written as relatively prime ordered pairs, meaning we write p/q as (p, q), then going left gives the linear transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p + q \\ q \end{bmatrix}$$

and going right gives

$$\begin{bmatrix} p \\ q \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q + p \end{bmatrix}.$$

Thus, to generalize our work to triples, we attempted to find three 3×3 matrices which could be used to produce every $3-$dimensional vector without repeats. We checked a number of different possibilities for matrices, but were unable to find a valid generalization since every linear combination we tried either resulted in repeated values or absent values. Whether such matrices, or even a non-linear rule, exist to generate every coprime triple in a ternary tree thus remains an unsolved problem of our group.

6 Acknowledgements

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