# Counting the Rationals

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# Introduction

In this presentation, we describe a recursive function $f : \mathbb{N} \to \mathbb{Q}^+$ and an infinite binary tree $T$ of positive rational numbers. We show that the function and tree have the same values and that the function bijectively maps from the natural numbers to the positive rational numbers. We describe a non-recursive algorithm for calculating $f(n)$ and $f^{-1}(p/q)$ and summarize our progress on generalizing our results to triplets of positive coprime integers.

# The Function

Define $f : \mathbb{N} \to \mathbb{Q}^+$ by the following recursive formula:

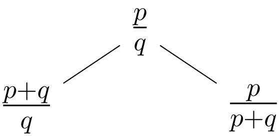
$$f(1) = \frac{1}{1} \text{ and } f(n+1) = \frac{1}{[f(n)] + 1 - \{f(n)\}} (n > 1).$$

Here, for $x \in \mathbb{R}$, $\{x\}$ represents the fractional part of $x$ while $[x]$ represents the greatest integer less than or equal to $x$. The first $16$ values of this function are:

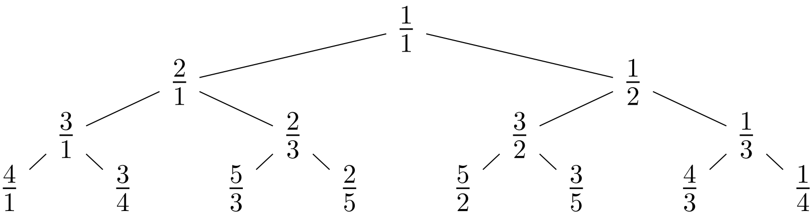
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(n)$ | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{2}{1}$ | $\frac{1}{3}$ | $\frac{3}{2}$ | $\frac{2}{3}$ | $\frac{3}{1}$ | $\frac{1}{4}$ | $\frac{4}{3}$ | $\frac{3}{5}$ | $\frac{5}{2}$ | $\frac{2}{5}$ | $\frac{5}{3}$ | $\frac{3}{4}$ | $\frac{4}{1}$ | $\frac{1}{5}$ |

# The Tree

Define the infinite binary tree $T$ as having a root of $\frac{1}{1}$ with the children of a node $\frac{p}{q}$ being $\frac{p+q}{q}$ and $\frac{p}{p+q}$.




- Every right node is less than 1 and left node is greater than 1
- The first four rows of $T$ are:

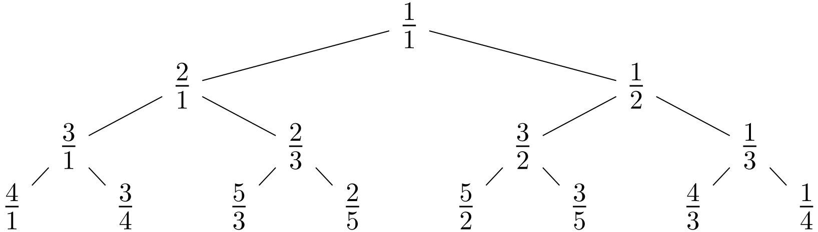


# Connections between $f$ and $T$

Again, the values of $f$ are:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(n)$ | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{2}{1}$ | $\frac{1}{3}$ | $\frac{3}{2}$ | $\frac{2}{3}$ | $\frac{3}{1}$ | $\frac{1}{4}$ | $\frac{4}{3}$ | $\frac{3}{5}$ | $\frac{5}{2}$ | $\frac{2}{5}$ | $\frac{5}{3}$ | $\frac{3}{4}$ | $\frac{4}{1}$ | $\frac{1}{5}$ |

The values of $T$ are:




We see that $f$ enumerates a right-to-left level-order traversal of $T$. Also, notice that every node $\frac{p}{q}$ in $T$ is in simplest form. To see this, note that for coprime $p$ and $q$, $p+q$ is coprime to both $p$ and $q$.

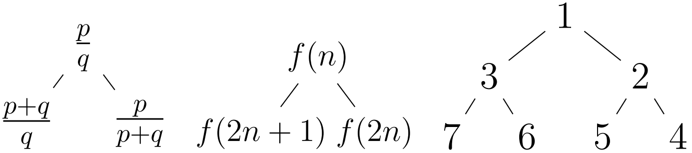
# Connections between $f$ and $T$

Formalizing this right to left traversal,

$$f(2n+1) = f(n) + 1$$

$$f(2n) = \frac{f(n)}{f(n)+1}.$$

Moving to the left in the tree is represented by $f(2n+1)$ while moving to the right is represented by $f(2n)$. We proved each of these identities inductively. Since both the tree and the function have the same initial value of $\frac{1}{1}$, $f$ traces though the entire tree $T$.



# Injectivity of $f$

## Proposition

*For* $n, m \in \mathbb{N}$, $f(n) = f(m) \iff n = m$

## Proof

FTSOC let $n \in \mathbb{N}$ be the smallest counterexample where there exists an $m \in \mathbb{N}$ such that $f(n) = f(m)$ and $m < n$. If $m = 1$, then $n > 1$. We have

$$f(n) = f(m)$$

$$\frac{1}{[f(n-1)] + 1 - \{f(n-1)\}} = 1$$

Rearranging, we have

$$[f(n-1)] = \{f(n-1)\}$$

Which means $f(n-1) = 0$, a contradiction. (continued)

# Injectivity of $f$

## Proof.

Otherwise, if $m > 1$, we have

$$f(n) = f(m)$$
$$\frac{1}{[f(n-1)] + 1 - \{f(n-1)\}} = \frac{1}{[f(m-1)] + 1 - \{f(m-1)\}}$$
$$[f(n-1)] - [f(m-1)] = \{f(n-1)\} - \{f(m-1)\}.$$

This is possible only if both sides of the above equality are zero, so

$$[f(n-1)] = [f(m-1)] \text{ and } \{f(n-1)\} = \{f(m-1)\}$$
$$f(n-1) = f(m-1)$$

Which is a contradiction since $n$ was the minimal counterexample. □

# Surjectivity of $f$

## Proposition

*Any positive rational $\frac{p}{q} \in \mathbb{Q}^+$ is a node in $T$.*

## Proof.

FTSOC let $\frac{p}{q}$ where $p, q \in \mathbb{N}$ and $\mathrm{GCD}(p,q) = 1$ be a counterexample where $p+q$ is minimal.

- If $p = q$, then $\frac{p}{q} = \frac{1}{1} \in T$.
- If $p > q$, then $\frac{p-q}{q} \in \mathbb{Q}^+$ with a smaller sum of numerator and denominator, so $\frac{p-q}{q}$ is in the tree. However, the left child of $\frac{p-q}{q}$ is $\frac{(p-q)+q}{q} = \frac{p}{q}$, so we have a contradiction.
- Similarly, if $p < q$, then $\frac{p}{q-p}$ is a positive rational in the tree, and $\frac{p}{q}$ is its right child. So $\frac{p}{q}$ is always in $T$.

$\square$

# Bijectivity of $f$

We showed inductively that $f$ is injective and constructed surjectivity by relating the function with the tree.

## Proposition

*$f$ is a bijection.*

## Proof.

As shown previously every value in $T$ is a value of $f(n)$. By surjectivity of $T$ on $\mathbb{Q}^+$, any $\frac{p}{q} \in \mathbb{Q}^+$ is in $T$. For any $\frac{p}{q} \in \mathbb{Q}^+$, there exists a $n \in \mathbb{N}$ such that $f(n) = \frac{p}{q}$, so $f$ is surjective. Observe $f$ was also proven injective, hence $f$ is a bijection. $\square$

Thus $f$ maps the natural numbers to the rational numbers bijectively.

# A Non-Recursive Algorithm for $f$

- Compare the tree of the continued fraction representation of $f$, and the tree of binary numbers.
- The tree of the continued fractions is gives the value of f(n) where n is the binary number in the same spot of the binary tree.
- For example, $f(1010) = [0, 1, 1, 2]$

- 1
  - 11
    - 111
      - 1111
      - 1110
    - 110
      - 1101
      - 1100
  - 10
    - 101
      - 1011
      - $*1010*$
    - 100
      - 1001
      - 1000

- [1]
  - [2]
    - [3]
      - [4]
      - [0,1,3]
    - [0,1,2]
      - [1,1,2]
      - [0,2,2]
  - [0,2]
    - [1,2]
      - [2,2]
      - $*[0,1,1,2]*$
    - [0,3]
      - [1,3]
      - [0,4]

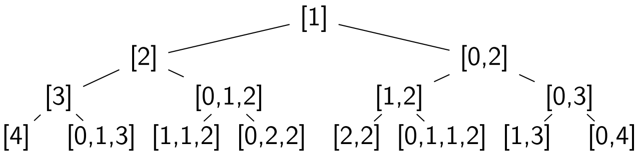
# A Non-Recursive Algorithm for $f$

- Going right is equivalent to adding a $0$ to the end of the binary representation of $n$ which is going from $f(n) \to f(2n) = \frac{f(n)}{1+f(n)}$.
- Going left is equivalent to adding a $1$ to the end of $n$ which is $f(n) \to f(2n+1) = 1 + f(n)$
- Say $f(n) = [a_1, a_2, ..., a_k]$
- Then going left results in a continued fraction where $a_1$ is increased by 1 because $f(2n+1) = 1 + f(n) = 1 + [a_1, a_2, ..., a_k] = [1 + a_1, a_2, ..., a_k]$ so $a_1$ is increased by 1

- [1]
  - [2]
    - [3]
      - [4]
      - [0,1,3]
    - [0,1,2]
      - [1,1,2]
      - [0,2,2]
  - [0,2]
    - [1,2]
      - [2,2]
      - [0,1,1,2]
    - [0,3]
      - [1,3]
      - [0,4]

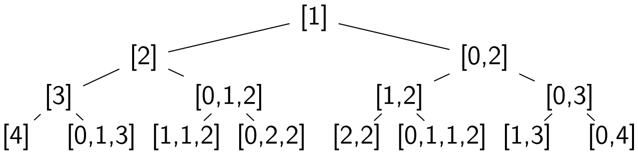
# Going right on $f$ part 1

- Going right is $f(n) \to f(2n)$
- Suppose $f(n) = [a_1, ..., a_k]$ where $a_1 \neq 0$
- $f(n) \to f(2n) = \frac{f(n)}{1+f(n)} = \frac{1}{\frac{1+f(n)}{f(n)}} = \frac{1}{1+\frac{1}{f(n)}} = 0 + \frac{1}{1+\frac{1}{[a_1,...,a_k]}} = [0, 1, a_1, ..., a_k]$
- So going right adds a 0,1 to the beginning of the continued fraction



# Going Right on $f$ Part 2

- Going right is $f(n) \to f(2n)$
- Suppose $f(n) = [0, a_1, ..., a_k]$
- $f(n) \to f(2n) = \frac{f(n)}{1+f(n)} = \frac{1}{\frac{1+f(n)}{f(n)}} = \frac{1}{1+\frac{1}{f(n)}} = \frac{1}{1+\frac{1}{[0,a_1,...,a_k]}} = \frac{1}{1+[a_1,...,a_k]} = \frac{1}{[a_1+1,...,a_k]} = [0, a_1 + 1, ..., a_k]$
- So going right increases $a_1$, the value after the first 0, by 1



# Non-Recursive $f$

- To better understand this formula, we look at the following example:
- Find $f(22_{10}) = f(10110_2)$ by looking at $f(1_2), f(10_2), f(101_2), f(1011_2), f(10110_2)$
- $f(1_2) = [0, 1]$
- $f(10_2) = [0, 2]$
- $f(101_2) = [1, 2]$
- $f(1011_2) = [2, 2]$
- $f(10110_2) = [0, 1, 2, 2]$

# A Non-Recursive Algorithm for $f$

1. To calculate $f(n)$, start by converting $n$ to binary.
2. Create a list of numbers: the number of 1's at the right end of $n$'s binary representation (which could be 0), then the number of consecutive $0$'s in a row before the 1's, then the number of consecutive 1's before those 0's, and so on until you get to the beginning of the binary representation of n.
3. If the second digit of the binary representation of n is a 0, treat the first digit as a 0 when generating the list.
4. This list is the continued fraction of of the value of $f(n)$.

# A Non-Recursive Algorithm for $f$

Example: Find $f(22)$

1. $22_{10} = 10110_2$
2. Starting from the end, there are 0 ones, 1 zero, 2 ones, 1 zero, and 1 one. Thus far the list is, $0, 1, 2, 1, 1$.
3. The second digit is a 0 so first 1 is treated as a 0. So the list becomes, $0, 1, 2, 2$.
4. So $f(22) = [0, 1, 2, 2]$.

Also, you can see that the sum of the numbers in the continued fraction for $f(n)$ is the same as the number of digits in the binary representation of $n$ since each step in the tree adds a digit to n and increases the sum of the digits of the continued fraction by 1

# Areas for Further Exploration

By representing each element of $\frac{p}{q}$ of $T$ instead by an ordered pair of coprime natural numbers $(p, q)$, it is natural to ask whether it is possible to enumerate all of the coprime natural number triples with a ternary tree. We had a number of problems though while trying to create a formula to move from parent to child nodes in such a way that the tree includes every triple of coprime natural numbers precisely once. For example, we tried the following formula:

$$(p, q, r)$$

$$(p + q, q, r) \quad (p, q + r, r) \quad (p, q, p + r)$$

This though (and every other rule we attempted) results in repeated nodes.

# Areas for Further Exploration

Further attempting to generalize $T$, we represented each ordered pair in our original tree by a vector $\begin{bmatrix} p \\ q \end{bmatrix}$. Then, moving to left and right in $T$ respectively become

$$\begin{bmatrix} p \\ q \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p+q \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ p+q \end{bmatrix}.$$

To generalize our work to triples, we attempted to find three $3 \times 3$ matrices which could be used to produce every $3-$dimensional vector without repeats. Trying many matrices, we were unable to find a feasible generalization. Whether such matrices or perhaps non-linear rules exist remains unsolved for our group.

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