

Correction of midterm exam n°2

Exercise 1 (4,5 points)

1. Using an integration by parts, by setting $u(x) = \sin(\ln(x))$ and $v'(x) = 1$, we get :

$$\begin{aligned} I &= \left[x \sin(\ln(x)) \right]_1^e - \int_1^e \cos(\ln(x)) \, dx \\ &= e \sin(1) - \int_1^e \cos(\ln(x)) \, dx \end{aligned}$$

Using a second integration by parts, by setting $u(x) = \cos(\ln(x))$ and $v'(x) = 1$, we get :

$$I = e \sin(1) - \left(\left[x \cos(\ln(x)) \right]_1^e + I \right)$$

thus $2I = 1 + e(\sin(1) - \cos(1))$, that is to say $I = \frac{1}{2}(1 + e(\sin(1) - \cos(1)))$.

2. Using an integration by parts, by setting $u(x) = \arctan(x)$ and $v'(x) = 1$, we get :

$$\begin{aligned} J &= \left[x \arctan(x) \right]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \\ &= \frac{\pi}{4} - \frac{1}{2} [\ln(1+x^2)]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2} \ln(2) \end{aligned}$$

3. Using the substitution $u = \sqrt{x}$, $x = u^2$ thus $dx = 2u \, du$.

We get : $K = 2 \int_0^\pi u \cos(u) \, du$.

Using an integration by parts, by setting $\varphi(u) = u$ and $\psi'(u) = \cos(u)$, we get :

$$K = 2 \left(\left[u \sin(u) \right]_0^\pi - \int_0^\pi \sin(u) \, du \right) = -2 \int_0^\pi \sin(u) \, du$$

Thus $K = -4$.

Exercise 2 (3 points)

1. From the hypothesis we deduce that

$$\left\{ \begin{array}{l} \frac{u_n}{u_{n-1}} \leq \frac{v_n}{v_{n-1}} \\ \vdots \\ \frac{u_1}{u_0} \leq \frac{v_1}{v_0} \end{array} \right.$$

By multiplying these inequalities of positive numbers, we get :

$$\frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \cdot \dots \cdot \frac{u_1}{u_0} \leq \frac{v_n}{v_{n-1}} \cdot \frac{v_{n-1}}{v_{n-2}} \cdot \dots \cdot \frac{v_1}{v_0}$$

from which we deduce that $\frac{u_n}{u_0} \leq \frac{v_n}{v_0}$, that is to say $0 < u_n \leq \frac{u_0}{v_0} v_n$.

Therefore, using the squeeze theorem (sandwich theorem), if $v_n \xrightarrow[n \rightarrow +\infty]{} 0$ then $u_n \xrightarrow[n \rightarrow +\infty]{} 0$.

2. Similarly, $v_n \geq \frac{v_0}{u_0} u_n$.

Thus, still using the squeeze theorem, if $u_n \xrightarrow[n \rightarrow +\infty]{} +\infty$ then $v_n \xrightarrow[n \rightarrow +\infty]{} +\infty$.

Exercise 3 (3 points)

- a. Let (u_n) be a sequence of real numbers, and $\ell \in \mathbb{R}$. Then the assertion « if (u_n) converges towards ℓ then, for every $n \in \mathbb{N}$, $u_n \leq \ell$ » is equivalent to the assertion « if there exists $n \in \mathbb{N}$ such that $u_n > \ell$, then (u_n) does not converge towards ℓ ».
- b. If (u_n) is a nonzero geometric sequence with common ratio $q \in \mathbb{R}^*$, then $\left(\frac{1}{u_n}\right)$ is a geometric sequence with common ratio $\frac{1}{q}$.
- c. If (u_n) is a bounded numerical sequence, there exists a subsequence of (u_n) that is convergent.
- d. Let (u_n) be a numerical sequence. Then (u_{6n}) is a subsequence of (u_{2n}) .
- e. Let (u_n) be a numerical sequence. Then $(u_{3 \cdot 2^{n+1}})$ is a subsequence of (u_{6n}) .
- f. none of the above

Exercise 4 (3 points)

Let us prove that (u_n) is (strictly) increasing.

$$u_{n+1} - u_n = \sum_{k=0}^{2n+3} \frac{(-1)^k}{(2k)!} - \sum_{k=0}^{2n+1} \frac{(-1)^k}{(2k)!} = \frac{(-1)^{2n+2}}{(2(2n+2))!} + \frac{(-1)^{2n+3}}{(2(2n+3))!} = \frac{1}{(4n+4)!} - \frac{1}{(4n+6)!}$$

Yet

$$(4n+6)! > (4n+4)!$$

hence

$$\frac{1}{(4n+6)!} < \frac{1}{(4n+4)!}$$

Therefore $u_{n+1} - u_n > 0$, thus (u_n) is strictly increasing.

Let us prove that (v_n) is (strictly) decreasing.

$$v_{n+1} - v_n = u_{n+1} + \frac{1}{(4(n+1)+4)!} - u_n - \frac{1}{(4n+4)!}$$

By reusing the already calculated expression of $u_{n+1} - u_n$, we get :

$$v_{n+1} - v_n = \frac{1}{(4n+4)!} - \frac{1}{(4n+6)!} + \frac{1}{(4n+8)!} - \frac{1}{(4n+4)!} = \frac{1}{(4n+8)!} - \frac{1}{(4n+6)!}$$

Yet

$$(4n+8)! > (4n+6)!$$

hence

$$\frac{1}{(4n+8)!} < \frac{1}{(4n+6)!}$$

Therefore $v_{n+1} - v_n < 0$ thus (v_n) is strictly decreasing.

Eventually, $v_n - u_n = \frac{1}{(4n+4)!} \rightarrow 0$ thus (u_n) and (v_n) are adjacent sequences.

Exercise 5 (2 points)

- $\ln(n!) = \ln(1) + \dots + \ln(n) = \sum_{k=1}^n \ln(k) \leq n \ln(n)$ as the function \ln is increasing.
- Thus, for every $n \in \mathbb{N}^*$, $0 \leq u_n \leq \frac{\ln(n)}{n}$.

Using the squeeze theorem together with a result of compared growth, $(u_n)_{n \in \mathbb{N}^*}$ converges towards 0.

Exercise 6 (5,5 points)

1. As $q \neq 1$, $\sum_{k=1}^n q^{k-1} = \frac{1-q^n}{1-q}$
2. Using the previous question,

$$\sum_{k=1}^n \frac{1}{2^{k-1}} = \sum_{k=1}^n \left(\frac{1}{2}\right)^{k-1} = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 \left(1 - \frac{1}{2^n}\right)$$

that is to say

$$\sum_{k=1}^n \frac{1}{2^{k-1}} = 2 - \frac{1}{2^{n-1}}$$

$$3. \text{ Let } k \geq 2. \text{ Then : } \begin{cases} 2 \geq 2 \\ 3 \geq 2 \\ \vdots \\ k \geq 2 \end{cases} \text{ hence } \begin{cases} \frac{1}{2} \leq \frac{1}{2} \\ \frac{1}{3} \leq \frac{1}{2} \\ \vdots \\ \frac{1}{k} \leq \frac{1}{2} \end{cases}$$

$$\text{Thus, } \frac{1}{k!} = \frac{1}{2 \times 3 \times \dots \times k} = \frac{1}{2} \times \frac{1}{3} \times \dots \times \frac{1}{k} \leq \frac{1}{2^{k-1}}$$

This property still holds when $k = 1$ as $\frac{1}{1!} = 1 \leq \frac{1}{2^0} = 1$.

4. Let $n \in \mathbb{N}$. Then $u_{n+1} - u_n = \frac{1}{(n+1)!} \geq 0$ thus (u_n) is increasing.
5. Using questions 2 and 3,

$$u_n = 1 + \sum_{k=1}^n \frac{1}{k!} \leq 1 + \sum_{k=1}^n \frac{1}{2^{k-1}} \leq 1 + 2 - \frac{1}{2^{n-1}} \leq 3$$

6. (u_n) is an increasing numerical sequence that is bounded above, thus it is convergent.