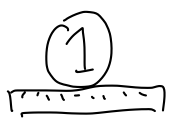
$\boxed{\text{Lesson 1}}$ 递归算法

Optimal Recursive Data Processing Algorithm
最优 递归 数据处理 算法

卡尔曼滤波更像是一种观测器而不是滤波器

不确定性：
① 不存在完美的数学模型
② 系统的扰动不可控，也很难建模
③ 测量传感器存在误差

例：测量一枚硬币的直径：



测量结果 z_k ← 第k次

$z_1 = 50.1mm$
$z_2 = 50.4mm$
$z_3 = 50.2mm$

估计真实数据 ⟶ 取平均值：

Hat ← 估计值

$$\hat{x}_k = \frac{1}{k}(z_1 + z_2 + \cdots + z_k)$$

$$= \frac{1}{k}(z_1 + z_2 + \cdots + z_{k-1}) + \frac{1}{k}z_k$$

$$= \frac{1}{k}\cdot\frac{k-1}{k-1}(z_1 + z_2 + \cdots + z_{k-1}) + \frac{1}{k}z_k$$

k-1次时的平均值 $\hat{x}_{k-1}$

$$= \frac{k-1}{k}\hat{x}_{k-1} + \frac{1}{k}z_k$$

$$= \hat{x}_{k-1} - \frac{1}{k}\hat{x}_{k-1} + \frac{1}{k}z_k$$

$$\Rightarrow \boxed{\hat{x}_k = \hat{x}_{k-1} + \frac{1}{k}(z_k - \hat{x}_{k-1})}$$

令 $K_k = \frac{1}{k}$

$k \uparrow$, $\frac{1}{k} \to 0$, $\hat{x}_k \to \hat{x}_{k-1}$

随着k增加，测量结果不再重要

$k \downarrow$, $\frac{1}{k}$ 大, z_k 作用较大

$$\boxed{\hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})}$$

递归思想

当前的估计值 = 上一次的估计值 + 系数 × (当前测量值 − 上一次的估计值)

K_k: Kalman Gain 卡尔曼增益/因数

估计误差: e_{EST}

Estimate 估计

测量误差: e_{MEA}

Measurement 测量

$$K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$$

在k时刻: ① $e_{EST_{k-1}} \gg e_{MEA_k}$: $K_k \to 1$, $\hat{x}_k = \hat{x}_{k-1} + z_k - \hat{x}_{k-1} = z_k$

相信测量值

② $e_{EST_{k-1}} \ll e_{MEA_k}$: $K_k \to 0$; $\hat{x}_k = \hat{x}_{k-1}$

相信估计值

例：

Step 1：计算 Kalman Gain $K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$

Step 2：计算 $\hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$

Step 3、更新 $e_{EST_k} = (1 - K_k) e_{EST_{k-1}}$ $\longrightarrow$ 后面详细推导

实际长度 $x = 50mm$

$\hat{x}_0 = 40mm$

$e_{EST_0} = 5mm$

$z_1 = 51mm$

$e_{MEA_k} = 3mm$

k	z_k	e_{MEA_k}	$\hat{x}_k$	K_k	e_{EST_k}
0			40		5
1	51	3	46.875	0.625	1.875
2	48	3	47.308	0.3846	1.154
3					

$k=1$：

$K_k = \frac{5}{5+3} = 0.625$

$\hat{x}_k = 40 + 0.625(51 - 40) = 46.875$

$e_{EST} = (1 - 0.625)5 = 1.875$

$k=2$：

$K_k = \frac{1.875}{1.875+3} = 0.3846$

$\hat{x}_k = 46.875 + 0.3846(48 - 46.875) = 47.308$

$e_{EST} = (1 - 0.3846)1.875 = 1.154$

Lesson 2　数学基础

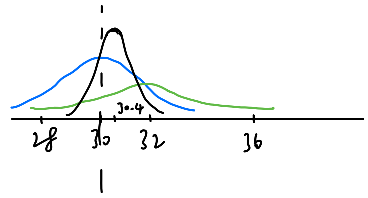
卡尔曼滤波器、数据融合、协方差矩阵、状态空间方程、观测器

Data Fusion 数据融合

标准差

$z_1 = 30g \qquad \sigma_1 = 2g$

$z_2 = 32g \qquad \sigma_2 = 4g$　　正态分布



$\hat{z} = 30.4g \qquad \sigma = 1.79g$

估计真实值：$\hat{z} = ?$

Kalman Gain

$$\hat{z} = z_1 + K(z_2 - z_1)$$

$K \in [0,1]$

$K=0$：$\hat{z} = z_1$

$K=1$：$\hat{z} = z_2$

求 K 使得 $\sigma_{\hat{z}}^2$ 最小 $\Rightarrow$ 方差 $Var(\hat{z})$ 最小

相互独立

$$\sigma_{\hat{z}}^2 = Var(z_1 + K(z_2 - z_1)) = Var(z_1 + Kz_2 - Kz_1) = Var((1-K)z_1 + Kz_2)$$

$$= Var((1-K)z_1) + Var(Kz_2) = (1-K)^2 Var(z_1) + K^2 Var(z_2)$$

$$= (1-K)^2\sigma_1^2 + K^2\sigma_2^2$$

$$\frac{d\sigma_{\hat{z}}^2}{dK} = -\cancel{2}(1-K)\sigma_1^2 + \cancel{2}K\sigma_2^2 = 0 \Rightarrow -\sigma_1^2 + K\sigma_1^2 + K\sigma_2^2 = 0$$

$$K(\sigma_1^2 + \sigma_2^2) = \sigma_1^2 \qquad K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \frac{4}{4+16} = 0.2$$

$$\therefore \hat{z} = z_1 + 0.2(z_2 - z_1) = 30 + 0.2(32 - 30) = 30.4$$

$$\sigma_{\hat{z}}^2 = (1-0.2)^2 2^2 + 0.2^2 4^2 = 3.2 \qquad \sigma_{\hat{z}} = \sqrt{3.2} = 1.79$$

Covariance Matrix 协方差矩阵

方差、协方差在一个矩阵中表现出来

↓

变量间的联动关系

例：

球员	身高	体重	年龄
A	179	74	33
B	187	80	31
C	175	71	28
平均	180.3	75	30.7

方差：$\sigma_x^2 = \frac{1}{3}\left((179-180.3)^2 + (187-180.3)^2 + (175-180.3)^2\right) = 24.89$

$\sigma_y^2 = 14$

$\sigma_z^2 = 4.22$

协方差：$\sigma_x\sigma_y = \frac{1}{3}\left((179-180.3)(74-75) + (187-180.3)(80-75) + (175-180.3)(71-75)\right) = \underset{\sim\sim}{18.7} = \sigma_y\sigma_x$ >0 正相关

$\sigma_x\sigma_z = 4.4 = \sigma_z\sigma_x$

$\sigma_y\sigma_z = 3.3 = \sigma_z\sigma_y$

协方差矩阵 $P = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y & \sigma_x\sigma_z \\ \sigma_y\sigma_x & \sigma_y^2 & \sigma_y\sigma_z \\ \sigma_z\sigma_x & \sigma_z\sigma_y & \sigma_z^2 \end{bmatrix} = \begin{bmatrix} 24.89 & 18.7 & 4.4 \\ 18.7 & 14 & 3.3 \\ 4.4 & 3.3 & 4.22 \end{bmatrix}$

如何通过矩阵的运算计算协方差（对编程有帮助）

过渡矩阵：$a = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} - \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}}_{平均值}$

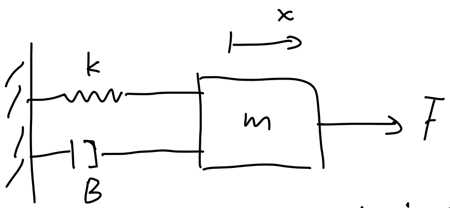
$P = \frac{1}{3}a^T a$

State Space Representation
状态空间表达

state
状态变量 $x_1 = x$

$x_2 = \dot{x}$

例:



$m\ddot{x} + B\dot{x} + kx = F$ ↗ u: input

$m\ddot{x} = u - B\dot{x} - kx$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x} = \frac{1}{m}u - \frac{B}{m}\dot{x} - \frac{k}{m}x = \frac{1}{m}u - \frac{B}{m}x_2 - \frac{k}{m}x_1$$

测量 $z_1 = x = x_1$ 位置

$z_2 = \dot{x} = x_2$ 速度

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$\dot{X}(t) = \underbrace{A}_{\text{状态转移矩阵}} X(t) + B u(t)$$

state space

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

连续的表达形式

$$Z(t) = \underbrace{H}_{\text{状态观测矩阵}} X(t)$$

$\boxed{\text{不确定性}}$

离散: $X_k = AX_{k-1} + Bu_{k-1} + w_{k-1}$ → 过程噪音

⇕ 融合

$Z_k = HX_k + v_k$ → 测量噪音

下标 k、k-1、k+1 时间单位

sample time 采样

模型不准确、测量的值也不准确

⇓

问题:都不准确的情况下,如何精确的估计 $\hat{x}$? ⇒ Kalman Filter

Lesson 3 卡尔曼增益/因数详细推导

状态空间方程

$$X_k = AX_{k-1} + Bu_{k-1} + \underset{\text{过程噪声}}{W_{k-1}}$$

$$Z_k = HX_k + V_k \quad \leftarrow \text{测量噪声}$$

但是实际过程中噪声都不可知：

先验估计

$$\hat{X}_k^- = A\hat{X}_{k-1} + Bu_{k-1} \qquad \text{算出来的}$$

$$Z_k = HX_k \longrightarrow \hat{X}_{k_{mea}} = H^- Z_k \qquad \text{测出来的}$$

根据数据融合：

后验 $\hat{X}_k = \hat{X}_k^- + G(H^- Z_k - \hat{X}_k^-)$

$G=0$: $\hat{X}_k = \hat{X}_k^-$ 相信估计的

$G=1$: $\hat{X}_k = H^- Z_k$ 相信测量的

一般书中会： $G = K_k H$，得：

卡尔曼滤波器 $\boxed{\hat{X}_k = \hat{X}_k^- + K_k(Z_k - H\hat{X}_k^-)}$

$K_k \in [0, H^-]$

$K_k = 0$: $\hat{X}_k = \hat{X}_k^-$

$K_k = H^-$: $\hat{X}_k = H^- Z_k$

目标：寻找 K_k，使得 $\hat{X}_k \rightarrow \underset{\text{实际值}}{X_k}$

$$e_k = X_k - \hat{X}_k$$

$$P_k = E[ee^T] = E[(X_k - \hat{X}_k)(X_k - \hat{X}_k)^T]$$

则 $X_k - \hat{X}_k^-$ 为 e_k^-，先验误差

$$= E[[(I-K_kH)e_k^- - K_kV_k][(I-K_kH)e_k^- - K_kV_k]^T]$$

$\because (AB)^T = B^TA^T$

$(A+B)^T = A^T + B^T$

$\therefore = [(I-K_kH)e_k^-]^T - (K_kV_k)^T$

$= e_k^{-T}(I-K_kH)^T - V_k^TK_k^T$

已知分布

$$P(W) \sim (\underset{\text{期望}}{0}, \underset{\text{过程误差协方差矩阵}}{Q})$$

$Q = E[ww^T]$ 则 $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$= E\left[[w_1\ w_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right]$$

$$= E\left[\begin{bmatrix} w_1^2 & w_1w_2 \\ w_2w_1 & w_2^2 \end{bmatrix}\right]$$

$$= \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$$

$VAR(X) = E(X^2) - E^2(X)$

$$= \begin{bmatrix} \sigma w_1^2 & \sigma w_1 \sigma w_2 \\ \sigma w_2 \sigma w_1 & \sigma w_2^2 \end{bmatrix} \text{ 协方差矩阵}$$

不是标准差相乘，而是 $w_1 w_2$ 的协方差

同理 已知分布 $P(V) \sim (\underset{\text{期望}}{0}, \underset{\text{测量误差协方差矩阵}}{R})$

即 $E(VV^T) = R$

$P(e_k) \sim (0, P)$

$$P_k = E[ee^T] = \begin{bmatrix} \sigma e_1^2 & \sigma e_1 \sigma e_2 \\ \sigma e_2 \sigma e_1 & \sigma e_2^2 \end{bmatrix}$$

$tr(P) = \sigma e_1^2 + \sigma e_2^2$ 最小 $\Rightarrow$ 方差最小

迹

$$X_k - \hat{X}_k = X_k - (\hat{X}_k^- + K_k(Z_k - H\hat{X}_k^-))$$

$$= X_k - \hat{X}_k^- - K_k \underset{HX_k + V_k}{Z_k} + K_kH\hat{X}_k^-$$

$$= E\left[\left[(I-K_kH)e_k^- - K_kv_k\right]\left[e_k^{-T}(I-K_kH)^T - v_k^TK_k^T\right]\right]$$

$$= E\left[E(I-K_kH)e_k^- e_k^{-T}(I-K_kH)^T \cancel{- E(I-K_kH)e_k^- v_k^T K_k^T} \to 0\right.$$

线性关系

$$\left.\cancel{- E K_kv_k e_k^{-T}(I-K_kH)^T} \to 0 + E K_kv_kv_k^TK_k^T\right]$$

$$= E\left[(I-K_kH)e_k^- e_k^{-T}(I-K_kH)^T\right] + E(K_kv_kv_k^TK_k^T)$$

$$= (I-K_kH)\underbrace{E(e_k^- e_k^{-T})}_{P_k^-}(I-K_kH)^T + K_k\underbrace{E(v_kv_k^T)}_{R}K_k^T$$

详见右侧草纸最开始

$$= (P_k^- - K_kHP_k^-)(I^T - H^TK_k^T) + K_kRK_k^T$$

互为转置

$$P_k = P_k^- - K_kHP_k^- - \underline{P_k^-H^TK_k^T} + K_kHP_k^-H^TK_k^T + K_kRK_k^T$$

误差的协方差矩阵

若两个矩阵互为转置，它们的迹相同

$$tr(P_k) = tr(P_k^-) - 2\,tr(K_kHP_k^-) + tr(K_kHP_k^-H^TK_k^T) + tr(K_kRK_k^T)$$

想使迹最小，即 $\dfrac{d\,tr(P_k)}{dK_k} = 0$

$$\frac{d\,tr(P_k)}{dK_k} = 0 - 2(HP_k^-)^T + 2K_kHP_k^-H^T + 2K_kR = 0$$

$$= -(HP_k^-)^T + K_kHP_k^-H^T + K_kR$$

$$= -P_k^{-T}H^T + K_k(HP_k^-H^T + R) = 0$$

$$= -P_k^-H^T + K_k(HP_k^-H^T + R) = 0$$

$$\therefore \boxed{K_k = \frac{P_k^-H^T}{HP_k^-H^T + R}}$$

Kalman Gain

$R = E(vv^T)$ 是噪声矩阵协方差

$R\uparrow$ $K_k \to 0$: $\hat{x}_k = \hat{x}_k^-$ 相信计算结果

$R\downarrow$ $K_k = H^{-1}$: $\hat{x}_k = H^{-1}z_k$ 相信测量结果

$$= x_k - \hat{x}_k^- - K_kHx_k - K_kv_k + K_kH\hat{x}_k^-$$

$$= (x_k - \hat{x}_k^-) - K_kH(x_k - \hat{x}_k^-) - K_kv_k$$

$$= (I - K_kH)\underbrace{(x_k - \hat{x}_k^-)}_{e_k^-} - K_kv_k$$

$$E\left[(I-K_kH)e_k^- v_k^T K_k^T\right]$$

$$= (I-K_kH)\underbrace{E(e_k^- v_k^T)}K_k^T$$

若独立 $E(AB) = E(A)E(B)$

$$\Downarrow$$

$E(e_k^-)E(v_k^T) = 0$　　$E(e_k^-) = 0$, $E(v_k^T) = 0$

$= 0$

$$(P_k^-H^TK_k^T)^T = ((P_k^-H^T)K_k^T)^T = K_k(P_k^-H^T)^T$$

$$= K_kHP_k^{-T}$$

$\because P_k^-$ 是协方差矩阵 $\therefore P_k^{-T} = P_k^-$

$$\therefore (P_k^-H^TK_k^T)^T = K_kHP_k^-$$

$$\boxed{\frac{d\,tr(AB)}{dA} = B^T}$$

例：

$$tr\left[\overset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}\overset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}\right]$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & \cdots \\ \cdots & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$\therefore tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$$

$$\frac{d\,tr(AB)}{dA} = \begin{bmatrix} \frac{\partial tr(AB)}{a_{11}} & \frac{\partial tr(AB)}{a_{12}} \\ \frac{\partial tr(AB)}{a_{21}} & \frac{\partial tr(AB)}{a_{22}} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = B^T$$

$$\boxed{\frac{d(ABA^T)}{dA} = 2AB}$$

同样可通过上述方式推导

Lesson 4 误差协方差矩阵

$$x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1} \qquad w \sim p(0, Q)$$

$$z_k = Hx_k + v_k \qquad v \sim p(0, R)$$

先验估计

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_{k-1}$$

后验估计

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

卡尔曼增益

$$K_k = \frac{P_k^- H^T}{HP_k^- H^T + R}$$

求 P_k^-

$$P_k^- = E[e_k^- e_k^{-T}]$$

$$\begin{aligned} e_k^- &= x_k - \hat{x}_k^- \\ &= Ax_{k-1} + Bu_{k-1} + w_{k-1} - A\hat{x}_{k-1} - Bu_{k-1} \\ &= A(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} \\ &= Ae_{k-1} + w_{k-1} \end{aligned}$$

$$\begin{aligned} \therefore P_k^- = E[e_k^- e_k^{-T}] &= E[(Ae_{k-1} + w_{k-1})(Ae_{k-1} + w_{k-1})^T] \\ &= E[(Ae_{k-1} + w_{k-1})[(Ae_{k-1})^T + w_{k-1}^T]] \\ &= E[(Ae_{k-1} + w_{k-1})(e_{k-1}^T A^T + w_{k-1}^T)] \\ &= \cancel{E}[EAe_{k-1}e_{k-1}^T A^T + EAe_{k-1}w_{k-1}^T + Ew_{k-1}e_{k-1}^T A^T + Ew_{k-1}w_{k-1}^T] \end{aligned}$$

$e_{k-1} = x_{k-1} - \hat{x}_{k-1}$ 为上一时刻误差

而由 $x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1}$ 可知 w_{k-1} 作用于此时刻

$\therefore$ e_{k-1} 与 w_{k-1} 相互独立

$$\therefore E[A e_{k-1} w_{k-1}^T] = A E[e_{k-1} w_{k-1}^T] = A \underbrace{E[e_{k-1}] E[w_{k-1}^T]}_{\text{期望都为}0} = 0$$

同理 $E[w_{k-1} e_{k-1}^T A^T] = 0$

$$\therefore = E[A e_{k-1} e_{k-1}^T A^T] + E[w_{k-1} w_{k-1}^T]$$

$$= A \underbrace{E[e_{k-1} e_{k-1}^T]}_{P_{k-1}} A^T + \underbrace{E[w_{k-1} w_{k-1}^T]}_{Q}$$

$$\therefore \boxed{P_k^- = A P_{k-1} A^T + Q}$$

由此我们就可以根据卡尔曼滤波器估计状态变量的值了.

分为预测和校正两个过程

预测

∴每次预测都会用到上一次的结果

∴还要给初值

校正

先验 $\hat{x}_k^- = A \hat{x}_{k-1} + B u_{k-1}$ ① (x_0 初始)

先验误差协方差矩阵 $P_k^- = A P_{k-1} A^T + Q$ ② (P_0 初始)

用到了上一次的误差协方差

卡尔曼增益 $K_k = \dfrac{P_k^- H^T}{H P_k^- H^T + R}$ ③

后验估计 $\hat{x}_k = \hat{x}_k^- + K_k (z_k - H \hat{x}_k^-)$ ④

∴每次校正完后,

更新误差协方差矩阵 $P_k = (I - K_k H) P_k^-$ ⑤

由上一节可知: $P_k = P_k^- - K_k H P_k^- - P_k^- H^T K_k^T + \underline{K_k H P_k^- H^T K_k^T + K_k R K_k^T}$

$$= K_k (H P_k^- H^T + R) K_k^T$$

$$= \frac{P_k^- H^T}{\cancel{H P_k^- H^T + R}} \left(\cancel{H P_k^- H^T + R}\right) K_k^T = P_k^- H^T K_k^T$$

$$= P_k^- - K_k H P_k^- - P_k^- \cancel{H^T K_k^T} + P_k^- \cancel{H^T K_k^T}$$

$$= P_k^- - K_k H P_k^- = (I - K_k H) P_k^-$$

$\boxed{\text{Lesson 5}}$ 直观理解与二维实例

$z_1 = 6.5\,mm$ $\sigma_1 = 0.2\,mm$ $\hat{z}$?

$z_2 = 7.3\,mm$ $\sigma_2 = 0.4\,mm$

$$\hat{z} = z_1 + k(z_2 - z_1)$$

$$Var(\hat{z}) = Var[z_1 + k(z_2 - z_1)] = Var(z_1 + kz_2 - kz_1)$$

$$= Var(\underbrace{z_1(1-k) + kz_2}_{\text{独立}})$$

$$= Var((1-k)z_1) + Var(kz_2)$$

$$= (1-k)^2 Var(z_1) + k^2 Var(z_2)$$

$$= (1-k)^2 \sigma_1^2 + k^2 \sigma_2^2$$

$$\therefore \frac{d\,Var(\hat{z})}{dk} = 2k\sigma_1^2 - 2\sigma_1^2 + 2k\sigma_2^2 = 0$$

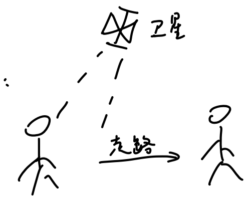
$$\therefore k\sigma_1^2 - \sigma_1^2 + k\sigma_2^2 = 0$$

$$\therefore k = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \frac{0.04}{0.04 + 0.16} = 0.2$$

$$\therefore \hat{z} = z_1 + k(z_2 - z_1) = 6.5 + 0.2(7.3 - 6.5) = 6.66$$

课件密码，与本节课内容无关

例2：



states: $\longmapsto$ x_1：位置

x_2：速度

位置：$x_{1,k} = x_{1,k-1} + \Delta T\, x_{2,k-1} + \underline{w_{1,k-1}}$

速度：$x_{2,k} = x_{2,k-1} + \underline{w_{2,k-1}}$ 不确定性

采样时间 ΔT：k时刻与k-1时刻的间隔

W：过程噪声 process noise

$P(w) \sim N(\underset{\text{期望}}{0}, \underset{\text{协方差矩阵}}{Q})$ 正态分布

卫星的测量：

$z_{1,k} = x_{1,k} + \underline{v_{1,k}}$ 不确定性

$z_{2,k} = x_{2,k} + \underline{v_{2,k}}$

V：测量噪声 $P(v) \sim N(0, R)$

$$x_k = A x_{k-1} + w_{k-1}$$

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \end{bmatrix} + \underbrace{\begin{bmatrix} w_{1,k-1} \\ w_{2,k-1} \end{bmatrix}}_{\text{模型不准确}}$$

$$z_k = H x_k + v_k$$

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \underbrace{\begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix}}_{\text{测量也不准确}}$$

用两个都不准确去估计一个最优值 $\longrightarrow \hat{x}_k$

这个估计的过程就是卡尔曼滤波器

Kalman Filter

预测

$$\hat{x}_k^- = A\hat{x}_{k-1}$$

$$P_k^- = A P_{k-1} A^T + Q$$

校正

$$K_k = \frac{P_k^- H^T}{H P_k^- H^T + R}$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

$$P_k = (I - K_k H) P_k^-$$

Lesson 6 扩展卡尔曼滤波器 Extened Kalman Filter (EKF)

(将非线性系统线性化)

Linear System

$x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1}$ $\qquad P(w) \sim N(0, Q)$

$z_k = Hx_k + v_k$ $\qquad P(v) \sim N(0, R)$

预测

先验 $\hat{x}_k^- = A\hat{x}_{k-1}$

$P_k^- = AP_{k-1}A^T + Q$

校正

$K_k = \dfrac{P_k^- H^T}{HP_k^-H^T + R}$

后验 $\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

$P_k = (I - K_kH)P_k^-$

Data Fusion 思想

Nonlinear

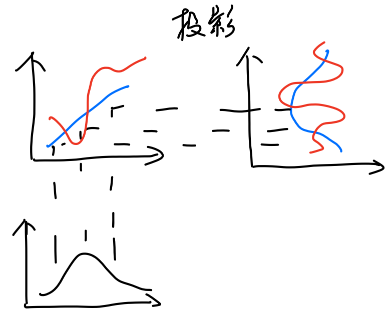
非线性

$x_k = f(x_{k-1}, u_{k-1}, w_{k-1})$ $\qquad P(w) \sim N(0, Q)$

$z_k = h(x_k, v_k)$ $\qquad P(v) \sim N(0, R)$

f.h 非线性

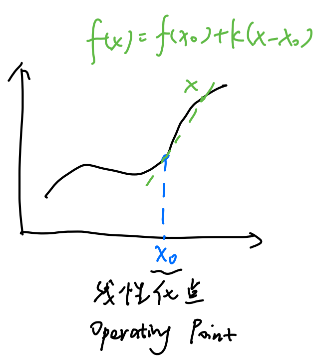
投影



正态分布的随机变量通过非线性系统后就不再是正态的了.

∴需要线性化 Linearization

Taylor Series 泰勒级数 $\qquad f(x) = f(x_0) + \dfrac{\partial f}{\partial x}(x - x_0)$



线性化点最好是真实点

但系统有误差，无法在真实点线性化

$f(x_k)$ 在 $\hat{x}_{k-1}$ 处线性化

k-1时刻的后验估计（上一次）

过程方程线性化

误差假设为0 $\Rightarrow f(\hat{x}_{k-1}, u_{k-1}, 0) = \tilde{x}_k$

$$x_k = f(\hat{x}_{k-1}, u_{k-1}, \cancel{w_{k-1}}\ 0) + A(x_{k-1} - \hat{x}_{k-1}) + W_k \cdot w_{k-1}$$

雅各比矩阵 $A = \frac{\partial f}{\partial x}\Big|_{\hat{x}_{k-1}, u_{k-1}}$　　　$W_k = \frac{\partial f}{\partial w}\Big|_{\hat{x}_{k-1}, u_{k-1}}$

再代入 $\hat{x}_{k-1}, u_{k-1}$

例：$x_1 = x_1 + \sin x_2 = f_1$
$x_2 = x_1^2 = f_2$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & \cos x_2 \\ 2x_1 & 0 \end{bmatrix}\Bigg|_{\hat{x}_{1,k-1}, \hat{x}_{2,k-1}}$$

$$A_k = \begin{bmatrix} 1 & \cos \hat{x}_{2,k-1} \\ 2\hat{x}_{1,k-1} & 0 \end{bmatrix}$$

$\therefore A_k$ 随 k 变化
即每一步都要重新计算 A_k

测量过程线性化

z_k 在 $\tilde{x}_k$ 线性化

误差假设0 $\Rightarrow h(\tilde{x}_k, 0) = \tilde{z}_k$

$$z_k = h(\tilde{x}_k, \cancel{v_k}\ 0) + H(x_k - \tilde{x}_k) + V \cdot v_k$$

$$H = \frac{\partial h}{\partial x}\Big|_{\tilde{x}_k}$$

$$V = \frac{\partial h}{\partial v}\Big|_{\tilde{x}_k}$$

$\therefore$

$$\boxed{\begin{aligned} x_k &= \tilde{x}_k + A(x_{k-1} - \hat{x}_{k-1}) + W \cdot w_{k-1} \\ z_k &= \tilde{z}_k + H(x_k - \tilde{x}_k) + V \cdot v_{k-1} \end{aligned}}$$

$P(w) \sim N(0, Q)$　　$P(Ww) \sim N(0, WQW^T)$

$P(v) \sim N(0, R)$　　$P(Vv) \sim N(0, VRV^T)$

$\Downarrow$

线性

预测

先验 $x_k^- = A\hat{x}_{k-1}$

$P_k^- = AP_{k-1}A^T + Q$

校正

$K_k = \frac{P_k^- H^T}{HP_k^- H^T + R}$

后验 $\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

$P_k = (I - K_k H)P_k^-$

非线性

预测

先验 $x_k^- = f(\hat{x}_{k-1}, u_{k-1}, 0)$

$P_k^- = AP_{k-1}A^T + WQW^T$

校正

$K_k = \frac{P_k^- H^T}{HP_k^- H^T + VRV^T}$

后验 $\hat{x}_k = \hat{x}_k^- + K_k(z_k - h(\hat{x}_k^-, 0))$

$P_k = (I - K_k H)P_k^-$