growth functions

how to find a good algorithm?

1. Precisely define the input

2. Precisely define the output

3. Choose a suitable data structure to store the input and intermediate data

4. Start with a brute-force (but correct) algorithm that obtains the output from the input in not-so-efficient way

5. Analyze the time and space complexity of the algorithm, also analyze the **time-space tradeoff** and identify which of the resources (time or space) is more valuable to you

6. Try to update the not-so-efficient algorithm either by adopting a suitable data structure (in step 3) and/or by using a suitable **algorithm design pattern** (in step 4)

7. Repeat step 3, 4, and 5 until you are satisfied with the efficiency of the algorithm

asymptotic behavior

- Generally, $T(n)$ and $S(n)$ are monotonic, non-decreasing functions
- We are interested in the behavior of $T(n)$ (or $S(n)$) for large input sizes, i.e., $n \rightarrow \infty$
- Focus on what's important, by ignoring lower-order terms and constants in the growth function
- Different notions of analyzing the asymptotic behavior:

$\Theta \approx =$

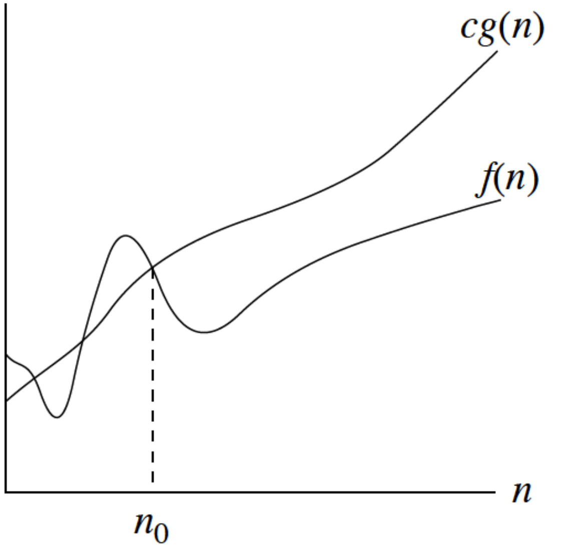
$O \approx \leq$

$o \approx <$

$\Omega \approx \geq$

$\omega \approx >$

O-notation



- $O(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that
  $0 \le f(n) \le cg(n)$ for all $n \ge n_0$ $\}$

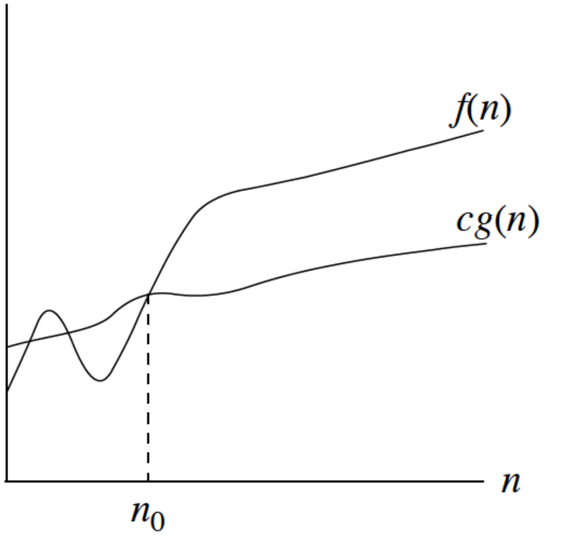
- g(n) is an <u>asymptotic **upper** bound</u> for f(n)

Two examples:

1. $T(n) = (3/2)n^2 + (5/2)n - 3$
   It grows similarly to n^2: $O(n^2)$

2. Show that for $a > 0$ and $b > 0$, any linear function $an+b = O(n^2)$
   $0 \le f(n) \le cg(n) \Rightarrow 0 \le an+b \le cn^2 \Rightarrow 0/n^2 \le an/n^2+b/n^2 \le cn^2/n^2 \Rightarrow$
   $0 \le a/n + b/n^2 \le c \Rightarrow \{ 0 \le a + b \le c$, for $n_0=1$ and $c=a+b$ $\}$

Ω-notation



- $\Omega(g(n))$ = { f(n): there exist positive constants c and n_0 such that
  $0 \le cg(n) \le f(n)$ for all $n \ge n_0$ }

- g(n) is an asymptotic **lower** bound for f(n)

Example:

Show that $2n^2 + n$ is in $\Omega(n^2)$ (find c and n_0 that satisfy the definition)

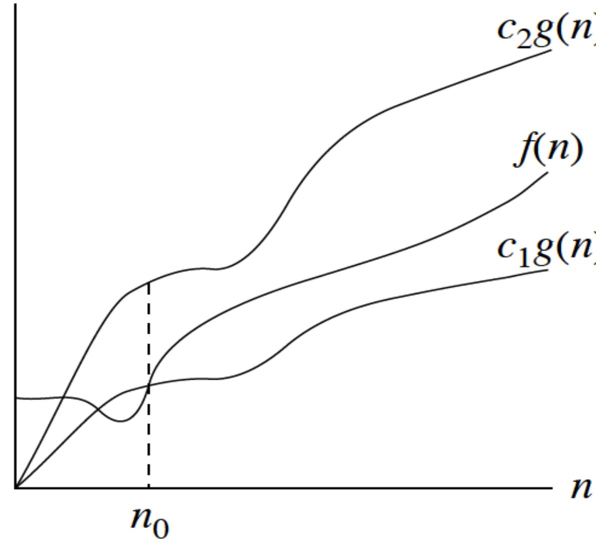
$0 \le cg(n) \le f(n) \Rightarrow 0 \le cn^2 \le 2n^2 + n \Rightarrow 0/n^2 \le cn^2/n^2 \le 2n^2/n^2 + n/n^2 \Rightarrow$

$0 \le c \le 2 + 1/n \Rightarrow 0 \le c \le 2$ (1)

$0 \le c \le 2 + 1/n_0 \Rightarrow -2 \le c-2 \le 1/n_0$ (2)

(1)^(2) => { c = 2 and $n_0 = 1$ }

Θ-notation



- $\Theta(g(n))$ = { f(n): there exist positive constants c_1, c_2, and n_0 such that
  $0 \le c_1g(n) \le f(n) \le c_2g(n)$ for all $n \ge n_0$ }

- g(n) is an <u>asymptotic **tight** bound</u> for f(n)

Example:

Consider $f(n) = an^2 + bn + c$. For $a > 0$, show that $f(n) = \Theta(n^2)$

$0 \le c_1g(n) \le f(n) \le c_2g(n) \Rightarrow c_1n^2 \le an^2 + bn + c \le c_2n^2 \Rightarrow$

$c_1n^2/n^2 \le an^2/n^2 + bn/n^2 + c/n^2 \le c_2n^2/n^2 \Rightarrow c_1 \le a + b/n + c/n^2 \le c_2$

Remember asymptotic behavior: as $n \rightarrow \infty \Rightarrow b/n \rightarrow 0$ and $c/n^2 \rightarrow 0$

– $a + b/n + c/n^2$ is max for $n=n_0=1$

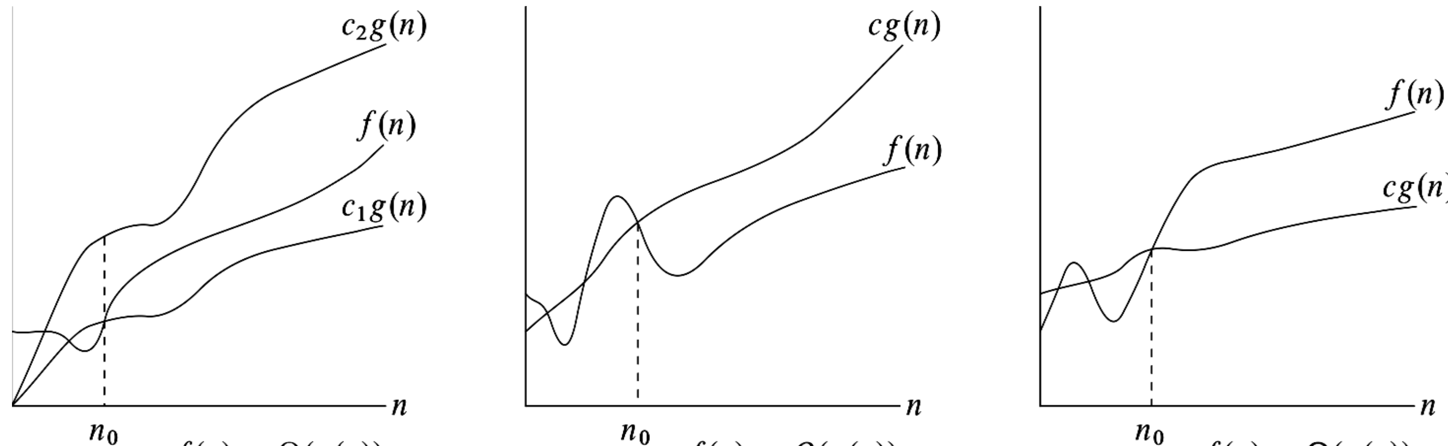
– $c_1 \le a + b/n + c/n^2 \Rightarrow c_1 = a$ and $a + b/n + c/n^2 \le c_2 \Rightarrow c_2 = a + b + c$

o- and ω-notation

- $o(g(n)) = \{ f(n)$: for all $c > 0$ there exists a constant $n_0 > 0$ such that
  $0 \leq f(n) < cg(n)$ for all $n \geq n_0 \}$

- $\omega(g(n)) = \{ f(n)$: for all $c > 0$ there exists a constant $n_0 > 0$ such that
  $0 \leq cg(n) < f(n)$ for all $n \geq n_0 \}$

summary



$f(n) = \Theta(g(n))$ (a) $f(n) = O(g(n))$ (b) $f(n) = \Omega(g(n))$ (c)

- $o\big(g(n)\big) = \{f(n) \mid \forall c > 0, \exists n_0 > 0 \; such\ that\ 0 \leq f(n) < cg(n), \forall n > n_0\}$
- Alternative definition: $o\big(g(n)\big) = \left\{f(n): \lim\limits_{n\to\infty} \frac{f(n)}{g(n)}\right\} = 0$, i.e., $f(n)$ becomes insignificant with respect to $g(n)$ for large n
- $\omega(g(n) = \{f(n) \mid \forall c > 0, \exists n_0 > 0, such\ that\ 0 \leq cg(n) < f(n), \forall n > n_0\}$
- Alternative definition: $o\big(g(n)\big) = \left\{f(n): \lim\limits_{n\to\infty} \frac{f(n)}{g(n)}\right\} = \infty$, i.e., $g(n)$ becomes insignificant with respect to $f(n)$ for large n

a schema to remember

$$\mathit{lim}_{n \to \infty} f(n)/g(n)$$

= 0	**= c > 0**	**= ∞**
$f(n) = O(g(n))$	$f(n) = \Theta(g(n))$	$f(n) = \Omega(g(n))$
$f(n) = o(g(n))$	$f(n) = O(g(n))$	$f(n) = \omega(g(n))$
	$f(n) = \Omega(g(n))$	

notations in equations

- $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$

  It is $2n^2 + 3n + 1 = 2n^2 + f(n)$, for some function $f(n)$ in the family of $\Theta(n)$

- $2n^2 + \Theta(n) = \Theta(n^2)$

  It is $2n^2 + f(n) = g(n)$ for some function $f(n)$ in the family of $\Theta(n)$ and some function $g(n)$ in the family of $\Theta(n^2)$

properties

$lim_{n \to \infty}$ f(n)/g(n)		
= 0	= c > 0	= ∞
f(n) = O(g(n))	f(n) = Θ(g(n))	f(n) = Ω(g(n))
f(n) = o(g(n))	f(n) = O(g(n))	f(n) = ω(g(n))
	f(n) = Ω(g(n))	

Transitivity:

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$.

Same for O, Ω, o, and ω.

Reflexivity:

$f(n) = \Theta(f(n))$.

Same for O and Ω.

Symmetry:

$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

Transpose symmetry:

$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

$f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

- $f(n)$ is ***asymptotically smaller*** than $g(n)$ if $f(n) = o(g(n))$.
- $f(n)$ is ***asymptotically larger*** than $g(n)$ if $f(n) = \omega(g(n))$.

	$lim_{n\to\infty} f(n)/g(n)$	
= 0	**= c > 0**	**= ∞**
$f(n) = O(g(n))$	$f(n) = \Theta(g(n))$	$f(n) = \Omega(g(n))$
$f(n) = o(g(n))$	$f(n) = O(g(n))$	$f(n) = \omega(g(n))$
	$f(n) = \Omega(g(n))$	

transitivity same for o and ω

- **$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$**

$lim_{n\to\infty} f(n)/g(n) = c_1 > 0$

$lim_{n\to\infty} g(n)/h(n) = c_2 > 0$

$lim_{n\to\infty} [f(n)/g(n)]\,[g(n)/h(n)] = c_1c_2 > 0 \Rightarrow lim_{n\to\infty} f(n)/h(n) = c > 0 \Rightarrow f(n) = \Theta(h(n))$

- **$f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$**

$lim_{n\to\infty} f(n)/g(n) = c_1 > 0$ or $=0$

$lim_{n\to\infty} g(n)/h(n) = c_2 > 0$ or $=0$

$lim_{n\to\infty} [f(n)/g(n)]\,[g(n)/h(n)] = c_1c_2 > 0$ or $=0 \Rightarrow lim_{n\to\infty} f(n)/h(n) = c > 0$ or $0 \Rightarrow f(n) = O(h(n))$

- **$f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$, then $f(n) = \Omega(h(n))$**

$lim_{n\to\infty} f(n)/g(n) = c_1 > 0$ or $=\infty$

$lim_{n\to\infty} g(n)/h(n) = c_2 > 0$ or $=\infty$

$lim_{n\to\infty} [f(n)/g(n)]\,[g(n)/h(n)] = c_1c_2 > 0$ or $\infty \Rightarrow lim_{n\to\infty} f(n)/h(n) = c > 0$ or $\infty \Rightarrow f(n) = \Omega(h(n))$

$\lim_{n \to \infty} f(n)/g(n)$		
= 0	**= c > 0**	**= ∞**
$f(n) = O(g(n))$	$f(n) = \Theta(g(n))$	$f(n) = \Omega(g(n))$
$f(n) = o(g(n))$	$f(n) = O(g(n))$	$f(n) = \omega(g(n))$
	$f(n) = \Omega(g(n))$	

reflexivity **same for O and Ω**

- **$f(n) = \Theta(f(n))$**

$\lim_{n \to \infty} f(n)/f(n) = c > 0$

$\lim_{n \to \infty} f(n)/f(n) = c = 1 > 0$

symmetry **for Θ only**

- **$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$**

$\lim_{n \to \infty} f(n)/g(n) = c_1 > 0$

$\lim_{n \to \infty} g(n)/f(n) = c_2 > 0$

$\lim_{n \to \infty} [f(n)/g(n)]\, [g(n)/f(n)] = c_1 c_2 = 1 > 0$

transpose symmetry

- **$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$**

$\lim_{n \to \infty} f(n)/g(n) = c > 0$ or $=0$ => $\lim_{n \to \infty} g(n)/f(n) = 1/c_1 > 0$ or $=\infty$ => $g(n) = \Omega(f(n))$

- **$f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$**

$\lim_{n \to \infty} f(n)/g(n) = 0$ => $\lim_{n \to \infty} g(n)/f(n) = \infty$ => $g(n) = \omega(f(n))$

example#1

Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions.

Prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$

example#1

Let f(n) and g(n) be asymptotically nonnegative functions. Prove that max(f(n), g(n)) = Θ(f(n) + g(n))

$$\text{Define } h(n) = \begin{cases} f(n), & \text{if } f(n) \geq g(n) \\ g(n), & \text{if } f(n) < g(n) \end{cases}$$

f(n) and g(n) are asymptotically nonnegative =>

there exists n_0 such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$ =>

$$\left.\begin{array}{l} \text{for all } n \geq n_0,\ f(n) + g(n) \geq 0 \\ \\ \text{For a particular } n\ h(n) \text{ is either } f(n) \text{ or } g(n) \end{array}\right\} \mathbf{f(n) + g(n) \geq h(n) \geq 0}$$

For any particular n, h(n) is the larger of f(n) and g(n) =>

for all $n \geq n_0$ it is $0 \leq f(n) \leq h(n)$ and $0 \leq g(n) \leq h(n)$ =>

for all $n \geq n_0$ it is $0 \leq f(n) + g(n) \leq 2h(n)$ => $\mathbf{0 \leq (f(n) + g(n))/2 \leq h(n)}$

That is: for all $n \geq n_0$ it is

$\mathbf{0 \leq (f(n) + g(n))/2 \leq h(n) \leq f(n) + g(n)}$, and in the definition it is $c_1=1/2$ and $c_2=1$

example#2

Show that for any real constants a and b, with $b > 0$, $(n+a)^b = \Theta(n^b)$

We want to find constants c_1, c_2 and $n_0 > 0$, such that
$0 \le c_1 n^b \le (n+a)^b \le c_2 n^b$, for all $n \ge n_0$.

$n + a \le n + |a| \le 2n$, for $|a| \le n$
$n + a \ge n - |a| \ge n/2$, for $|a| \le n/2$

When $n \ge 2|a|$:

$0 \le n/2 \le n + a \le 2n \Rightarrow_{b>0} 0 \le (n/2)^b \le (n + a)^b \le (2n)^b \Rightarrow$

$0 \le (1/2)^b n^b \le (n + a)^b \le 2^b n^b \Rightarrow c_1 = (1/2)^b$, $c_2 = 2^b$, and $n_0 = 2|a|$

example#3

Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

To show $2^{n+1} = O(2^n)$ we must find constants c and $n_0 > 0$ such that

$0 \le 2^{n+1} \le c2^n$ for all $n \ge n_0$

$2^{n+1} = 2.2^n$ for all $n \ge n_0$ => we satisfy the definition for c=2 and $n_0=1$

Similarly for the second case:

$0 \le 2^{2n} \le c2^n$ for all $n \ge n_0$

$2^{2n} = 2^n 2^n \le c2^n \Rightarrow 2^n \le c$

There is no constant greater than all 2^n and so $2^{2n} = O(2^n)$ <u>does not hold</u>

more examples

- **Prove that $2n^2$ in $O(n^3)$**

Assume $f(n) = 2n^2$ and $g(n) = n^3$

We need to find c and $n_0 > 0$ that satisfy the definition:

$0 \leq f(n) \leq cg(n) \Rightarrow 0 \leq 2n^2 \leq cn^3 \Rightarrow_{(n \geq 1)} 0 \leq 2 \leq cn \Rightarrow 2 \leq cn \Rightarrow$ satisfied for e.g., $c=1$ and $n_0=2$

- **Prove that n^2 in $O(n^2)$**

Assume $f(n) = n^2$ and $g(n) = n^2$

We need to find c and $n_0 > 0$ that satisfy the definition:

$0 \leq f(n) \leq cg(n) \Rightarrow 0 \leq n^2 \leq cn^2 \Rightarrow_{(n \geq 1)} 0 \leq 1 \leq c \Rightarrow 1 \leq c \Rightarrow$ satisfied for e.g., $c=1$ and $n_0 \geq 1$

- **Prove that $1000n^2+1000n$ in $O(n^2)$**

Assume $f(n) = 1000n^2+1000n$ and $g(n) = n^2$

We need to find c and $n_0 > 0$ that satisfy the definition:

$0 \leq f(n) \leq cg(n) \Rightarrow 0 \leq 1000n^2+1000n \leq cn^2 \Rightarrow_{(n\geq1)} 1000+1000/n \leq c$

As $n \rightarrow \infty$, $1000+1000/n \rightarrow 1000$ (and the max value is at n=1)

For $n_0 = 1000$ and $c = 1001$, the definition holds

- **Prove that $5n^2$ in $\Omega(n)$**

Assume $f(n) = 5n^2$ and $g(n) = n$

We need to find c and $n_0 > 0$ that satisfy the definition:

$0 \leq cg(n) \leq f(n) \Rightarrow 0 \leq cn \leq 5n^2 \Rightarrow_{(n\geq1)} c \leq 5n$

It is satisfied for c=5 and $n_0=1$

- **Prove that $(1/2)n^2 - (1/2)n = \Theta(n^2)$**

Assume $f(n) = (1/2)n^2 - (1/2)n$ and $g(n) = n^2$

We need to find c_1, c_2 and $n_0 > 0$ that satisfy the definition:

$c_1 g(n) \le f(n) \le c_2 g(n) \Rightarrow c_1 n^2 \le (1/2)n^2 - (1/2)n \le c_2 n^2 \Rightarrow_{(n \ge 1)}$

$c_1 \le 1/2 - 1/(2n) \le c_2$

$1/2 - 1/(2n) \le c_2$ satisfied for $c_2 = ½$

$c_1 \le 1/2 - 1/(2n)$, and (so far) $c_1 > 0$, $n \ge 1$

For $n_0=1$ it becomes $c_1 \le 0$: not what we want

The condition is satisfied for $n_0=2$ and $c_1=1/4$

- **Prove that n^2 NOT in $o(n^2)$**

Assume $f(n) = n^2$ and $g(n) = n^2$

For <u>any</u> $c > 0$ we need to show that

$f(n) < cg(n) \Rightarrow n^2 < cn^2 \Rightarrow_{(n \ge 1)} 1 < c \Rightarrow$ not *for any c*

- **Prove that $5n^2$ in ω(n)**

Assume $f(n) = 5n^2$ and $g(n) = n$
For <u>any</u> $c > 0$ we need to show that
$cg(n) < f(n) \Rightarrow cn < 5n^2 \Rightarrow_{(n \geq 1)} c \leq 5n$ which is satisfied for any $c > 0$
Why? For any $c > 0$ we choose, we can choose an n_0 that satisfies
$c \leq 5n_,$ for $n \geq n_0$

- **Prove that 5n+10 NOT in ω(n)**

Assume $f(n) = 5n+10$ and $g(n) = n$
For <u>any</u> $c > 0$ we need to show that
$cg(n) < f(n) \Rightarrow cn < 5n+10 \Rightarrow (c-5)n \leq 10 \Rightarrow n \leq 10/(c-5)$
n must be a positive integer => the condition does not hold in general