Cette fois-ci c'est la bonne !

X et Y iid $\hookrightarrow \mathcal{N}(0,1)$

Lois marginales :
densités :

$X(\Omega)=Y(\Omega)=\mathbb{R}$

$$f_X : \begin{array}{l}\mathbb{R}\to\mathbb{R}\\ x\mapsto \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\end{array} \qquad f_Y : \begin{array}{l}\mathbb{R}\to\mathbb{R}\\ y\mapsto \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\end{array}$$

Calcul de $\mathbb{E}(|X-Y|)$

Loi conjointe du couple (X,Y) densité $f_{X,Y}$ $\qquad (X,Y)(\Omega)=\mathbb{R}^2$

X et Y sont indépendantes donc
$\forall (x,y)\in\mathbb{R}^2 \quad f_{X,Y}(x,y) = f_X(x)\,f_Y(x)$ (vrai mais H.P. pour le CAPES !)

$$\mathbb{E}(|X-Y|) = \iint_{\mathbb{R}^2} |x-y|\, f_{X,Y}(x,y)\,dx\,dy = \frac{1}{2\pi}\iint_{\mathbb{R}^2} |x-y|\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy$$

$$= \underbrace{\frac{1}{2\pi}\iint_{\substack{x\geq y\\(x,y)\in\mathbb{R}^2}} (x-y)\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy}_{①} + \underbrace{\frac{1}{2\pi}\iint_{\substack{x\leq y\\(x,y)\in\mathbb{R}^2}} (y-x)\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy}_{②}$$

Par un double changement de variable dans ②, on montre que ② = ①

(en posant $\begin{cases} u=-x\\ v=-y\end{cases}$ on a $dx\,dy = du\,dv$, $(y-x) = -v-(-u) = (u-v)$, $e^{-x^2/2}e^{-y^2/2} = e^{-u^2/2}e^{-v^2/2}$)

$$\text{donc } \mathbb{E}(|X-Y|) = \frac{1}{\pi}\iint_{\substack{x\geq y\\(x,y)\in\mathbb{R}^2}} (x-y)\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy = \underbrace{\frac{1}{\pi}\iint_{x\geq y} x\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy}_{③} \underbrace{- \frac{1}{\pi}\iint_{x\geq y} y\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy}_{④}$$

$$③ = \frac{1}{\pi}\int_{y=-\infty}^{+\infty} e^{-y^2/2}\left(\int_{x=y}^{+\infty} x\, e^{-x^2/2}\,dx\right)dy = \frac{1}{\pi}\int_{y=-\infty}^{+\infty} e^{-y^2/2}\left[-e^{-x^2/2}\right]_y^{+\infty} dy$$

$\lim_{x\to+\infty} e^{-x^2/2} = 0$ dc CV.

$$③ = \frac{1}{\pi}\int_{y=-\infty}^{+\infty} e^{-y^2/2}\left(-0+e^{-y^2/2}\right)dy = \frac{1}{\pi}\int_{y=-\infty}^{+\infty} e^{-y^2}\,dy = \frac{1}{\pi}\int_{u=-\infty}^{+\infty} e^{-u^2/2}\,\frac{du}{2}$$

(chg^t variable : $u=2y$)

$$③ = \frac{1}{\sqrt{2\pi}} \times \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-u^2/2}\,du}_{=1 \quad \mathcal{N}(0,1)} = \frac{1}{\sqrt{2\pi}}$$

$$④ = -\frac{1}{\pi}\iint_{y\leq x} y\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy = -\frac{1}{\pi}\int_{x=-\infty}^{+\infty} e^{-x^2/2}\left(\int_{y=-\infty}^{x} y\, e^{-y^2/2}\,dy\right)dx = -\frac{1}{\pi}\int_{x=-\infty}^{+\infty} e^{-x^2/2}\left[-e^{-y^2/2}\right]_{-\infty}^{x} dy$$

$\lim_{y\to-\infty} e^{-y^2/2} = 0$ dc CV

$$④ = -\frac{1}{\pi}\int_{x=-\infty}^{+\infty} e^{-x^2/2}\left(-e^{-x^2/2}+0\right)dx = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-x^2}\,dx = ③ = \frac{1}{\sqrt{2\pi}}$$

donc au final $\mathbb{E}(|X-Y|) = \dfrac{2}{\sqrt{2\pi}} = \sqrt{\dfrac{2}{\pi}}$ ^^