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Distributed decision-making algorithms with multiple manipulative actors[☆]

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ABSTRACT

Distributed decision-making in the presence of multiple manipulative actors is studied, in the context of a linear distributed-consensus algorithm which has been enhanced to represent feedback controls enacted by the actors. The main contribution of the work is to evaluate the interplay among the manipulative actors in deciding: (1) the asymptotic decisions reached by the agents and (2) network's transient dynamics. In particular, the dependence of the asymptotic opinions on the network's topology and the manipulative actors' control schemes is characterized. Then, the impacts of the actors' control schemes on the intrinsic settling dynamics of the network, as well on each others' transfer functions, is studied. The analyses show that the asymptotics and transients both show a complex spatial dependence on the relative locations of manipulated agents within the network. Two examples are used to illustrate the results.

1. Introduction

Distributed decision-making algorithms are widely implemented in engineered systems such as computer and sensor networks (Lynch, 1996; Pratt & Nguyen, 1995), and also are descriptive of myriad interactive processes that occur in human groups (Holley & Liggett, 1975). With growing cyber-enablement and networking, the use of distributed decision-making algorithms is becoming increasingly common.

This growing interest in distributed decision-making has motivated a research effort on consensus and opinion dynamics in networks in the controls community (Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005; Blondel, Hendrickx, & Tsitsiklis, 2010; Mesbahi & Egerstedt, 2010; Olfati-Saber, Fax, & Murray, 2007; Proskurnikov, Matveev, & Cao, 2016; Proskurnikov & Tempo, 2017; Ren & Beard, 2005). One theme of this research has been to understand the influence of selfish stakeholders or actors, which aim to manipulate the decision process via their local actions (Ghaderi & Srikant, 2013; Parsegov, Proskurnikov, Tempo, & Friedkin, 2017; Pirani & Sundaram, 2014; Sundaram & Hadjicostis, 2011). Manipulation has been modeled in several ways, including using stubborn agents which are unresponsive to neighbors, as feedback controls (Chen, Chen, Xiang, Liu, & Yuan, 2009), or as open-loop actuations (Dhal & Roy, 2013; Ghaderi & Srikant, 2014; Mesbahi & Egerstedt, 2010; Pasqualetti, Zampieri, & Bullo, 2014).

Decision-making in many settings, such as voting in human groups and market clearing processes, may involve multiple selfish actors which seek to manipulate network opinions toward divergent goals. Based on this motivation, a few recent studies have considered distributed decision-making algorithms with multiple stubborn agents that have different opinions (Ghaderi & Srikant, 2014). In a related direction, some studies have also considered algorithms that lead to group consensus, with different sets of agents being guided to different decisions or consensus values (Sundaram & Hadjicostis, 2011; Yu & Wang, 2010).

This study is concerned with distributed decision-making algorithms involving multiple selfish or manipulative actors with divergent goals. The focus of the work is to examine how the manipulative actors interact in the decision-making process. Specifically, a linear distributed-consensus algorithm defined on a network is considered (Olfati-Saber et al., 2007; Ren & Beard, 2005; Xiao, Boyd, & Kim, 2007), and manipulative actors are broadly modeled as enacting feedback controls at one or a small set of agents. Our main goal is to understand how the control capabilities of the different manipulative actors interact in deciding the outcome of the decision-making algorithm, and hence how the actions of some manipulative actors modulate the capabilities of others. The following specific contributions are made:

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(1) The influences of the manipulative actors on the asymptotic decision reached by each agent are examined, and shown to exhibit a structured dependence on the control gains and on spatial proximity in the network.

(2) The impacts of the manipulative actors' control gains on the settling rate and on each others' control channels are characterized, as a means to understand how actors may alter algorithm performance and limit other actors' influence.

While we motivate the technical analyses from the perspective of opinion dynamics with manipulative actors, the analyses abstractly apply to synchronization/consensus processes with multiple imposed feedback channels, which have different reference signals and gains. Our results translate to other settings where such models are of interest, such as in containment control in leader–follower networks (Li, Xie, & Zhang, 2015; Liu, Xie, & Wang, 2012), and analysis of vehicle networks (Ji, Ferrari-Trecate, Egerstedt, & Buffa, 2008).

The paper is organized as follows. The model and problem formulation are developed in Section 2. The asymptotic decisions reached by the agents are characterized in Section 3, while the network's dynamics are studied in Section 4. Two examples are given in Section 5.

Preliminary results in this direction were presented in the conference paper Koorehdavoudi, Roy and Xue (2018). Relative to Koorehdavoudi, Roy, Xue et al. (2018), this article considers a more general model (with local and targeting actors), examines the approach to steady state, considers dynamic compensation, includes proofs, and gives a broader motivation for the work.

2. Problem formulation

A distributed decision-making algorithm specifies rules by which *agents* in a network update their *opinions* via local interactions, with the goal of reaching a common opinion or enabling an action (e.g. voting for a leader) after some time (Olfati-Saber & Murray, 2004; Roy, Herlugson, & Saberi, 2006). Here, we consider a standard continuous-time algorithm (termed a distributed consensus algorithm (Olfati-Saber et al., 2007; Ren & Beard, 2005; Xiao et al., 2007)), wherein agents update continuous-valued opinions to equilibrate with graphical neighbors. Here, the model is enhanced to capture manipulation by multiple actors who seek to achieve varying decision values.

Formally, a network with n agents, specified by the set $\mathcal{N} = \{1, \dots, n\}$, is considered. Each agent i has an opinion $x_i(t)$ which evolves in continuous time ($t \in \mathbb{R}^+$). The interactions among the agents are defined by a weighted digraph $\Gamma = (V, E : W)$, where the weights W are assumed to be positive. We assume throughout the article that Γ is strongly connected. In addition, we model m independent manipulative *actors*, specified by the set $\mathcal{M} = \{1, \dots, m\}$. The manipulative actors are viewed as being able to access (actuate and measure) subsets or projections of the agents' opinions. Each actor then applies a linear feedback control, with the goal of guiding the agents' opinions to a desired opinion or reference trajectory.

The full dynamics of the decision-making algorithm with manipulative actors is given by:

$$\begin{aligned}
\dot{\mathbf{x}} &= -L(\Gamma)\mathbf{x} + \sum_{i=1}^{m} B_i \mathbf{u}_i, \qquad (1)\\
\mathbf{y}_i &= C_i \mathbf{x} \quad i = 1, \dots, m\\
\dot{\mathbf{z}}_i &= Q_i \mathbf{z}_i + R_i \mathbf{r}_i(t) + S_i \mathbf{y}_i \quad i = 1, \dots, m\\
\mathbf{u_i} &= T_i \mathbf{z_i}. \quad i = 1, \dots, m
\end{aligned}$$

Here, $\mathbf{x} = [x_1, \dots, x_n]'$ is the opinion vector or state vector of the model. Also, $L(\Gamma)$ is the (asymmetric) Laplacian matrix associated with the directed graph Γ. Specifically, $L(\Gamma)$ is defined as follows: each off-diagonal entry L_{ij} is set equal to the negative of the weight from vertex j to vertex i in the graph Γ if there is an edge (and to zero otherwise); the diagonal entries are selected so that each row sums to zero. Meanwhile, $\mathbf{u}_i$ is the (vector) actuation signal provided by the actor i, and the input matrix B_i specifies how the actor i actuates the agents. Next, $\mathbf{y}_i$ are the opinion measurements made by actor i, and C_i specifies the projections of the state which are measured. The final two equations in (1) specify the linear state–space feedback applied by Actor i. Here, $\mathbf{z}_i(t)$ is the actor's internal state, $\mathbf{r}_i(t)$ is the reference signal vector for the actor, and Q_i, R_i, and S_i are controller gains. We refer to the dynamics as a whole as the *distributed-decision-making algorithm with manipulative actors*.

In this study, we focus on two specific models for manipulative actors that fall within this framework, which capture common types of manipulative behaviors in distributed decision-making algorithms. Specifically, we model *local actors* which apply local feedback at a subset of the agents (i.e., the agent's feedback input depends only on its own opinion and the reference signal), with the goal of driving network opinions to a fixed reference. Conceptually, the local-actor model captures manipulators that influence individual agents toward their preferred opinion, based on an understanding of their local opinions. We also represent targeting actors, which each seek to remotely target the opinion of one measured agent, by applying a feedback to another (actuated) agent in the network. The targeting-actor model captures manipulators that have the ability to directly influence one agent, and seek to use this ability to indirectly manipulate a different agent.

Formally, a model with a mixture of local and targeting actors is considered. Specifically, the full set of manipulative actors $\mathcal{M}$ is partitioned into a set $\mathcal{M}_1$ containing m_1 independently-acting local actors, and a set $\mathcal{M}_2$ containing m_2 independently-acting targeting actors. Each actor j ($j \in \mathcal{M}_1$) seeks to manipulate the agents in the set $\mathcal{N}_j$ where $j \in \mathcal{M}$ via local feedback. Meanwhile, each targeting actor $j \in \mathcal{M}_2$ can apply an input to one actuated agent $a(j)$, with the goal of manipulating a second measured/targeted agent $b(j)$.

The distributed decision-making dynamics with local and targeting actors is:

$$\begin{aligned}
\dot{\mathbf{x}} &= -L(\Gamma)\mathbf{x} + \sum_{i \in \mathcal{N}} \mathbf{e}_i (u_i + v_i) \qquad (2)\\
u_i &= \sum_{j \in \mathcal{M}_1} k_{ij}(\bar{r}_j - \mathbf{e}_i'\mathbf{x}) \quad \forall i \in \mathcal{N},\\
v_i &= \sum_{j \in \mathcal{M}_2 \, s.t. \, a(j)=i} l_j(\bar{r}_j - \mathbf{e}_{b(j)}'\mathbf{x}) \quad \forall i \in \mathcal{N}.
\end{aligned}$$

Here, u_i is the total actuation provided to agent i by the local actors, and v_i is the total actuation provided to agent i by the targeting actors. Also, $r_j(t) = \bar{r}_j$ is the fixed reference (opinion goal) of actor j, k_{ij} is control gain applied by local actor j at agent i if $i \in \mathcal{N}_j$ and is zero otherwise, l_j is the control gain applied by targeting actor j, and $\mathbf{e}_q$ is a 0–1 indicator vector whose qth entry is unity.

We notice that, in our formulation, the feedback provided by the actors is modeled as a static linear (proportional) feedback. We also generalize the model (2) to represent linear dynamics in the actors' influences on agents (e.g., sluggishness in the feedback actuation, or predictive controls), by replacing the static feedback gain with a transfer function. In this case, the control applied by the local actors at each agent is given in the Laplace domain by $U_i(s) = \sum_{j \in \mathcal{M}_1} H_{ij}(s)(\bar{r}_j - \mathbf{e}_i'\mathbf{x}(s))$, where $H_{ij}(s)$ represents the controller transfer function for the feedback applied by actor j at agent i. Similarly, the control applied by the targeting actors at each agent is given in the Laplace domain by $V_i(s) = \sum_{j\in\mathcal{M}_2\, s.t.\, a(j)=i} H_j(s)(\bar{r}_j - \mathbf{e}'_{b(j)}\mathbf{x}(s))$, where $H_j(s)$ is the transfer function of the controller applied by targeting actor j. A specialization of the dynamic actors model, in which lead-compensation strategies are applied by the actors, is also considered in this study.

Our main goal in this work is to analyze how the multiple manipulative actors interact. The control enacted by each actor modifies the evolution of the distributed decision-making algorithm, and hence modulates the ability of other actors to manipulate the dynamics. Our aim is to analyze these dependencies, focusing particularly on their impact on (1) the asymptotic opinions achieved by the agents (Section 3), and (2) the network's transient dynamics, including global settling properties and control-channel transfer functions (Section 4).

Conceptual Example: A conceptual example is briefly developed to illustrate the problem formulation. Consider advertisers (actors) trying to sell two competing products 0 and 1 to people in a city (agents). Each person in the city maintains an opinion between 0 and 1 which indicates their proclivity for buying each product. People gradually change their opinions based on the opinions of their friends, but with different influences (weights). On the other hand, the advertisers have influence on some subgroups within the population. This problem can be modeled as distributed-decision-making algorithm with manipulative actors. Consider the following specific models for manipulation: (1) The advertisers distribute individuals in certain populations (e.g., among 18–20 year olds at colleges to sell sneakers), who try to convince their contacts to buy one of the products. In this case, the advertiser serves as a local actor which measures/actuates a subpopulation. (2) Alternately, consider the case that an advertiser wants to target a specific audience (e.g., convince an office worker to buy a certain pen), but has influence over another individual who then can convince the target audience (e.g. a stay-at-home spouse who watches television advertisements about the pen). Then the actor is a targeting actor.

3. Analysis of asymptotic opinions

We study how the interactions among the manipulative actors decide the asymptotic opinions achieved by the agents. If the network had no manipulative actors, the agents would asymptotically achieve a common opinion or *decision* provided that the network graph is strongly connected (e.g. Olfati-Saber et al. (2007)). However, the presence of multiple manipulative actors may either prevent the formation of a steady state, or yield a steady-state with a gradation of opinions. Here, we seek to understand how the controls enacted by the different actors impact the asymptote, and thus to understand how one actors' efforts constrain the influence of the other actors. Some algebraic results on the asymptotic opinions are developed first, and then used to obtain the main graph-theoretic results.

Our development depends on the eigenanalysis of what we call *diffusive matrices*. For our purposes here, a diffusive matrix is an irreducible non-singular M-matrix. A diffusive matrix can also be viewed as a grounded Laplacian matrix, whose associated digraph is strongly connected. A number of properties of diffusive matrices and their inverses are used in our analysis, see e.g. Berman and Plemmons (1994) for a compendium of properties.

3.1. Algebraic results

We analyze the distributed decision-making algorithm with local and targeting actors (2) under the following assumptions: (1) all local actors' control gains k_{ij} are nonnegative, reflecting the typical circumstance that the actors apply negative feedback to draw agents toward their desired reference signal; (2) all targeting actors' control gains l_j satisfy $0 \le l_j < -L_{a(j)b(j)}$, reflecting that the targeting actor applies a negative feedback which is no larger than the native influence between the actuated and targeted agent; (3) one of the control gains l_j or k_{ij} is positive, reflecting that there is at least one actor that is actively engaged in manipulation. Under these assumptions, the state matrix of the controlled system dynamics is in the form of the negative of a diffusive matrix. From properties of diffusive matrices, the closed-loop system is thus asymptotically stable in the sense of Lyapunov. Since the reference inputs are constant signals, the opinion of each agent reaches a steady-state that is independent of the initial opinions of the agents. Further, from linearity, the steady-state opinion of each agent is a linear combination of the references of all local and targeting actors:

$$\bar{x}_i = \sum_{j\in\mathcal{M}} \lambda_{ij}\bar{r}_j, \tag{3}$$

where $\bar{x}_i$ is the asymptotic value for the opinion of the agent i ($x_i(t)$), i.e. $\lim_{t\to\infty} x_i(t) = \bar{x}_i$. The weightings λ_{ij}, which we formally refer to as the *contributions* of each actor j to the asymptotic opinion of each agent i, delineate how the actors influence the agents' asymptotic opinions. Our goal is to characterize these contributions. To present these characterizations, we also find it convenient to denote $-L_{ij}$ as the *link-weight* from agent j to agent i. We notice that the link weight from j to i is equal to the network graph's edge weight if there is an edge from j to i, and is zero otherwise.

An initial result, developed in Lemma 1, shows that the asymptotic opinions are weighted averages of the manipulative actors' reference signals:

Lemma 1. *Consider the distributed-decision-making algorithm with local and targeting actors (2). The contribution of the manipulative actor j to the asymptotic opinion of agent i, i.e. λ_{ij}, satisfies the following conditions: (1) $0 \le \lambda_{ij} \le 1$ and (2) $\sum_{j\in\mathcal{M}} \lambda_{ij} = 1$ for $\forall i \in \mathcal{N}$. In the case where each actor uses at least one strictly positive gain k_{ij} (or l_j for targeting actors), the contributions λ_{ij} are strictly positive.*

Proof of Lemma 1. Without loss of generality, we label local actors as $\mathcal{M}_1 = \{1, 2, \dots, m_1\}$ and label targeting actors as $\mathcal{M}_2 = \{m_1+1, m_1+2, \dots, m_1+m_2\}$. Eq. (2) can be simplified to: $\dot{\mathbf{x}} = A\mathbf{x} + \sum_{i\in\mathcal{N}} \mathbf{e}_i(\sum_{j\in\mathcal{M}_1} k_{ij}\bar{r}_j) + \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}l_j\bar{r}_j$, where $A = -L - diag(K\mathbf{1}_{m_1}) - \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j$, the $n\times m_1$ matrix K is defined as $K = [k_{ij}]$ (where k_{ij} is set to 0 if $i \notin \mathcal{N}_j$), and $\mathbf{1}_{m_1}$ is an m_1-element column vector with all entries equal to 1. The asymptotic opinions of the agents can be obtained by solving:

$$\mathbf{0} = A\bar{\mathbf{x}} + \sum_{i\in\mathcal{N}} \mathbf{e}_i(\sum_{j\in\mathcal{M}_1} k_{ij}\bar{r}_j) + \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}l_j\bar{r}_j \tag{4}$$

By substituting the expressions for $\bar{\mathbf{x}}$ from Eq. (3) and canceling the reference signal terms $\bar{r}_j$, we reach the following equation:

$$(L + diag(K\mathbf{1}_{m_1}) + \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)\Lambda = [K, \mathbf{0}_{(n,m_2)}] + Z \tag{5}$$

where the $n\times m$ matrix Λ is defined as $\Lambda = [\lambda_{ij}]$, the $n\times m_2$ matrix Z is defined as $Z = \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}\mathbf{e}'_j l_j$, and $\mathbf{0}_{(n,m_2)}$ is defined as $n\times m_2$ zero matrix. To simplify further, assuming that all targeting actors' control gains l_j satisfy $0 \le l_j < -L_{a(j)b(j)}$, we notice that the matrix $(L + diag(K\mathbf{1}_{m_1}) + \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)$ is a diffusive matrix, since it is assumed that Γ is strongly connected and the matrix K has at least one non-zero entry. From properties of diffusive matrices, it follows immediately that $(L+diag(K\mathbf{1}_{m_1})+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)^{-1}$ is a strictly positive matrix (Berman & Plemmons, 1994). Since the matrix K is also nonnegative, the matrix $\Lambda=(L+diag(K\mathbf{1}_{m_1})+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)^{-1}K$ is nonnegative, and hence $\lambda_{ij}\geq 0$. In addition, from Eq. (5), we have: $(L+diag(K\mathbf{1}_{m_1})+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)\Lambda\mathbf{1}_m = ([K,\mathbf{0}_{(n,m_2)}]+Z)\mathbf{1}_m$. Rearranging this equation yields $\Lambda\mathbf{1}_m=\mathbf{1}_n$, so $\sum_{j\in\mathcal{M}}\lambda_{ij}=1$ for $\forall i\in\mathcal{N}$. Thus, it also follows that $0\leq\lambda_{ij}\leq 1$. In the case where each actor uses at least one strictly positive gain, each column of $([K,\mathbf{0}_{(n,m_2)}]+Z)$ has at least one strictly positive entry, and hence Λ is strictly positive. ■

Remark. Lemma 1 is similar in flavor to the steady-state analysis of distributed decision-making algorithms with stubborn agents (Ghaderi & Srikant, 2014) and of leader–follower consensus processes (Liu et al., 2012). However, the formulation differs in that the reference inputs enter via a feedback mechanism, and also in the presence of non-local (targeting) actors. As a further contrast, our analysis is a starting point for relating the asymptotics to the actors' control gains, as developed next. Like the related results in Ghaderi and Srikant (2014) and Liu et al. (2012), the proof of Lemma 1 primarily draws on properties of diffusive matrices, however a somewhat different argument is needed because of the differences noted above.

In the following theorem, we characterize the dependence of the contributions λ_{ij} on the actors' control gains:

Theorem 1. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Suppose that there is more than one manipulative actor using a strictly positive gain. The contribution of the manipulative actor j to the asymptotic opinion of agent i, i.e. λ_{ij}, is a concave strictly-increasing function of the manipulative actor's control gains (i.e. l_j or k_{pj} for $p\in\mathcal{N}_j$). Meanwhile, each contribution λ_{ij} is a convex strictly-decreasing function of the other actors' control gains (i.e. l_q or k_{pq} for $q\neq j$ and $p\in\mathcal{N}_q$). In other words (1) $\frac{\partial\lambda_{ij}}{\partial k_{pj}}>0$, $\frac{\partial^2\lambda_{ij}}{\partial k_{pj}^2}<0$; or $\frac{\partial\lambda_{ij}}{\partial l_j}>0$, $\frac{\partial^2\lambda_{ij}}{\partial l_j^2}<0$; (2) $\frac{\partial\lambda_{ij}}{\partial k_{pq}}<0$, $\frac{\partial^2\lambda_{ij}}{\partial k_{pq}^2}>0$; or $\frac{\partial\lambda_{ij}}{\partial l_q}<0$, $\frac{\partial^2\lambda_{ij}}{\partial l_q^2}>0$ for $q\neq j$ and $p\in\mathcal{N}_q$.*

Proof of Theorem 1. By applying the operator $\frac{\partial}{\partial k_{pq}}$ on both sides of Eq. (5) we have:

$$(L+diag(K\mathbf{1}_{m_1})+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)\frac{\partial}{\partial k_{pq}}\Lambda = [D,\mathbf{0}_{(n,m_2)}] \tag{6}$$

where $[\frac{\partial}{\partial k_{pq}}\Lambda]_{ij}=\frac{\partial\lambda_{ij}}{\partial k_{pq}}$, and the $n\times m_1$ matrix D is given by $D_{ij}=0$ for $i\neq p$, $D_{pj}=-\lambda_{pj}$ for $j\neq q$, and $D_{pq}=1-\lambda_{pq}$. Since the inverse of $(L+diag(K\mathbf{1}_{m_1}+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j))$ is a strictly positive matrix and $0<\lambda_{ij}<1$ (since there are more than one manipulative actor where each actor uses at least one strictly positive gain k_{ij} or l_j), it follows that (1) $\frac{\partial\lambda_{ij}}{\partial k_{pj}}>0$ for $\forall i\in\mathcal{N}$ and (2) $\frac{\partial\lambda_{ij}}{\partial k_{pq}}<0$ for $\forall q\neq j\in\mathcal{M}$ and $\forall i\in\mathcal{N}$. Similarly, by applying the operator $\frac{\partial}{\partial k_{pq}}$ on both sides of Eq. (6) the concave and convex behavior can be proved. By applying the operator $\frac{\partial}{\partial l_q}$ on both sides of Eq. (5) we have:

$$(L+diag(K\mathbf{1}_{m_1})+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)\frac{\partial}{\partial l_q}\Lambda = \mathbf{e}_{a(q)}(\mathbf{e}'_q-\Lambda_{a(q),:}) \tag{7}$$

where $\Lambda_{a(q),:}$ is the $a(q)$-th row of matrix Λ. Since the inverse of $(L+diag(K\mathbf{1}_{m_1}+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j))$ is a strictly positive matrix and $0<\lambda_{ij}<1$ (since there are more than one manipulative actor where each actor uses at least one strictly positive gain k_{ij} or l_j), it follows that (1) $\frac{\partial\lambda_{ij}}{\partial l_j}>0$ for $\forall i\in\mathcal{N}$ and (2) $\frac{\partial\lambda_{ij}}{\partial l_q}<0$ for $\forall q\neq j\in\mathcal{M}$ and $\forall i\in\mathcal{N}$. Similarly, by applying the operator $\frac{\partial}{\partial l_q}$ on both sides of Eq. (7) the concave and convex behavior can be proved. ■

The above theorem shows that, if a manipulative actor j increases any of its control gains k_{pj} or l_j, its contribution to all agents' asymptotic opinions (i.e. λ_{ij} for $\forall i\in\mathcal{N}$) increases. Meanwhile, all other actors' contributions (i.e. λ_{iq} for $\forall q\neq j$ and $i\in\mathcal{N}$) decrease. Thus, by increasing a control gain, an actor increases his own influence on the agents and uniformly decreases the contributions of all other actors. It is interesting to ask whether an actor can gain absolute influence (i.e., achieve a contribution λ_{ij} of 1), by using a high gain. The following lemma clarifies that local actors can gain absolute influence on the agents which they directly actuate and measure.

Lemma 2. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Consider the contribution of a local actor q on an agent p that it directly actuates ($p\in\mathcal{N}_q$), as defined in (3). If q is a local actor, then for sufficiently large control gain k_{pq}, the contribution is arbitrarily close to 1, i.e. $\lim_{k_{pq}\to\infty}\lambda_{pq}=1$. Alternately, consider that q is a targeting actor which can apply a large control gain l_j (i.e. the gain is not bounded by $L_{a(j)b(j)}$). Then the asymptotic opinion of the targeted agent b(j) is arbitrarily close to the reference signal of the targeting actor j. i.e. $\lim_{l_j\to\infty}\bar{x}_{b(j)}=\bar{r}_j$, provided that asymptotic stability is maintained.*

Proof of Lemma 2. Consider the entry in row p and column q of Λ. In order for Eq. (5) to hold as k_{pq} is made large, it is necessary that λ_{pq} approaches 1, since all other terms in the expression become negligible. An entirely analogous argument can be used to demonstrate that the opinion of the targeted agent is arbitrarily close to the reference signal of the targeted actor, provided that asymptotic stability is maintained. ■

Although local actors can gain absolute influence on agents that they directly actuate/measure, their influence on other agents in the network is generally not absolute no matter how large a control gain they use. Thus, for agents that are not directly being actuated by a local actor, other actors can ensure a minimum level of influence (minimum contribution to the asymptotic opinion of the actor) by applying a control anywhere in the network. Per Theorem 1, by increasing their control gains, these actors can also reduce the maximum influence that can be achieved by the first actor.

3.2. Graph-theoretic results

A main contribution of this study is to characterize the spatial pattern of impact of an actor. Specifically, in the following theorem, we show the contribution of a local or targeting actor on the agents' asymptotic opinion is, in a certain sense, a monotonically decreasing function of the graphical distance of the agent from the actuation sphere of the actor. The theorem requires some terminology and notation relating to cutsets of the network's graph. To develop this terminology, let us first consider a particular local actor j, and assume without loss of generality that the actor applies feedback to all agents that it actuates/measures (i.e. $k_{ij}\neq 0$ for $\forall i\in\mathcal{N}_j\subset\mathcal{N}$). Let us also define $\bar{\mathcal{N}_j}\subset\mathcal{N}$ as the complement set of $\mathcal{N}_j$. We define a *separating cutset* for actor j as a vertex cutset $\mathcal{Q}(j)$ such that all the vertices in subset $\mathcal{N}_j$

are contained in only one partition formed by the cutset, or on the cutset itself, i.e., all actuated vertices are on "one side" of the cutset or the cutset itself. (Notice that the vertices in subset $\bar{\mathcal{N}}_j$ may be present in all partitions formed by the cutset or on the cutset itself). We also refer to the corresponding group of the agents as *separating agents*. Let us use the label $\mathcal{S}(j)$ as the set of the vertices in the partition that includes vertices from set $\mathcal{N}_j$, and refer to this set as the actor-close vertices (agents). Similarly, we use the label $\mathcal{T}(j)$ for the set of vertices in the partition(s) that do not include vertices from set $\mathcal{N}_j$, and refer to this set as the actor-far vertices (agents). We note that any path from a vertex in set $\mathcal{S}(j)$ to a vertex in set $\mathcal{T}(j)$ passes through at least one of the vertices in set $\mathcal{Q}(j)$. We also define separating cutsets for each targeting actor in exactly the same way, but with the definition being based on the targeting actor's actuation location in Γ.

Now the theorem is presented:

Theorem 2. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Consider a particular manipulative actor j and any corresponding separating cutset $\mathcal{Q}(j)$, which forms an actor-close set $\mathcal{S}(j)$ and an actor-far set $\mathcal{T}(j)$. The contribution of the manipulative actor j to the asymptotic opinion of any agent in the actor-far set $\mathcal{T}(j)$ is less than its contribution to at least one separating agent specified in $\mathcal{Q}$, i.e. $\lambda_{ij} \le \max_{\{q\in\mathcal{Q}(j)\}} \lambda_{qj}$ for $\forall i \in \mathcal{T}(j)$.*

Proof of Theorem 2. From Eq. (5), we have $\hat{L}\Lambda = \hat{K}$ where $\hat{L} = (L + diag(K\mathbf{1}_{m_1}) + \sum_{j\in\mathcal{M}_2} \mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)$ and $\hat{K} = [K, \mathbf{0}_{(n,m_2)}] + Z$. Without loss of generality, the agents are ordered such that $\mathbf{x}$ can be partitioned as $\mathbf{x} = \begin{bmatrix}\mathbf{x}_\mathcal{S}\\ \mathbf{x}_\mathcal{Q}\\ \mathbf{x}_\mathcal{T}\end{bmatrix}$. Considering the notation used in the proof of Lemma 1, the contribution of actor j to all agents' asymptotic opinion is $\bar{\lambda}$ which is the jth column of matrix Λ, and similarly the control gains of the actor j are specified by the vector $\hat{K}_{:,j}$ which is the jth column of matrix $\hat{K}$. The matrices $\hat{L}$, $\bar{\lambda}$, and $\hat{K}$, can be partitioned commensurately with the state vector, as

$$L = \begin{bmatrix} L_{\mathcal{SS}} & L_{\mathcal{SQ}} & L_{\mathcal{ST}} \\ L_{\mathcal{QS}} & L_{\mathcal{QQ}} & L_{\mathcal{QT}} \\ L_{\mathcal{TS}} & L_{\mathcal{TQ}} & L_{\mathcal{TT}} \end{bmatrix},$$

$$\bar{\lambda} = \begin{bmatrix} \bar{\lambda}_\mathcal{S} \\ \bar{\lambda}_\mathcal{Q} \\ \bar{\lambda}_\mathcal{T} \end{bmatrix},$$

$$\hat{K}_{:,j} = \begin{bmatrix} \hat{K}_{\mathcal{S}_{:,j}} \\ \hat{K}_{\mathcal{Q}_{:,j}} \\ \hat{K}_{\mathcal{T}_{:,j}} \end{bmatrix},$$

and $\hat{K} = \begin{bmatrix} \hat{K}_\mathcal{S} \\ \hat{K}_\mathcal{Q} \\ \hat{K}_\mathcal{T} \end{bmatrix}$. From the definition of a separating cutset, we know $\hat{L}_{\mathcal{ST}} = \mathbf{0}$, $\hat{L}_{\mathcal{TS}} = \mathbf{0}$, and $\hat{K}_{\mathcal{T}_{:,j}} = \mathbf{0}$. From Eq. (5) and using the partitioned matrices, we find:

$$\hat{L}_{\mathcal{TQ}}\bar{\lambda}_{\mathcal{Q}_{:,j}} + \hat{L}_{\mathcal{TT}}\bar{\lambda}_{\mathcal{T}_{:,j}} = 0 \tag{8}$$

From this equation, one can find:

$$\bar{\lambda}_{\mathcal{T}_{:,j}} = -(\hat{L}_{\mathcal{TT}})^{-1}\hat{L}_{\mathcal{TQ}}\bar{\lambda}_{\mathcal{Q}_{:,j}} \tag{9}$$

Considering that $\hat{L}_{\mathcal{TT}}$ is a diffusive matrix and $-\hat{L}_{\mathcal{TQ}}$ has nonnegative entries, $-(\hat{L}_{\mathcal{TT}})^{-1}\hat{L}_{\mathcal{TQ}}$ is a nonnegative matrix and its row sum is less than or equal to 1 (see proof of Lemma 1). Therefore,

$$\bar{\lambda}_{\mathcal{T}_{:,j}} \le -(\hat{L}_{\mathcal{TT}})^{-1}\hat{L}_{\mathcal{TQ}}\mathbf{1}_{n_\mathcal{Q}} \max(\bar{\lambda}_{\mathcal{Q}_{:,j}}),$$

where $n_\mathcal{Q}$ as the number of separating agents corresponding to separating cutset $\mathcal{Q}$.

$$\bar{\lambda}_{\mathcal{T}_{:,j}} \le \max(\bar{\lambda}_{\mathcal{Q}_{:,j}})\mathbf{1}_{n_\mathcal{T}} \tag{10}$$

where $n_\mathcal{T}$ is the number of agents in set $\mathcal{T}$, '$\le$' is an elementwise operator, and the max operator is over all entries of the vector $\bar{\lambda}_\mathcal{Q}\mathbf{1}_{n_\mathcal{T}}$. The previous inequality is equivalent to $\lambda_{ij} \le \max_{\{q\in\mathcal{Q}(j)\}} \lambda_{qj}$ for $\forall i \in \mathcal{T}(j)$. ■

Theorem 2 shows that the influence of an actor is smaller at a remote agent, as compared to at least one agent on a cutset that separates the remote agent from its actuation set. This main result indicates that there is a spatial degradation in the contributions of actors, along sequential cutsets away from the agent's actuated set. The following corollary formalizes this notion of a spatial degradation, beginning with a general case and then some specializations. For the corollary, we define $\mathcal{D}(j, d)$ as the set of vertices at distance d from vertex j.

Corollary 1. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Then:*

- *Consider an arbitrary actor j and two separating cutsets $\mathcal{Q}_1(j)$ and $\mathcal{Q}_2(j)$, such that any path from vertices in set $\mathcal{N}_j$, to the vertices in set $\mathcal{Q}_2(j)$ passes through at least one of the vertices in set $\mathcal{Q}_1(j)$. The contribution of actor j to the asymptotic opinion of all separating agents in $\mathcal{Q}_2(j)$ is less than or equal to its contribution to at least one of the separating agents in $\mathcal{Q}_1(j)$, i.e. $\lambda_{ij} \le \max_{\{q\in\mathcal{Q}_1(j)\}} \lambda_{qj}$ for $\forall i \in \mathcal{Q}_2(j)$.*
- *Consider a particular actor j and two agents p and q such that any path between a vertex in set $\mathcal{N}_j$ to the vertex q passes through the vertex p. The contribution of the local actor j to the asymptotic opinion of agent q is less than or equal to its contribution to the agent p, i.e. $\lambda_{qj} \le \lambda_{pj}$.*
- *Assume that the actor $\hat{j}$ only actuates the agent j. The maximum contribution of actor $\hat{j}$ to the asymptotic opinion of the agents at distance d from vertex j (i.e. agents in set $\mathcal{D}(j, d)$), is a monotonically decreasing function of d. That is, $\max_{p\in\mathcal{D}(j,d)} \lambda_{p\hat{j}} \ge \max_{q\in\mathcal{D}(j,\hat{d})} \lambda_{q\hat{j}}$ if $\hat{d} > d$.*

The corollary follows readily from Theorem 2, hence a detailed proof is not given. Theorem 2 and Corollary 1 hold regardless of the control gains used by the different actors. Hence, the results also hold for the maximum contribution that can be achieved by an actor on the agents through design of the control gains.

The influences of the manipulative actors have a further special structure for portions of the network that are isolated, in the sense that they are immune to actuation and connected to the remainder of the network via a single vertex cutset. To formalize this notion, we define an *isolated partition* $\mathcal{I}_p$ as a set of vertices with the following properties: (1) none of the vertices are actuated by any manipulative actors and (2) the vertex p is a single-vertex cutset which partitions $\mathcal{I}_p$ from the remainder of the graph. (Note here that we include p within the set $\mathcal{I}_p$.) The following corollary shows that each manipulative actor has identical contribution on the asymptotic opinion of all agents corresponding to the isolated partition $\mathcal{I}_p$.

Corollary 2. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Assume that there is an isolated partition $\mathcal{I}_p$ corresponding to vertex p. The contribution of the local actor j to the asymptotic opinion of all agents in the isolated partition $\mathcal{I}_p$ are identical, i.e. $\lambda_{ij} = \lambda_{qj}$ for $\forall i, q \in \mathcal{I}_p$ and $\forall j \in \mathcal{M}$.*

Next, we investigate how changes in the link weight between two agents affect the asymptotic opinions in the network. These results give further intuition about how the network's graph modulates the level of influence that different actors can achieve.

Theorem 3. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Suppose that more than one manipulative actor is using a strictly positive gain (l_j or k_{ij}). The contribution of a manipulative actor q on all agents' asymptotic opinions is a strictly-increasing function of the link weight from agent i to agent j, i.e. $-L_{ji}$, if $\lambda_{iq} > \lambda_{jq}$ (i.e., $\frac{\partial\lambda_{pq}}{-\partial L_{ji}} > 0$ in this case). Conversely, it is a strictly-decreasing function of $-L_{ji}$ if $\lambda_{iq} < \lambda_{jq}$.*

Proof of Theorem 3. By applying the operator $\frac{\partial}{-\partial L_{ji}}$ on both sides of Eq. (5), we have:

$$(L + diag(K\mathbf{1}_{m_1}) + \sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)\frac{\partial}{-\partial L_{ji}}\Lambda = D \tag{11}$$

where $[\frac{\partial}{-\partial L_{ji}}\Lambda]_{uv} = \frac{\partial\lambda_{uv}}{-\partial L_{ji}}$, the matrix D is defined as follows: $D_{uv} = 0$ for $u \neq j$, and $D_{jv} = \lambda_{iv} - \lambda_{jv}$ for $\forall v$. Again recall that the inverse of $(L+diag(K\mathbf{1}_{m_1})+\sum_{j\in\mathcal{M}_2}\mathbf{e}_{a(j)}\mathbf{e}'_{b(j)}l_j)$ is a strictly-positive matrix. Hence for a particular local manipulative actor q, (1) if $\lambda_{iq} > \lambda_{jq}$ then $\frac{\partial\lambda_{uq}}{-\partial L_{ji}} > 0$, and (2) if $\lambda_{iq} < \lambda_{jq}$ then $\frac{\partial\lambda_{uq}}{-\partial L_{ji}} < 0$. ■

Theorem 3 shows that the contribution of an actor to all agents' asymptotic opinions is increased if link weights from high-contribution agents to low-contribution agents are augmented. In other words, the actor's influence can be spread widely if links are built from highly-influenced agents to other agents. Conversely, if less-influenced agents have stronger links to more influenced agents, then the influence of the actor globally decreases.

The next result specializes Theorem 3 to the case where each actor is only actuating one agent:

Corollary 3. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). Consider two actors $\hat{i}$ and $\hat{j}$ which each actuate only a single agent, i and j respectively. The contribution of the actor $\hat{i}$ to all agents' asymptotic opinions is a monotonically increasing function of $-L_{ji}$, while the contribution of actor $\hat{j}$ to the asymptotic opinions is a monotonically decreasing function of $-L_{ji}$.*

Proof of Corollary 3. This corollary is the immediate result of Corollary 1 and Theorem 3. ■

A further result is that, if the link weight from one agent to another is sufficiently increased, the contributions of the actors for these agents and hence their asymptotic opinions become close:

Theorem 4. *Consider the distributed-decision-making algorithm with local actors and targeting actors (2). The contribution of any manipulative actor q to the asymptotic opinion of the agents i and j, i.e. $\lambda_{i,q}$ and $\lambda_{j,q}$ for any $q \in \mathcal{M}$, become close to each other for sufficiently large $-L_{ij}$ or $-L_{ji}$. That is, $\lim_{(-L_{ij})\to\infty}(\lambda_{iq} - \lambda_{jq}) = 0$ and $\lim_{(-L_{ji})\to\infty}(\lambda_{iq} - \lambda_{jq}) = 0$ for any $q \in \mathcal{M}$.*

Proof of Theorem 4. Consider jth row in Eq. (4). For sufficiently large values of $-L_{ji}$, the solution of the equation becomes close to the solution of the equation $L_{ji}(\bar{x}_j - \bar{x}_i) = 0$, from a continuity argument. The theorem statement is thus proved. ■

The above analyses can be generalized to the case where the actors apply dynamic compensation. In particular, consider the generalized model for manipulative actors given in the problem formulation, where in general dynamic compensations are applied by the actors. The analyses of the asymptotics readily generalize to this dynamic setting, provided that (1) the asymptotic stability of the system is maintained, and (2) the dynamic compensation schemes have finite positive DC gain. Given these two conditions, the results presented in Lemmas 1–2, Theorems 1–4, and their associated corollaries hold, with the DC control gain ($H_{ij}(0)$ for local actors, $H_j(0)$ for targeting actors) replacing the static gain in the various analyses of the asymptotic opinions. Based on this characterization, it is of interest to understand what dynamic compensation schemes applied by manipulative actors achieve asymptotic stability. In the following theorem, we consider specifically the case that the actors use a lead-compensation strategy rather than a static control. A lead compensator is an appropriate model of manipulation in some application domains, in that it captures forecasting of opinions in applying feedback (i.e., proportional-derivative-type control), while also representing filtering of noise or intrinsic sluggishness in measurement/actuation. The following theorem verifies that local actors and targeting actors using lead compensation maintain asymptotic stability:

Theorem 5. *Consider the distributed-decision-making algorithm with local actors and targeting actors which apply lead compensation (i.e., each local actor j applies a compensation with transfer function $H_{ij}(s) = k_{ij}\frac{s+\alpha_{ij}}{s+\beta_{ij}}$, and similarly each targeting actor j applies a compensation with transfer function $H_j(s) = l_j\frac{s+\alpha_j}{s+\beta_j}$). Assume that the controller parameters satisfy $k_{ij} > 0$, $0 < \alpha_{ij} < \beta_{ij}$, $0 \leq l_j < -L_{a(j)b(j)}$, and $0 < \alpha_j < \beta_j$. Then the closed-loop system is asymptotically stable.*

Proof of Theorem 5. The internal dynamics of the closed-loop system can be written in state–space form as $\dot{\mathbf{x}} = -L(\Gamma)\mathbf{x} + \sum_{i\in\mathcal{N}}\mathbf{e}_i(u_i + v_i)$ where $u_i = \sum_{j\in\mathcal{M}_1}u_{ij}$, $u_{ij} = -k_{ij}\mathbf{e}'_i\mathbf{x} + k_{ij}z_{ij}$, $\dot{z}_{ij} = -\beta_{ij}z_{ij} + (\beta_{ij} - \alpha_{ij})\mathbf{e}'_i\mathbf{x}$ for all $i \in \mathcal{N}$ and $j \in \mathcal{M}_1$. Similarly $v_i = \sum_{j\in\mathcal{M}_2\, s.t.\, a(j)=i}v_{ij}$, $V_{ij} = -l_j\mathbf{e}'_i\mathbf{x} + l_jz_j$, $\dot{z}_j = -\beta_jz_j + (\beta_j - \alpha_j)\mathbf{e}'_i\mathbf{x}$ for all $j \in \mathcal{M}_2$ and $i = a(j)$. Consider the state matrix for the closed loop system, whose extended state vector includes $\mathbf{x}$ and the controllers' internal states z_{ij} and z_j. Under the assumptions on the controller parameters, this state matrix is the negative of a diffusive matrix. From the properties of diffusive matrices, it is immediate that the closed-loop system is asymptotically stable. ■

Since asymptotic stability is maintained, the topological results in the above lemmas, theorems, and corollaries hold in the case of lead-compensated manipulative actors.

4. Dynamic effects

Each manipulative actor alters the dynamical (transient) properties of the distributed decision-making algorithm, including the algorithm's global approach to steady-state and the dynamics seen by other actors' control systems. These dynamic effects are central to understanding manipulation of decision-making algorithms, as they modulate the global performance of the algorithm (e.g. the time required to reach a decision) and place constraints on the achievable influence of other manipulators. Here, an exploratory study is undertaken of the impacts on dynamics incurred by the manipulative actors, focusing on: (1) how settling properties are altered by local manipulative actors, and (2) limits placed on other actors' controls via the dynamics.

The settling properties of distributed decision-making or consensus algorithms have been extensively studied (Pirani & Sundaram, 2014; Xiao et al., 2007), primarily through analysis of a dominant eigenvalue of the state matrix. Our interest here is understanding whether manipulative actors slow down or speed up decision-making. Thus, we compare settling properties of distributed decision-making algorithms with and without manipulative actors. Following on the settling analysis in the literature, we focus here on networks with symmetric state matrix and local actors, which allows analysis using techniques for Hermitian matrices.

The settling rate of the nominal model (without manipulators) is governed by the subdominant eigenvalue (closest non-zero eigenvalue to the imaginary axes) of the state matrix, which is a symmetric Laplacian matrix (Xiao et al., 2007). On the other hand, when local manipulative actors are included in the model, the distributed decision-making algorithm (2) becomes asymptotically stable, with the settling rate governed by the dominant eigenvalue (the closest eigenvalue to the imaginary axis) of the symmetric negative-grounded-Laplacian state matrix. To permit comparison, we formally define the settling rate R_s of the decision-making algorithm as the absolute value of the non-zero state-matrix eigenvalue that is closest to the imaginary axis. The following theorem shows that the presence of a single actor which actuates a single agent necessarily leads to slower settling compared to the nominal model; however, actuation of other agents by manipulative actors or an increase in the control gains then causes a comparative increase in the settling rate.

Theorem 6. *Consider the distributed-decision-making algorithm with only local actors (2), in the case that the state matrix $-L$ is symmetric. The settling rate R_s with a single local actor actuating a single agent is less than or equal to the settling rate for the nominal model. However, actuation of further agents by local actors causes an increase in the settling rate. Additionally, the settling rate is a strictly-increasing function of the local actor's control gains k_{pj} for all positive k_{pj}, $p \in \mathcal{N}_j$.*

Proof of Theorem 6. First, let us consider the settling rate with a single local actor actuating a single agent. Noticing that the state matrix with the local actor included is a symmetric rank-1 perturbation of the original state matrix, it follows from Weyl's inequalities that its dominant eigenvalue interlaces the dominant and subdominant eigenvalues of the nominal model. Thus, the settling rate R_s is seen to be less than or equal to that of the nominal model.

Now consider actuation at further agents and/or increase in the local feedback gains. As discussed in the proof of Lemma 1, the state matrix of the controlled system is the negative of a diffusive matrix. Increasing any control gain k_{pj} modifies the state matrix by decreasing (making more negative) the diagonal entry (p, p) of the state matrix; all other entries remain unchanged. To prove the result, we thus need to prove that the dominant eigenvalue of a symmetric diffusive matrix is a monotonically nondecreasing function of any diagonal entry. However, this is a well-known property of diffusive matrices (Berman & Plemmons, 1994). ■

Remark. Theorem 6 can be readily generalized to the case of diagonally symmetrizable state matrices, which encompasses e.g. network graphs with a tree topology. For general network topologies, it is possible to find examples where the majorization does not hold.

Beyond impacting settling properties, the actions of one actor can place limits on the other actors' ability to control the decision-making dynamics, by making these actors' control channels prone to instability or oscillation. To understand why, we recall that actors' abilities to modulate the steady-state decision are conditioned on stability of the closed-loop dynamics being maintained; thus, if actors' control gains are subject to constraints to maintain stability (or avoid oscillations), their effective influence in the network is limited. One means to characterize limitations on an actor's control capability is to examine the *zeros* of the transfer function for the actor's control channel. In particular, the occurrence of nonminimum phase (right-half-plane) zeros indicates that the actor cannot apply a strong (high-gain) feedback without incurring instability, i.e. the actor is limited in its control capability. Further, the presence of right-half-plane zeros places fundamental limits on the actor's best achievable control performance (e.g., on the reference tracking error, and the possibility for cheap control), no matter what controller the actor uses (Qiu & Davison, 1993). Thus, to understand interactions among manipulative actors, we are motivated to study how the zeros of an actor's control channel depend on the controls applied by another actor. The following results distinguish circumstances where the actions of manipulative actors preserve minimum-phase dynamics at other actors' channels, from cases where nonminimum-phase behaviors result. These results extend the analysis of channel properties of diffusive network models (Abad Torres & Roy, 2014), toward assessment of control-channel interactions. We note that the results hold for general asymmetric network topologies.

The first result verifies that each local actor's control channel is minimum phase under broad conditions on other actors' controls:

Theorem 7. *Consider the distributed decision-making algorithm with local actors and targeting actors (2). For this model, assume that all targeting actors' control gains l_j for $\forall j \in \mathcal{M}_2$ satisfy the condition $0 \leq l_j < -L_{a(j)b(j)}$. The control channel seen by any local actor is minimum phase.*

Proof of Theorem 7. The result can be proved by transforming the input–output model to the *special coordinate basis* (Sannuti & Saberi, 1987), as was done in Abad Torres and Roy (2014). Specifically, it was shown in Abad Torres and Roy (2014) that, for a diffusive network model, the finite zeros of any control channel with collocated inputs and outputs (e.g., a local actor's channel) are the eigenvalues of the principal submatrix of the state matrix which is formed by deleting the rows and columns corresponding to the input/output nodes. Applying this result to the opinion-dynamics model, we immediately see that the zeros are eigenvalues of a principal submatrix of the closed-loop system's state matrix (the state matrix with the manipulative actors' controllers included), provided that the closed-loop state matrix is diffusive. Given the constraints on the control gains, we know that the controlled system dynamics has a state matrix in the form of the negative of a diffusive matrix, and hence the result from Abad Torres and Roy (2014) holds. Further, any principal submatrix of the state matrix is also in form of negative of a diffusive matrix, which has all eigenvalues in the left half plane. Thus, the control channel is minimum phase. ■

Remark. We stress here that the local actors may act on multiple agents, so in general the local actors' channels are multi-input multi-output (MIMO).

The above theorem shows that each local actor's control channel remains minimum phase, regardless of the feedback gains used by other local actors. The channel also remains minimum phase in the presence of targeting actors, provided that the targeting actors' gains do not exceed the native influences between the actuated and targeted agents. Thus, local actors are seen to have considerable flexibility to impose controls, regardless of the controls used by other actors.

The following three results show that, in contrast, the phase property of targeting actors' control channels depend on the network's topology and the other actors' controls in a sophisticated way. The results are presented without proof or with an abbreviated proof, as they use a similar special-coordinate-basis argument as Theorem 7. The first result is that targeting actors' control channels are minimum-phase for tree-like networks, regardless of the other controls used in the network:

Lemma 3. *Consider the distributed decision-making algorithm with local actors and targeting actors (2). For this model, assume that all targeting actors' control gains l_j for $\forall j \in \mathcal{M}_2$ satisfy the condition $0 \le l_j < -L_{a(j)b(j)}$, while the local actors may use any nonnegative control gain. Consider a particular targeting actor q with actuation vertex $a(q)$ and measurement/target vertex $b(q)$. Suppose that the graph Γ has only a single path from vertex $a(q)$ to $b(q)$. The control channel for targeting actor q is minimum phase.*

A targeting actor's control channel might be either minimum phase or nonminimum phase, if the network graph Γ has multiple paths between the targeted agent and actuated agent. The following theorem shows that properly-located local manipulative actors which use large gains can guarantee that a targeting actor's channel is minimum phase. The theorem requires some further terminology. Consider a subset V_g of the vertices V in Γ: (1) the notation d_{pq} is used for the distance from the vertex p to q, and (2) the notation $d_{pq}^{(V_g)}$ is used for the length (number of edges) of the shortest path from vertex p to q which does not pass through any vertices in V_g. Now, we present the theorem:

Theorem 8. *Consider the distributed decision-making algorithm with local actors and targeting actors (2). For this model, assume that all targeting actors' control gains l_j for $\forall j \in \mathcal{M}_2$ satisfy the condition $0 \le l_j < -L_{a(j)b(j)}$, while the local actors may use any nonnegative control gain. Consider a particular targeting actor q with actuation vertex $a(q)$ and measurement/target vertex $b(q)$. Consider set G as a subset of local actors' control gains and V_g as its corresponding set of actuation vertices where vertices $a(q)$ and $b(q)$ are remote from vertices in set V_g, i.e. $a(q), b(q) \notin V_g$. Suppose that the graph Γ has only a single path from vertex $a(q)$ to $b(q)$ which does not pass through any vertices in set V_g. When the control gains in set G are sufficiently scaled up, the control channel for targeting actor q is minimum phase.*

Abbreviated Proof of Theorem 8: Proving the theorem requires some further terminology. Let us write the dynamics of the control channel of targeting actor q as $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{e}_{a(q)}u$, $y = \mathbf{e}_{b(q)}{}^T\mathbf{x}$, where u and y are the input and output of the SISO channel respectively and all reference signals ($\bar{r}_j$) for local or targeting manipulative actors have been ignored. Let us also define the vectors $\mathbf{e}_p^{(V_g)}$ as a modified version of the vector $\mathbf{e}_p$, where the entries $i \in V_g$ are omitted. Similarly, $A^{(V_g)}$ is defined as a submatrix of A obtained by deleting the rows and columns specified in V_g. Then, the *deletion subsystem* is defined as:

$$\dot{\mathbf{x}}^{(V_g)} = A^{(V_g)}\mathbf{x}^{(V_g)} + \mathbf{e}_{a(q)}^{(V_g)}u \tag{12}$$
$$y = \mathbf{e}_{b(q)}^{(V_g)T}\mathbf{x}^{(V_g)}$$

where $\mathbf{x}^{(V_g)}$, u, and y are the state, input, and output, respectively. The deletion system (12) is associated with a weighted directed *deletion graph* $\Gamma^{(V_g)}$, which is formed by removing from Γ all vertices in V_g and their incident edges.

The zeros of the targeting actor's control channel can be related to those of the deletion system, using the special coordinate basis (see Koorehdavoudi, Roy, Torres, and Xue (2017)). It follows from this analysis that, if $d_{a(q)b(q)}^{(V_g)} = d_{a(q)b(q)}$, then for sufficiently scaled-up control gains in set G, the zeros of the control channel of targeting actor q approach the zeros of the deletion system (12) while all other zeros are in open left half plane (OLHP). Next, from Lemma 3, if there is a single path between vertex $a(q)$ and $b(q)$ in graph $\Gamma^{(V_g)}$ (i.e. there is a single path from vertex $a(q)$ to $b(q)$ which does not pass through any vertices in set V_g), the zeros of the deletion system are in open left half plane, so the input–output channel of interest is minimum phase. ■

The above theorem suggests an interesting approach for collusion among actors: local actors can deliberately ensure that a targeting actor has a minimum-phase channel, thus allowing it to manipulate the target agent at will. Conversely, properly-placed local manipulative actors with sufficiently large control gains can also ensure that targeting actor's channel is nonminimum phase, as formalized next:

Theorem 9. *Consider the distributed decision-making algorithm with local actors and targeting actors (2). For this model, assume that local actors may use any nonnegative control gain. Consider a particular targeting actor q with actuation vertex $a(q)$ and measurement/target vertex $b(q)$. Consider set G as a subset of local actors' control gains and V_g as its corresponding set of actuation vertices where vertices $a(q)$ and $b(q)$ are remote from vertices in set V_g, i.e. $a(q), b(q) \notin V_g$. Suppose that $d_{a(q)b(q)}^{(V_g)} \ge d_{a(q)b(q)} + 2$. When the control gains in set G are sufficiently scaled up, the control channel for targeting actor q is nonminimum phase.*

The above theorem shows that high-gain local actors can cause targeting actors' channels to have nonminimum-phase dynamics, if they act on the shortest input–output path between the actuated and targeted vertices in the network graph.

Remark. Technical details of the special coordinate basis for channels with remote actuation and measurement, which underlies the analysis of targeting actors' zeros, can be found in Abad Torres and Roy (2014), Koorehdavoudi et al. (2017) and Koorehdavoudi, Roy, Torres and Xue (2018). These details are omitted here to save space. The abbreviated proof of Theorem 9 is included, however, to show how the control-channel-interactions analysis can be related to the channel zeros analysis.

5. Examples

Two examples are used to illustrate impacts of manipulative actors on the asymptotics and dynamics of the distributed decision-making algorithm.

In the first example, distributed decision making in a network of 100 agents is considered, see the network graph in Fig. 1; each edge is assumed to have unity weight in both directions. Two local actors 1 and 2 are defined. The contributions of the two actors to the agents' asymptotic opinions (i.e., λ_{ij}) have been determined, when the gains $k_{11} = 10$ and $k_{22} = 10$ are used. The contributions of local actor 1 are illustrated in Fig. 1. Per Theorem 2, the contributions show a spatial degradation. Also, per Corollary 2, the contribution of actor 1 to the asymptotic opinion of the agents inside the isolated partition are found to be identical.

In the second example, a network with 7 agents and two local manipulative actors is considered. The network's graph is shown in Fig. 2: the edge weights are assumed to be identical in both direction. The two local actors, called Actors 1 and 2, measure/actuate Agents 1 and 2 respectively.

The contribution of Actor 2 to Agent 5 is plotted as a function of the feedback gain k_{22} for several values of k_{11}, see Fig. 3. The actor's influence on the agent is amplified as the feedback gain is increased, however absolute influence is not achieved even with a large gain.

For this example, the impact of the manipulative actors on the algorithm's settling is also examined. For the model without manipulative actors, the settling rate is $R_s = 0.1613$. When a

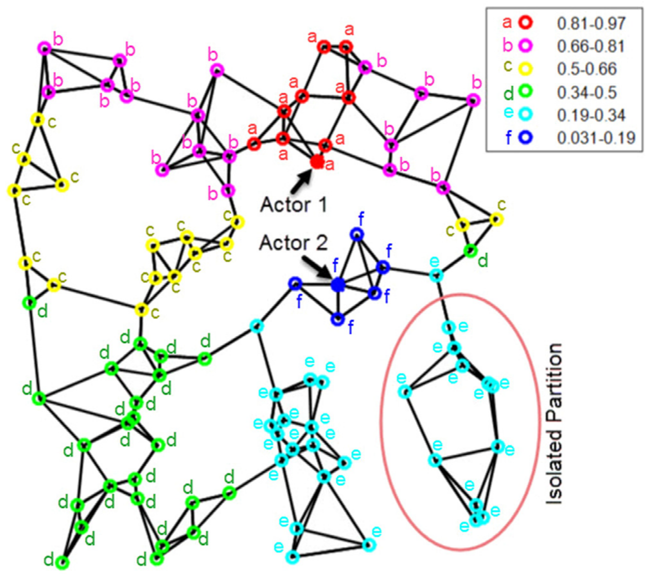

Fig. 1. The graph Γ is shown for a network with 100 agents and two actors. The color and corresponding single letter of each node shows the level of contribution of the local actor 1 to the asymptotic opinion of that agent. The contribution of actor 1 to the asymptotic opinion of each agent inside the isolated partition is identical and equal to 0.23.

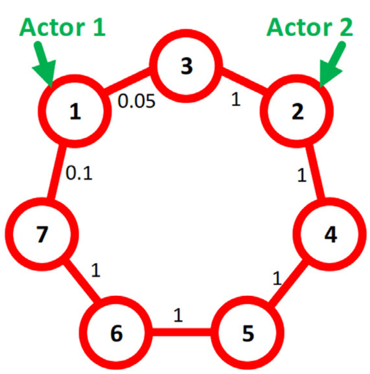

Fig. 2. The graph Γ is shown for a network with 7 agents and two local actors.

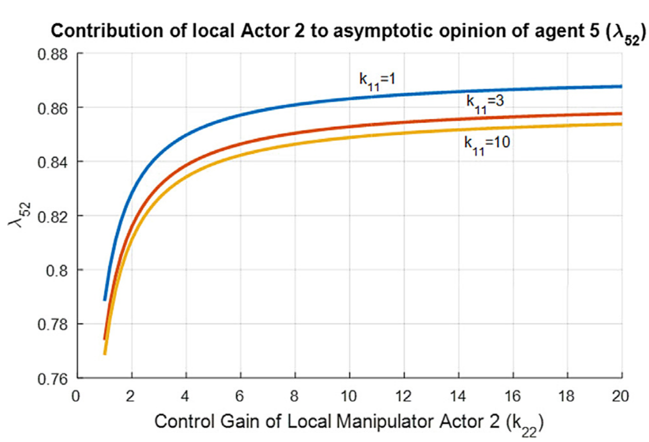

Fig. 3. The dependence of the contribution of local actor 2 on the asymptotic opinion of agent 5 for different values of the control gains of the local actor 1 and 2, i.e. k_{11} and k_{22}.

single manipulative actor is included (Actor 1, using a gain $k_{11} = 1$), the settling rate is $R_s = 0.0206$. With both actors included

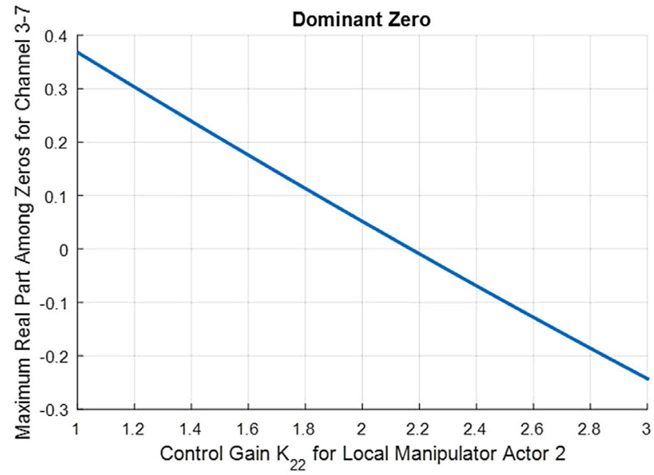

Fig. 4. The dependence of the dominant zero location (the largest real part among the zeros) for transfer function channel 3–7 (i.e., channel seen by actor 3) on the control gain k_{22} of actor 2.

($k_{11} = 1$, $k_{22} = 1$), the settling rate is $R_s = 0.1079$. Thus, we see that the presence of a single manipulative actor slows down the settling of the distributed decision-making algorithm, while the inclusion of the second manipulative actor comparatively speeds up the settling.

We also study impacts of one actor on another actor's control channel, for a slight modification of the example above. In particular, we also include a targeting actor (Actor 3) which measures/targets Agent 3 and actuates agent 7. We study how the zeros of the transfer function for Actor 3 vary with the control gain k_{22} used by Actor 2 (Actor 1's control gain is fixed at $k_{11} = 1$). The dependence of the dominant zero location (the largest real part among the zeros) on k_{22} is shown in Fig. 4. The plot shows that the transfer function seen by Actor 3 is nonminimum phase for small gains k_{22} (i.e. has zeros in the right half plane), but becomes minimum-phase for large gains. Thus, in this example, Actor 2 can support actor 3's efforts to manipulate the network by using a higher gain – i.e., the two actors can collude.

6. Conclusions

A model has been introduced for distributed decision-making processes with multiple manipulative actors, which are represented as enacting linear feedback controls with different reference signals. The ability of the actors to manipulate agents' opinions across the network has been examined from a graph-theoretic perspective: a main finding is that actors' influences degrade along cutsets away from their actuation locations. In addition, the impacts of the actors on the settling behavior of the decision-making algorithm, and the possibility that each actor can constrain other actors' control capabilities, has been studied.

While this effort has been focused on distributed decision-making processes, the model considered in the work is representative of various dynamical network processes which have multiple independently-operated embedded control systems (e.g., building thermal processes, complex industrial systems, transportation systems with multiple control authorities). As future work, we expect to study interactions among control channels in these various application domains, both by enhancing the modeling framework to capture practical constraints in these applications, and by examining how controllers can be designed to improve interactions.

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