Option-Valuation and Application

Least-Squares Monte Carlo Approach

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Tree Method versus Simulation Method

- Tree Method
  - **Backward induction**
  - Convergence to the BS formula according to the central limit theory (CLT)
  - Suitable for American-style derivatives (early performance allowed)
- Monte Carlo
  - **Forward induction**
  - Convergence to the BS formula according to the law of large numbers (LLN)
  - Suitable for path-dependent derivatives

Outline

Introduction

American Option Pricing by Simulation

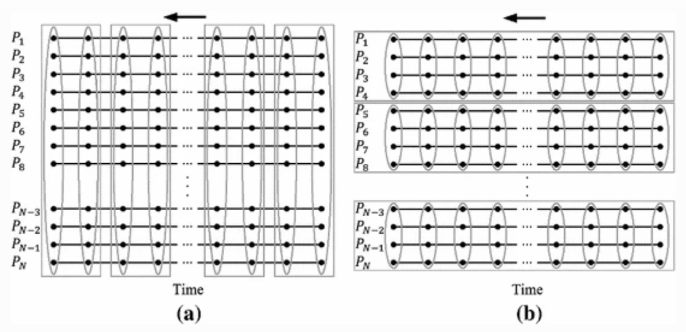
- The option holder must compare the immediate exercise value and the continuation value.
- In standard Monte Carlo simulation, each path is treated **independently** of other paths.
- But the decision to exercise the option cannot be reached by looking at one path alone.

Introduction

- Longstaff and Schwartz (2001) think that **continuation value** can be estimated from the cross-sectional information in the simulation by using least squares.
- The result is a function (of the state) for estimating the continuation values.
- Use the function to estimate the continuation value for each path to determine its cash flow
- This is called the **least-squares Monte Carlo (LSMC)** method.

Introduction

- provably convergent (see Clément et al., 2002; Stentoft, 2004)
- be easily parallelized (see Doan et al., 2010; Lyuu et al., 2014; Wan et al., 2006; Zhang et al., 2011)



A Numerical Example

- Consider a 3-year American put on a non-dividend-paying stock.
- The put is exercisable at years 0, 1, 2, and 3.
- The strike price $K = 105$.
- The annualized riskless rate is $r = 5\%$.
- The current stock price is 101.

A Numerical Example

- Stock Price paths

Path	Year 0	Year 1	Year 2	Year 3
1	**101**	**97.6424**	**92.5815**	107.5178
2	**101**	**101.2103**	105.1763	**102.4524**
3	**101**	105.7802	**103.6010**	124.5115
4	**101**	**96.4411**	**98.7120**	108.3600
5	**101**	124.2345	**101.0564**	**104.5315**
6	**101**	**95.8375**	**93.7270**	**99.3788**
7	**101**	108.9554	**102.4177**	**100.9225**
8	**101**	**104.1475**	113.2516	115.0994

A Numerical Example

- Cash-flow matrix at Year 3

Path	Year 1	Year 2	Year 3
1	–	–	0
2	–	–	2.5476
3	–	–	0
4	–	–	0
5	–	–	0.4685
6	–	–	5.6212
7	–	–	40775
8	–	–	0

A Numerical Example

- For each state that is in the money at year 2, we must decide whether to exercise it.
- There are 6 paths for which the put is in the money: 1,3, 4, 5, 6, 7.
- Let X denote the stock prices at year 2 for those 6 paths.
- Let Y denote the corresponding discounted future cash flows (at year 3) if the put is not exercised at year 2.

A Numerical Example

- Regression at year 2

Path	X	Y
1	92.5815	0×0.951229
2	–	–
3	103.6010	0×0.951229
4	98.7120	0×0.951229
5	101.0564	0.4685×0.951229
6	93.7270	5.6212×0.951229
7	102.4177	4.0775×0.951229
8	–	–

A Numerical Example

- We regress Y on $1, X,$ and X^2.
- The result is

$$\mathbb{E}(Y|X) = 22.08 - 0.313114X + 0.00106918X^2$$

- $\mathbb{E}(Y|X)$ estimates the continuation value conditional on the stock price at year 2.

A Numerical Example

- Optimal early exercise decision at year 2

Path	Exercise	Continuation
1	12.4185✓	$\mathbb{E}(Y \mid X = 92.5815) = 2.2558$
2	–	–
3	1.3990✓	$\mathbb{E}(Y \mid X = 103.6010) = 1.1168$
4	6.2880✓	$\mathbb{E}(Y \mid X = 98.7120) = 1.5901$
5	3.9436✓	$\mathbb{E}(Y \mid X = 101.0564) = 1.3568$
6	11.2730✓	$\mathbb{E}(Y \mid X = 93.7270) = 2.1253$
7	2.5823✓	$\mathbb{E}(Y \mid X = 102.4177) = 0.3326$
8	–	–

A Numerical Example

- When the option is exercised at time 2, the cash flow in the final column becomes zero.
- Once the option is exercised there are no further cash flows since the option can only be exercised once.

Path	Year 0	Year 1	Year 2	Year 3
1	–	–	12.4185	0
2	–	–	0	2.5476
3	–	–	1.3990	0
4	–	–	6.2880	0
5	–	–	3.9436	0
6	–	–	11.2730	0
7	–	–	2.5823	0
8	–	–	0	0

A Numerical Example

- Regression as year 1

Path	X	Y
1	97.6424	12.4185×0.951229
2	101.2103	2.5476×0.951229^2
3	–	–
4	96.4411	6.2880×0.951229
5	–	–
6	95.8375	11.2730×0.951229
7	–	–
8	104.1475	0

A Numerical Example

- Optimal early exercise decision at year 1

Path	Exercise	Continuation
1	7.3576	$\mathbb{E}(Y\|X = 97.6424) = 8.2230$✓
2	3.7897	$\mathbb{E}(Y\|X = 101.2103) = 3.9882$✓
3	–	–
4	8.5589	$\mathbb{E}(Y\|X = 96.4411) = 9.3329$✓
5	–	–
6	9.1625	$\mathbb{E}(Y\|X = 95.8375) = 9.8304$✓
7	–	–
8	0.8525✓	$\mathbb{E}(Y\|X = 104.1475) = -0.5519$

A Numerical Example

- Cash flows at years 1, 2, & 3

Path	Year 0	Year 1	Year 2	Year 3
1	–	0	12.4185	0
2	–	0	0	2.5476
3	–	0	1.3990	0
4	–	0	6.2880	0
5	–	0	3.9436	0
6	–	0	11.2730	0
7	–	0	2.5823	0
8	–	0.8252	0	0

A Numerical Example

- Discount to Year 0

Path	Year 0
1	$12.4185 \times 0.951229^2$
2	2.5476×0.951229^3
3	1.3990×0.951229^2
4	6.2880×0.951229^2
5	3.9436×0.951229^2
6	$11.2730 \times 0.951229^2$
7	2.5823×0.951229^2
8	0.8252×0.951229^1
Average	4.66263

Summary

- Generate stock price simulation path
- Calculate payoff at expiry
- Regression Analysis
- Estimate option value

Basis Regression Selection

Alternative regression: Laguerre polynomials

$$
\begin{aligned}
L_0(X) &= \exp\left(-\tfrac{X}{2}\right) \\
L_1(X) &= \exp\left(-\tfrac{X}{2}\right)(1 - X) \\
L_2(X) &= \exp\left(-\tfrac{X}{2}\right)(1 - 2X + \tfrac{X^2}{2}) \\
L_n(X) &= \exp\left(-\tfrac{X}{2}\right) \tfrac{e^X}{n!} \tfrac{d^n}{dX^n}(X^n e^{-X})
\end{aligned}
$$

With this specification, the conditional expectation can be represented as:

$$
\mathbb{E}(Y|X) = \sum_{j=1}^{n} a_j L_j(X)
$$

Implementation by Python

See code

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