# Option-Valuation and Application
## Extended Issues

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# Outline

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# Introduction

## Volatility

- Volatility is a measure of the magnitude of an asset's price fluctuation, reflecting the level of market uncertainty and risk.
- In financial markets, volatility is widely used to assess risk and is an important basis for investment decisions and risk management.
- Common Method:
  - Historical Volatility
  - Implied Volatility
  - VIX

# Historical Volatility

- Historical volatility (HV) is a statistical measure of the dispersion of returns for a given security or market index over a given period of time.
- Common Method:
  - Simple Moving Average (SMA)

$$\sigma = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(R_t - \bar{R})^2}$$

  - Exponential Weighted Average Moving Average (EWMA)

$$\sigma = \sqrt{(1-\lambda)\sum_{t=1}^{N}\lambda^{t-1}(R_t - \bar{R})^2}$$

  where $\lambda$ is a decay factor, usually set to 0.99, 0.97 or 0.94

# Historical Volatility

- Historical volatility (HV) is a statistical measure of the dispersion of returns for a given security or market index over a given period of time.
- Common Method:
  - High Low Range Volatility (Parkinson, 1980)

$$\sigma = \sqrt{\frac{0.361}{N} \sum_{t=1}^{N} \left( \ln \frac{H_t}{L_t} \right)^2}$$

  - Modified High Low Range Volatility (Garman and Klass, 1980)

$$\sigma = \sqrt{\frac{0.361}{N} \sum_{t=1}^{N} \left[ \left( \frac{1}{2} \ln \frac{H_t}{L_t} \right)^2 - (2 \ln 2 - 1) \left( \ln \frac{C_t}{O_t} \right)^2 \right]}$$

# Implied Volatility

- Implied volatility (IV) is estimated from observed market prices of options.
- It represents the market's forecast of the likely future volatility of the underlying asset's price.

$$C = f(S_0, K, r, T, \sigma)$$

$$\sigma_{imp} = f^{-1}(S_0, K, r, T, C)$$

# Implied Volatility

Compared to HV, IV offers several advantages:

- forward-looking nature
- superior predictive power
- standardization in quotation

# Implied Volatility

## Standardization in Quotation

JOURNAL ARTICLE

Delta-Hedged Gains and the Negative Market Volatility Risk Premium

Gurdip Bakshi and Nikunj Kapadia


*The Review of Financial Studies*
Vol. 16, No. 2 (Summer, 2003), pp. 527-566 (40 pages)
Published by: Oxford University Press. Sponsor: The Society for Financial Studies.

Both the cross-sectional and time-series tests provide evidence that support the hypothesis of a nonzero volatility risk premium.[1] In particular, the results suggest that option prices reflect a *negative* market volatility risk premium. To confirm the hedging rationale underlying a negative volatility risk premium, we empirically estimate the option vega, and verify that it is strictly positive. Moreover, we show that options become more expensive (as measured by implied volatility) after extreme market declines.

## Standardization in Quotation

The Journal of FINANCE
The Journal of THE AMERICAN FINANCE ASSOCIATION

Volume 71, Issue 5
October 2016
Pages 2417-2480

ARTICLE | Full Access

### The Price of Political Uncertainty: Theory and Evidence from the Option Market

BRYAN KELLY, ĽUBOŠ PÁSTOR, PIETRO VERONESI

First published: 01 March 2016 | https://doi.org/10.1111/jofi.12406 | Citations: 249

Figures References Related Information

We find strong empirical support for the model's predictions. First, the unconditional means of all three variables are significantly positive, for both elections and summits. The average (mean-adjusted) implied volatility is 1.43% per year, which implies that one-month ATM put options whose lives span political events tend to be 5.1% more expensive than neighboring options.[2]

## Standardization in Quotation

ELSEVIER

Journal of Financial Economics

Volume 94, Issue 2, November 2009, Pages 310-326

# Cross-section of option returns and volatility ☆

Amit Goyal [a], Alessio Saretto [b]

Note that we are not suggesting that mean-reversion implies that IV should be the same as realized (historical or current) volatility. Indeed the stochastic nature of volatility and the existence of a volatility risk premium necessarily results in differences between IV and HV.[10] However, high autocorrelation of volatility implies that large deviations between IV and HV are unlikely to persist. Therefore, we speculate that, if there is volatility mispricing, it is more likely to manifest itself in extreme temporary deviations between HV and IV. Stocks for which IV is much lower than HV have cheap options, and stocks for which IV is much higher than HV have expensive options.

# Volatility Index

- The CBOE volatility index (VIX) is a popular measure of the stock market's expectation of volatility implied by S&P 500 index options
- It's calculated and disseminated on a real time basis by the CBOE, and is commonly refereed to as the fear index.

# Volatility Index

**VXO**

- The first version of the volatility index was introduced by the CBOE in 1993.
- It was based on the implied volatility of near-the-money options of the S&P 100 Index.

**VIX**

- In 2003, the CBOE revised its calculation methodology.
- It is based on the price of all out-of-the-money S&P500 options and is calculated using a specific formula.

# Volatility Index

**Why Change VIX's Methodology**

- change in underlying index
- improvement in calculation method
- greater sensitivity to market dynamics

# Volatility Index

The CBOE's revised VIX formula for 2003 is as follows

$$\left(\frac{VIX}{100}\right)^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2$$

where

- $Q(K_i)$ denotes the OTM options price with strike price $K_i$
- $F$ denotes forward price
- $K_0$ is the first strike price below $F$

proof

# Outline

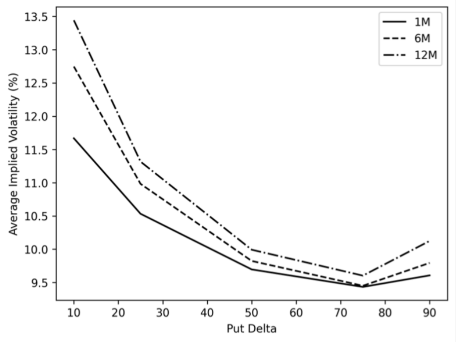
# Introduction

**Foreign-exchange Options Market**

- The foreign-exchange (FX) options market is one of the largest and most liquid OTC derivatives markets in the world.
- The market has developed its own way to quote options, which differs significantly from other markets.
- A fact that is often ignored in the academic literature is that there are a number of different delta and at-the-money conventions.

### Volatility Smile/ Smirk

- Previous studies have found that option market quotes violate the Black and Scholes (1973); Merton (1973) assumption: **constant volatility** (see Canina and Figlewski, 1993; Rubinstein, 1994)




Average implied volatility smiles on USD/JPY options (2006/01-2023/06).

### Quotation of FX Option Market

- FX options are quoted using implied volatility based on different in- and out-of-the-money values.

$$C_{BS}(\sigma^{imp}, S, K, \tau, r) = C_{mkt}$$

$$\Rightarrow \sigma^{imp} = C_{BS}^{-1}(C_{mkt}, , S, K, \tau, r)$$

- For a single time to maturity, the implied volatility as a function of moneyness.
- To be more precise, the volatility smile is a mapping,

$$X \mapsto \sigma(X) \in [0, \infty)$$

with $X$ being the moneyness variable.

# Introduction

**Why use implied volatility**

- The use of implied volatility as a quoting metric better *reflects overall market expectations and risk perception* than direct quoting
- Traders can more directly assess and manage their risk exposure

*Note that although the market is quoted using implied volatility, this does not imply that these options strictly follow the assumptions of the BS model.*

# FX Options Quotation

In practice, practitioners use volatility quotes for risk reversal (RR), butterfly (BF) and at-the-money (ATM) positions.

# FX Options Quotation

In practice, practitioners use volatility quotes for risk reversal (RR), butterfly (BF) and at-the-money (ATM) positions.

| | EURUSD (%) | USDJPY (%) |
|---|---|---|
| $\sigma_{ATM}$ | 21.622 | 21.001 |
| $\sigma_{25-RR}$ | -0.500 | -5.300 |
| $\sigma_{25-BF}$ | 0.738 | 0.184 |

FX market data for 1-month, Jan. 20, 2009

# FX Options Quotation

In practice, practitioners use volatility quotes for risk reversal (RR), butterfly (BF) and at-the-money (ATM) positions.

- ATM ($\Delta_{Call} = \Delta_{Put}$):

$$\sigma_{ATM}$$

For given specific x-delta ($x = 10$ or $25$)

- RR:

$$\sigma_{x-RR} = \sigma_{x-Call} - \sigma_{x-Put}$$

- BF:

$$\sigma_{x-BF} = \frac{1}{2}(\sigma_{x-Call} + \sigma_{x-Put}) - \sigma_{ATM}$$

# FX Options Quotation

**Once you get a quote from the market**

- Derive the implied volatility of the call/put by moneyness:

$$\sigma_{x-Call} = \sigma_{x-BF} + \sigma_{ATM} + \frac{1}{2}\sigma_{x-RR}$$

$$\sigma_{x-Put} = \sigma_{x-BF} + \sigma_{ATM} - \frac{1}{2}\sigma_{x-RR}$$

- Derive the strike price and plug it into the BS pricing formula to get the option price
- More detail on the PREMIUM QUOTING CONVENTIONS, please refer to Reiswich, D., & Wystup, U. (2010). A guide to FX options quoting conventions. *Journal of Derivatives*, 18(2), 58.

**Integrated Volatility**

Our goal is to find the mathematical formula for $\mathbb{E}(\int_t^T \sigma_u^2 du)$ , visualized as the forward price of the realized cumulative variance over $[t, T]$.

$$\int_t^T \sigma_u^2 du = 2\left[\underbrace{\int_t^T \frac{dF_u}{F_u}}_{\text{Dynamic Replication}} - \underbrace{\ln\frac{F_T}{F_t}}_{\text{Static Replication}}\right]$$

proof

$$\begin{cases} \dfrac{dF_t}{F_t} = \sigma_t dW_t \\ d\ln F_t = -\dfrac{\sigma_t^2}{2}dt + \sigma_t dW_t \end{cases} \Rightarrow \frac{dF_t}{F_t} - d\ln F_t = \frac{\sigma_t^2}{2}$$

# VIX formula

**Lemma** For any twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$, and $K \geq 0$, we have

$$\begin{aligned} f(F_T) =& f(K_0) + f'(K_0)(F_T - K_0) + \int_0^{K_0} f''(K)(K - F_T)^+ \, dK \\ &+ \int_{K_0}^{\infty} f''(K)(F_T - K)^+ \, dK \end{aligned}$$

The sum of the two integrals is the integral representation of the remainder in the Taylor expansion of $f(F_T)$ up to the first power term.

# VIX formula

Let $f(F_T) = \ln F_T$, then

$$\ln \frac{F_T}{K_0} = \frac{F_T - K_0}{K_0} - \int_0^{K_0} \frac{(K - F_T)^+}{K^2} dK - \int_{K_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK$$

Recall

$$\int_t^T \sigma_u^2 du = 2 \left[ \int_t^T \frac{dF_u}{F_u} - \ln \frac{F_T}{F_t} \right]$$

Note that:

$$\mathbb{E}\left[ \int_t^T \frac{dF_u}{F_u} \middle| \mathcal{F}_t \right] = 0, \ \ln \frac{F_T}{F_t} = \ln \frac{F_T}{K_0} + \ln \frac{K_0}{F_t}, \text{ and } \mathbb{E}\left[F_T | \mathcal{F}_t\right] = F_t$$

# VIX formula

Hence

$$\mathbb{E}\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right] = 2\mathbb{E}\left[\int_0^{K_0} \frac{(K-F_T)^+}{K^2} dK + \int_{K_0}^{\infty} \frac{(F_T-K)^+}{K^2} dK \middle| \mathcal{F}_t\right]$$
$$- 2\left[\frac{F_t - K_0}{K_0} + \ln \frac{K_0}{F_t}\right]$$

By the risk-neutral condition, we have

$$C_t(k) = e^{-r(T-t)}\mathbb{E}\left[(F_T - K)^+ \middle| \mathcal{F}_t\right] \text{ and } P_t(k) = e^{-r(T-t)}\mathbb{E}\left[(K - F_T)^+ \middle| \mathcal{F}_t\right]$$

# VIX formula

Then

$$\mathbb{E}\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right] = 2e^{r(T-t)}\left[\int_0^{K_0} \frac{P_t(k)}{K^2} dK + \int_{K_0}^{\infty} \frac{C_t(k)}{K^2} dK\right] - 2\left[\frac{F_t - K_0}{K_0} - \ln\frac{F_t}{K_0}\right]$$

where

$$\frac{F_t - K_0}{K_0} - \ln\frac{F_t}{K_0} \approx \frac{1}{2}\left(\frac{F_t - K_0}{K_0}\right)^2$$

# VIX formula

Hence

$$\frac{1}{T-t}\mathbb{E}\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right] = \frac{2}{T-t}e^{r(T-t)}\left[\int_0^{K_0}\frac{P_t(k)}{K^2}dK + \int_{K_0}^{\infty}\frac{C_t(k)}{K^2}dK\right] - \frac{1}{T-t}\left(\frac{F_t-K_0}{K_0}\right)^2$$

Discretization

$$\sigma^2 = \left(\frac{VIX}{100}\right)^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0}-1\right]^2$$

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