Chapter 3

Basic principles of strapdown inertial navigation systems

3.1 Introduction

The previous chapter has provided some insight into the basic measurements that are necessary for inertial navigation. For the purposes of the ensuing discussion, it is assumed that measurements of specific force and angular rate are available along and about axes which are mutually perpendicular. Attention is focused on how these measurements are combined and processed to enable navigation to take place.

3.2 A simple two-dimensional strapdown navigation system

We begin this chapter by describing a simplified two-dimensional strapdown navigation system. Although functionally identical to the full three-dimensional system discussed later, the computational processes which must be implemented to perform the navigation task in two dimensions are much simplified compared with a full strapdown system. Therefore, through this introductory discussion, it is hoped to provide the reader with an appreciation of the basic processing tasks which must be implemented in a strapdown system without becoming too deeply involved in the intricacies and complexities of the full system computational tasks.

For the purposes of this discussion, it is assumed that a system is required to navigate a vehicle which is constrained to move in a single plane. A two-dimensional strapdown system capable of fulfilling this particular navigation task was introduced very briefly in Chapter 2 and is shown diagrammatically in Figure 3.1.

The system contains two accelerometers and a single axis rate gyroscope, all of which are attached rigidly to the body of the vehicle. The vehicle body is represented, in the figure, by the block on which the instruments shown are mounted. The sensitive axes of the accelerometers, indicated by the directions of the arrows in the diagram, are at right angles to one another and aligned with the body axes of the

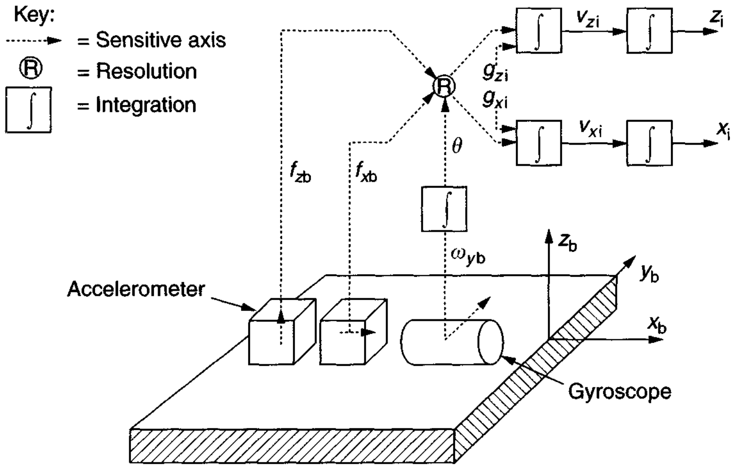

Figure 3.1 Two-dimensional strapdown inertial navigation system

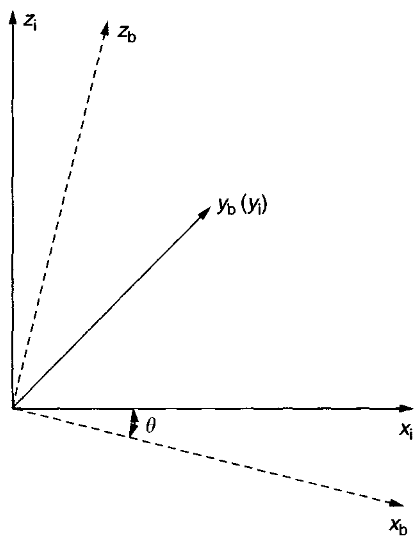

Figure 3.2 Reference frames for two-dimensional navigation

vehicle in the plane of motion; they are denoted as the x_b and z_b axes. The gyroscope is mounted with its sensitive axis orthogonal to both accelerometer axes allowing it to detect rotations about an axis perpendicular to the plane of motion; the y_b axis. It is assumed that navigation is required to take place with respect to a space-fixed reference frame denoted by the axes x_i and z_i. The reference and body axis sets are shown in Figure 3.2, where θ represents the angular displacement between the body and reference frames.

$$\dot{\theta} = \omega_{yb}$$
$$f_{xi} = f_{xb}\cos\theta + f_{zb}\sin\theta$$
$$f_{zi} = -f_{xb}\sin\theta + f_{zb}\cos\theta$$
$$\dot{v}_{xi} = f_{xi} + g_{xi}$$
$$\dot{v}_{zi} = f_{zi} + g_{zi}$$
$$\dot{x}_i = v_{xi}$$
$$\dot{z}_i = v_{zi}$$

Figure 3.3 Two-dimensional strapdown navigation system equations

Referring now to Figure 3.1, body attitude, θ, is computed by integrating the measured angular rate, ω_{yb}, with respect to time. This information is then used to resolve the measurements of specific force, f_{xb} and f_{zb}, into the reference frame. A gravity model, stored in the computer, is assumed to provide estimates of the gravity components in the reference frame, g_{xi} and g_{zi}. These quantities are combined with the resolved measurements of specific force, f_{xi} and f_{zi}, to determine true accelerations, denoted by $\dot{v}_{xi}$ and $\dot{v}_{zi}$. These derivatives are subsequently integrated twice to obtain estimates of vehicle velocity and position. The full set of equations which must be solved are given in Figure 3.3.

Having defined the basic functions which must be implemented in a strapdown inertial navigation system, consideration is now given to the application of the two-dimensional system, described above, for navigation in a rotating reference frame. For instance, consider the situation where it is required to navigate a vehicle moving in a meridian plane around the Earth, as depicted in Figure 3.4. Hence, we are concerned here with a system which is operating in the vertical plane alone. Such a system would be required to provide estimates of velocity with respect to the Earth, position along the meridian and height above the Earth.

Whilst the system mechanisation as described could be used to determine such information, this would entail a further transformation of the velocity and position, derived in space fixed coordinates, to a geographic frame. An alternative and often used approach is to navigate directly in a local geographic reference frame, defined in this simplified case by the direction of the local vertical at the current location of the vehicle. In order to provide the required navigation information, it now becomes necessary to keep track of vehicle attitude with respect to the local geographic frame denoted by the axes x and z. This information can be extracted by differencing the successive gyroscopic measurements of body turn rate with respect to inertial space, and the current estimate of the turn rate of the reference frame with respect to inertial space. For a vehicle moving at a velocity, v_x, in a single plane around a perfectly spherical Earth of radius R_0, this rate is given by $v_x/(R_0 + z)$ where z is the height of the vehicle above the surface of the Earth. This is often referred to as the transport rate.

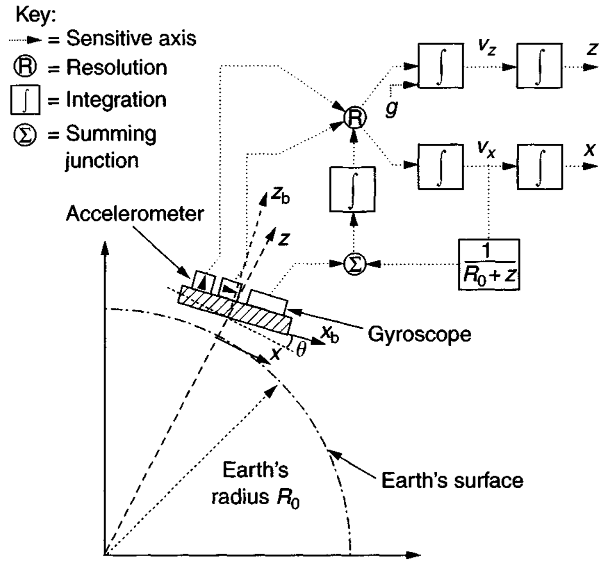

Figure 3.4 Two-dimensional strapdown inertial system for navigation in a rotating reference frame

$$\dot{\theta} = \omega_{yb} - v_x / (R_0 + z)$$
$$f_x = f_{xb} \cos\theta + f_{zb} \sin\theta$$
$$f_z = -f_{xb} \sin\theta + f_{zb} \cos\theta$$
$$\dot{v}_x = f_x + v_x v_z / (R_0 + z)$$
$$\dot{v}_z = f_z + g - v_x^2 / (R_0 + z)$$
$$\dot{x} = v_x$$
$$\dot{z} = v_z$$

Figure 3.5 Simplified two-dimensional strapdown system equations for navigation in a rotating reference frame

Figure 3.4 shows a modified two-dimensional strapdown system for navigation in the moving reference frame. As shown in the figure, an estimate of the turn rate of the reference frame is derived using the estimated component of horizontal velocity.

The equations which must be solved in this system are given in Figure 3.5.

Comparison with the equations given in Figure 3.3, relating to navigation with respect to a space-fixed axis set, reveals the following differences. The attitude computation is modified to take account of the turn rate of the local vertical reference frame as described above. Consequently, the equation in θ is modified by the subtraction of the term $v_x/(R_0 + z)$ in Figure 3.4. The terms $v_x v_z/(R_0 + z)$ and $v_x^2/(R_0 + z)$ which appear in the velocity equations are included to take account of the additional forces acting as the system moves around the Earth (Coriolis forces, see Section 3.4). The gravity term (g) appears only in the v_z equation as it is assumed that the Earth's gravitational acceleration acts precisely in the direction of the local vertical.

This section has outlined the basic form of the computing tasks to be implemented in a strapdown navigation system using a much simplified two-dimensional representation. In the remainder of this chapter the extension of this simple strapdown system to three dimensions is described in some detail. It will be appreciated that this entails a substantial increase in the complexity of the computing tasks involved. In particular, attitude information in three dimensions can no longer be obtained by a simple integration of the measured turn rates.

3.3 Reference frames

Fundamental to the process of inertial navigation is the precise definition of a number of Cartesian co-ordinate reference frames. Each frame is an orthogonal, right-handed, co-ordinate frame or axis set.

For navigation over the Earth, it is necessary to define axis sets which allow the inertial measurements to be related to the cardinal directions of the Earth, that is, frames which have a physical significance when attempting to navigate in the vicinity of the Earth. Therefore, it is customary to consider an inertial reference frame which is stationary with respect to the fixed stars, the origin of which is located at the centre of the Earth. Such a reference frame is shown in Figure 3.6, together with an Earth-fixed reference frame and a local geographic navigation frame defined for the purposes of terrestrial inertial navigation.

The following co-ordinate frames are used in the text:

The inertial frame (i-frame) has its origin at the centre of the Earth and axes which are non-rotating with respect to the fixed stars, defined by the axes Ox_i, Oy_i, Oz_i, with Oz_i coincident with the Earth's polar axis (which is assumed to be invariant in direction).

The Earth frame (e-frame) has its origin at the centre of the Earth and axes which are fixed with respect to the Earth, defined by the axes Ox_e, Oy_e, Oz_e with Oz_e along the Earth's polar axis. The axis Ox_e lies along the intersection of the plane of the Greenwich meridian with the Earth's equatorial plane. The Earth frame rotates, with respect to the inertial frame, at a rate Ω about the axis Oz_i.

The navigation frame (n-frame) is a local geographic frame which has its origin at the location of the navigation system, point P, and axes aligned with the directions of north, east and the local vertical (down). The turn rate of the navigation frame, with respect to the Earth-fixed frame, ω_{en}, is governed by the motion of the point P with respect to the Earth. This is often referred to as the transport rate.

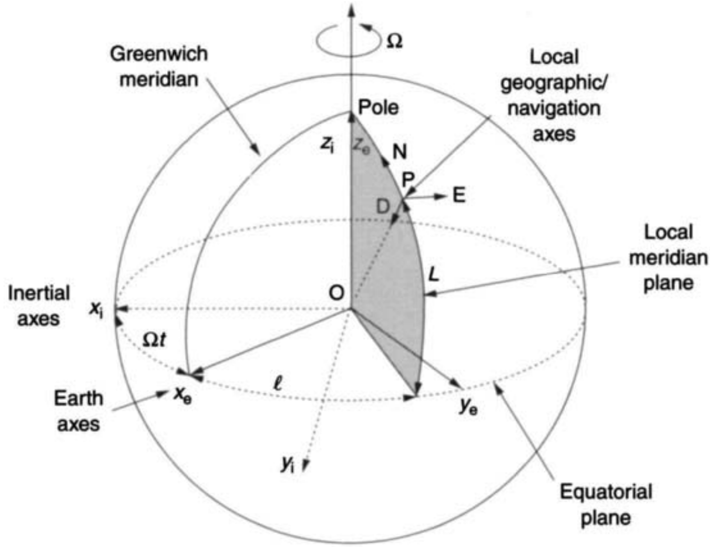

Figure 3.6 Frames of reference

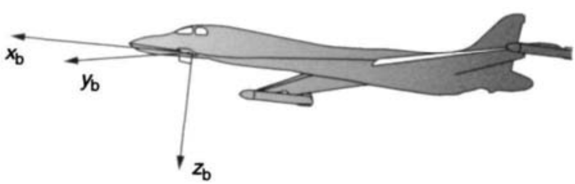

Figure 3.7 Illustration of a body reference frame

The wander azimuth frame (w-frame) may be used to avoid the singularities in the computation which occur at the poles of the navigation frame. Like the navigation frame, it is locally level but is rotated through the wander angle about the local vertical. Its use is described in Section 3.5.

The body frame (b-frame), depicted in Figure 3.7, is an orthogonal axis set which is aligned with the roll, pitch and yaw axes of the vehicle in which the navigation system is installed.

3.4 Three-dimensional strapdown navigation system – general analysis

3.4.1 Navigation with respect to a fixed frame

Consider the situation where it is required to navigate with respect to a fixed, or non-accelerating, and non-rotating set of axes. The measured components of specific

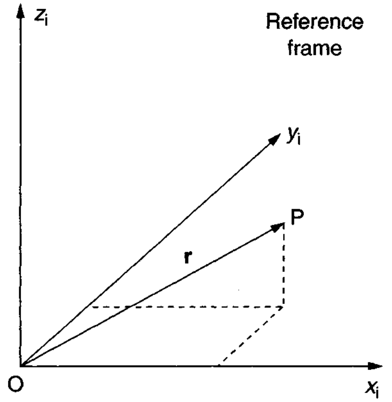

Figure 3.8 Position vector with respect to reference frame

force and estimates of the gravitational field are summed to determine components of acceleration with respect to a space-fixed reference frame. These quantities can then be integrated twice, giving estimates of velocity and position in that frame.

This process may be expressed mathematically in the following manner.[1] Let $\mathbf{r}$ represent the position vector of the point P with respect to O, the origin of the reference frame shown in Figure 3.8.

The acceleration of P with respect to a space-fixed axis set, termed the i-frame and denoted by the subscript i, is defined by:

$$\mathbf{a}_\mathrm{i} = \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} \tag{3.1}$$

A triad of perfect accelerometers will provide a measure of the specific force ($\mathbf{f}$) acting at point P where

$$\mathbf{f} = \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} - \mathbf{g} \tag{3.2}$$

in which $\mathbf{g}$ is the mass attraction gravitation vector.

Rearranging eqn. (3.2) yields the following equation:

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_\mathrm{i} = \mathbf{f} + \mathbf{g} \tag{3.3}$$

This is called the navigation equation since, with suitable integration, it yields the navigational quantities of velocity and position. The first integral gives the velocity

[1] Vector and matrix notation is widely used throughout the text for the mathematical representation of strapdown inertial system processes. This notation is adopted both in the interests of brevity and to be consistent with other texts on the subject. Vector and matrix quantities are written in boldface type.

of point P with respect to the i-frame, viz.

$$\mathbf{v}_{\mathrm{i}} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{i}} \tag{3.4}$$

whilst a second integration gives its position in that frame.

3.4.2 *Navigation with respect to a rotating frame*

In practice, one often needs to derive estimates of a vehicle's velocity and position with respect to a rotating reference frame, as when navigating in the vicinity of the Earth. In this situation, additional apparent forces will be acting which are functions of reference frame motion. This results in a revised form of the navigation equation which may be integrated to determine the ground speed of the vehicle, $\mathbf{v}_{\mathrm{e}}$, directly. Alternatively, $\mathbf{v}_{\mathrm{e}}$ may be computed from the inertial velocity, $\mathbf{v}_{\mathrm{i}}$, using the theorem of Coriolis, as follows,

$$\mathbf{v}_{\mathrm{e}} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{e}} = \mathbf{v}_{\mathrm{i}} - \boldsymbol{\omega}_{\mathrm{ie}} \times \mathbf{r} \tag{3.5}$$

where $\boldsymbol{\omega}_{\mathrm{ie}} = [0 \quad 0 \quad \Omega]^{\mathrm{T}}$ is the turn rate of the Earth frame with respect to the i-frame and $\times$ denotes a vector cross product.

Revised forms of the navigation equation suitable for navigation with respect to the Earth are the subject of Section 3.5.

3.4.3 *The choice of reference frame*

The navigation equation, eqn. (3.3), may be solved in any one of a number of reference frames. If the Earth frame is chosen, for example, then the solution of the navigation equation will provide estimates of velocity with respect to either the inertial frame or the Earth frame, expressed in Earth coordinates, denoted $\mathbf{v}_{\mathrm{i}}^{\mathrm{e}}$ and $\mathbf{v}_{\mathrm{e}}^{\mathrm{e}}$, respectively.[2]

In Section 3.5, a number of different strapdown system mechanisations for navigating with respect to the Earth are described. In each case, it will be shown that the navigation equation is expressed in a different manner depending on the choice of reference frame.

3.4.4 *Resolution of accelerometer measurements*

The accelerometers usually provide a measurement of specific force in a body fixed axis set, denoted $\mathbf{f}^{\mathrm{b}}$. In order to navigate, it is necessary to resolve the components of the specific force in the chosen reference frame. In the event that the inertial frame is selected, this may be achieved by pre-multiplying the vector quantity $\mathbf{f}^{\mathrm{b}}$ by the direction cosine matrix, $\mathbf{C}_{\mathrm{b}}^{\mathrm{i}}$, using,

$$\mathbf{f}^{\mathrm{i}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{i}}\mathbf{f}^{\mathrm{b}} \tag{3.6}$$

[2] Superscripts attached to vector quantities denote the axis set in which the vector quantity coordinates are expressed.

where $\mathbf{C}_{\mathrm{b}}^{\mathrm{i}}$ is a 3×3 matrix which defines the attitude of the body frame with respect to the i-frame. The direction cosine matrix $\mathbf{C}_{\mathrm{b}}^{\mathrm{i}}$ may be calculated from the angular rate measurements provided by the gyroscopes using the following equation:

$$\dot{\mathbf{C}}_{\mathrm{b}}^{\mathrm{i}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{i}} \mathbf{\Omega}_{\mathrm{ib}}^{\mathrm{b}} \tag{3.7}$$

where $\mathbf{\Omega}_{\mathrm{ib}}^{\mathrm{b}}$ is the skew symmetric matrix:

$$\mathbf{\Omega}_{\mathrm{ib}}^{\mathrm{b}} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{3.8}$$

This matrix is formed from the elements of the vector $\boldsymbol{\omega}_{\mathrm{ib}}^{\mathrm{b}} = [p \quad q \quad r]^{\mathrm{T}}$ which represents the turn rate of the body with respect to the i-frame as measured by the gyroscopes. Equation (3.7) is derived in Section 3.6.

The attitude of the body with respect to the chosen reference frame, which is required to resolve the specific force measurements into the reference frame, may be defined in a number of different ways. For the purposes of the discussion of navigation system mechanisations in this and the following section, the direction cosine method will be adopted. Direction cosines and some alternative attitude representations are described in some detail in Section 3.6.

3.4.5 System example

Consider the situation in which it is required to navigate with respect to inertial space and the solution of the navigation takes place in the i-frame. Equation (3.3) may be expressed in i-frame coordinates as follows:

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{i}} = \mathbf{f}^{\mathrm{i}} + \mathbf{g}^{\mathrm{i}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{i}}\mathbf{f}^{\mathrm{b}} + \mathbf{g}^{\mathrm{i}} \tag{3.9}$$

It is clear from the preceding discussion that the integration of the navigation equation involves the use of information from both the gyroscopes and the accelerometers contained within the inertial navigation system. A block diagram representation of the resulting navigation system is given in Figure 3.9.

The diagram displays the main functions to be implemented within a strapdown navigation system; the processing of the rate measurements to generate body attitude, the resolution of the specific force measurements into the inertial reference frame, gravity compensation and the integration of the resulting acceleration estimates to determine velocity and position.

3.5 Strapdown system mechanisations

Attention is focused here on inertial systems which may be used to navigate in the vicinity of the Earth. It has been shown in Section 3.4 how estimates of position and velocity are derived by integrating a navigation equation of the form given in

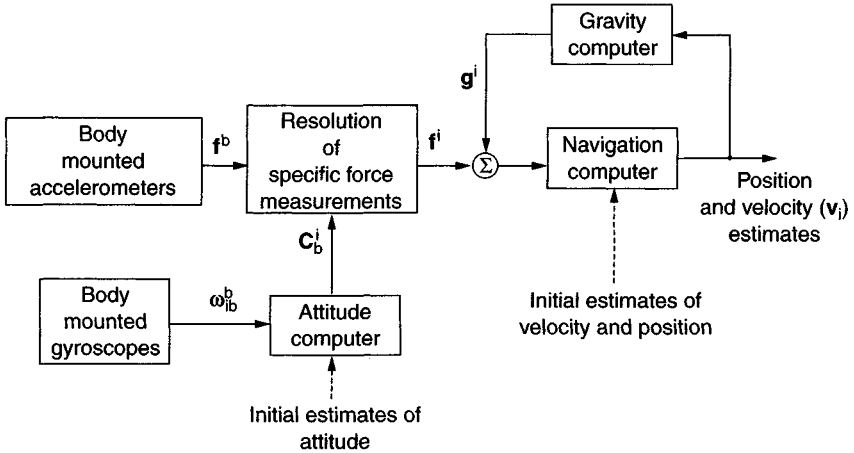

Figure 3.9 Strapdown inertial navigation system

eqn. (3.3). In systems of the type described later, in which it is required to derive estimates of vehicle velocity and position with respect to an Earth fixed frame, additional apparent forces will be acting which are functions of the reference frame motion. In this section, further forms of the navigation equation are derived, corresponding to different choices of reference frame [1].

The resulting system mechanisations are described together with their applications. As will become apparent, the variations in the mechanisations described here are in the strapdown computational algorithms and not in the arrangement of the sensors or the mechanical layout of the system.

3.5.1 *Inertial frame mechanisation*

In this system, it is required to calculate vehicle speed with respect to the Earth, the ground speed, in inertial axes, denoted by the symbol $\mathbf{v}_e^i$. This may be accomplished by expressing the navigation equation (eqn. (3.3)) in inertial axes and deriving an expression for $\left.\frac{d^2\mathbf{r}}{dt^2}\right|_e$ in terms of ground speed and its time derivatives with respect to the inertial frame.

Inertial velocity may be expressed in terms of ground speed using the Coriolis equation, viz.

$$\left.\frac{d\mathbf{r}}{dt}\right|_i = \left.\frac{d\mathbf{r}}{dt}\right|_e + \boldsymbol{\omega}_{ie} \times \mathbf{r} \tag{3.10}$$

Differentiating this expression and writing $\left.\frac{d\mathbf{r}}{dt}\right|_e = \mathbf{v}_e$, we have,

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_i = \left.\frac{d\mathbf{v}_e}{dt}\right|_i + \left.\frac{d}{dt}[\boldsymbol{\omega}_{ie} \times \mathbf{r}]\right|_i \tag{3.11}$$

Applying the Coriolis equation in the form of eqn. (3.10) to the second term in eqn. (3.11) gives:

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_i = \left.\frac{d\mathbf{v}_e}{dt}\right|_i + \boldsymbol{\omega}_{ie} \times \mathbf{v}_e + \boldsymbol{\omega}_{ie} \times [\boldsymbol{\omega}_{ie} \times \mathbf{r}] \tag{3.12}$$

In generating the above equation, it is assumed that the turn rate of the Earth is constant, hence $\frac{d\boldsymbol{\omega}_{ie}}{dt} = 0$.

Combining eqns. (3.3) and (3.12) and rearranging yields:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_i = \mathbf{f} - \boldsymbol{\omega}_{ie} \times \mathbf{v}_e - \boldsymbol{\omega}_{ie} \times [\boldsymbol{\omega}_{ie} \times \mathbf{r}] + \mathbf{g} \tag{3.13}$$

In this equation, $\mathbf{f}$ represents the specific force acceleration to which the navigation system is subjected, while $\boldsymbol{\omega}_{ie} \times \mathbf{v}_e$ is the acceleration caused by its velocity over the surface of a rotating Earth, usually referred to as the Coriolis acceleration. The term $\boldsymbol{\omega}_{ie} \times [\boldsymbol{\omega}_{ie} \times \mathbf{r}]$, in eqn. (3.13), defines the centripetal acceleration experienced by the system owing to the rotation of the Earth, and is not separately distinguishable from the gravitational acceleration which arises through mass attraction, $\mathbf{g}$. The sum of the accelerations caused by the mass attraction force and the centripetal force constitutes what is known as the local gravity vector, the vector to which a 'plumb bob' would align itself when held above the Earth (Figure 3.10). This is denoted here by the symbol $\mathbf{g}_l$, that is:

$$\mathbf{g}_l = \mathbf{g} - \boldsymbol{\omega}_{ie} \times [\boldsymbol{\omega}_{ie} \times \mathbf{r}] \tag{3.14}$$

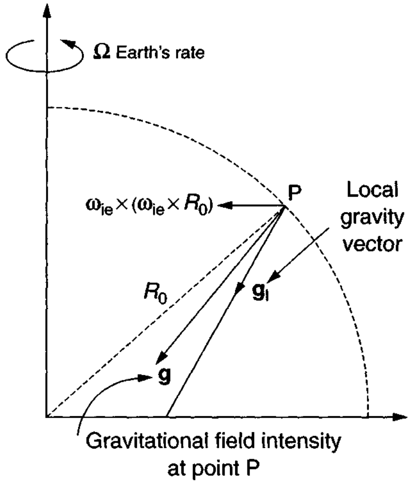

Figure 3.10 Diagram showing the components of the gravitational field

Combining eqns. (3.13) and (3.14) gives the following form of the navigation equation:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_i = \mathbf{f} - \boldsymbol{\omega}_{ie} \times \mathbf{v}_e + \mathbf{g}_l \tag{3.15}$$

This equation may be expressed in inertial axes, as follows, using the superscript notation mentioned earlier.

$$\dot{\mathbf{v}}_e^i = \mathbf{f}^i - \boldsymbol{\omega}_{ie}^i \times \mathbf{v}_e^i + \mathbf{g}_l^i \tag{3.16}$$

The measurements of specific force provided by the accelerometers are in body axes, as denoted by the vector quantity $\mathbf{f}^b$. In order to set up the navigation eqn. (3.16), the accelerometer outputs must be resolved into inertial axes to give $\mathbf{f}^i$. This may be achieved by pre-multiplying the measurement vector $\mathbf{f}^b$ by the direction cosine matrix $\mathbf{C}_b^i$ as described in Section 3.4.4 (eqn. (3.6)). Given knowledge of the attitude of the body at the start of navigation, the matrix $\mathbf{C}_b^i$ is updated using eqns. (3.7) and (3.8) based on measurements of the body rates with respect to the i-frame which may be expressed as follows:

$$\boldsymbol{\omega}_{ib}^b = [p \quad q \quad r]^T \tag{3.17}$$

Substituting for $\mathbf{f}^i$ from eqn. (3.6) in eqn. (3.16) gives the following form of the navigation equation:

$$\dot{\mathbf{v}}_e^i = \mathbf{C}_b^i\mathbf{f}^b - \boldsymbol{\omega}_{ie}^i \times \mathbf{v}_e^i + \mathbf{g}_l^i \tag{3.18}$$

The final term in this equation represents the local gravity vector expressed in the inertial frame.

A block diagram representation of the resulting inertial frame mechanisation is shown in Figure 3.11.

3.5.2 *Earth frame mechanisation*

In this system, ground speed is expressed in an Earth-fixed co-ordinate frame to give $\mathbf{v}_e^e$. It follows from the Coriolis equation, that the rate of change of $\mathbf{v}_e$, with respect to Earth axes, may be expressed in terms of its rate of change in inertial axes using:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_e = \left.\frac{d\mathbf{v}_e}{dt}\right|_i - \boldsymbol{\omega}_{ie} \times \mathbf{v}_e \tag{3.19}$$

Substituting for $\left.\frac{d\mathbf{v}_e}{dt}\right|_i$ from eqn. (3.15), we have:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_e = \mathbf{f} - 2\boldsymbol{\omega}_{ie} \times \mathbf{v}_e + \mathbf{g}_l \tag{3.20}$$

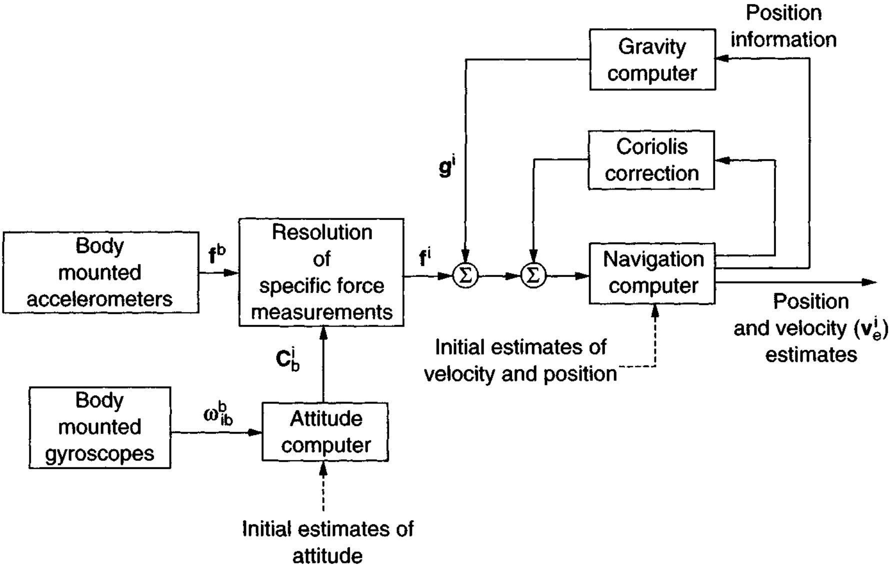

Figure 3.11 Strapdown inertial navigation system – inertial frame mechanisation

This may be expressed in Earth axes as follows:

$$\dot{\mathbf{v}}_e^e = \mathbf{C}_b^e \mathbf{f}^b - 2\boldsymbol{\omega}_{ie}^e \times \mathbf{v}_e^e + \mathbf{g}_l^e \tag{3.21}$$

where $\mathbf{C}_b^e$ is the direction cosine matrix used to transform the measured specific force vector into Earth axes. This matrix propagates in accordance with the following equation:

$$\dot{\mathbf{C}}_b^e = \mathbf{C}_b^e \boldsymbol{\Omega}_{eb}^b \tag{3.22}$$

where $\boldsymbol{\Omega}_{eb}^b$ is the skew symmetric form of $\boldsymbol{\omega}_{eb}^b$, the body rate with respect to the Earth-fixed frame. This is derived by differencing the measured body rates, $\boldsymbol{\omega}_{ib}^b$, and estimates of the components of Earth's rate, $\boldsymbol{\omega}_{ie}$, expressed in body axes as follows:

$$\boldsymbol{\omega}_{eb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{C}_e^b \boldsymbol{\omega}_{ie}^e \tag{3.23}$$

in which $\mathbf{C}_e^b = \mathbf{C}_b^{eT}$, the transpose of the matrix $\mathbf{C}_b^e$.

A block diagram representation of the Earth frame mechanisation is shown in Figure 3.12.

A variation on this system may be used when it is required to navigate over relatively short distances, with respect to a fixed point on the Earth. A mechanisation of this type may be used for a tactical missile application in which navigation is required with respect to a ground based tracking station. In such a system,

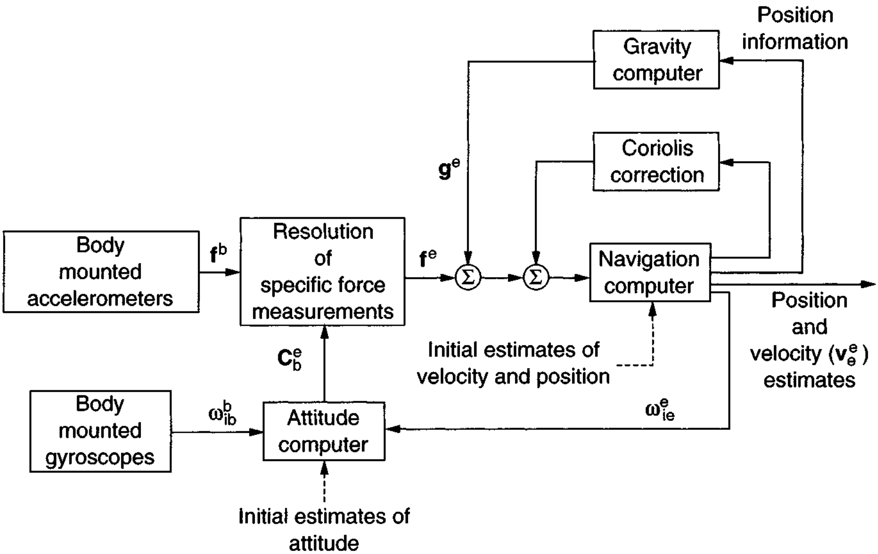

Figure 3.12 Strapdown inertial navigation system – Earth frame mechanisation

target tracking information provided by the ground station may need to be combined with the outputs of an on-board inertial navigation system to provide missile mid-course guidance commands. In order that the missile may operate in harmony with the ground systems, all information must be provided in a common frame of reference.

In this situation, an Earth-fixed reference frame may be defined, the origin of which is located at the tracking station, its axes aligned with the local vertical and a plane which is tangential to the Earth's surface, as illustrated in Figure 3.13.

For very short term navigation, as required for some tactical missile applications, further simplifications to this system mechanisation may be permitted. For instance, where the navigation period is short, typically 10 minutes or less, the effects of the rotation of the Earth on the attitude computation process can sometimes be ignored, and Coriolis corrections are no longer essential in the velocity equation to give sufficiently accurate navigation. In this situation, attitude is computed solely as a function of the turn rates measured by the gyroscopes, and eqn. (3.21) reduces to the following:

$$\dot{\mathbf{v}}_e^e = \mathbf{C}_b^e \mathbf{f}^b + \mathbf{g}_l^e \tag{3.24}$$

It is stressed, that such simplifications can only be allowed in cases where the navigation errors, induced by the omission of Earth rate and Coriolis terms, lie within the error bounds in which the navigation system is required to operate. This situation arises when the permitted gyroscopic errors are in excess of the rotation rate of the Earth, and allowable accelerometer biases are in excess of the acceleration errors introduced by ignoring the Coriolis forces.

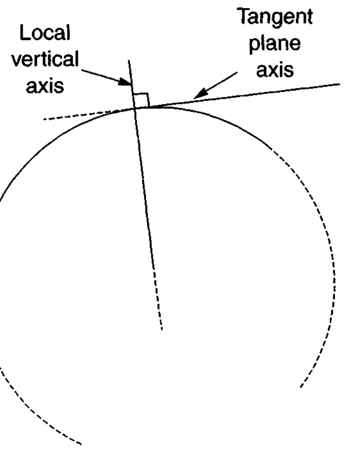

Figure 3.13 Tangent plane axis set

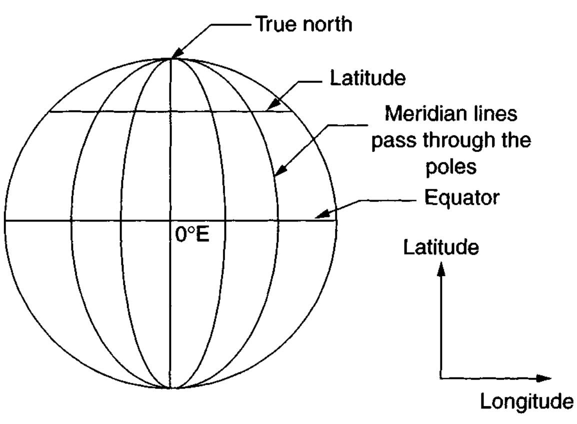

Figure 3.14 Geographic co-ordinate system

3.5.3 Local geographic navigation frame mechanisation

In order to navigate over large distances around the Earth, navigation information is most commonly required in the local geographic or navigation axis set described earlier. Position on the Earth may be specified in terms of latitude (degrees north or south of a datum) and longitude (degrees east or west of a datum). Figure 3.14 shows this geographic co-ordinate system on a globe. Lines of constant latitude and longitude are called parallels and meridians, respectively.

Navigation data are expressed in terms of north and east velocity components, latitude, longitude and height above the Earth. Whilst such information can be computed using the position estimates provided by the inertial or Earth frame mechanisations described before, this involves a further transformation of the vector quantities $\mathbf{v}_e^i$ or $\mathbf{v}_e^e$. Further, difficulties arise in representing the Earth's gravitational field precisely in a computer. For these reasons, the navigation frame mechanisation, described here, is often used when navigating around the Earth.

In this mechanisation, ground speed is expressed in navigation coordinates to give $\mathbf{v}_e^n$. The rate of change of $\mathbf{v}_e^n$ with respect to navigation axes may be expressed in terms of its rate of change in inertial axes as follows:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_n = \left.\frac{d\mathbf{v}_e}{dt}\right|_i - [\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}] \times \mathbf{v}_e \tag{3.25}$$

Substituting for $\left.\frac{d\mathbf{v}_e}{dt}\right|_i$, from eqn. (3.15), we have:

$$\left.\frac{d\mathbf{v}_e}{dt}\right|_n = \mathbf{f} - [2\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}] \times \mathbf{v}_e + \mathbf{g}_l \tag{3.26}$$

This may be expressed in navigation axes as follows:

$$\dot{\mathbf{v}}_e^n = \mathbf{C}_b^n \mathbf{f}^b - [2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n] \times \mathbf{v}_e^n + \mathbf{g}_l^n \tag{3.27}$$

where $\mathbf{C}_b^n$ is a direction cosine matrix used to transform the measured specific force vector into navigation axes. This matrix propagates in accordance with the following equation.

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \boldsymbol{\Omega}_{nb}^b \tag{3.28}$$

where $\boldsymbol{\Omega}_{nb}^b$ is the skew symmetric form of $\boldsymbol{\omega}_{nb}^b$, the body rate with respect to the navigation frame. This is derived by differencing the measured body rates, $\boldsymbol{\omega}_{ib}^b$, and estimates of the components of navigation frame rate, $\boldsymbol{\omega}_{in}$. The latter term is obtained by summing the Earth's rate with respect to the inertial frame and the turn rate of the navigation frame with respect to the Earth, that is, $\boldsymbol{\omega}_{in} = \boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}$. Therefore,

$$\boldsymbol{\omega}_{nb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{C}_n^b[\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n] \tag{3.29}$$

A block diagram representation of the navigation frame mechanisation is shown in Figure 3.15.

It is instructive to consider the physical significance of the various terms in the navigation equation (3.27). From this equation, it can be seen that the rate of change of the velocity, with respect to the surface of the Earth, is made up of the following terms:

1. The specific force acting on the vehicle, as measured by a triad of accelerometers mounted within it.
2. A correction for the acceleration caused by the vehicle's velocity over the surface of a rotating Earth, usually referred to as the Coriolis acceleration. The effect in two dimensions is illustrated in Figure 3.16. As the point P moves away

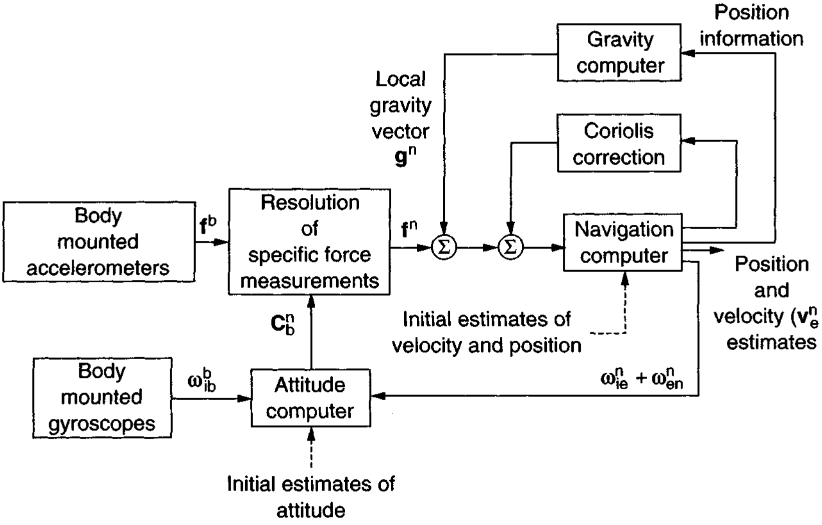

Figure 3.15 Strapdown inertial navigation system – local geographic navigation frame mechanisation

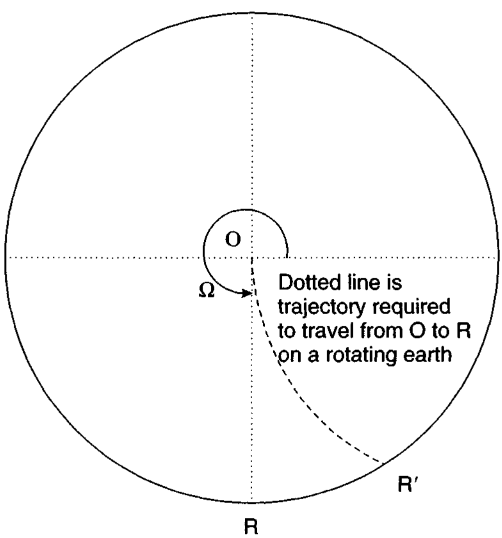

Figure 3.16 Illustration of the effect of Coriolis acceleration

from the axis of rotation, it traces out a curve in space as a result of the Earth's rotation.

3. A correction for the centripetal acceleration of the vehicle, resulting from its motion over the Earth's surface. For instance, a vehicle moving due east over

the surface of the Earth will trace out a circular path with respect to inertial axes. To follow this path, the vehicle is subject to a force acting towards the centre of the Earth of magnitude equal to the product of its mass, its linear velocity and its turn rate with respect to the Earth.

4. Compensation for the apparent gravitational force acting on the vehicle. This includes the gravitational force caused by the mass attraction of the Earth, and the centripetal acceleration of the vehicle resulting from the rotation of the Earth. The latter term arises even if the vehicle is stationary with respect to the Earth, since the path which its follows in space is circular.

A simple example serves to illustrate the importance of the Coriolis effect. Consider a vehicle launched from the north pole with the intention of flying to New York city. The vehicle is assumed to travel at an average speed of 3600 miles/h. During the flight, of approximately 1 h, the Earth will have rotated by about 15°, a distance of approximately 900 miles at the latitude of New York. Consequently, if no Coriolis correction was made to the on-board inertial guidance system during the course of the flight, the vehicle would arrive in the Chicago area rather than New York as originally intended.

3.5.4 *Wander azimuth navigation frame mechanisation*

In the local geographic navigation frame mechanisation described in the previous section, the n-frame is required to rotate continuously as the system moves over the surface of the Earth in order to keep its x-axis parallel to true north. In order to achieve this condition worldwide, the n-frame must rotate at much greater rates about its z-axis as the navigation system moves over the surface of the Earth in the polar regions, compared to the rates required at lower latitudes. This effect is illustrated in Figure 3.17 which shows a polar view of a near polar crossing. It should be clear from the diagram that the rate at which the local geographic navigation frame must rotate about its z-axis in order to maintain the x-axis pointing at the pole becomes very large, the heading direction slewing rapidly through 180° when moving past the pole. In the most extreme case, a direct crossing of the pole, the turn rate becomes infinite when passing over the pole.

The effect is illustrated mathematically as follows. The turn rate of the navigation frame, the transport rate, may be expressed in component form as:

$$\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}} = \left[\frac{v_{\mathrm{E}}}{R_0 + h} \quad \frac{-v_{\mathrm{N}}}{R_0 + h} \quad \frac{-v_{\mathrm{E}} \tan L}{R_0 + h} \right]^{\mathrm{T}} \tag{3.30}$$

where v_{N} is the north velocity, v_{E} the east velocity, R_0 the radius of the Earth, L the latitude and h the height above ground.

It will be seen that the third component of the transport rate becomes indeterminate at the geographic poles.

One way of avoiding the singularity, and so providing a navigation system with world-wide capability, is to adopt a wander azimuth mechanisation in which the z-component of $\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}}$ is set to zero. A wander axis system is a locally level frame

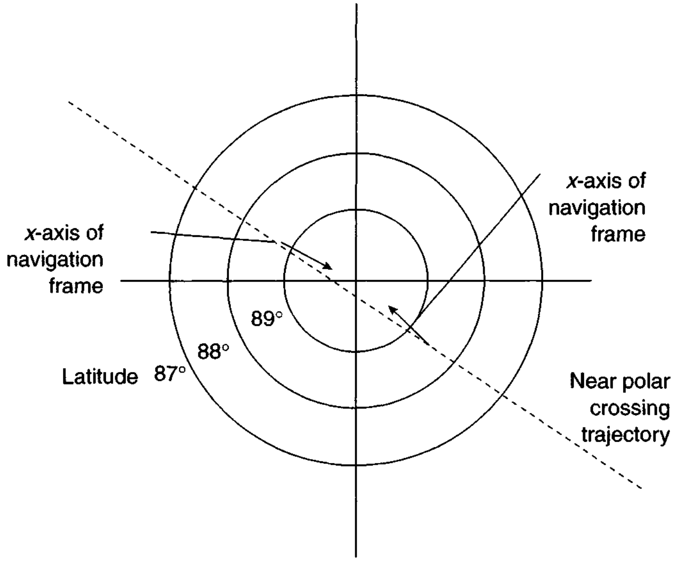

Figure 3.17 Geographic reference singularity at pole crossings

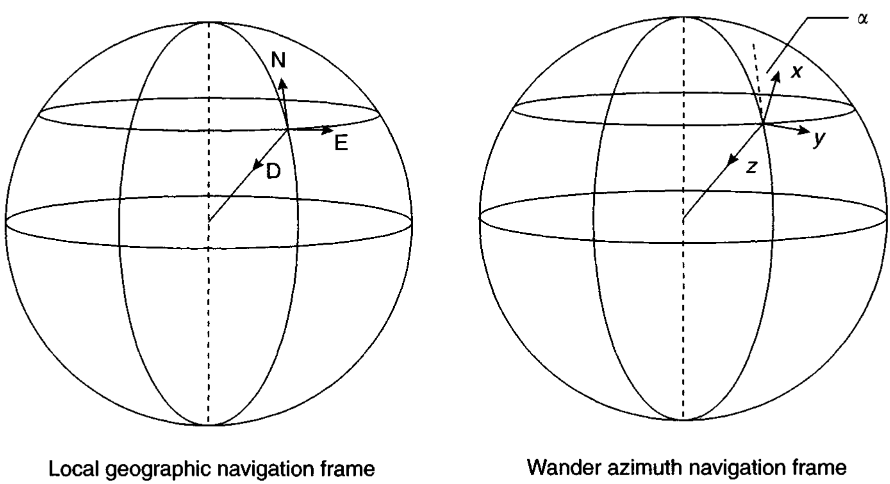

Figure 3.18 Illustration of wander azimuth frame

which moves over the Earth's surface with the vehicle, as depicted in Figure 3.18. However, as the name implies, the azimuth angle between true north and the x-axis of the wander axis frame varies with vehicle position on the Earth. This variation is chosen in order to avoid discontinuities in the orientation of the wander frame with respect to the Earth as the vehicle passes over either the north or south poles.

A navigation equation for a wander azimuth system, which is similar in form to eqn. (3.27), may be constructed as follows:

$$\dot{\mathbf{v}}_{\mathrm{e}}^{\mathrm{w}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{w}}\mathbf{f}^{\mathrm{b}} - [2\mathbf{C}_{\mathrm{e}}^{\mathrm{w}}\boldsymbol{\omega}_{\mathrm{e}}^{\mathrm{e}} + \boldsymbol{\omega}_{\mathrm{ew}}^{\mathrm{w}}] \times \mathbf{v}_{\mathrm{e}}^{\mathrm{w}} + \mathbf{g}_{\mathrm{l}}^{\mathrm{w}} \tag{3.31}$$

This equation is integrated to generate estimates of vehicle ground speed in the wander azimuth frame, $\mathbf{v}_{\mathrm{e}}^{\mathrm{w}}$. This is then used to generate the turn rate of the wander frame with respect to the Earth, $\boldsymbol{\omega}_{\mathrm{ew}}^{\mathrm{w}}$. The direction cosine matrix which relates the wander frame to the Earth frame, $\mathbf{C}_{\mathrm{e}}^{\mathrm{w}}$, may be updated using the equation

$$\dot{\mathbf{C}}_{\mathrm{e}}^{\mathrm{w}} = \mathbf{C}_{\mathrm{e}}^{\mathrm{w}}\boldsymbol{\Omega}_{\mathrm{ew}}^{\mathrm{w}} \tag{3.32}$$

where $\boldsymbol{\Omega}_{\mathrm{ew}}^{\mathrm{w}}$ is a skew symmetric matrix formed from the elements of the angular rate vector $\boldsymbol{\omega}_{\mathrm{ew}}^{\mathrm{w}}$. This process is implemented iteratively and enables any singularities to be avoided. Further details concerning wander azimuth systems and the mechanisations described earlier appear in Reference 1.

3.5.5 Summary of strapdown system mechanisations

This section has provided outline descriptions of a number of possible strapdown inertial navigation system mechanisations. Further details are given in Reference 1. The choice of mechanisation is dependent on the application. Whilst any of the schemes described may be used for navigation close to the Earth, the local geographic navigation frame mechanisation is commonly employed for navigation over large distances. The wander azimuth system provides a world-wide navigation capability. These mechanisations provide navigation data in terms of north and east velocity, latitude and longitude and allow a relatively simple gravity model to be used. For navigation over shorter distances, an Earth fixed reference system may be applicable.

3.6 Strapdown attitude representations

3.6.1 Introductory remarks

Consider now ways in which a set of strapdown gyroscopic sensors may be used to instrument a reference co-ordinate frame within a vehicle which is free to rotate about any direction. The attitude of the vehicle with respect to the designated reference frame may be stored as a set of numbers in a computer within the vehicle. The stored attitude is updated as the vehicle rotates using the measurements of turn rate provided by the gyroscopes.

The co-ordinate frames referred to during the course of the discussion which follows are orthogonal, right-handed axis sets in which positive rotations about each axis are taken to be in a clockwise direction looking along the axis from the origin, as indicated in the Figure 3.19. A negative rotation acts in an opposite sense, that is, in an anti-clockwise direction. This convention is used throughout this book.

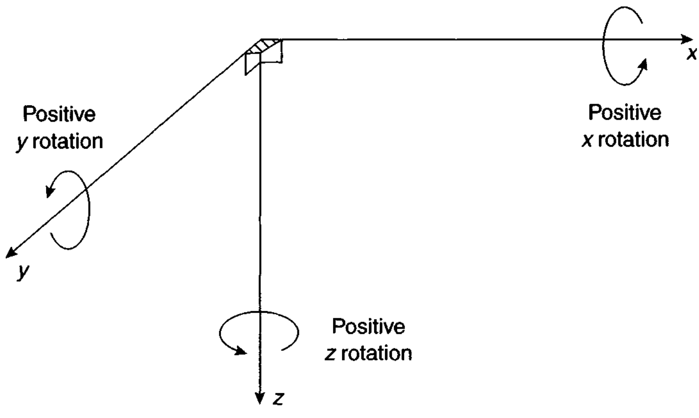

Figure 3.19 *Definition of axis rotations*

It is important to remember that the change in attitude of a body, which is subjected to a series of rotations about different axes, is not only a function of the angles through which it rotates about each of those axes, but the order in which the rotations occur. The illustration given in Figure 3.20, although somewhat extreme, shows quite clearly that the order in which a sequence of rotations occurs is most important.

Rotations are defined here with respect to the orthogonal right-handed axis set, Oxyz, indicated in the figure. The sequence of rotations shown in the left half of the figure is made up of a 90° pitch, or y-axis rotation, followed by a 90° yaw, or z-axis rotation, and a further pitch rotation of −90°. On completion of this sequence of turns, it can be seen that a net rotation of 90° about the roll (x) axis has taken place. In the right hand figure, the order of the rotations has been reversed. Although the body still ends up with its roll axis aligned in the original direction, it is seen that a net roll rotation of −90° has now taken place. Hence, individual axis rotations are said to be non-commutative. It is clear that failure to take account of the order in which rotations arise can lead to a substantial error in the computed attitude.

Various mathematical representations can be used to define the attitude of a body with respect to a co-ordinate reference frame. The parameters associated with each method may be stored within a computer and updated as the vehicle rotates using the measurements of turn rate provided by the strapdown gyroscopes. Three attitude representations are described here, namely:

1. *Direction cosines.* The direction cosine matrix, introduced in Section 3.5, is a 3×3 matrix, the columns of which represent unit vectors in body axes projected along the reference axes.
2. *Euler angles.* A transformation from one co-ordinate frame to another is defined by three successive rotations about different axes taken in turn. The Euler angle representation is perhaps one of the simplest techniques in terms of physical

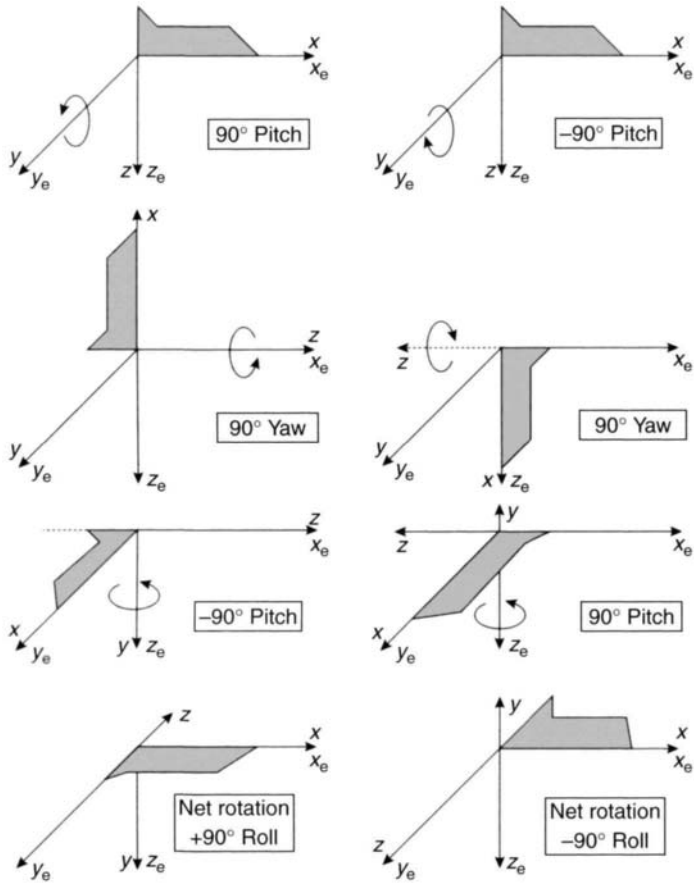

Figure 3.20 Illustration of effect of order of body rotations

appreciation. The three angles correspond to the angles which would be measured between a set of mechanical gimbals,[3] which is supporting a stable element, where the axes of the stable element represent the reference frame, and with the body being attached via a bearing to the outer gimbal.

3. *Quaternions*. The quaternion attitude representation allows a transformation from one co-ordinate frame to another to be effected by a single rotation about a vector defined in the reference frame. The quaternion is a four-element vector representation, the elements of which are functions of the orientation of this vector and the magnitude of the rotation.

[3] A gimbal is a rigid mechanical frame which is free to rotate about a single-axis to isolate it from angular motion in that direction. A stable platform can be isolated from body motion if supported by three such frames with their axes of rotation nominally orthogonal to each other.

In the following sections, each of these attitude representations is described in detail.

3.6.2 *Direction cosine matrix*

3.6.2.1 *Introduction*

The direction cosine matrix, denoted here by the symbol $\mathbf{C}_b^n$, is a 3×3 matrix, the columns of which represent unit vectors in body axes projected along the reference axes. $\mathbf{C}_b^n$ is written here in component form as follows:

$$\mathbf{C}_b^n = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{3.33}$$

The element in the ith row and the jth column represents the cosine of the angle between the i-axis of the reference frame and the j-axis of the body frame.

3.6.2.2 *Use of direction cosine matrix for vector transformation*

A vector quantity defined in body axes, $\mathbf{r}^b$, may be expressed in reference axes by pre-multiplying the vector by the direction cosine matrix as follows:

$$\mathbf{r}^n = \mathbf{C}_b^n \mathbf{r}^b \tag{3.34}$$

3.6.2.3 *Propagation of direction cosine matrix with time*

The rate of change of $\mathbf{C}_b^n$ with time is given by:

$$\dot{\mathbf{C}}_b^n = \lim_{\delta t \to 0} \frac{\delta \mathbf{C}_b^n}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{C}_b^n(t+\delta t) - \mathbf{C}_b^n(t)}{\delta t} \tag{3.35}$$

where $\mathbf{C}_b^n(t)$ and $\mathbf{C}_b^n(t+\delta t)$ represent the direction cosine matrix at times t and $t+\delta t$, respectively. $\mathbf{C}_b^n(t+\delta t)$ can be written as the product of two matrices as follows:

$$\mathbf{C}_b^n(t+\delta t) = \mathbf{C}_b^n(t)\mathbf{A}(t) \tag{3.36}$$

where $\mathbf{A}(t)$ is a direction cosine matrix which relates the b-frame at time t to the b-frame at time $t+\delta t$. For small angle rotations, $\mathbf{A}(t)$ may be written as follows:

$$\mathbf{A}(t) = [\mathbf{I} + \delta\boldsymbol{\Psi}] \tag{3.37}$$

where $\mathbf{I}$ is a 3×3 identity matrix and

$$\delta\boldsymbol{\Psi} = \begin{bmatrix} 0 & -\delta\psi & \delta\theta \\ \delta\psi & 0 & -\delta\phi \\ -\delta\theta & \delta\phi & 0 \end{bmatrix} \tag{3.38}$$

in which $\delta\psi$, $\delta\theta$ and $\delta\phi$ are the small rotation angles through which the b-frame has rotated over the time interval δt about its yaw, pitch and roll axes, respectively.

In the limit as δt approaches zero, small angle approximations are valid and the order of the rotations becomes unimportant.

Substituting for $\mathbf{C}_b^n(t + \delta t)$ in eqn. (3.35) we obtain:

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \lim_{\delta t \to 0} \frac{\delta \boldsymbol{\Psi}}{\delta t} \tag{3.39}$$

In the limit as $\delta t \to 0$, $\delta\boldsymbol{\Psi}/\delta t$ is the skew symmetric form of the angular rate vector $\boldsymbol{\omega}_{nb}^b = [\omega_x \quad \omega_y \quad \omega_z]^T$, which represents the turn rate of the b-frame with respect to the n-frame expressed in body axes, that is,

$$\lim_{\delta t \to 0} \frac{\delta \boldsymbol{\Psi}}{\delta t} = \boldsymbol{\Omega}_{nb}^b \tag{3.40}$$

Substituting in eqn. (3.39) gives:

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \boldsymbol{\Omega}_{nb}^b \tag{3.41}$$

where

$$\boldsymbol{\Omega}_{nb}^b = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3.42}$$

An equation of the form of eqn. (3.41) may be solved within a computer in a strapdown inertial navigation system to keep track of body attitude with respect to the chosen reference frame. It may be expressed in component form as follows:

$$\begin{aligned} \dot{c}_{11} &= c_{12}\omega_z - c_{13}\omega_y & \dot{c}_{12} &= c_{13}\omega_x - c_{11}\omega_z & \dot{c}_{13} &= c_{11}\omega_y - c_{12}\omega_x \\ \dot{c}_{21} &= c_{22}\omega_z - c_{23}\omega_y & \dot{c}_{22} &= c_{23}\omega_x - c_{21}\omega_z & \dot{c}_{23} &= c_{21}\omega_y - c_{22}\omega_x \\ \dot{c}_{31} &= c_{32}\omega_z - c_{33}\omega_y & \dot{c}_{32} &= c_{33}\omega_x - c_{31}\omega_z & \dot{c}_{33} &= c_{31}\omega_y - c_{32}\omega_x \end{aligned} \tag{3.43}$$

3.6.3 Euler angles

3.6.3.1 Introduction

A transformation from one co-ordinate frame to another can be carried out as three successive rotations about different axes. For instance, a transformation from reference axes to a new co-ordinate frame may be expressed as follows:

rotate through angle ψ about reference z-axis
rotate through angle θ about new y-axis
rotate through angle ϕ about new x-axis

where ψ, θ and ϕ are referred to as the Euler rotation angles. This type of representation is popular because of the physical significance of the Euler angles which correspond to the angles which would be measured by angular pick-offs between a set of three gimbals in a stable platform inertial navigation system.

3.6.3.2 Use of Euler angles for vector transformation

The three rotations may be expressed mathematically as three separate direction cosine matrices as defined below:

$$\text{rotation } \psi \text{ about } z\text{-axis, } \mathbf{C}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.44}$$

$$\text{rotation } \theta \text{ about } y\text{-axis, } \mathbf{C}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3.45}$$

$$\text{rotation } \phi \text{ about } x\text{-axis, } \mathbf{C}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{3.46}$$

Thus, a transformation from reference to body axes may be expressed as the product of these three separate transformations as follows:

$$\mathbf{C}_n^b = \mathbf{C}_3\mathbf{C}_2\mathbf{C}_1 \tag{3.47}$$

Similarly, the inverse transformation from body to reference axes is given by:

$$\mathbf{C}_b^n = \mathbf{C}_n^{bT} = \mathbf{C}_1^T\mathbf{C}_2^T\mathbf{C}_3^T \tag{3.48}$$

$$\begin{aligned} \mathbf{C}_b^n &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\psi & \begin{matrix} -\cos\phi\sin\psi \\ +\sin\phi\sin\theta\cos\psi \end{matrix} & \begin{matrix} \sin\phi\sin\psi \\ +\cos\phi\sin\theta\cos\psi \end{matrix} \\ \cos\theta\sin\psi & \begin{matrix} \cos\phi\cos\psi \\ +\sin\phi\sin\theta\sin\psi \end{matrix} & \begin{matrix} -\sin\phi\cos\psi \\ +\cos\phi\sin\theta\sin\psi \end{matrix} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \end{aligned} \tag{3.49}$$

This is the direction cosine matrix given by eqn. (3.33) expressed in terms of Euler angles.

For small angle rotations, $\sin\phi \to \phi$, $\sin\theta \to \theta$, $\sin\psi \to \psi$ and the cosines of these angles approach unity. Making these substitutions in eqn. (3.49) and ignoring products of angles which also become small, the direction cosine matrix expressed in terms of the Euler rotations reduces approximately to the skew symmetric form shown below:

$$\mathbf{C}_b^n \approx \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} \tag{3.50}$$

This form of matrix is used in Chapter 11 to represent the small change in attitude which occurs between successive updates in the real time computation of body attitude, and in Chapters 10 and 12 to represent the error in the estimated direction cosine matrix.

3.6.3.3 *Propagation of Euler angles with time*

Following the gimbal analogy mentioned earlier, ϕ, θ and ψ are the gimbal angles and $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ are the gimbal rates. The gimbal rates are related to the body rates, ω_x, ω_y and ω_z as follows:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_3 \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{C}_3\mathbf{C}_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \tag{3.51}$$

This equation can be rearranged and expressed in component form as follows:

$$\begin{aligned} \dot{\phi} &= (\omega_y \sin\phi + \omega_z \cos\phi)\tan\theta + \omega_x \\ \dot{\theta} &= \omega_y \cos\phi - \omega_z \sin\phi \\ \dot{\psi} &= (\omega_y \sin\phi + \omega_z \cos\phi)\sec\theta \end{aligned} \tag{3.52}$$

Equations of this form may be solved in a strapdown system to update the Euler rotations of the body with respect to the chosen reference frame. However, their use is limited since the solution of the $\dot{\phi}$ and $\dot{\psi}$ equations become indeterminate when $\theta = \pm 90°$.

3.6.4 *Quaternions*

3.6.4.1 *Introduction*

The quaternion attitude representation is a four-parameter representation based on the idea that a transformation from one co-ordinate frame to another may be effected by a single rotation about a vector $\boldsymbol{\mu}$ defined with respect to the reference frame. The quaternion, denoted here by the symbol $\mathbf{q}$, is a four element vector, the elements of which are functions of this vector and the magnitude of the rotation:

$$\mathbf{q} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \cos(\mu/2) \\ (\mu_x/\mu)\sin(\mu/2) \\ (\mu_y/\mu)\sin(\mu/2) \\ (\mu_z/\mu)\sin(\mu/2) \end{bmatrix} \tag{3.53}$$

where μ_x, μ_y, μ_z are the components of the angle vector $\boldsymbol{\mu}$ and μ the magnitude of $\boldsymbol{\mu}$.

The magnitude and direction of $\boldsymbol{\mu}$ are defined in order that the reference frame may be rotated into coincidence with the body frame by rotating about $\boldsymbol{\mu}$ through an angle μ.

A quaternion with components a, b, c and d may also be expressed as a four-parameter complex number with a real component a, and three imaginary components,

b, c and d, as follows:

$$\mathbf{q} = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \tag{3.54}$$

This is an extension of the more usual two parameter complex number form with one real component and one imaginary component, $x = a + \mathbf{i}b$, with which the reader is more likely to be familiar.

The product of two quaternions, $\mathbf{q} = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ and $\mathbf{p} = e + \mathbf{i}f + \mathbf{j}g + \mathbf{k}h$ may then be derived as shown below applying the usual rules for products of complex numbers, viz:

$$\mathbf{i} \cdot \mathbf{i} = -1 \qquad \mathbf{i} \cdot \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \cdot \mathbf{i} = -\mathbf{k} \qquad \ldots \text{ etc.}$$

Hence,

$$\begin{aligned}\mathbf{q} \cdot \mathbf{p} &= (a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d)(e + \mathbf{i}f + \mathbf{j}g + \mathbf{k}h) \\ &= ea - bf - cg - dh + (af + be + ch - dg)\mathbf{i} \\ &\quad + (ag + ce - bh + df)\mathbf{j} + (ah + de + bg - cf)\mathbf{k}\end{aligned} \tag{3.55}$$

Alternatively, the quaternion product may be expressed in matrix form as:

$$\mathbf{q} \cdot \mathbf{p} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \tag{3.56}$$

3.6.4.2 Use of quaternion for vector transformation

A vector quantity defined in body axes, $\mathbf{r}^{\mathrm{b}}$, may be expressed in reference axes as $\mathbf{r}^{\mathrm{n}}$ using the quaternion directly. First define a quaternion, $\mathbf{r}^{\mathrm{b}'}$, in which the complex components are set equal to the components of $\mathbf{r}^{\mathrm{b}}$, and with a zero scalar component, that is, if:

$$\mathbf{r}^{\mathrm{b}} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

$$\mathbf{r}^{\mathrm{b}'} = 0 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

This is expressed in reference axes as $\mathbf{r}^{\mathrm{n}'}$ using:

$$\mathbf{r}^{\mathrm{n}'} = \mathbf{q}\mathbf{r}^{\mathrm{b}'}\mathbf{q}^* \tag{3.57}$$

where $\mathbf{q}^* = (a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d)$, the complex conjugate of $\mathbf{q}$.

Hence,

$$\begin{aligned}\mathbf{r}^{\mathrm{n}'} &= (a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d)(0 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z)(a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d) \\ &= 0 + \{(a^2 + b^2 - c^2 - d^2)x + 2(bc - ad)y + 2(bd + ac)z\}\mathbf{i} \\ &\quad + \{2(bc + ad)x + (a^2 - b^2 + c^2 - d^2)y + 2(cd - ab)z\}\mathbf{j} \\ &\quad + \{2(bd - ac)x + 2(cd + ab)y + (a^2 - b^2 - c^2 + d^2)z\}\mathbf{k}\end{aligned} \tag{3.58}$$

Alternatively, $\mathbf{r}^{n'}$ may be expressed in matrix form as follows:

$$\mathbf{r}^{n'} = \mathbf{C}'\mathbf{r}^{b'}$$

where

$$\mathbf{C}' = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{C} \end{bmatrix} \qquad \mathbf{r}^{b'} = \begin{bmatrix} 0 \\ \mathbf{r}^{b} \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2) \end{bmatrix} \tag{3.59}$$

which is equivalent to writing:

$$\mathbf{r}^{n} = \mathbf{C}\mathbf{r}^{b}$$

Comparison with eqn. (3.34) reveals that $\mathbf{C}$ is equivalent to the direction cosine matrix $\mathbf{C}_b^n$.

3.6.4.3 Propagation of quaternion with time

The quaternion, $\mathbf{q}$, propagates in accordance with the following equation:

$$\dot{\mathbf{q}} = 0.5\mathbf{q}\cdot\mathbf{p}_{nb}^{b} \tag{3.60}$$

This equation may be expressed in matrix form as a function of the components of $\mathbf{q}$ and $\mathbf{p}_{nb}^{b} = [0, \quad \boldsymbol{\omega}_{nb}^{b\,T}]^T$ as follows:

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = 0.5\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}\begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{3.61}$$

that is,

$$\begin{aligned} \dot{a} &= -0.5(b\omega_x + c\omega_y + d\omega_z) \\ \dot{b} &= 0.5(a\omega_x - d\omega_y + c\omega_z) \\ \dot{c} &= 0.5(d\omega_x + a\omega_y - b\omega_z) \\ \dot{d} &= -0.5(c\omega_x - b\omega_y - a\omega_z) \end{aligned} \tag{3.62}$$

Equations of this form may be solved in a strapdown navigation system to keep track of the quaternion parameters which define body orientation. The quaternion parameters may then be used to compute an equivalent direction cosine matrix, or used directly to transform the measured specific force vector into the chosen reference frame (see eqn. (3.57)).

3.6.5 *Relationships between direction cosines, Euler angles and quaternions*

As shown in the preceding sections, the direction cosines may be expressed in terms of Euler angles or quaternions, viz:

$$\mathbf{C}_{\mathrm{b}}^{\mathrm{n}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \begin{array}{l} -\cos\phi\sin\psi \\ \quad +\sin\phi\sin\theta\cos\psi \end{array} & \begin{array}{l} \sin\phi\sin\psi \\ \quad +\cos\phi\sin\theta\cos\psi \end{array} \\ \cos\theta\sin\psi & \begin{array}{l} \cos\phi\cos\psi \\ \quad +\sin\phi\sin\theta\sin\psi \end{array} & \begin{array}{l} -\sin\phi\cos\psi \\ \quad +\cos\phi\sin\theta\sin\psi \end{array} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2) \end{bmatrix} \tag{3.63}$$

By comparing the elements of the above equations, the quaternion elements may be expressed directly in terms of Euler angles or direction cosines. Similarly, the Euler angles may be written in terms of direction cosines or quaternions. Some of these relationships are summarised in the following sections.

3.6.5.1 *Quaternions expressed in terms of direction cosines*

For small angular displacements, the quaternion parameters may be derived using the following relationships:

$$\begin{aligned} a &= \tfrac{1}{2}(1 + c_{11} + c_{22} + c_{33})^{1/2} \\ b &= \frac{1}{4a}(c_{32} - c_{23}) \\ c &= \frac{1}{4a}(c_{13} - c_{31}) \\ d &= \frac{1}{4a}(c_{21} - c_{12}) \end{aligned} \tag{3.64}$$

A more comprehensive algorithm for the extraction of quaternion parameters from the direction cosines, which takes account of the relative magnitudes of the direction cosine elements, is described by Shepperd [2].

3.6.5.2 Quaternions expressed in terms of Euler angles

$$
\begin{aligned}
a &= \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\
b &= \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\
c &= \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\
d &= \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}
\end{aligned}
\tag{3.65}
$$

3.6.5.3 Euler angles expressed in terms of direction cosines

The Euler angles may be derived directly from the direction cosines as described below. For conditions where θ is not equal to 90° the Euler angles can be determined using

$$
\begin{aligned}
\phi &= \arctan\left[\frac{c_{32}}{c_{33}}\right] \\
\theta &= \arcsin\left[-c_{31}\right] \\
\psi &= \arctan\left[\frac{c_{21}}{c_{11}}\right]
\end{aligned}
\tag{3.66}
$$

For situations in which θ approaches $\pi/2$ radians, the equations in ϕ and ψ become indeterminate because the numerator and the denominator approach zero simultaneously. Under such conditions, alternative solutions for ϕ and ψ are sought based upon other elements of the direction cosine matrix. This difficulty may be overcome by using the direction cosine elements c_{12}, c_{13}, c_{22} and c_{23}, which do not appear in eqn. (3.66), to derive the following relationships:

$$
\begin{aligned}
c_{23} + c_{12} &= (\sin\theta - 1)\sin(\psi + \phi) \\
c_{13} - c_{22} &= (\sin\theta - 1)\cos(\psi + \phi) \\
c_{23} - c_{12} &= (\sin\theta + 1)\sin(\psi - \phi) \\
c_{13} + c_{22} &= (\sin\theta + 1)\cos(\psi - \phi)
\end{aligned}
\tag{3.67}
$$

For θ near $+\pi/2$:

$$
\psi - \phi = \arctan\left[\frac{c_{23} - c_{12}}{c_{13} + c_{22}}\right]
$$

For θ near $-\pi/2$:

$$
\psi + \phi = \arctan\left[\frac{c_{23} + c_{12}}{c_{13} - c_{22}}\right]
\tag{3.68}
$$

Equations (3.67) and (3.68) provide values for the sum and difference of ϕ and ψ under conditions where θ approaches $\pi/2$. Separate solutions for ϕ and ψ cannot be obtained when $\theta = +\pi/2$ because both become measures of angle about parallel axes (about the vertical), that is, a degree of rotational freedom is lost. This is equivalent to the 'gimbal lock' (or nadir) condition which arises with a set of mechanical gimbals when the pitch, or inner, gimbal is rotated through 90°.

When θ approaches $+\pi/2$, either ϕ or ψ may be selected arbitrarily to satisfy some other condition while the unspecified angle is chosen to satisfy eqn. (3.68). To avoid 'jumps' in the values of ϕ or ψ between successive calculations when θ is in the region of $+\pi/2$, one approach would be to 'freeze' one angle, ϕ, for instance, at its current value and to calculate ψ in accordance with eqn. (3.68). At the next iteration, ψ would be frozen and ϕ determined using eqn. (3.68). This process of updating ϕ or ψ alone at successive iterations would continue until θ is no longer in the region of $+\pi/2$.

3.7 Detailed navigation equations

3.7.1 Navigation equations expressed in component form

For a terrestrial navigation system operating in the local geographic reference frame, it has been shown (Section 3.5.3) that the navigation equation may be expressed as follows:

$$\dot{\mathbf{v}}_{\mathrm{e}}^{\mathrm{n}} = \mathbf{f}^{\mathrm{n}} - (2\boldsymbol{\omega}_{\mathrm{ie}}^{\mathrm{n}} + \boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}}) \times \mathbf{v}_{\mathrm{e}}^{\mathrm{n}} + \mathbf{g}_{\mathrm{l}}^{\mathrm{n}} \tag{3.69}$$

where, $\mathbf{v}_{\mathrm{e}}^{\mathrm{n}}$ represents velocity with respect to the Earth expressed in the local geographic frame defined by the directions of true north, east and the local vertical, in component form:

$$\mathbf{v}_{\mathrm{e}}^{\mathrm{n}} = [v_{\mathrm{N}} \quad v_{\mathrm{E}} \quad v_{\mathrm{D}}]^{\mathrm{T}} \tag{3.70}$$

$\mathbf{f}^{\mathrm{n}}$ is the specific force vector as measured by a triad of accelerometers and resolved into the local geographic reference frame;

$$\mathbf{f}^{\mathrm{n}} = [f_{\mathrm{N}} \quad f_{\mathrm{E}} \quad f_{\mathrm{D}}]^{\mathrm{T}} \tag{3.71}$$

$\boldsymbol{\omega}_{\mathrm{ie}}^{\mathrm{n}}$ is the turn rate of the Earth expressed in the local geographic frame;

$$\boldsymbol{\omega}_{\mathrm{ie}}^{\mathrm{n}} = [\Omega \cos L \quad 0 \quad -\Omega \sin L]^{\mathrm{T}} \tag{3.72}$$

$\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}}$ represents the turn rate of the local geographic frame with respect to the Earth-fixed frame; the transport rate. This quantity may be expressed in terms of the rate of change of latitude and longitude as follows:

$$\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}} = [\dot{\ell} \cos L \quad -\dot{L} \quad -\dot{\ell} \sin L]^{\mathrm{T}} \tag{3.73}$$

Writing $\dot{\ell} = v_{\mathrm{E}}/(R_0 + h)\cos L$ and $\dot{L} = v_{\mathrm{N}}/(R_0 + h)$ yields:

$$\boldsymbol{\omega}_{\mathrm{en}}^{\mathrm{n}} = \left[\frac{v_{\mathrm{E}}}{R_0 + h} \quad -\frac{v_{\mathrm{N}}}{R_0 + h} \quad -\frac{v_{\mathrm{E}} \tan L}{R_0 + h}\right]^{\mathrm{T}} \tag{3.74}$$

where R_0 is the radius of the Earth and h is the height above the surface of the Earth.

$\mathbf{g}_l^n$ is the local gravity vector which includes the combined effects of the mass attraction of the Earth ($\mathbf{g}$) and the centripetal acceleration caused by the Earth's rotation ($\boldsymbol{\omega}_{ie} \times \boldsymbol{\omega}_{ie} \times \mathbf{R}$). Hence, we may write

$$\mathbf{g}_l^n = \mathbf{g} - \boldsymbol{\omega}_{ie} \times \boldsymbol{\omega}_{ie} \times \mathbf{R} = \mathbf{g} - \frac{\Omega^2(R_0 + h)}{2} \begin{pmatrix} \sin 2L \\ 0 \\ 1 + \cos 2L \end{pmatrix} \tag{3.75}$$

The navigation equation may be expressed in component form as follows:

$$\begin{aligned} \dot{v}_N &= f_N - v_E(2\Omega + \dot{\ell}) \sin L + v_D \dot{L} + \xi g \\ &= f_N - 2\Omega v_E \sin L + \frac{v_N v_D - v_E^2 \tan L}{R_0 + h} + \xi g \end{aligned} \tag{3.76}$$

$$\begin{aligned} \dot{v}_E &= f_E + v_N(2\Omega + \dot{\ell}) \sin L + \dot{v}_D(2\Omega + \dot{\ell}) \cos L - \eta g \\ &= f_E + 2\Omega(v_N \sin L + v_D \cos L) + \frac{v_E}{R_0 + h}(v_D + v_N \tan L) - \eta g \end{aligned} \tag{3.77}$$

$$\dot{v}_D = f_D - v_E(2\Omega + \dot{\ell}) \cos L - v_N \dot{L} + g = f_D - 2\Omega v_E \cos L - \frac{v_E^2 + v_N^2}{R_0 + h} + g \tag{3.78}$$

where ξ and η represent angular deflections in the direction of the local gravity vector with respect to the local vertical owing to gravity anomalies, as discussed in Section 3.7.4.

Latitude, longitude and height above the surface of the Earth are given by:

$$\dot{L} = \frac{v_N}{R_0 + h} \tag{3.79}$$

$$\dot{\ell} = \frac{v_E \sec L}{R_0 + h} \tag{3.80}$$

$$\dot{h} = -v_D \tag{3.81}$$

It is assumed, in the equations given above, that the Earth is perfectly spherical in shape. Additionally, it is assumed that there is no variation in the Earth's gravitational field with changes in the position of the navigation system on the Earth or its height above the surface of the Earth.

The modifications which must be applied to the navigation equations in order to take account of the errors introduced by these assumptions and so permit accurate navigation over the surface of the Earth are summarised briefly in the following sections. The reader requiring a more detailed analysis of these effects is referred to the texts by Britting [3] and Steiler and Winter [4] in which such aspects are discussed in detail.

3.7.2 The shape of the Earth

It is clear from the preceding analysis that, in order to determine position on the Earth using inertial measurements, it is necessary to make some assumptions regarding the shape of the Earth. The spherical model assumed so far is not sufficiently representative for very accurate navigation. Owing to the slight flattening of the Earth at the poles, it is customary to model the Earth as a reference ellipsoid which approximates more closely to the true geometry. Terrestrial navigation involves the determination of velocity and position relative to a navigational grid which is based on the reference ellipsoid; see illustration in Figure 3.21.

In accordance with this model, the following parameters may be defined:

$$
\begin{array}{ll}
\text{the length of the semi-major axis,} & R \\
\text{the length of the semi-minor axis,} & r = R(1-f) \\
\text{the flattening of the ellipsoid,} & f = (R-r)/R \\
\text{the major eccentricity of the ellipsoid,} & e = [f(2-f)]^{1/2}
\end{array}
\tag{3.82}
$$

By modelling the Earth in accordance with a reference ellipsoid as defined here, a meridian radius of curvature (R_N) and a transverse radius of curvature (R_E) may be derived in accordance with the following equations:

$$R_N = \frac{R(1-e^2)}{(1-e^2\sin^2 L)^{3/2}} \tag{3.83}$$

$$R_E = \frac{R}{(1-e^2\sin^2 L)^{1/2}} \tag{3.84}$$

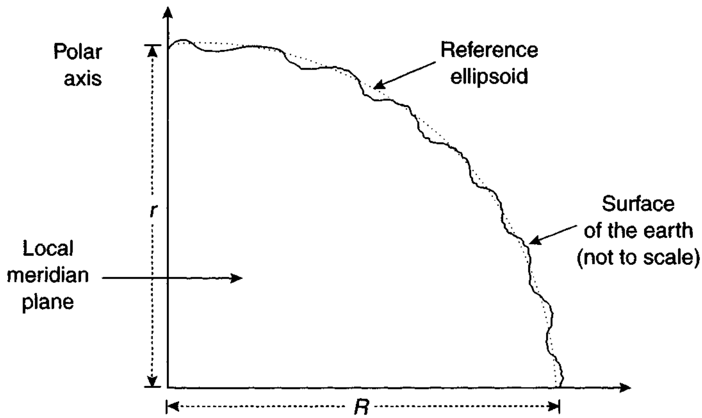

Figure 3.21 Reference ellipsoid

The rates of change of latitude and longitude may then be expressed in terms of a R_N and R_E as follows:

$$\dot{L} = \frac{v_N}{R_N + h} \tag{3.85}$$

$$\dot{\ell} = \frac{v_E \sec L}{R_E + h} \tag{3.86}$$

The mean radius of curvature used in the earlier equations, $R_0 = (R_E R_N)^{1/2}$. The flattening of the Earth at the poles gives rise to a difference of approximately 20 km between the mean radius used for a spherical earth model and the measured polar radius of approximately 20 km.

Similarly, the transport rate now takes the following form:

$$\boldsymbol{\omega}_{en}^{n} = \begin{bmatrix} \dfrac{v_E}{R_E + h} & \dfrac{-v_N}{R_N + h} & \dfrac{-v_E \tan L}{R_N + h} \end{bmatrix}^T \tag{3.87}$$

Further discussion regarding the shape of the Earth and the choice of datum reference frames appears in the following section.

With the aid of Figure 3.22, the distinction is drawn here between geocentric and geodetic latitude (see following section).

Geocentric latitude at a point on the surface of the Earth is the angle between the equatorial plane and a line passing through centre of Earth and the surface location point. Geodetic latitude at a point on the surface of the Earth is the angle between the equatorial plane and a line normal to the reference ellipsoid which passes through the point.

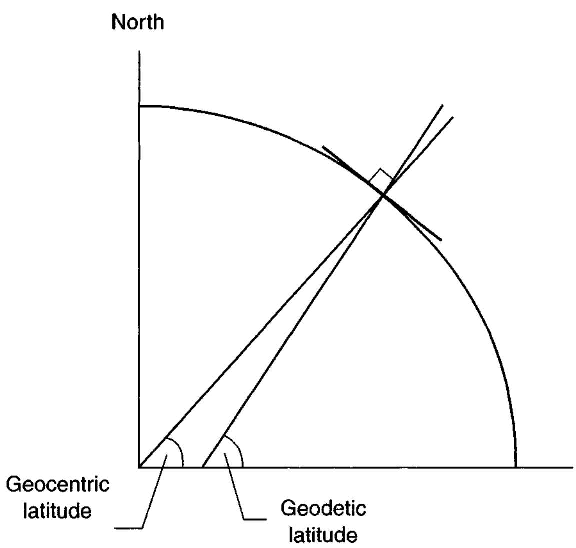

Figure 3.22 Geocentric and geodetic latitude

3.7.3 Datum reference models

The surface of the Earth is highly irregular in shape and can be modelled in various ways.

Topographic models represent the physical shape of the Earth and the mean level of the oceans.

Geodetic models yield a surface which is perpendicular to the local gravity vector at all points; an 'equipotential surface'. The resulting shape is referred to as a geoid.

Geodesy is the name given to the study of the size and shape of the Earth, and the term geodetic navigation is used here to refer to navigation which takes proper account of this shape. Aspects of geodesy and the use of geodetic datum points for mapping, surveying and navigation are discussed in the following paragraphs. For a more detailed discussion of the subject and a full definition of the terminology, the reader is referred to the standard text book by Bomford [5].

Local variations in gravity, caused by variations in the Earth's core and surface materials, cause the surface of the geoid, the gravity surface, to be irregular. Whilst it is much smoother than the physical surface of the Earth, as represented by a topographic model, it is too irregular to be used as a surface in which to specify spatial coordinates. For terrestrial navigation, a geometrical shape that approximates closely to the geoid model is used; an ellipsoid, which in this context is a three-dimensional (3-D) shape formed by rotating an ellipse about its minor axis. The term oblate spheroid is sometimes used in place of ellipsoid.

The term geodetic datum is used to define the ellipsoid and its positional relationship with respect to the solid Earth. In combination with an axis definition, a geodetic datum defines a 3-D geographic co-ordinate system, the dimensions being geodetic latitude and longitude and ellipsoidal height (height above the surface of the ellipsoid).

In practice, vertical position is not defined with respect to the surface of the ellipsoid because this surface offers no physical reference point for measurement. The geoid, which corresponds approximately to mean sea level, offers a much more convenient vertical reference. For this reason, height above mean sea level is most commonly used. For land based surveys [6], the reference level used as a zero datum is defined by mean sea level at a selected coastal location, or an average value of mean sea level at several locations, over a specified period of time. Land surveys should reference the vertical datum chosen. In the United Kingdom, the vertical datum used is Ordnance Datum Newlyn (ODN) and in the United States, the North American Vertical Datum of 1988 (NAVD88).

It is possible to define a geodetic datum which approximates to the shape of the Earth over the entire globe. The figures given in Figure 3.23 were defined for such a datum by the World Geodetic System Committee in 1984, the WGS-84 model [7]. The value of Earth's rate is discussed in Figure 3.24.

Many geodetic datum points used for mapping, surveying and navigation are defined to provide a more precise fit over a restricted geographical area, the

Length of the semi-major axis,	R	= 6378137.0 m
Length of the semi-minor axis,	$r = R(1-f)$	= 6356752.3142 m
Flattening of the ellipsoid,	$f = (R-r)/R$	= 1/298.257223563
Major eccentricity of the ellipsoid,	$e = [f(2-f)]^{1/2}$	= 0.0818191908426
Earth's rate (see Figure 3.24),	Ω	= 7.292115×10^{-5} rad/s (15.041067°/h)

Figure 3.23 WGS-84 model[4]

The duration of a solar day is 24h, the time taken between successive rotations for an Earth-fixed object to point directly at the sun. The Sidereal day represents the time taken for the Earth to rotate to the same orientation in space and is of slightly shorter duration than the solar day, 23h, 56min, 4.1s. The Earth rotates through one geometric revolution each Sidereal day, not in 24h, which accounts for the slightly strange value of Earth's rate.

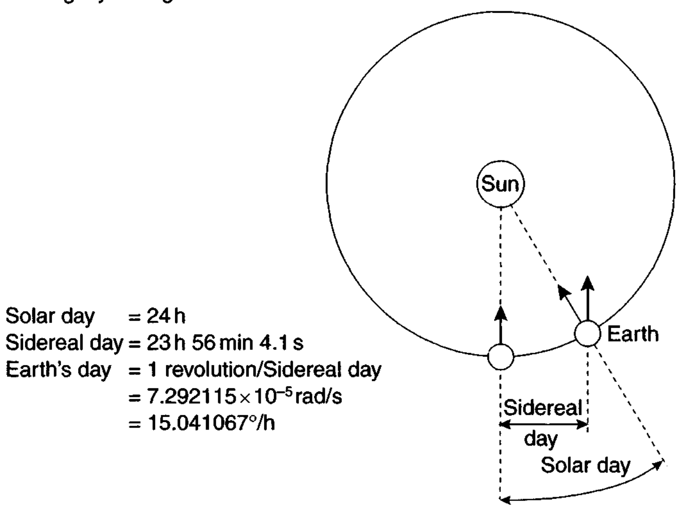

Figure 3.24 The solar day and the sidereal day

UK Ordnance Survey (OSGB 1936), the British National Grid for example. Regional datum points such as this have proliferated over time with the result that their areas of application may overlap. As a consequence, it is necessary when referring to a positional location on the Earth in terms of its latitude and longitude, to specify also

[4] The former Soviet Union devised a similar model, SGS-90 or PZ-90, discussed in Appendix D.

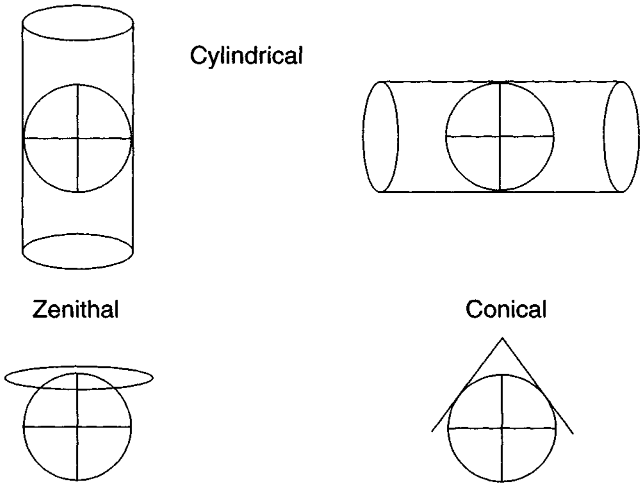

Figure 3.25 Examples of orthomorphic projection schemes used for mapping

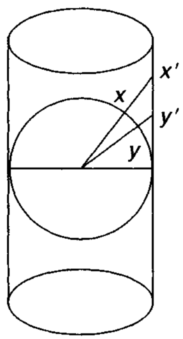

Figure 3.26 Standard Mercator projection

the corresponding geodetic datum or geographic co-ordinate reference. Contrary to common belief, the coordinates alone do not adequately define a particular location.

Lines of constant latitude and longitude are curved in three dimensions, but may be represented on a plane by means of a projection. The resulting rectangular co-ordinate system on the plane is called a grid. Various projections of the Earth's surface into two dimensions have been used, using a geodetic reference ellipsoid as the basis for projection.

A flat grid system can be obtained by projecting the reference ellipsoid onto a cylindrical, conical or flat shape as indicated in Figure 3.25. It is noted that the x and y axes must be orthomorphic, that is, of equal scale.

The Standard Mercator projection, illustrated in Figure 3.26, is generated by placing a cylinder over the Earth so that the contact point is around the equator. A point

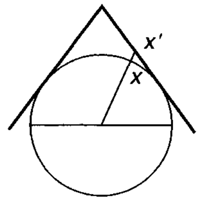

Figure 3.27 Lambert conical projection

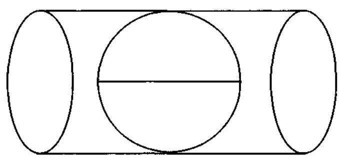

Figure 3.28 Universal Transverse Mercator projection

on the Earth is projected onto the inside of the cylinder by taking a line from the centre of the Earth through the point (e.g. x to x', y to y' as shown in Figure 3.26). When all points have been projected onto the cylinder, the cylinder is unwrapped and laid flat. This type of projection produces the most commonly observed map of the world.

The Mercator is a useful projection for navigation because a bearing direction on the globe is very similar to a direction on the projection. However, distances and areas will be distorted on small scale maps. Mercator and other projections do not preserve scale or area. For instance, with Standard Mercator, distance along the equator is represented exactly, but distance at higher latitude is magnified. As a result, Greenland appears to be the same size as South America, whereas it is actually about one-third of the size.

Various projection techniques are used to overcome the effects of distortion in regional maps where it is required to have a map or rectangular grid system that is appropriate to the locality of interest. For example, the Lambert conical projection, illustrated in Figure 3.27, provides an accurate representation of the area around the point of contact between the cone and the reference ellipsoid. Alternatively, the use of a cylinder or cone that cuts the reference ellipsoid in two places close to the area of interest may be used. This allows a reduction in the distortion to be achieved adjacent to and between the points of contact, providing two horizontal parallels and minimum vertical distortion.

One of the most commonly used projections is the Universal Transverse Mercator (UTM) projection. This uses the same principle as the Standard Mercator, with the exception that the cylinder is rotated through 90° so that the contact point of the cylinder with the Earth is along a meridian line; see Figure 3.28. To minimise distortion and preserve accuracy, the technique is employed of using only a small strip on either side of a designated central meridian. The UTM projection is used worldwide.

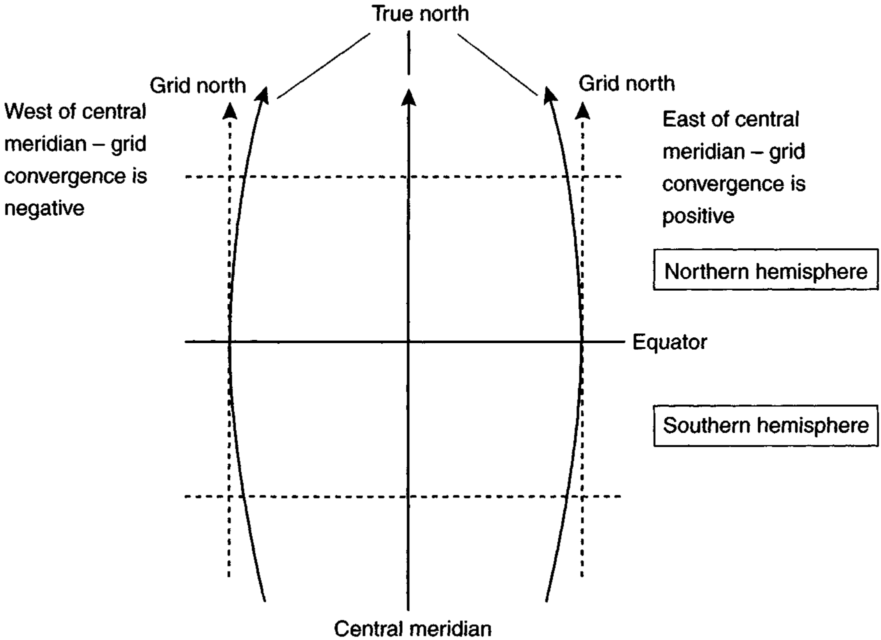

Figure 3.29 Universal Transverse Mercator grid zone

The UTM system uses the Transverse Mercator in separate zones each 6° wide. UTM zones are identified alphanumerically; numbering from 1 to 60, starting with 1 in the 180°W–174°W zone and increasing eastwards to 60 in the 174°E–180°E zone. Hence grid zone 32 is from 6° to 12° longitude. The UTM grid is further subdivided into blocks of 8° of latitude from −80° to +80° which are identified by the use of the letters C to X, excluding I and O (e.g. 64°S–56°S is E).

Any UTM grid zone can be represented as shown in Figure 3.29 where the central meridian is an odd number and a multiple of 3. The grid north and the true north will only be coincident along the central meridian and on the equator. At all other points there is a difference referred to as 'convergence'. It is noted that to the true north is west of grid north to the right of the central meridian and true north is east of grid north to the left of the central meridian.

3.7.4 *Variation of gravitational attraction over the Earth*

As described earlier, accelerometers provide measurements of the difference between the acceleration with respect to inertial space and the gravitational attraction acting at the location of the navigation system. In order to extract the precise estimates of true acceleration needed for very accurate navigation in the vicinity of the Earth, it is necessary to model accurately the Earth's gravitational field. This of course is also true for navigation close to any other body with a gravitational field.

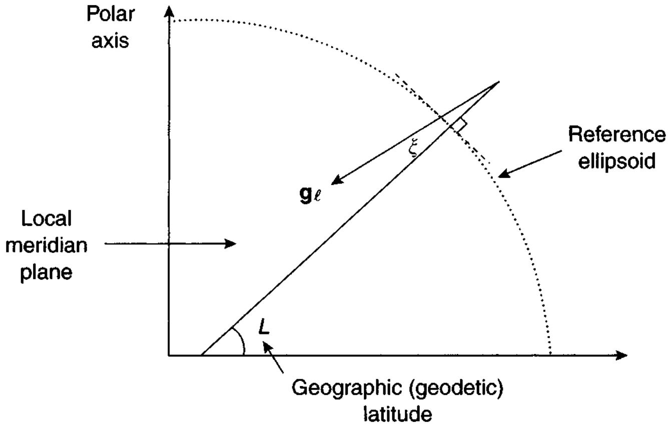

Figure 3.30 Deflection of local vertical owing to gravity anomalies

It is assumed in the earlier derivation of the navigation equation that the gravity vector acts vertically downwards, that is, normal to the referenced ellipsoid. In practice, both the magnitude and the direction of the gravity vector vary with position on the Earth's surface and height above it. Variations occur because of the variation between the mass attraction of the Earth and gravity vector; the centrifugal acceleration being a function of latitude. In addition, gravity varies with position on the Earth because of the inhomogenous mass distribution of the Earth. Such deviations in the magnitude and direction of the gravity vector from the calculated values are known as gravity anomalies.

Mathematical representations of the Earth's gravitational field are discussed in some depth by Britting [3]. The deflection of the local gravity vector from the vertical may be expressed as angular deviations about the north and east axes of the local geographic frame as follows:

$$\mathbf{g}_l = [\xi g, \ -\eta g, \ g]^{\mathrm{T}} \tag{3.88}$$

where ξ is the meridian deflection and η is the deflection perpendicular to the meridian. The deflection in the meridian plane is illustrated in Figure 3.30.

The resulting deviation of the vertical over the surface of the Earth varies by up to 30 arc s.

The precise knowledge of the gravity vector becomes important for certain high accuracy applications, such as for marine navigation where the deflection of the vertical becomes an important factor. Exact knowledge of the magnitude of gravity is also vital for the testing of very precise accelerometers, that is, sensors having a measurement bias of less than $10^{-5}g$. Similarly, it is important for surveying and gravity gradiometry, where attempts are made to measure the gravity vector very accurately.

Various international models for the variation of gravity with latitude are given in the literature. Steiler and Winter [4] give the following expressions for the variation of the magnitude of the gravity vector with latitude at sea level ($h = 0$) and its rate of change with height above ground:

$$g(0) = 9.780318(1 + 5.3024 \times 10^{-3} \sin^2 L - 5.9 \times 10^{-6} \sin^2 2L)\,\mathrm{m/s^2} \tag{3.89}$$

$$\frac{\mathrm{d}g(0)}{\mathrm{d}h} = -0.0000030877(1 - 1.39 \times 10^{-3} \sin^2 L)\,\mathrm{m/s^2/m} \tag{3.90}$$

For many applications, precise knowledge of gravity is not required and it is sufficient to assume that the variation of gravity with altitude is as follows:

$$g(h) = \frac{g(0)}{(1 + h/R_0)^2} \tag{3.91}$$

where $g(0)$ is derived from eqn. (3.89).

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