Signal Processing
Fourier Analysis on Finite Abelian Groups

6.S096

January 30, 2024

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- We will mostly follow Bao Luong's eponymous textbook.
- Classes at MIT?

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We also introduce the notion

Inner Product Space

Let V be a complex vector space. An *inner product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following properties for all vectors $\mathbf{u}$, $\mathbf{v}$, and scalars $a, b \in \mathbb{C}$:

1. Conjugate Symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.
2. Linearity in the First Argument: $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$.
3. Positive Definiteness: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

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$$\langle T(u), T(v) \rangle_W = \langle u, v \rangle_V.$$

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Another special property of inner-products is that they allow one to compute coordinates against a basis. If $b_1, \cdots, b_n$ is an orthonormal basis then

$$\boxed{v = \sum^{n} \langle v, b_i \rangle b_i.}$$

Functions on a Set

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Let Δ_S be the set of such δ_s functions. This implies that V_S is a $|S|$-dimensional complex vector space.

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$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}$$

where $n_1, n_2, \ldots, n_k$ are positive integers whose product is $|G|$.

[a] is isomorphic

Finishing the Proof

Moreover, one can verify that the group of characters on $\widehat{G_1 \times G_2}$ is isomorphic $\widehat{G_1} \times \widehat{G_2}$, so this boils down to the following result.

Lemma

The dual $\widehat{\mathbb{Z}_n}$ is isomorphic to $\mathbb{Z}_n$ for all $n \geq 1$.

How do characters relate to each other?

Theorem (Orthogonality Relation I)

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

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This gives rise to an orthogonality result.

Corollary (Orthogonality of Characters)

$$\langle \chi, \chi' \rangle = \begin{cases} |G| & \text{if } \chi = \chi', \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

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$$\sum_{g \in G} \chi(g)\overline{\chi}(g') = \begin{cases} |G| & \text{if } g = g' \\ 0 & \text{otherwise.} \end{cases}$$

Character Matrix

What we just proved can be put together in a concise way. Let us enumerate the characters $\chi_1, \cdots, \chi_n$.

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Example (Character Matrix of $\mathbb{Z}_n$)

Let $\zeta_n = e^{2\pi i/n}$. Then the character matrix of $\mathbb{Z}_n$ is given by

$$\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{n-1} \\ 1 & \zeta_n^2 & \zeta_n^4 & \cdots & \zeta_n^{2(n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \zeta_n^{(n-1)} & \zeta_n^{2(n-1)} & \cdots & \zeta_n^{(n-1)(n-1)} \end{bmatrix}$$

Orthogonality, altogether

Theorem

If G is a finite abelian group of order n, then matrix $\frac{1}{\sqrt{n}}X_G$ is a unitary matrix. That is, $X_G X_G^* = nI_n$, where I_n is the $n \times n$ identity matrix.

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This operator is denoted as $\mathcal{F}$. We denote the Fourier transform (FT) of a function $f \in V_G$ as either $\mathcal{F}(g)$ or as $\hat{f}$.

Properties of the Fourier Transform

We can prove some elegant properties of the FT.

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3. *The FT of the FT is just the reversed signal: $\hat{\hat{f}}(x) = f(-x)$.*
4. *The FT at the identity corresponds to the signal's average value: $\hat{f}(1_G) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g)$.*
5. *The FT on product of groups can be computed iteratively.*

[a] this is known as *Plancherel's identity*

Periods

Let us put this into context. Here is one natural way this construction arises. Consider a function $f : \mathbb{Z} \to \mathbb{C}$ which is periodic modulo n for some $n > 1$. This function descends uniquely to a function on $\mathbb{Z}_n$.

Lemma (Periods dividing Periods)

If a function on $\mathbb{Z}_n$ is periodic, then its period divides n.

Theorem

If $f \in V_{\mathbb{Z}_n}$ is a nonconstant peropdic function with period σ, and $f_\sigma \in V_{\mathbb{Z}_\sigma}$ is the corresponding restriction, then

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If $f \in V_{\mathbb{Z}_n}$ is a nonconstant peropdic function with period σ, and $f_\sigma \in V_{\mathbb{Z}_\sigma}$ is the corresponding restriction, then

$$\hat{f}(s) = \begin{cases} \sqrt{\frac{n}{\sigma}}\hat{f}_\sigma(m) & \text{if } s = m\frac{n}{\sigma} \text{ for some } 0 \leq m < \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Periods

Let us put this into context. Here is one natural way this construction arises. Consider a function $f : \mathbb{Z} \to \mathbb{C}$ which is periodic modulo n for some $n > 1$. This function descends uniquely to a function on $\mathbb{Z}_n$.

Lemma (Periods dividing Periods)

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It follows that $\hat{f}$ has at most σ nonzero values.

Additional Topics

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3. **Iterated Fourier Transforms**: (See Luong's Section 4.6)