Lecture 16: Determinants (Section 4.1-4.2)

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For the next few lectures, all matrices are square!

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The determinant of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([a]) = a$.

We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

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    Reverses the sign of the determinant.
  - (scaling) Multiply all entries in a row by s.
    Multiplies the determinant by s.

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$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix} \overset{R_2 \to \frac{1}{2} R_2}{=} 2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{vmatrix} \overset{R_3 \to \frac{1}{7} R_3}{=} 14 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 14.$$

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- Example. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

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- $\det(A^T) = \det(A)$

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- Example. Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?
- Solution. A has three rows. Multiplying all 3 of them by 2 produces $2A$. Hence, $\det(2A) = 2^3 \det(A) = 40$.

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$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -2(-1) + (-1)1 - 0 = 1.$$

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- Solution. We expand by the third column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 0 - 2(-4) + 1(-7) = 1.$$

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- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$.
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add.
$\Longrightarrow$ too much work.