Lecture 22: Complex Matrices (Sections 5.5)

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Complex Numbers and Their Conjugates

- Complex conjugate of $a + ib$ is $a - ib$ and is denoted by

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- Multiplying $a + ib$ by its conjugate produces $a^2 + b^2$:

$$(a + ib)(a - ib) = a^2 + b^2.$$

Lengths and Inner Products in the Complex Case

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- Example.

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ i \end{bmatrix} \Longrightarrow \|\boldsymbol{x}\| = \sqrt{|1|^2 + |i|^2} = 2.$$

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- Inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$

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- Example. Find inner products $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ and $\langle \boldsymbol{y}, \boldsymbol{x} \rangle$ of $\boldsymbol{x} = (1 + 3i, 3i)$ and $\boldsymbol{y} = (6 + 3i, 4 + i)$.

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$$\boxed{\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}}$$

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- **Example.** If $\boldsymbol{x} = \begin{bmatrix} 1+i \\ 2-i \\ 3+4i \end{bmatrix}$, then

$$\boldsymbol{x}^H = (1-i, 2+i, 3-4i).$$

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- $(AB)^H = B^H A^H$.

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The diagonal entries of a Hermitian matrix must be real. And $a_{ij} = \overline{a_{ji}}$.

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- Property. If $A = A^H$, then for all complex vectors $\boldsymbol{x}$, $\boldsymbol{x}^H A \boldsymbol{x}$ is real.
- Property. If $A = A^H$, every eigenvalue is real.
- Property. Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

Hermitian Matrices

- Example.

$$(A-8I)\boldsymbol{x}=\begin{bmatrix}-6 & 3-i\\ 3+3i & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Longrightarrow \boldsymbol{x}=\begin{bmatrix}1\\ 1+i\end{bmatrix}$$

$$(A+I)\boldsymbol{y}=\begin{bmatrix}3 & 3-i\\ 3+3i & 6\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Longrightarrow \boldsymbol{y}=\begin{bmatrix}1-i\\ -1\end{bmatrix}.$$

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These two eigenvectors are orthogonal

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▶ Property. (Spectral Theorem) A real symmetric matrix can be factored into $A=Q\Lambda Q^T$. Its orthonormal eigenvectors are in the orthogonal matrix Q and its eigenvalues are in Λ.

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- Property. $(U\boldsymbol{x})^H(U\boldsymbol{y}) = \boldsymbol{x}^H U^H U\boldsymbol{y} = \boldsymbol{x}^H\boldsymbol{y}$ and lengths are perserved by U:

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- Property. Every eigenvalue of U has absolute value $|\lambda| = 1$.
- Property. Eigenvectors corresponding to different eigenvalues are orthonormal.

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- Example. (normalized) Fourier matrix

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & w^{n-1} & \cdots & w^{(n-1)^2} \end{bmatrix}$$

where $w = e^{2\pi i/n}$.

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Row i of U^H times column j of U is

$$\frac{1}{n}(1 + W + W^2 + \cdots + W^{n-1}) = \frac{W^n - 1}{W - 1} = 0,$$

where $W = w^{j-i}$.