Lecture 26: Singular Value Decompositions (Sections 6.3)

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- columns of U (m by m) are eigenvectors of AA^T
- columns of V (n by n) are eigenvectors of A^TA
- The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

SVD Theory

$$AV = U\Sigma \Longrightarrow A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j, \qquad j = 1, 2, \ldots, r.$$

- If $\sigma_j = 0, A\boldsymbol{v}_j = 0$ and $\boldsymbol{v}_j$ is in $N(A)$.
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- Else, $\boldsymbol{v}_j$ is in $C(A^T)$ and the corresponding $\boldsymbol{u}_j$ is in $C(A)$.
- Number of nonzero σ_j = rank of A.

Singular Value Decompositions

- Example. Find SVD of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

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- Solution. Step 1: Find eigenvalues and eigenvectors of A^TA:

$$A^TA = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \Longrightarrow \lambda_1 = 8 \text{ and } \boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \lambda_2 = 2 \text{ and } \boldsymbol{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

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Step 2: compute $A\boldsymbol{v}_i$. If $\sigma_i \neq 0$, get $\boldsymbol{u}_i$. Else, find $\boldsymbol{u}_i$ from $N(A^T)$.

$$A\boldsymbol{v}_1 = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} = \sigma_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \sigma_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

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$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

- $\lambda_1 = 3 \Longrightarrow \boldsymbol{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

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$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

Least squares and the pseudoinverse

A least squares solution $\hat{\boldsymbol{x}}$ of the linear system

$$A\boldsymbol{x} = \boldsymbol{b}$$

is the one minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|$. To solve this, we can solve the associate normal system

$$A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}.$$

Least squares and the pseudoinverse

- If A has full rank, $A^T A$ is invertible and the system $A\boldsymbol{x} = \boldsymbol{b}$ has unique least squares solution

$$\hat{\boldsymbol{x}} = (A^T A)^{-1} A^T \boldsymbol{b}.$$

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- But how if A does not have full rank? The least squares problem still has a solution, but it is no longer unique, i.e., there are many $\boldsymbol{x}$ that minimize $\|A\boldsymbol{x} - \boldsymbol{b}\|$. The optimal solution of the $A\boldsymbol{x} = \boldsymbol{b}$ is the vector $\hat{\boldsymbol{x}}$ that has the minimum length. We denote this solution by $\boldsymbol{x}^\dagger$, and the matrix which produces $\boldsymbol{x}^\dagger$ from $\boldsymbol{b}$ is called the pseudoinverse of A and is denoted by $A^\dagger$.

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$$\boldsymbol{x}^\dagger = A^\dagger \boldsymbol{b}.$$

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The singular values $\sigma_1, \dots, \sigma_r$ are on the diagonal of Σ, and their reciprocals $\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_r}$ are on the diagonal of $\Sigma^\dagger$.

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▶ Example. Find the pseudoinverse of $A = [-1\ 2\ 2]$.

The SVD of A is

$$A = [1][3\ 0\ 0]\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

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Then the pseudoinverse of A is

$$A^\dagger = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}[1] = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9} \end{bmatrix}.$$