MATH 102 - PRACTICE PROBLEMS FOR MIDTERM I

1. Find the LU-decomposition of the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 3 & 1 \\ 3 & 1 & -2 \end{pmatrix}.$$

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ -1 & 0 & -1 & -2 \end{pmatrix}$$

(i) Give a basis for the null space of A. What is the nullity of A?

(ii) Show that the columns $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}$ of A are linearly dependent by exhibiting explicit relations between them.

(iii) Give the equation for $C(A)$ as a subspace of $\mathbb{R}^3$.

(iv) Give a basis for $C(A)$. What is the rank of A?

(v) Give a basis for the row space of A.

(vi) What is the dimension of the left null space of A?

(vii) Write down the general solution to the following system of equations

$$Ax = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}.$$

(viii) Does A admit a left inverse? How about a right inverse?

3. If the vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis of $\mathbb{R}^3$, show that $\{\mathbf{u}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - 2\mathbf{w}\}$ also form a basis of $\mathbb{R}^3$.

4. Which of the following are vector spaces?

(i) The set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1 \right\}.$

(ii) The set $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ in } \mathbb{R}^4 : x - 2y + 3z - 4w = 0 \text{ and } x - y - z + w = 0 \right\}.$

(iii) The set of vectors in $\mathbb{R}^4$ equidistant from the three points $(3, -4, 0, 0)$, $(0, -3, 4, 0)$ and $(0, 4, 3, 0)$.

(iv) The set of polynomials P of degree less or equal to 4 such that

$$P(0) = P'(0) = P''(0) = 0.$$

(v) The set of skew-symmetric matrices.

(vi) The set of solutions to the differential equation

$$y'' + 4y = \sin t.$$

5. A matrix A and its row reduced form are shown below. What is the second column of A?

$$A = \begin{bmatrix} 1 & ? & 5 & 9 \\ 2 & ? & 6 & 10 \\ 3 & ? & 7 & 11 \\ 4 & ? & 8 & 13 \end{bmatrix}, \ \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

Calculate $T\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right)$.

7. Consider $\mathcal{P}_2$ the vector space of polynomials of degree at most equal to 2 and let

$$T : \mathcal{P}_2 \to \mathcal{P}_2$$

be the linear transformation

$$T(f) = xf'' + f.$$

(i) Show that $\{1, x-1, x^2\}$ is a basis of $\mathcal{P}_2$.

(ii) Find the matrix of the linear transformation T in the basis in (i).

8. Assume A is a square $k \times k$ matrix, B is a $k \times (k-1)$ matrix and C is a $(k-1) \times k$ matrix. Furthermore, assume that

$$A = BC.$$

Is it true that $N(A)$ always contains a non-zero vector?

9. Let A be a 2×2 matrix. Explain why there must exist non-zero constants c_0, c_1, c_2, c_3, c_4 such that

$$c_0 I + c_1 A + c_2 A^2 + c_3 A^3 + c_4 A^4 = 0.$$