MATH 20E, MIDTERM 1 SOLUTIONS
(VERSION C)

PROBLEM 1

(a) The tangent plane at the point $(0,0,1)$ is given by

$$\begin{aligned} z &= f(0,0) + f_x(0,0)x + f_y(0,0)y \\ &= 1 + 2x, \end{aligned}$$

where the final equality follows since

$$\begin{array}{ll} f_x(x,y) = 2e^{2x+y^2} & f_y(x,y) = 2ye^{2x+y^2} \\ f_x(0,0) = 2 & f_y(0,0) = 0 \end{array}$$

and $z = 1 = f(0,0) = f(x,y)$ at the given point $(0,0,1)$.

(b) It seems students used a variety of formulas for this problem with differing notations. I accepted any notation as long as the solution was correct. One acceptable method was to use

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) + R_2(\mathbf{x}_0, \mathbf{h})$$

from page 160 in the textbook (section 3.2). In the problem, $\mathbf{x}_0 = (0,0)$ is given, so the final solution only contains h_1 and h_2 terms. It was also fine if you used x and y instead of h_1 and h_2. Alternatively, you could have expressed the above equation in matrix form (I'll use x and y here),

$$g(x,y) = f(0,0) + \begin{bmatrix} f_x & f_y \end{bmatrix}\Big|_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\Bigg|_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix},$$

where each of the matrices are evaluated at the point $(0,0)$. The values of the partial derivatives are

$$\begin{array}{ll} f_x(x,y) = 2e^{2x+y^2} & f_x(0,0) = 2 \\ f_y(x,y) = 2ye^{2x+y^2} & f_y(0,0) = 0 \\ f_{xx}(x,y) = 4e^{2x+y^2} & f_{xx}(0,0) = 4 \\ f_{yy}(x,y) = 2e^{2x+y^2} + 4y^2e^{2x+y^2} & f_{yy}(0,0) = 2 \\ f_{xy}(x,y) = f_{yx}(x,y) = 4ye^{2x+y^2} & f_{xy}(0,0) = 0 \end{array}$$

For some reason, many students forgot to use the product rule when finding $f_{yy}(x, y)$, so they missed a term and erroneously concluded $f_{yy}(0, 0) = 0$ (wrong). Plugging these values into either the summation or matrix forms yields

$$f(h_1, h_2) = 1 + 2h_1 + 2h_1^2 + h_2^2 + R_2(\mathbf{0}, \mathbf{h})$$

or

$$g(x, y) = 1 + 2x + 2x^2 + y^2,$$

depending on which form you used, respectively. Although the first expression only approximately holds if the remainder term is not included (i.e., you should replace the "=" with an "≈"), no points were deducted if this term wasn't included.

PROBLEM 2

(a) Differentiating $g(x, y) = (ye^{x^2}, xe^{y^2})$ yields

$$Dg(x, y) = \begin{bmatrix} 2xye^{x^2} & e^{x^2} \\ e^{y^2} & 2xye^{y^2} \end{bmatrix}.$$

(b) In order to compute $D(f \circ g)(1, 0)$, it is easiest to find the partial derivatives of $f(u, v)$ with respect to u and v, the partial derivatives of $g(x, y)$ with respect to x and y, and use the chain rule:

$$D(f \circ g)(1, 0) = Df(g(1, 0))\ Dg(1, 0).$$

The derivative of f is evaluated at $g(1, 0) = (0, 1)$. Each of $Df(0, 1)$ and $Dg(1, 0)$ are 2×2 matrices, so their product should also be a 2×2 matrix. Using

$$Df(u, v) = \begin{bmatrix} 2u & -2v \\ -v\sin(uv) & -u\sin(uv) \end{bmatrix},$$

we find

$$Df(g(1, 0))\ Dg(1, 0) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & e \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}.$$

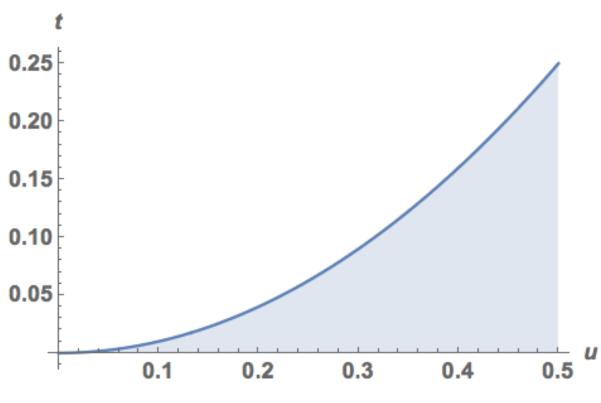
Several students attempted to write down the full function composition, and then differentiate with respect to x and y. Although this method is still correct, the arithmetic is far more tedious.

PROBLEM 3

The calculation is below.

$$\begin{aligned}
\int_0^{1/4}\int_{\sqrt{t}}^{1/2}\frac{e^u}{u}\,du\,dt &= \int_0^{1/2}\int_0^{u^2}\frac{e^u}{u}\,dt\,du \\
&= \int_0^{1/2}\frac{e^u}{u}t\Big|_{t=0}^{t=u^2}\,du \\
&= \int_0^{1/2} ue^u\,du \\
&= ue^u\big|_{u=0}^{u=1/2} - \int_0^{1/2} e^u\,du \\
&= \tfrac{1}{2}e^{1/2} - e^u\big|_{u=0}^{u=1/2} \\
&= \tfrac{1}{2}e^{1/2} - \left(e^{1/2}-1\right) \\
&= 1 - \tfrac{1}{2}e^{1/2}.
\end{aligned}$$

Students mostly had trouble with determining the bounds on the iterated integrals. The easiest approach is to start by drawing $u = \sqrt{t}$ in the $t - u$ plane, then shade the region between this curve and the line $u = \frac{1}{2}$. Since t ranges from 0 to $tfrac14$, the integral is taken over the entire shaded region. Several students either shaded the wrong region even though they determined the correct bounds or they drew the curve $u = t^2$ instead of $u = \sqrt{t}$. Some simply forgot to label their axis. The correct region is shown below.



Also, don't forget about integration by parts!

$$\int_a^b \left(\frac{df(x)}{dx}\right) g(x)\,dx = f(x)g(x)\big|_{x=a}^{x=b} - \int_a^b f(x)\left(\frac{dg(x)}{dx}\right)\,dx.$$

PROBLEM 4

The most direct method to use for this problem is to evaluate the area of D by changing variables to D^*. The transformation is provided,

$$T(u,v) = (u^2v, uv^2) = (x, y),$$

so that

$$\begin{aligned}
\text{Area}(D) &= \iint_D dxdy \\
&= \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \\
&= \int_0^1 \int_0^1 \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv \\
&= \int_0^1 \int_0^1 \left| 4u^2v^2 - u^2v^2 \right| dudv \\
&= \int_0^1 \int_0^1 3u^2v^2 dudv \\
&= \frac{1}{3}.
\end{aligned}$$

There seemed to be a lot of confusion on this problem with students trying to determine the correct limits of integration in the $x - y$ plane, then perform the integral there. It can be done (very few students did this *correctly*), but is probably more difficult. It also doesn't utilize any of the machinery that you have learned in 20E. For those that tried it, the correct region in the $x - y$ plane was just the "almond-shaped" area bounded by the two curves $y = x^2$ and $y = \sqrt{x}$, shown below.