Section 2.6 : Gradients & Directional Derivatives

Recall: If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable its gradient is the vector given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

e.g.

$$f(x,y,z) = xyz + e^x$$

$$\Rightarrow \nabla f = (yz + e^x, xz, xy)$$

Directional Derivatives

Recall that if we have $f(x,y,z)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{i}) - f(\vec{x})}{h}$$

So $\frac{\partial f}{\partial x}$ is the derivative in the direction of $\vec{i}$

Similarly $\frac{\partial f}{\partial y}$ is the derivative in the direction of $\vec{j}$.

What if we want the derivative in the direction of some unit vector $\vec{v}$?

$$\lim_{h \to 0} \frac{f(\vec{x}+h\vec{v}) - f(\vec{x})}{h}$$

Directional Derivative in the direction of the unit vector $\vec{v}$.

Theorem: The Directional Derivative (in the dir. of $\vec{v}$, $\|\vec{v}\|=1$) of a differentiable function $f: \mathbb{R}^3 \to \mathbb{R}$ is

$$\text{Dir. Der.: } (Df)(\vec{x})\vec{v} = \underbrace{\nabla f}_{\text{vector}} \cdot \underbrace{\vec{v}}_{\text{vector}}$$

dot product

$$= \frac{\partial f}{\partial x}(\vec{x})v_1 + \frac{\partial f}{\partial y}(\vec{x})v_2 + \frac{\partial f}{\partial z}(\vec{x})v_3$$

Proof: The dir. derivative is

$$\lim_{t\to 0} \frac{f(\vec{x}+t\vec{v}) - f(\vec{x})}{h}$$

defining $\vec{c}(t) = \vec{x} + t\vec{v}$

the directional der. can be written

as $$\lim_{t\to 0} \frac{f(\vec{c}(t)) - f(\vec{c}(0))}{t} = g'(0)$$

where $g(t) = f(\vec{c}(t))$

but by the chain rule

$$g'(t) = (\nabla f)\Big|_{\vec{c}(t)} \cdot \vec{c}\,'(t)$$

$$\Rightarrow g'(0) = (\nabla f)\Big|_{\underbrace{\vec{c}(0)}_{\vec{c}(0)=x}} \cdot \underbrace{\vec{c}\,'(0)}_{\vec{v}}$$

$$= (\nabla f)(x) \cdot \vec{v}$$

Example: $f(x,y,z) = x^2 e^{-yz}$

Find the rate of change of f in the direction of $\vec{v} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ at the point $(1,0,0)$

Solution: We need

$$\nabla f \cdot \vec{v} = \left(2x e^{-yz}, -x^2 z e^{-yz}, -x^2 y e^{-yz}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow (\nabla f \cdot \vec{v})\Big|_{(1,0,0)} = (2,0,0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}.$$

Remark: If $\vec{v}$ is not a unit vector, use $\dfrac{\vec{v}}{\|\vec{v}\|}$.

Directions of Fastest Increase

If $\vec{n}$ is a unit vector, the rate of change of f in the direction of $\vec{n}$ is

$$(\nabla f)(\vec{x}) \cdot \vec{n} = \|(\nabla f)(\vec{x})\| \underbrace{\|\vec{n}\|}_{=1} \underbrace{\cos\theta}_{|\cos\theta| \leq 1}$$

So if $\nabla f(\vec{x}) \neq 0$, the rate of change is maximized when $\cos\theta = 1$, i.e., $\theta = 0$ $\iff$ when $\vec{n}$ & ∇f are parallel.

In summary, If $\nabla f(x) \neq 0$, then $\nabla f(x)$ points in the direction along which f is increasing the fastest

<u>Example:</u> Find the direction of fastest increase (maximum rate of change) of $f(x,y) = x^2 + y^3$ at the point $P = (2,3)$

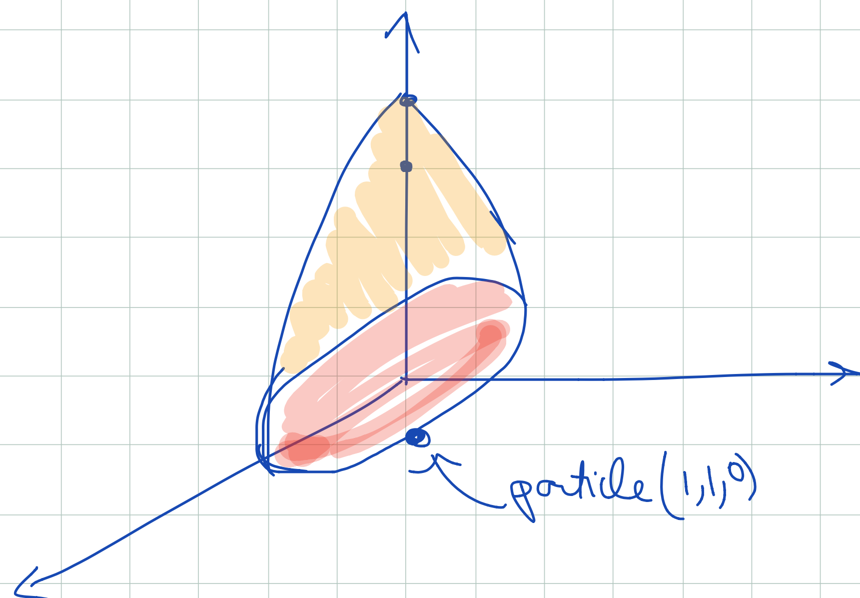
<u>Sol'n:</u> $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, 3y^2)$

$$\Rightarrow (\nabla f)(2,3) = (4, 27)$$

So the direction of maximum increase is $4\vec{i} + 27\vec{j}$

Example:

A particle is at the point $(1,1,0)$ on the surface given by $f(x,y)=4-x^2-3y^2$



- Find ∇f.

ans: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-2x, -6y)$

- What is the direction of maximal increase for the particle

ans. $(\nabla f)(1,1) = (-2,-6)$

(So if the particle wanted to climb the moutain it should do so by moving in this direction.)

- What is the directional derivative of f in the direction (1,1) ?

$$(\nabla f)(\vec{x}) \cdot \frac{\vec{n}}{\|\vec{n}\|} = (2x, -6y) \cdot \frac{(1,1)}{\sqrt{2}}$$

$$= \frac{2x - 6y}{\sqrt{2}}$$

and in the direction of the gradient?

$$(\nabla f)(\vec{x}) \cdot \frac{(\nabla f)(\vec{x})}{\|(\nabla f)(\vec{x})\|} = \|(\nabla f)(\vec{x})\| = \sqrt{4x^2 + 36y^2}$$