Section 2.5 Autonomous Equations and Population Dynamics.

The first order ODE $\frac{dy}{dt} = \underbrace{f(y)}_{}$ is called

autonomous. f only depends on y not on t.

r is a constant.

* Exponential Growth: $y' = ry$. (What's its solution?)
  - simple model that works well for population with no impediments to growth.
  - of course growth cannot occur indefinitely.

* Logistic Growth: $y' = h(y)\,y$.

Let's modify the exponential growth eq. so that
1. $y' \approx ry$ when y is small.
2. $y' \to 0$ as $y \to K$ where K is the population that can be sustained.

$$y' = ry\underbrace{\left(1 - \frac{y}{K}\right)}_{}$$

- close to 1 when y is close to zero.
- close to 0 when y is close to K.
- positive if $y < K$, negative is $y > K$.

This gives the logistic growth equation.

$$\boxed{\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y}$$

the rate at which a population increases in size if there are no density-dependent forces regulating the population $\Leftarrow$ r is the intrinsic growth rate

K is the saturation ~~the~~ level or environment carrying capacity

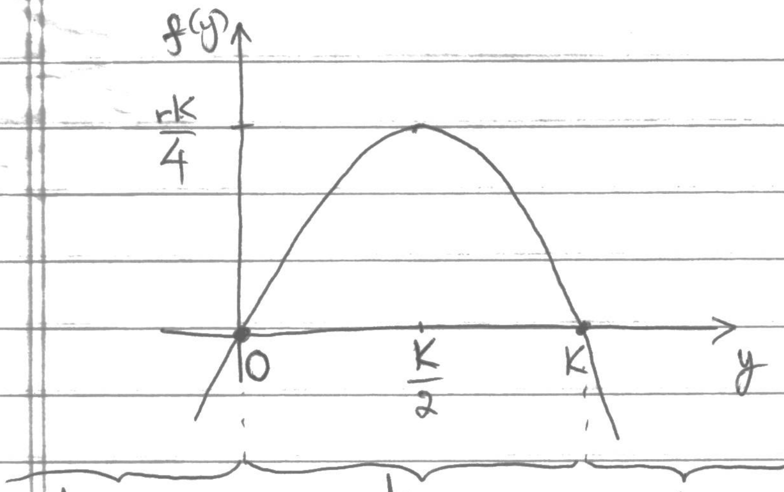
Example: $\frac{dy}{dt} = \frac{1}{2}(1 - y)\,y$ $\qquad r = 1/2$, $K = 1$.

E.g. Consider $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$, $r>0$, $K>0$.

If $y>0$ and $y<K$, then $r\left(1-\frac{y}{K}\right)y > 0 \Rightarrow y$ increases

If $y>0$ and $y>K$, then $r\left(1-\frac{y}{K}\right)y < 0 \Rightarrow y$ decreases

Given $\frac{dy}{dt} = f(y) = r\left(1-\frac{y}{k}\right)y$, we can plot $f(y)$ vs. y. ~~vs. y~~



$\frac{dy}{dt} < 0$ $\qquad$ $\frac{dy}{dt} > 0 \Rightarrow y(t)$ increasing $\qquad$ values of y for which $\frac{dy}{dt}$ is negative meaning $y(t)$ is decreasing.

Equilibrium points: $\frac{dy}{dt} = 0$

(same as critical points)

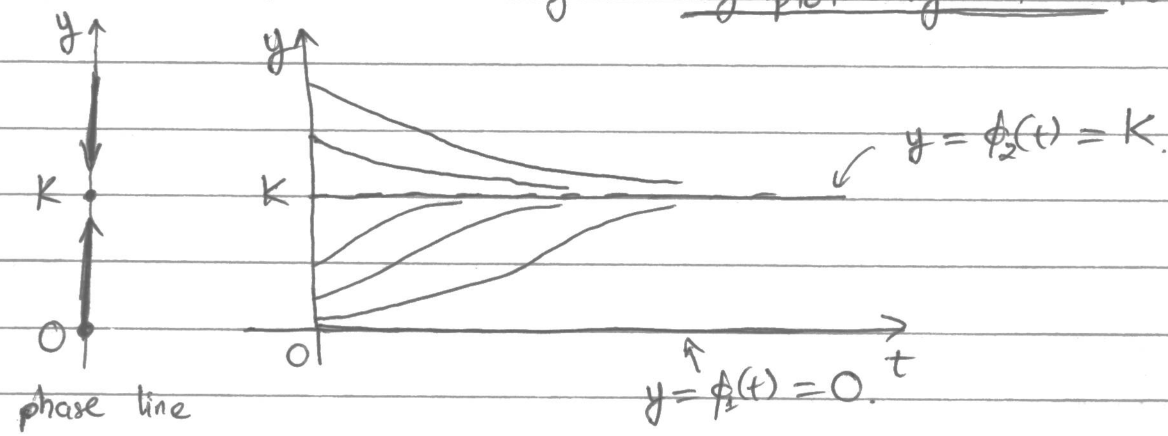
Here, $y = \phi_1(t) = 0$ and $y = \phi_2(t) = K$ are equilibrium points.

This is why K is called the saturation level. If y is above it, y will decrease towards it. If y is below it, y will increase towards it.

(equilibrium)

The points 0 and K are called critical points.

The critical points in these equations can be characterized as being asymptotically stable solution or unstable solution.



To solve the logistic equation $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$ we separate ~~them~~: ~~vara~~ variables:

$$\frac{dy}{\left(1 - \frac{y}{K}\right)y} = r\,dt.$$

To integrate the LHS, we can use partial fractions:

$$\frac{1}{\left(1 - \frac{y}{K}\right)y} = \frac{A}{1 - \frac{y}{K}} + \frac{B}{y} \Rightarrow A = \frac{1}{K}, \quad B = 1.$$

Hence,

$$\frac{dy}{K\left(1 - \frac{y}{K}\right)} + \frac{dy}{y} = r\,dt.$$

$$\int \frac{dy}{K - y} + \int \frac{dy}{y} = \int r\,dt$$

$$-\ln|K - y| + \ln|y| = rt + C$$

$$\ln\left|\frac{y}{K - y}\right| = rt + C.$$

$$\left|\frac{y}{K - y}\right| = C e^{rt}.$$

$y > 0$

$K - y > 0.$

If $y(0) = y_0$, then $\dfrac{y_0}{K-y_0} = C \Rightarrow \dfrac{y}{K-y} = \dfrac{y_0}{K-y_0} e^{rt}$.

Solving for y,

$$\frac{1}{\frac{K}{y} - 1} = \frac{y_0}{K-y_0} e^{rt}$$

$$\frac{K}{y} - 1 = \frac{K-y_0}{y_0} e^{-rt}$$

$$\frac{K}{y} = \frac{(K-y_0)e^{-rt} + y_0}{y_0}$$

$$\frac{y}{K} = \frac{y_0}{(K-y_0)e^{-rt} + y_0}$$

$$\boxed{y = \frac{Ky_0}{y_0 + (K-y_0)e^{-rt}}}$$

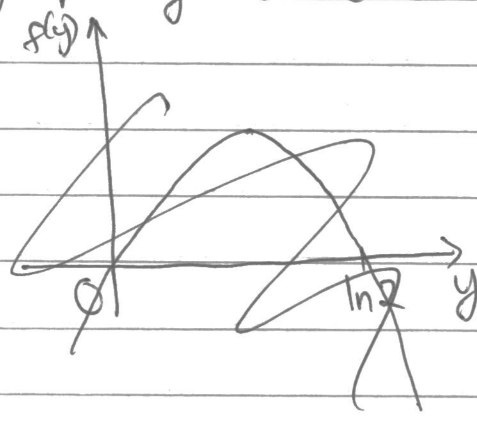
Notice that as $t \to \infty$, $y \to K$. (if $r>0$).

Example. Let $\dfrac{dy}{dt} = f(y)$ ~~and~~ be $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{e^{y}}{2}\right)$

$-\infty < y_0 < \infty$.

1) Sketch $f(y)$ vs. y.

2) Find the critical points and classify them as asymptically stable or unstable.



1) Critical point at $2y\left(1 - \dfrac{e^y}{2}\right) = 0$.

$\Rightarrow y = 0$

or $1 = \dfrac{e^y}{2} \Rightarrow y = \ln 2$.

are two equilibrium points.

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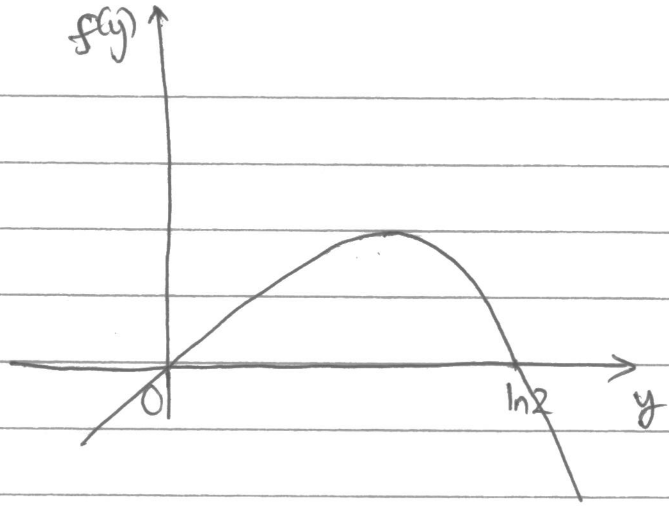

$y = 0$: unstable critical points since
$f(y) > 0$ when $y > 0$, $f(y) < 0$ when $y < 0$.

$y = \ln 2$: asymptotically stable critical point
since $f(y) > 0$ when $y < \ln 2$.
and $f(y) < 0$ when $y > \ln 2$.

Section 2.6. Exact Equations and Integrating Factors.

Recall that a 1st-order linear ODE:

$$N(x)\frac{dy}{dx} + M(x)\,y = L(x)$$

Can be solved by using an integrating factor

$$u(x) = e^{\int \frac{M(x)}{N(x)}dx}$$

to obtain

$$\left(u(x)\,y\right)' = u(x)\,\frac{L(x)}{N(x)}.$$

Now, consider a general 1st order ODE:

(*) $$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

We will try a similar approach ~~to~~ with (*).

Recall that if $\psi(x,y)$ is a function of x&y:

$$\frac{d}{dx}\psi(x,y) = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial\psi}{\partial y}\cdot\frac{\partial y}{\partial x}$$

$$= \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\cdot\frac{dy}{dx}.$$

If we can find $\psi(x,y)$ so that

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\cdot\frac{dy}{dx} = M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

then we can write $\frac{d\psi}{dx} = 0$, so $\psi(x,\phi(x)) = C$

and we will have solved the ODE implicitly for y.

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Our task is to find $\psi(x,y)$ so that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

Since $\dfrac{\partial \psi}{\partial x} = M(x,y)$, $\quad \psi(x,y) = \int M(x,y)\,dx + h(y)$.

Now differentiate w.r.t. y

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx + h(y)\right]$$

$$= \int \frac{\partial}{\partial y} M(x,y)\,dx + h'(y).$$

so $\quad N(x,y) = \displaystyle\int \frac{\partial M}{\partial y}\,dx + h'(y).$

or $\quad h'(y) = N(x,y) - \displaystyle\int \frac{\partial M}{\partial y}\,dx.$

This says that the RHS must only be a function of y (since the LHS is only a function of y)

so $\frac{\partial}{\partial x}(\text{RHS})$ must be zero

$$\frac{\partial}{\partial x}\left[N - \int \frac{\partial M}{\partial y}\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \qquad (**).$$

If $(**)$ is true, then we need only find $h(y)$ and then

$$\psi(x,y) = \int M\,dx + h(y) = C$$

yields the solution.

E.g. $2xy^2 + (2x^2y - 3y^2)\frac{dy}{dx} = 0$

Sol. $M(x,y) = 2xy^2$ $\qquad N(x,y) = 2x^2y - 3y^2.$

Does $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4xy \\ \frac{\partial N}{\partial x} &= 4xy \end{aligned}\right\} \text{same!}$$

So
$$\begin{aligned} \psi &= \int M(x,y)\,dx + h(y) \\ &= \int 2xy^2\,dx + h(y) \\ &= x^2y^2 + h(y). \end{aligned}$$

And
$$N(x,y) = \frac{\partial \psi}{\partial y}.$$

$$\Rightarrow \quad 2x^2y - 3y^2 = 2x^2y + h'(y).$$
$$\Rightarrow \quad h'(y) = -3y^2.$$
$$h(y) = -y^3 + C.$$

$\therefore \quad \psi(x,y) = x^2y^2 - y^3.$

[C can be dropped since we'll account for it in the final solution].

$\Rightarrow$ Sol: $\boxed{x^2y^2 - y^3 = C}$

* Remark: Equations which have the property $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ are called Exact.

* Basic technique:

1) Put ODE into $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ form.
2) Check for exactness, i.e. does $M_y = N_x$?
3) Integrate M w.r.t. x and add an arbitrary function ~~at~~ of y to get $\psi = \int M dx + h(y)$.
4) Use $\frac{\partial \psi}{\partial y} = N$ to find $h(y)$.
5) Solution is $\cancel{\psi = 0}$ $\psi = C$.

E.g. $(ye^{2xy} + x)\,dx + xe^{2xy}\,dy = 0$

$M(x,y) = ye^{2xy} + x$ and $N(x,y) = xe^{2xy}$.

$\cancel{\frac{\partial M}{\partial x} = 2y^2e^{2xy} + 1}$

same $\Rightarrow$ exact!

$$\begin{cases} \frac{\partial M}{\partial y} = e^{2xy} + 2xy\,e^{2xy} \\ \frac{\partial N}{\partial x} = e^{2xy} + 2xy\,e^{2xy} \end{cases}$$

$$\Rightarrow \psi = \int (ye^{2xy} + x)\,dx + h(y)$$

$$= \frac{1}{2}e^{2xy} + \frac{x^2}{2} + h(y)$$

And $N(x,y) = \frac{\partial \psi}{\partial y}$.

$$\Rightarrow xe^{2xy} = xe^{2xy} + h'(y)$$

$$\Rightarrow h'(y) = 0.$$

$-$ h is constant.

$$\frac{1}{2}e^{2xy} + \frac{x^2}{2} = C$$

$$\Rightarrow e^{2xy} = C - x^2$$

$$y = \frac{\ln(C - x^2)}{2x}$$

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If the eq. is not exact, we can try and find an integrating factor $u(x,y)$ which makes it exact.

$$u(x,y)\,M(x,y) + u(x,y)\,N(x,y)\,\frac{dy}{dx} = 0.$$

We need

$$\frac{\partial}{\partial y}\left[u(x,y)M(x,y)\right] = \frac{\partial}{\partial x}\left[u(x,y)\,N(x,y)\right]$$

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)(N) + u\frac{\partial N}{\partial x}.$$

$$\Rightarrow\ N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} + \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]u = 0.$$

But this one is as hard ~~as~~ to solve as the original ODE. If however u ~~$u(x,y)$~~ is a function of x only then

$$N\frac{du}{dx} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)u = 0.$$

which ~~not~~ may be solvable for u

[Similarly, if ~~$u(x,y)$~~ u is a function of y only,

$$-M\frac{\partial u}{\partial y} + \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]u = 0.$$

E.g. $2xy + (2x^2 - 3y)\frac{dy}{dx} = 0.$

Sol. $M(x,y) = 2xy$

$N(x,y) = 2x^2 - 3y.$

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Is it exact?

$$\frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 4x.$$

$\Rightarrow$ no!

is $u(x)$ an integrating factor? Is it easy to solve?

$$\frac{du}{dx} + \frac{1}{2x^2 - 3y} (4x - 2x) u = 0.$$

$$\frac{du}{dx} + \underbrace{\frac{2x}{2x^2 - 3y}}_{\text{depends on } y} u = 0.$$

depends on y, $\Rightarrow$ won't work!

How about $u(y)$?

$$\frac{du}{dy} - \frac{1}{2xy} (4x - 2x) u = 0.$$

$$\frac{du}{dy} - \frac{1}{y} u = 0 \Rightarrow$$

$$\frac{du}{u} = \frac{dy}{y}$$

$$\ln u = \ln y + C.$$

$$u = y e^C$$

Take $u(y) = y$.

Thus,

$$2xy^2 + (2x^2 y - 3y^2) \frac{dy}{dx} = 0 \quad \text{is exact.}$$

(our example before).

Section 3.1 Second order linear equations.

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

Linear: $y'' + p(t)y' + q(t)y = g(t)$.

Initial Conditions: $y(t_0) = y_0$, $y'(t_0) = y_0'$.

homogeneous: $g(t) = 0$.

homogeneous constant coefficient second order linear different equations:

$$ay'' + by' + cy = 0.$$

Example: 1: i) Find general solution.

$$\begin{cases} y'' + y' - 2y = 0 \\ y(0) = 0, \quad y'(0) = -3. \end{cases}$$

Try $y(t) = e^{rt}$. $\Rightarrow$ $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$.

Hence,

$$r^2e^{rt} + re^{rt} - 2e^{rt} = 0$$

$$(r^2 + r - 2)e^{rt} = 0$$

$$r^2 + r - 2 = 0$$

$$(r-1)(r+2) = 0$$

$$r_1 = 1 \quad \text{or} \quad r_2 = -2.$$

$$y_1(t) = e^{r_1t} = e^t, \quad y_2(t) = e^{r_2t} = e^{-2t}.$$

Let $y(t) = c_1y_1(t) + c_2y_2(t)$.

Then $y'' + y' - 2y$

$$= c_1y_1'' + c_2y_2'' + c_1y_1' + c_2y_2' - 2c_1y_1 - 2c_2y_2$$

$$= c_1(y_1'' + y_1' - 2y_1) + c_2(y_2'' + y_2' - 2y_2)$$

$$= c_1 \cdot 0 + c_2 \cdot 0$$

$$= 0.$$

$\Rightarrow y(t) = c_1 e^{t} + c_2 e^{-2t}$ solves the ODE.

ii) Find particular solution cor. to initial condition.

$y(t) = c_1 e^{t} + c_2 e^{-2t}$ $\quad y'(t) = c_1 e^{t} - 2c_2 e^{-2t}$

$\Rightarrow \begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = c_1 - 2c_2 = -3 \end{cases}$

$\Rightarrow c_1 = -1$ and $c_2 = 1 \Rightarrow y(t) = e^{-2t} - e^{t}$.

In general,

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_0'.$$

Find characteristic eq:

$$ar^2 + br + c = 0$$

Solve for r:

$$r = r_1, \; r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (r_1 \neq r_2).$$

$\Rightarrow y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the general solution.

$\Rightarrow y(0) = c_1 + c_2 = y_0$

$y'(0) = c_1 r_1 + c_2 r_2 = y_0'.$

Solve for c_1 and c_2:

$$c_1 = \frac{r_2 y_0 - y_0'}{r_2 - r_1}, \quad c_2 = \frac{r_1 y_0 - y_0'}{r_1 - r_2}.$$

$$y(t) = \frac{r_2 y_0 - y_0'}{r_2 - r_1} e^{r_1 t} + \frac{r_1 y_0 - y_0'}{r_1 - r_2} e^{r_2 t}.$$

~~Consider variable coefficient~~

E.g. $\begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$

Sol. Consider characteristic eq:

$$r^2 + 4r + 3 = 0.$$

$$\Rightarrow (r+1)(r+3) = 0$$

$$r_1 = -1 \quad \text{and} \quad r_2 = -3.$$

$$\Rightarrow \quad y = c_1 e^{-t} + c_2 e^{-3t}.$$

Find c_1, c_2: $\begin{cases} y(0) = c_1 + c_2 = 2. \\ y'(0) = -c_1 - 3c_2 = -1. \end{cases}$

$$\Rightarrow c_1 = \frac{5}{2} \quad \text{and} \quad c_2 = -\frac{1}{2}$$

$$\boxed{y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}}$$