Math 20B: Calculus for Science and Engineering 2.

- Instructor: Thang Huynh
- Course Webpage: thanghuynh.org/ → teaching → this course.

Contains/will contain:

- Link to the main course webpage: (Syllabus, Exam Schedule, HW.)
- Office Hours: 4-5 PM Monday and Friday.
- at AP&M 6341.
- TA Information.
- Book.
- Lecture Notes.

- Grading: There are two methods:

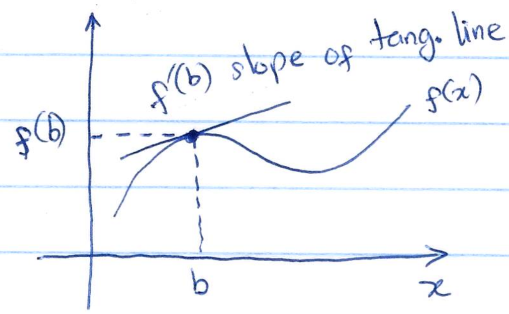
~~M1~~:	~~20~~ (HW,	MT1,	MT2,	Final)
M1:	20%	20%	20%	40%
M2:	20%	Best 20%		60%.

whichever is higher.

* Review:

Calculus can be divided into 2 parts: (Suppose we have a continuous function)

Differential Calculus

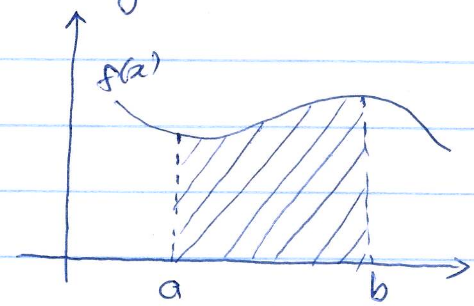


$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

def.

both are computed using limits

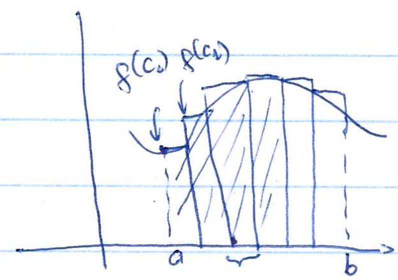
Integral Calculus



$\int_a^b f(x)\,dx := \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$

def. (Riemann Sum)

Recall:



$\|P\|$ = largest gap.

$\int_a^b f(x)\,dx$ is the signed area under the graph of $f(x)$.

The two (rates of change and area under a curve) seem unrelated, except for the Fundamental Theorem of Calculus.

Assume that $f(x)$ is cont. on $[a,b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a,b]$. (i.e. $F'(x) = f(x)$)
Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In words: If you "sum up" the rate of change of $F(x)$ between a and b (the left hand side) you get $F(b) - F(a)$.

E.g: Let $f(x) = x^2$. Find the area under the curve of $f(x)$ between $x = 1$ and $x = 3$.

Sol. $\text{Area} = \int_1^3 x^2\,dx$

$F(x) = \frac{x^3}{3}$ is an antiderivative of x^2 (Why?)

Then

$$\text{Area} = \int_1^3 x^2\,dx = \left.\frac{x^3}{3}\right|_1^3 = \frac{3^3}{3} - \frac{1}{3} = 9 - \frac{1}{3}$$

E.g: Let $v(t) = 60 + 10\cos t$.
Find $\int_a^b v(t)\,dt$.

In fact, let's give this "some meaning".

E.g. A car travels along a straight highway with velocity $v(t) = 60 + 10\cos(t)$ km/h.

Find the car's displacement between $t=0$ and $t=1$ hr.

③

Sol: Recall that velocity is the rate of change of position. Let's call the position $F(t)$, then $F'(t) = v(t)$. The question is asking us for $F(1) - F(0)$ = displacement ~~along the straight line.~~ over a time interval $[0,1]$

final pos. initial pos.

By FTOC, $F(1) - F(0) = \int_0^1 F'(t)\,dt$

$$= \int_0^1 60 + 10\cos(t)\,dt$$

$$= 60t + 10\sin(t)\Big|_0^1$$

$$= 60 + 10\sin(1).$$

The FTOC part (II): If $f(x)$ is continuous then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

So we have $\frac{d}{dx}A(x) = f(x)$

i.e. $$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

E.g. $\frac{d}{dx}\int_a^x e^{t^2} t^2 \cos t\,dt = e^{x^2} x^2 \cos x$

E.g. $\frac{d}{dx}\int_x^a t^3 \ln t\,dt \underset{\text{why?}}{=} -\frac{d}{dx}\int_a^x t^3 \ln t\,dt = -x^3 \ln x.$

E.g. $\frac{d}{dx}\int_0^{x^2} te^t\,dt$ ← problem

defining $A(x) = \int_0^x te^t\,dt$

④

We are asked to find $\frac{dA(x^2)}{dx}$.

Chain Rule! $\frac{d}{dx}A(x^2) = 2x \frac{d}{dx}A(x)$

$= 2x \frac{d}{dx}\int_0^x te^t\,dt$

$= 2x \cdot xe^x$

In general, $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}A(g(x))$

$= A'(g(x))\,g'(x)$

So $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$.

(5)

* Fundamental Theorem of Calculus, Part II:

Assume that f is continuous on an open interval.
Then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of f on I, i.e. $A'(x) = f(x)$,

or $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

E.g. $$\frac{d}{dx}\int_x^a t^3 \ln t\,dt \underset{\text{why?}}{=} -\frac{d}{dx}\int_a^x t^3\ln t\,dt = -x^3\ln x.$$

But,

E.g. $$\frac{d}{dx}\int_0^{x^2} t e^t\,dt = ?$$

FTOC II does not apply directly because the upper limit is x^2 rather than x.

Let $A(x) = \int_0^x t e^t\,dt$, then $A'(x) = xe^x$.

Then $G(x) = A(x^2) = \int_0^{x^2} te^t\,dt$.

By the Chain Rule
$$\begin{aligned} G'(x) &= A'(x^2)\cdot(x^2)' \\ &= x^2e^{x^2}\cdot 2x \\ &= 2x^3e^{x^2} \end{aligned}$$

~~Insights:~~ In general,
To differentiate the function $G(x) = \int_a^{g(x)} f(t)\,dt$,
write $G(x) = A(g(x))$ where $A(x) = \int_a^x f(t)\,dt$.

Then use the Chain Rule:
$$\begin{aligned} G'(x) &= A'(g(x))\cdot g'(x) \\ &= f(g(x))\cdot g'(x). \end{aligned}$$

⑥

* <u>Insight</u>: FTOC shows that integration and differentiation are <u>inverse operations</u>. That is,

$$\underset{\text{continuous}}{f(x)} \xrightarrow{\text{integrate}} \int_a^x f(t)\,dt \xrightarrow{\text{differentiate}} \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

and

$$f(x) \xrightarrow{\text{differentiate}} f'(x) \xrightarrow{\text{integrate}} \int_a^x f'(t)\,dt = f(x) - f(a).$$

* Some Examples (Section 5.6)

The integral of velocity:

If $s(t)$ is the position at time t of an object in linear motion (like a car on a straight road), then $v(t) = s'(t)$ is its ~~vect~~ velocity.

and $\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} s'(t)\,dt = \underbrace{s(t_2) - s(t_1)}_{\text{displacement}}$

(net change in position)

On the other hand: $\int_{t_1}^{t_2} |v(t)|\,dt =$ Distance travelled during $[t_1, t_2]$.

Remark: $-$ velocity means moving backwards.

E.g. Let $v(t) = \cos t$. Find the displacement & distance traveled from $[0, \pi]$

Ex: Let ~~$s(t) = 1 + 2\cos(\pi t)$~~. $s(t) = 1 + 2\cos(\pi t)$.

- Find the distance traveled between $t_1 = 0$ & $t_2 = 5$
- Find the displacement.
- Are they different? Why?

* Total vs. Marginal Cost.

$C(x)$: cost of producing x units of a particular product.

$C'(x)$: marginal cost.

$\int_a^b C'(x)\,dx = C(b) - C(a) =$ cost of increasing products from a to b units.

E.g. The marginal cost of producing (in units of 1000) x papers clip is $1000 + 3x^2 - 50x$ dollars.

What is the cost of increasing production from 1000 to 2000?

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Sol: $C(2) - C(1) = \int_1^2 C'(x)\,dx$

$= \int_1^2 1000 + 3x^2 - 50x \, dx$

$= 1000x + x^3 - 25x^2 \Big|_1^2$

$= 2(1000) + 2^3 - 25(2^2)$
$- 1000 - 1^3 + 25$

$= \$932.$

* In Summary:

1) The net change in a quantity $s(t)$ is equal to the integral of its rate of change:

$$\underbrace{s(t_2) - s(t_1)}_{\text{Net change over } [t_1, t_2]} = \int_{t_1}^{t_2} s'(t)\,dt.$$

* Substitution Method (Section 5.7)

Integration can be a little trickier than differentiation, but there are important general techiniques which are very helpful.

"Substitution method = Chain rule in reverse".

E.g. $f(x) = \sin x^2$

$f'(x) = \cos x^2 \cdot 2x$ by the Chain rule.

So $\int 2x \cos(x^2)\,dx = \sin x^2 + C$.

* Recall: (Chain Rule) If $F'(u) = f(u)$

$$\frac{d}{dx} F(u(x)) = F'(u(x)) \cdot u'(x).$$
$$= f(u(x)) \cdot u'(x).$$

$$\Rightarrow \boxed{\int f(u(x))\, u'(x)\, dx = F(u(x)) + C.}$$

substitution method.

E.g. $\frac{d}{dx} \sin(x^3) = 3x^2 \cos(x^3)$. (by Chain rule).

So $\int 3x^2 \cos(x^3)\,dx = \sin(x^3) + C$.

der.^{n} of inside function. inside function.

Substitution using differentials:

If $u(x)$ is a function of x, then $\frac{du}{dx} = u'(x)$.

$\Rightarrow \quad du = u'(x)\,dx$.

So now $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

$= F(u) + C$.

E.g. Evaluate $\int 3x^2 \cos(x^3)\,dx$.

~~Sol: Let $u = x^3$~~

Sol: Step 1: let $u = x^3$

Step 2: $du = 3x^2\,dx$.

Step 3: $\int 3x^2 \cos(x^3)\,dx = \int \cos(\underbrace{x^3}_{u})\,\underbrace{3x^2\,dx}_{du}$.

$= \int \cos(u)\,du$.

$= \sin(u) + C$

Step 4: $= \sin(x^3) + C$.

E.g. ~~Find~~ Evaluate $\int (x^2+3)^4\,x\,dx$

Sol: Step 1: Let $u = x^2 + 3$

Step 2: $du = 2x\,dx$.

Step 3: $\int (x^2+3)^4 x\,dx = \int (\underbrace{x^2+3}_{u})^4 \,\frac{1}{2}\,\underbrace{2x\,dx}_{du}$

$= \frac{1}{2}\int u^4\,du$

$= \frac{u^5}{10} + C$

Step 4: $= \frac{(x^2+3)^5}{10} + C$.

E.g. Evaluate $\int \tan\theta \, d\theta$

Sol. We observe that $\int \frac{\sin\theta}{\cos\theta} \, d\theta$

Step 1: let $u = \cos\theta$

Step 2: $du = -\sin\theta \, d\theta$

Step 3: $\int \tan\theta \, d\theta = -\int -\frac{\sin\theta}{\cos\theta} \, d\theta$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos\theta| + C.$$

E.g. Evaluate $\int x\sqrt{2x+1} \, dx$

Sol. $u = 2x+1$

$du = 2dx.$

$$\int x\sqrt{2x+1} \, dx = \int \frac{u-1}{2} \sqrt{u} \cdot \frac{1}{2} \, du.$$

$$= \frac{1}{4} \int (u-1)\sqrt{u} \, du$$

$$= \frac{1}{4} \int u^{3/2} - u^{1/2} \, du$$

$$= \frac{1}{4} \left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} \right) + C$$

check ! $\longrightarrow$ $$= \frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6} + C$$

E.g. Evaluate $\int \frac{x^2}{(x+3)^7} \, dx.$

* Change of variables formula for definite integrals

Suppose we want

$$\int_a^b f(u(x))\, u'(x)\, dx$$

and suppose that $F(x)$ is an antiderivative of $f(x)$.

Then $\left[F(u(x))\right]' = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$].

so by the FTOC,

$$\int_a^b f(u(x))\,u'(x)\,dx = F(u(b)) - F(u(a)).$$

$$= \int_{u(a)}^{u(b)} f(u)\,du.$$

E.g. Evaluate $\int_0^2 x^3\sqrt{x^4+1}\,dx$

Sol: let $u = x^4 + 1$.

so $du = 4x^3\,dx$.

$$\int_0^1 x^3\sqrt{x^4+1}\,dx = \frac{1}{4}\int_{u(0)}^{u(1)} \sqrt{u}\,du = \frac{1}{4}\int_1^2 u^{1/2}\,du$$

x	0	[illegible]	1
u	1	1	2

$$= \frac{1}{6} u^{3/2}\Big|_1^2$$

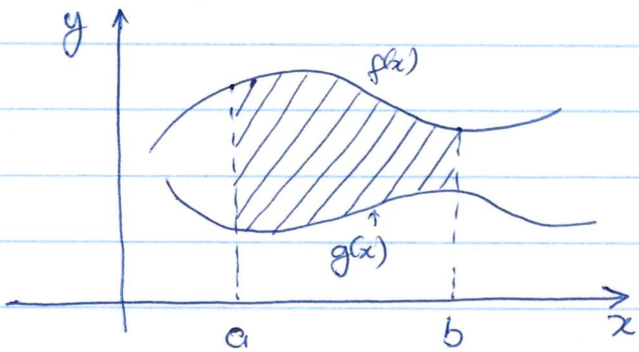
$$= \frac{1}{6}\left(17^{3/2} - 1\right).$$

Chapter 6: Applications of the integrals

(Volume, averages, work, total mass, ...)

6.1. Area between two curves:

Suppose you want to find the shaded area.



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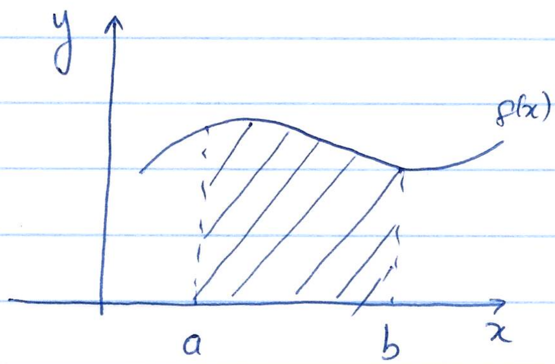

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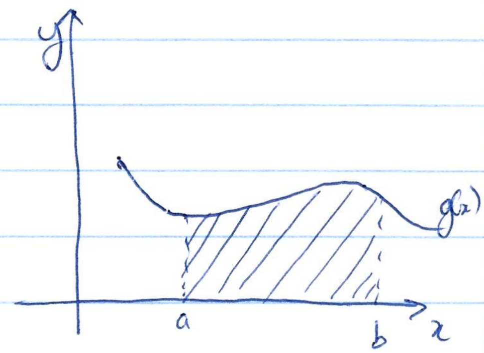

$\Rightarrow$ Area between the graphs = Area under graph of $f(x)$
− Area under graph of $g(x)$.

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

$$= \int_a^b (f(x) - g(x))\,dx$$

This is true when $f(x) \geq g(x)$.

[Area between the graphs $= \int_a^b y_{top} - y_{bottom}\,dx$.

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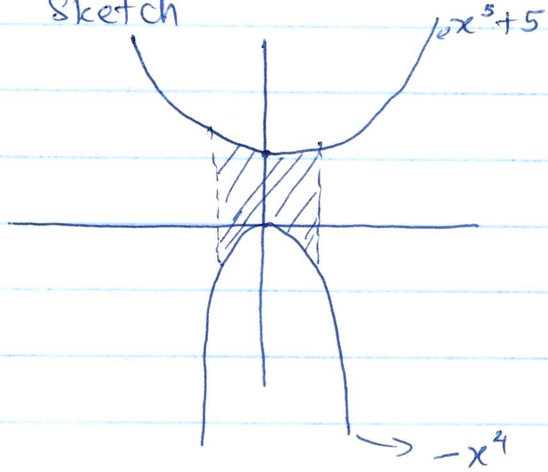
Example 1:

~~Eg.~~ Find the area between the graphs of

$f(x) = x^2+5$ & $g(x) = -x^4$, $-1 \leq x \leq 1$.

Sol: Step 1: Sketch



Step 2: Setup the integral.

$$\text{Area} = \int_{-1}^{1} f(x) - g(x)\,dx = \int_{-1}^{1} x^2+5 - (-x^4)\,dx$$

$$= \int_{-1}^{1} x^2+5+x^4\,dx$$

$$= \frac{x^3}{3} + 5x + \frac{x^5}{5} \Big|_{-1}^{1}$$

$$= \left(\frac{1}{3}+5+\frac{1}{5}\right) - \left(-\frac{1}{3}-5-\frac{1}{5}\right)$$

$$= 10 + \frac{16}{15}$$

Example 2: Find the area between the graph of

$f(x) = x^2 - 5x - 7$, $g(x) = x - 12$, $x \in [-2, 5]$.

(15)

Step 1: Sketch (and find points of intersection)

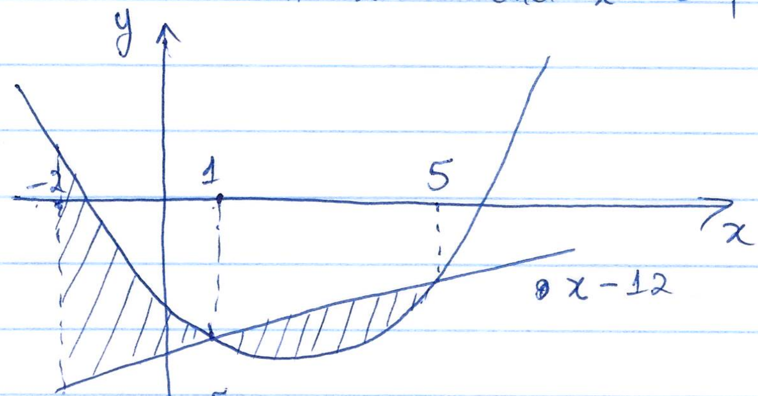
If there are points of intersection, we find them by solving $f(x) = g(x)$, or $f(x) - g(x) = 0$.

In this case: $x^2 - 5x - 7 - x + 12 = 0$

$$\Leftrightarrow x^2 - 6x + 5 = 0$$

$$\Leftrightarrow (x-1)(x-5) = 0$$

At $x = 1$ and $x = 5$, the graphs intersect.



$$\text{Area} = \int_{-2}^{5} y_{Top} - y_{bottom}\, dx.$$

$$= \int_{-2}^{1} f(x) - g(x)\, dx + \int_{1}^{5} g(x) - f(x)\, dx$$

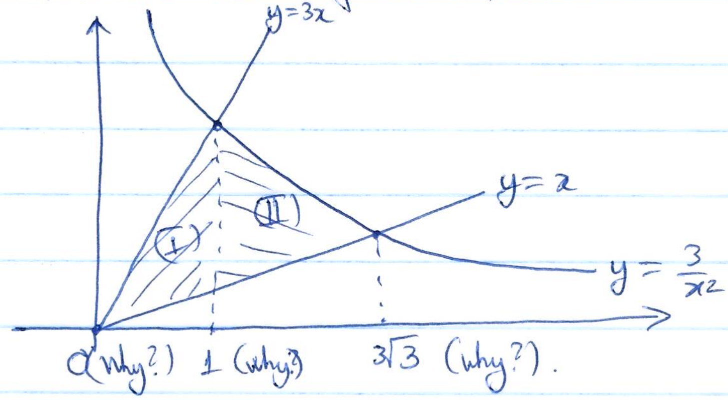
$$= \int_{-2}^{1} x^2 - 6x + 5\, dx + \int_{1}^{5} -x^2 + 6x - 5\, dx$$

$$= \frac{1}{3}x^3 - 3x^2 + 5x \Big|_{-2}^{1} + -\frac{x^3}{3} + 3x^2 - 5x \Big|_{1}^{5}$$

$$= \overset{\text{Check!}}{\ldots} = \frac{113}{3}.$$

Example 3: Area of the region bounded by the graphs of $y = \frac{3}{x^2}$, $y = 3x$ & $y = x$.

Step 1: Sketch and find intersection



Sol. Required Area $=$ Area (I) $+$ Area (II)

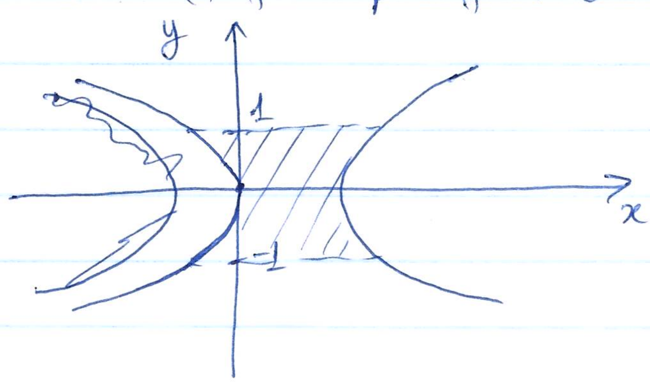
$$= \int_0^1 (3x - x)\, dx + \int_1^{\sqrt[3]{3}} \frac{3}{x^2} - x \, dx.$$

$$= \ldots$$

* Integration along y-axis:

Example: Find the area between the graphs of
$f(y) = y^2 + 1$, & $g(y) = -y^2$, $-1 \leq y \leq 1$.

Sol: sketch (+ find pts of intersection)



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$$\text{Area} = \int_{-1}^{1} x_{right} - x_{left}\, dx = \int_{-1}^{1} y^2 + 1 - (-y^2)\, dy$$

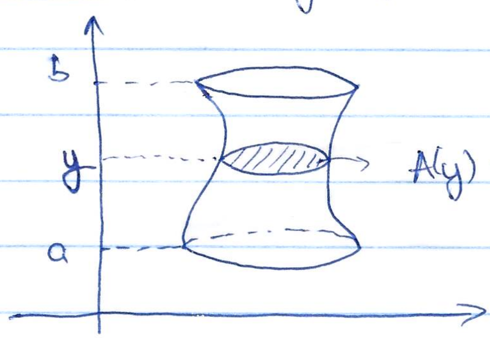
$$= \int_{-1}^{1} 2y^2 + 1\, dy$$

$$= \frac{2}{3}y^3 + y \Big|_{-1}^{1}$$

$$= \frac{2}{3} + 1 - \left(-\frac{2}{3} - 1\right)$$

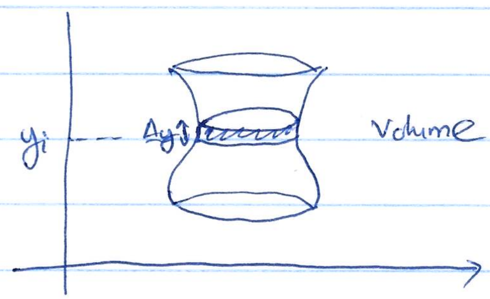
$$= \frac{10}{3}.$$

Section 6.2. Setting up integrals: Volume/ Density/ Ave. Volume.



$A(y) =$ area of thin slice at height y.

Claim: $V = \int_a^b A(y)\, dy.$



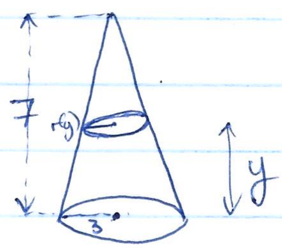
Volume of this slice is $\approx \underbrace{A(y_i)}_{\text{Area of base}} \times \underbrace{\Delta y}_{\text{height}}$

Total volume $\approx \sum_{i=1}^{N} A(y_i)\, \Delta y$ ($\approx$ sum of volume of N slices)

$\xrightarrow[\text{slice thinner (take } N \to \infty)]{\text{make the}}$ $V = \int_a^b A(y)\, dy.$

Example: Find the volume of a right circular cone with base radius = 3 and height = 7.

Sol: 1) Sketch



2) Find the cross-sectional area as a function of y.

By similar triangles: $\frac{3}{r} = \frac{7}{7-y}$. (why?)

$$\Rightarrow \quad 3(7-y) = 7r.$$

$$21 - 3y = 7r$$

$$r = 3 - \frac{3}{7}y.$$

$$\Rightarrow \quad A(y) = \pi r^2 = \pi\left(3 - \frac{3}{7}y\right)^2.$$

3) Integrate: The volume is $V = \int_0^7 A(y)\,dy$.

$$= \int_0^7 \pi\left(3 - \frac{3}{7}y\right)^2 dy.$$

$$= \dots = 21\pi.$$

(Check: $\frac{\pi r^2 h}{3} = \pi \cdot \frac{9 \cdot 7}{3} = 21\pi$).

* Average/ mean value:

Ave. value of $f(x)$ over $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$

E.g. Find the average value of $x+\sin x$ over the interval $[0, 2\pi]$.

Sol. Ave. value $= \frac{1}{2\pi-0}\int_0^{2\pi} x+\sin x \, dx$

$$= \frac{1}{2\pi}\left(\frac{x^2}{2} - \cos x\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{(2\pi)^2}{2} - \cos 2\pi - \left(\frac{0}{2} - \cos 0\right)\right]$$

$$= \frac{1}{2\pi}\left(2\pi^2 - 1 + 1\right)$$

$$= \pi.$$

* Mean Value theorem for integrals:

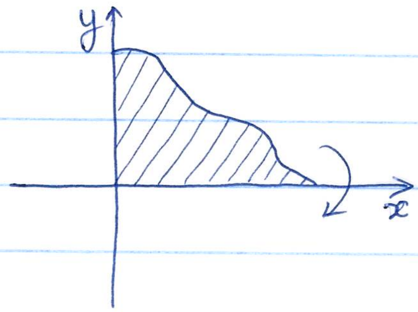
In words: A continuous function always takes its average value somewhere in the interval.

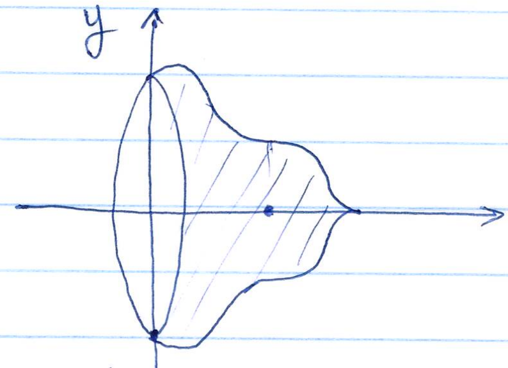
In math: If $f(x)$ is continuous on $[a,b]$, then there is a point $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Section 6.3. Volume of Revolution.

Idea.





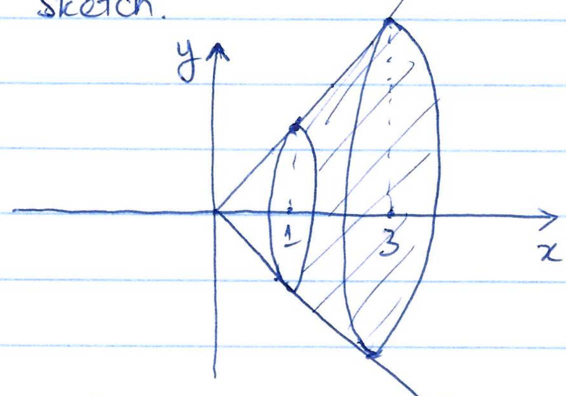
From the previous section.

$$V = \int_a^b A(x)\,dx$$ but the cross sections are circles of radius $f(x)$.

So $$\boxed{V = \int_a^b \pi f^2(x)\,dx}$$ Volume of Revolution (Disk method).

E.g. Calculate the volume obtained by rotating the region $y = 2x$ about the x-axis for $1 \leq x \leq 3$.

Sol: 1) Sketch.



2) $$V = \pi \int_1^3 (2x)^2\,dx = 4\pi \int_1^3 x^2\,dx = 4\pi \left.\frac{x^3}{3}\right|_1^3 = 4\pi\left(9 - \frac{1}{3}\right)$$

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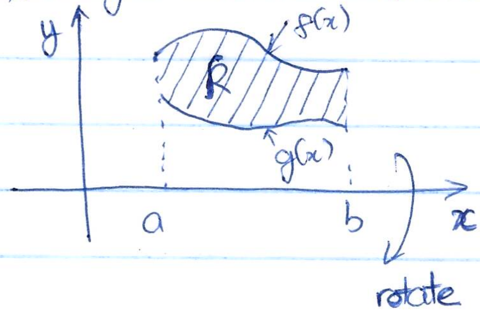
~~Example~~ * (Region between 2 curves)



$f(x) \geqslant g(x)$

What is the volume of the region obtained by rotating R about the x-axis

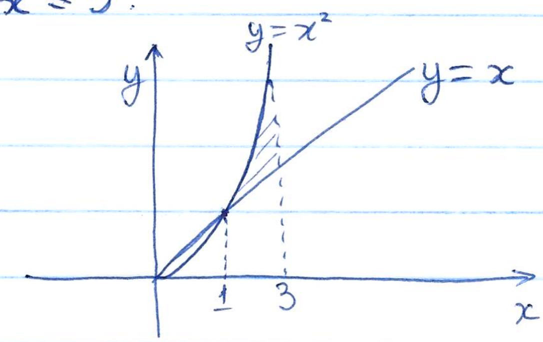
Ans: Volume = Volume generated by $f(x)$
− Volume generated by $g(x)$
(outer vol. − inner vol.)

$$= \pi \int_a^b f^2(x)\,dx - \pi \int_a^b g^2(x)\,dx.$$

$$\Rightarrow \quad V = \pi \int_a^b f^2(x) - g^2(x)\,dx.$$

Example: Find the volume V obtained by rotating the region between $y = x^2$ & $y = x$ about the x-axis for $1 \leqslant x \leqslant 3$.

Sol.



$$V = V_{outer} - V_{inner}$$

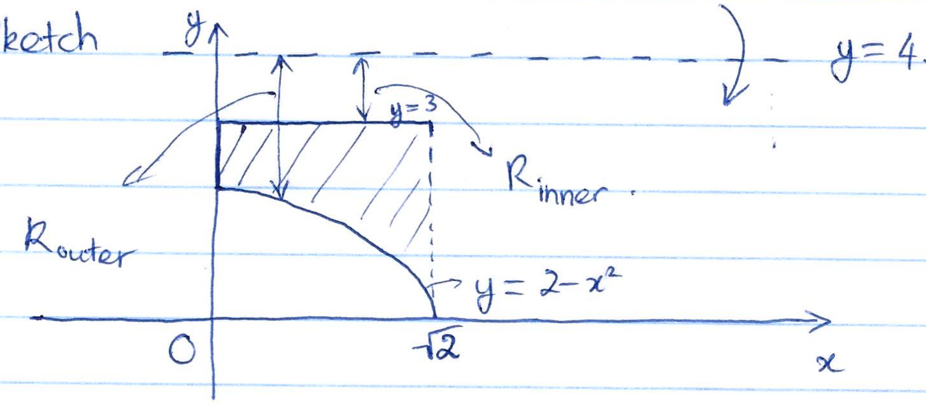
$$= \pi \int_1^3 (x^2)^2 - x^2\,dx$$

$$= \pi \int_1^3 x^4 - x^2\,dx = \pi \left(\frac{x^5}{5} - \frac{x^3}{3}\right)\Big|_1^3 = \pi \,\frac{596}{15}.$$

(22)

Example: Find the volume obtained by rotating the region between $f(x) = 2-x^2$ & $g(x) = 3$ about the line $y=4$ where $0 \leq x \leq \sqrt{2}$.

Sol: 1) Sketch



2) Volume $=$ $V_{outer} - V_{inner}$.

$$= \pi \int_0^{\sqrt{2}} R_{outer}^2 - R_{inner}^2 \, dx.$$

$$= \pi \int_0^{\sqrt{2}} (4-2+x^2)^2 - (4-3)^2 \, dx$$

$$= \pi \int_0^{\sqrt{2}} (2+x^2)^2 - 1 \, dx$$

$$= \pi \int_0^{\sqrt{2}} x^4 + 4x^2 + 3 \, dx.$$

$$= \pi \left(\frac{x^5}{5} + 4\frac{x^3}{3} + 3x \right)_0^{\sqrt{2}}.$$

$$= \pi \left(\frac{(\sqrt{2})^5}{5} + 4\frac{(\sqrt{2})^3}{3} + 3\sqrt{2} \right).$$

* Section 7.1. Integrations by Parts.

This is the reverse product rule.

Recall that $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$.

Integrating both sides.

$$\int (u(x)v(x))'\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

$$\Rightarrow u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

$$\Rightarrow \boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.}$$

Integrations by parts.

Example: Evaluate $\int \underbrace{x}_{u}\,\underbrace{\cos x}_{v'}\,dx$.

Tip: • pick u which's easy to differentiate
• pick v' which is easy to integrate.

Let $u(x) = x$ and $v'(x) = \cos(x)$

$u'(x) = 1$ $\quad v(x) = \sin(x)$

$$\int \underbrace{x\cos x}_{uv'}\,dx = \underbrace{x\sin x}_{uv} - \int \underbrace{\sin x}_{u'v}\,dx$$
$$= x\sin x + \cos x + C.$$

Example: Evaluate $\int xe^x\,dx$

Sol: $u = x$ $\quad v' = e^x$

$u' = 1$ $\quad v = e^x$.

$$\Rightarrow \int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Tips: To integrate xe^x or $x\sin x$ or $x\cos x$

or $x^n e^x$ or $x^n \sin x$ or $x^n \cos x$.

Pick $u = x$ (or x^n) and $v' = e^x$ (or $\sin x$ or $\cos x$).

- Integration by parts more than once:

Evaluate $\int x^2 \sin(x)\,dx$.

Pick $u = x^2$ $\quad v' = \sin x$

$u' = 2x$ $\quad v = -\cos x$.

$$\int \underbrace{x^2}_{u} \underbrace{\sin x}_{v'}\,dx = \underbrace{-x^2\cos x}_{uv} + \int \underbrace{2x}_{u'} \underbrace{\cos x}_{v}\,dx.$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

$$\Rightarrow \int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

E.g. Evaluate $\int_2^7 \ln x\,dx$. (later)

Sol. $u = \ln x$ $\quad v' = 1$.

$u' = \frac{1}{x}$ $\quad v = x$.

$$\int_2^7 \ln x\,dx = x\ln x\Big|_2^7 - \int_2^7 x \cdot \frac{1}{x}\,dx$$

$$= 7\ln 7 - 2\ln 2 - x\Big|_2^7$$

$$= 7\ln 7 - 2\ln 2 - (7-2)$$

(25)

Tip: To integrate $\int x^n \ln x \, dx$ or $\int x^n (\ln x)^m \, dx$. integer

pick $u = \ln x$ or $(\ln x)^m$.

Example: (Important)

Evaluate $\int e^x \sin x \, dx$

Let's try $u = \sin x$ $\quad v' = e^x$

$u' = \cos x$ $\quad v = e^x$

(1) $\Rightarrow \int e^x \sin x \, dx = e^x \sin x - \underbrace{\int e^x \cos x \, dx}_{\text{by parts. again.}}$

$\int e^x \cos x \, dx = ?$

pick $u = \cos x$ $\quad v' = e^x$

$u' = -\sin x$ $\quad v = e^x$

(2) $\Rightarrow \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$.

sub. (2) back into (1).

$\int e^x \sin x \, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx\right)$.

$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$.

$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C$.

* Summary:

- $\int u(x)\, v'(x)\, dx = uv - \int u'(x)\, v(x)\, dx$.

  use when you don't know how to integrate uv' but also $u'v$ is easier.

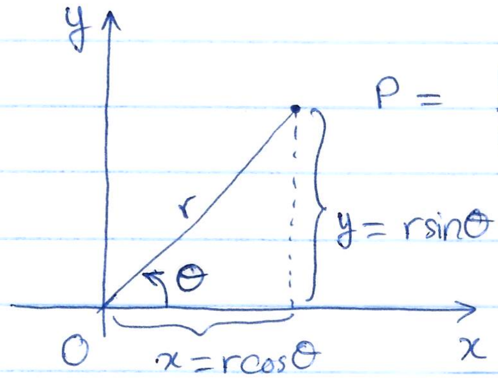
- Some tips & tricks.

  In general (but not always) want u' to be simpler/easier than u.

  Sometimes $v' = 1$ is a good choice.

Exercise: show $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$.

Section 11.3 Polar Coordinates.



$$P = \begin{cases} (x,y) \rightsquigarrow \text{cartesian/rectangular} \\ (r,\theta) \rightsquigarrow \text{Polar.} \end{cases}$$

$r =$ distance from the origin (radial coordinate)
$\theta =$ angle between $\overline{OP}$ and the positive x-axis
(angular coordinate).

Convention: the angle is positive if the corresponding rotation is counter clockwise.

Rectangular to Polar

$$r = \sqrt{x^2 + y^2} \quad \& \quad \tan\theta = y/x.$$

Polar to rectangular

$$x = r\cos\theta \quad \& \quad y = r\sin\theta.$$

E.g. 1) Convert $(r,\theta) = (7, \pi/3)$ to Cartesian coordinates

Sol. $x = r\cos\theta = 7\cos\frac{\pi}{3} = 7 \cdot \frac{1}{2} = \frac{7}{2}$

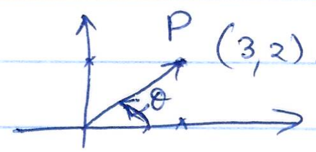
$y = r\sin\theta = 7\sin\frac{\pi}{3} = 7\,\frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{2}.$

$\rightarrow (x,y) = \left(\frac{7}{2}, \frac{7\sqrt{3}}{2}\right).$

2) Find the polar coordinates of $P = (x,y) = (3,2)$.

Sol: $r = \sqrt{x^2+y^2} = \sqrt{3^2+2^2} = \sqrt{13}$

$\tan\theta = \frac{y}{x} = \frac{2}{3}$



$\Rightarrow \theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.588.$

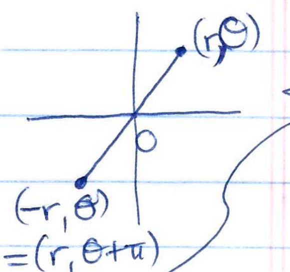
(P is in the first quadrant)

By depinition, $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$.
If $r>0$, a coordinate θ of $P=(x,y)$ is

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x}, & \text{if } x>0 \\ \tan^{-1}\frac{y}{x} + \pi, & \text{if } x<0 \\ \pm\frac{\pi}{2}, & \text{if } x=0 \end{cases}$$

Important Comments:

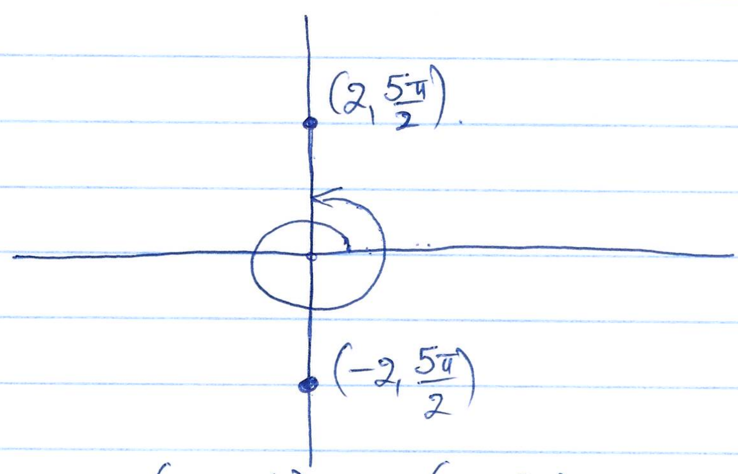
- Angles are not unique.
  $(r,\theta) = (r, \theta+2\pi) = (r, \theta-2\pi)$.
  $= (r, \theta + 2\pi n)$ for any $n \in \mathbb{Z}$.
- Origin $= (0,\theta)$ for any angle ~~theta~~. $\cancel{\theta}$ θ
- Aside from the origin, every pt has unique polar coordinate if we insist on $r>0$, & $0 \le \theta < 2\pi$.
- Convention: Allow negative r coordinates.



$(-r,\theta)$ is the reflection of (r,θ) through the origin

E.g. Find a polar representation of $P = (r_0,\theta_0) = \left(-2, \frac{5\pi}{2}\right)$ with $\begin{cases} r>0 \\ 0 \le \theta \le 2\pi \end{cases}$.

$\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$



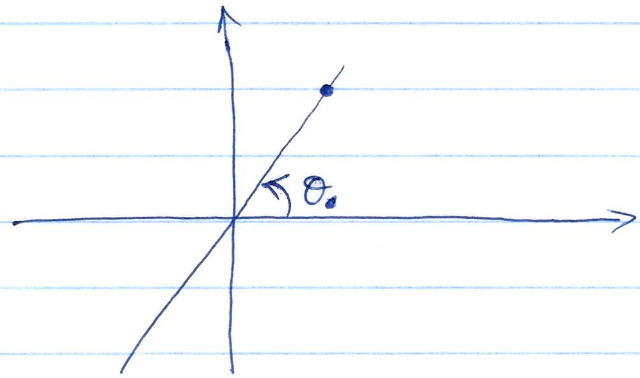
$\Rightarrow \left(-2, \frac{5\pi}{2}\right) = \left(2, \frac{3\pi}{2}\right).$

Exer Find two polar representations of $P=(-1,1)$, one with $r>0$ and one with $r<0$.

* Equations in polar coordinates:

E.g. Find the polar ~~coordinate~~ equation, of the line passing through the origin with slope $\frac{3}{2}$.

Sol:



$$\text{slope} = \frac{3}{2} \Rightarrow \tan\theta_0 = \frac{3}{2}$$

$\Rightarrow$ Polar equation is $\theta_0 = \tan^{-1}\frac{3}{2}$.

E.g. Identify the curve with polar equation $r = 2a\cos\theta$

$\uparrow$ constant.

Sol: we know that $x^2 + y^2 = r^2$.

and $r = 2a\cos\theta$

$$\Rightarrow r^2 = 4a^2\cos^2\theta$$

$$x^2 + y^2 = 2a \cdot \underbrace{r\cos\theta}_{x}$$

completing the square

$$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$$

$$(x-a)^2 + y^2 = a^2$$

$\Rightarrow$ circle of radius a and center $(a, 0)$.

Similarly, $r = 2a\sin\theta$ is a circle of radius a and center $(0, a)$.

~~20~~ 30

<u>Exercise</u>: Write the polar eq. of a circle with radius 2 and center $(x, y) = (2,3)$.

<u>Sol</u>: Cartesian equation: $(x-2)^2 + (y-3)^2 = 4$.

$$\Rightarrow x^2 - 4x + 4 + y^2 - 6y + 9 = 4.$$

$$(r\cos\theta)^2 - 4r\cos\theta + (r\sin\theta)^2 - 6r\sin\theta + 9 = 0.$$

$$r^2\cos^2\theta - 4r\cos\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 0.$$

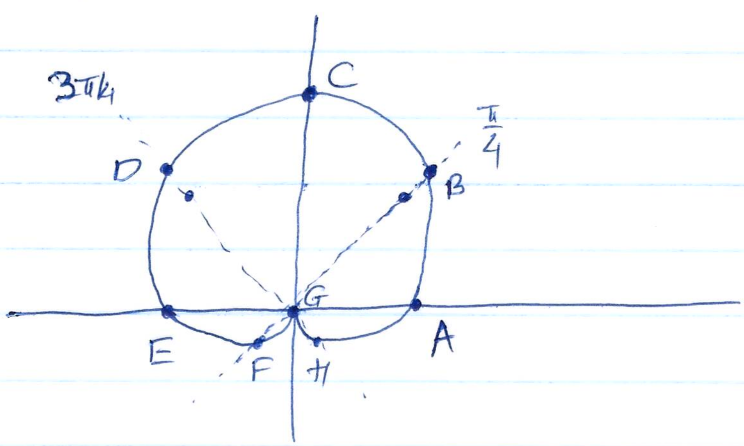
$$r^2 - r(4\cos\theta + 6\sin\theta) + 9 = 0.$$

<u>E.g.</u> Convert $r = 3\cos\theta + 4\sin\theta$ to an equation of the circle in rectangular coordinates in terms of x & y.

<u>E.g.</u> Sketch $r = 1 + \sin\theta$. $\Rightarrow$ Find $r(\theta)$ for θ varying from 0 to 2π.

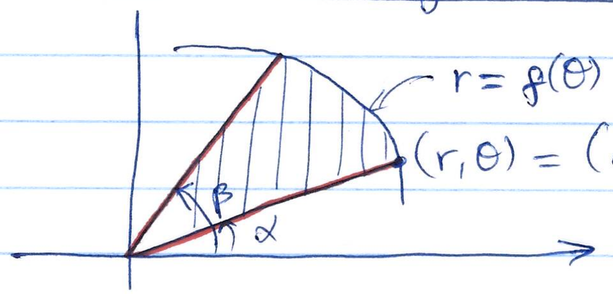
	A	B	C	D	E	F	G	H
θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r = 1 + \sin\theta$	1	1.707	2	1.707	1	0.293	0	0.293

(cardioid curve).

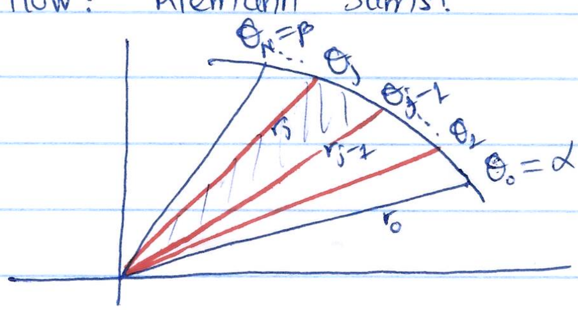


Section 11.4: Area & Arc length in polar coordinate:

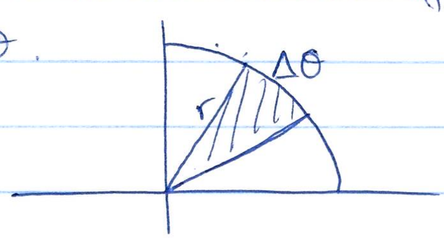
Polar coordinates are convenient for finding the area of a sector bounded by a curve.



How? Riemann Sums!



Area of a circular sector of radius r and angle $\Delta\theta$ is $\frac{r^2}{2}\Delta\theta$.



Why? ~~A(circle disk) = πr^2~~ $A(\text{disk}) = \pi r^2 = \frac{r^2}{2} \underbrace{2\pi}_{\text{full revolution}}$.

you can slice a circle into $\frac{2\pi}{\Delta\theta}$ pieces of angle $\Delta\theta$

$\Rightarrow$ area of each is $\left(\frac{r^2}{2} \cdot 2\pi\right) \Big/ \left(\frac{2\pi}{\Delta\theta}\right) = \frac{r^2}{2}\Delta\theta$.

(32)

Area of non-circular sector $\approx \sum_{j=1}^{N} \frac{1}{2} r_j^2 \Delta\theta$

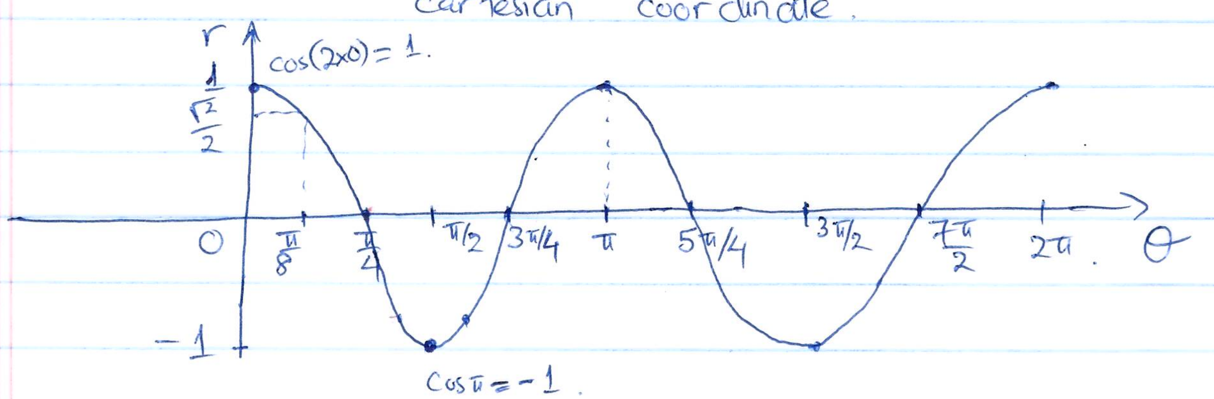
$$= \sum_{j=1}^{N} \frac{1}{2} f(\theta_j)^2 \Delta\theta$$

$$\Rightarrow \boxed{A(\text{region}) = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 \, d\theta}.$$

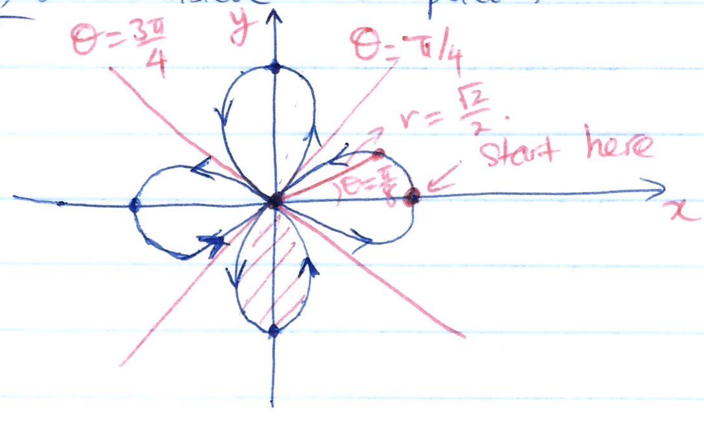
Example: The area of a circle of radius 2 is $\frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} 4 \, d\theta = 2(2\pi - 0) = 4\pi$.

E.g. Sketch $r = \cos(2\theta)$ and find the area of one petal.

Sol: Step 1: Sketch r as a function of θ in cartesian coordinate.



Step 2: "Translate" to polar.



$(-1, \frac{\pi}{2})$

$(1, \frac{\pi}{2})$

Step 3: Compute the area of a petal

$$A = \frac{1}{2}\int_{\pi/4}^{\frac{3\pi}{4}} (\cos(2\theta))^2\, d\theta$$

Recalling that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\Rightarrow \cos^2 2\theta = \frac{1}{2}(1 + \cos 4\theta).$$

$$A = \frac{1}{4}\int_{\pi/4}^{\frac{3\pi}{4}} 1 + \cos 4\theta \, d\theta$$

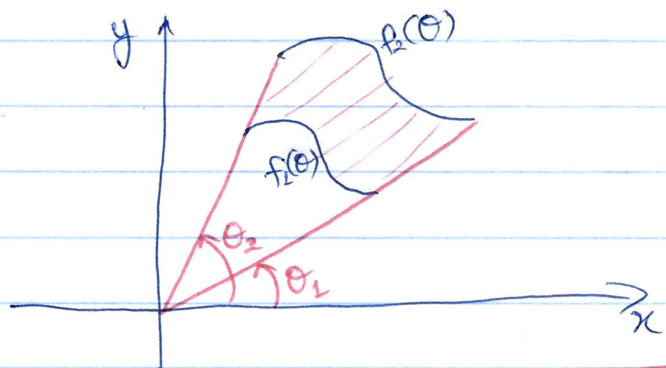
$$= \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_{\pi/4}^{\frac{3\pi}{4}}$$

$$= \frac{1}{4}\left[\frac{3\pi}{4} + \frac{\sin 3\pi}{4} - \frac{\pi}{4} - \frac{\sin \pi}{4}\right].$$

$$= \frac{1}{4}\left[\frac{2\pi}{4}\right]$$

$$= \frac{\pi}{8}.$$

* Area between two curves:



$$\boxed{\text{Area between two curves} = \frac{1}{2}\int_{\theta_1}^{\theta_2} f_2(\theta)^2 - f_1(\theta)^2 \, d\theta.}$$

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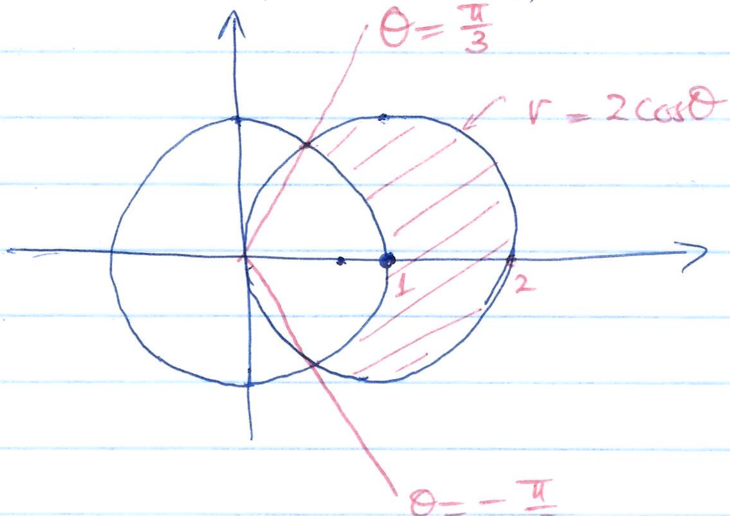
E.g. Find the area of the region inside the circle $r = 2\cos\theta$. and outside the circle $r = 1$.

Sol: ① Sketch & find pts of intersection.



$2\cos\theta = 1.$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$ are points of intersection.

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\theta)^2 - 1^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2\theta - 1 \, d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2(1+\cos 2\theta) - 1 \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\pi/3} 2\cos 2\theta + 1 \, d\theta$$

$$= \frac{\sin 2\theta}{2} + \frac{\theta}{2} \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{3}.$$