HW 7 — Solution

1). $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad \det A_2 = 0 - 1 = -1.$

- $A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

expanding in cofactors along the first row

$$\det(A_3) = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 0 - 1(-1) + 1 = 2$$

- $A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$\det A_4 = 0\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} + \underbrace{1(-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1(-1)^2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}_{\text{exchanging rows changes the sign of the determinant}}$$

$$= -3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -3\left(1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right)$$

$$= -3\left(1(-1) - 1(-1) + 1\right)$$

$$= -3.$$

In general,

$$\det(A_n) = (-1)^{n-1}(n-1).$$

2) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Compute the determinant of A using cofactors:

$$\det A = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1(-1) - 1(1)$$

$$= -2.$$

Since $\det(A) \neq 0$, A is invertible.

$\Rightarrow$ columns of A are linearly independent.

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$\det(B) = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7)$$

$$= 0.$$

Since $\det(B) = 0$, B is not invertible

$\Rightarrow$ its columns are linearly dependent.

For matrix C, since the columns of B are linearly dependent, the last three columns of C must also be linearly dependent.

$\Rightarrow \det C = 0.$

3) a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix} = 3$

$C_{12} = (-1)^{1+2} \det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0.$

$C_{13} = (-1)^{4} \det \begin{bmatrix} 0 & 3 \\ 0 & 4 \end{bmatrix} = 0$

$C_{21} = (-1)^{3} \det \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = -2$

$C_{22} = (-1)^{4} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$

$C_{23} = (-1)^{5} \det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = -4.$

$C_{31} = (-1)^{4} \det \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} = 0.$

$C_{32} = (-1)^{5} \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0.$

$C_{33} = (-1)^{6} \det \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = 3.$

$\Rightarrow C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{bmatrix}$ and $\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 1 \cdot 3 + 2 \cdot 0 + 0 \cdot 0$

$= 3$

$\Rightarrow A^{-1} = \frac{1}{\det A} C^{T} = \frac{1}{3} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}.$

b) We use the fact that A is symmetric, which implies

$$A_{ij} = A^T_{ji} \quad \text{and so}$$

$$C_{ij} = (-1)^{i+j} \det A_{ij} = (-1)^{i+j} \det A^T_{ji} = (-1)^{j+i} \det A_{ji} = C_{ji}$$

$\Rightarrow$ we only need to compute C_{ij} for which $i \leq j$:

$$C_{11} = (-1)^2 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$C_{12} = (-1)^3 \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2$$

$$C_{13} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$C_{22} = (-1)^4 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$C_{23} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$$

$$C_{33} = (-1)^6 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3.$$

$$\Rightarrow \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad \det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 2 \cdot 3 + (-1) \cdot 2 + 0 \cdot 1$$

$$= 4.$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det A} C^T = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

4) If we form the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

then the columns of A form the edges of the box, and so the determinant of A will give the volume of the box:

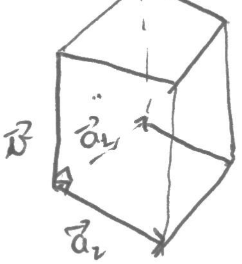
$$\begin{aligned} \det A &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= 3 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\ &= 3(8) \quad - 1(2) \quad + 1(-2) \\ &= 20. \end{aligned}$$

$\Rightarrow$ Volume of the box is 20.

Notice that, if we can find a vector $\vec{v}$ of length 1 which is perpendicular to the side spanned by the first two colums of A (call them $\vec{a_1}$ and $\vec{a_2}$), then the volume of the box spanned by $\vec{a_1}, \vec{a_2}$, and $\vec{v}$ will be the same as the area of the parallelogram spanned by $\vec{a_1}$ and $\vec{a_2}$.

Such $\vec{v}$ is in the nullspace of

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$ ~~whose~~



$$\Rightarrow \quad \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

$$\Rightarrow N\left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}\right) = \text{span}\left\{ \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix} \right\}.$$

$$\Rightarrow \vec{v} = \frac{1}{\sqrt{\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 + 1^2}} \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/6 \\ -\sqrt{2}/6 \\ \frac{2\sqrt{2}}{3} \end{bmatrix}$$

$\Rightarrow$ the area of the parallelogram spanned by $\vec{a}_1$ $\vec{a}_2$ is equal to

$$\det \begin{bmatrix} 3 & 1 & -\sqrt{2}/6 \\ 1 & 3 & -\sqrt{2}/6 \\ 1 & 1 & 2\sqrt{2}/3 \end{bmatrix} = 6\sqrt{2}.$$

The other ~~paral~~ parallelgrams also have the area $6\sqrt{2}$ (Try it!).