HW6 - Solution.

1) $A = [a_1 \; a_2] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$

<u>G-S process:</u>

$$b_1 = a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow q_1 = \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{9}} b_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$b_2 = a_2 - \langle a_1, q_1 \rangle q_1$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{1}{3}(1+6+2)\left(\frac{1}{3}\right)\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow q_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(Check $\langle q_2, q_1 \rangle = 0$!).

$$\Rightarrow Q = [q_1 \; q_2] = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}.$$

$$R = Q^T A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}.$$

2) a) The solutions of the given equantion are solutions of the matrix equantion

$$\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

$\Rightarrow$ S is the nullspace of $A = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}$.
Since A is already in reduced echelon form, (with three free variables x_2, x_3, x_4), the solution to the above matrix equation are the vectors of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the basis for $S = N(A)$ is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b) Since $S = N(A)$, it must be the case that $S^\perp$ = row space of $A = C(A^T)$.

$\Rightarrow$ the basis for $S^\perp$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

c) Given $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we want to decompose b as $b = b_1 + b_2$ where $b_1 \in S$ and $b_2 \in S^\perp$.

First, we project b onto $S^\perp$ to find b_2

$$b_2 = \text{proj}_{S^\perp} b = \frac{\left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \right\rangle} \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

$$= \frac{2}{4}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

$$\Rightarrow b_2 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

$$\Rightarrow b_1 = b - b_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\3\end{bmatrix}.$$

3) a)

$$\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} \underset{R_3+R_1}{\overset{R_2-2R_1}{=}} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix}$$

$$\overset{R_4+2R_2}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix}$$

$$\overset{R_4+\frac{5}{2}R_3}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

$$= 1(-1)(-2)(10)$$

$$= 20$$

b) $\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} \overset{R_2+\frac{1}{2}R_1}{=} \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$

$$\overset{R_3+\frac{2}{3}R_2}{=} \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$\overset{R_4+\frac{3}{4}R_3}{=} \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{vmatrix}$$

$$= 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)$$

$$= 5.$$

4) If $Q^T Q = I$,

$$\det(Q^T Q) = \det(I)$$

$$\det(Q^T)\det(Q) = 1.$$

Since $\det(Q^T) = \det(Q)$, we obtain

$$\det(Q)^2 = 1$$

$$\det(Q) = \pm 1.$$

$\Rightarrow$ The columns of Q form a box of volume 1.

In fact, they form a cubical box.