20A PRACTICE FINAL # 2

1. (6 points) Find the each of the following limits, or state that it does not exist.

(a) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$; this is approaching indeterminate $\frac{0}{0}$.

Multiply top/bottom by conjugate $\sqrt{x}+2$:

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}} = \lim_{x\to 4} \sqrt{x}+2 = \boxed{2}$$

(b) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$; this is approaching indeterminate $\frac{0}{0}$.

Apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin(x)-x}{x^3} \underset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{\cos(x)-1}{3x^2} \underset{\text{L'Hôpital (still approaching } \frac{0}{0}!)}{=} \lim_{x\to 0} \frac{-\sin(x)}{6x}$$

$$\underset{\text{L'Hôpital (still approaching } \frac{0}{0}!)}{=} \lim_{x\to 0} \frac{-\cos(x)}{6} \underset{\text{continuity}}{=} \boxed{\frac{-1}{6}}$$

(c) $\lim_{x\to\infty} \frac{5x^4-6x^2+3}{3x^4-x}$;

apply asymptotics of rational functions (or L'Hôpital) to see that

$$\lim_{x\to\infty} \frac{5x^4-6x^2+3}{3x^4-x} = \boxed{\frac{5}{3}}$$

2. (6 points) Find the derivative of the following functions. You need not simplify the resulting expressions.

(a) $f(x) = \sin^2(\cos(5x))$; Apply the chain rule twice:

$$f'(x) = 2\sin(\cos(5x))\frac{d}{dx}(\sin(\cos(5x)))$$

$$= 2\sin(\cos(5x))\cos(\cos(5x))(-5\sin(5x))$$

$$= \boxed{-10\sin(5x)\sin(\cos(5x))\cos(\cos(5x))}$$

(b) $F(x) = \int_{-1}^{2x^2} \sin^2(5\theta)\,d\theta$; Apply the chain rule and FTC II:

$$F'(x) = \frac{d}{dx}\int_{-1}^{2x^2} \sin^2(5\theta)\,d\theta = \underbrace{\sin^2(10x^2)}_{\substack{\text{derivative of} \\ \text{"outside"} \\ \text{by FTC II}}} \cdot \underbrace{(4x)}_{\substack{\text{derivative of} \\ \text{"inside"} \\ \text{by power rule.}}}$$

$$= \boxed{4x\sin^2(10x^2)}$$

(c) $g(x) = (x^2-1)^2(2x^3-5x)$; apply the product and chain rules:

$$g'(x) = \frac{d}{dx}((x^2-1)^2)\cdot(2x^3-5x) + (x^2-1)^2\cdot\frac{d}{dx}(2x^3-5x)$$

$$= 2(x^2-1)(2x)(2x^3-5x) + (x^2-1)^2(6x^2-5)$$

$$= \boxed{(x^2-1)(4x(2x^3-5x)+(x^2-1)(6x^2-5))}$$

(simplification not necessary...)

3. (8 points) If $\int_0^1 f(x)\,dx = 5$, $\int_0^2 f(x)\,dx = 2$, and $\int_0^2 g(x)\,dx = -3$, find

(a) $\int_1^2 f(x)\,dx$ $= \int_1^0 f(x)dx + \int_0^2 f(x)\,dx$

$= -\int_0^1 f(x)\partial x + \int_0^2 f(x)\partial x$

$= -5 + 2 = \boxed{3}$

(b) $\int_0^2 3f(u)\,du$ $= 3\int_0^2 f(u)du = 3(2) = \boxed{6}$

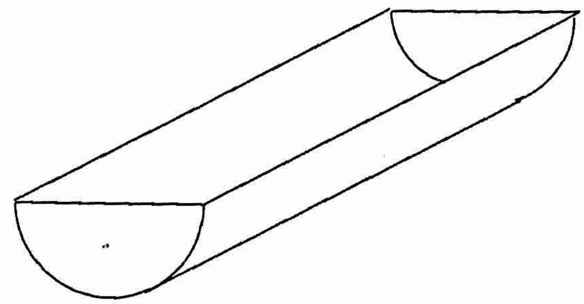
(c) $\int_1^0 f(x)\,dx$ $= -\int_0^1 f(x)dx = \boxed{-5}$

(d) $\int_0^2 \{2g(x) - 3f(x)\}\,dx$ $= \int_0^2 2g(x)dx - \int_0^2 3f(x)dx$

$= 2\int_0^2 g(x)dx - 3\int_0^2 f(x)dx$

$= 2(-3) - 3(2) = -6 - 6 = \boxed{-12}$

4. (6 points) A metal water trough with equal semicircular ends and open top needs to have a capacity of 64π cubic feet. Determine its radius r and length h if the trough is to require the least material for its construction.



we did not cover optimization

5. (6 points) A spherical balloon is being inflated at the rate of 12 cubic inches per minute. What is the radius of the balloon when the rate of change of its surface area is 3 square inches per minute? (Note: the volume V and surface area A of a sphere of radius r are given by the the formulas $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$)

Given:

spherical balloon: $V(t) = \frac{4}{3}\pi r(t)^3$

$A(t) = 4\pi r(t)^2$

inflated: $V'(t) = 12 \text{ in}^3/\text{min}$

wanted:

$r(a)$ such that $A'(a) = 3 \text{ in}^2/\text{min}$

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① find a formula for $r(a)$:

$V(t) = \frac{4}{3}\pi r(t)^3 \Rightarrow V'(t) = 4\pi r(t)^2 r'(t)$

$\Rightarrow 12 = 4\pi r(t)^2 r'(t) \Rightarrow r(a)^2 = \dfrac{3}{\pi r'(a)}$

$\Rightarrow r(a) = \sqrt{\dfrac{3}{\pi r'(a)}}$

② now find $r'(a)$:

$A(t) = 4\pi r(t)^2 \Rightarrow A'(t) = 8\pi r(t) r'(t)$

Now, we know $A'(a) = 3$, so:

$3 = 8\pi r(a) r'(a) \Rightarrow \dfrac{3}{8\pi r(a)} = r'(a)$

③ solve the resulting system

$r(a) = \sqrt{\dfrac{3}{\pi r'(a)}} = \sqrt{\dfrac{3}{\pi\left(\frac{3}{8\pi r(a)}\right)}} = \sqrt{\dfrac{8\cancel{\pi} r(a)}{\cancel{\pi}}} \rightarrow$

6. (6 points) Calculate the following definite integrals.

(a) $\int_{-3}^{4} |x^2 - 4|\, dx$; we must treat $|x^2-4|$ as a precewise function:

$$|x^2-4| = \begin{cases} (x^2-4) & \text{when } x^2-4 \geq 0 \\ -(x^2-4) & \text{when } x^2-4 < 0 \end{cases} = \begin{cases} (x^2-4) & \text{when } |x| \geq 2 \\ -(x^2-4) & \text{when } |x| < 2 \end{cases}$$

$$= \begin{cases} x^2-4 & \text{for } x \in (-\infty,-2] \cup [2,\infty) \\ -(x^2-4) & \text{for } x \in (-2,2) \end{cases}$$; now we integrate:

antiderivative: $G(x) = \frac{-x^3}{3} + 4x$

$$\int_{-3}^{4} |x^2-4|\, dx = \int_{-3}^{-2} (x^2-4)\,dx + \int_{-2}^{2} -(x^2-4)\,dx + \int_{2}^{4} (x^2-4)\,dx$$

antiderivative: $F(x) = \frac{x^3}{3} - 4x$

$$= (F(-2)-F(-3)) + (G(2)-G(-2)) + (F(4)-F(2)) = \left(\frac{-8}{3}+8-\left(\frac{-9}{3}+12\right)\right) + \left(\left(\frac{-8}{3}+8\right)-\left(\frac{8}{3}-8\right)\right) + \left(\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right)$$

(b) $\int_{-3}^{-1} \frac{3+2x^2}{x}\, dx$

$$= \int_{-3}^{-1} \frac{3}{x}\,dx + \int_{-3}^{-1} 2x\,dx$$

$$= 3(\ln|-1| - \ln|-3|) + ((-1)^2 - (-3)^2)$$

FTC I

$$= 3(\cancel{\ln(1)}^{0} - \ln(3)) + (1-9)$$

$$= \boxed{-3\ln(3) - 8}$$

7. (8 points) Consider the graph of $f(x) = \frac{1}{1-x^2}$. $= \frac{1}{(1-x)(1+x)}$ factor

(a) Determine the vertical asypmtote(s), if any.

$\mathrm{dom}(f) = \{x \in \mathbb{R} \mid x \neq \pm 1\}$

hence there are vertical asymptotes at $\boxed{x=-1 \text{ and } x=1}$

(b) Determine the horizontal asymptote(s), if any.

See that $\lim_{x\to\pm\infty} \frac{1}{1-x^2} = 0$ by asymptotics of rational functions.

hence a horizontal asymptote at $\boxed{y=0}$

(c) Determine the interval(s) of increase and the interval(s) of decrease.

First, we find the critical points. $f'(x) = -(1-x^2)^{-2}(-2x)$ $= \frac{2x}{(1-x^2)^2}$

hence the f and f' have same domain

and $f'(x)=0 \iff x=0$.

Now we'll check the signs:

x	$f'(x)$	sign
-1	$\frac{-2}{+}$	$-$
0	0	CP
1	$\frac{1}{+}$	$+$

hence $\boxed{\begin{array}{l}\text{decrease on } (-\infty,0) \\ \text{increase on } (0,\infty)\end{array}}$

(d) Determine the intervals of concavity.

we check possible inflections:

$f''(x) = \frac{(2)(1-x^2)^2 - (2x)(2)(1-x^2)(-2x)}{(1-x^2)^4}$

$= \frac{2(1-x^2) + 8x^2}{(1-x^2)^3} = \frac{2-2x^2+8x^2}{(1-x^2)^3} = \frac{2-6x^2}{(1-x^2)^3}$; $f''(x)$ DNE at $x=\pm 1$

and $f''(x)=0$ at $2-6x^2=0 \iff \pm\sqrt{\frac{1}{3}} = x$

now we'll check the signs:

x	$f''(x)$	sign
-2	$\frac{-}{-}$	$+$
-1	undef	PI
$-\frac{2}{3}$	$\frac{-}{+}$	$-$
$-\sqrt{\frac{1}{3}}$	0	PI
0	$\cancel{0}$ 2	$+$
$\sqrt{\frac{1}{3}}$	0	PI
$\frac{2}{3}$	$\frac{-}{+}$	$-$
1	undef	PI
2	$\frac{-}{-}$	$+$

hence we are

Concave up on $(-\infty,-1)$, $(-\sqrt{\frac{1}{3}},\sqrt{\frac{1}{3}})$, $(1,\infty)$
Concave down on $(-1,-\sqrt{\frac{1}{3}})$, $(\sqrt{\frac{1}{3}},1)$

8. (6 points) Find the linear approximation to $f(x) = x^{\frac{3}{2}}$ at $x = 25$ and use it to estimate $(25.06)^{\frac{3}{2}}$. (Note: $25^{\frac{3}{2}} = 125$.)

First we'll find the linear approx of f at $x=25$:

$$f'(x) = \frac{3}{2} x^{\frac{1}{2}}$$

$$f'(25) = \frac{3}{2}\sqrt{25} = \frac{15}{2}$$

$$L(x) = f'(25)(x-25) + f(25)$$

$$= \frac{15}{2}(x-25) + 125$$

$$\Rightarrow \boxed{L(x) = \frac{15}{2}x - \frac{15 \cdot 25}{2} + 125}$$

hence now

since 25.06 near 25

$$(25.06)^{3/2} = f(25.06) \approx L(25.06)$$

$$= \boxed{\frac{15}{2}(25.06) - \frac{15 \cdot 25 + 250}{2}}$$