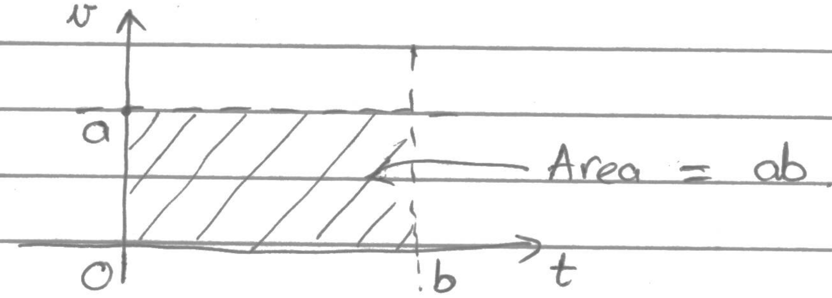
Approximating and Computing Area

Q: How far does a particle with constant velocity $v(t) = a$ m/s go in b sec?

A: $(a \text{ m/s}) \cdot (b \text{ sec}) = ab$ meters.

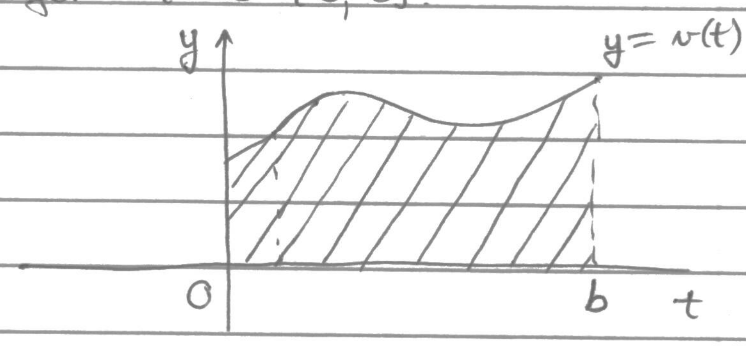


The area under the graph $v(t) = a$ for $0 \le t \le b$ is the distance travelled by the particle.

$\Rightarrow$ ~~Moore g~~ More general question.

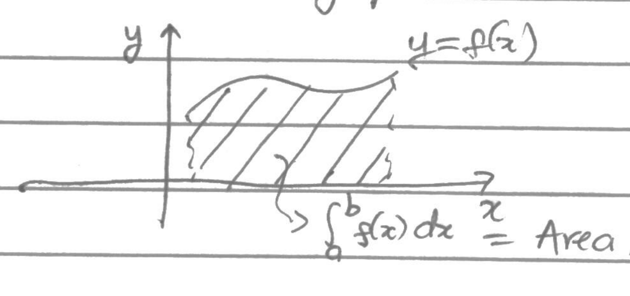
How far does a particle with velocity $v(t)$ go in b sec?

A: The area under the graph $y = v(t)$ for $t \in [0, b]$.



$\Rightarrow$ Goal: Study the area under a graph

$$\int_a^b f(x)\,dx$$



As in the case of the <u>derivative</u>, we can define this quantity as a <u>limit</u>. We will approach this by considering approximations of the area via Riemann sums.

- <u>Summation notation</u>:

$$\boxed{\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.}$$

<u>E.g.</u> $\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2$.

$$\sum_{k=2}^{5} f(k) = f(2) + f(3) + f(4) + f(5).$$

* <u>Important properties</u>:

- <u>Linearity</u>: $\sum_{i=n}^{m} (a_i + b_i) = \sum_{i=n}^{m} a_i + \sum_{i=n}^{m} b_i$.

$$\sum_{i=n}^{m} k a_i = \underset{\substack{\uparrow \\ \text{constant}}}{k} \sum_{i=n}^{m} a_i$$

- <u>Sum of constants</u>: $\sum_{i=n}^{m} k = (m-n+1)k$.

adding up k $(m-n+1)$ times

- <u>Power sums</u>:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}$$

$$\sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4}.$$

E.g. 1) $\sum_{j=1}^{200} j = \frac{200(201)}{2} = 20100.$

2) $\sum_{j=101}^{200} j \neq \sum_{j=1} = ?$

We see that $\sum_{j=1}^{200} j = \sum_{j=1}^{100} j + \sum_{j=101}^{200} j.$

$$20100 = \frac{100(101)}{2} + \sum_{j=101}^{200} j.$$

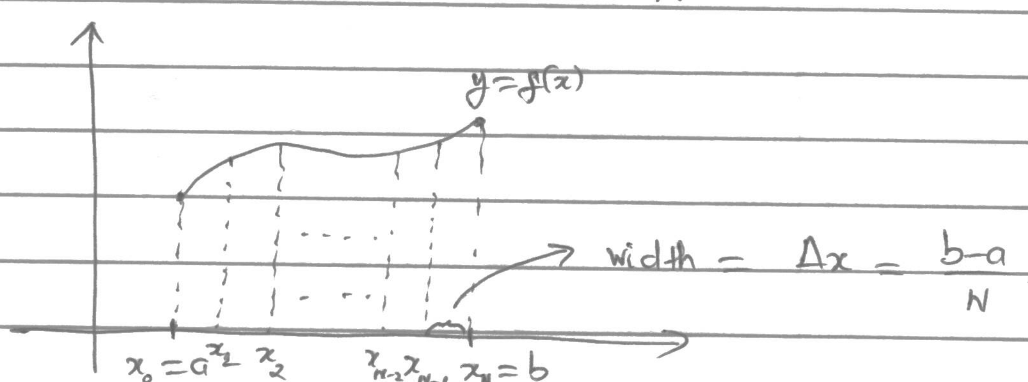
$$\Rightarrow \sum_{j=101}^{200} j = 20100 - 50(101).$$

3) $\sum_{j=101}^{200} j^3 = \sum_{j=1}^{200} j^3 - \sum_{j=1}^{100} j^3$

$$= \frac{200^2(201)^2}{4} - \frac{100^2(101)^2}{4}.$$

* Riemann Sums: The Riemann sums provides estimations of the area $\int_a^b f(x)\,dx$.

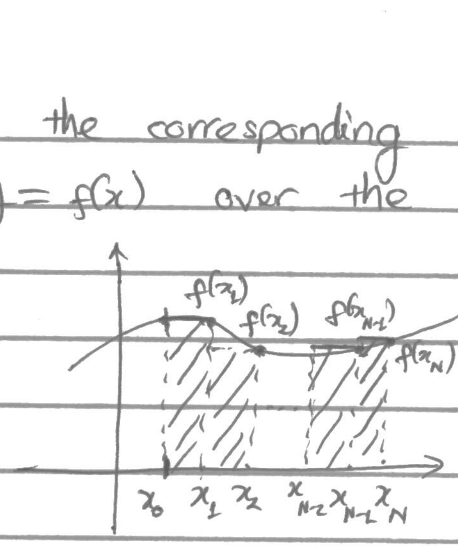
Consider an interval $[a, b]$ and a partition
$$x_j = a + \frac{j(b-a)}{N} \quad \text{for } j = 0, \ldots, N.$$



Then we can define the corresponding Riemann sums for the area under $y = f(x)$ over the interval $[a,b]$

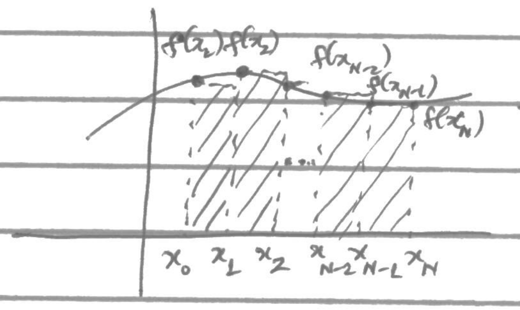
- Right:

$$R_N = \frac{(b-a)}{N} \sum_{j=1}^{N} f(x_j)$$



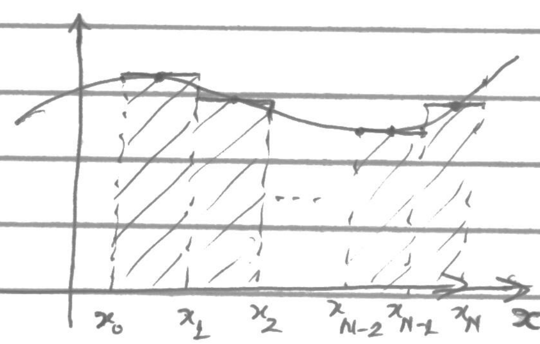
- Left:

$$L_N = \frac{(b-a)}{N} \sum_{j=0}^{N-1} f(x_j)$$



- Midpoint:

$$M_N = \frac{b-a}{N} \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$



Thm: (Riemann)

f is continuous on $[a, b]$, then

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = \int_a^b f(x)\,dx.$$

Eg. 1) $f(x) = \sqrt{x}$ over $[0,4]$.

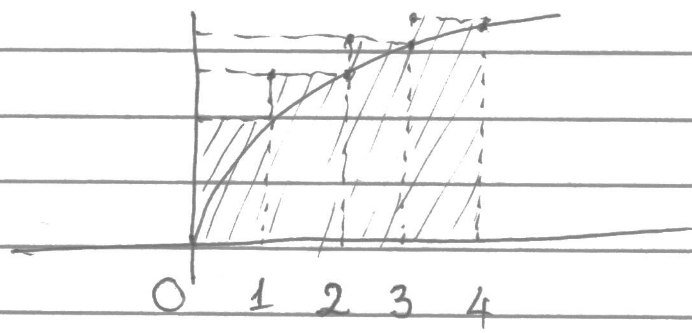
partition $x_j = j\frac{4}{N}$.

$$R_4 = \frac{4}{4}\sum_{j=1}^{4} f\left(j\frac{4}{4}\right) = \sum_{j=1}^{4} f(j) = 1 + \sqrt{2} + \sqrt{3} + 2.$$

$$L_4 = \frac{4}{4}\sum_{j=0}^{3} f(j) = \sum_{j=0}^{3} f(j) = \cancel{f(0)} + 0 + \sqrt{1} + \sqrt{2} + \sqrt{3}$$

$$M_4 = \frac{4}{4}\sum_{j=0}^{3} f\left(\frac{1}{2}\left(j\frac{4}{4} + (j+1)\frac{4}{4}\right)\right)$$

$$= \sum_{j=0}^{3} f\left(\frac{2j+1}{2}\right) = \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}} + \sqrt{\frac{5}{2}} + \sqrt{\frac{7}{2}}.$$



$$L_4 < \int_0^4 f(x)\,dx < R_4.$$

Fact: $f(x)$ increasing, then $L_N \leq M_N \leq R_N$.

2) $f(x) = 3x^2 - 2x$, find $\int_1^2 f(x)\,dx$ of a limit of right Riemann sums.

See that length of $[1,2]$ is 1 and partition is $x_j = 1 + \frac{j}{N}$.

Then $R_N = \frac{1}{N} \sum_{j=1}^{N} f(x_j)$

$= \frac{1}{N} \sum_{j=1}^{N} \left[3\left(1 + \frac{j}{N}\right)^2 - 2\left(1 + \frac{j}{N}\right)\right]$.

$= \dots$ (exercise)

$= 1 + \frac{4(N+1)}{2N} + \frac{3(N+1)(2N+1)}{N^2}$

Then

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} 1 + \frac{4N+1}{2N} + \frac{6N^2+9N+3}{N^2}$$

$$= 1 + \frac{4}{2} + 6$$

$$= 9.$$

$\Rightarrow \int_1^2 f(x)\,dx = 9.$

Properties:

1) Constant: $\int_a^b M\,dx = M(b-a)$.

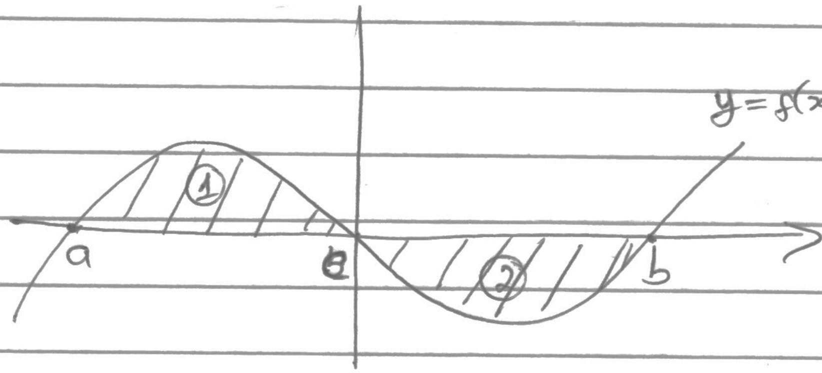
2) Linearity: $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$.

3) Interval splitting: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4) signed intervals: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

<u>Note:</u> $\int_a^b f(x)\,dx$ = signed area for functions that aren't ≥ 0.

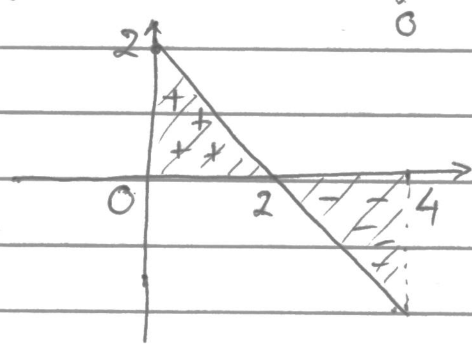


$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= +\text{ Area } 1 + (-\text{ Area } 2)$$

<u>E.g.</u> Calculate $\int_0^4 (2-x)\,dx$.

$$= \frac{1}{2}(2\cdot 2) - \frac{1}{2}(4) = 0.$$



<u>Thm:</u> (~~Capari~~ Comparison theorem) If f and g are integrab
and $g(x) \leq f(x)$ for $x \in [a,b]$, then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx.$$

<u>E.g.</u> Prove $\int_1^3 \frac{1}{x^3}\,dx \leq \int_1^3 \frac{1}{x}\,dx$.

Antiderivatives

Def. A function F is an antiderivative of f on (a, b) if $F'(x) = f(x)$.

Given an antiderivative of f, we may form the general antiderivative:

$$F(x) + C \qquad (\text{because } (F(x)+C)' = f(x))$$

where C is an arbitrary constant.
The general antiderivative is often denoted by $\int f(x)dx$

Def. A function F is the specific antiderivative of f if it satisfies

$$F'(x) = f(x) \quad \text{and} \quad F(a) = b$$

for some $a, b \in \mathbb{R}$.

E.g. Find the specific antiderivative F for $f(x) = 2x$ where $F(0) = 1$.

Clearly, $F(x) = x^2 + C$ is a general antiderivative.
Since $F(0) = 1$,

$$0 + C = 1$$

$$C = 1$$

$$\Rightarrow F(x) = x^2 + 1.$$

Note that antiderivatives follow the linearity rules:
If F is an antiderivative of f and G is ~~als~~ an antiderivative of g, then $aF + bG$ is an antiderivative of $af + bg$.

In $\int$ notation:

$$\int (af + bg)\, dx = a\int f\, dx + b\int g\, dx.$$

Basic rules:

- Power: $(n \neq -1)$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

- Exponent: $\int e^x\, dx = e^x + C$

- Trig: $\int \sin(x)\, dx = -\cos(x) + C$ $\quad\left\{ \int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C \right.$

  $\int \cos(x)\, dx = \sin(x) + C$ $\quad\left| \int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C \right.$

- Antiderivative of $\frac{1}{x}$:

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

E.g. Evaluate

1) $\int (2x^3 - 3x^{1/2} + x^{-2})\, dx.$

$$= \int 2x^3\,dx - \int 3x^{1/2}\, dx + \int x^{-2}\, dx \quad \text{(Sum rule)}$$

$$= 2\int x^3\,dx - 3\int x^{1/2}\, dx + \int x^{-2}\, dx \quad \text{(Multiple rule)}$$

$$= 2\left(\frac{x^4}{4}\right) - 3\,\frac{x^{1/2+1}}{\frac{1}{2}+1} + \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{x^4}{2} - \cancel{2\tfrac{1}{3}x^{3/2}}\ 2x^{3/2}\ \cancel{+} - x^{-1} + C.$$

Verify:

$$\frac{d}{dx}\left(\frac{x^4}{2} - 2x^{3/2} - x^{-1} + C\right)$$

$$= 2x^2 - 3x^{1/2} + x^{-2}$$

Eg. 2) Evaluate $\int (\sin(7t) + 13\cos(6t))\,dt$.

$= \int \sin(7t)\,dt + \int 13\cos(6t)\,dt$.

$= -\frac{1}{7}\cos(7t) + 13\ \frac{1}{6}\sin(6t) + C$.

3) Evaluate $\int (3e^x + 3)\,dx$

$= \int 3e^x\,dx + \int 3\,dx$

$= 3\int e^x\,dx + 3\int dx$

$= 3e^x + 3x + C$.

4) $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$.

The Fundamental Theorem of Calculus.

Thm (FTC I)

Suppose f is continuous on $[a,b]$ and F is an antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

E.g. $\int_{-3}^{2} x^2\,dx$

First, let's do it with Riemann sums. The interval is length 5 and the partition is

$$x_j = -3 + \frac{5j}{N}.$$

$$\Rightarrow R_N = \frac{5}{N}\sum_{j=1}^{N}\left(\frac{5j}{N} - 3\right)^2$$

$$= \frac{5}{N}\sum_{j=1}^{N}\left(\frac{5^2 j^2}{N^2} - \frac{30j}{N} + 9\right).$$

$$= \frac{5^3}{N^3}\sum_{j=1}^{N} j^2 - \frac{150}{N^2}\sum_{j=1}^{N} j + \frac{5}{N}\sum_{j=1}^{N} 9.$$

$$= \frac{5^3}{N^3}\frac{N(N+1)(2N+1)}{6} - \frac{150}{N^2}\cdot\frac{N(N+1)}{2} + \frac{5}{N}(9N)$$

$$= \frac{125}{6}\frac{(N+1)(2N+1)}{N^2} - 75\,\frac{N+1}{N} + 45.$$

$$\Rightarrow \lim_{N\to\infty} R_N = \frac{2(125)}{6} - 75 + 45 = \frac{125}{3} - 30$$

$$\Rightarrow \int_{-3}^{2} x^2\,dx = \frac{125}{3} - 30 = \frac{8}{3} + 9.$$

With FTCI, it's much easier:

Consider $f(x) = x^2 \Rightarrow F(x) = \frac{x^3}{3}$ is an antiderivative. Hence,

$$\int_{-3}^{2} x^2\,dx = F(2) - F(-3) = \frac{8}{3} - \frac{(-3)^3}{3} = 9 + \frac{8}{3}.$$

E.g. If $f(x) = \begin{cases} 12 - x^2 & \text{for } x \le 2 \\ x^3 & \text{for } x > 2. \end{cases}$

Compute $\int_{-2}^{3} f(x)\,dx$

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= \int_{-2}^{2} (12 - x^2)\,dx + \int_{2}^{3} x^3\,dx$$

$F(x) = 12x - \frac{x^3}{3}$ is an antiderivative of $12 - x^2$.

$G(x) = \frac{x^4}{4}$ is an anti. of x^3.

$$= F(2) - F(-2) + G(3) - G(2)$$

$$= \left[12(2) - \frac{2^3}{3}\right] - \left[12(-2) - \frac{(-2)^3}{3}\right] + \frac{3^4}{4} - \frac{2^4}{4}.$$

$$= \ldots = 44 - \frac{16}{3} + \frac{3^4}{4}.$$

E.g. $\int_{0}^{3} |x^2 - 1|\,dx$

Observe that $|x^2 - 1| = \begin{cases} x^2 - 1 & \text{for } x^2 - 1 \ge 0 \\ -(x^2 - 1) & \text{for } x^2 - 1 < 0. \end{cases}$

$$= \begin{cases} x^2 - 1 & \text{for } x^2 \ge 1 \\ -(x^2 - 1) & \text{for } x^2 < 1. \end{cases}$$

$$= \begin{cases} x^2 - 1 & \text{for } |x| \ge 1 \\ -(x^2 - 1) & \text{for } |x| < 1. \end{cases}$$

$$= \begin{cases} x^2 - 1 & \text{for } x \in (-\infty, -1] \cup [1, \infty). \\ -x^2 + 1 & \text{for } -1 < x < 1. \end{cases}$$

Now split the integral:

$$\int_0^3 |x^2 - 1|\, dx = \int_0^1 |x^2 - 1|\, dx + \int_1^3 |x^2 - 1|\, dx$$

$$= \int_0^1 \underbrace{-x^2 + 1}\, dx + \int_1^3 \underbrace{x^2 - 1}\, dx.$$

antiderivatives $\quad F(x) = -\frac{x^3}{3} + x \qquad G(x) = \frac{x^3}{3} - x.$

$$= F(1) - F(0) + G(3) - G(1)$$

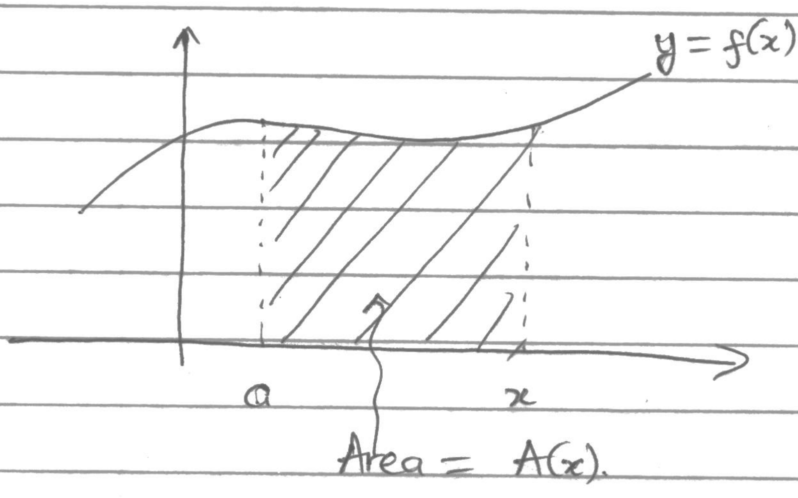
$$= \left[\left(-\frac{1^3}{3} + 1\right) - 0\right] + \left[\left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)\right].$$

Thm (FTC II): Let $f(t)$ be a continuous function and define the area function

$$A(x) = \int_a^x f(t)\, dt.$$

Then:

$$A'(x) = f(x)$$



E.g. 1) $A(x) = \int_2^x u^4\,du$

Compute $A'(x)$ with both FTC I and FTC II.

- FTC I,

$$A(x) = \int_2^x u^4\,du = \frac{x^5}{5} - \frac{2^5}{5}$$

then $A'(x) = x^4$.

- FTC II: $A'(x) = x^4$.

Note that $A(x)$ is the antiderivative of $f(x)$ such that $A(2) = 0$.

2) $A(x) = \int_4^x e^{3u}\,du$. Find $A'(x)$.

$$A'(x) = e^{3x}$$

The area function $A(x) = \int_a^x f(t)\,dt$ is precisely the specific antiderivative of $f(x)$ such that $A(a) = 0$.

* Apply the chain rule with area functions:

Let $G(x) = \int_a^{g(x)} f(t)\,dt$.

Then $G(x) = A(g(x))$, where $A(x) = \int_a^x f(t)\,dt$.

Now apply the chain rule:

$$G'(x) = A'(g(x))\, g'(x)$$

$$= \underbrace{f(g(x))}_{\text{FTC II}}\, g'(x).$$

E.g. $A(x) = \int_{3x}^{9x+2} e^{-u}\, du$. Find $A'(x)$.

$$A(x) = \int_{3x}^{9x+2} e^{-u}\, du = \int_{3x}^{0} e^{-u}\, du + \int_{0}^{9x+2} e^{-u}\, du.$$

$$= -\int_{0}^{3x} e^{-u}\, du + \int_{0}^{9x+2} e^{-u}\, du.$$

$$\Rightarrow A'(x) = -\frac{d}{dx}\left(\int_{0}^{3x} e^{-u}\, du\right) + \frac{d}{dx}\left(\int_{0}^{9x+2} e^{-u}\, du\right).$$

$$= -e^{-3x}(3) + e^{-(9x+2)}(9).$$

~~E.g.~~