2) a) $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

b) Suppose that Q is an orthogonal matrix, i.e., $Q^T = Q^{-1}$.
Then

$$\begin{aligned}
\|Qx\|_2^2 &= \langle Qx, Qx\rangle. \\
&= (Qx)^T Qx. \\
&= x^T \underbrace{Q^T Q}\, x. \\
&= x^T I x. \\
&= x^T x. \\
&= \|x\|_2^2.
\end{aligned}$$

$$\Rightarrow \quad \|Qx\|_2 = \|x\|_2.$$

c) Since Q is an orthogonal matrix (why?)

$$\|Qx\|_2 = \|x\|_2 = \sqrt{1^2 + 2^2 + 2^2 + (-1)^2} = \sqrt{10}.$$

3) For $\vec{u}, \vec{v} \in \mathbb{R}^m$, $\vec{u} \neq 0$, $\vec{v} \neq 0$, let $A = I + uv^T$.
Suppose that A is nonsingular.
Consider $B = I + \alpha uv^T$ for some scalar α, Then

$$\begin{aligned}
BA &= (I + \alpha uv^T)(I + uv^T) \\
&= I + uv^T + \alpha uv^T + \alpha u(v^T u)v^T. \\
&= I + \left(1 + \alpha + \alpha\langle v, u\rangle\right) uv^T.
\end{aligned}$$

- If $\langle v, u\rangle \neq -1$, then take $\alpha = \dfrac{-1}{1 + \langle v, u\rangle}$

we obtain

$$BA = I.$$

$$\Rightarrow \ A \text{ is nonsingular and } A^{-1} = I + \frac{-1}{1 + \langle v, u\rangle} uv^T.$$

- If $\langle v, u\rangle = -1$, then $u = -\frac{v}{\|v\|_2^2}$ (why?)

$\Rightarrow A = I + \vec{u}\vec{v}^T = I - \frac{1}{\|v\|_2^2} v v^T$.

and A is singular since

$$A\vec{v} = v - \frac{1}{\|v\|_2^2} v v^T v = v - v = 0.$$

$$\Rightarrow \quad v \in \text{Null}(A).$$

And $\text{Null}(A) \cancel{= sp\{e_1}} = \text{span}\{v\}$.

since for any vector $\vec{x} \neq t\vec{v}$ (i.e., x is not in the direction of v).

$$A\vec{x} = \vec{x} - \underbrace{\frac{1}{\|v\|_2^2}\langle\vec{v},\vec{x}\rangle}\vec{v}$$

$$= \vec{x} - s\vec{v}. \neq 0.$$

$$\Rightarrow \quad \vec{x} \notin \text{Null}(A).$$

4) Let $E = uv^T$. By definition,

$$\|E\|_2 = \sup_{\|x\|_2 = 1} \|Ex\|_2$$

$$= \sup_{\|x\|_2 = 1} \|u v^T x\|_2.$$

$$= \sup_{\|x\|_2 = 1} |\langle v, x\rangle| \; \|u\|_2.$$

By the Cauchy-Schwartz inequality:

$$|\langle v, x\rangle| \leq \|v\|_2 \|x\|_2.$$

$$\therefore \quad \|E\|_2 \leq \sup_{\|x\|_2 = 1} \|v\|_2 \|x\|_2 \|u\|_2 = \|v\|_2 \|u\|_2.$$

③.

$\Rightarrow \quad \|E\|_2 \leq \|v\|_2 \|u\|_2$.

we can achieve "=" by

Taking $x = \dfrac{v}{\|v\|_2}$. then $\|x\|_2 = 1$,

and $\|Ex\|_2 = \|uv^T x\|_2 = \left\|uv^T \dfrac{v}{\|v\|_2}\right\|_2$

$= \|u \|v\|_2\|_2$.

$= \|v\|_2 \|u\|_2$.

$\therefore \quad \|E\|_2 = \|v\|_2 \|u\|_2$.

It's also true for $\|E\|_F = \|v\|_F \|u\|_F$.

Pf: We observe that

$$E = [v_1\vec{u} \quad v_2\vec{u} \quad \ldots \quad v_n\vec{u}].$$

Then,

$$\|E\|_F^2 = \|v_1\vec{u}\|_2^2 + \|v_2\vec{u}\|_2^2 + \ldots + \|v_n\vec{u}\|_2^2.$$

$$= |v_1|^2 \|u\|_2^2 + |v_2|^2 \|\vec{u}\|_2^2 + \ldots + |v_n|^2 \|\vec{u}\|_2^2$$

$$= (|v_1|^2 + |v_2|^2 + \ldots + |v_n|^2) \|\vec{u}\|_2^2.$$

$$= \|\vec{v}\|_2^2 \|\vec{u}\|_2^2.$$

$\Rightarrow \quad \|E\|_F = \|\vec{v}\|_2 \|\vec{u}\|_2 = \|\vec{v}\|_F \|\vec{u}\|_F$.

Note that $\|v\|_2 = \|v\|_F$ and $\|u\|_2 = \|u\|_F$.

4.

5) $A = \begin{bmatrix} -2 & 3 & 2 \\ -4 & 5 & 1 \\ 1 & -2 & 4 \end{bmatrix}$

$$\|A\|_1 = \max\{|-2| + |-4| + 1,\ |3| + |5| + |-2|,\ |2| + |1| + |4|\}$$
$$= \max\{7, 10, 7\}$$
$$= 10$$

$$\|A\|_\infty = \max\{|-2| + |3| + |2|,\ |-4| + |5| + |1|,\ |1| + |-2| + |4|\}.$$
$$= \max\{7, 10, 7\}.$$
$$= 10.$$

$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)} \Rightarrow$ ~~will not~~ will learn how to find it later.

$$\|A\|_F = \sqrt{|-2|^2 + |3|^2 + |2|^2 + |-4|^2 + |5|^2 + |1|^2 + |1|^2 + |-2|^2 + |4|^2}$$
$$= 80.$$

6) Given $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, show that A^TA is nonsingular if and only if A has full rank.

Pf: ($\Rightarrow$) Suppose A^TA is nonsingular.

If $x \in \mathbb{R}^n$ such that $Ax = 0$, then

$$A^TAx = A^T0.$$
$$A^TAx = 0.$$
$$\Rightarrow x \in Null(A^TA).$$

Since A^TA is nonsingular, $x = 0$.

$\Rightarrow$ $Null(A) = \{0\}$. $\Rightarrow$ A is of full rank.

5

($\Leftarrow$) Suppose A is of full rank.

Let $x \in \mathrm{Null}(A^TA)$, then $A^TAx = 0$.

$\Rightarrow A^Ty = 0$, where $y = Ax$.

$\Rightarrow$ y is orthogonal to columns of A.

But $y \in \mathrm{range}(A)$.

$\Rightarrow y = 0$.

$\Rightarrow Ax = 0$

$\Rightarrow x = 0$.

$\therefore \mathrm{Null}(A^TA) = \{0\}$.

$\therefore A^TA$ is nonsingular.

7) Will learn this week (Week 3).