Lecture 5: Vector spaces and subspaces

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Vector spaces

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In other words, a vector space is a collection of vectors which can be added and scaled; subject to the usual rules you would hope for, namely associativity, commutativity, distributivity.

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- Example. Let $\mathscr{P}_n$ be the set of all polynomials of degree at most $n \geq 0$. Is $\mathscr{P}_n$ a vector space?
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- Example. Let V be the set of all polynomials of degree exactly 3. Is V a vector space?

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- Example. Let V be the set of all polynomials of degree exactly 3. Is V a vector space?
  No! For example, take $f(x) = x^3 + x$ and $g(x) = -x^3 + 5x^2 + 2$. Both f and g are in V, but their sum $f(x) + g(x) = 5x^2 + x + 2$ *is not in* V.

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How to check if a subset W is a subspace?

- W contains the zero vector 0.
- W is closed under addition, i.e. if $\boldsymbol{u}, \boldsymbol{v} \in W$, then $\boldsymbol{u} + \boldsymbol{v} \in W$.
- W is closed under scaling, i.e. if $\boldsymbol{u} \in W$ and $c \in \mathbb{R}$, then $c\boldsymbol{u} \in W$.

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Spans of vectors are subspaces

The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all their linear combinations. We denote it by span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

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Theorem. If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are in a vector space V, then $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is a subspace.

Spans of vectors are subspaces

- Example. Is $W = \left\{ \begin{bmatrix} a+3b \\ 2a-b \end{bmatrix} \;:\; a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

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- Example. Is $W = \left\{ \begin{bmatrix} -a & 2b \\ a+b & 3a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of $\mathscr{M}_{2\times 2}$, the vector space of 2×2 matrices?

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Spans of vectors are subspaces

- Exercise. Are the following vector spaces?
  - $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + 3b = 0, 2a - c = 1 \right\}$.
  - $W_2 = \left\{ \begin{bmatrix} a + c \\ -2b \\ b + 3c \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.
  - $W_1 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : ab \geq 0 \right\}$.

Null spaces

- Definition. The **null space** of a matrix A is

$$N(A) = \{\boldsymbol{v} \ : \ A\boldsymbol{v} = 0\}.$$

In other words, if A is $m \times n$, then its null space consists of those vectors $\boldsymbol{v} \in \mathbb{R}^n$ which solve the *homogeneous* equation $A\boldsymbol{x} = 0$.

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Theorem. If A is $m \times n$, then $N(A)$ is subspace of $\mathbb{R}^n$.

- Proof.

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- Example. Find an explicit description of $N(A)$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

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$$\left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & \boxed{1} & -6 & -15 & 0 \end{array}\right].$$

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Hence, x_1 and x_3 are pivot variables. Let $x_2 = u, x_4 = v$, and $x_5 = w$.

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The solutions of $A\boldsymbol{x} = \mathbf{0}$ is of the form

$$\begin{bmatrix} -2u - 13v - 33w \\ u \\ 6v + 15w \\ v \\ w \end{bmatrix} = u\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v\begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

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$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Column spaces

- Definition. The **column space**, $C(A)$, of A is the span of the vector columns of A, i.e., if $A = [\ \boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_n\ ]$, then $C(A) = \text{span}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$.

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  - $\boldsymbol{b}$ is in $C(A)$ if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. (Why?)

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  - $\boldsymbol{b}$ is in $C(A)$ if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. (Why?)
  - If A is $m \times n$, then $C(A)$ is a subspace of $\mathbb{R}^m$. (Why?).

Column spaces

- Example. Find a matrix A such that $W = C(A)$ where

$$W = \left\{ \begin{bmatrix} 2x - y \\ 3y \\ 7x + y \end{bmatrix} \ : \ x, y \in \mathbb{R} \right\}.$$

$C(A)$ and solutions to $A\boldsymbol{x} = \boldsymbol{b}$

Theorem. Let $\boldsymbol{x}_p$ be a solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$. Then every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is a solution to the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$.

- Example. Find a parametric description of the solutions to

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$