Section 3.4. Repeated Roots; Reduction of Order.

(*) $\quad ay'' + by' + c = 0.$

Characteristic Eq:

$$ar^2 + br + c = 0.$$

$\Rightarrow$ Sols. $\quad r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

When $b^2 - 4ac = 0$, we have repeated roots

$$r = -\frac{b}{2a}$$

$\Rightarrow \quad y_1(t) = e^{rt} = e^{-bt/2a}$ is one solution.

How to find another solution? $\left(y_2(t) = te^{rt} = te^{-bt/2a} \right)$.

Use Reduction of order method (originated by d'Alembert in the 18th century).

Let $\quad y(t) = v(t)\, e^{rt} \quad = v(t)\, y_1(t).$

(If $y(t)$ is a sol to (*), it has to satisfy (*).
$\Rightarrow$ find $v(t)$.)

$$y' = v'(t)e^{rt} + v(t)\, re^{rt}.$$
$$= (v' + vr)e^{rt}.$$

and
$$y'' = (v'' + v'r)e^{rt} + (v' + vr)\, re^{rt}.$$
$$= (v'' + 2v'r + vr^2)\, e^{rt}.$$

Suppose that $y = v(t)e^{rt}$ is a sol. to (*), then

$$0 = ay'' + by' + cy$$
$$= a(v'' + 2v'r + vr^2)e^{rt} + b(v' + vr)e^{rt} + cve^{rt}.$$
$$= [av'' + 2arv' + ar^2 v + bv' + brv + cv)e^{rt}.$$
$$= [av'' + (2ar + b)v' + \overset{0}{(ar^2 + \cancel{br + c})}v]e^{rt}.$$

$$\Rightarrow \quad av'' + (2ar + b)v' = 0.$$

Since $r = -\frac{b}{2a}$, $\quad av'' = 0$.

$\therefore \quad v'' = 0$.

Let $u = v' \Rightarrow \quad u' = 0$.

$u = C_1$, where C_1 is a constant.

$\Rightarrow \quad v' = C_1$.

$v(t) = C_1 t + C_2$, for arbitrary constants C_1, C_2

Take $C_1 = 1$ and $C_2 = 0$.

Another solution to $(*)$ is $y_2(t) = t e^{-bt/2a}$.

$\Rightarrow$ General Solution:

$$y(t) = c_1 \underbrace{e^{-bt/(2a)}}_{y_1(t)} + c_2 \underbrace{t e^{-bt/2a}}_{y_2(t)}.$$

Q: Does y include all sols to $(*)$?

Check with the Wronskian:

$$W[y_1, y_2](t) = \begin{vmatrix} e^{-bt/(2a)} & t e^{-bt/(2a)} \\ -\frac{b}{2a} e^{-bt/(2a)} & \left(1 - \frac{bt}{2a}\right) e^{-bt/(2a)} \end{vmatrix} = \cdots = e^{-bt/a}.$$

$\Rightarrow W[y_1, y_2](t) \neq 0$

$\Rightarrow$ y_1 and y_2 are fundamental set of solutions and y includes all solutions.

E.g. $\quad y'' + 6y' + 9y = 0$.

Find general sol.

Sol. Characteristic Eq:

$$r^2 + 6r + 9 = 0.$$

$$(r+3)^2 = 0$$

$$r = -3.$$

$\Rightarrow y_1(t) = e^{-3t}$, $y_2(t) = t e^{-3t}$, and $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$.

* Summary for second-order linear homogeneous eq. with constant coefficients:

$$ay'' + by' + cy = 0.$$

Characteristic Eq.

$$ar^2 + br + c = 0.$$

with roots r_1 and r_2.

1) $r_1 \neq r_2$ are real :

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

2) r_1 and r_2 are Complex : $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$.

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).$$

3) $r_1 = r_2$:

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}.$$

E.g. Find the solution of the initial value problem.

$$y'' - y' + \frac{y}{4} = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}.$$

Sol. Char. Eq:

$$r^2 - r + \frac{1}{4} = 0.$$

$$\left(r - \frac{1}{2}\right)^2 = 0$$

$$\Rightarrow \quad r_1 = r_2 = \frac{1}{2}.$$

General sol. $y(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$

$$2 = y(0) = c_1 \Rightarrow \cancel{c_1 = 0}.\ c_1 = 2$$

And $y'(t) = \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} + c_2 \frac{t}{2} e^{t/2}$.

$$\frac{1}{3} = y'(0) = \frac{c_1}{2} + c_2 \Rightarrow c_2 = -\frac{2}{3}.$$

$$\Rightarrow y(t) = 2e^{t/2} - \frac{2}{3} t e^{t/2}.$$

Q: Can we use Reduction of order method for

(*) $$y'' + p(t)\, y' + q(t)\, y = 0 \quad ?$$

A: Yes!

Suppose y_1 is a sol to (*).

and $y(t) = v(t)\, y_1(t)$ is another sol.

Then $y' = v' y_1 + v y_1'$

and $y'' = v'' y_1 + v' y_1' + v' y_1' + v y_1''$

$= v'' y_1 + 2 v' y_1' + v y_1''$.

Then

$$\begin{aligned} 0 &= y'' + p(t)\, y' + q(t)\, y \\ &= v'' y_1 + 2v' y_1' + v y_1'' + p(t)(v' y_1 + v y_1') + q(t)\, v y_1 \\ &= v'' y_1 + v'(2y_1' + p(t)\, y_1) + v\underbrace{(y_1'' + p(t)\, y_1' + q(t)\, y_1)}_{0} \end{aligned}$$

$$\Rightarrow y_1(t)\, v'' + (2y_1'(t) + p(t)\, y_1(t))\, v' = 0.$$

Let $u(t) = v'$

$$\Rightarrow y_1(t)\, u' + (2y_1'(t) + p(t)\, y_1(t))\, u = 0.$$

This is a first-order ODE.

$\Rightarrow$ solve for u.

then solve for v from $v' = u$.

E.g. Given that $y_1(t) = \frac{1}{t}$ is a solution of

(3) $$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

find a fundamental set of solutions.

Sol. Let $y(t) = v(t)\frac{1}{t} = v(t)\,t^{-1}$.

Then $y' = v't^{-1} - vt^{-2}$

and $y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$.

Substituting y, y', and y'' into (3), we obtain

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} = 0$$

$$= 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v$$

$$= 2tv'' - v' = 0.$$

Let $u = v'$. Then

$$2tu' - u = 0.$$

$$2tu' = u$$

$$\int \frac{du}{u} = \int \frac{1}{2t}\,dt$$

$$\ln u = \frac{1}{2}\ln t$$

$$\Rightarrow u = t^{1/2}.$$

Hence, $v'(t) = t^{1/2}$.

$$v(t) = \int t^{1/2}\,dt = \frac{2}{3}t^{3/2} \cancel{(+c)}$$

$\Rightarrow y_2(t) = t^{3/2}\cdot t^{-1} = t^{1/2}$.

General sol: $y(t) = c_1t^{-1} + c_2t^{1/2}$.

$$W[y_1, y_2](t) = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2}t^{-1/2} \end{vmatrix} = \frac{1}{2}(t^{-1})(t^{-1/2}) + (t^{-2})t^{1/2} \neq 0$$

as $t > 0$.

$\Rightarrow \{y_1, y_2\}$ is a fundamental set of solutions.

Section 3.5. Nonhomogeneous Eqs; Method of Undetermined Coefficients.

(1) $\quad y'' + p(t)y' + q(t)y = g(t).$

Note that if Y_1 and Y_2 are sols. to (1),

$Y_2 - Y_1$ is sol. of (why?).

(2) $\quad y'' + p(t)y' + q(t)y = 0$

Hence, $Y_2 - Y_1 = c_1 y_1 + c_2 y_2$, where y_1 & y_2 form a fundamental set of solutions of (2).

$\Rightarrow$ General sol. of (1) is of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where Y is any sol. of (1).

$\Rightarrow$ To solve (1):

1) Find general sol. $\underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{is called the complementary sol}}$ of (2)

2) Find any solution $y_p(t)$ of (1). This is called the particular solution.

3) General sol. to (1) is

$$y(t) = y_c(t) + y_p(t).$$

Section 3.5 Nonhomogeneous Eqs: Method of Undetermined coefficients.

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

Step 1: Find the general solution y_c of the homogeneous eq:

$$ay'' + by' + cy = 0.$$

Step 2: guess a particular sol.

$$y_p(t) = A_1 Y_1(t) + \dots + A_n Y_n(t)$$

Step 3: plug $y_p(t)$ back into the original ODE

$$ay'' + by' + cy = g(t)$$

Step 4: Solve for the undetermined coefficients

$$A_1, A_2, \dots, A_n.$$

Step 5: General sol of the original ODE is

$$y(t) = y_c(t) + y_p(t).$$

Step 6: Use initial condition to determine the constants and find the solution to the I.V.P.

E.g. Solve $y'' + 4y' + 4y = e^{3t}$, $y(0) = 0$, $y'(0) = 1$.

Sol. Step 1. Consider

$$y'' + 4y' + 4y = 0.$$

$$\Rightarrow \quad r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2.$$

$$\Rightarrow \quad y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Step 2: Guess $y_p(t) = Ae^{3t}$.

[Since the exponential function reproduces itself through differentiation, the most plausible way to achieve the desire result is to assume $y_p(t)$ is some multiple of e^{3t}.]

Step 3: $y_p'(t) = 3Ae^{3t}$, $y_p''(t) = 9Ae^{3t}$.

$$\Rightarrow \quad y_p'' + 4y_p' + 4y_p$$
$$= 9Ae^{3t} + 12Ae^{3t} + 4Ae^{3t}$$
$$= 25Ae^{3t} = e^{3t}.$$

Step 4: $A = \frac{1}{25} \rightarrow y_p(t) = \frac{1}{25}e^{3t}$.

Step 5: $y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{25}e^{3t}$.

Step 6: ~~y(t)~~ use I.V.C to find c_1 and c_2.

$$\Rightarrow c_1 = -\frac{1}{25} \text{ and } c_2 = -\frac{1}{5}$$

$$\Rightarrow y(t) = -\frac{1}{25}e^{-2t} - \frac{1}{5}te^{-2t} + \frac{1}{25}e^{3t}.$$

E.g. 2. $y'' + 4y' + 4y = e^{-2t}$, $y(0) = 0$, $y'(0) = 1$.

Step 1: Consider

$$y'' + 4y' + 4y = 0.$$
$$\Rightarrow r^2 + 4r + 4 = 0.$$
$$r = -2$$
$$\Rightarrow y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Step 2: Guess $y_p(t) = At^2e^{-2t}$ (Will tell you the trick!).

Step 3: $y_p' = 2Ate^{-2t} + At^2(-2)e^{-2t}$

$$y_p'' = 2Ae^{-2t} + 2At(-2)e^{-2t} + 4At^2e^{-2t} - 4Ate^{-2t}$$
$$= 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t}$$

Then

$$y_p'' + 4y_p' + 4y_p$$
$$= 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t} + 8Ate^{-2t} - 8At^2e^{-2t} + 4At^2e^{-2t}$$
$$= e^{-2t}(2A - 8At + 4At^2 + 8At - 8At^2 + 4At^2)$$
$$= 2Ae^{-2t} = e^{-2t}$$

Step 4: $\Rightarrow \quad A = \frac{1}{2}$.

Step 5: $y(t) = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^2e^{-2t}$.

Step 6: Find c_1 and c_2 (Exercise: $c_1 = 0$ and $c_2 = 1$)

~~Q. Given a nonhomogeneous Eq, which particular sol. y_p should we guess?~~

E.g. Find a particular solution of
$$y'' - 3y' - 4y = 2\sin t.$$

Sol. Let's assume $y_p(t) = A\sin t$. Then
$$y'' - 3y' - 4y = -A\sin t - 3A\cos t - 4A\sin t$$
$$= 2\sin t$$

$$\Rightarrow (2+5A)\sin t + 3A\cos t = 0.$$

We want this to hold for all t. Impossible! (Why?)

Try $y(t) = A\sin t + B\cos t$.

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$y'(t) = A\cos t - B\sin t.$

$y''(t) = -A\sin t - B\cos t.$

Then $y'' - 3y' - 3y = (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t$

$= 2\sin t.$

$\Rightarrow (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t.$

holds for all t

$$\Rightarrow \begin{cases} -A + 3B - 4A = 2 \\ -B - 3A - 4B = 0. \end{cases}$$

$\Rightarrow \quad A = -\frac{5}{17}$ and $B = \frac{3}{17}.$

In general, if:

$g(t) =$	Assume $y_p =$
$\lambda e^{\mu t}$	$Ae^{\mu t}$
$\lambda_1 \sin(\mu t) + \lambda_2 \cos(\mu t)$	$A\sin(\mu t) + B\cos(\mu t)$
$\lambda_n t^n + \lambda_{n-1} t^{n-1} + \dots + \lambda_0$	$A_n t^n + A_{n-1} t^{n-1} + \dots + A_0$

- If $g(t)$ is a product of any of these types, assume $y_p(t)$ as the product of the corresponding functions.

Particular solution of $ay'' + by' + cy = g_i(t)$.

$g_i(t) =$	Assume $y_p =$
1) $P_n(t) = a_n t^n + \dots + a_1 t + a_0$	$t^s(A_n t^n + \dots + A_1 t + A_0)$ $s =$ number of times 0 is a root of $ay'' + by' + cy = 0$
2) $P_n(t) e^{\alpha t}$	$t^s(A_n t^n + \dots + A_1 t + A_0) e^{\alpha t}$ $s =$ number of times α is a root of $ay'' + by' + cy = 0$.
3) $P_n(t) e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$	$t^s(A_n t^n + \dots + A_1 t + A_0) e^{\alpha t} \cos\beta t$ $+ \; t^s(B_n t^n + \dots + B_1 t + B_0) e^{\alpha t} \sin\beta t$ $s =$ number of times $\alpha + i\beta$ is a root of $ay'' + by' + cy = 0$.

Section 3.6. Variation of Parameters.

E.g. (1) $y'' + 9y = \underbrace{9\sec^2(3t)}$ $\quad 0 < t < \frac{\pi}{6}$.

not a sum or product of poly.,
sin and cos, exp.
$\Rightarrow$ undetermined coeff. will not work.

Let solve $y'' + 9y = 0$.

$\Rightarrow y_c = c_1 \sin(3t) + c_2 \cos(3t)$ (why).

Lagrange found a way to find the general sol. of the ODE by replacing c_1 and c_2 by $u_1(t)$ and $u_2(t)$.

~~Assume~~ (like reduction of order method).

Assume $y = u_1(t)\sin(3t) + u_2(t)\cos(3t)$

$\cancel{y' = u_1 \cdot 3\cos(3t) - u_2 \cdot 3\sin(3t)}$

$y' = u_1 \cdot 3\cos(3t) - u_2 \cdot 3\sin(3t) + u_1' \sin(3t) + u_2' \cos(3t)$

before we find y'', consider that we will end up with $y'' + 9y = 9\sec^2(3t)$ which will have 2 unknown functions, $u_1(t)$ and $u_2(t)$.

Since we will have a single eq. with 2 unknowns, we expect infinitely many solutions. Let's pick one that make our jobs easier:

$$u_1' \sin(3t) + u_2' \cos(3t) = 0.$$

So

$$y' = 3u_1 \cos(3t) - 3u_2 \sin(3t).$$

$$y'' = -9u_1 \sin(3t) - 9u_2 \cos(3t) + 3u_1' \cos(3t) - 3u_2' \sin(3t).$$

Then sub. y', y'' into (1)

$$y'' + 9y = 3u_1' \cos(3t) - 3u_2' \sin(3t) = 9\sec^2(3t).$$

So

$$\begin{cases} u_1' \cos(3t) - u_2' \sin(3t) = 3\sec^2(3t). \\ u_1' \sin(3t) + u_2' \cos(3t) = 0. \end{cases}$$

Thus We have 2 eqs. with 2 unknowns.

$$u_1' = \frac{\begin{vmatrix} 3\sec^2(3t) & -\sin(3t) \\ 0 & \cos(3t) \end{vmatrix}}{\begin{vmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{vmatrix}} = \frac{3\sec(3t)}{\cos^2(3t) + \sin^2(3t)} = 3\sec(3t)$$

$$u_2' = \frac{\begin{vmatrix} \cos(3t) & 3\sec^2(3t) \\ \sin(3t) & 0 \end{vmatrix}}{1} = -3\sec^2(3t)\sin(3t)$$

$$= -3\sec(3t)\tan(3t).$$

Now we integrate each to find u_1 and u_2.

$$u_1(t) = \int 3\sec(3t)\,dt = \ln|\sec(3t) + \tan(3t)| + C_1$$

and $u_2(t) = \int -3\sec^2(3t)\sin(3t)\,dt = -\sec(3t) + C_2.$

Take $C_1 = C_2 = 0$.

$$\Rightarrow \quad y_p(t) = \left(\ln|\sec(3t) + \tan(3t)|\right)\sin(3t) + (-\sec(3t))\cos 3t.$$

$\Rightarrow$ General sol of (1) is

$$y(t) = c_1 \sin(3t) + c_2 \cos(3t) + \ln|\sec(3t) + \tan(3t)|\sin(3t) - 1.$$

General approach.

Given $\underbrace{y'' + p(t)y' + q(t)y}_{\text{denote by } L[y]} = g(t)$.

Step 1: Solve $L[y]=0$ to find $y_c = c_1 y_1 + c_2 y_2$.

Step 2: Replace c_1 and c_2 by functions of t;
$u_1(t)$ and $u_2(t)$.

Step 3: Require that $u_1'(t)\,y_1(t) + u_2'(t)\,y_2(t) = 0$ $\boxed{1}$
and substitute y, y', y'' into $L[y] = g(t)$ to get
$u_1'(t)\,y_1'(t) + u_2'(t)\,y_2'(t) = g(t)$ (why?) $\boxed{2}$

Step 4: Solve $\boxed{1}$ and $\boxed{2}$ to find $u_1'(t)$ and $u_2'(t)$.

Step 5: Integrate to find $u_1(t)$ & $u_2(t)$.

General sol:
$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$.

Q: Why not use this all the time?
- integrals are sometimes difficult!
- can be tedious.

E.g. (1) $t^2 y'' - t(t+2)y' + (t+2)y = 2t^3$
1) Verify $y_1 = t$ and $y_2 = te^t$ are sol. to
$t^2 y'' - t(t+2)y' + (t+2)y = 0$.
2) Find a sol. to (1).

(51)

Sol. $y_p = u_1(t)\,t + u_2(t)\,te^t.$

$$\Rightarrow \begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \cancel{te}\ 2t^3 \end{cases}$$

$$\Rightarrow \quad u_1 = -\int \frac{2t \cdot te^t}{t^2 e^t}\, dt = -2t.$$

$$u_2 = \int \frac{2t \cdot t}{t^2 e^t}\, dt = -2e^{-t}.$$

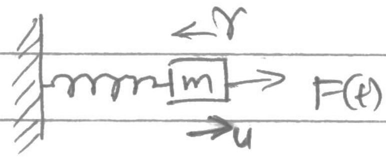
$$\Rightarrow y(t) = c_1 t + c_2 te^t - 2t^2 - 2e^{-t}\, te^t.$$

$$= c_1 t + c_2 te^t - 2t^2.$$

Section 7.1–7.2: Review of Matrices

(I) 1. Systems of ODEs

Eg. 1



$$mu'' + \gamma u' + ku = F(t) \qquad (*)$$

Let $x_1 = u$, $x_2 = u'$ $\Rightarrow$ $x_1' = x_2$

$\Rightarrow$ $(*)$ can be rewritten as

$$m x_2' + \gamma x_2 + k x_1 = F(t)$$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -\dfrac{\gamma}{m} x_2 - \dfrac{k}{m} x_1 + \dfrac{F(t)}{m} \end{cases}.$$

2) Consider $4y''' + 8y^2 y'' - \sin y' = 0$.

$x_1 = y$, $x_2 = y'$, and $x_3 = y''$

Then $4x_3' + 8x_1^2 x_3 - \sin x_2 = 0$.

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -2x_1^2 x_3 + \dfrac{1}{4}\sin x_2 \end{cases}.$$

Generally,

$$\begin{cases} x_1' = F_1(t, x_1, \ldots, x_n) \\ x_2' = F_2(t, x_1, \ldots, x_n) \\ \vdots \\ x_n' = F_n(t, x_1, \ldots, x_n) \end{cases} \qquad \text{systems of ODEs.}$$

$$\text{if } \begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + g_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + g_n(t) \end{cases}$$

then the system of ODEs is called linear; otherwise it's nonlinear.

If $g_i(t)$'s all $= 0$, it is call homogeneous; otherwise it's nonhomogeneous.

For linear ODE systems it can be written as

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \quad A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}$$

and $$\vec{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

(II) Review of Matrices

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1+i & 0 & 1 \\ 1 & i & -2 \\ 0 & 2+i & 0 \end{bmatrix}$$

$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \rightsquigarrow$ transpose of A: $(A^T)_{ij} = A_{ji}$.

$\overline{C} = \begin{bmatrix} 1-i & 0 & 1 \\ 1 & -i & -2 \\ 0 & 2-i & 0 \end{bmatrix} \rightarrow$ conjugate $(\overline{C})_{ij} = \overline{C_{ij}}$.

$I =$ identity matrix: $I_{ij} = 0$ for $i \neq j$.

$I_{ii} = 1$. for all i's.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$O =$ zero matrix $\Rightarrow$ all entries are zero.

2) $2A - B = 2\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4 & 6 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 6 \\ 3 & 6 \end{bmatrix}$

$3A = 3\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 3 & 12 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 3\cdot(-1) & 2\cdot 0 + 3\cdot 2 \\ 1\cdot(3) + 4\cdot(-1) & 1\cdot 0 + 4\cdot 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 \\ -1 & 8 \end{bmatrix}$.

$$BA = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + 0(1) & 3\cdot 3 + 0\cdot 4 \\ -1\cdot 2 + 2\cdot 1 & -1\cdot 3 + 2\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix}.$$

$AB \neq BA$.

3) $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \Rightarrow \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i.$

4) If $AB = I$, then $BA = I$ and $B = A^{-1}$.

* Gaussian elimination:
  - i) swap two rows.
  - ii) multiply one row by α.
  - iii) subtract one row from another.

E.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{swap}]{R_2 \& R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{-1 \times R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

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$$\xrightarrow{R_1-2R_2}\left[\begin{array}{ccc|ccc}1&0&-3&-5&0&2\\0&1&3&3&0&-1\\0&0&-1&-2&1&0\end{array}\right]\xrightarrow{-1\times R_3}\left[\begin{array}{ccc|ccc}1&0&-3&-5&0&2\\0&1&3&3&0&-1\\0&0&1&2&-1&0\end{array}\right].$$

$$\xrightarrow[-R_3]{R_1+3R_3}\left[\begin{array}{ccc|ccc}1&0&0&1&-3&2\\0&1&3&3&0&-1\\0&0&1&2&-1&0\end{array}\right]\xrightarrow{R_2-3R_3}\left[\begin{array}{ccc|ccc}1&0&0&1&-3&2\\0&1&0&-3&3&-1\\0&0&1&2&-1&0\end{array}\right]$$

$$\rightarrow \quad A^{-1}=\begin{bmatrix}1&-3&2\\-3&3&-1\\2&-1&0\end{bmatrix}$$

5) $\vec{x}(t)=\begin{bmatrix}2t\\\sin t\end{bmatrix}$ $\quad \vec{x}'(t)=\begin{bmatrix}2\\\cos t\end{bmatrix}$

$A=[a_{ij}(t)]$ $\quad \frac{dA}{dt}=(a'_{ij}(t))$

$=\begin{bmatrix}\sin t & t\\1&\cos t\end{bmatrix}$ $\quad A'=\begin{bmatrix}\cos t & 1\\0&-\sin t\end{bmatrix}.$

$$\int_0^{\pi}A(t)\,dt=\begin{bmatrix}\int_0^{\pi}\sin t\,dt & \int_0^{\pi}t\,dt\\ \int_0^{\pi}1\,dt & \int_0^{\pi}\cos t\,dt\end{bmatrix}$$

$$=\begin{bmatrix}2 & \pi^2/2\\ \pi & 0\end{bmatrix}.$$