HW09

1) $K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$

Find eigenvalues of K:

$$\det(K - \lambda I) = \begin{vmatrix} i-\lambda & i \\ i & i-\lambda \end{vmatrix} = (i-\lambda)^2 - i^2 = -\lambda(2i-\lambda)$$

$\Rightarrow \lambda_1 = 2i$ and $\lambda_2 = 0$ are the eigenvalues.

. For $\lambda_1 = 2i$: $(A - 2iI)\vec{x} = 0$.

$$\begin{bmatrix} -i & i \\ i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ is the eigenvect

. For $\lambda_2 = 0$: $A\vec{x} = 0$.

$$\begin{bmatrix} i & i \\ i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ is the eigenvector.

$$\Rightarrow K = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} 2i & 0 \\ 0 & 0 \end{bmatrix}}_{e^{\Lambda t}}\underbrace{\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}}_{S^{-1}}$$

$$\Rightarrow e^{Kt} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2ti} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

$$= \begin{bmatrix} e^{2ti} & -1 \\ e^{2ti} & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^{2ti}+1) & \frac{1}{2}(e^{2ti}-1) \\ \frac{1}{2}(e^{2ti}-1) & \frac{1}{2}(e^{2ti}+1) \end{bmatrix}.$$

$$(e^{Kt})^H = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}^H \begin{bmatrix} e^{2ti} & 0 \\ 0 & 1 \end{bmatrix}^H \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^H$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} e^{-2ti} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= (S^{-1})^H \quad (e^{\Lambda t})^H \quad S^H$$

$\Rightarrow$

$$e^{Kt}(e^{Kt})^H = (S e^{\Lambda t} S^{-1})(S^{-1})^H (e^{\Lambda t})^H S^H$$

$$= S e^{\Lambda t} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} (e^{\Lambda t})^H S^H$$

$$= \frac{1}{2} S e^{\Lambda t} (e^{\Lambda t})^H S^H$$

$$= \frac{1}{2} S \begin{bmatrix} e^{2ti} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-2ti} & 0 \\ 0 & 1 \end{bmatrix} S^H$$

$$= \frac{1}{2} S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^H$$

$$= \frac{1}{2} S S^H$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= I.$$

Similarly, $(e^{Kt})^H e^{Kt} = I$.

$\therefore$ e^{Kt} is unitary.

b) $\frac{d}{dt} e^{Kt} \Big|_{t=0} = K e^{Kt} \Big|_{t=0} = K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$.

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2) $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Find eigenvalues: $\det(P - \lambda I) = 0$.

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0. \Rightarrow \lambda^3 - 1 = 0.$$

$$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0.$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = \frac{-1 + i\sqrt{3}}{2} = e^{\frac{2\pi}{3}i}, \quad \lambda_3 = \frac{-1 - i\sqrt{3}}{2} = e^{-\frac{2\pi}{3}i}$$

- For $\lambda_1 = 1$: $(P - \lambda I)\vec{x} = 0$.

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}$$ is a unit eigenvector.

- For $\lambda_2 = e^{\frac{2\pi}{3}i}$:

$$\begin{bmatrix} -e^{\frac{2\pi}{3}i} & 1 & 0 \\ 0 & -e^{\frac{2\pi}{3}i} & 1 \\ 1 & 0 & -e^{\frac{2\pi}{3}i} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$\Rightarrow \vec{x}_2 = \cancel{\frac{1}{\sqrt{3}} \begin{bmatrix} e^{-\frac{2\pi}{3}i} \\ e^{-\frac{2\pi}{3}i} \\ e^{-\frac{2\pi}{3}i} \end{bmatrix}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ e^{2\pi i/3} \\ e^{4\pi i/3} \end{bmatrix}$$ is a unit eigenvector.

- For $\lambda_3 = e^{-\frac{2\pi}{3}i}$:

$$\begin{bmatrix} -e^{-\frac{2\pi}{3}i} & 1 & 0 \\ 0 & -e^{-\frac{2\pi}{3}i} & 1 \\ 1 & 0 & -e^{-\frac{2\pi}{3}i} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \vec{x}_3 = \cancel{\frac{1}{\sqrt{3}} \begin{bmatrix} e^{\frac{2\pi}{3}i} \\ e^{\frac{2\pi}{3}i} \\ e^{\frac{2\pi}{3}i} \end{bmatrix}}$$ is a unit eigenvector.

$$\vec{x}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ e^{-2\pi i/3} \\ e^{-4\pi i/3} \end{bmatrix}$$

$$U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{4\pi i/3} & e^{-4\pi i/3} \end{bmatrix}.$$

(Check: to see if this is a unitary matrix, i.e., $UU^H = I$)

b) It's easy to see that

$$PP^T = P^T P = I.$$

$\Rightarrow$ P is an orthogonal matrix.

$\Rightarrow$ P is a normal matrix.

$\Rightarrow$ That's why these above eigenvectors are orthogonal.

and

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\vec{w}_1 + 4\vec{w}_2,$$

so the desired matrix is

$$M = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

8. Problem 5.6.38. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (find them). But the block sizes don't match and J *is not similar to* K.

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For any matrix M, compare JM and MK. If they are equal, show that M is not invertible. Then $M^{-1}JM = K$ is impossible.

Answer: First, we find the eigenvectors of J and K. Since all eigenvalues of both are 0, we're just looking for vectors in the nullspace of J and K. First, for J, we note that J is already in reduced echelon form and that $J\vec{v} = \vec{0}$ implies that $\vec{v}$ is a linear combination of

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Hence, these are the eigenvectors of J.

Likewise, K is already in reduced echelon form and $K\vec{v} = \vec{0}$ implies that $\vec{v}$ is a linear combination of

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence, these are the eigenvectors of K.

Now, suppose

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

such that $JM = MK$. Then

$$JM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$MK = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}.$$

Therefore $JM = MK$ means that

$$\begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

and so we have that

$$m_{21} = m_{24} = m_{22} = m_{41} = m_{44} = m_{42}0.$$

Plugging these back into M, we see that

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & 0 & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & m_{43} & 0 \end{bmatrix}.$$

Clearly, the second and fourth rows are multiples of each other, so M cannot possibly have rank 4. However, M not having rank 4 means that M cannot be invertible. Therefore, $M^{-1}JM = K$ is impossible, so it cannot be the case that J and K are similar.

9. Problem 5.6.40. Which pairs are similar? Choose a, b, c, d to prove that the other pairs aren't:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

Answer: The second and third are clearly similar, since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

Likewise, the first and fourth are similar, since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d & c \\ b & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

There are no other similarities, as we can see by choosing

$$a = 1, \quad b = c = d = 0.$$

Then the matrices are, in order

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Each of these is already a diagonal matrix, and clearly the first and fourth have 1 as an eigenvalue, whereas the second and third have only 0 as an eigenvalue. Since similar matrices have the same eigenvalues, we see that neither the first nor the fourth can be similar to either the second or the third.

10. (**Bonus Problem**) Problem 5.6.14. Show that every number is an eigenvalue for $Tf(x) = df/dx$, but the transformation $Tf(x) = \int_0^x f(t)dt$ has no eigenvalues (here $-\infty < x < \infty$).

Proof. For the first T, note that, if $f(x) = e^{ax}$ for any real number a, then

$$Tf(x) = \frac{df}{dx} = ae^{ax} = af(x).$$

Hence, any real number a is an eigenvalue of T.

Turning to the second T, suppose we had that $Tf(x) = af(x)$ for some number a and some function f. Then, by the definition of T,

$$\int_0^x f(t)dt = af(x).$$

Now, use the fundamental theorem of calculus to differentiate both sides:

$$f(x) = af'(x).$$

Solving for f, we see that

$$\int \frac{f'(x)dx}{f(x)} = \int \frac{1}{a}dx,$$

so

$$\ln|f(x)| = \frac{x}{a} + C.$$

Therefore, exponentiating both sides,

$$|f(x)| = e^{x/a+C} = e^C e^{x/a}.$$

We can get rid of the absolute value signs by substituting A for e^C (allowing A to possibly be negative):

$$f(x) = Ae^{x/a}.$$

Therefore, we know that

$$Tf(x) = \int_0^x f(t)dt = \int_0^x Ae^{t/a}dt = aAe^{t/a}\Big]_0^x = aAe^{x/a} - aA = a(Ae^{x/a} - A) = a(f(x) - A).$$

On the other hand, our initial assumption was that $Tf(x) = af(x)$, so it must be the case that

$$af(x) = a(f(x) - A) = af(x) - aA.$$

Hence, either $a = 0$ or $A = 0$. However, either implies that $f(x) = 0$, so T has no eigenvalues. $\square$