20A PRACTICE FINAL # 3

1. (6 points) Evaluate the following limits or state that they do not exist (DNE).

(a) $\lim_{\theta \to \frac{\pi}{2}} \left[\sec(\theta) - \tan(\theta)\right]$ (b) $\lim_{x \to 0} \dfrac{\ln(1+x^2)}{x\sin(3x)}$

Rearrange + L'Hôpital L'Hôpital x 2

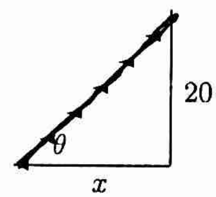
2. (6 points) Compute the derivative of each of the following functions.

(a) $f(x) = 10^{\sqrt{x+\sqrt{x}}} + e^3$ (b) $g(x) = \dfrac{\arctan(x)}{\ln(x)} + \dfrac{x^4}{4}$

3. (6 points) Use the Linear Approximation to determine which is larger of $\sqrt{4.1} - \sqrt{4}$ or $\sqrt{9.1} - \sqrt{9}$.

4. (6 points) The base x of a right triangle is increasing in length at a rate of 4 cm/s, while the height remains constant at 20 cm. How fast is the angle θ changing when the base x is 20 cm in length?

related rates



5. (6 points) Let $f(x) = x^3 - x$. check end points!

(a) What are the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-1, 1]$?

(b) What are the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-2, 2]$?

why can we find absolute min/max here?

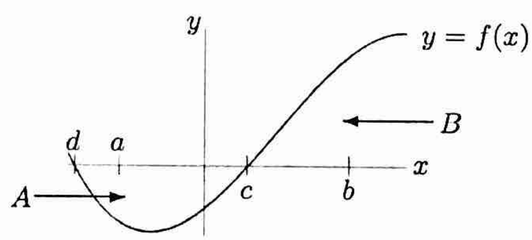
6. (6 points) Compute the following integrals.

(a) $\displaystyle\int_{-\pi}^{\pi} \left[3 + \cos(x)\right] dx$ (b) $\displaystyle\int \frac{1 + 4x + x^2}{x}\, dx$

antiderivative

7. (6 points) Compute $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} e^{t^2}\, dt$.

8. (4 points) The following is a graph of the function f.



(a) If $\displaystyle\int_a^b f(x)\, dx > 0$, then which is larger: The area of region A or the area of region B?

(b) Let $F(x) = \displaystyle\int_a^x f(t)\, dt$. On what interval or intervals is F decreasing? Either give your answer using interval notation, or state that there is not in enough information given to solve the problem and explain why not.

9. (6 points) Suppose that $F(x) = e^{\sin(x)}$ and suppose that $F'(x) = f(x)$.

(a) Compute $\dfrac{d}{dx}F(x)$. (b) Compute $\displaystyle\int_0^x f(s)\, ds$.

10. (6 points) Find the equation of the tangent line to $(x^2 + y^2)^2 = 9(x^2 - y^2)$ at the point $(\sqrt{2}, 1)$.

20A PRACTICE FINAL #3 SOL'N

1) Evaluate the limits

(a) $\lim_{\theta \to \frac{\pi}{2}} [\sec(\theta) - \tan(\theta)]$

notice this approaches $\infty - \infty$, hence we must do more work to evaluate:

see that $\sec(\theta) - \tan(\theta) = \frac{1}{\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)}$

$= \frac{1 - \sin(\theta)}{\cos(\theta)}$

now we have:

$\lim_{\theta \to \frac{\pi}{2}} [\sec(\theta) - \tan(\theta)] = \lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin\theta}{\cos\theta}$ which approaches $\frac{0}{0}$, so apply L'Hôpital

$= \lim_{\theta \to \frac{\pi}{2}} \frac{-\cos\theta}{-\sin\theta}$

continuity $= \frac{-\cos(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = \frac{-0}{-1} = \boxed{0}$

(b) $\displaystyle\lim_{x \to 0} \frac{\ln(1+x^2)}{x\sin(3x)}$

This approaches $\frac{0}{0}$ so apply L'Hôpital:

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{x\sin(3x)} = \lim_{x \to 0} \frac{\frac{1}{1+x^2}\frac{d}{dx}(1+x^2)}{\sin(3x) + 3x\cos(3x)}$$

L'Hôpital

$$= \lim_{x \to 0} \frac{\frac{2x}{1+x^2}}{\sin(3x) + 3x\cos(3x)}$$

$$= \lim_{x \to 0} \frac{2x}{(1+x^2)(\sin(3x) + 3x\cos(3x))}$$ ($\to$ again approaches $\frac{0}{0}$)

L'Hôpital

$$= \lim_{x \to 0} \frac{2}{(2x)(\sin(3x)+3x\cos(3x)) + (1+x^2)(3\cos(3x) + 3\cos(3x) + 9x\sin(3x))}$$

$$= \frac{2}{2(0)(\underline{\qquad}) + (1+0^2)(3\cos(0) + 3\cos(0) - 9(0) -)}$$

$$= \frac{2}{(1)(3+3-0)} = \boxed{\frac{1}{3}}$$

2 (a) $f(x) = 10^{\sqrt{x+\sqrt{x}}} + e^3$; $f'(x) = ?$

First, see that

$$f(x) = 10^{\sqrt{x+\sqrt{x}}} + e^3$$

$a = e^{\ln(a)}$

$$= e^{\ln(10)\sqrt{x+\sqrt{x}}} + e^3$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left[e^{\ln(10)\sqrt{x+\sqrt{x}}} \right]$$

chain rule

$$= e^{\ln(10)\sqrt{x+\sqrt{x}}} \frac{d}{dx}\left[\ln(10)\sqrt{x+\sqrt{x}}\right]$$

$$= \ln(10) e^{\ln(10)\sqrt{x+\sqrt{x}}} \frac{d}{dx}\left(\sqrt{x+\sqrt{x}}\right)$$

chain rule

$$= \ln(10) e^{\ln(10)\sqrt{x+\sqrt{x}}} \left(\frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right)$$

simplification

$$= \boxed{\frac{\ln(10)\, 10^{\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)}{2\sqrt{x+\sqrt{x}}}}$$

(b) $g(x) = \dfrac{\arctan(x)}{\ln(x)} + \dfrac{x^4}{4}$; $g'(x) = ?$

$$g'(x) = \frac{d}{dx}\left[\frac{\arctan(x)}{\ln(x)}\right] + \frac{d}{dx}\left[\frac{x^4}{4}\right]$$

$$= \frac{\frac{d}{dx}(\arctan(x))\ln(x) - \arctan(x)\frac{d}{dx}(\ln(x))}{\ln(x)^2} + x^3$$

$$= \frac{\left(\frac{1}{1+x^2}\right)\ln(x) - \frac{\arctan(x)}{x}}{\ln(x)^2} + x^3$$

$$= \boxed{\left(\frac{1}{1+x^2}\right) - \frac{\arctan(x)}{\ln(x)^2 x} + x^3}$$

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③ Use Linear approximation to determine which is larger:

$$\sqrt{4.1} - \sqrt{4} \quad \text{or} \quad \sqrt{9.1} - \sqrt{9}$$

We'll use ~~tan~~ approximation of $f(x) = \sqrt{x}$ at both $x = 4$ and $x = 9$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{4}, \quad f'(9) = \frac{1}{6}$$

and then

$$\left.\begin{aligned} L_1(x) &= f'(4)(x-4) + f(4) \\ &= \frac{1}{4}(x-4) + \cancel{\sqrt{4}} \end{aligned}\right) \text{approx. @ } x=4$$

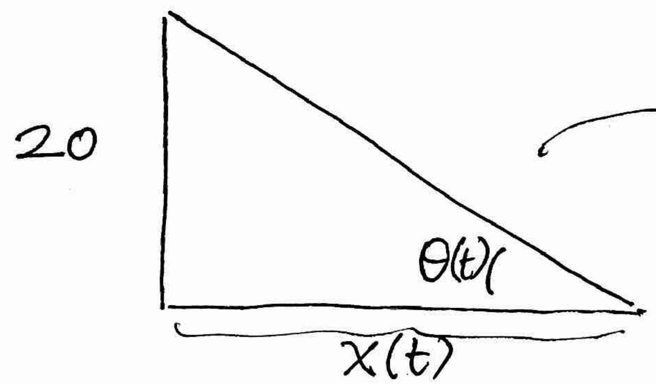
$$\left.\begin{aligned} L_2(x) &= f'(9)(x-9) + f(9) \\ &= \frac{1}{6}(x-9) + \sqrt{9} \end{aligned}\right) \text{approx. @ } x=9$$

hence $\sqrt{4.1} - \sqrt{4} \approx L_1(4.1) - \sqrt{4} = \frac{1}{4}(0.1)$

$\sqrt{9.1} - \sqrt{9} \approx L_2(9.1) - \sqrt{9} = \frac{1}{6}(0.1)$

hence $\boxed{\sqrt{4.1} - \sqrt{4} \text{ is lager}}$

(4) Right triangle has base length x increasing at a rate of 4 cm/sec, while the height remains constant at 20 cm.
How fast is the angle θ changing when the base x is 20 cm in length?



• Given:

$(*)\ \tan(\theta(t)) = \frac{20}{x(t)}$

$(**)\ x'(t) = 4$

• wanted: $\theta'(a) = ?$ for $x(a) = 20$ $(***)$

(1) Find a formula for $\theta'(a)$:

$(*)$ $\tan(\theta(t)) = \frac{20}{x(t)} \xrightarrow{\text{apply } \frac{d}{dt}} \sec^2(\theta(t))\theta'(t) = \frac{-20x'(t)}{x(t)^2}$

now set $t=a$ to see:

$$\theta'(a) = \frac{-20x'(a)}{\sec^2(\theta(a))x(a)^2} = \frac{-20(4)}{\sec^2(\theta(a))x(a)^2}$$

use $(**)$

$$= \frac{-20(4)}{\sec^2(\theta(a))(20)^2}$$

use $(***)$

$$\theta'(a) = \frac{-20(4)}{\sec^2(\theta(a))(20)^2} = \frac{-4}{20\sec^2(\theta(a))}$$

hence we need $\theta(a)$.

② find $\theta(a)$:

recall $\tan(\theta(t)) = \frac{20}{x(t)}$,

hence $\tan(\theta(a)) = \frac{20}{x(a)} = \frac{20}{20} = 1$

$$\Rightarrow \theta(a) = \frac{\pi}{4}$$

③ substitute into equation from ①:

$$\theta'(a) = \frac{-4}{20\sec^2(\frac{\pi}{4})} = \frac{-1}{5\frac{1}{\cos^2(\frac{\pi}{4})}}$$

$$= \frac{\cos^2(\frac{\pi}{4})}{5} = \left(\frac{\sqrt{2}}{2}\right)^2 \frac{1}{5}$$

$$= \frac{2}{20} = \boxed{\frac{1}{10}}$$

hence it is changing at $\boxed{\frac{1}{10} \text{ radians/sec}}$

8

(5) $f(x) = x^3 - x$

(a) abs min/max on $[-1,1]$

$$f'(x) = 3x^2 - 1 = 0 \iff x = \pm\sqrt{\tfrac{1}{3}}$$ critical points

So we check at the critical points and endpoints:

$$f(-1) = -1 - (-1) = 0$$

$$f\left(-\sqrt{\tfrac{1}{3}}\right) = -\tfrac{1}{3}\sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{1}{3}}\left(1 - \tfrac{1}{3}\right)$$

$$f\left(\sqrt{\tfrac{1}{3}}\right) = \tfrac{1}{3}\sqrt{\tfrac{1}{3}} - \sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{1}{3}}\left(\tfrac{1}{3} - 1\right)$$

$$f(1) = 1 - 1 = 0$$

So

$$\boxed{\begin{array}{l} \text{max is } \sqrt{\tfrac{1}{3}}\left(1 - \tfrac{1}{3}\right) \\ \text{min is } \sqrt{\tfrac{1}{3}}\left(\tfrac{1}{3} - 1\right) \end{array}}$$

(b) abs min/max on $[-2,2]$

check again w/ new endpoints:

$$f(-2) = -8 + 2 = -6$$

$$f\left(-\sqrt{\tfrac{1}{3}}\right) = \sqrt{\tfrac{1}{3}}\left(\tfrac{2}{3}\right) = \frac{2}{3\sqrt{3}}$$

$$f\left(\sqrt{\tfrac{1}{3}}\right) = \sqrt{\tfrac{1}{3}}\left(\tfrac{-2}{3}\right) = \frac{-2}{3\sqrt{3}}$$

$$f(2) = 8 - 2 = 6$$

$$\boxed{\begin{array}{l} \text{max is } 6 \\ \text{min is } -6 \end{array}}$$

6 (a) Integrate $\int_{-\pi}^{\pi}(3+\cos(x))\,dx$

$3+\cos(x)$ has antiderivative

$$F(x) = 3x + \sin(x)$$

so apply FTCI:

$$\int_{-\pi}^{\pi}(3+\cos(x))\,dx = F(\pi) - F(-\pi)$$

$$= 3\pi + \cancel{\sin(\pi)}^{0} - 3(-\pi) - \cancel{\sin(-\pi)}^{0}$$

$$= \boxed{6\pi}$$

(remark: this can also done ~~[illegible]~~ by splitting the integral and using geometry on $\int_{-\pi}^{\pi} 3\,dx$ and symmetry of cosine on $\int_{-\pi}^{\pi}\cos(x)\,dx$)

(b) General antiderivative of $\frac{1+4x+x^2}{x}$

First, simplify with algebra:

$$\frac{1+4x+x^2}{x} = \frac{1}{x} + \frac{4x}{x} + \frac{x^2}{x}$$

$$= \frac{1}{x} + 4 + x$$

log rule power rule

and now

$$\boxed{F(x) = \ln|x| + 4x + \frac{x^2}{2} + C}$$

is the general antiderivative.

7. $\frac{d}{dx}\int_{-x}^{x} e^{t^2} dt$

See that:

$$\frac{d}{dx}\left(\int_{-x}^{x} e^{t^2} dt\right) = \frac{d}{dx}\left(\int_{-x}^{0} e^{t^2} dt + \int_{0}^{x} e^{t^2} dt\right)$$

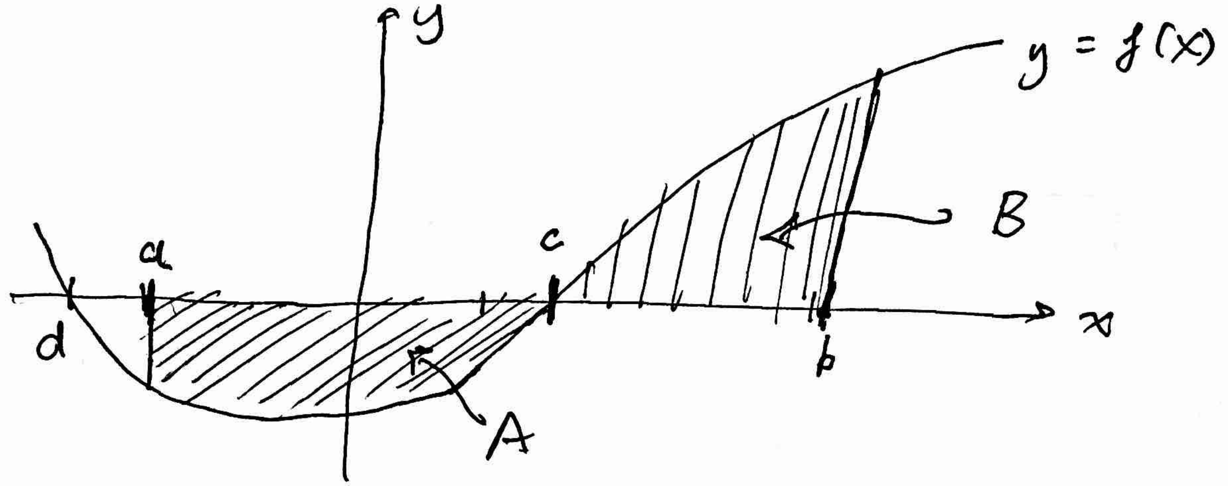
$$= \frac{d}{dx}\left(-\int_{0}^{-x} e^{t^2} dt\right) + \frac{d}{dx}\left(\int_{0}^{x} e^{t^2} dt\right)$$

$$= -\frac{d}{dx}\left(\int_{0}^{-x} e^{t^2} dt\right) + e^{x^2}$$

$$= -e^{x^2}\left(\frac{d}{dx}(-x)\right) + e^{x^2}$$

$$= e^{x^2} + e^{x^2} = \boxed{2e^{x^2}}$$

(8) Consider the graph of f:



(a) If $\int_a^b f(x)dx > 0$, then is A or B larger?

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= -\text{Area}(A) + \text{Area}(B) > 0$$

$$\Rightarrow \text{Area}(B) > \text{Area}(A)$$

$$\Rightarrow \boxed{B \text{ is larger}}$$

(b) $F(x) = \int_a^x f(t)\,dt$ increase/decrease intervals?

$\boxed{\text{decreasing on } (a, c)}$ ← getting more negative!

$\boxed{\text{increasing on } (c, \infty)}$ ← adding positive area!

for $x < a$, see that:

$$F(x) = -\int_x^a f(t)\,dt$$

hence F is $\boxed{\text{decreasing on } (-\infty, d)}$

↳ getting more negative!

$\boxed{\text{increasing on } (d, a)}$

(9) Suppose $F(x) = e^{\sin(x)}$
and suppose $F'(x) = f(x)$.

(a) Compute $\frac{d}{dx} F(x)$,

$$\frac{d}{dx}\left[e^{\sin(x)}\right] = e^{\sin(x)} \frac{d}{dx}(\sin(x))$$
$$= \boxed{\cos(x)\, e^{\sin(x)}}$$

(b) $\int_0^x f(s)\,ds$

$$= F(x) - F(0) \quad \text{by FTC I}$$
since $F' = f$

$$= e^{\sin(x)} - e^{\sin(0)}$$
$$= \boxed{e^{\sin(x)} - 1}$$

(10) Find equation for tangent line to $(x^2+y^2)^2 = 9(x^2-y^2)$ at the point $(\sqrt{2}, 1)$

Apply $\frac{d}{dx}$ to both sides:

$$\frac{d}{dx}\left((x^2+y^2)^2\right) = \frac{d}{dx}\left(9(x^2-y^2)\right)$$

$$\Rightarrow 2(x^2+y^2)\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(9x^2) - \frac{d}{dx}(9y^2)$$

$$\Rightarrow 2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 18x - 18y\frac{dy}{dx}$$

$$\Rightarrow (2x^2+2y^2)\left(2x+2y\frac{dy}{dx}\right) = 18x - 18y\frac{dy}{dx}$$

$$\Rightarrow 4x^3 + 4yx^2\frac{dy}{dx} + 4xy^2 + 4y^3\frac{dy}{dx} = 18x - 18y\frac{dy}{dx}$$

$$\Rightarrow 4yx^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} + 18y\frac{dy}{dx} = 18x - 4x^3 - 4xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{18x-4x^3-4xy}{4yx^2+4y^3+18y}$$

then $$\left.\frac{dy}{dx}\right|_{\substack{x=\sqrt{2}\\y=1}} = \frac{18(\sqrt{2})-4(\sqrt{2})^3-4(\sqrt{2})(1)}{4(1)(2)+4(1)^3+18(1)}$$

$$= \frac{18\sqrt{2} - 8\sqrt{2} - 4\sqrt{2}}{8 + 4 + 18}$$

$$= \frac{6\sqrt{2}}{30} = \frac{\sqrt{2}}{5}$$

now the tangent line is given by,

$$y = \left.\frac{dy}{dx}\right|_{\substack{x=\sqrt{2}\\ y=1}} (x - \sqrt{2}) + 1$$

$$= \frac{\sqrt{2}}{5}(x - \sqrt{2}) + 1$$

$$= \frac{\sqrt{2}}{5}x - \frac{2}{5} + 1$$

$$y = \boxed{\frac{\sqrt{2}}{5}x + \frac{3}{5}}$$