Math 20E: Vector Calculus.

Instructor: Thang Huynh → teaching

Course Webpage: thanghuynh.org~~/teaching/math20e_f16.html~~. → this course.

* Contains/will contain: Syllabus
  Exam schedule
  HW
  Office Hours.
  TA Information.
  etc.

* Grading scheme: HW, 2 MTs, Final.

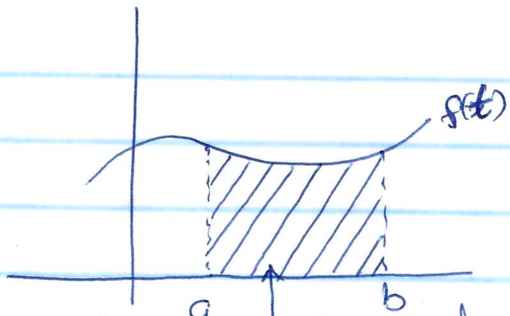
  20%, 20%, 20%, 40% Final

  or 20% (HW), 20% (highest MT), 60% Final.

  (whichever is higher)

* Plan: (for the first part of the course).
  - Review:
    Differentiation
    Double Integrals
    Triple Integrals
  - Linear Maps and the change of variable formula.
    ~~Cycli~~ Cylindrical & Spherical Coordinates.
  - Vector fields.
  - Integrals over paths and surfaces.

- Recall:



How to find this area?

$$\text{Area} = \int_a^b f(t)\,dt \overset{\text{FTOC}}{=} F(b) - F(a).$$

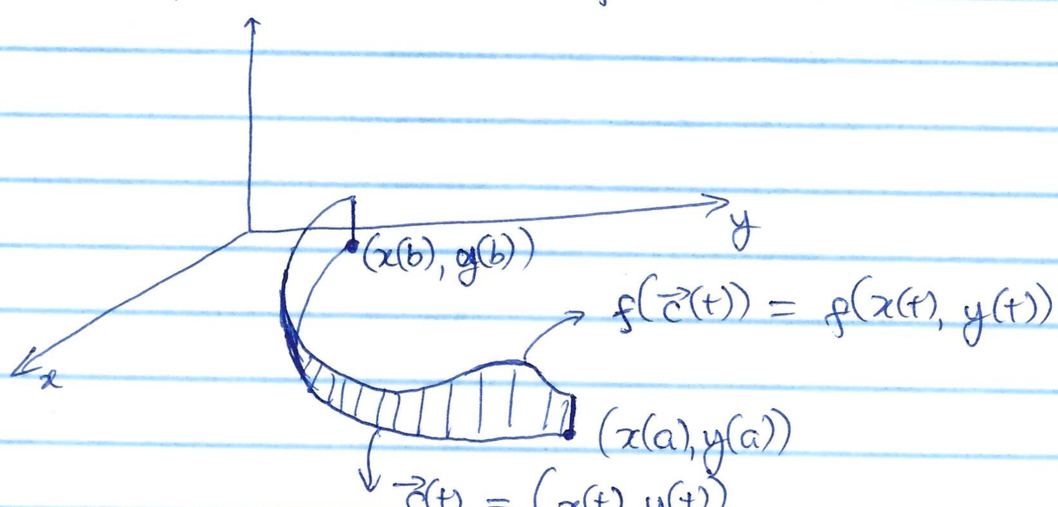
(FTOC)

Fundamental theorem of Calculus: If f is continuous and F is an antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

~~⇒ In this course, we~~

Area of a fence. Let $f: \mathbb{R}^2 \to \mathbb{R}$.



How to find the area of the fence?

⇒ Fund. Thm of Line Integrals.

$$\int_{\vec{c}} (\nabla f)\cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a)).$$

⇒ In this course, we will build up the mathematical technology to generalize the FTOC.

2.3 Differentiation.

Recall: $f: \mathbb{R} \to \mathbb{R}$

$$\boxed{\frac{df(x)}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

$\Rightarrow$ Higher dimensions? $\boxed{\text{Partial derivatives}}$

E.g. If $f: \mathbb{R}^2 \to \mathbb{R}$

then $$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

and $$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

~~(What's ∂f~~

E.g. $f(x,y) = xy^3 + \sin(xy)$

Find $\frac{\partial f}{\partial y}(0,1)$.

Sol: $$\frac{\partial f}{\partial y}(x,y) = 3xy^2 + \cos(xy) \cdot x$$

$$\frac{\partial f}{\partial y}(0,1) = 0.$$

Recall: Defn of differentiability (2 variables).

Let $f: \mathbb{R}^2 \to \mathbb{R}$. We say that f is differentiable at (x_0, y_0) if $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist and if

$$\frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\|(x,y) - (x_0,y_0)\|} \to 0$$

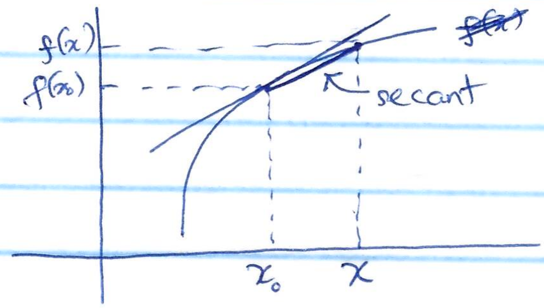
as $(x,y) \to (x_0, y_0)$

What does this mean?

* 1-variable

$$\frac{f(x) - f(x_0) - \left[\frac{df}{dx}(x_0)\right](x - x_0)}{x - x_0} \to 0 \quad \text{as } x \to x_0$$

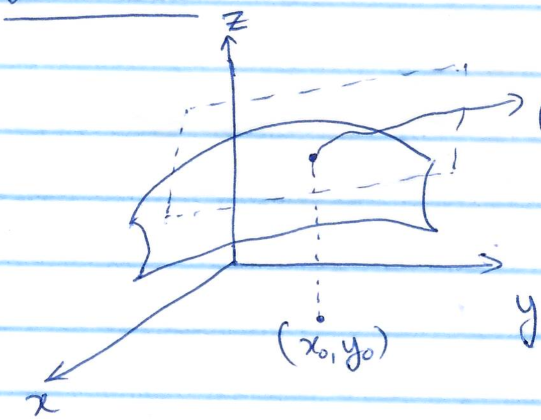
$$\Rightarrow \quad \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\text{slope of secant}} \to \underbrace{\frac{df}{dx}(x_0)}_{\text{slope of tangent}} \quad \text{as } x \to x_0.$$



$$f(x) \to \overbrace{f(x_0) + \left[\frac{df}{dx}(x_0)\right](x - x_0)}^{\text{lin. approx. at } x_0}.$$

as $x \to x_0$.

* 2-variables:



when we're close to (x_0, y_0), the graph of the tangent plane is close to the graph of f.

we have

$$(*) \quad z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

the linear approx of f at (x_0, y_0) (plane) is a good approx. of $f(x)$ when $(x, y) \to (x_0, y_0)$.

Def: (tangent plane) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0). The plane in $\mathbb{R}^3$ given by $(*)$ is called the tangent plane to the graph of f at $(x_0, y_0, f(x_0, y_0))$

⑤

E.g. Find the plane tangent to the graph of
$$f(x,y) = x + y^2 + \cos(xy) \quad \text{at } (0,1).$$

Sol: The equation of the tangent plane (at $(0,1)$) is given by
$$z = \underset{?}{f(0,1)} + \underset{?}{\left[\frac{\partial f}{\partial x}(0,1)\right]}(x-0) + \underset{?}{\left[\frac{\partial f}{\partial y}(0,1)\right]}(y-1).$$

$$\cancel{\frac{\partial f}{\partial x}(x} \quad f(0,1) = 0 + 1^2 + \cos(0)$$
$$= 1 + 1$$
$$= 2$$

$$\frac{\partial f}{\partial x}(x,y) = 1 + 0\cancel{\partial y^2} + - y\sin(xy)$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,1) = 1 \cancel{+} \cancel{+} - \sin(0) = \cancel{0}\ 1.$$

$$\frac{\partial f}{\partial y}(x,y) = 0 + 2y - x\sin(xy)$$

$$\Rightarrow \frac{\partial f}{\partial y}(0,1) = 2.$$

Hence,
$$z = 2 + (x-0) + 2(y-1).$$
$$= x + 2y.$$

* Differentiability: The general case

The derivative of $f: \mathbb{R}^n \to \mathbb{R}^m$ at $\vec{x}_0$ denoted by $Df(\vec{x}_0)$ is a matrix T whose elements are $t_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\vec{x}_0}$.

i.e. if $f = (f_1, \dots, f_m)$

$$\underset{\uparrow}{T} = \underset{\nearrow}{Df(\vec{x}_0)} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\vec{x}_0} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{\vec{x}_0} \\ \vdots & & & \\ \left.\frac{\partial f_m}{\partial x_1}\right|_{\vec{x}_0} & \left.\frac{\partial f_m}{\partial x_2}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_{\vec{x}_0} \end{bmatrix}$$

derivative or differential of f at $\vec{x}_0$.

Def: Let U be an open set in $\mathbb{R}^n$ and $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Then f is differentiable at $\vec{x}_0 \in U$ if the partial derivatives of f exist at $\vec{x}_0$ and if

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} = 0$$

E.g. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (\underbrace{x + e^z + \sin y}_{f_1}, \underbrace{yx^2}_{f_2})$

Find $Df(x,y,z)$.

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial(x+e^z+\sin y)}{\partial x} & \frac{\partial(x+e^z+\sin y)}{\partial y} & \frac{\partial(x+e^z+\sin y)}{\partial z} \\ \frac{\partial(yx^2)}{\partial x} & \frac{\partial(yx^2)}{\partial y} & \frac{\partial(yx^2)}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos y & e^z \\ 2yx & x^2 & 0 \end{bmatrix}$$

Remark: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ then $Df(x_0)$ is a $1 \times n$ matrix. The corresponding vector $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ is called ~~to~~ the gradient and denoted by ∇f.

Thm: If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0 \in U$, then f is continuous at $\vec{x}_0$.

Thm: If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is such that

- the partials $\frac{\partial f_i}{\partial x_j}$ all exist
- and are continuous in a neighborhood of $\vec{x} \in U$.

then f is differentiable at $\vec{x}$.

$\to$ much easier to check than the def. of diff.

E.g. $f(x,y,z) = (x + e^z + \sin y,\ yx^2)$ is differentiable because the partial derivatives exist and ~~on~~ are continuous.

* 2.5. Properties of the derivative

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{x}_0$. Then:

(1) Constant multiple rule:

If $c \in \mathbb{R}$ and $h(\vec{x}) = cf(\vec{x})$, then h is differentiable at $\vec{x}_0$ and $Dh(\vec{x}_0) = cDf(\vec{x}_0)$.

(2) Sum rule: If $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$, then h is differentiable at $\vec{x}_0$ and

$$Dh(\vec{x}_0) = Df(\vec{x}_0) + Dg(\vec{x}_0)$$

(3) Product rule: ~~If $h(\vec{x}) = f(\vec{x})g$~~

If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ and $g: U \subset \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{x}_0$ and $h(\vec{x}) = f(\vec{x})g(\vec{x})$, then h is differentiable at $\vec{x}_0$ and

$$Dh(\vec{x}_0) = \underbrace{g(\vec{x}_0)}_{\in \mathbb{R}} \underbrace{Df(\vec{x}_0)}_{1\times n \text{ matrix}} + \underbrace{f(\vec{x}_0)}_{\in \mathbb{R}} \underbrace{Dg(\vec{x}_0)}_{1\times n \text{ matrix}}.$$

~~Eg.~~

(4) Quotient Rule:

(4) With the same assumption as in rule (3), suppose further that $g(\vec{x}) \neq 0 \quad \forall \vec{x} \in U$

if $h(\vec{x}) = \dfrac{f(\vec{x})}{g(\vec{x})}$, then h is differentiable at $\vec{x}_0$

and

$$Dh(\vec{x}_0) = \frac{g(\vec{x}_0)Df(\vec{x}_0) - f(\vec{x}_0)Dg(\vec{x}_0)}{[g(\vec{x}_0)]^2}.$$

E.g. $f(x,y) = x^2 + y^2$ and $g(x,y) = \sin^2 x + 1$

$$h(x,y) = \frac{x^2+y^2}{\sin^2 x + 1}$$

Find $Dh(x,y)$

Sol: $$Dh(x,y) = \frac{g(x,y)\,Df(x,y) - f(x,y)\,Dg(x,y)}{[g(x,y)]^2}$$

$$\cancel{= (\sin^2 x+1)[2x, 2y] - (x^2+y^2)[}$$

$$= \frac{(\sin^2 x + 1)\left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] - (x^2+y^2)\left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right]}{[g(x,y)]^2}$$

$$= \frac{(\sin^2 x + 1)[2x \quad 2y] - (x^2+y^2)[2\sin x \cos x \quad 0]}{(\sin^2 x + 1)^2}$$

$$= \left[\frac{(\sin^2 x + 1)\cdot 2x - (x^2+y^2)\,2\sin x\cos x}{(\sin^2 x+1)^2} \quad \frac{\cancel{(\sin^2 x+1)}\cdot 2y}{(\sin^2 x+1)^{\cancel{2}}}\right].$$

(5) Chain rule:

Let $f: \underset{\substack{\uparrow \\ \text{open set}}}{V} \subset \mathbb{R}^m \to \mathbb{R}^p$ and $g: \underset{\substack{\uparrow \\ \text{open set}}}{U} \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$

Let g be diff. at $\vec{x}_0$ and f be diff. at $\vec{y}_0 = g(\vec{x}_0)$.

Then

$$D(f\circ g)(\vec{x}_0) = [\underbrace{Df(\vec{y}_0)}_{\substack{p\times m\\ \text{matrix}}}][\underbrace{Dg(\vec{x}_0)}_{\substack{m\times n\\ \text{matrix}}}].$$

$p \times n$ matrix

Compare to single variable calculus $\dfrac{df(g(x_0))}{dx} = f'(\underbrace{g(x)}_{y})g'(x)$.

E.g. $g(x,y) = (x^2+1,\ y^2)$ and $f(u,v) = (u+v,\ u,\ v^2)$

Find $D(f\circ g)(1,1)$ using the chain rule.

Soln:

$\cancel{D(f\circ g)(1,y)}$ By the chain rule

$$D(f\circ g)(1,1) = [Df(g(1,1))][Dg(1,1)]$$
$$= \underset{?}{[Df(2,1)]}\ \underset{?}{[Dg(1,1)]} \quad \text{since } g(1,1)=(2,1)$$

$$Df(u,v) = \begin{bmatrix} \frac{\partial(u+v)}{\partial u} & \frac{\partial(u+v)}{\partial v} \\ \frac{\partial(u)}{\partial u} & \frac{\partial(u)}{\partial v} \\ \frac{\partial(v^2)}{\partial u} & \frac{\partial(v^2)}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix} \Rightarrow \cancel{Df(2,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}}$$

$$Df(2,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and $$Dg(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} \Rightarrow Dg(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$D(f\circ g)(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

* Special case of the chain rule:

$c: \mathbb{R} \to \mathbb{R}^3$ is a diff. path, $f: \mathbb{R}^3 \to \mathbb{R}$

$\vec{c}(t) = (x(t), y(t), z(t))$

and $h(t) = f(c(t)) = f(x(t), y(t), z(t))$.

then

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$
$$= \nabla f(\vec{c}(t)) \cdot \overrightarrow{c'(t)}$$
$$= \underbrace{[Df(\vec{c}(t))]}_{1\times 3}\ \underbrace{[D\vec{c}(t)]}_{3\times 1}.$$

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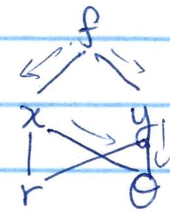
E.g. Let $f(x, y)$ be given and make the substitution $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial r}$.

Sol:



$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \left(\frac{\partial f}{\partial x}\right)\cdot(-r\sin\theta) + \left(\frac{\partial f}{\partial y}\right)(r\cos\theta)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial f}{\partial x} + (\sin\theta)\frac{\partial f}{\partial y}.$$

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* 3.2 Taylor's Theorem:

- Single variable:

$$f(x_0+h) = \underbrace{\overbrace{f(x_0) + f'(x_0)h}^{\text{linear approx.}} + f''(x_0)\frac{h^2}{2}}_{\text{quadratic approx}} + \dots + \frac{f^{(k)}(x_0)}{k!}h^k + R_k(x_0, h).$$

$R_k(x_0, h)$ is a remainder and

$$\lim_{h \to 0} \frac{R_k(x_0, h)}{h^k} = 0.$$

- Multi-variables: $f: \mathbb{R}^n \to \mathbb{R}$.

- linear approx:

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot \vec{h} + R_1(\vec{x}_0, \vec{h}).$$

$$= f(\vec{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0) h_i + R_1(\vec{x}_0, \vec{h}).$$

$$\lim_{h \to 0} \frac{R_1(\vec{x}_0, \vec{h})}{\|\vec{h}\|} = 0$$

- quadratic approx:

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot \vec{h} +$$

$$+ \frac{1}{2}(h_1, \dots, h_n) \underbrace{\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\vec{x}_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\vec{x}_0) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\vec{x}_0) \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\vec{x}_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\vec{x}_0) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\vec{x}_0) \end{bmatrix}}_{\substack{\text{Hessian} \\ n \times n \text{ matrix of second derivatives.}}} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}.$$

$$+ \underbrace{R_2(\vec{x}_0, \vec{h})}$$

$$\lim_{\vec{h} \to 0} \frac{R_2(\vec{x}_0, \vec{h})}{\|\vec{h}\|^2} = 0.$$

(13)

$$= f(\vec{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0)\, h_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)\, h_i h_j +$$

$$+ R_2(\vec{x}_0, \vec{h}).$$

E.g. $f(x,y) = \sin(x + 2y)$.

Find the second-order Taylor formula about $\vec{x}_0 = (0,0)$.

Need: $f(0,0)$ $\quad \frac{\partial f}{\partial x}(0,0)$ $\quad \frac{\partial f}{\partial y}(0,0)$ $\quad \frac{\partial^2 f}{\partial x^2}(0,0)$ $\quad \frac{\partial^2 f}{\partial y^2}(0,0)$

and $\frac{\partial^2 f}{\partial x \partial y}(0,0)$.

- $f(0,0) = \sin(0) = 0$
- $\frac{\partial f}{\partial x}(x,y) = \cos(x+2y) \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1.$
- $\frac{\partial f}{\partial y}(x,y) = 2\cos(x+2y) \Rightarrow \frac{\partial f}{\partial y}(0,0) = 2.$
- $\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin(x+2y) \Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 0.$
- $\frac{\partial^2 f}{\partial y^2}(x,y) = -4\sin(x+2y) \Rightarrow \frac{\partial^2 f}{\partial y^2}(0,0) = 0.$
- $\frac{\partial^2 f}{\partial x \partial y}(x,y) = -2\sin(x+2y) \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0.$

so $\quad H\big|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

$$\Rightarrow f(\vec{h}) = f(h_1, h_2) = 1 \cdot h_1 + 2h_2 + 0 + \underbrace{R_2(\vec{0}, \vec{h})}_{}.$$

$$\lim_{\vec{h} \to \vec{0}} \frac{R_2(\vec{0}, \vec{h})}{\|\vec{h}\|^2} = 0$$

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E.g. Determine the second order Taylor formula for $f(x,y) = e^{x+y}$ about $(0,0)$.

- $f(0,0) = 1$.
- $\frac{\partial f}{\partial x}(x,y) = e^{x+y} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1$.
- $\frac{\partial f}{\partial y}(x,y) = e^{x+y} \Rightarrow \frac{\partial f}{\partial y}(0,0) = 1$.
- ~~$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \partial^2$~~
- $\frac{\partial^2 f}{\partial x^2}(x,y) = e^{x+y} = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$.

$\Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$.

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$f(\vec{h}) = f(h_1,h_2) = 1 + h_1 + h_2 + \frac{1}{2}(h_1,h_2)H\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R_2(\vec{0},\vec{h})$$

$$= 1 + h_1 + h_2 + \frac{1}{2}(h_1,h_2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R_2(\vec{0},\vec{h})$$

$$= 1 + h_1 + h_2 + \frac{1}{2}\begin{bmatrix} h_1+h_2 & h_1+h_2 \end{bmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R_2(\vec{0},\vec{h})$$

$$= 1 + h_1 + h_2 + \frac{1}{2}h_1^2 + \frac{1}{2}h_2^2 + h_1h_2 + \underbrace{R_2(\vec{0},\vec{h})}$$

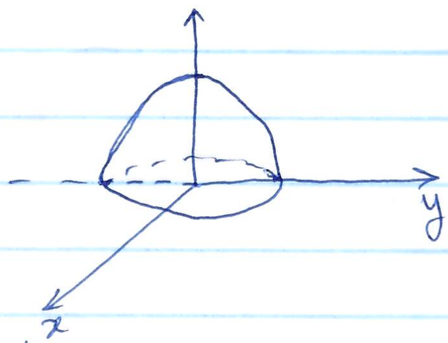
$$\lim_{h\to 0}\frac{R_2(0,h)}{\|h\|^2} = 0$$

* Double Integrals: (5.1 – 5.4).

$$\iint_R f(x,y)\,\underbrace{dA}_{\substack{dx\,dy \\ \text{or } dy\,dx}} = $$ Volume of the region above R and under the graph of f when f is non-negative.

Example: $f(x,y) = 1 - x^2 - y^2$

$R: -1 \le x \le 1,\ -1 \le y \le 1.$



$$\int_{-1}^{1}\left[\int_{-1}^{1} (1 - x^2 - y^2)\,dy\right]dx = \int_{-1}^{1} \left(y - x^2y - \frac{y^3}{3}\right)\Big|_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left(1 - x^2 - \frac{1}{3} - \left(-1 + x^2 + \frac{1}{3}\right)\right) dx$$

$$= \int_{-1}^{1} 2 - 2x^2 - \frac{2}{3}\,dx$$

$$= \int_{-1}^{1} \left(\frac{4}{3} - 2x^2\right) dx$$

$$= \frac{4}{3}x - \frac{2x^3}{3}\Big|_{-1}^{1}$$

$$= \frac{4}{3} - \frac{2}{3} - \left(-\frac{4}{3} + \frac{2}{3}\right)$$

$$= \frac{8}{3} - \frac{4}{3}$$

$$= \frac{4}{3}$$

iterated integral because it is obtained by integrating with respect to y and then w.r.t. x.

Sometimes evaluating an iterated integral can be hard so we may need to change the order of integration.

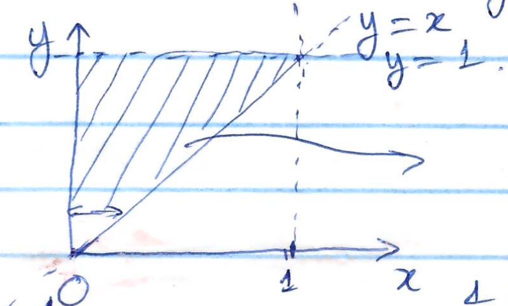
E.g. $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$

e^{y^2} doesn't have an antiderivative that we know

$\Rightarrow$ cannot solve directly.

$\Rightarrow$ try changing the order of integration.

$0 \le x \le 1$ and $x \le y \le 1$.



equivalently $0 \le y \le 1$.
$0 \le x \le y$.

$$\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

$$= \int_0^1 e^{y^2} x \Big|_0^y \,dy$$

$$= \int_0^1 y e^{y^2} - 0 \,dy.$$

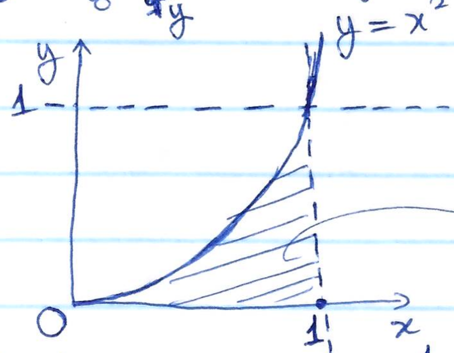
$$= \int_0^1 y e^{y^2}\,dy.$$

$$= \frac{1}{2} e^{y^2} \Big|_0^1$$

$$= \frac{1}{2} e^1 - \frac{1}{2} \cdot e^0$$

$$= \frac{1}{2}(e - 1).$$

E.g. $\displaystyle\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$



$0 \le y \le x^2$

$0 \le x \le 1.$

$$\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy = \int_0^1\int_0^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_0^1 e^{x^3}\, y\Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 e^{x^3}(x^2 - 0)\,dx.$$

$$= \int_0^1 x^2 e^{x^3}\,dx$$

or let $u = x^3 \Rightarrow du = 3x^2dx.$

$$= \frac{e^{x^3}}{3}\Big|_0^1$$

$$= \frac{1}{3}(e-1).$$

* <u>Triple Integrals:</u> (5.5)

$$\iiint_R f(x,y,z)\, \underbrace{dV}_{dx\,dy\,dz}$$

$R \to$ solid in space

<u>E.g.</u> Let B be the box given by

$0 \leq x \leq 1, \quad 0 \leq y \leq 2, \quad -1 \leq z \leq 0.$

evaluate $\iiint_B x^2 + xy + z^2 y \, dV$

$$= \int_0^1 \int_0^2 \int_{-1}^0 x^2 + xy + z^2 y \; dz\,dy\,dx \qquad \text{(iterated integrals).}$$

$$= \int_0^1 \int_0^2 x^2 z + xyz + \frac{z^3}{3} y \Big|_{z=-1}^{z=0} dy\,dx$$

$$= \int_0^1 \int_0^2 0 - \left(-x^2 - xy - \frac{y}{3}\right) dy\,dx.$$

$$= \int_0^1 \int_0^2 x^2 + xy + \frac{y}{3} \; dy\,dx$$

$$= \int_0^1 x^2 y + \frac{xy^2}{2} + \frac{y^2}{6} \Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 2x^2 + \frac{4}{2}x + \frac{4}{6} \; dx.$$

$$= \int_0^1 2x^2 + 2x + \frac{2}{3} \; dx$$

$$= \frac{2x^3}{3} + x^2 + \frac{2}{3}x \Big|_0^1$$

$$= \frac{2}{3} + 1 + \frac{2}{3}$$

$$= \frac{7}{3}.$$

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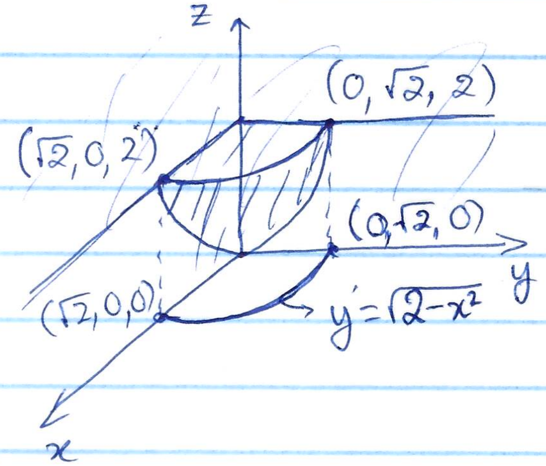
<u>Exercise:</u> Verify that you get the same ~~number~~ answer if you change the order of integration.

<u>E.g.</u> Let W be the region bounded by the planes $x=0$, $y=0$, $z=2$, and the surface $z = x^2+y^2$. lying in the quadrant $x \geq 0$, $y \geq 0$.

Compute $\iiint_W x\,dx\,dy\,dz$.

$W = \{x^2+y^2 \le z \le 2,\ \overset{0}{\cancel{\sqrt{2-x^2}}} \le y \le \sqrt{2-x^2},\ 0 \le x \le \sqrt{2}\}$



$$\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x \; dz\,dy\,dx$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xz \Big|_{z=x^2+y^2}^{\cancel{z=2-2x^2}\ z=2} dy\,dx$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 2x - x(x^2+y^2)\,dy\,dx$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 2x - x^3 - xy^2 \; dy\,dx$$

$$= \int_0^{\sqrt{2}} 2xy - x^3 y - x\frac{y^3}{3} \Big|_{y=0}^{y=\sqrt{2-x^2}} dx$$

$$= \int_0^{\sqrt{2}} 2x(\sqrt{2-x^2}) - x^3\sqrt{2-x^2} - x\frac{(\sqrt{2-x^2})^3}{3}\,dx.$$

$$= \int_0^{\sqrt{2}} 2x(2-x^2)^{1/2} - x^3\sqrt{2-x^2} - \frac{x(2-x^2)\sqrt{2-x^2}}{3}\,dx.$$

$$= \int_0^{\sqrt{2}} 2x\sqrt{2-x^2} - x^3\sqrt{2-x^2} - \frac{2x\sqrt{2-x^2}}{3} + \frac{x^3\sqrt{2-x^2}}{3}\,dx.$$

$$= \int_0^{\sqrt{2}} \frac{4x}{3}\sqrt{2-x^2}$$

(20)

$$= \int_0^{\sqrt{2}} (2-x^2)\left(x\sqrt{2-x^2}\right) - \frac{x(2-x^2)^{3/2}}{3}\, dx.$$

$$= \int_0^{\sqrt{2}} \frac{2}{3} x\,(2-x^2)^{3/2}\, dx.$$

Let $u = 2-x^2 \Rightarrow du = -2x\,dx.$

$$= \int_2^0 \frac{2}{3} u^{3/2} \cdot \frac{du}{-2}$$

$$= -\int_2^0 \frac{u^{3/2}}{3}\, du.$$

$$= \int_0^2 \frac{u^{3/2}}{3}\, du$$

$$= \frac{1}{3} \left.\frac{u^{3/2+1}}{3/2+1}\right|_0^2$$

$$= \frac{1}{3} \frac{2^{5/2}}{5/2}$$

$$= \frac{2}{15} \cdot 2^{5/2}$$

$$= \frac{2^{7/2}}{15}$$

$$= \frac{8\sqrt{2}}{15}.$$

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E.g. Calculate the area of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

Sol: $A(D) = \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} dx\,dy$.

$u = \frac{x}{a}$

$v = \frac{y}{b}$.

$A(D) = \iint\limits_{u^2+v^2\leq 1} ab\,du\,dv$

$= ab \iint\limits_{u^2+v^2\leq 1} du\,dv$

$= \pi ab$.

Solve: $\int_0^1 xe^{x^2}\,dx \rightarrow$ change of variable/substitution.

Let $u = x^2 \Rightarrow du = 2x\,dx$.

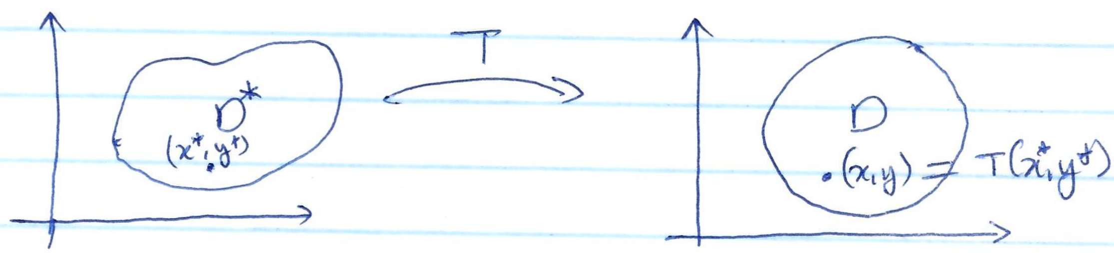
$$\int_0^1 xe^{x^2}\,dx = \int_0^1 \frac{e^u}{2}\,du = \frac{e^u}{2}\Big|_0^1 = \frac{e}{2} - \frac{1}{2}.$$

When we have several variables, we also need to do something similar. In this chapter, we develop the multidimensional change of variables formula, ~~which~~ -i

6.1. The geometry of maps from $\mathbb{R}^2 \to \mathbb{R}^2$.

Let T be a map from $D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$

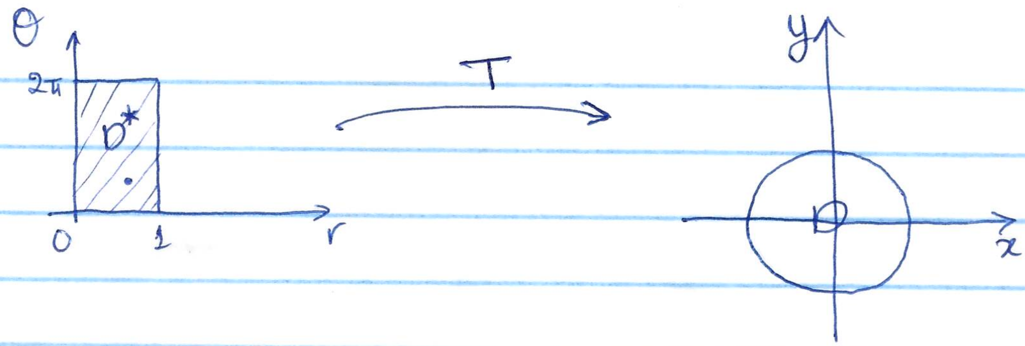
We call $D = T(D^*)$ the set of image points of T (so every point (x,y) in D must be equal to $T(x^*, y^*)$ for some (x^*, y^*) in D^*).



E.g. (Polar coordinates)

Let $D^* \subset \mathbb{R}^2$ be the rectangle $D^* = [0,1] \times [0, 2\pi]$, i.e. all points in D^* are of the form (r, θ), where $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$$T(r, \theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y}).$$



$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2 \le 1.$$

$\Rightarrow$ $D = T(D^*)$ is contained in a unit disk.

<u>Q:</u> Is it the whole unit disk? Yes!

Because for any (x,y) in the unit disk, (x,y) can be written as $(r\cos\theta, r\sin\theta)$ for some $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

<u>Thm:</u> Let $\overset{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{\overset{\shortparallel}{A}}$ be a 2×2 matrix with $\overset{ad-bc}{\overset{\shortparallel}{\det(A)}} \ne 0$.
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = A\vec{x}$.
(in other words, $T(x,y) = (ax+by,\ cx+dy)$).

Then T transforms parallelogram into parallelogram and vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.

<u>E.g.</u> Let $T(x,y) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix}$.

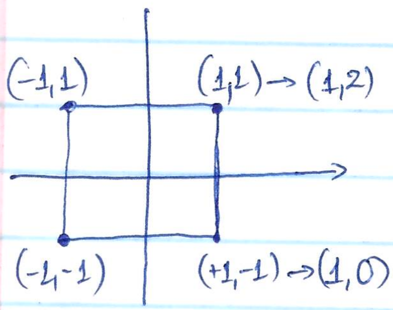
and let $D^* = [-1,1]\times[-1,1]$.

Find D?

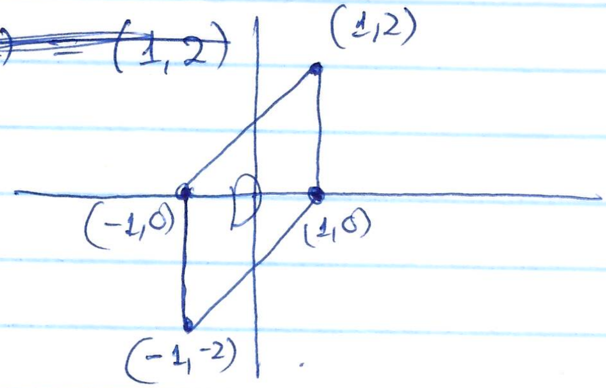
(23) $T(x,y) = \begin{pmatrix} x \\ x+y \end{pmatrix}$

Sol. $\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) = 1 \neq 0$. And D^* is a rectangle.

$\Rightarrow$ we only need to map the vertices and connect them.



~~$T(1,1) = (1,2)$~~



$T(1,1) = (1,2)$
$T(1,-1) = (1,0)$
$T(-1,1) = (-1,0)$
$T(-1,-1) = (-1,-2)$

Def: A mapping T is one-to-one on D^* if for (u,v) and $(u',v') \in D^*$,
$T(u,v) = T(u',v') \Rightarrow u = u',\ v = v'$.

(24)

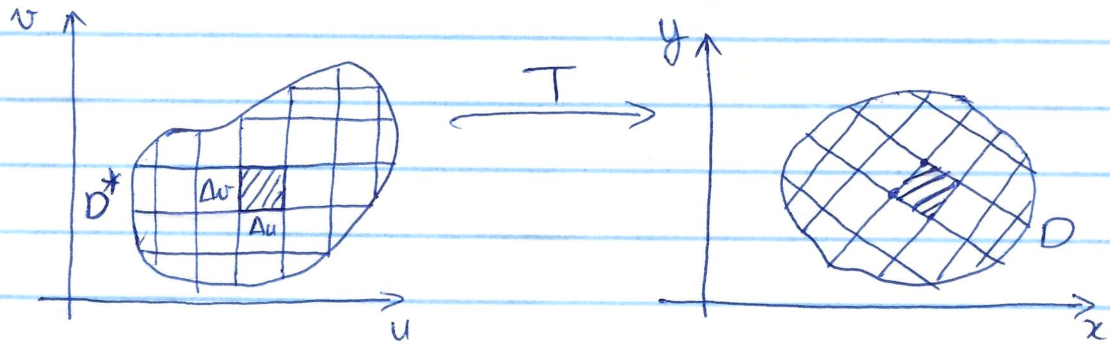
* 6.2. Change of Variables Theorem.

Want to evaluate $\iint_D f(x,y)\,dx\,dy$

but too hard in some cases.

$\Rightarrow$ use a chage of variable.

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(u,v) = (x(u,v),\ y(u,v)).$$



Find the area
$$\text{Area}(D) = \iint_D dx\,dy.$$

Area(D) = sum of areas of the little "almost" ~~paral~~ parallelogram.

$\approx$ approximately sum of areas of parallelograms obtained by linear approximation of T.

Taking limit as the parallelograms become very small,
$$\text{Area}(D) = \iint_D dx\,dy.$$

$\Rightarrow$ we need the areas of the parallelograms.

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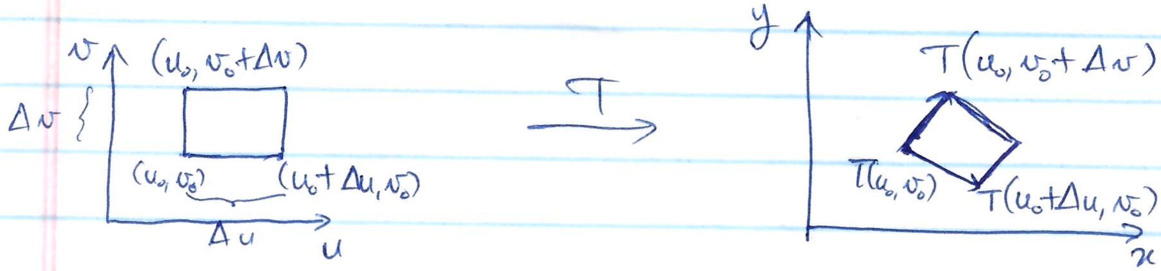

linear approximation:

$$T(u,v) \approx T(u_0, v_0) + T'\begin{pmatrix} \cancel{\Delta u}\ u-u_0 \\ v-v_0 \end{pmatrix}.$$

$$\Rightarrow T(u_0+\Delta u, v_0) \approx T(u_0, v_0) + T'\begin{pmatrix} \Delta u \\ 0 \end{pmatrix}$$

$$= T(u_0, v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{pmatrix} \Delta u \\ 0 \end{pmatrix}$$

Similarly,

$$T(u_0, v_0+\Delta v) \approx T(u_0, v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{pmatrix} \cancel{\Delta} 0 \\ \Delta v \end{pmatrix}.$$

$\Rightarrow$ the sides of the parallelogram are

$$T(u_0+\Delta u, v_0) - T(u_0, v_0) = \begin{pmatrix} \frac{\partial x}{\partial u} \cdot \Delta u \\ \frac{\partial y}{\partial u} \Delta u \end{pmatrix}$$

and

$$T(u_0, \cancel{+\Delta u}\ v_0+\Delta v) - T(u_0, v_0) = \begin{pmatrix} \frac{\partial x}{\partial v} \Delta v \\ \frac{\partial y}{\partial v} \Delta v \end{pmatrix}.$$

$\Rightarrow$ The area of the parallelogram.

$$\begin{vmatrix} \frac{\partial x}{\partial u} \cdot \Delta u & \frac{\partial x}{\partial v} \Delta v \\ \frac{\partial y}{\partial u} \cdot \Delta u & \frac{\partial y}{\partial v} \Delta v \end{vmatrix} \leftarrow \text{determinant.}$$

$$= \left| \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \right) \Delta u \Delta v \right|$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v.$$

$\Rightarrow$ Area$(D) \approx \sum\sum_{\text{all } (u_0, v_0)}$ areas of the parallelogram

$$= \sum\sum_{\text{all } (u_0, v_0)} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}_{(u_0, v_0)} \Delta u \Delta v.$$

$$\xrightarrow[\text{shrinks}]{\text{as parallelogram}} \iint_{D^*} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \cancel{dx\,dy} \; du\,dv$$

$$\Rightarrow \text{Area}(D) = \iint_{D^*} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv.$$

In general, the change of variable formula is

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\substack{\text{Jacobian} \\ \text{determinant}}} du\,dv$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right|.$$

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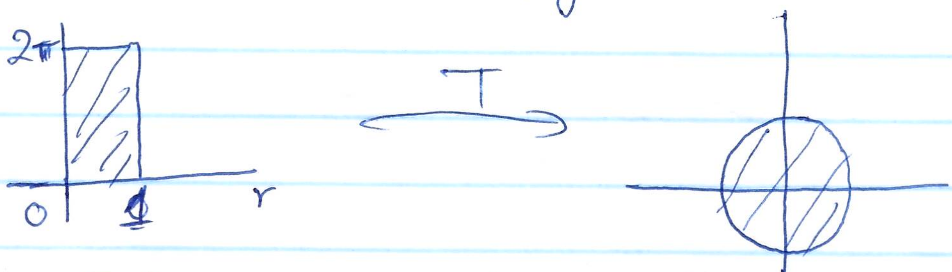
E.g. Use polar coordinates to find

$$\underbrace{\iint_{x^2+y^2\le 1}}_{D} e^{-(x^2+y^2)}\,dx\,dy.$$

Sol: $x = r\cos\theta$

$y = r\sin\theta.$

$T(r,\theta) = (x,y)$



Find Jacobian determinant.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin\theta)\sin\theta$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

$$\cancel{\iint_D e^{-r^2}} r\,dr \Rightarrow \iint_D e^{-(x^2+y^2)}\,dx\,dy = \iint_{D^*} e^{-r^2}\, r\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^{1} r e^{-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r e^{-r^2}\,dr$$

$$= 2\pi \cdot \left(-e^{-r^2}\right)\Big|_0^1$$

$$= 2\pi\left(1 - e^{-1}\right).$$

In general, polar coordinates change of variable formu

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

---

* Change of variables formula for triple integrals.

Let T be a function from $\mathbb{R}^3 \to \mathbb{R}^3$.

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w).$$

$\Rightarrow$ The Jacobian determinant of T:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

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<u>Two important cases</u>

1) <u>Cylindrical coordinates:</u> $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

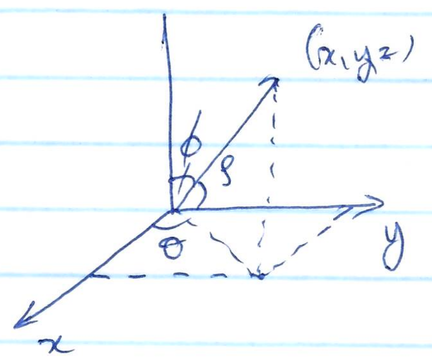
$$\Rightarrow \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$$

2) ~~Sp~~ <u>Spherical coordinates:</u>

$x = \rho\cos\theta\sin\phi$

$y = \rho\, \cancel{\cos\theta\cos\phi}\, \sin\theta\sin\phi$.

$z = \rho\cos\phi$.



$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \dots = \rho^2\sin\phi.$$

$$\Rightarrow \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} \cancel{f(x(\rho,\theta,\phi), y(\rho,\theta,\phi))}.$$

$$= \iiint_{W^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

30.

E.g. Compute the volume of the solid W between the paraboloids: $z = x^2 + y^2$ and $z = 1 - x^2 - y^2$.