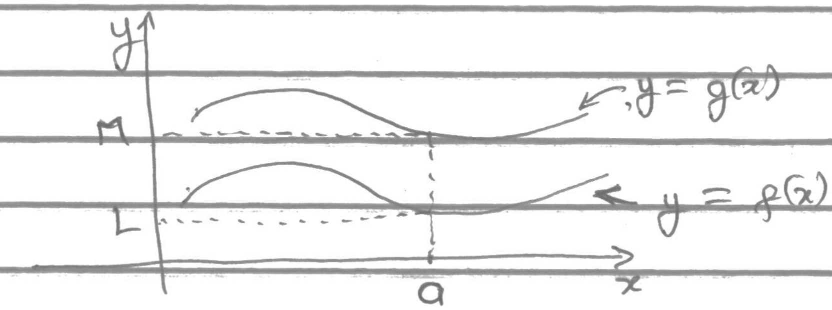
* Limits and Inequalities:



From graph, $\lim_{x \to a} f(x) = L \leq \lim_{x \to a} g(x) = M$.

. **Thm:** Let f and g be functions whose limits for $x \to a$ exist, and assume that

$$f(x) \leq g(x) \quad \text{holds for all } x.$$

Then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

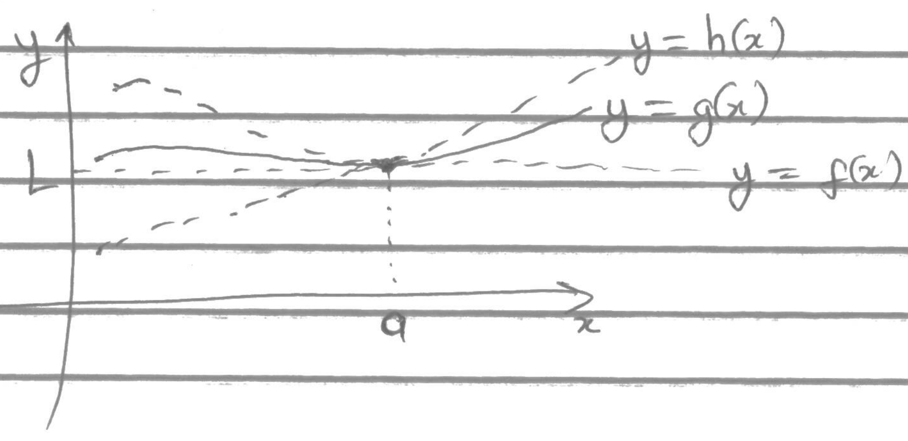
. **Thm:** (The squeeze or sandwich theorem).

Suppose that

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x$$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then $\lim_{x \to a} g(x) = L$



E.g. 1) Evaluate $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$

By properties of sine functions:

$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1 \quad \text{for all } x \ne 0$$

$\Rightarrow$ For $x > 0$, multiplying through by x:

$$-|x| \le x\sin\left(\frac{1}{x^2}\right) \le |x|.$$

For $x < 0$,

$$x \le x\sin\left(\frac{1}{x^2}\right) \le -x.$$

$$\Rightarrow -|x| \le x\sin\left(\frac{1}{x^2}\right) \le |x|$$

As a result, for $x \ne 0$.

$$-|x| \le x\sin\left(\frac{1}{x^2}\right) \le |x|$$

and $\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$

By the squeeze theorem, $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = 0$.

Important Trig limits:

$$\boxed{\lim_{t\to 0} \frac{\sin(t)}{t} = 1 \quad \text{and} \quad \lim_{t\to 0} \frac{1-\cos(t)}{t} = 0.}$$

Hint: For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $x \ne 0$

$$\cos(x) \le \frac{\sin(x)}{x} \le 1.$$

22

2) (Change of variables)

Evaluate $\lim_{x \to 0} \frac{\tan(5x)}{\tan(3x)}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\sin(5x)}{\cos(5x)} \cdot \frac{\cos(3x)}{\sin(3x)}$$

$$= \lim_{x \to 0} \frac{\sin(5x)}{\cos(5x)} \cdot \frac{\cos(3x)}{\sin(3x)} \cdot \frac{x}{x}$$

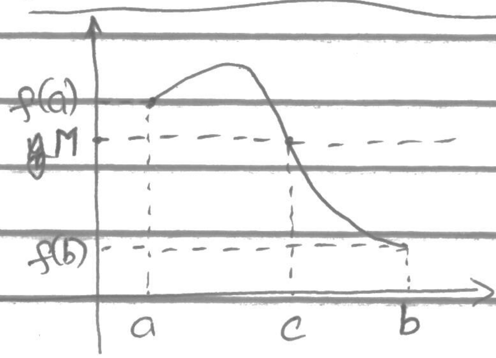
$$= \lim_{x \to 0} \frac{5}{3}\left(\frac{\sin(5x)}{5x}\right) \cdot \left(\frac{3x}{\sin(3x)}\right) \left(\frac{\cos(3x)}{\cos(5x)}\right)$$

$$= \frac{5}{3} \cdot 1 \quad 1 \cdot \frac{\cos(0)}{\cos(0)}$$

$$= \frac{5}{3}.$$

* The Intermediate Value theorem:

$\boxed{\text{a continuous function cannot skip values.}}$



The IVT says that a continuous function must attain any given value y between f(a) and f(b) at least once. In the graph, there is a value of c for which $f(c) = M$ holds.

(23)

Thm (IVT) If f is continuous on a closed interval $[a, b]$, for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a,b)$ such that $f(c) = M$.

E.g. Show that $2^x + 3^x = 4^x$ has a solution.

Sol. Consider $f(x) = 2^x + 3^x - 4^x$.

Then $f(0) = 1 + 1 - 1 = 1 > 0$

and $f(2) = 2^2 + 3^2 - 4^2 = -3 < 0$

and f is continuous (why?)

$\Rightarrow$ by the IVT,

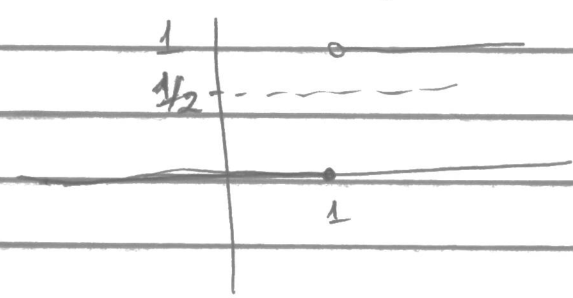
there exist $c \in (0,2)$ such that

$f(c) = 0$.

That is, $2^c + 3^c - 4^c = 0$.

$\Rightarrow$ c is a solution of $2^x + 3^x = 4^x$.

Remark: If f is discontinuous, the thm may not work. For example.



$$f(x) = \begin{cases} 1, & \text{if } x > 1. \\ 0, & \text{if } x \leq 1. \end{cases}$$

We cannot find c such that $f(c) = \frac{1}{2}$.

Derivatives.

Def: Let f be a function which is defined on some interval (c, d) and let a be some number in this interval.

The derivative of the function f at a is the value of the limit

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

f is said to be differentiable at a if this limit exists.

f is said to be differentiable on the interval (c, d) if it's differentiable at every point a in (c, d).

- Other notation: substitute $x = a + h$ in the limit and let $h \to 0$, then

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Remark: The numerator $f(a+h) - f(a)$ represents the amount by which the function value of f changes if one increases its argument a by a small amount h.

Eg. 1) The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$= 2x.$$

2) The derivative of a linear function $f(x) = mx + b$ is $f'(x) = m$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[m(x+h) + b] - (mx+b)}{h}$$

$$= \lim_{h\to 0} m$$

$$= m.$$

3) The derivative of any constant function is zero.

$g(x) = c \Rightarrow g'(x) = 0.$

Show:

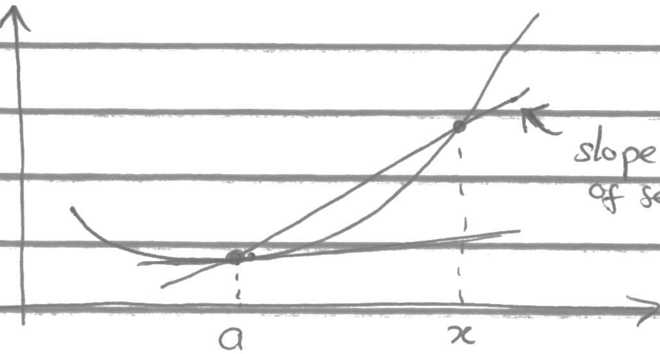
$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{c - c}{h}$$

$$= \lim_{h\to 0} 0 = 0.$$

Def: (Tangent line) Assume that f is differentiable at a. The tangent line to the graph $y = f(x)$ at $P = (a, f(a))$ is the line through P of slope $f'(a)$. The equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x-a)$$



slope of secant line $= \dfrac{f(x)-f(a)}{x-a}$.

$\Rightarrow$ slope of tangent line at $x=a$ = limit of slopes of secant line.

E.g. Find an equation of the tangent line to the graph of $f(x) = x^3$ at $x = 1$.

Sol. Find the slope:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^3 - 1^3}{h}$$

$$= \lim_{h\to 0} \frac{h\left[(1+h)^2 + (1+h) + 1\right]}{h}$$

$$= \lim_{h\to 0} \left[(1+h)^2 + (1+h) + 1\right] = 3.$$

(27).

$\Rightarrow$ The equation of the tangent line is

$$y - f(1) = f'(1)(x-1)$$

$$\Rightarrow y - 1 = 3(x-1)$$

$$y = 3x - 2.$$

E.g. $f(x) = \dfrac{1}{\sqrt{2x+1}}$, $a = 4$.

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2(4+h)+1}} - \frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{3-\sqrt{9+2h}}{3\sqrt{2(4+h)+1}} = \lim_{h\to 0} \frac{9-(9+2h)}{3h\,(3+\sqrt{9+2h})\sqrt{2(4+h)+1}}$$

conjugation.

$$= \lim_{h\to 0} \frac{-2}{3(3+\sqrt{9+2h})\sqrt{2(4+h)+1}}$$

$$= -1/27$$

The Derivative as a function.

We have just shown how to take a value $a \in \mathbb{R}$ and obtain another value $f'(a) \in \mathbb{R}$, hence we can treat f' as a function.

$$\text{Domain}(f') = \{x \in \mathbb{R} \mid f'(x) \text{ is defined}\}$$
$$= \{x \in \mathbb{R} \mid f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}\}$$
$$= \text{set of values } x \in \mathbb{R} \text{ such that } f \text{ is differentiable.}$$

. Important examples:

1) $f(x) = c \Rightarrow f'(x) = 0$ (constant rule)
2) $f(x) = x \Rightarrow f'(x) = 1$
3) $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ (power rule)
~~4) $f(x) = b^x \Rightarrow f'(x) =$~~

Thm: (the derivative is linear)
f, g differentiable and $c \in \mathbb{R}$, then:
1) $(f+g)' = f' + g'$
2) $(cf)' = cf'$

Examples:

1) $f(x) = \sqrt[3]{x}$, $a = 8$

$f'(x) = \frac{1}{3} x^{-2/3}$ by the power rule.

$\Rightarrow f'(8) = \frac{1}{3} 8^{-2/3} = \frac{1}{\cancel{3\cdot 4}} \cdot \frac{1}{12}$

Tangent line: slope $= \frac{1}{12}$, intersept $= (8, f(8))$

$$\Rightarrow \boxed{y = \frac{1}{12}(x-8) + 2}$$

2) When does the graph of $f(x) = 12x - x^3$ have a horizontal tangent line?

i.e. when does $f'(x) = 0$?

$\underbrace{f'(x) = 12 - 3x^2}_{\text{by power rule}}$, then $f'(x) = 0 \iff x = \pm 2$

Exercise: Find all x such that the tangent line to $f(x) = 4x^2 + 11x + 2$ is steeper than the tangent line to $g(x) = x^3$.

3) $f(x) = e^x \Rightarrow f'(x) = e^x$.

$$\boxed{\text{Differentiable} \Rightarrow \text{Continuous}}$$

Thm: if f is differentiable at a then f is continuous at a.

continuous $\not\Rightarrow$ differentiable.

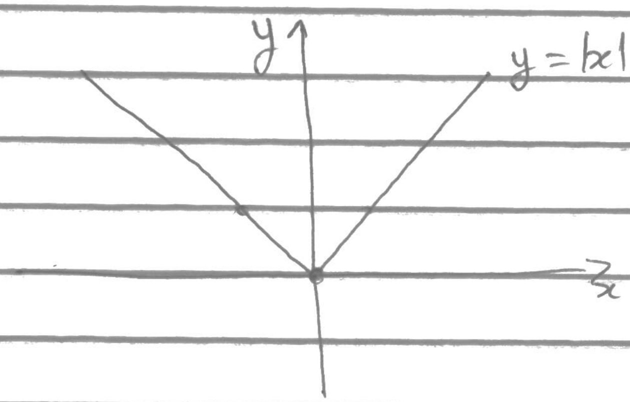
Eg. Some non-differentiable functions:

1) the following function is continuous at 0 but not differentiable

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

(30)

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$ DNE



the graph has a corner at the origin.

* The Differentiation Rule:

Remark: We also denote $f'(x)$ by $\frac{df}{dx}$.

Suppose f and g are differentiable at $x=a$:

1) Constant rule:

$$(c)' = 0$$

2) Sum rule:

$$(f(a)+g(a))' = f'(a) + g'(a)$$

3) Product rule:

$$\cancel{(f(a)\cdot g(a))' = f'(a)g(a) + f(a)g'(a)}$$

$$(f(a)\cdot g(a))' = f'(a)g(a) + f(a)g'(a).$$

4) Quotient rule: Suppose $g(a) \neq 0$.

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(x)}{g^2(a)}.$$

Examples:

1) Find $f'(x)$ of $f(x) = x^2(3+x^{-1})$.

Let $g(x) = x^2$ and $h(x) = 3 + x^{-1}$

$\Rightarrow$ $f(x) = g(x)h(x)$.

$$f'(x) = (gh(x))' \overset{\text{product rule}}{=} g'(x)h(x) + g(x)h'(x).$$

$$= \underset{\uparrow\ \text{power rule}}{(2x)}(3+x^{-1}) + (x^2)(\underset{\uparrow\ \text{constant rule}}{0} - x^{-1-1})$$

$$= 6x + 2 \quad - 1$$

$$= 6x.$$

2) Find the derivative of the function

$$g(x) = \frac{2x^4 - x^3 + 7}{1 + x^2}$$

Let $u(x) = 2x^4 - x^3 + 7$ and $v(x) = 1 + x^2$.

Then,

$$g'(x) = \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$= \frac{(1+x^2)(2\cdot 4x^3 - 3x^2 + 0) - (2x^4 - x^3 + 7)(0 + 2x)}{(1+x^2)^2}$$

$$= \frac{6x^5 - x^4 + 8x^3 - 3x^2 - 14x}{(1+x^2)^2}$$

3) $$\frac{d}{dx}\left(\frac{e^x}{e^x + x}\right) = \frac{(e^x + x)(e^x)' - (e^x)(e^x + x)'}{(e^x + x)^2}$$

$(e^x)' = e^x$

$$= \frac{(e^x + x)e^x - e^x(e^x + 1)}{(e^x + x)^2}$$

$$= \frac{e^x(x - 1)}{(e^x + x)^2}$$

* Using the derivative:
  - Find the rate of change of $f(x)$ at $x = a$
  - Find (eq. of) the tangent line at $x = a$.

  $\Rightarrow$ estimate $f(x)$ near $x = a$ using the tangent line at $x = a$:

$$f(x) \approx f'(a)(x - a) + f(a)$$

for x near a.

E.g. Estimate $\sqrt{145}$?