# Lecture 9: Linear Transformations (Section 2.6)

Thang Huynh, UC San Diego

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## Linear transformations

- A map $T : V \to W$ between vector spaces is **linear** if

$$T(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \alpha T(\boldsymbol{x}) + \beta T(\boldsymbol{y})$$

equivalently, $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x} + \boldsymbol{y})$ and $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$.

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- Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}. \text{ What is } T\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)?$$

# Representing linear maps by matrices

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for $V$. A linear map $T : V \to W$ is determined by the values $T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)$.

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▶ Definition. (From linear maps to matrices)
Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for $V$, and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ a basis for $W$.
The **matrix representing** $T$ with respect to these bases

- has $n$ columns (one for each of the $\boldsymbol{x}_j$),
- the $j$-th column has $m$ entries $a_{1j}, \ldots, a_{mj}$ determined by

$$T(\boldsymbol{x}_j) = a_{1j}\boldsymbol{y}_1 + \ldots + a_{mj}\boldsymbol{y}_m.$$

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- Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$.
  - Is $T$ a linear transformation?

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$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 0 & ? \\ 1 & ? \end{bmatrix}.$$

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  - Which matrix $B$ represents $T$ with respect to the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

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## Representing linear maps by matrices

- Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix $A$ representing $T$ w.r.t. the following bases?

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{x}_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\boldsymbol{x}_2} \right\} \text{ for } \mathbb{R}^2, \quad \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_3} \right\} \text{ for } \mathbb{R}^3.$$

# Representing linear maps by matrices

- Solution.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}5\\2\\10\end{bmatrix}=5\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}-3\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}+5\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\Longrightarrow B=\begin{bmatrix}5 & ?\\-3 & ?\\5 & ?\end{bmatrix}.$$

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$$T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}7\\-2\\11\end{bmatrix} = 7\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} - 9\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} \Longrightarrow B = \begin{bmatrix}5 & 7\\-3 & -9\\5 & 4\end{bmatrix}.$$

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$$T(p) = p'.$$

Find the matrix $A$ representing $P$ w.r.t. the standard bases.

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$$\text{Differentiation matrix } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

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- Example. Let $T : \mathscr{P}_3 \to \mathscr{P}_4$ defined by

$$T(p) = \int_0^t p(x)\,dx.$$

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$$\text{Integration matrix } A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

## Representing linear maps by matrices

▶ Example. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let $T$ be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

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# Important geometric examples

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reflects every vector in $\mathbb{R}^2$ through the line $y = x$.

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gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$, i.e.

rotates every vector in $\mathbb{R}^2$ counter-clockwise by $90^o$.