Section 2.5 Properties of the Derivative

Goal: To differentiate products, quotients, sums of functions, and compositions of functions (a.k.a chain rule).

Sum, Product, and Quotient Rules

Let's start with sums, products, and quotients. Recall that if $f : \mathbb{R}^n \to \mathbb{R}^m$, its derivative is $T = Df$, the matrix of partial derivatives.

Let us assume that the functions we have are differentiable.

1. If $h(\vec{x}) = cf(\vec{x})$ where c is some scalar, then $(Dh)(\vec{x}) = c(Df)(\vec{x})$.
   Example. Let $f(x, y, z) = 3x^2 + 2yz$, and $h(x, y, z) = 6x^2 + 4yz = 2f(x, y, z)$. Then
   $$(Df)(x, y, z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [6x, 2z, 2y]$$
   and
   $$(Dh)(x, y, z) = 2(Df)(x, y, z) = [12x, 4z, 4y].$$
2. *Sum Rule.* If $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$, then $(Dh)(\vec{x}) = (Df)(\vec{x}) + (Dg)(\vec{x})$.
   Example. Let $f(x, y, z) = 3x^2 + 2yz, g(x, y, z) = e^{xyz}$, and $h(x, y, z) = f(x, y, z) + g(x, y, z) = 3x^2 + 2yz + e^{xyz}$. Then
   $$(Dh)(x, y, z) = \underbrace{(6x, 2z, 2y)}_{Df} + \underbrace{(yze^{xyz}, xze^{xyz}, xye^{xyz})}_{Dg}.$$
3. *Product Rule.* Let $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}$, and $h(\vec{x}) = f(\vec{x})g(\vec{x})$. Then
   $$(Dh)(\vec{x}) = g(\vec{x})(Df)(\vec{x}) + f(\vec{x})(Dg)(\vec{x}).$$
   Example. Let $f(x, y) = x^2 + y^2$ and $g(x, y) = xy$. Let $h(x, y) = f(x, y)g(x, y)$. Then
   $$\begin{aligned}(Dh)(x, y) &= g(x, y)(Df)(x, y) + f(x, y)(Dg)(x, y)\\ &= (xy)(2x, 2y) + (x^2 + y^2)(y, x)\\ &= (2x^2y, 2xy^2) + (x^2y + y^3, x^3 + xy^2)\\ &= (3x^2y + y^3, x^3 + 3xy^2).\end{aligned}$$
4. *Quotient Rule.* Let $q(\vec{x}) = \frac{f(\vec{x})}{g(\vec{x})}$ (where $g(\vec{x}) \neq 0$). Then
   $$(Dq)(\vec{x}) = \frac{g(\vec{x})(Df)(\vec{x}) - f(\vec{x})(Dg)(\vec{x})}{[g(\vec{x})]^2}.$$

Chain Rule

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are both differentiable. If we define the composition function $h(\vec{x})$ as $h(\vec{x}) = (g \circ f)(\vec{x}) = (g \circ f)(\vec{x})$, what is $(Dh)(\vec{x})$? Answer:

$$(Dh)(\vec{x}) = \underbrace{(Dg)(f(\vec{x}))}_{\text{derivative of } g \text{ evaluated at } f(\vec{x})} \overbrace{(Df)(\vec{x})}_{\text{derivative of } f \text{ evaluated at } \vec{x}}.$$

Recall that for one-variable functions $\frac{dg(f(x))}{dx} = g'(f(x))f'(x)$.
In this course, we will be interested in two special cases of the chain rule.

Special Case I

Let $\vec{c} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable path and $f : \mathbb{R}^3 \to \mathbb{R}$. Let

$$h(t) = (f \circ \vec{c})(t) = f(\vec{c}(t)).$$

What is $\frac{dh}{dt}$?

We recall the following definition:

Definition. If $f : \mathbb{R}^n \to \mathbb{R}, (Df)(\vec{x})$ is a $1 \times n$ matrix $\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}$. We can form the corresponding vector $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n})$ called the *gradient* of f and denoted by ∇f. Note that ∇f is a vector.

By the Chain Rule, $\frac{dh}{dt} = (Df)(\vec{c}(t))(D\vec{c})(t)$. We then can rewritte it as

$$\begin{aligned}\frac{dh}{dt} &= \nabla f(\vec{c}(t)) \cdot \vec{c}(t)' \\ &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \cdot (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})\end{aligned}$$

which leads to

$$\boxed{\tfrac{dh}{dt} = \tfrac{\partial f}{\partial x}\tfrac{dx}{dt} + \tfrac{\partial f}{\partial y}\tfrac{dy}{dt} + \tfrac{\partial f}{\partial z}\tfrac{dz}{dt}.}$$

Example. Suppose that a particle moves along the path $\vec{c}(t) = (t, 2t, 3t)$ and that the temperature of a point (x, y, z) is given by

$$f(x, y, z) = \cos x + \sin y + \cos z.$$

1. What is the temperature experienced by the particle as a function of time?
   Solution. The temperature experienced by the particle is given by

   $$g(t) = f(\vec{c}(t)) = f(t, 2t, 3t) = \cos(t) + \sin(2t) + \cos(3t).$$

2. Use the Chain Rule to determine the rate of change of the temperature experienced by the partile
   Solution. We want to find $g'(t) = \frac{dg}{dt}$. So,

   $$\begin{aligned}g'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= -\sin(x)(1) + \cos(y)(2) - \sin(z)(3) \\ &= -sin(t) + 2\cos(2t) - 3\sin(3t).\end{aligned}$$

Special Case II

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$. We write

$$g(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$$

and define $h : \mathbb{R}^3 \to \mathbb{R}$ by $h = f \circ g$, i.e.

$$h(x, y, z) = f(g(x, y, z)) = f(u(x, y, z), v(x, y, z), w(x, y, z)).$$

Then

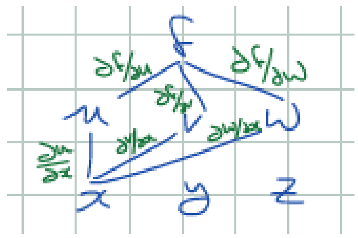
$$(Dh)(x, y, z) = Df|_{(u,v,w)}Dg$$

$$\begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}.$$

What this really means is

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\
\frac{\partial h}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \\
\frac{\partial h}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}.
\end{aligned}$$

An easy way to remember how to get $\frac{\partial h}{\partial x}$ if $h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$.



Informally, there are 3 routes to get from f to x.

$$\frac{\partial h}{\partial x} = \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}}_{\text{route 1}} + \underbrace{\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}}_{\text{route 2}} + \underbrace{\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}}_{\text{route 3}}.$$

Example. Let $u(x, y, z) = x^2 y, v(x, y, z) = y^2, w(x, y, z) = e^{-xz}$, and $f(u, v, w) = u^2 + v^2 - w$. Verify the chain rule for computing $\frac{\partial h}{\partial x}$ where $h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$.
Solution. By the chain rule,

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\
&= (2u)(2xy) + (2v)(0) + (-1)(-ze^{-xz}) \\
&= (2x^2 y)(2xy) + 0 + ze^{-xz} \\
&= 4x^3 y^2 + ze^{-xz}.
\end{aligned}$$

Now we compare it to directly differentiate. Since $h(x, y, z) = (x^2 y)^2 + y^4 + -e^{-xz}$,

$$\frac{\partial h}{\partial x} = 4x^3 y^2 + ze^{-xz}.$$

Example. (Polar coordinates) Let $f(x, y)$ be some function and make the substitution $x = r\cos\theta, y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$.
Solution.

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.$$

Example. Find $\frac{dy}{dx}$, given the implicit equation $x\cos(3y) + x^3 y^5 = 3x - e^{xy}$.
Solution. First, let's rewrite the implicit equation as $x\cos(3y) + x^3 y^5 - 3x + e^{xy} = 0$, and let the LHS be $F(x, y)$. That is,

$$F(x, y) = x\cos(3y) + x^3 y^5 - 3x + e^{xy}.$$

We think of y as a function of x. Since $F(x, y(x)) = 0$, differentiating both sides w.r.t x we obtain

$$\begin{aligned}
\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} &= 0 \\
\frac{\partial y}{\partial x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.
\end{aligned}$$

Since $\frac{\partial F}{\partial x} = \cos(3y) + 3x^2y^5 - 3 + ye^{xy}$ and $\frac{\partial F}{\partial y} = -3x\sin(3y) + 5x^3y^4 + xe^{xy}$,

$$\frac{\partial y}{\partial x} = -\frac{\cos(3y) + 3x^2y^5 - 3 + ye^{xy}}{-3x\sin(3y) + 5x^3y^4 + xe^{xy}}.$$