Rates of change

Consider a function $y = f(x)$. We want to know how much $f(x)$ changes if x changes.

if we change x to $x + \Delta x$

then y change from $f(x)$ to $f(x+\Delta x)$

$\Rightarrow$ The change in y is

$$\Delta y = f(x+\Delta x) - f(x)$$

and the average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

(The average rate of change of f over the interval from x to $x + \Delta x$.)

To define the rate of change of the function f at x, let the length Δx of the interval be smaller and smaller. $\Rightarrow$ the average rate of change over the shorter and shorter time intervals will get closer and closer to some number.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Examples:

1) Acceleration as the rate at which velocity changes

$v(t)$ = velocity at time t (mile per hour)

you want to see how fast your velocity is ~~chag~~ changing.

$\Rightarrow$ you measure it at time t $\Rightarrow$ you get $v(t)$.
than ~~at~~ a little bit later at $t+\Delta t$ $\Rightarrow$ get $v(t+\Delta t)$
$\Rightarrow$ your velocity changed by $\Delta v = v(t+\Delta t) - v(t)$
during a time interval of length Δt.

$\Rightarrow$ ~~averg~~ $\underbrace{\text{average rate at which your velocity changed}}_{\text{average acceleration}} = \dfrac{\Delta v}{\Delta t} = \dfrac{v(t+\Delta t) - v(t)}{\Delta t}$.

And

$\{$ instantaneous acceleration at time t $\}$
$=$ limit of average acceleration.

$$\Rightarrow a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

~~2) To produce x bagels, we need~~
~~$C(x) = 300 + 0.25x + 0.5\left(\frac{x}{1000}\right)^3$ dollars.~~

E.g. A truck enters the off-ramp of a highway at $t=0$. Its position after t seconds is $s(t) = 25t - 0.3t^3$ for $0 \le t \le 5$.

i) How fast is the truck going at the moment it enters the off-ramp?

~~ii) Is the truck speeding up~~

ii) And what is its acceleration at that moment?

Sol. i) $v(t) = s'(t) = 25 - 0.3(3t^2) = 25 - 0.9t^2.$

$\Rightarrow v(0) = 25$ m/s.

ii) $a(t) = v'(t) = -1.8t.$

$\Rightarrow a(0) = 0.$

* **Higher derivatives:**

Recall:

$$f(x) \text{ real function } \xrightarrow{\frac{d}{dx}} f'(x) \text{ real function.}$$

If we iterate:

$$f'(x) \rightsquigarrow f''(x) \rightsquigarrow \dots \rightsquigarrow f^{(n)}(x)$$

$$\frac{df}{dx} \rightsquigarrow \frac{d^2f}{dx^2} \rightsquigarrow \dots \rightsquigarrow \frac{d^nf}{dx^n}$$

Examples:

1) $f(x) = x^3 + 2x^2 + 1.$

$\Rightarrow f'(x) = 3x^2 + 2(2x) + 0. = 3x^2 + 4x.$

$\Rightarrow f''(x) = 3(2x) + 4 = 6x + 4$

$\Rightarrow f'''(x) = 6.$

$f^{(4)}(x) = 0.$

In general, polynomials

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$

$\Rightarrow p^{(n)}(x) = (n!)\, a_n$ (why?).

and $p^{(n+1)}(x) = 0$

Recall: The nth derivative of f is denoted by $f^{(n)}$

E.g. $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$, ...

Leibniz's notation for the nth derivative of $y = f(x)$ is

$$\frac{d^n y}{dx^n} = f^{(n)}(x).$$

E.g. Find $\frac{d^2}{dx^2}(x^2 e^x)$.

$$\frac{d}{dx}(x^2 e^x) = \frac{d}{dx}(x^2) \cdot e^x + x^2 \frac{d}{dx}(e^x)$$

$$= 2xe^x + x^2 e^x$$

$$\frac{d^2}{dx^2}(x^2 e^x) = \frac{d}{dx}(2xe^x + x^2 e^x)$$

$$= \frac{d}{dx}(2xe^x) + \underbrace{\frac{d}{dx}(x^2 e^x)}$$

$$= \frac{d}{dx}(2x) \cdot e^x + (2x)\frac{d}{dx}(e^x) + 2xe^x + x^2 e^x$$

$$= 2e^x + 2x e^x + 2xe^x + x^2 e^x$$

$$= 2e^x + 4xe^x + x^2 e^x$$

Remark: Be careful to distinguish the second derivative from the square of the first derivative.

$$\frac{d^2 y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2.$$

2) Consider $f(x) = \dfrac{1}{1-x}$.

$$\Rightarrow \quad f'(x) = \frac{1}{(1-x)^2}$$

$$\Rightarrow \quad f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

3) Consider $f(x) = x e^x$. Find $f^{(n)}(x)$?

$$f'(x) = e^x + xe^x$$

$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x.$$

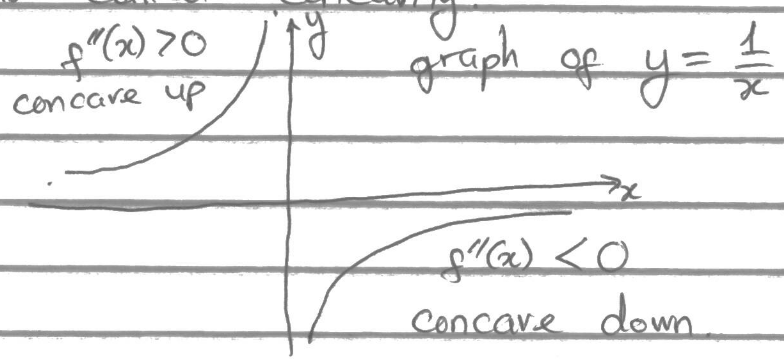
$$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x.$$

$$\Rightarrow \quad f^{(n)}(x) = ne^x + xe^x.$$

- Geometry of $f''(x)$:

$f''(x)$ tells how fast the slope of the tangent line to $y = f(x)$ is changing.

That is, how fast the graph is bending. This property is called concavity.



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Differentiating Trigonometric Functions

Sine, Cosine and tangent are differentiable.

1) $\dfrac{d\sin x}{dx} = \cos(x)$

2) $\dfrac{d\cos x}{dx} = -\sin(x)$ (Note the minus sign).

3) $\dfrac{d\tan x}{dx} = \dfrac{1}{\cos^2(x)} = \sec^2 x$

pf of (3): Recall that $\tan x = \dfrac{\sin x}{\cos x}$

$$\Rightarrow \frac{d\tan x}{dx} \underset{\substack{\uparrow \\ \text{quotient rule}}}{=} \frac{(\sin x)'\cos x - (\sin x)(\cos x)'}{\cos^2 x}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

E.g. 1) Find the derivative of the following function:

1) $f(x) = x\cos(x) - \sin x$.

Sol. $f'(x) = (x)'\cos(x) + x(\cos x)' - (\sin x)'$

$= \cos x + x(-\sin x) - \cos x$

2) $g(x) = \cot(x) = \dfrac{\cos x}{\sin x}$

$$g'(x) = \frac{(\cos x)' \sin x - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

$$\Rightarrow \boxed{\frac{d}{dx} \cot x = -\csc^2 x.}$$

We also have:

$$\boxed{\begin{aligned} \frac{d}{dx} \sec x &= \sec x \tan x. \\ \frac{d}{dx} \csc x &= -\csc x \cot x. \end{aligned}}$$

E.g. Find a and b so that the function

$$f(x) = \begin{cases} \cos x & \text{for } x \leq \frac{\pi}{4} \\ a + bx & \text{for } x > \frac{\pi}{4} \end{cases}$$

a) is continuous at $x = \frac{\pi}{4}$

b) is differentiable at $x = \frac{\pi}{4}$.

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Sol. a) Need: $\lim_{x\to\frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} f(x) = \frac{\sqrt{2}}{2}.$$

That is, $\lim_{x\to\frac{\pi}{4}^+} f(x) = \frac{\sqrt{2}}{2}$ and $\lim_{x\to\frac{\pi}{4}^-} f(x) = \frac{\sqrt{2}}{2}$.

$$\Rightarrow \lim_{x\to\frac{\pi}{4}^+} a + bx = \frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{x\to\frac{\pi}{4}^-} \cos(x) = \frac{\sqrt{2}}{2} \checkmark.$$

Need: $a + b\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ ~~or~~

or $a = \frac{\sqrt{2}}{2} - b\frac{\pi}{4}$.

b) Need: $a = \frac{\sqrt{2}}{2} - b\frac{\pi}{4}$ (to make sure f is cts. at $x=\frac{\pi}{4}$).

and $\lim_{x\to\frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$ exists.

$$\Rightarrow \lim_{x\to\frac{\pi}{4}^+} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}^+} \frac{a + bx - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$$

$$\cancel{= \lim \frac{\sqrt{2}}{2} - b\frac{\pi}{4} x - \frac{\sqrt{2}}{2}}$$

$$= \lim_{x\to\frac{\pi}{4}^+} \frac{\frac{\sqrt{2}}{2} - b\frac{\pi}{4} + bx - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$$

$$= \lim_{x\to\frac{\pi}{4}^+} \frac{b\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$$

$$= b.$$

and $\lim_{x\to\frac{\pi}{4}^-} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}^-} \frac{\cos x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}} = -\sin\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

$$\Rightarrow b = \frac{-\sqrt{2}}{2} \quad \text{and} \quad a = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\pi}{4}.$$

* **Chain Rule:** to find derivatives of composition functions.

Composition functions:

$$x \longrightarrow \boxed{\ f\ } \longrightarrow f(x) \longrightarrow \boxed{\ g\ } \longrightarrow g(f(x)).$$

The composition of g and f is the function $g\circ f$ defined by $\cancel{f(}$ $(g\circ f)(x) = g(f(x))$.

E.g. Find $f\circ g$ and $g\circ f$ where $f(x) = \sqrt{x}$ and $g(x) = 1-x$.

a) $(f\circ g)(x) = f(g(x)) = f(1-x) = \sqrt{1-x}$

b) $(g\circ f)(x) = g(f(x)) = g(\sqrt{x}) = 1-\sqrt{x}$.

Chain rule:

Suppose $g(x)$ is differentiable at $x=a$
and $f(x)$ is differentiable at $x=g(a)$
Then $(f\circ g)(x) := f(g(x))$ is differentiable at $x=a$
and

$$\boxed{(f\circ g)'(x) = f'(g(a))\, g'(a)}$$

In Leibniz notation:

$$\frac{d(f\circ g)}{dx}(a) = \frac{d(f\circ g)}{dx}\,\frac{dg}{dx}(a).$$

Examples:

1) $f(x) = (2x+1)^2$

Note: cannot use power rule directly, i.e.,

$$f'(x) \neq 2(2x+1)^1.$$

Why? Let's expand $f(x)$ first:

$$f(x) = 4x^2 + 4x + 1.$$

$$\Rightarrow f'(x) = 8x + 4 \neq 2(2x+1).$$

Now, use the chain rule:

$f(x) = h(g(x))$ where $h(x) = x^2 \Rightarrow h'(x) = 2x$

and $g(x) = 2x+1 \Rightarrow g'(x) = 2$

chain rule

$$\Rightarrow f'(x) = \underbrace{h'(g(x))}\,\underbrace{g'(x)}.$$

$$= 2(g(x)) \cdot 2.$$

$$= 2(2x+1) \cdot 2.$$

$$= 4(2x+1).$$

2) $y = e^{8x+9}$.

Let $f(x) = e^x \Rightarrow f'(x) = e^x$.

$h(x) = 8x+9 \Rightarrow h'(x) = 8$.

and $y = f(h(x)) = \cancel{(f\circ g)}\ (f \circ h)(x)$.

$\Rightarrow$ Chain rule:

$$\frac{dy}{dx} = (f\circ h)'(x) = f'(h(x))\, h'(x)$$

$$= e^{h(x)} \cdot 8$$

$$= e^{8x+9} \cdot 8.$$

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3) $f(x) = \tan(x)\cos(e^{x})$.

$$f'(x) = \frac{d}{dx}(\tan(x)) \cdot \cos(e^{x}) + \tan(x) \cdot \frac{d}{dx}(\cos(e^{x}))$$

$$= \sec^2(x)\,\cancel{\tan(x)}\,\cos(e^{x}) + \tan(x)\left[-\cancel{\cos}\,\sin(e^{x})\cdot e^{x}\right]$$

$$= \sec^2(x)\cos(e^{x})\ \cancel{+} - \tan(x)\sin(e^{x})\cdot e^{x}.$$

4) $y = \sqrt{\sqrt{x+1} + 1}$

Let $f(x) = x+1$.

and $g(x) = \sqrt{x}$.

Then $y = \sqrt{\sqrt{f(x)} + 1} = \sqrt{g(f(x)) + 1}$

$= \sqrt{(g\circ f)(x) + 1}$

$= \sqrt{(f\circ g\circ f)(x)}$

$= (g\circ f\circ g\circ f)(x)$.

4) $y = \sqrt{x^2+1}$ $\cancel{\neq}$ $\cancel{x}$.

Let $f(x) = x^2 + 1$. and $g(x) = \sqrt{x}$.

Then $y = (g\circ f)(x)$

$$\Rightarrow \quad \frac{dy}{dx} = g'(f(x))\, f'(x).$$

$$= \frac{1}{2}\,[f(x)]^{-1/2}\,(2x)$$

$$= \frac{1}{2}\,(x^2+1)^{-1/2}\,(2x)$$

$$= \frac{x}{\sqrt{x^2+1}}.$$

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5) $y = \cos^2(5t)$.

$$\begin{cases} f(t) = t^2 \\ g(t) = \cos(t) \\ h(t) = 5t \end{cases} \rightarrow \begin{cases} f'(t) = 2t \\ g'(t) = -\sin(t) \\ h'(t) = 5. \end{cases}$$

and $y = (f \circ g \circ h)(t)$.

$$\Rightarrow \frac{dy}{dt} = (f \circ g \circ h)'(t).$$

$$= f'((g \circ h)(t)) \cdot (g \circ h)'(t).$$

$$= f'((g \circ h)(t))\ g'(h(t))\ h'(t).$$

$$= 2(\cos(5t))(-\sin(5t))\ 5.$$

$$= -10 \cos(5t) \sin(5t).$$