HW03

Due Week 5 and 6

Reading: fundamental theorem of linear algebra, orthogonality, projections.

1. Suppose $P \in \mathbb{R}^{m\times m}$ is a projector. Show that $\mathrm{Null}(I-P) = \mathrm{range}(P)$.
2. Consider the matrices
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1,2,3)^T$?
3. Consider $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathbb{R}^2$. Show that P is an orthogonal projector onto the x-axis, and its complementary is an orthogonal projector onto y-axis.
4. Show that for any real invertible matrix A, A^TA is a positive definite matrix.
5. Let $M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Show that M is a positive definite matrix.
6. Let $V = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.
   (a) Find the orthogonal projector onto the subspace V.
   (b) Project $\begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$ onto V.
   (c) Recall that the *reconstruction error* is the distance between the original data point and its projection onto a lower-dimensional subspace. What is the reconstruction error of the vector $\begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$?
7. Consider 3 data points $(-2,-2), (0,0)$, and $(2,2)$.
   (a) What is the first principal component?
   (b) If we project the original data points into the 1-D subspace by the principal you choose, what are their coordinates in the 1-D subspace? What is the variance of the projected data?
   (c) For the projected data you just obtained above, now if you represent them in the original 2-D space and consider them as the reconstruction of the original data points, what is the reconstruction error?
8. Let $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 1, -1, 1 \end{bmatrix}$, and $\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Let
$$f(\alpha_1, \alpha_2) = \sum_{i=1}^{3} \|\boldsymbol{x}_i - \alpha_1 \boldsymbol{b}_1 - \alpha_2 \boldsymbol{b}_2\|_2^2,$$

where $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{b}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Find $\frac{\partial J}{\partial \alpha_1}$ and $\frac{\partial J}{\partial \alpha_2}$.