71

In general, • if $c \neq 0$ and $|r| < 1$.

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \dots = \frac{c}{1-r}$$

and $$\sum_{n=M}^{\infty} cr^n = cr^M + cr^{M+1} + \dots = \frac{cr^M}{1-r}.$$

• If $|r| \geq 1$, then the geometric series diverges.

E.g. Evaluate $S = \sum_{n=1}^{\infty} \frac{2+3^n}{5^n}$.

Sol. Write S as a sum of two geometric series.

$$S = \sum_{n=1}^{\infty} \frac{2+3^n}{5^n} = \sum_{n=1}^{\infty} \frac{2}{5^n} + \sum_{n=1}^{\infty} \frac{3^n}{5^n}.$$

$$= \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$$

$$= 2 \cdot \frac{\frac{1}{5}}{1-\frac{1}{5}} + \frac{\frac{3}{5}}{1-\frac{3}{5}}.$$

$$= \frac{2}{5} \cdot \frac{1}{\frac{4}{5}} + \frac{3}{5} \cdot \frac{1}{\frac{2}{5}}$$

$$= \frac{1}{2} + \frac{3}{2}$$

$$= 2.$$

• Divergence Test.

Recall that $$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \underbrace{\sum_{n=1}^{N} a_n}_{S_N}.$$

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

E.g. Show that $\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2}$ diverges.

Sol. Since $\lim_{n\to\infty} \frac{n^2+1}{3n^2} = \frac{1}{3} \neq 0$,

$\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2}$ diverges by the divergence test.

Remark: If $\lim_{n\to\infty} a_n = 0$, this does not mean that $\sum_{n=1}^{\infty} a_n$ necessarily ~~div~~ converges. It could either converge or diverge.

E.g. Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverge or converge ?

Sol. Even though $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$,

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

$$S_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}}$$

$$\geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}}$$

$$= \sqrt{N}$$

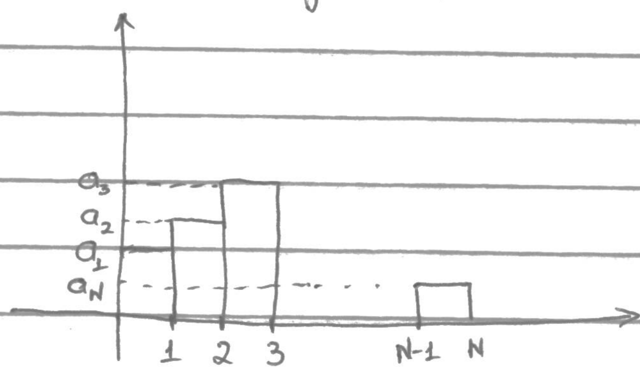
$$\Rightarrow \quad S_N \geq \sqrt{N} \quad \Rightarrow \quad \lim_{N\to\infty} S_N = \infty .$$

10.3. Convergence of Series with positive terms.

Let $a_n \geq 0$ for all n.

Idea: $S_N = a_1 + a_2 + \dots + a_N$
$= a_1 \cdot 1 + a_2 \cdot 1 + \dots + a_N \cdot 1$

Think of these as "areas of rectangles of height a_n & width 1.



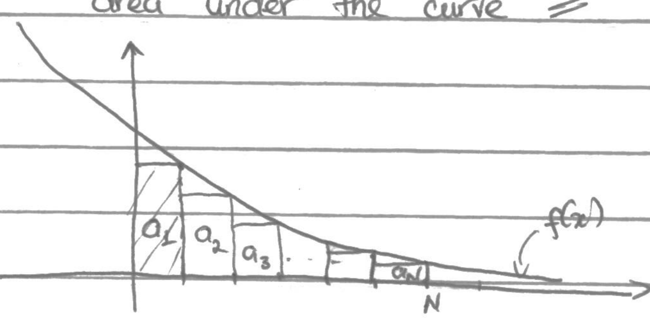
So S_N is the area of the first N rectangles and S_{N+1} is the area of the first $N+1$ rectangles.

$\Rightarrow \quad S_N \leq S_{N+1}$

- Integral test: let $a_n = f(n)$ where $\begin{cases} f(x) \geq 0 \\ f(x) \text{ decreasing} \\ f(x) \text{ continuous.} \end{cases}$

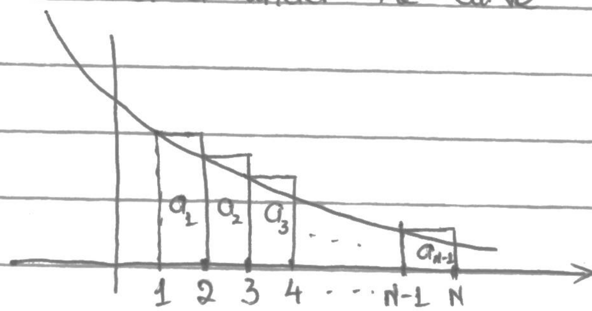
i) If $\underbrace{\int_1^\infty f(x)\,dx \text{ converges}}_{\text{area under the curve}}$, then $\sum_{n=1}^{\infty} a_n$ converges.

area under the curve $\geq$ areas of rectangles.



ii) If $\underbrace{\int_1^\infty f(x)\,dx}_{\text{area under the curve}}$ diverges, then $\underbrace{\sum_{n=1}^\infty a_n}_{\text{area of rectangles}}$ diverges.

area under the curve $\leq$ area of rectangles.



E.g. Does the harmonic series $\sum_{n=1}^\infty \underbrace{\frac{1}{n}}_{a_n}$ converge or diverge?

Sol. let $f(x) = \frac{1}{x}$, which is positive, continuous, and decreasing.

and $\int_1^\infty \frac{1}{x}\,dx = \infty$

$\Rightarrow \sum_{n=1}^\infty \frac{1}{n}$ diverges by the integral test.

Thm: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p>1$ and diverges otherwise.

E.g. $\sum_{n=1}^\infty \frac{1}{n^3}$ converges and $\sum_{n=1}^\infty \frac{1}{n^{2/3}}$ diverges.

~~Sol. let f(x) =~~

E.g. Does $\sum_{n=1}^\infty \frac{n}{(n^2+1)^2}$ converges?

Sol. Let $f(x) = \frac{x}{(x^2+1)^2}$,

$f(x) \geqslant 0$

$f(x)$ continuous

decreasing (why?)

$(f'(x) < 0$ for $x \geqslant 1)$

75

$$\int_1^\infty \frac{x}{(x^2+1)^2}\,dx \overset{u=x^2+1}{=} \int \ldots = \frac{1}{4}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges by the integral test.

Thm: (Comparison Test) Assume that $0 \le a_n \le b_n$ for $n \ge M$

i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

E.g. Does $\sum_{n=1}^{\infty} \frac{(0.9)^n}{\sqrt{n}}$ converge?

Sol: $0 \le \underbrace{\frac{0.9^n}{\sqrt{n}}}_{a_n} \le \underbrace{0.9^n}_{b_n}$ for $n \ge 1$.

and $\sum_{n=1}^{\infty} 0.9^n = \frac{0.9}{1-0.9} = 9$ converges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{0.9^n}{\sqrt{n}}$ converges by the comparison test.

E.g. Does $\sum_{n=3}^{\infty} \frac{1}{(n^2+7)^{1/3}}$ converge?

Sol. Idea: $\sum_{n=3}^{\infty} \frac{1}{(n^2+7)^{1/3}}$ feels like a p-series with $p < 1$.

Observe that

$$7 < n^2 \quad \text{for} \quad n \ge 3.$$

$$\Rightarrow n^2 + 7 < 2n^2$$

$$(n^2+7)^{1/3} < 2^{1/3} n^{2/3}.$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{1/3}\, n^{2/3}}$ diverges

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{(n^2+7)^{1/3}}$ diverges by the comparison test.

- **Limit comparison test:**

Let $a_n \geq 0$ and $b_n \geq 0$ for all n.
and suppose that the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \quad \text{exists.}$$

Then,

1) If $L > 0$, $\sum a_n$ converges if and only if $\sum b_n$ converges.

2) If $\cancel{L=0}$ $L=\infty$, and $\sum a_n$ converges, then $\sum b_n$ converges.

3) If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

E.g. let's revisit $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{(n^2+7)^{1/3}}}_{a_n}$

Let $b_n = \dfrac{1}{n^{2/3}}$.

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^2+7)^{1/3}}{1/\,n^{2/3}} = \lim_{n\to\infty} \frac{n^{2/3}}{(n^2+7)^{1/3}} = 1.$

Moreover, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges,

$\Rightarrow \displaystyle\sum \frac{1}{(n^2+7)^{1/3}}$ diverges.

77

10.4 Absolute and Conditional convergence.

Def: Absolute convergence

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

E.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ ~~is~~ converges absolutely.

because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Thm: $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges.

E.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges.

E.g. How about $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$?

Does it converge absolutely?

No! Because

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$.

It diverges.

$\Rightarrow$ $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ does not converge absolutely.

But does $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ converge?

Def: Conditional Convergence

~~$\sum |a_n|$~~ $\sum a_n$ converges conditionally

if $\sum a_n$ converges but $\sum |a_n|$ diverges.

<u>Thm:</u> (Alternating series test)

$\{b_n\}$, $b_n > 0$ and decreasing and converges to 0

$b_1 > b_2 > b_3 > \dots > 0$, $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

<u>E.g.</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

Let $b_n = \frac{1}{\sqrt{n}}$, then $b_n > 0$ ~~and~~

b_n is decreasing.

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

then $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the alternating series test.

but $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right|$ diverges,

$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ <u>converges conditionally</u>.

$\sum a_n$	$\sum \lvert a_n \rvert$	Conclusion	Examples.
conv.	conv.	abs. conv.	$\sum \frac{(-1)^n}{n^2}$ and $\sum \frac{1}{n^2}$.
conv.	div.	conditional conv.	$\sum \frac{(-1)^n}{\sqrt{n}}$ and $\sum \frac{1}{\sqrt{n}}$.
div.	conv.	never happen.	
div.	div.		

10.5. Ratio and Root tests.

Thm (Ratio Test): Assume that the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

1) If $\rho < 1$, then Σa_n converges absolutely.
2) If $\rho > 1$, then Σa_n diverges.
3) If $\rho = 1$, the test is inconclusive (Σa_n may conv. or di

E.g. Does $\sum_{n=1}^{\infty} \frac{7^n}{n!}$ converge?

Tip: seeing a factorial should tell you to try the ratio test.

Sol: $a_n = \frac{7^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{7^{n+1}/(n+1)!}{7^n/n!}\right| = \lim_{n\to\infty}\frac{7}{n+1} = 0$$

So $\sum_{n=1}^{\infty}\frac{7^n}{n!}$ converges abs. by the ratio test.

E.g. Does $\sum_{n=1}^{\infty}(-1)^n\frac{n!}{3^n}$ converge?

Sol. $a_n = (-1)^n\frac{n!}{3^n}$

So $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!/3^{n+1}}{n!/3^n}\right| = \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

$\Rightarrow$ the series diverges.

E.g. What does the ratio test say about $\sum_{n=1}^{\infty} n^{-2}$? How about $\sum_{n=1}^{\infty} n^2$?

Sol. The ratio test is inconclusive in both cases (why?). But $\sum_{n=1}^{\infty} n^{-2}$ converges (why?) and

$\sum_{n=1}^{\infty} n^2$ diverges (why?)

Thm (Root test) Assume that the limit

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} \text{ exists.}$$

1) If $L<1$, then $\sum a_n$ converges absolutely.
2) If $L>1$, then $\sum a_n$ diverges.
3) If $L=1$, the test is inconclusive.

E.g. Does $\sum_{n=1}^{\infty} \left(\frac{2n+1}{5n+7}\right)^n$ converge?

(Tip: when you see nth-power, try root test).

Sol: $a_n = \left(\frac{2n+1}{5n+7}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+1}{5n+7} = \frac{2}{5} < 1.$$

$\Rightarrow$ Conv. by root test.

E.g. Does $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n}$ conv. or div. ?

Sol. $a_n = \left(1+\frac{1}{n}\right)^{-n}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-1} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

$\Rightarrow$ inconclusive.

E.g. Does $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ converge?

Sol. $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} < 1.$$

$\Rightarrow$ conv.

Section 10.6: Power Series.

A power series with center c is an infinite series.

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x\cancel{+}-c)^2 + \dots$$

E.g. $F(x) = 1 + 2(x-7) + 3(x-7)^2 + \cdots$

$$= \sum_{n=0}^{\infty} (n+1)(x-7)^n$$

Since a power series is a series that depends on x, it may converge for certain values of x and diverge for others

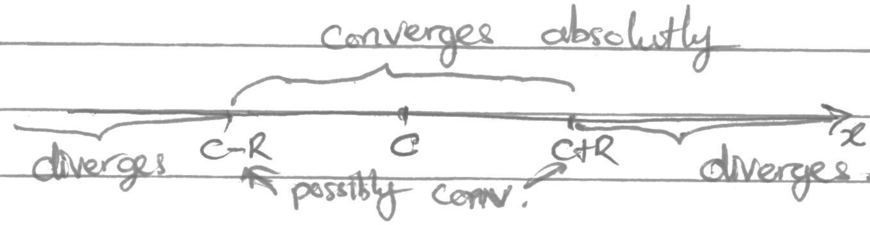
E.g. $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$ ~~and diverges~~
diverges when $|x| \geq 1$.
(why?)

Thm: (Radius of convergence)
Every Power Series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of convergence R, where either $R \geq 0$ or $R = \infty$.

- If R is finite, $F(x)$ converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$.
- If $R = \infty$, then $F(x)$ converges absolutely for all x.



E.g. $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (x-0)^n$

$a_n = 1$ $\quad c = 0$.

converges when $|x-0| < 1$ and diverges otherwise.

Exercise: Read the proof in the book.

Finding the interval of convergence of $F(x)$.

Step 1: Find the Radius of convergence R.
(usually with ratio test).

Step 2: Check the endpoints (when $R \neq 0$ and $R \neq \infty$)

E.g. Where does $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

Sol: Let $b_n = \frac{x^n}{2^n}$ and compute.

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/2^{n+1}}{x^n/2^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{2}\right|$$

$$= \left|\frac{x}{2}\right|$$

So $\rho < 1$ whenever $\frac{|x|}{2} < 1 \iff |x| < 2$.

$\Rightarrow$ $F(x)$ converges when $|x| < 2$ and diverges when $|x| > 2$.

$\Rightarrow$ Radius of conv. $R = 2$.

2) Check the endpoints $x = \pm 2$.

$F(2) = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1 \Rightarrow$ div.

$F(-2) = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n \Rightarrow$ div.

$\Rightarrow$ $F(x)$ conv. when $|x| < 2$.

E.g. Where does

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n} (x-5)^n \quad \text{converge?}$$

Sol: Step 1. Let $a_n = \frac{(-1)^n}{4^n n} (x-5)^n$

and ~~let~~ consider $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \to \infty} \left| \frac{(x-5)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(x-5)^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(x-5)n}{4(n+1)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{n}{\cancel{4}(n+1)} \right| \cdot \frac{|x-5|}{4}$$

$$= \frac{|x-5|}{4}.$$

To have $\rho < 1$, $\frac{|x-5|}{4} < 1 \iff |x-5| < 4$.

So $F(x)$ conv. absolutely when $1 < x < 9$

and diverges when $x > 9$ and $x < 1$.

Step 2: At the end points.

$$F(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n} \cdot (-4)^n = \sum \frac{1}{n} \to \text{div.}$$

$$F(9) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \to \text{conv. (why?)}$$

E.g. Show that $\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$ when $|x| < 1/3$.

Sol: Since $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ when $|r| < 1$.
we can plug in $3x$ in place of r to get

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n \quad \text{when } |x| < \frac{1}{3}.$$

E.g. Find a power series with center $c=0$ for $F(x) = \frac{1}{2+x^2}$.
Sol Again $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$

$$-r = \frac{x^2}{2} \Rightarrow r = -\frac{x^2}{2}$$

$$F(x) = \frac{1}{2} \cdot \frac{1}{1+\frac{x^2}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{2n} \quad \text{when } \left|\frac{x^2}{2}\right| < 1 \Leftrightarrow |x| < \sqrt{2}.$$

so $R = \sqrt{2}$.

E.g. ~~Ex~~ Find the interval ~~of~~ and radius of convergence:

$$\sum_{n=2}^{\infty} \frac{(x-7)^n}{n\, 10^n}.$$

Sol: Ratio test: $\lim_{n\to\infty} \left| \frac{(x-7)^{n+1}}{(n+1)10^{n+1}} \cdot \frac{n\, 10^n}{(x-7)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-7)}{10} \frac{n}{n+1} \right|$

$$= \left| \frac{x-7}{10} \right| < 1.$$

$$\Rightarrow |x-7| < 10 \Rightarrow \text{radius} = 10.$$

Check endpoints: $|x-7| < 10 \Leftrightarrow -3 < x < 17$.

$x = 17 \Rightarrow \sum_{n=2}^{\infty} \frac{10^n}{n 10^n} = \sum_{n=2}^{\infty} \frac{1}{n}$ div.

$x = -3 \Rightarrow \sum_{n=2}^{\infty} \frac{(-10)^n}{n 10^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ conv. by alt. test

$\Rightarrow$ interval of conv: $-3 \leq x < 17$.

* Differentiation and Integration:

Say $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has a radius of convergence R. Then:

$F(x)$ is differentiable on $(c-R, c+R)$. Furthermore, we can integrate and differentiate $F(x)$ term-by-term.

So, for $c - R < x < c + R$.

$$\int F(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$$

and
$$F'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

and these series have the same radius of convergence.

E.g. Find a power series for $\frac{1}{(1-x)^2}$ where $-1 < x < 1$. What is its radius of convergence?

Sol:
$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

and
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \cancel{\frac{x^{n+1}}{n+1}}\; n x^{n-1}.$$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$ with radius of convergence $R = 1$.

E.g. Find a power series for $\tan^{-1} x$, $-1<x<1$.

Sol: We know that $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt$.

and $\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$

$$\Rightarrow \tan^{-1} x = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt.$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Radius of convergence: same as for $\sum_{n=0}^{\infty} (-1)^n t^{2n}$ which is 1.

Section 10.7 Taylor Series.

Goal: We want to a general method for finding power series.

Recall: If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for $c-R < x < c+R$

then we can compute the derivative of $f(x)$ by differentiating term by term.

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots$$

$$\Rightarrow f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots$$

$$f''(x) = 0 + 2a_2 + 2\cdot 3 a_3(x-c) + (3)(4)a_4(x-c)^2 + \cdots$$

$$\cancel{f'''(x) = 2\cdot 3!\, a_3 +}$$

$$f'''(x) = 3!\, a_3 + 2\cdot 3\cdot 4\, a_4(x-c) + 3\cdot 4\cdot 5\, a_5(x-c)^2 + \cdots$$

$\Rightarrow$ In general,

$$f^{(k)}(x) = k!\, a_k + 2\cdot 3 \cdots (k+1) a_{k+1}(x-c) + \cdots$$

$\uparrow$ kth derivative.

$$\Rightarrow \quad f^{(k)}(c) = k!\, a_k \quad \Rightarrow \quad \boxed{a_k = \frac{f^{(k)}(c)}{k!}}$$

Since we ~~start~~ started with

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n,$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n} \leftarrow \text{Taylor series of } f(x) \text{ centered at } c.$$

That is,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

Sometimes, we denote the Taylor series by $T(x)$.

Special case: (Maclaurin series)
In the special case $c = 0$, $T(x)$ is called the Maclaurin series.

Warning: If you start with a function $f(x)$ and you compute $T(x)$, there is no guarantee that $T(x)$ converges to $f(x)$.

Thm: (Taylor series Expansion)
If $f(x)$ is represented by a power series centered at c, with $|x-c| < R$, then that power series is the Taylor series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

E.g. Find the Taylor series for $f(x) = x^{-3}$ at $x=1$.
Sol. 1) Find the derivatives

$$f'(x) = -3x^{-4}$$
$$f''(x) = (-3)(-4)\,x^{-5}$$
$$f'''(x) = (-3)(-4)(-5)\,x^{-6}$$
$$f^{(n)}(x) = (-3)(-4)(-5)\ldots(-n-2)\,x^{-(n+3)}$$
$$= (-1)^n \frac{(n+2)!}{2} x^{-(n+3)}$$

2) Evaluate at c:

$$f^{(n)}(1) = (-1)^n \frac{(n+2)!}{2}$$

and find $a_n = \dfrac{f^{(n)}(1)}{n!} = \dfrac{(-1)^n (n+2)!}{n!\;2} = \dfrac{(-1)^n(n+1)(n+2)}{2}$

3) Write ~~the~~ $T(x)$

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (x-1)^n$$

But how do we know if $T(x) = f(x)$?

Define the kth partial sum

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

and the Remainder $R_k(x) = f(x) - T_k(x)$.

$T(x)$ converges to $f(x)$ if and only if

$$\lim_{k \to \infty} R_k(x) = 0.$$

Thm: Let $I = (c-R, c+R)$, where $R>0$. Suppose there exists $K>0$ such that all derivatives of f are bounded by K on I:

$$|f^{(k)}(x)| \leq K \quad \text{for all } k \geq 0 \text{ and } x \in I.$$

Then f is represented by its Taylor series in I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \quad \text{for all } x \in I.$$

E.g. Expansions of Sine and Cosine.

1) $f(x) = \sin x$.

	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$x=0 \Rightarrow$	0	1	0	-1	0

In other words, the even derivatives are zero and the odd derivatives alterate in sign

$$f^{(2n+1)}(0) = (-1)^n$$

Therefore, $a_{2n} = 0$

and $a_{2n+1} = \dfrac{(-1)^n}{(2n+1)!}$

$$\Rightarrow \quad T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Since $|f^{(n)}(x)| \leq 1$, the Taylor series converges to $f(x)$. That is,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Similarly,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

E.g. Find the Maclaurin series for $f(x) = e^x$.

$f'(x) = e^x$

$f''(x) = e^x$

$\Rightarrow f^{(n)}(x) = e^x.$

$\Rightarrow f^{(n)}(0) = 1.$

and $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{n!}.$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

E.g. The $\frac{\sin t}{t}$ is difficult to find its antiderivative. But its power series can help us.

Let $F(x) = \int_0^x \frac{\sin t}{t}\, dt$. Show that

$$F(x) = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots$$

Sol. $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}$

$$\Rightarrow F(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{t^{2n+1}}{t}\, dt$$

$$= \sum_{n=0}^{\infty} \cancel{\int_0^x} \frac{(-1)^n}{(2n+1)!} \int_0^x t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left.\frac{t^{2n+1}}{2n+1}\right|_{t=0}^{t=x}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3\cdot 3!} + \cdots$$

Some important functions:

	Maclaurin series	radius / interval of conv.
e^x	?	
$\sin x$	?	
$\cos x$		
$\frac{1}{1-x}$		
$\ln(1+x)$		
$\tan^{-1} x$.		