Lecture 15: QR decomposition and Fourier Series (Section 3.4)

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Gram-Schmidt

- Gram-Schmidt orthonormalization:
  - Input: basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ for V
  - Output: orthonormal basis $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ for V.

$$
\begin{aligned}
\boldsymbol{b}_1 &= \boldsymbol{a}_1, & \boldsymbol{q}_1 &= \frac{\boldsymbol{b}_1}{\|\boldsymbol{b}_1\|} \\
\boldsymbol{b}_2 &= \boldsymbol{a}_2 - \langle \boldsymbol{a}_2, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1, & \boldsymbol{q}_2 &= \frac{\boldsymbol{b}_2}{\|\boldsymbol{b}_2\|} \\
\boldsymbol{b}_3 &= \boldsymbol{a}_3 - \langle \boldsymbol{a}_3, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 - \langle \boldsymbol{a}_3, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2, & \boldsymbol{q}_3 &= \frac{\boldsymbol{b}_3}{\|\boldsymbol{b}_3\|} \\
&\vdots
\end{aligned}
$$

The QR decomposition

Let A be an $m \times n$ matrix of rank n. Then we have the **QR decomposition** $A = QR$,

- where Q is $m \times n$ and has orthonormal columns, and
- R is upper triangular, $n \times n$ and invertible.

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▶ Example. Find the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

The QR decomposition

- We apply Gram-Schmidt to the columns of A:

$$\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{q}_1,$$

$$\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \left\langle \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \boldsymbol{q}_1 \right\rangle \boldsymbol{q}_1 = \boldsymbol{q}_1 \Longrightarrow \boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{b}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \left\langle \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \boldsymbol{q}_1 \right\rangle \boldsymbol{q}_1 - \left\langle \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \boldsymbol{q}_2 \right\rangle \boldsymbol{q}_2 \Longrightarrow \boldsymbol{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

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$$Q = [\ \boldsymbol{q}_1\ \boldsymbol{q}_2\ \boldsymbol{q}_3\ ] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Why does this process ensure R is an upper triangular matrix?

Practice Problems

- Example. Complete $\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}-2\\-1\\2\end{bmatrix}$ to an orthonormal basis of $\mathbb{R}^3$.

  a) by using the FTLA to determine the orthogonal complement of the span you already have
  b) by using Gram-Schmidt after throwing in an independent vector such as $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

- Example. Find the QR decomposition of $A = \begin{bmatrix}1 & 1 & 2\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$.

The QR decomposition

▶ Example. The QR decomposition is very useful for solving least squares problems:

$$\begin{aligned} A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b} &\Longleftrightarrow (QR)^T QR\hat{\boldsymbol{x}} = (QR)^T\boldsymbol{b} \\ &\Longleftrightarrow R^T R\hat{\boldsymbol{x}} = R^T Q^T\boldsymbol{b} \end{aligned}$$

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$\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $R\hat{\boldsymbol{x}} = Q^T\boldsymbol{b}$, where $A = QR$.

Application: Fourier series

Given an orthogonal basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots,$ we express a vector $\boldsymbol{x}$ as

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots, \text{ where } c_i\boldsymbol{v}_i = \frac{\langle \boldsymbol{x}, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle}\boldsymbol{v}_i.$$

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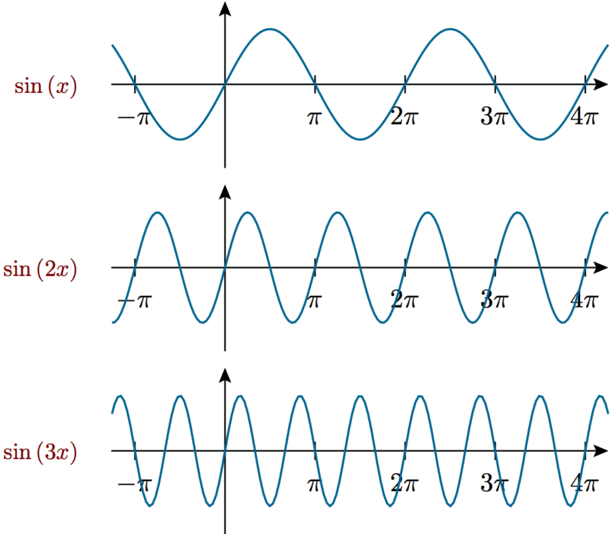
A **Fourier series** of a function $f(x)$ is an infinite expansion:

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots$$

Application: Fourier series

- Example.



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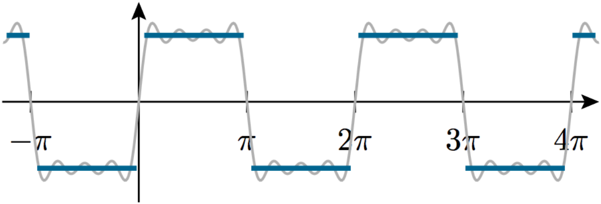
- Example.

$$\text{blue function} = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \dots\right)$$



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- We are working in the vector space of functions $\mathbb{R} \to \mathbb{R}$.
  - More precisely, "nice" (e.g., piecewise continuous) functions that have period 2π.
  - There are infinite dimensional vector spaces.

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- The functions

$$1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots$$

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- Functions: $\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$.
- Example. Show that $\cos(x)$ and $\sin(x)$ are orthogonal.
- Example. What is the norm of $\cos(x)$?

Application: Fourier series

Fourier series of $f(x)$:

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots$$

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How can we find a_k and b_k?

$$a_0 = \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,$$

$$a_k = \frac{\langle f(x), \cos(kx)\rangle}{\langle \cos(kx), \cos(kx)\rangle} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx,$$

$$b_k = \frac{\langle f(x), \sin(kx)\rangle}{\langle \sin(kx), \sin(kx)\rangle} = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx,$$

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▶ Example. Find the Fourier series of the 2π-periodic function $f(x)$ defined by

$$f(x) = \begin{cases} 1, & \text{for } x \in (0, \pi), \\ -1, & \text{for } x \in (\pi, 2\pi). \end{cases}$$