(15)

Last time

$$\vec{x}' = A\vec{x}$$

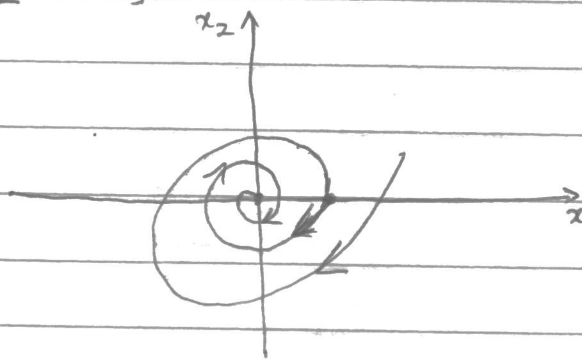
where A has complex eigenvalues.

$$\lambda = \alpha \pm \beta i$$

α: Decay/grow, βi: Rotation (oscillation).

* $A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix} \Rightarrow$ eigenvalues $\lambda = -2 \pm 3i$.

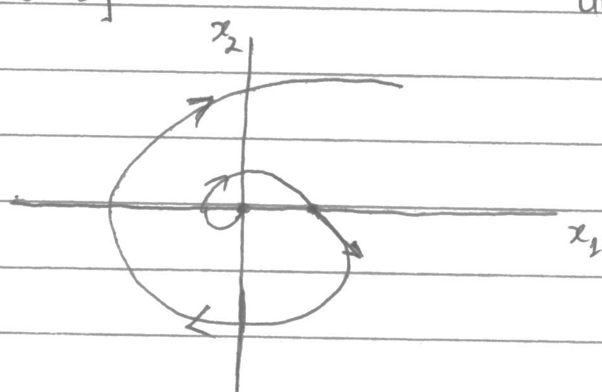
$\Downarrow$ $\boxed{\text{stable spiral}}$



$$\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

* $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \Rightarrow$ eigenvalues $\lambda = 2 \pm 3i$.

unstable spiral



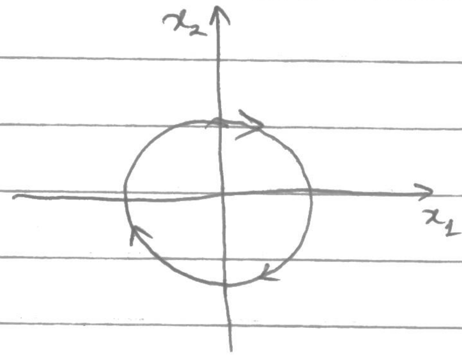
$$\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

(7b)

* $A = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ $\Rightarrow$ eigenvalues $\lambda = \pm\sqrt{5}\,i$.

real part of λ is 0

$\Rightarrow$ center critical point.



(19)

Section 7.8: Repeated Eigenvalues.

Consider $\vec{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \vec{x}$

Find eigenvalues:

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda) + 1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0.$$

$$\lambda = 3 \quad \text{(repeated)}$$

$\Rightarrow$ $\lambda = 3$ is an eigenvalue of multiplicity 2.

Now find its corresponding eigenvector.

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\xi_1 + \xi_2 = 0$$

Take $\xi_1 = 1$ and $\xi_2 = 1$.

$$\Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \vec{x}^{(1)} = e^{\lambda t} \vec{\xi}^{(1)} = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

But we don't have a second linearly independent solution.

Let $\vec{x} = \vec{\xi}^{(1)} t e^{\lambda t} + \vec{\xi}^{(2)} e^{\lambda t}$ for some $\vec{\xi}^{(2)}$. Then

~~If $\vec{x}' = A\vec{x}$~~

$$\vec{x}' = \vec{\xi}^{(1)} e^{\lambda t} + \lambda \vec{\xi}^{(1)} t e^{\lambda t} + \lambda \vec{\xi}^{(2)} e^{\lambda t}.$$

If $\vec{x}' = A\vec{x}$,

$$\vec{\xi}^{(1)} + \lambda \vec{\xi}^{(1)} t + \lambda \vec{\xi}^{(2)} = A\left(\vec{\xi}^{(1)} t + \vec{\xi}^{(2)}\right).$$

$$\lambda \vec{\xi}^{(1)} t + \vec{\xi}^{(1)} + \lambda \vec{\xi}^{(2)} = A\vec{\xi}^{(1)} t + A\vec{\xi}^{(2)}.$$

Equating coefficients, we obtain

$$\lambda \vec{\xi}^{(1)} = A\vec{\xi}^{(1)} \quad \checkmark$$

and $\vec{\xi}^{(1)} + \lambda \vec{\xi}^{(2)} = A\vec{\xi}^{(2)}$.

$\Rightarrow$ We need to find $\vec{\xi}^{(2)}$ such that

$$(A-\lambda I)\vec{\xi}^{(2)} = \vec{\xi}^{(1)}.$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$-\xi_1 + \xi_2 = 1.$$

$$\Rightarrow \quad \xi_1 = \xi_2 - 1.$$

$$\Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} \xi_2 - 1 \\ \xi_2 \end{bmatrix} = \xi_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

for arbitrary ξ_2.

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t}}_{\vec{x}^{(2)}} + \xi_2 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}_{\vec{x}^{(1)}}.$$

So general solution:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) e^{3t}.$$

In general, if A has a repeated eigenvalue λ.
Find its eigenvector $\vec{\xi}^{(1)}$.
The find $\vec{\xi}^{(2)}$ such that

$$(A - \lambda I)\vec{\xi}^{(2)} = \vec{\xi}^{(1)}.$$

the general solution is

$$\vec{x} = c_1 \vec{\xi}^{(1)} e^{\lambda t} + c_2\left(\vec{\xi}^{(1)} t + \vec{\xi}^{(2)}\right) e^{\lambda t}.$$

E.g. $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{x}$.

$\det(A - \lambda I) = (2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = 0$.

$\lambda = 3$. eigenvalue.

$\Rightarrow$ Find eigenvector:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$-\xi_1 - \xi_2 = 0$.

$\Rightarrow$ Take $\xi_1 = 1$ and $\xi_1 = -1$.

$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector.

$\Rightarrow$ First solution: $\cancel{\xi}$ $\vec{x}^{(1)} = \vec{\xi}^{(1)} e^{\lambda t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$.

How to find $\vec{x}^{(2)}$?

First, find $\vec{\xi}^{(2)}$ such that

$$(A - \lambda I)\vec{\xi}^{(2)} = \vec{\xi}^{(1)},$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$-\xi_1 - \xi_2 = 1$.

~~let $\xi_1 = a$, then $\xi_2 = -a-1$~~

Take $\xi_1 = 0$, then $\xi_2 = -1$.

$\vec{\xi}^{(2)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$\Rightarrow \vec{x}^{(2)} = \vec{\xi}^{(1)} t e^{\lambda t} + \vec{\xi}^{(2)} e^{\lambda t}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{3t}.$$

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$$W[x^{(1)}, x^{(2)}] = \det\begin{bmatrix} e^{3t} & te^{3t} \\ -e^{3t} & -te^{3t}-e^{3t} \end{bmatrix}$$

$$= -te^{6t} - e^{6t} - te^{6t}.$$

$$= -e^{6t} \neq 0.$$

$\Rightarrow$ General solution:

$$\vec{x} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{3t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{3t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{3t}\right).$$

Review of Power Series.

A power series has the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ where x_0 is a fixed point.

The power series converges at x_0 if

$$\lim_{n\to\infty} \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

exists.

The power series converges absolutely at x_0 if the series

$$\sum_{n=0}^{\infty} |a_n (x-x_0)^n|$$

converges.

Remark: converges absolutely at x_0 $\underset{\not\Leftarrow}{\Longrightarrow}$ converges at x_0.

Tests for convergences: nth root test and ratio test.

- nth root test: the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely at x_0 if

$$\lim_{n\to\infty} \sqrt[n]{|a_n (x-x_0)^n|} = L < 1.$$

$$\therefore \quad |x-x_0| \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1.$$

- Ratio test: the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely at x_0 if

$$\lim_{n\to\infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = L < 1.$$

or $|x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1.$

If $L > 1$, the series diverges.

$L = 1$, the test is inconclusive.

For any power series, there is a nonnegative number ρ called the radius of convergence such that if

$$|x - x_0| < \rho$$

then the series converges absolutely, and if

$$|x - x_0| > \rho$$

the series diverges.

E.x. Determine the radius of convergence of

a) $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$, b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$,

Assume $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

and $g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$.

1) $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$

Why series?

- needed for non-constant coefficients.
- get solutions as accurate as we desire.

E.g. $y'' - xy' - y = 0$.

Sol. We assume a solution centered at $x_0 = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

To identify a_n, we substitute this back into ODE.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

So
$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} \overset{n a_n}{\cancel{(n+1)a_{n+1}}}\, x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \underbrace{\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n\right] x^n}_{} = 0.$$

To be true with all values of x, the coefficients must be zero.

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0.$$

$$(n+2)(n+1) a_{n+2} = (n+1) a_n.$$

$$a_{n+2} = \frac{1}{n+2} a_n.$$

$a_0 = a_0$ $\qquad\qquad$ $a_1 = a_1$

$a_2 = \frac{1}{2} a_0$ $\qquad\qquad$ $a_3 = \frac{1}{3} a_1$

$a_4 = \frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0$ $\qquad\qquad$ $a_5 = \frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1$

$a_6 = \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} a_0$ $\qquad\qquad$ $a_7 = \frac{1}{7} a_5 = \frac{1}{7 \cdot 5 \cdot 3} a_1$

$\vdots$

So $y(x) = a_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{384} + \cdots \right]$

$+ a_1 \left[1 + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \frac{x^9}{945} + \cdots \right]$

$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n! \, x^{2n+1}}{(2n+1)!}$

E.g. Find a series solution in powers of x of Airy's eq.

$$y'' - xy = 0, \qquad -\infty < x < \infty$$

Sol. Assume that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

(and the series converges in some interval $|x| < \rho$).

$$\Rightarrow y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$$

$$y'' = \sum_{n=0}^{\infty} a_{n+1}(n+1)n x^{n-1} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

Sub. y and y'' into the original eq:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - x\sum_{n=0}^{\infty} a_{n}\cancel{(n+1)}x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

I want x has the same power $\Rightarrow$ reindex the second term.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2(1)a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n = 0$$

$$\Rightarrow \begin{cases} \cancel{2a_2 = 0} \\ \cancel{(n+2)(n+1)a_{n+2} - a_{n-1} = 0} \end{cases}$$

$$\Rightarrow \begin{cases} 2a_2 = 0 \Rightarrow a_2 = 0 \\ (n+2)(n+1)a_{n+2} - a_{n-1} = 0 \end{cases}$$

Recurrence Relation

for $n = 1, 2, 3, \ldots$

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_{n-1} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Since $a_2 = 0$, $\Rightarrow a_5 = a_8 = a_{11} = \ldots = 0$.

1) a_0 determines a_3, which in turn determines a_6, ...

$$a_3 = \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_3}{6 \cdot 5} = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0$$

$$a_9 = \frac{a_6}{9 \cdot 8} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} a_0, \ldots$$

$$\Rightarrow \quad a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n-1)(3n)}, \quad n \geq 4$$

2) a_1 determines a_4, which in turn determines a_7, ...

$$a_4 = \frac{a_1}{3 \cdot 4}, \quad a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}$$

$$a_{10} = \frac{a_7}{10 \cdot 9} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \cancel{10} \cdot 9 \cdot 10},$$

$$\Rightarrow \quad a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}, \quad n \geq 4$$

$\Rightarrow$ Sol. of Airy's eq:

$$y(x) = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \ldots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \ldots \right]$$

$$+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \ldots + \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)} + \ldots \right]$$

Section 6.1. Definition of the Laplace Transform

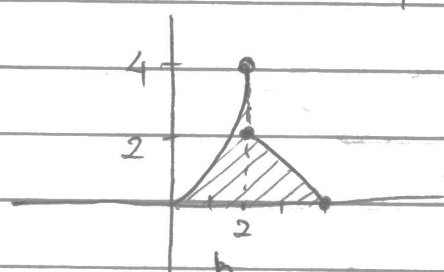
1) Improper integrals

$$\int_a^\infty f(t)\,dt = \lim_{b\to\infty}\int_a^b f(t)\,dt \quad \begin{cases}\text{converges if limit exists}\\ \text{diverges if limit doesn't exist}\end{cases}$$

E.g. $$\int_1^\infty \frac{1}{t^2}\,dt = \lim_{b\to\infty}\int_1^b \frac{1}{t^2}\,dt = \lim_{b\to\infty} -\frac{1}{t}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b}+1 = 1$$

E.g. $$\int_1^\infty \frac{1}{t^p}\,dt = \begin{cases}\frac{1}{p-1} & \text{if } p>1.\\ \text{does not exist} & \text{if } p\le 1.\end{cases}$$

E.g. Let $$f(t) = \begin{cases} t^2 & 0\le t < 2\\ 4-t & 2\le t\le 4.\end{cases}$$



$$\int_0^4 f(t)\,dt = \lim_{b\to 2^-}\int_0^b f(t)\,dt + \int_2^4 f(t)\,dt$$

$$= \lim_{b\to 2^-}\int_0^b t^2\,dt + \int_2^4 (4-t)\,dt$$

$$= \lim_{b\to 2^-} \frac{t^3}{3}\Big|_0^b + \left(4t-\frac{t^2}{2}\right)_2^4$$

$$= \lim_{b\to 2^-}\frac{b^3}{3} + 2$$

$$= \frac{8}{3} + 2$$

$$= \frac{14}{3}$$

This is an example of a piecewise continuous function on $[\alpha, \beta]$

- continuous between α and β except at a finite number of points.
- discontinuities are restricted to jump discontinuities, i.e. $f \to$ finite limit as $x \to$ endpoint from interior of interval.

Why study transform methods?

- hard problem $\to$ easier problem.
- unknown problem $\to$ known problem.

We define the Laplace transform of $f(t)$ to be:

$$\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt.$$

(We are using the integral to transform from functions $f(t)$ to functions in a different space containing functions $F(s)$ of s.)

Thm: If f is piecewise continuous and $|f(t)| \leq Ke^{at}$ when $t \geq M$, then

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

exists for $s > a$.

E.g. Compute $\mathcal{L}\{f(t)\}$ if $f(t) = 1$.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} -\frac{e^{-st}}{s}\Big|_0^b = \frac{1}{s}, \quad s > 0.$$

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E.g. Compute $\mathcal{L}\{f(t)\}$ if $f(t) = t$. (Ex. Ans: $\frac{1}{s^2}$).

E.g. Compute $\mathcal{L}\{f(t)\}$ for $f(t) = \cos(at)$.

$$F(s) = \int_0^\infty e^{-st}\cos(at)\,dt$$

Let $I = \int_0^b e^{-st}\cos(at)\,dt$.

$u = e^{-st}$ $dv = \cos(at)dt$

$$= e^{-st}\frac{\sin(at)}{a}\Big|_0^b + \int_0^b \frac{s}{a}e^{-st}\sin(at)\,dt.$$

$$= \dots = e^{-sb}\frac{\sin(ab)}{a} - \frac{s}{a^2}e^{-sb}\cos(ab) + \frac{s}{a^2} - \frac{s^2}{a^2}I.$$

$$\Rightarrow \left(1+\frac{s^2}{a^2}\right)I = \frac{1}{a}e^{-sb}\sin(ab) - \frac{s}{a^2}e^{-sb}\cos(ab) + \frac{s}{a^2}.$$

Taking limit both sides as $b \to \infty$.

$$\Rightarrow \left(1+\frac{s^2}{a^2}\right)F(s) = \frac{s}{a^2}.$$

$$F(s) = \frac{s}{a^2+s^2}, \quad s > 0.$$

* Remark: $\mathcal{L}\{\ \}$ is linear.

$$\mathcal{L}\{cf(t) + dg(t)\} = c\,\mathcal{L}\{f(t)\} + d\,\mathcal{L}\{g(t)\}.$$

Section 6.2: Solutions of Initial Value Problem.

$\mathcal{L}\{f(t)\} = ?$

$$\mathcal{L}\{f'(t)\} = \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\, dt.$$

$$I = \int_0^b e^{-st} f'(t)\, dt.$$

let $u = e^{-st}$ $\quad dv = f'(t)dt$

$du = -se^{-st}\, dt$ $\quad v = f(t).$

$$I = e^{-st} f(t)\Big|_0^b + \int_0^b s e^{-st} f(t)\, dt$$

$$= e^{-sb} f(b) - f(0) + \cancel{sI}\ s\int_0^b e^{-st} f(t)\, dt.$$

$$\Rightarrow \lim_{b\to\infty} I = 0\cdot f(b) - f(0) + s\,\mathcal{L}\{f(t)\}.$$

$$\Rightarrow \boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$
$$= s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$
$$= s^2\,\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

In general,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

<u>Thm</u>: Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. If $\exists$ K, a, M such that $|f(t)| \le Ke^{at}$ for $t \ge M$ then $\mathcal{L}\{f'(t)\}$ exists for $s>0$ and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

E.g. $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = -1$.

$$\mathcal{L}\{y''\} = s^2\underbrace{\mathcal{L}\{y\}}_{Y(s)} - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY(s) - 1$$

$$\mathcal{L}\{y\} = Y(s).$$

Then

$$0 = \mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\}$$

$$= s^2Y(s) - s + 1 - (sY(s) - 1) - 6Y(s)$$

$$= (s^2 - s - 6)Y(s) - s + 2.$$

$$\Rightarrow \quad Y(s) = \frac{s-2}{s^2 - s - 6} = \frac{s-2}{(s-3)(s+2)}.$$

Q: How to find the solution from $Y(s)$?

A: inverse ~~of~~ Laplace transform.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

In other words, we need to ~~find~~ figure out what function has $Y(s)$ as its Laplace transform.

$$Y(s) = \frac{s-2}{(s-3)(s+2)} = \frac{1/5}{s-3} + \frac{4/5}{s+2} \quad \text{(partial fraction)}$$

use table in section 6.2

$$\boxed{y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}.}$$

E.g. $y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\}.$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{2}{s^2 + 4}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{2}{s^2+4}.$$

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$$(s^2+1)\,\mathcal{L}\{y\} - 2s - 1 = \frac{2}{s^2+4}$$

$$\mathcal{L}\{y\} = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$

Partial fraction:

$$\frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$\begin{aligned} 2s^3+s^2+8s+6 &= (As+B)(s^2+1) + (Cs+D)(s^2+4) \\ &= As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D \\ &= (A+C)s^3 + (B+D)s^2 + (A+4C)s + B+4D. \end{aligned}$$

$$\begin{cases} A+C = 2 \\ B+D = 1 \\ A+4C = 8 \\ B+4D = 6 \end{cases} \qquad C = 2 \text{ and } A = 0 \qquad D = \frac{5}{3} \text{ and } B = -\frac{2}{3}$$

$$\Rightarrow \mathcal{L}\{y\} = -\frac{2}{3}\cdot\frac{1}{s^2+4} + \frac{2s+\frac{5}{3}}{s^2+1}$$

$$y = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{1}{3}\sin(2t) + 2\cos(t) + \frac{5}{3}\sin(t).$$

* A list of few known transforms:

(Ex)

$\mathcal{L}\{1\} = \frac{1}{s},\ s>0$

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s>a$

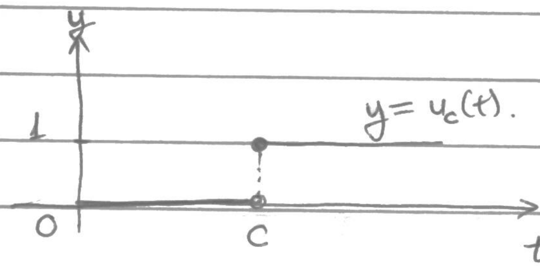
$\mathcal{L}\{\sin(mt)\} = \frac{m}{s^2+m^2}$

$\mathcal{L}\{\cos(mt)\} = \frac{s}{s^2+m^2}$

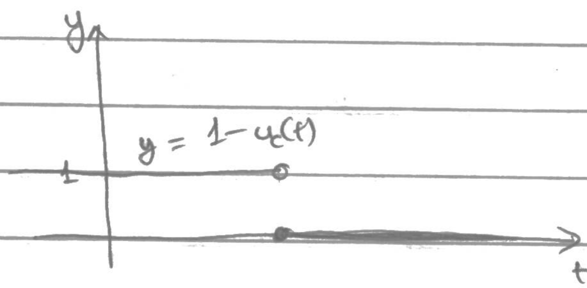
Section 6.3: Step Functions.

The unit step function

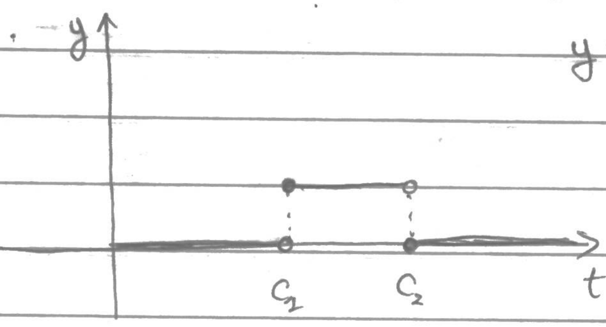
$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \qquad \text{for } c \geq 0.$$



$y = 1 - u_c(t)$.

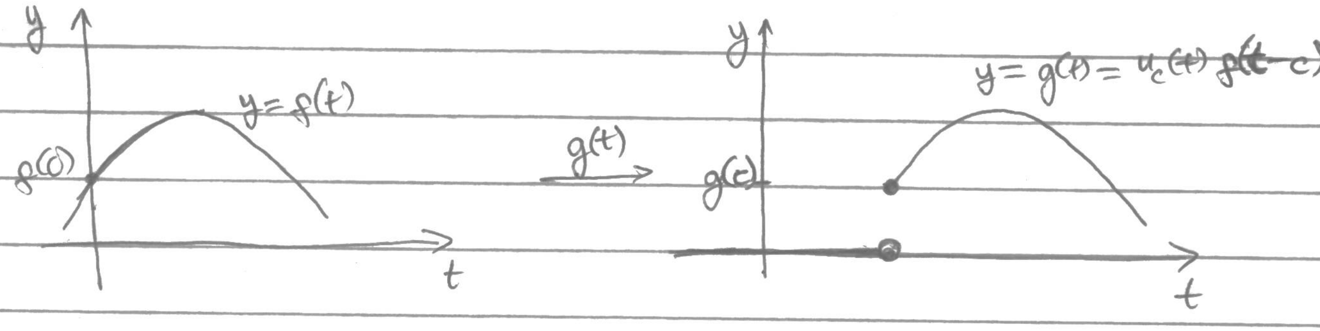


$y = u_{c_1}(t) - u_{c_2}(t)$, $c_1 \leq c_2$.



Suppose we have a function $f(t)$, $t \geq 0$, and want to translate it to the right an amount c. If we also want our new function to be zero from $t = 0$ to $t = c$, we can accomplish all this with

$$g(t) = \underbrace{u_c(t)}_{\text{zero until } t=c} \underbrace{f(t-c)}_{\text{shift to the right}}$$



What is $\mathcal{L}\{g(t)\}$?

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt.$$

let $w = t - c$, then $dw = dt$

so $$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-s(w+c)} f(w)\,dw.$$

$$= e^{-sc} \int_0^\infty e^{-sw} f(w)\,dw.$$

Thm: $\boxed{\mathcal{L}\{g(t)\} = e^{-sc}\, \mathcal{L}\{f(t)\}}$

and $g(t) = \mathcal{L}^{-1}\{e^{-sc} F(s)\} = u_c(t) f(t-c)$

E.g. Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t, & 1 \le t. \end{cases}$

Note you can use the definition to find $\mathcal{L}\{f(t)\}$ directly.
Here, let's try to use the above theorem:

$$f(t) = u_1(t)\,t = u_1(t)\cdot(t-1) + u_1(t).$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)\cdot(t-1)\} + \mathcal{L}\{u_1(t)\}$$

$$= e^{-s}\mathcal{L}\{t\} + e^{-s}\mathcal{L}\{1\}$$

$$= e^{-s}\left(\frac{1}{s^2}\right) + e^{-s}\left(\frac{1}{s}\right).$$

Ex. Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ t^2, & 2 \le t. \end{cases}$

~~Eg. Find $\mathcal{L}^{-1}\{F(s)\}$ i~~

Eg. Find $\mathcal{L}^{-1}\{F(s)\}$ if $F(s) = \left(1 - e^{-\pi s}\right) \dfrac{s}{s^2+4}$.

$$F(s) = \frac{s}{s^2+4} - e^{-\pi s} \frac{s}{s^2+4}.$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s} \frac{s}{s^2+4}\right\}.$$

$$= \cos(2t) - u_\pi(t) \cos[2(t-\pi)]$$

Thm: If $F(s) = \mathcal{L}\{f(t)\}$, $s > a \ge 0$, then

$$\boxed{\mathcal{L}\{e^{ct} f(t)\} = F(s-c)} \qquad s > a+c.$$

and $e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$.

Eg. Find $\mathcal{L}^{-1}\{F(s)\}$ if $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$.

$$F(s) = \frac{2(s-1)e^{-2s}}{(s-1)^2+1} = 2e^{-2s} \frac{s-1}{(s-1)^2+1}.$$

Note that $\dfrac{s-1}{(s-1)^2+1} \xrightarrow{\mathcal{L}^{-1}} e^t \cos(t)$

$$\mathcal{L}^{-1}\{F(s)\} = 2\, u_2(t)\, e^{t-2} \cos(t-2).$$

Section 6.4. Differential Equations with ~~discontinuities~~ discontinuous Forcing Functions

E.g. $y' + 2y = g(t)$, $y(0) = 0$, $g(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$.

so $y' + 2y = 1 - u_1(t)$.

Applying Laplace transform, we obtain

$$sY(s) - y(0) + 2Y(s) = \mathcal{L}\{1\} - \mathcal{L}\{u_1(t)\}$$
$$= \frac{1}{s} - e^{-s}\frac{1}{s}$$
$$(2+s)\,Y(s) = \frac{1}{s}\left(1 - e^{-s}\right)$$
$$Y(s) = \frac{1-e^{-s}}{s(2+s)}.$$

Let $H(s) = \dfrac{1}{s(s+2)}$, then $Y(s) = (1-e^{-s})\,H(s)$
$$= H(s) - e^{-s}H(s).$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{H(s)\} - \mathcal{L}^{-1}\{e^{-s}H(s)\}$$
$$= h(t) - u_1(t)\,h(t-1)$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

So we have to find ~~$H(s)$~~ $h(t)$.

$$H(s) = \frac{1}{s(s+2)} = \frac{1/2}{s} - \frac{1/2}{s+2}$$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{2} - \frac{1}{2}e^{-2t}.$$

Hence, $y(t) = \dfrac{1}{2} - \dfrac{1}{2}e^{-2t} - u_1(t)\left(\dfrac{1}{2} - \dfrac{1}{2}e^{-2(t-1)}\right)$.

$$= \begin{cases} \frac{1}{2}(1 - e^{-2t}), & 0 \le t \le 1 \\ \frac{1}{2}e^{-2t}(e^2 - 1), & t > 1. \end{cases}$$

E.g $y'' - 2y' + 2y = g(t)$, $y(0) = 0$, $y'(0) = 0$.

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \quad \text{or} \quad t \ge 2. \\ 1 & 1 \le t < 2. \end{cases}$$

Sol. $\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\}$

$$= s^2 Y(s) - s\,y(0) - y'(0) - 2(sY(s) - y(0)) + 2Y(s)$$

$$= s^2 Y(s) - 2sY(s) + 2Y(s)$$

$$= (s^2 - 2s + 2)Y(s).$$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_1(t) - u_2(t)\} = \dfrac{e^{-s}}{s} - \dfrac{e^{-2s}}{s}.$

$$\Rightarrow (s^2 - 2s + 2)\,Y(s) = \frac{e^{-s} - e^{-2s}}{s}$$

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s(s^2 - 2s + 2)}$$

$$= e^{-s} \cdot \frac{1}{s(s^2 - 2s + 2)} - e^{-2s} \frac{1}{s(s^2 - 2s + 2)}$$

$$= e^{-s} \cdot F(s) - e^{-2s} F(s)$$

where $F(s) = \dfrac{1}{s(s^2 - 2s + 2)}.$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-s} F(s)\} - \mathcal{L}^{-1}\{e^{-2s} F(s)\}.$

$$= u_1(t) f(t-1) - u_2(t) f(t-2).$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

How to find $f(t)$?

$$\frac{1}{s(s^2 - 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 2}$$

$$1 = A(s^2 - 2s + 2) + (Bs + C)s.$$

$$= (A + B)s^2 + (-2A + C)s + 2A.$$

$$\begin{cases} A + B = 0 \\ -2A + C = 0 \\ 2A = 1 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = -1/2 \\ C = 1 \end{cases}$$

$$\frac{1}{s(s^2-2s+2)} = \frac{1}{2}\frac{1}{s} + \frac{-\frac{1}{2}s+1}{s^2-2s+2}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{(s-1)}{(s-1)^2+1} + \frac{1}{2}\frac{1}{(s-1)^2+1}.$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s^2-2s+2)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$\cancel{y(t)}\ g(t) = \frac{1}{2} - \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t.$$

Hence,

$$y(t) = u_1(t)\left(\frac{1}{2} - \frac{1}{2}e^{t-1}\cos(t-1) + \frac{1}{2}e^{t-1}\sin(t-1)\right)$$

$$+ u_2(t)\left(\frac{1}{2} - \frac{1}{2}e^{t-2}\cos(t-2) + \frac{1}{2}e^{t-2}\sin(t-2)\right)$$