1.2 The Inner (Dot) Product, Length, and Distance

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If we want to determine the angle between two vectors $\vec{a}$ and $\vec{b}$, what should we do? *Inner products* (or Dot product) will help us to do this.

Definition. The inner product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Note that sometimes we write $\vec{a} \cdot \vec{b}$ as $\langle \vec{a}, \vec{b} \rangle$.

Example. Find the inner product of $\vec{a} = 3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + \vec{k}$.

Solution. $\vec{a} \cdot \vec{b} = 3(1) + 1(-1) + (-2)(1) = 0$.

Properties of Inner Products. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$ and α, β be real numbers in $\mathbb{R}$. Then

1) $\vec{a} \cdot \vec{a} \geq 0$; $\vec{a} \cdot \vec{a} = 0$ *if and only if* $\vec{a} = 0$

2) $\alpha \vec{a} \cdot \vec{b} = \alpha(\vec{a} \cdot \vec{b})$, and $\vec{a} \cdot \alpha \vec{b} = \alpha(\vec{a} \cdot \vec{b})$

3) $\vec{a}(\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$, and $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

4) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

The length/norm of a vector.

In 2D, the length of $\vec{a} = (a_1, a_2)$ is $\sqrt{a_1^2 + a_2^2}$. But $\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 = a_1^2 + a_2^2$. So $\vec{a} \cdot \vec{a} = (\text{length of } \vec{a})^2$.

We write this as $\vec{a} \cdot \vec{a} = \|a\|^2$, and $\|\vec{a}\|$ is called a norm of $\vec{a}$.

In 3D, if $\vec{a} = (a_1, a_2, a_3)$, then

$$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = (\text{length of } \vec{a})^2.$$

Unit Vectors. For any non-zero vector $\frac{\vec{a}}{\|\vec{a}\|}$ is a unit vector (i.e. its length is 1).

Example. Normalize the vector (i.e. make it unit length) $\vec{v} = 2\vec{i} + 3\vec{j} + 4\vec{k}$.

Solution. $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2\vec{i} + 3\vec{j} + 4\vec{k}}{\sqrt{2^2 + 3^2 + 4^2}} = \frac{2}{\sqrt{29}}\vec{i} + \frac{3}{\sqrt{29}}\vec{j} + \frac{4}{\sqrt{29}}\vec{k}$.

Summary. Let $\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = (b_1, b_2, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$. Then

1) $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. 2) $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{a} \cdot \vec{a}}$

3) The vector $\frac{\vec{a}}{\|\vec{a}\|}$ is normalized, i.e. it has unit norm.

4) The distance between the endpoints of $\vec{a}$ and $\vec{b}$ is $\|\vec{b} - \vec{a}\|$.

The angle between 2 vectors.

Theorem. Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbb{R}^3$ and let θ, where $0 \leq \theta \leq \pi$, be the angle between them. Then

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta.$$

Exercise. Read the proof in the book.

Example. Find the angle between the vectors $(1, 1, 2)$ and $(1, -1, 1)$.

Solution. Since $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{1(1) + 1(-1) + 2(1)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{1^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{18}} = \frac{2}{3\sqrt{2}}.$$

Hence, $\theta = \arccos\left(\frac{2}{3\sqrt{2}}\right)$.

Corollary. (Cauchy-Schwarz Inequality) $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$

with equality "=" if and only if $\vec{a}$ is a scalar multiple of $\vec{b}$ (or one of them is 0).

Proof.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\|\|\vec{b}\| \cos\theta \\ |\vec{a} \cdot \vec{b}| &= \|\vec{a}\|\|\vec{b}\||\cos\theta| \\ &\leq \|\vec{a}\|\|\vec{b}\| \end{aligned}$$

since $|\cos\theta| \leq 1$. Moreover, equality can only happen if $\vec{a} = 0, \vec{b} = 0$, or $\cos\theta = 0$.

Remark. Suppose $\vec{a}$ and $\vec{b}$ are nonzero.

If $\vec{a} \cdot \vec{b} = 0$, then $\cos\theta = 0$ or $\vec{a}$ and $\vec{b}$ are perpendicular.

If $\vec{a}$ and $\vec{b}$ are perpendicular

In other words, two vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if their dot product is zero.

Definition. If $\vec{a} \cdot \vec{b} = 0$, we say that they are *orthogonal*.

Definition. If $\vec{a} \cdot \vec{b} = 0$ and $\|\vec{a}\| = \|\vec{b}\| = 1$, we say that $\vec{a}$ and $\vec{b}$ are orthonormal.

Example. Let $\vec{a}$ and $\vec{b}$ be two orthogonal vectors. Let $\vec{c}$ be a vector in the plane spanned by $\vec{a}$ and $\vec{b}$. We can write $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ for some scalars α and β. Use the inner product to determine α and β.

Solution. We observe that

$$\vec{a} \cdot \vec{c} = \vec{a} \cdot (\alpha\vec{a} + \beta\vec{b}) = \alpha\vec{a} \cdot \vec{a} + \beta\vec{a} \cdot \vec{b} = \alpha\|\vec{a}\|^2$$

since $\vec{a} \cdot \vec{b} = 0$. Then we have

$$\alpha = \frac{\vec{a} \cdot \vec{c}}{\|a\|^2}.$$

Similarly, $\beta = \frac{\vec{b} \cdot \vec{c}}{\|\vec{b}\|^2}$.

Remark. We call $\alpha\vec{a}$ the projection of $\vec{c}$ along $\vec{a}$ and $\beta\vec{b}$ the projection of $\vec{c}$ along $\vec{b}$.

Orthogonal Projection.

$\vec{p}$ is the orthogonal projection of $\vec{v}$ on $\vec{a}$ if $\vec{p} = \frac{\vec{a} \cdot \vec{v}}{\|\vec{a}\|^2}\vec{a}$.

Example. The orthogonal projection of $\vec{i} + \vec{j}$ on $\vec{i} - 2\vec{j}$ is

$$\vec{p} = \frac{(\vec{i} + \vec{j}) \cdot (\vec{i} - 2\vec{j})}{(\vec{i} - 2\vec{j})(\vec{i} - 2\vec{j})}(\vec{i} - 2\vec{j}) = \frac{1 - 2}{1 + 4}(\vec{i} - 2\vec{j}) = -\frac{1}{5}(\vec{i} - 2\vec{j}).$$

Triangle Inequality.

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\|+\|\vec{b}\|.$$

Proof.

$$\begin{aligned}
\|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) \\
&= \|\vec{a}\|^2 + 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 \\
&\leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2 \\
&= (\|\vec{a}\|+\|\vec{b}\|)^2.
\end{aligned}$$