Lecture 17: Cofactor Expansion (Section 4.3)

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Review

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  - Multiply by t in row 1: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Cofactor expansion

- Definition. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Denote by M_{ij} the submatrix of A obtained by deleting its row and column containing a_{ij}. Then $\det(M_{ij})$ is called the minor of a_{ij}.

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- **Example.** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then

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The minor of a_{11} is $\det(M_{11}) = 5(9) - 8(6) = -3$ and the minor of a_{23} is $\det(M_{23}) = -6$.

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- **Example.** From the previous example, $A_{11} = (-1)^2(-3) = -3$ and $A_{23} = (-1)^5(-6) = 6$.

Cofactor expansion

The method of cofactor expansion is given by the formulas

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$$

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$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$$

(expansion of $\det(A)$ along jth column).

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$$\det(A) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = 1(-3) + 2(6) + 3(-3) = 0.$$

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▶ Example. Compute the determinant of

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▶ Solution.

$$\begin{vmatrix} 2 & -1 & 1 & 0 \\ 3 & 5 & 0 & -2 \\ 1 & 1 & 0 & -3 \\ 4 & 0 & 3 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 5 & -2 \\ 1 & 1 & -3 \\ 4 & 0 & -1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 3 & 5 & -2 \\ 1 & 1 & -3 \\ 4 & 0 & -1 \end{vmatrix} = -50 + 99 = 49.$$

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Because to compute a large $n \times n$ determinant,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$.
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add.
$\Longrightarrow$ too much work.