Lecture 7: Linear independence and Four Fundamental Subspaces (Section 2.3-2.4)

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A basis of a vector space

- ▶ Definition. A set of vectors $\{v_1, ..., v_m\}$ in V is a **basis** of V if
 - $V = \operatorname{span}\{v_1, \dots, v_m\}$, and
 - the vectors v_1, \dots, v_m are linearly independent.

Theorem. Suppose that V has dimension d.

- A set of d vectors in V are a basis if they span V.
- A set of d vectors in V are a basis if they are linearly independent.

A basis of a vector space

- ▶ Example. Let \mathcal{P}_2 be the space of polynomials of degree at most 2.
 - a) What is the dimension of \mathcal{P}_2 ?
 - b) Is $\{t, 1 t, 1 + t t^2\}$ a basis of \mathcal{P}_2 ?

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Example. Produce a basis of \mathbb{R}^2 from the vectors

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► Example. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$

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• What is the dimension of this subspace of \mathbb{R}^3 ?

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- What is the dimension of this subspace of \mathbb{R}^3 ?
- Extend it to a basis of \mathbb{R}^3 .

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- find the parametric form of the solutions to Ax = 0,
- express solutions x as a linear combination of vectors with the free variables as coefficients,
- these vectors form a basis of N(A).

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$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

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The solutions of Ax = 0 is of the form

$$\begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

$$N(A) = \operatorname{span} \left\{ \begin{bmatrix} -2\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -13\\0\\6\\1\\0 \end{bmatrix}, \begin{bmatrix} -33\\0\\15\\0\\1 \end{bmatrix} \right\}.$$

These above three vectors form a basis of N(A). (Why?)

Recall that the columns of A are independent $\iff Ax = 0$ has only the trivial solution (namely, x = 0), $\iff A$ has no free variables.

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A basis for C(A) is given by the pivot columns of A.

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$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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▶ Solution. Find row reduction form of *A*

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence,
$$\left\{ \begin{bmatrix} 1\\2\\3\\4 \end{bmatrix}, \begin{bmatrix} 0\\-1\\2\\0 \end{bmatrix} \right\}$$
 form a basis for $C(A)$.

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- ▶ Remark. Row operations do not preserve the column space.
- ▶ Remark. Row operations do preserve the null space.

Dimension of C(A) **and** N(A)

Theorem. Let A be an $m \times n$ matrix. Then

- $\dim C(A)$ is the number of pivots of A.
- $\dim N(A)$ is the number of free variables of A.
- $\dim C(A) + \dim N(A) =$
- ▶ Proof.???

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- $\dim C(A) + \dim N(A) = n$
- ▶ Proof.???

Dimension of C(A) and N(A)

► Example. Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find the dimension of C(A) and N(A).

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- ightharpoonup Definition. The **rank** of a matrix *A* is the number of its pivots.

Example. Find a basis for C(A) and $C(A^T)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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▶ Solution. To find C(A), we can just use the echelon form of A. Likewise, we can also obtain $C(A^T)$ for an echelon form of A^T . But, it's not necessary!

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The rank of
$$A$$
 is 2. And $\left\{ \begin{bmatrix} 1\\2\\3\\4 \end{bmatrix}, \begin{bmatrix} 0\\-1\\2\\0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

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Recall that $C(A) \neq C(B)$. (We performed row operations). However, $C(A^T) = C(B^T)$.

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Hence,
$$\left\{ \begin{bmatrix} 1\\2\\0\\-1\\-5 \end{bmatrix} \right\}$$
 form a basis for $C(A)$.

Theorem. (Fundamental Theorem of Linear Algebra, Part I) Let A be an $m \times n$ matrix of rank r.

- $\dim C(A) = r$
- dim $C(A^T) = r$
- $\dim N(A) = n r$
- $\dim N(A^T) = m r$