Lecture 10: Orthogonality (Section 3.1)

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Important geometric examples

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maps $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$

reflects every vector in $\mathbb{R}^2$ through the line $y = x$.

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gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$, i.e.

rotates every vector in $\mathbb{R}^2$ counter-clockwise by 90°.

Inner product and distances

- Definition. The **inner product** or **dot product** of $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is defined by

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

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- Example. Find the inner product of the following two vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Inner product and distances

- Definition. The **norm** or **length** of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

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- Example. In $\mathbb{R}^2$,

$$\text{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

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$$\begin{aligned}
&\iff \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2 \\
&\iff \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w})^2 \\
&\iff \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} \\
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- **Example.** Are the following vectors orthogonal?
  - $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$
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Orthogonal vectors

Theorem. Suppose that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal. Then $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent.

- Proof.

Orthogonal vectors

- Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are pairwise orthogonal and have length 1.

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- Example. Consider $A = \begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix}$. Find $N(A)$ and $C(A^T)$.

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- Example. Consider $A = \begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix}$. Find $N(A)$ and $C(A^T)$.
- Solution. $N(A) = \text{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$ and $C(A^T) = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

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- Example. Consider $A = \begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix}$. Find $N(A)$ and $C(A^T)$.
- Solution. $N(A) = \text{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$ and $C(A^T) = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$. Note that the two basis vectors are orthogonal $\begin{bmatrix}-2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} = 0$.

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the vector in $N(A)$ is orthogonal to the vectors in $C(A^T)$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

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- Example. Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. Find $N(A)$ and $C(A^T)$.
- Solution. $N(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ and $C(A^T) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

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Vectors in $N(A)$ are orthogonal to vectors in $C(A^T)$.

The fundamental theorem, part II

- Definition. Let W be a subspace of $\mathbb{R}^n$, and $\boldsymbol{v}$ in $\mathbb{R}^n$.
  - $\boldsymbol{v}$ is **orthogonal** to W, if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ for all $\boldsymbol{w}$ in W.

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  - Another subspace V is **orthogonal** to W, if every vector in V is orthogonal to W.

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  - The **orthogonal complement** of W is the space $W^\perp$ of all vectors in $\mathbb{R}^n$ that are orthogonal to W.

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  - The **orthogonal complement** of W is the space $W^\perp$ of all vectors in $\mathbb{R}^n$ that are orthogonal to W. (Is $W^\perp$ a vector space?)

- Example. In previous example, $N(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ and

$C(A^T) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ are orthogonal subspaces.

The fundamental theorem, part II

Theorem. (Fundamental Theorem of Linear Algebra, Part II)

- $N(A)$ is orthogonal to $C(A^T)$. (The two spaces are orthogonal complements)
- $N(A^T)$ is orthogonal to $C(A)$. (The two spaces are orthogonal complements)

▶ Proof.

The fundamental theorem, part II

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▶ Proof.

Theorem. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of rank r.

- $\dim C(A) = \dim C(A^T) = r$.
- $\dim N(A) = n - r$
- $\dim N(A^T) = m - r$.

▶ Example. Find all vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

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▶ Example. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \, : \, a + b = 2c \right\}$. Find a basis for the orthogonal complement of V.

A new perspective on $A\boldsymbol{x} = \boldsymbol{b}$

$A\boldsymbol{x} = \boldsymbol{b}$ is solvable

$\iff \boldsymbol{b}$ is in $C(A)$

$\iff \boldsymbol{b}$ is orthogonal to $N(A^T)$

A new perspective on $Ax = b$

$$\begin{aligned} &A\boldsymbol{x} = \boldsymbol{b} \text{ is solvable} \\ &\iff \boldsymbol{b} \text{ is in } C(A) \\ &\iff \boldsymbol{b} \text{ is orthogonal to } N(A^T) \end{aligned}$$

- **Example.** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?