1. (6 points) Evaluate each of the following limits, or state that it does not exist.

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + 3x - 10}$. $\left(\frac{0}{0} \right.$ indeterminate form $\left. \right)$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x+5)(x-2)}$$

$$= \lim_{x \to 2} \frac{x+2}{x+5}$$

$$= \frac{2+2}{2+5} = \frac{4}{7}.$$

(b) $\lim_{x \to 0} x^2 \sin\left(\frac{2}{x}\right)$.

~~1st method~~:

Since $-1 \leq \sin\left(\frac{2}{x}\right) \leq 1$,

$$-x^2 \leq x^2 \sin\left(\frac{2}{x}\right) \leq x^2.$$

and $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$

by the squeeze theorem:

$$\lim_{x \to 0} x^2 \sin\left(\frac{2}{x}\right) = 0$$

~~2nd method: use change of variables. let $u = \frac{2}{x}$. Then $u \to \infty$ lim~~

(c) $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} \frac{2|x|}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{2|x|}{x} = \lim_{x \to 0^+} \frac{2x}{x} = \lim_{x \to 0^+} 2 = 2.$$

and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{2|x|}{x} = \lim_{x \to 0^-} \frac{-2x}{x} = \lim_{x \to 0^-} -2 = -2.$

$\Rightarrow \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) \Rightarrow \lim_{x \to 0} f(x)$ D.N.E.

2. (6 points) Show that the equation $e^x = \frac{1}{4}x^2 + 2$ has at least one solution in the interval $[0, 1]$. (Note: $e \simeq 2.7$)

The equation is equivalent to

$$e^x - \frac{1}{4}x^2 - 2 = 0.$$

Consider $f(x) = e^x - \frac{1}{4}x^2 - 2$.

Then $f(0) = 1 - 0 - 2 = -1 < 0$

and $f(1) = e - \frac{1}{4} - 2 = e - 2.25 > 0$.

$\Rightarrow \quad f(0) < 0 < f(1)$.

and since $f(x)$ is continuous

$\Rightarrow$ By the Intermediate Value Theorem,

there exists ~~ez con~~ $c \in [0, 1)$ such that

$$f(c) = 0.$$

or $\quad e^c - \frac{1}{4}c^2 - 2 = 0.$

$\Rightarrow$ The equation has at least one solution in $[0, 1]$.

3. (6 points) Let $f(x) = \sqrt{2x+1}$. Compute $f'(x)$ using the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \cancel{f}\sqrt{2x+1}}{h}$$

Conjugation $= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$

$$= \lim_{h \to 0} \frac{[2(x+h)+1] - (2x+1)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{2x + 2h + 1 - 2x - 1}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \frac{2}{\sqrt{2(x+0)+1} + \sqrt{2x+1}}$$

$$= \frac{2}{2\sqrt{2x+1}}.$$

$$= \frac{1}{\sqrt{2x+1}}$$

4. (6 points) Let $g(x) = \sqrt{5 - x} + 3$.

(a) Determine the domain and range of g.

Domain: $x \leq 5$ or $(-\infty, 5]$.

Range: $[3, +\infty)$.

Will not test on this kind of problem.

(b) Find a formula for the inverse $g^{-1}(x)$ and state its domain and range.

5. (4 points) Let f be a function such that $f(2) = 2$ and $f'(2) = -5$.

(a) Find an equation for the line tangent to the graph of f at the point $(2, 2)$.

$$y - f(2) = f'(2)(x-2).$$

$$y - 2 = -5(x-2).$$

(b) Find the value of $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$ and justify your answer.

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = f'(2) = -5.$$