Midterm 1A

NAME: PID:

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DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO
(This exam is worth 40 points)

Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

Name:

Problem 1.(9 points.) Find LU-decomposition of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

and $U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

NAME:

Problem 2.(10 points.) Given a system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$, forward elimination changes $A\boldsymbol{x} = \boldsymbol{b}$ to a reduced row form $R\boldsymbol{x} = \boldsymbol{c}$: the complete solution is

$$\boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

2a. (5 points) What is the 3 by 3 reduced row echelon matrix R and what is $\boldsymbol{c}$?

2b. (5 points) If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what is the matrix A and what is the vector $\boldsymbol{b}$?

a) $\vec{x} \in \mathbb{R}^3$ and there are two free variables.

$\Rightarrow$ R is 3×3 matrix.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + 2u + 5v \\ u \\ v \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 4 + 2x_2 + 5x_3 \\ x_1 - 2x_2 - 5x_3 = 4. \end{array}$$

$$R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \vec{c} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

b) We see that $[A|b] \rightsquigarrow [EA|E\vec{b}] = [R|\vec{c}]$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} A = R. \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}^{-1} R$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}$$

$$\text{and } \vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}.$$

NAME:

Problem 3.(10 points.) Let $\mathcal{P}_2$ be the vector space of all polynomials of degree at most 2. Let $T : \mathcal{P}_2 \to \mathbb{R}^3$ be defined by

$$T(p(t)) = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix}$$

for every polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ in $\mathcal{P}_2$.

3a. (4 points) Show that T is a linear transformation.

3b. (4 points) Find a matrix A representing T with respect to the standard bases, i.e.,

$$\{1, t, t^2\} \text{ for } \mathcal{P}_2 \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \mathbb{R}^3.$$

3a) For any scalar c and d in $\mathbb{R}$ and any polynomials $p(t), g(t) \in \mathcal{P}_2$.

$$T(cp(t) + dg(t)) = \begin{bmatrix} cp(1) + dg(1) \\ cp(2) + dg(2) \\ cp(3) + dg(3) \end{bmatrix} = c\begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix} + d\begin{bmatrix} g(1) \\ g(2) \\ g(3) \end{bmatrix}$$

$$\Rightarrow \quad T(cp(t) + dg(t)) = cT(p(t)) + dT(g(t)).$$

Thus, T is a linear transformation.

b) $$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(t^2) = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 9 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

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3c. (2 points) Find a polynomial $p \in \mathcal{P}_2$ such that

$$T(p(t)) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

For any polynomial $p \in \mathbb{P}_2$, $p(t) = a_0 + a_1 t + a_2 t^2$.
The vector representing $p(t)$ is $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

Hence, if $T(p(t)) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$\Rightarrow$ $a_0 = 2$, $a_1 = 1$, $a_2 = 0$.

Thus, $p(t) = 2 + t$.

NAME:

Problem 4.(10 points.) Let A be the following matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & k \end{bmatrix},$$

where k is a fixed constant. Clearly label the statements as **TRUE** or **FALSE**. In all cases, *briefly explain* your claim.

4a. (2 points) For $k = 1$, the matrix A has 4 pivot columns.

$\boxed{\text{False}}$. Because row reduction has 3 pivots.

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 2 \\ 0 & 0 & \boxed{3} & 0 & 3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4b. (2 points) The matrix A is invertible.

$\boxed{\text{False.}}$ A is not a square matrix.

NAME:

4c. (2 points) For $k = 1$, the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ belongs to the column space of A.

$\boxed{\text{False}}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ belongs to $C(A)$ if $A\vec{x} = \vec{b}$ is solvable.

Row reduction implies that $A\vec{x} = \vec{b}$ is not solvable.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-1} \end{array}\right]$$

4d. (2 points) For $k \neq 1, \dim N(A) = 1$.

$\boxed{\text{True}}$ Since there is 1 free variable.

4e. (2 points) For $k \neq 1, \dim C(A) = 4$.

$\boxed{\text{True}}$. $\dim C(A) = 5 - 1 = 4.$

$\uparrow$ # of columns of A.

Midterm 1B

MATH 102, WINTER 2018

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Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1.(9 points.) Find LU-decomposition of the following matrix

$$A = \begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}.$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$E_2 E_1 A = U \Rightarrow A = E_1^{-1} E_2^{-1} U.$$

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

and $U = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

NAME:

Problem 2.(10 points.) Given a system of linear equation $A\boldsymbol{x} = \boldsymbol{b}$, forward elimination changes $A\boldsymbol{x} = \boldsymbol{b}$ to a reduced row form $R\boldsymbol{x} = \boldsymbol{c}$: the complete solution is

$$\boldsymbol{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}.$$

2a. (5 points) What is the 3 by 3 reduced row echelon matrix R and what is $\boldsymbol{c}$?

2b. (5 points) If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what is the matrix A and what is the vector $\boldsymbol{b}$?

(See Ver A for explanation)

a) $R = \begin{bmatrix} 1 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & -1 & 6 \\ 3 & -3 & 18 \\ 5 & -5 & 30 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 18 \\ 30 \end{bmatrix}$

NAME:

Problem 3.(10 points.) Let $\mathcal{P}_2$ be the vector space of all polynomials of degree at most 2. Let $T : \mathcal{P}_2 \to \mathbb{R}^3$ be defined by

$$T(p(t)) = \begin{bmatrix} p(2) \\ p(3) \\ p(1) \end{bmatrix}$$

for every polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ in $\mathcal{P}_2$.

3a. (4 points) Show that T is a linear transformation.

3b. (4 points) Find a matrix A representing T with respect to the standard bases, i.e.,

$$\{1, t, t^2\} \text{ for } \mathcal{P}_2 \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \mathbb{R}^3.$$

a) See Ver. A.

b) $T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad T(t) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ $\quad T(t^2) = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 1 & 1 \end{bmatrix}$$

NAME:

3c. (2 points) Find a polynomial $p \in \mathcal{P}_2$ such that

$$T(p(t)) = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}.$$

c) $p(t) = 2 + t.$

NAME:

Problem 4.(10 points.) Let A be the following matrices

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & k \end{bmatrix},$$

where k is a fixed constant. Clearly label the statements as **TRUE** or **FALSE**. In all cases, *briefly explain* your claim.

4a. (2 points) For $k \neq 1$, the matrix A has 4 pivot columns.

$\boxed{\text{True}}$

$$A \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & k-1 \end{bmatrix}$$

4b. (2 points) The matrix A is invertible.

$\boxed{\text{False}}$.

NAME:

4c. (2 points) For $k \neq 1$, the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ belongs to the column space of A.

$\boxed{\text{True}}$ Because

$$[A|\vec{b}] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & k-1 & -1 \end{array}\right].$$

$A\vec{x} = \vec{b}$ is solvable.

4d. (2 points) For $k = 1, \dim N(A) = 1$.

$\boxed{\text{False}}$ $\dim N(A) = 2$

4e. (2 points) For $k = 1, \dim C(A) = 4$.

$\boxed{\text{False}}$ $\dim C(A) = 5 - 2 = 3$.