MATH 102 - PRACTICE PROBLEMS FOR FINAL

Main topics for the final.

- 1. LU, LDU, PLU decomposition.
- 2. Null space, column space, row space, left null space, rank, nullity, left/right inverse, systems.
- 3. The matrix of linear transformations.
- 4. Abstract vector spaces and abstract linear maps.
- 5. Orthogonal complements. Relationships between the four subspaces of a matrix.
- 6. Orthogonal/orthonormal bases and projections when an orthonormal basis is given.
- 7. Gram-Schmidt. QR decomposition.
- 8. Projections onto subspaces. Left inverse. Least squares.
- 9. Gram-Schmidt for abstract inner product spaces.
- 10. Determinants and their applications.
- 11. Similar matrices. Diagonalizable matrices. Powers of matrices and exponentials.
- 12. Difference and differential equations. Stability.
- 13. Complex vectors and complex matrices. Unitarily diagonalizable matrices. Hermitian, skew Hermitian, unitary, normal matrices.
- 14. Symmetric matrices, Cholesky decomposition, positive decomposition.
- 15. Quadratic forms.
- 16. SVD decomposition. Pseudoinverses. Applications to projections and least squares.

The 22 *sample problems listed below illustrate the above* 16 *topics. Going through all questions in detail is the equivalent of* $2-3$ *final exams. This is a bit time consuming, but different people need practice with different topics. You don't need to solve every question below, just focus on the topics you feel you need more practice with. The problem numbers match the list above.*

1.

(i) Find the LU-decomposition of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix}.$$

Also write down the LDU decomposition of the matrix.

(ii) Using the LU decomposition, find the determinant of A.

(iii) Using the LU decomposition, solve the system

$$Ax = \begin{bmatrix} 2 \\ 9 \\ 8 \end{bmatrix}.$$

2.A. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \\ 2 & 6 & 4 & -12 \end{bmatrix}.$$

After carrying out several row operations, we arrive at the matrix

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. \end{bmatrix}.$$

(i) Give a basis for the null space of A. What is the nullity of A?

(ii) Show that the columns $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}$ of A are linearly dependent by exhibiting explicit relations between them.

(iii) Give a basis for $C(A)$. What is the rank of A?

(iv) Give a basis for the row space of A.

(v) What is the dimension of the left null space of A?

(vi) Does A admit a left inverse? How about a right inverse?

2.B. Consider a matrix A such that $\operatorname{rref} A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$.

(i) Find the set of solutions to the system $A\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$. Is this set of solutions a vector space?

(ii) It is known that the second column of A is $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and the fourth column of A is $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

Find the matrix A.

3. Consider Rot_θ the rotation in $\mathbb{R}^3$ around the y axis by angle θ in the direction that takes the positive x axis to the positive z axis. Find the matrix of the linear transformation Rot_θ in the standard basis.

4. Let $\mathcal{P}_n$ be the space of polynomials of degree at most equal to n. Let

$$T : \mathcal{P}_n \to \mathcal{P}_{n+1}, \;\; f \mapsto Tf$$

where Tf is defined as

$$(Tf)(x) = \int_{-x}^{0} f(t)\, dt.$$

(i) Show that

$$T : \mathcal{P}_n \to \mathcal{P}_{n+1}$$

is well defined, i.e. T transforms polynomials of degree at most equal to n into polynomials of degree at most equal to $n+1$.

(ii) Show that T is a linear transformation.

(iii) Find the matrix of T using the standard basis $\{1, x, \ldots, x^n\}$ for the spaces $\mathcal{P}_n$ and the similar basis for $\{1, x, \ldots, x^{n+1}\}$ for $\mathcal{P}_{n+1}$.

5. 6. Consider the subspace $V \subset \mathbb{R}^4$ spanned by the vectors $V = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} \right\}$.

(i) Find the orthogonal complement $V^\perp$.

(ii) Find an orthonormal basis for V.

(iii) Calculate the projection of the vector $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 7 \\ 0 \end{bmatrix}$ onto V.

(iv) Find the orthogonal projection of the vector $\vec{u}$ onto $V^\perp$.

(v) Write down the matrix of the projection onto V as a product of a matrix and its transpose.

7. Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

8. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -3 \\ -1 & -2 \\ 1 & 3 \end{bmatrix}.$$

(i) Find the left inverse of A.

(ii) Find the projection matrix onto the column space of A.

(iii) Find the projection matrix onto the left null space of A.

(iv) Find the least squares solution of the system

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}.$$

9. Let $\mathcal{P}$ be the space degree at most equal to 2 polynomials with real coefficients. In this problem we will find the first three Hermite polynomials. (Hermite polynomials are important in solving for the eigenstates of the harmonic oscillator in physics, or for building portfolios in mathematical finance.)

For this problem, you may need to recall the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

and integration by parts. (It is possible to solve this problem even without knowing the above integral.)

(i) Check that

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx$$

defines an inner product on $\mathcal{P}$.

(ii) Starting with the basis $\{1, x, x^2\}$, obtain an orthogonal basis for $\mathcal{P}$. These are the Hermite polynomials.

(iii) How would you extend the inner product in (i) to a Hermitian product on the space of degree at most 2 polynomials with complex coefficients?

10.

(i) Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and the inverse of A.

(ii) If $AB = -BA$, can B be invertible?

11.A. For what values of a, b are the two matrices below similar

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

11.B. Consider the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

Calculate the exponential e^{tA}.

12.A. Find the solution of the Fibonacci-like recursion

$$G_{n+2} = 3G_{n+1} - 2G_n, G_0 = 0, G_1 = 1.$$

12.B. Let a be a real parameter and let

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}.$$

Determine the stability of the differential equation below in terms of the parameter a

$$\frac{dy}{dt} = Ay.$$

12.C. Consider the difference equation

$$Y_{n+1} = AY_n, \ Y_0 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

for the matrix

$$A = \begin{bmatrix} .5 & .2 & .3 \\ .5 & .2 & .3 \\ 0 & .6 & .4. \end{bmatrix}.$$

What is the limit $\lim_{n\to\infty} Y_n$?

13.A. Assume that A is unitarily diagonalizable i.e. $A = UDU^{-1}$ for a unitary U. Check that A is in fact normal, that is

$$AA^H = A^H A.$$

13.B. For matrices (i)-(v), determine which are Hermitian, skew-Hermitian, unitary, normal.

(i) $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}.$

(ii) $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0. \end{bmatrix}$

(iii) $A = \begin{bmatrix} i & 2+i \\ -2+i & 4i \end{bmatrix}.$

(iv) $A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(v) $A = \frac{1}{\sqrt{2}}\begin{bmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{bmatrix}$ where z is a complex number of modulus 1.

13.C. Unitarily diagonalize the Hermitian matrix

$$A = \begin{bmatrix} 4 & 2+2i \\ 2-2i & 6. \end{bmatrix}.$$

13.D. Show that if $\mathbf{v}$ is a complex column vector of length 1 in $\mathbb{C}^n$, then the matrix

$$H = I - 2\mathbf{v}\cdot\mathbf{v}^H$$

is both Hermitian and unitary.

13.E. True or false:

(i) A symmetric matrix can't be similar to a nonsymmetric matrix.

(ii) $A + I$ can't be similar to $A - I$.

(iii) An invertible matrix can't be similar to a singular matrix.
(iv) Product of diagonalizable matrices is diagonalizable.
(v) Product of Hermitian matrices is Hermitian.
(vi) The product AA^H is always a normal matrix.
(vii) The determinant of a 4×4 skew Hermitian matrix is always real.
(viii) The trace of a 4×4 unitary matrix could equal $3 + 4i$.
(ix) Two simultaneously diagonalizable matrices necessarily commute.

14. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

which has $\lambda = 2$ as a repeated eigenvalue.

(i) Orthogonally diagonalize A, that is write $A = QDQ^{-1}$ where Q is an orthogonal matrix.
(ii) Show that A is positive definite and find a positive decomposition $A = RR^T$ by any method you wish.
(iii) Write the polynomial

$$f = 3x^2 + 3y^2 + 3z^2 + 2xy + 2yz + 2zx$$

as sum of three squares.

15. Discuss the definiteness of the following quadratic forms:

(i) $Q(x, y) = 2x^2 + 3y^2 - 4xy$;
(ii) $Q(x, y, z) = x^2 + y^2 + z^2 + 6xy + 6xz + 6yz$.

16. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(i) Find the SVD of A.
(ii) Find the pseudoinverse of A.
(iii) Find the matrix of the projection onto the column space of A.
(iv) Find the shortest length least square solution of the system

$$Ax = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(v) From the SVD read off the four subspaces of A: column space, null space, row space, left null space.