Math 20E - Lecture A00 **Midterm #2, VERSION C**
Fall 2016 **Name:** ____________________
11/18/2016 PID: ____________________
Time Limit: 50 Minutes Section Time: ____________________

This exam contains 3 pages (including this cover page) and 4 questions.
Total of points is 100.

You may not use any notes (except your cheat sheet) or calculators during this exam.
Write your *Name, PID, and Section* on the front of your Blue Book.
Write the *Version* of your exam on the front of your Blue Book.
Write your solutions clearly in your Blue Book.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.

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1. (25 points) Evaluate $\int_C y\,dx - x^2\,dy$, where C is the boundary of the square $[-1,1]\times[-1,1]$ oriented in the counterclockwise direction, *using Green's theorem.*

   Solution. Note that traversing the boundary counterclockwise corresponds to the same orientation used in Green's theorem. Application of the theorem yields

$$\begin{aligned}\int_{\partial S} y\,dx - x^2\,dy &= \iint_S \left(\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(y)\right) dxdy \\ &= \int_{-1}^{1}\int_{-1}^{1}(-2x-1)\,dxdy \\ &= -4.\end{aligned}$$

2. (25 points) Let $\Phi(u,v) = (u^2, u\cos v, u\sin v)$ be a parametrized surface, where $0 \le u \le 3$ and $0 \le v \le 2\pi$.

   a. (10 points) Compute the tangent vectors $\mathbf{T}_u(u,v)$ and $\mathbf{T}_v(u,v)$ associated to this parametrization.
   b. (5 points) Find a unit vector $\mathbf{n}(u,v)$ orthogonal to the surface at the point $\Phi(u,v)$.
   c. (10 points) Compute the area of the surface.

   Solution. The parametrization is given by $\mathbf{\Phi}(u,v) = (u^2, u\cos(v), u\sin(v))$ with $0 \le u \le 3$ and $0 \le v \le 2\pi$.

   a.

$$\begin{aligned}\mathbf{T}_u(u,v) &= (2u, \cos(v), \sin(v)) \\ \mathbf{T}_v(u,v) &= (0, -u\sin(v), u\cos(v))\end{aligned}$$

   b.

$$\begin{aligned}(\mathbf{T}_u \times \mathbf{T}_v) &= (u\cos^2(v) + u\sin^2(v), -2u^2\cos(v), -2u^2\sin(v)) \\ &= (u, -2u^2\cos(v), -2u^2\sin(v)).\end{aligned}$$

Therefore,

$$\|\mathbf{T}_u \times \mathbf{T}_v\|^2 = u^2 + (2u^2)^2 \implies \|\mathbf{T}_u \times \mathbf{T}_v\| = u\sqrt{1+4u^2}.$$

The unit normal is then given by

$$\hat{\mathbf{n}} = \frac{(\mathbf{T}_u \times \mathbf{T}_v)}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \left(\frac{1}{\sqrt{1+4u^2}}, -\frac{2u\cos(v)}{\sqrt{1+4u^2}}, -\frac{2u\sin(v)}{\sqrt{1+4u^2}}\right)$$

c. The area of the surface is given by

$$\begin{aligned} A(S) &= \iint_S \|\mathbf{T}_u \times \mathbf{T}_v\|\, dudv \\ &= \int_0^{2\pi} \int_0^3 u\sqrt{1+4u^2}\, dudv \\ &= \frac{\pi}{6}\left(37\sqrt{37}-1\right). \end{aligned}$$

3. (25 points) Let C be the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

   a. (10 points) Write a parametrization $\mathbf{c}(t)$ that traces out the arc C for $-1 \le t \le 2$.
   b. (10 points) Compute the path integral $\int_{\mathbf{c}} \frac{y}{x}\, ds$.
   c. (5 points) Let $f(x, y) = \frac{e^x}{x^2+y^2}$. Compute the line integral $\int_{\mathbf{c}} (\nabla f) \cdot d\mathbf{s}$.

Solution.

a. Let $\mathbf{c}(t) = \left(t, t^2\right)$ with $t \in [-1, 2]$.

b.

$$\begin{aligned} \int_{\mathbf{c}} \frac{y}{x} ds &= \int_{\mathbf{c}} \frac{y(t)}{x(t)} \|\mathbf{c}'(t)\|\, dt \\ &= \int_{-1}^{2} t\sqrt{1+4t^2}\, dt \\ &= \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

c. The original intent of this problem was to test the Fundamental Theorem of Line Integrals.

$$\int_{\mathbf{c}} (\nabla f) \cdot d\mathbf{s} = \int_{t_1}^{t_2} \left(\frac{d}{dt} f(\mathbf{c}(t))\right) dt = f(\mathbf{c}(t_2)) - f(\mathbf{c}(t_1)),$$

so that for $f(x, y) = \frac{e^x}{x^2+y^2}$, we would have

$$\int_{\mathbf{c}} (\nabla f) \cdot d\mathbf{s} = f(2, 4) - f(-1, 1) = \frac{e^2}{20} - \frac{e^{-1}}{2}.$$

However, the function is not defined at the point $(0, 0)$ (at the origin, the function is infinity), and hence it is not a $\mathcal{C}^1$ function on $\mathbb{R}^2$. Therefore, the theorem cannot be applied. The line integral, as written, does not exist because $\mathbf{c}(t)$ passes through a point (the origin) that is not in the domain of $f(x, y)$. It was not at all the original intent of the problem to test these subtleties. Since the function is $\mathcal{C}^1$ on its domain, $\mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$, the theorem could be applied to other any curve with these two endpoints that does not pass through the origin. In lieu of this, students who applied the fundamental theorem still received full marks.

4. (25 points) Let S be the upper half of the unit sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Suppose the surface is oriented so that the outer normal points upward, i.e., $\mathbf{n} \cdot \mathbf{k} > 0$. Find the flux $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F} = \frac{z}{2}\mathbf{k}$ through S.

Solution. A parametrization of the upper half of a unit sphere is given by

$$\mathbf{\Phi}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

Then

$$\begin{aligned}
\text{Flux through } S &= \iint_S \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_S \mathbf{F}(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) \cdot \hat{\mathbf{n}} \, \|\mathbf{T}_\theta \times \mathbf{T}_\phi\| \, d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \left(0, 0, \tfrac{1}{2}\cos\phi\right) \cdot (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \sin\phi \, d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \tfrac{1}{2}\cos^2\phi \sin\phi \, d\phi d\theta \\
&= \pi \int_0^1 u^2 \, du \\
&= \frac{\pi}{3}.
\end{aligned}$$