Math 102 - Winter 2013 - Final Exam

Problem 1.

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix}.$$

(i) Find the left inverse of A.
(ii) Find the matrix of the projection onto the column space of A.
(iii) Find the matrix of the projection onto the left null space of A.
(iv) Find the QR decomposition of A.

`Solution:`

(i) *We have*

$$A^+ = (A^TA)^{-1}A^T.$$

In our case,

$$A^TA = \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$$

hence

$$(A^TA)^{-1} = \frac{1}{4}\begin{bmatrix} 4 & -6 \\ -6 & 10 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

We find

$$A^+ = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 2 & -1 & -2 & 1 \end{bmatrix}.$$

(ii) *The projection onto the column space of A equals*

$$AA^+ = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 2 & -1 & -2 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

(iii) *The left null space of A is the orthogonal complement of the column space. The two projections in (ii) and (iii) therefore add up to the identity. The matrix of the projection onto the left null space is*

$$I - AA^+ = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(iv) *We begin by normalizing the first column of A. The first column v_1 of A has length*

$$||v_1|| = \sqrt{10}.$$

The first column of R simply contains the entry $\sqrt{10}$ in the upper left corner and zero elsewhere. Then, the normalized first column of A equals

$$q_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}.$$

This is the first column of Q.

We now consider the second column v_2 of A. Using v_2, we produce a vector y_2 orthogonal to q_1. We have

$$q_1 \cdot v_2 = \frac{6}{\sqrt{10}}$$

hence

$$y_2 = v_2 - (q_1 \cdot v_2) q_1 = \frac{1}{5} \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

We have

$$||y_2|| = \frac{\sqrt{10}}{5}$$

hence the normalized second column is

$$q_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

We have

$$Q = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ -1 & -2 \\ -2 & 1 \end{bmatrix}$$

and

$$R = \begin{bmatrix} \sqrt{10} & 6/\sqrt{10} \\ 0 & \sqrt{10}/5 \end{bmatrix}.$$

Problem 2.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & 2 & 5 \\ -2 & -6 & -1 & 4 \end{bmatrix}.$$

(i) Find the LU decomposition of the matrix A.
(ii) Find a basis for the column space of A. What is the rank of A?
(iii) Find a basis for the null space of A.
(iv) Show that the columns of A are linearly dependent by exhibiting an explicit linear relation between them.
(v) Find a basis for the row space of A.
(vi) Find a basis for the orthogonal complement of the column space of A.
(vii) Does A admit either a left inverse or a right inverse?

`Solution:`

(i) *The row operations $R_2 \to R_2 - 3R_1$ followed by $R_3 \to R_3 + 2R_1$ yield the matrix*

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -1 & 2 \\ 0 & -2 & 1 & 6 \end{bmatrix}.$$

We continue with $R_3 \to R_3 - R_2$ thus yielding the upper triangular matrix

$$U = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

The lower triangular matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

We have

$$A = LU.$$

(ii) *We further row-reduce until we find*

$$\mathtt{rref} A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

There are three pivots, hence the rank is 3. A basis for the column space is provided by the first three columns of A, namely $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(iii) *To find the null space of A, we see that the free variable is the last one. From the row-reduced form we set up the system*

$$x + 3w = 0$$
$$y - 2w = 0$$
$$z + 2w = 0$$

which has a unique solution, up to multiples, given by

$$\begin{bmatrix} -3 \\ 2 \\ -2 \\ 1 \end{bmatrix}.$$

(iv) *From the entries of the vector spanning the null space, we can form a relation between the columns c_1, c_2, c_3, c_4 of A. We have*

$$-3c_1 + 2c_2 - 2c_3 + c_4 = 0.$$

(v) *Since the rank is 3, and the rank can be calculated as the dimension of the row space, it means that the rows of A span the row space and are linearly independent. The basis is given by the three row vectors* $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ -1 \\ 4 \end{bmatrix}.$

(vi) *The orthogonal complement of the column space of A is 0 since $C(A) = \mathbb{R}^3$.*

(vii) *The rank of A equals the number of rows. Thus A admits a right inverse.*

Problem 3.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

(i) Write down the SVD for the matrix A.

(ii) Find the pseudoinverse of A.

(iii) Find the matrix of the projection onto the row space of A.

(iv) From the SVD, write down orthonormal bases for

- the column space of A,
- the row space of A,
- the null space of A,
- the left null space of A.

(v) Consider the incompatible system

$$Ax = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Write down all least squares solutions, and the least squares solution of minimum length.

`Solution:`

(i) *We have*

$$A^T A = \begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix}$$

with eigenvalues 30 *and* 0. *The eigenvectors are*

$$v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, v_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}.$$

This gives already the matrix

$$V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \sqrt{30} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It remains to find U. *First*

$$AA^T = \begin{bmatrix} 5 & -5 & 10 \\ -5 & 5 & -10 \\ 10 & -10 & 20 \end{bmatrix}$$

with eigenvalues $30, 0, 0$. *The* $\lambda = 30$ *unit eigenvector is*

$$u_1 = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

The $\lambda = 0$ *eigenspace is the null space of the above matrix, that is*

$$x - y + 2z = 0.$$

A possible basis is

$$w_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, w_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

This basis is not orthonormal, so we apply Gram-Schmidt. We first normalize w_2 *to the vector*

$$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{bmatrix}.$$

Then we compute

$$y_3 = w_3 - (u_2 \cdot w_3)u_2 = w_3 - \frac{4}{\sqrt{5}}u_2 = \frac{1}{5}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

which we normalize to the vector

$$u_3 = \begin{bmatrix} 2/\sqrt{45} \\ 5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}.$$

We find

$$U = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 2/\sqrt{45} \\ -1/\sqrt{6} & 0 & 5/\sqrt{45} \\ 2/\sqrt{6} & -1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix}.$$

(ii) *The pseudoinverse is*

$$A^+ = V\Sigma^+U^T,$$

where

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sqrt{30}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find

$$A^+ = \frac{1}{30}\begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \end{bmatrix}.$$

(iii) *The matrix of the projection is*

$$AA^+ = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}.$$

(iv) *The rank of A is* 1*. The first column of U is spans the column space of A, the second and third columns of U span the left null space. The first column of V spans the row space of A, the second column of V spans the null space of A.*

(v) *The least square solutions solve the system*

$$A^T A x = A^T b$$

which becomes

$$\begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix} x = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

Solutions all satisfy

$$24x_1 + 12x_2 = 12 \implies x_1 = \frac{1}{2} - \frac{1}{2}x_2,$$

hence the general solution is

$$x = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

The least squares of minimum length is $A^+ b = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Problem 4.

Consider $\mathcal{P}$ the space of polynomials of degree at most equal to 2. Consider the basis $\mathcal{B}$ consisting of the polynomials $\{1, 1-x, x^2-x\}$.

(i) Find the matrix of the linear transformation $\mathcal{T} : \mathcal{P} \to \mathcal{P}$,

$$\mathcal{T}(f) = x^2 f'' + x f',$$

in the basis $\mathcal{B}$.

(ii) Using (i), find the rank of $\mathcal{T}$ and a basis for the column space of $\mathcal{T}$.

(ii) Endow $\mathcal{P}$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x).$$

Starting with the basis $\mathcal{B}$, obtain an orthogonal basis for $\mathcal{P}$ via Gram-Schmidt.

`Solution:`

(i) *We have*

$$\mathcal{T}(1) = 0$$
$$\mathcal{T}(1-x) = -x = 1 \cdot (1-x) + (-1) \cdot 1$$
$$\mathcal{T}(x^2 - x) = 2x^2 + x(2x-1) = 4x^2 - x = 4(x^2-x) - 3(1-x) + 3 \cdot 1.$$

Reading off the coefficients above, we obtain the matrix

$$T = \begin{bmatrix} 0 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}.$$

(ii) *Clearly, the matrix T has two independent columns, namely the second and third. Its rank is therefore 2. The basis for the column space is given by the second and third column, which correspond to*

$$\mathcal{T}(1-x) = -x, \mathcal{T}(x^2-x) = 4x^2 - x.$$

(iii) *We have $P_1 = 1, P_2 = 1-x, P_3 = x^2 - x$. Running Gram-Schmidt, we first set $Q_1 = P_1 = 1$. To find Q_2 we have*

$$Q_2 = P_2 - \frac{\langle P_2, Q_1 \rangle}{\langle Q_1, Q_1 \rangle} \cdot Q_1 = (1-x) - \frac{\langle 1-x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 = (1-x) - \frac{\int_{-1}^{1} 1 - x\, dx}{\int_{-1}^{1} dx} \cdot 1 = -x.$$

Finally, we have

$$Q_3 = P_3 - \frac{\langle P_3, Q_1 \rangle}{\langle Q_1, Q_1 \rangle} \cdot Q_1 - \frac{\langle P_3, Q_2 \rangle}{\langle Q_2, Q_2 \rangle} \cdot Q_2.$$

Note that

$$\langle Q_1, Q_1 \rangle = \langle 1, 1 \rangle = \int_{-1}^{1} dx = 2$$

$$\langle P_3, Q_1\rangle = \langle x^2 - x, 1\rangle = \int_{-1}^{1} x^2 - x\, dx = \frac{2}{3}$$
$$\langle P_3, Q_2\rangle = \langle x^2 - x, -x\rangle = \int_{-1}^{1} -x^3 + x^2\, dx = \frac{2}{3},$$
$$\langle Q_2, Q_2\rangle = \langle -x, -x\rangle = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$$

hence substituting we find

$$Q_3 = (x^2 - x) - \frac{2/3}{2} \cdot 1 - \frac{2/3}{2/3}(-x) = x^2 - \frac{1}{3}.$$

The orthogonal basis is $\{1, -x, x^2 - \frac{1}{3}\}$.

Problem 5.

Consider the quadratic form

$$Q(x, y, z) = 3x^2 + 3y^2 + 3z^2 + 4xy + 4yz + 4zx.$$

(i) Discuss the definiteness of the form Q, using any method you wish.

(ii) Using any method developed in this course, express Q as a sum of three squares.

`Solution:`

(i) *The associated matrix is*

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

We claim the matrix is positive definite, hence so is Q. *First, note that* $\lambda_1 = 1$ *is an eigenvalue. The eigenspace is*

$$E_1 = N(A - I)$$

which is immediately seen to the null space of the matrix

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

This is the space of vectors with

$$x + y + z = 0.$$

A basis for this eigenspace is

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Thus 1 *is an eigenvalue with multiplicity* 2. *Since the trace is* 9 *and is the sum of eigenvalues, and* 1 *is a repeated eigenvalue, we conclude that the eigenvalues of* A *are* $1, 1, 7$. *The matrix* A *has positive eigenvalues, hence it is positive definite.*

(ii) *We will write*

$$A = R^T R$$

first. There are many ways of finding R *here. One method is via the orthogonal diagonalization*

$$A = QDQ^T$$

by setting

$$R = \sqrt{D} Q^T.$$

We first find orthogonally diagonalize A. Note that a basis for the $\lambda_1 = \lambda_2 = 1$ eigenspace is

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

These vectors are not orthogonal so we need to run Gram-Schmidt for them. First

$$q_1 = \frac{1}{\sqrt{2}} u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

To find q_2, we first calculate the vector

$$y_2 = u_2 - (u_2 \cdot q_1) q_1 = u_2 - \frac{1}{\sqrt{2}} q_1 = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}.$$

We have $||y_2|| = \frac{\sqrt{6}}{2}$ hence

$$q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The eigenvector for $\lambda_3 = 7$ is found from the null space of

$$A - 7I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 3 & -4 \end{bmatrix}.$$

An eigenvector is

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We need to normalize this vector to have length 1:

$$q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

Then $R = \sqrt{D} Q^T$ becomes

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \sqrt{1/6} & -2/\sqrt{6} & 1/\sqrt{6} \\ \sqrt{7/3} & \sqrt{7/3} & \sqrt{7/3} \end{bmatrix}.$$

The three squares

$$f = f_1^2 + f_2^2 + f^3$$

are found by computing

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

This yields $f_1 = \frac{1}{\sqrt{2}}(x - z)$, $f_2 = \frac{1}{\sqrt{6}}(x - 2y + z)$, $f_3 = \sqrt{7/3}(x + y + z)$. *There are other possible answers.*

Problem 6.

TRUE OR FALSE (no explanation is necessary):

(T) The set of all skew symmetric $n \times n$ matrices is a vector space.

(T) Any complex normal matrix is diagonalizable.

(F) The product of two skew Hermitian matrices is Hermitian.

(T) For any matrix A with singular values $\sigma_1, \ldots, \sigma_r$

$$\text{Trace } (AA^T) = \sigma_1^2 + \ldots + \sigma_r^2.$$

(F) There exist skew-Hermitian matrices of determinant $1 + i$.

(T) The LU decomposition of an invertible matrix, if it exists, must be unique.

(F) The QR decomposition, if it exists, must be unique.

(F) The positive decomposition $A = R^T R$ of a symmetric matrix, if it exists, must be unique.

(T) If A and B are unitarily similar, then $\exp(A)$ and $\exp(B)$ are unitarily similar.

(T) Any positive definite quadratic form of n variables can be written as sum of n squares of linear terms.

(T) If Q is a unitary matrix, then $Q + 2I$ is invertible.

(T) For any complex vector $\mathbf{v}$, the matrix

$$R = I - 2\mathbf{v} \cdot \mathbf{v}^H$$

has real eigenvalues.

(T) A symmetric matrix has as many positive pivots as positive eigenvalues.

(T) All positive definite symmetric matrices admit LU decompositions.

(F) The rule

$$\langle f, g \rangle = f(0)\, g(0) + f(1)\, g(1)$$

defines an inner product on the space of polynomials of degree less or equal to 2.

Problem 7.

Consider the Fibonacci-type recursion

$$G_{n+2} = \frac{1}{3}G_{n+1} + \frac{2}{3}G_n,\ G_0 = 0,\ G_1 = 1.$$

(i) Let $\vec{x}_n = \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}$. Write down a difference equation that $\vec{x}_n$ satisfies in the form

$$\vec{x}_{n+1} = A\vec{x}_n.$$

(ii) Discuss the stability of the difference equation.

(iii) Solve the difference equation, and write down an explicit formula for G_n. What is the limit of G_n as $n \to \infty$?

`Solution:`

(i) *Write* $x_n = \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}$. *We have*

$$\begin{bmatrix} G_{n+2} \\ G_{n+1} \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}.$$

Thus

$$x_{n+1} = Ax_n,\ \text{for } A = \begin{bmatrix} 1/3 & 2/3 \\ 1 & 0 \end{bmatrix}.$$

(ii) *We find the eigenvalues and eigenvectors of the matrix A. First*

$$Tr(A) = \frac{1}{3}, \det A = -\frac{2}{3}$$

hence the eigenvalues solve the equation

$$\lambda^2 - \frac{1}{3}\lambda - \frac{2}{3} = 0 \implies \lambda_1 = 1, \lambda_2 = -\frac{2}{3}.$$

Thus, the difference equation is neutrally stable.

(iii) *We have*

$$x_n = A^n x_0 = A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The eigenvalue $\lambda_1 = 1$ has eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

while for $\lambda_2 = -\frac{2}{3}$ we find

$$v_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Thus, for

$$C = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \implies C^{-1} = \frac{-1}{5}\begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}.$$

We have

$$A = C\begin{bmatrix} 1 & 0 \\ 0 & -2/3 \end{bmatrix} C^-1 \implies A^n = C\begin{bmatrix} 1 & 0 \\ 0 & (-\frac{2}{3})^n \end{bmatrix} C^{-1}.$$

Thus

$$x_n = C \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{2}{3}\right)^n \end{bmatrix} C^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \cdot \left(1 - \left(-\frac{2}{3}\right)^{n+1}\right) \\ \frac{3}{5} \cdot \left(1 - \left(-\frac{2}{3}\right)^n\right) \end{bmatrix}$$

which yields

$$G_n = \frac{3}{5} \cdot \left(1 - \left(-\frac{2}{3}\right)^n\right).$$

We conclude

$$G_n \to \frac{3}{5}$$

as $n \to \infty$.

Problem 8.

Let $A = UDV^T$ be the singular value decomposition of the $n \times n$ matrix A with singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0.$$

Let $A^+ = VD^+U^T$ denote the pseudoinverse of A.

(i) Explain why
   - the null space of A equals the left null space of A^+;
   - the column space of A equals the row space of A^+.

(ii) Confirm (by direct calculation) that

$$DD^+D = D.$$

(iii) Using (ii), confirm that

$$AA^+A = A.$$

`Solution:`

(i)
   - *from the SVD decomposition of A, we see that the null space of A is spanned by the last $n-r$ columns of V. Since*

$$A^+ = VD^+U^T$$

   is the SVD decomposition for A^+, we read off the left null space of A^+ from the last $n-r$ columns of V as well;
   - *the column space of A is spanned by the first r columns of U; from the SVD decomposition of $A^+ = VD^+U^T$, we see that the row space of A^+ is spanned by the first r columns of U as well.*

(ii) *We have*

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & \sigma_r & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Then

$$D^+ = \begin{bmatrix} \sigma_1^{-1} & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2^{-1} & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & \sigma_r^{-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

and an immediate computation shows

$$DD^{+} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Note that this is not the identity matrix, because the last rows are rows of zeroes. Multiplying further by D immediately gives

$$DD^{+}D = D.$$

(iii) *We have*

$$AA^{+}A = (UDV^{T})(VD^{+}U^{T})(UDV^{T}) = UD(V^{T}V)D^{+}(U^{T}U)DV^{T}.$$

Using U, V are symmetric, we find

$$V^{T}V = I, U^{T}U = I$$

hence

$$AA^{+}A = UDD^{+}DV^{T}.$$

Using (ii), we find $DD^{+}D = D$ hence

$$AA^{+}A = UDV^{T} = A.$$