Practice Problems for Midterm II

Math 102, Winter 2018

1. Consider the vector subspace of $\mathbb{R}^4$ spanned by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) Find a basis for the orthogonal complement $V^\perp$.

   (b) Using Gramm-Schmidt, find an orthonormal basis for $V^\perp$.

   (c) Find the matrix of the orthogonal projection onto $V^\perp$ using the basis you found.

   (d) Find the projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto $V^\perp$. Derive from here the projection of the same vector onto V.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \\ 1 & 1 \\ 1 & -4 \end{bmatrix}, \text{ and the vector } \boldsymbol{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}.$$

   (a) Find the left inverse of the matrix A.

   (b) Using the calculation in part (a), find the matrix of the orthogonal projection onto the column space of A.

   (c) Find the least squares solution to the system

$$A\boldsymbol{x} = \boldsymbol{b}$$

   using the left inverse you calculated in part (a).

   (d) Find the QR decomposition of A.

   (e) Now redo part (c). That is, find the least squares solution to the system

$$A\boldsymbol{x} = \boldsymbol{b}$$

   using the QR decomposition you found in part (d).

3. Calculate the determinant of the matrix

$$\begin{bmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{bmatrix}$$

   (a) using either row or column operations;

   (b) using the method of cofactors.

4. Find the inverse of the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

using the method of cofactors.

5. The Laguerre polynomials are important in quantum mechanics, in writing down the solution of the Schrodinger equation for the hydrogen atom.

   (a) Show that
   $$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x}\, dx$$
   is an inner product on the space $\mathcal{P}$ of polynomials of degree at most 2.

   (b) Starting with the basis $\{1, x, x^2\}$, obtain an orthogonal basis for $\mathcal{P}$ using the Gram-Schmidt method. The resulting polynomials are the Laguerre polynomials.

   For this problem you may use the values of the integrals (called the gramma function):
   $$\int_0^\infty x^n e^{-x}\, dx = n!$$

6. (a) Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & 0 \\ -5 & 0 & 6 \\ 0 & 8 & 9 \end{bmatrix}$.

   (b) Find the area of the triangle with vertices at points $(1, 1), (2, 3), (-1, 5)$.

7. Use the Gram-Schmidt process to find 2 orthonormal vectors forming a basis for the column space of the matrix $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

8. True-False. Tell whether the following statements are true or false. If true, give a brief explanation and if false, give a counterexample.

   (a) Every matrix A is diagonalizable (i.e., A is of the form $A = PDP^{-1}$ with D diagonal).

   (b) $\det(AB) = \det(BA)$.