Math 20E - Lecture A00 **Name:** Key
Fall 2016
Midterm #1, VERSION B PID: ____________
10/21/2016
Time Limit: 50 Minutes Section Time: ____________

This exam contains 3 pages (including this cover page) and 4 questions.
Total of points is 100.

You may not use any notes (except your cheat sheet) or calculators during this exam.
Write your *Name, PID, and Section* on the front of your Blue Book.
Write the *Version* of your exam on the front of your Blue Book.
Write your solutions clearly in your Blue Book.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.

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1. (25 points) Let $f(x,y) = e^{x+y^2}$.

   a. (15 points) Find the tangent plane to the surface given by the graph of the function $z = f(x,y)$ at the point $(0,0,1)$.
   Solution. The tangent plane at the point $(0,0,1)$ is given by

   $$\begin{aligned} z &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 1 + x, \end{aligned}$$

   where the find equality follows from the following terms: $f(0,0) = 1$,

   $$f_x(x,y) = e^{x+y^2} \Rightarrow f_x(0,0) = 1$$

   and
   $$f_y(x,y) = 2ye^{x+y^2} \Rightarrow f_y(0,0) = 0.$$

   b. (10 points) Determine the second order Taylor polynomial of the function at this point.
   Solution. There are many possible way to write the second order Taylor polynomial (see the solution of Version C for discussion). One possible way is the following

   $$g(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\Bigg|_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix}.$$

   We already found the first order partial derivatives in part a. So we need to find the second order partial derivatives

   $$f_{xx}(x,y) = e^{x+y^2} \Rightarrow f_{xx}(0,0) = 1,$$

   $$f_{xy}(x,y) = 2ye^{x+y^2} \Rightarrow f_{xy}(0,0) = 0,$$

   $$f_{yy}(x,y) = 2e^{x+y^2} + 4y^2e^{x+y^2} \Rightarrow f_{yy}(0,0) = 2.$$

Hence,

$$\begin{aligned} g(x,y) &= 1 + x + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 1 + x + \frac{1}{2}x^2 + y^2. \end{aligned}$$

2. (25 points) Let $f(u,v) = (u^2 - v^2, \sin(uv))$ and $g(x,y) = (ye^{x^2}, xe^{y^2})$.

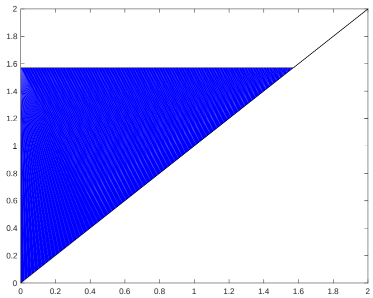
   a. (10 points) Compute $Dg(x,y)$ for any given point $(x,y) \in \mathbb{R}^2$.
   Solution.
   $$Dg(x,y) = \begin{bmatrix} \frac{\partial(ye^{x^2})}{\partial x} & \frac{\partial(ye^{x^2})}{\partial y} \\ \frac{\partial(xe^{y^2})}{\partial x} & \frac{\partial(xe^{y^2})}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xye^{x^2} & e^{x^2} \\ e^{y^2} & 2xye^{y^2} \end{bmatrix}.$$

   b. (15 points) Compute $D(f \circ g)(1,0)$.
   Solution. By the Chain Rule,
   $$D(f \circ g)(1,0) = Df(g(1,0))Dg(1,0).$$
   We observe that
   $$Dg(1,0) = \begin{bmatrix} 0 & e \\ 1 & 0 \end{bmatrix},$$
   and
   $$Df(u,v) = \begin{bmatrix} 2u & -2v \\ v\cos(uv) & u\cos(uv) \end{bmatrix} \Longrightarrow Df(g(1,0)) = Df(0,1) = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$
   Therefore,
   $$D(f \circ g)(1,0) = Df(g(1,0))Dg(1,0) = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & e \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & e \end{bmatrix}.$$

3. (25 points) Change the order and evaluate
$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx.$$
Be sure to clearly sketch the region of integration and indicate how you found the new limits of integration.

Solution.

$$\begin{aligned} \int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx &= \int_0^{\pi/2} \int_y^{0} \frac{\sin y}{y}\, dx\, dy \\ &= \int_0^{\pi/2} \frac{\sin y}{y}\, x\big|_{x=0}^{x=y}\, dy \\ &= \int_0^{\pi/2} \sin y\, dy \\ &= -\cos y\big|_0^{\pi/2} \\ &= -\cos\frac{pi}{2} + \cos(0) \\ &= 1. \end{aligned}$$



4. (25 points) Let $D^* = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$ and let D be the image of D^* under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(u, v) = (u^2 v, uv^2)$. Calculate the area of D.

   Solution. See Version C.