Lecture 25: Jordan form; Singular Value Decompositions (Sections 5.6-6.3)

Thang Huynh, UC San Diego

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The Jordan form

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How if A cannot be diagonalized?

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$$\text{Jordan form} \qquad J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}.$$

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Jordan form $$J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}.$$

Each J_i has only a single eigenvalue λ_i and one eigenvector:

Jordan block $$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}.$$

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The Jordan form

- Example. $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

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- Definition. If A is an $n \times n$ matrix, a generalized eigenvector of A corresponding to the eigenvalue λ is a nonzero vector $\boldsymbol{x}$ satisfying

$$(A - \lambda I)^p \boldsymbol{x} = 0$$

for some positive integer p.

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$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$M^{-1}TM = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

The Jordan form

▶ Example. For $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Which of the following matrices are the Jordan forms of A and B?

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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- The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

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$$A = U\Sigma V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}.$$

Least squares and the Pseudoinverse

A least squares solution $\hat{\boldsymbol{x}}$ of the linear system

$$A\boldsymbol{x} = \boldsymbol{b}$$

is the one minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|$. To solve this, we can solve the associate normal system

$$A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}.$$

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$$\boldsymbol{x}^\dagger = A^\dagger \boldsymbol{b}.$$

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The singular values $\sigma_1, \ldots, \sigma_r$ are on the diagonal of Σ, and their reciprocals $\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r}$ are on the diagonal of $\Sigma^\dagger$.

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▶ Example. Find the pseudoinverse of $A = [-1\ 2\ 2]$.

The SVD of A is

$$A = [1][3\ 0\ 0]\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

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Then the pseudoinverse of A is

$$A^\dagger = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}[1] = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9} \end{bmatrix}.$$