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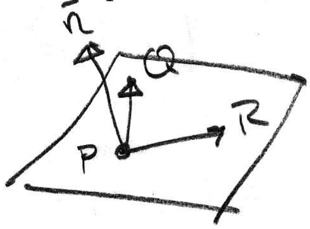
TA: Sec. No: Sec. Time:

Math 20E
Midterm Exam 1.
February 1, 2013

Turn off and put away your cell phone.
No electronic devices may be used during this exam.
You may use a one-sided page of notes, but no books or other assistance during this exam.
Read each question carefully, and answer each question completely.
Show all of your work, justify each step, and state any theorems or non-trivial results used from this class; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

1. (10 points) Determine an equation for the plane that contains the points: $P = (3, 2, 0)$, $Q = (-1, 3, 1)$, $R = (2, 0, 3)$.

$\vec{n} = \vec{PQ} \times \vec{PR}$ is normal to plane.



$$\vec{n} = \langle -4, 1, 1\rangle \times \langle -1, -2, 3\rangle$$

$$= \begin{vmatrix} i & j & k \\ -4 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = \langle 5, 11, 9\rangle$$

Plane $ax + by + cz = ax_0 + by_0 + cz_0$

$$\boxed{5x + 11y + 9z = 37}$$

$5 \cdot 2 + 9 \cdot 3 = 10 + 27 = 37$

#	Points	Score
1	10	
2	10	
3	10	
4	10	
4	10	
Σ	50	

2. (10 points) Let $f(x, y) = ye^{xy} + x^2e^{y^2}$.

$f(1,0) = 1$

(a) Find an equation for the tangent plane of f at the point $(1, 0)$.

$f_x|_{(1,0)} = (y^2e^{xy} + 2xe^{y^2})|_{(1,0)} = 2$

$f_y|_{(1,0)} = (e^{xy} + xye^{xy} + 2x^2ye^{y^2})|_{(1,0)} = 1$

Tangent Plane: $\boxed{z = 1 + 2(x-1) + y}$

(b) Write down the second order Taylor formula for f at the point $(1, 0)$

$f_{xx}|_{(1,0)} = (y^3e^{xy} + 2e^{y^2})|_{(1,0)} = 2$

$f_{yy}|_{(1,0)} = (xe^{xy} + xe^{xy} + x^2ye^{xy} + 2x^2e^{y^2} + 4x^2y^2e^{y^2})|_{(1,0)}$

$= (1 + 1 + 2) = 4$

$f_{xy}|_{(1,0)} = (2ye^{xy} + xy^2e^{xy} + 4xye^{xy}))|_{(1,0)} = 0$

Taylor's Formula:

$$f(x,y) = 1 + 2(x-1) + y + (x-1)^2 + 2y^2 + R_2(x,y)$$

where $\displaystyle\lim_{(x,y)\to(1,0)} \frac{R_2(x,y)}{\|(x,y)-(1,0)\|^2} = 0$

3. (10 points) Let $f(x, y, z) = xye^{xz}$ and $g(u, v) = (\overset{x}{u}, \overset{y}{u - v}, \overset{z}{v})$.

(a) Which of $f \circ g$ and $g \circ f$ is defined? Fill in the blanks.

$$f : \mathbb{R}^{(3)} \to \mathbb{R}^{(1)}, \quad g : \mathbb{R}^{(2)} \to \mathbb{R}^{(3)}$$

$$(f \circ g) : \mathbb{R}^{(2)} \to \mathbb{R}^{(1)}$$

(b) Compute the derivative of the composition from part (a) by using the chain rule. (The derivative will be a matrix)

$$\underset{1\times 2}{D(f\circ g)} = \underset{1\times 3}{Df(g)} \cdot \underset{3\times 2}{Dg}$$

$$= \left(ye^{xz} + xyze^{xz},\ xe^{xz},\ x^2ye^{xz}\right)\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \left((x+y+xyz)e^{xz} \quad (x^2y - x)e^{xz} \right)$$

$$= \left((u + u - v + uv(u-v))\, e^{uv} \quad (u^2(u-v) - u)\, e^{uv} \right)$$

$$= \left((2u - v + u^2v - uv^2)e^{uv} \quad (u^3 - u^2v - u)e^{uv} \right)$$

4. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (xy, \cos(x), 1 - y^2)$ and assume $c(t) : \mathbb{R} \to \mathbb{R}^2$ is a path so that $c(0) = (\pi, 1)$, $c'(0) = (2, 1)$.

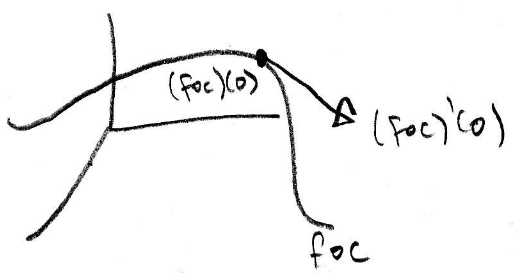
$Dc\circ f \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(a) Compute $D(f \circ c)(0)$.

$$D(f\circ c)(0) = Df(c(0)) \cdot Dc(0)$$

$$= \left.\begin{pmatrix} y & x \\ -\sin x & 0 \\ 0 & -2y \end{pmatrix}\right|_{(\pi,1)} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \pi \\ 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2+\pi \\ 0 \\ -2 \end{pmatrix}}$$

(b) The image of the path $f \circ c$ is a curve in $\mathbb{R}^3$. What is a vector tangent to the curve at $(f \circ c)(0)$? Parameterize the tangent line to the curve at this point.



$$(f\circ c)(0) = f(c(0)) = f(\pi, 1)$$
$$= (\pi, -1, 0)$$
$$(f\circ c)'(0) = (2+\pi, 0, -2) \text{ by (a)}$$

Tangent Line:

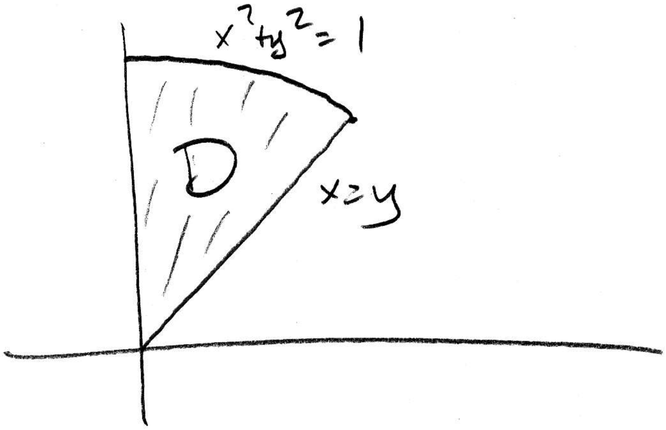
$$\boxed{L(t) = (\pi, -1, 0) + t(2+\pi, 0, -2)}$$

5. (10 Points) Let D be the domain in $\mathbb{R}^2$ determined by the conditions

$$y > x, \quad x > 0, \quad \text{and } x^2 + y^2 < 1$$

Compute the following integral (Hint: $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$)

$$\iint_D x^2(x^2+y^2)^{-1/2} dA$$



$x^2+y^2 = r^2$ $\quad x = r\cos\theta$

$y = r\sin\theta$

By the C.O.V. Formula:

$D: \frac{\pi}{4} < \theta < \frac{\pi}{2}$

$0 < r < 1$

$$\iint_D x^2(x^2+y^2)^{-\frac{1}{2}} dA$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r^2\cos^2\theta\,(r^2)^{-\frac{1}{2}} \cdot r\,dr\,d\theta$$

$$= \left(\int_0^1 r^2 dr\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos(2\theta))\,d\theta\right)$$

$$= \left(\frac{1}{3}r^3\Big|_0^1\right)\frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{1}{6}\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$$

$$= \frac{1}{6}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \boxed{\frac{\pi}{24} - \frac{1}{12}}$$