Section 3.2. Linear homogeneous equations.

(I) Interval of existence:

$$(*)\quad \begin{cases} y'' + p(t)\,y' + q(t)\,y = g(t) \\ y(t_0) = y_0\ , \quad y'(t_0) = y_0' \end{cases}$$

$\uparrow$ another constant.

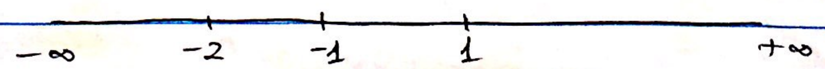
Thm: If $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval I that contains t_0, then $(*)$ has exactly one solution in I.

Ex 1. Find largest interval of existence.

$$\begin{cases} (t-1)y'' + (t+1)y' - \dfrac{t}{t+2}\,y = \dfrac{t+3}{t+1} \\ y(0) = 2\ , \quad y'(0) = 1. \end{cases}$$

$$p(t) = \frac{t+1}{t-1}\ , \quad q(t) = \frac{t}{(t+2)(t-1)} \qquad q(t) = \frac{t+3}{(t+1)(t-1)}$$

$\Rightarrow$ discont. points : $t = -2, -1, 1$.



$t_0 = 0 \in (-1, 1)$

so in $(-1, 1)$ sol. exists uniquely.

(II). Principle of Superposition.

If $y = y_1(t)$ and $y = y_2(t)$ are solutions to

$$y'' + p(t)\,y' + q(t)\,y = 0$$

then $y = c_1 y_1(t) + c_2 y_2(t)$ is also a solution.

III The Wronskian

$$y'' + p(t)y' + q(t)y = 0$$

Suppose sols $y = y_1(t)$ and $y = y_2(t)$

Then for I.V.P $\begin{cases} y'' + p(t)y' + q(t)y = \cancel{g(t)}\ 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$

Try $y = c_1 y_1(t) + c_2 y_2(t)$

Then ~~for~~ $\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y'(t_0) = y_0' \end{cases}$

Solve for c_1, c_2

Rewrite $\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{cases}$ (1)

The determinant of coefficient of (1) is

$$W = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

$$= \begin{vmatrix} y_1(t_0) & y_2(t) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}.$$

If $W \neq 0$, then (1) has <u>a unique solution (c_1, c_2)</u>

This solution is given by

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

and

$$c_2 = \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)} = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

W is called the <u>Wronskian determinant</u> or the Wronski[an] of the solutions y_1 and y_2.

Thm: Let $y_1(t)$ and $y_2(t)$ be solutions of

$$y'' + p(t)y' + q(t)y = 0$$

Then $c_1y_1(t) + c_2y_2(t)$ includes all the solutions if and only if $W(y_1, y_2) \neq 0$. at some t_0.

E.g. $t^2y'' + ty' - y = 0$

sol. $y_1 = \frac{1}{t}$ and $y_2 = t$.

does $y = c_1y_1(t) + c_2y_2(t)$ include all the sol'ns?

$y_1'(t) = -\frac{1}{t^2}$ and $y_2'(t) = 1$.

$$\cancel{W[y_1,} \; W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$= \frac{1}{t}(1) - t\left(\frac{-1}{t^2}\right)$$

$$= \frac{1}{t} + \frac{1}{t} = \frac{2}{t} \neq 0$$

for $t \neq 0$.

So $c_1y_1 + c_2y_2$ includes all the solutions.

Thm: (Abel's Thm)

The Wronskian of $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous on some interval I

is given by

$$W[y_1, y_2](t) = ce^{-\int p(t)\,dt}$$

where c is a certain constant depending on y_1, y_2 but not on t.

Proof. Suppose y_1 and y_2 are sols.

$$\begin{cases} y_1'' + p(t)\, y_1' + q(t)\, y_1 = 0 \\ y_2'' + p(t)\, y_2' + q(t)\, y_2 = 0. \end{cases}$$

Then

$$y_1'' y_2 + p(t)\, y_1' y_2 + q(t)\, y_1 y_2 - (y_1 y_2'' + \cancel{y}\, p(t) y_1 y_2' + q(t) y_1 y_2) = 0$$

$$(y_1'' y_2 - y_1 y_2'') + p(t)\,(y_1' y_2 - y_1 y_2') + q(t)\,(y_1 y_2 - y_1 y_2) = 0$$

$$\cancel{y} + (y_1' y_2 - y_1 y_2')' + p(t)\,(y_1' y_2 - y_1 y_2') = 0$$

$$W'(t) + p(t)\, W = 0.$$

$$\Rightarrow \quad W(t) = c e^{-\int p(t)\, dt}$$

E.g. $t^2 y'' + t y' - y = 0$

$$\Rightarrow \quad y'' + \frac{1}{t} y' - \frac{1}{t^2} y = 0.$$

$$W(t) = c e^{-\int \frac{1}{t}\, dt} = c e^{-\ln t} = \frac{c}{t}.$$

Section 3.3 Complex Roots

(*) $ay'' + by' + cy = 0$

Characteristic Equation:

$$ar^2 + br + c = 0$$

$$\Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac < 0$, we have complex roots.

Let $\lambda = -\frac{b}{2a}$ and $\mu = \frac{\sqrt{4ac - b^2}}{2a}$

then $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$.

Let $z_1 = e^{r_1 t} = e^{(\lambda + i\mu)t} = e^{\lambda t} \cdot e^{i\mu t}$

and $z_2 = e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t} \cdot e^{-i\mu t}$.

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Then

$$y_1(t) = \frac{1}{2}(z_1 + z_2) = e^{\lambda t}\cos(\mu t)$$

and $y_2(t) = \frac{1}{2i}(z_1 - z_2) = e^{\lambda t}\sin(\mu t)$

are solutions to (*) (Why?).

$\Rightarrow$ General solution:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$= e^{\lambda t}\left(c_1\cos(\mu t) + c_2\sin(\mu t)\right)$$

where c_1 and c_2 are arb. constants.

Ex. $y'' - 2y' + 4 = 0$, $y(0) = 0$, $y'(0) = 1$.

Characteristic Eq:

$$r^2 - 2r + 4 = 0.$$

$$r = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \cancel{2}\sqrt{3}\,i.$$

$\Rightarrow$ $\lambda = 1$ and $\mu = \sqrt{3}$.

General Solution

$$y(t) = e^t\left(c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\right)$$

I.V.P. $\Rightarrow$ Find c_1 and c_2.

$$0 = y(0) = c_1 \cos 0 + c_2 \sin 0$$

$$\Rightarrow \quad c_1 = 0$$

$$y'(t) = e^t\left(c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\right) + e^t\left(-\sqrt{3}c_1 \sin(\sqrt{3}t) + \sqrt{3}c_2 \cos(\sqrt{3}t)\right)$$

$$1 = y'(0) = 1(c_1 \cos 0 + 0) + 1(0 + \sqrt{3}c_2)$$

$$1 = \cancel{c_1 \cos}\; c_1 + \sqrt{3}c_2$$

$$\Rightarrow \quad c_2 = \frac{1}{\sqrt{3}}.$$

$$\Rightarrow \quad y(t) = \frac{e^t}{\sqrt{3}} \sin(\sqrt{3}t).$$