Lecture 24: Similarity Transformations; Singular Value Decompositions (Sections 5.6-6.3)

Thang Huynh, UC San Diego

3/12/2018

Similar transformations

- Definition. Two $n \times n$ matrices A and B are called similar if $B = M^{-1}AM$ for some invertible $n \times n$ matrix M.
- Example. If A can be diagonalized, Λ and A are similar.
- Property. If A and B are similar, they have the same eigenvalues. Every eigenvector $\boldsymbol{x}$ of A corresponds to an eigenvector $M^{-1}\boldsymbol{x}$ of B.

$A - \lambda I$ and $B - \lambda I$ have the same determinant.

Triangular forms with a unitary M

Theorem. (Schur's Theorem) There is a unitary matrix U such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.

Triangular forms with a unitary M

Theorem. (Schur's Theorem) There is a unitary matrix U such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.

- **Example.** $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ (twice).

$$U^{-1}AU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T.$$

Diagonalize symmetric and Hermitian matrices

Theorem. (Spectral Theorem) Every real symmetric A can be diagonalized by an orthogonal matrix Q. Every Hermitian matrix can be diagonalized by a unitary U:

(real) $\quad Q^{-1}AQ = \Lambda$ or $A = Q\Lambda Q^T$

(complex) $\quad U^{-1}AU = \Lambda$ or $A = U\Lambda U^H$

Diagonalize symmetric and Hermitian matrices

- Example. Consider $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Diagonalize symmetric and Hermitian matrices

- Example. Consider $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then A is symmetric with repeated eigevalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$.

Diagonalize symmetric and Hermitian matrices

- Example. Consider $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then A is symmetric with repeated eigevalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$.
  - For $\lambda_1 = \lambda_2 = 1$, eigenvectors $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
  - For $\lambda_3 = -1$, eigenvector $\boldsymbol{x}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

These are the columns of Q.

Normal matrices

- Definition. The matrix N is normal if it commutes with its conjugate transpose N^H: $NN^H = N^H N$.

Normal matrices

- Definition. The matrix N is normal if it commutes with its conjugate transpose N^H: $NN^H = N^H N$.
For such matrices, the triangular $T = U^{-1}NU$ is the diagonal Λ.
Normal matrices are exactly those that have a complete set of orthonormal eigenvectors.

Normal matrices

- **Definition.** The matrix N is normal if it commutes with its conjugate transpose N^H: $NN^H = N^H N$.
For such matrices, the triangular $T = U^{-1}NU$ is the diagonal Λ.
Normal matrices are exactly those that have a complete set of orthonormal eigenvectors.
- **Example.**
  - Symmetric and Hermitian matrices are normal.

Normal matrices

- **Definition.** The matrix N is normal if it commutes with its conjugate transpose N^H: $NN^H = N^H N$.
  For such matrices, the triangular $T = U^{-1}NU$ is the diagonal Λ.
  Normal matrices are exactly those that have a complete set of orthonormal eigenvectors.
- **Example.**
  - Symmetric and Hermitian matrices are normal.
  - Orthogonal and unitary matrices are also normal.

Normal matrices

- Property. If N is normal, then so is the triangular $T = U^{-1}NU$.

Normal matrices

- Property. If N is normal, then so is the triangular $T = U^{-1}NU$.
- Property. A triangular T that is normal must be diagonal.

(Why?)

The Jordan form

Recall that if A is diagonalized, A is similar to a diagonal matrix Λ whose entries are eigenvalues of A.

The Jordan form

Recall that if A is diagonalized, A is similar to a diagonal matrix Λ whose entries are eigenvalues of A.

How if A cannot be diagonalized?

The Jordan form

If A has s independent eigenvectors, it is similar to a matrix with s blocks:

$$\text{Jordan form} \qquad J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}.$$

▶ Property. Two matrices are similar if and only if they share the same Jordan form J.

The Jordan form

If A has s independent eigenvectors, it is similar to a matrix with s blocks:

Jordan form $$J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}.$$

Each J_i has only a single eigenvalue λ_i and one eigenvector:

Jordan block $$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}.$$

▶ Property. Two matrices are similar if and only if they share the same Jordan form J.

The Jordan form

- Example. $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

The Jordan form

- Example. $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

  These four matrices have eigenvalues 1 and 1 with only one eigenvector. How to find a matrix M such that $M^{-1}TM = J$?

The Jordan form

- Example. $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
These four matrices have eigenvalues 1 and 1 with only one eigenvector. How to find a matrix M such that $M^{-1}TM = J$?
- Definition. If A is an $n \times n$ matrix, a generalized eigenvector of A corresponding to the eigenvalue λ is a nonzero vector $\boldsymbol{x}$ satisfying

$$(A - \lambda I)^p \boldsymbol{x} = 0$$

for some positive integer p.

The Jordan form

$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue $\lambda = 1$ of multiplicity 2. We want to find generalized eigenvectors from the equations:

The Jordan form

$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue $\lambda = 1$ of multiplicity 2. We want to find generalized eigenvectors from the equations:

- $(A - I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and
- $(A - I)^2\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The Jordan form

$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue $\lambda = 1$ of multiplicity 2. We want to find generalized eigenvectors from the equations:

- $(A - I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and
- $(A - I)^2\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The Jordan form

$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue $\lambda = 1$ of multiplicity 2. We want to find generalized eigenvectors from the equations:

- $(A - I)\boldsymbol{x} = 0 \Rightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and
- $(A - I)^2\boldsymbol{x} = 0 \Rightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$M^{-1}TM = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

The Jordan form

▶ Example. For $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Which of the following matrices are the Jordan forms of A and B?

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Singular Value Decompositions

Recall we can decompose a matrix A as LU or QR.

Singular Value Decompositions

Recall we can decompose a matrix A as LU or QR.

(SVD) Any $m \times n$ matrix A can be factored into

$$A = U\Sigma V^T = (\text{orthogonal})(\text{diagonal})(\text{orthogonal}).$$

- columns of U (m by m) are eigenvectors of AA^T

Singular Value Decompositions

Recall we can decompose a matrix A as LU or QR.

(SVD) Any $m \times n$ matrix A can be factored into

$$A = U\Sigma V^T = (\text{orthogonal})(\text{diagonal})(\text{orthogonal}).$$

- columns of U (m by m) are eigenvectors of AA^T
- columns of V (n by n) are eigenvectors of A^TA

Singular Value Decompositions

Recall we can decompose a matrix A as LU or QR.

(SVD) Any $m \times n$ matrix A can be factored into

$$A = U\Sigma V^T = (\text{orthogonal})(\text{diagonal})(\text{orthogonal}).$$

- columns of U (m by m) are eigenvectors of AA^T
- columns of V (n by n) are eigenvectors of A^TA
- The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Singular Value Decompositions

- Example. Consider $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Singular Value Decompositions

- Example. Consider $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Then A has rank 2, and $AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ with eigenvalues $\lambda = 3$ and 1.

Singular Value Decompositions

▶ Example. Consider $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Then A has rank 2,

and $AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ with eigenvalues $\lambda = 3$ and 1.

$$A = U\Sigma V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}.$$