Lecture 23: Similarity Transformations (Sections 5.6)

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3/9/2018

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- **Example.** (normalized) Fourier matrix

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & w^{n-1} & \cdots & w^{(n-1)^2} \end{bmatrix}$$

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Row i of U^H times column j of U is

$$\frac{1}{n}(1 + W + W^2 + \cdots + W^{n-1}) = \frac{W^n - 1}{W - 1} = 0,$$

where $W = w^{j-i}$.

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If $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: triangular with $\lambda = 1$ and 0.

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- **Example.** $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ (twice).

$$U^{-1}AU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T.$$

Diagonalize symmetric and Hermitian matrices

Theorem. (Spectral Theorem) Every real symmetric A can be diagonalized by an orthogonal matrix Q. Every Hermitian matrix can be diagonalized by a unitary U:

(real) $\quad Q^{-1}AQ = \Lambda$ or $A = Q\Lambda Q^T$

(complex) $\quad U^{-1}AU = \Lambda$ or $A = U\Lambda U^H$

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- For $\lambda_1 = \lambda_2 = 1$, eigenvectors $\boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- For $\lambda_3 = -1$, eigenvector $\boldsymbol{x}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

These are the columns of Q.

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- **Example.**
  - Symmetric and Hermitian matrices are normal.
  - Orthogonal and unitary matrices are also normal.

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- Property. If N is normal, then so is the triangular $T = U^{-1}NU$.
- Property. A triangular T that is normal must be diagonal.

(Why?)