Section 2.2 Limits and Continuity

Open Sets. Let U be a subset of $\mathbb{R}^n$ (written $U \subset \mathbb{R}^n$). We say that U is an *open set* if for every x_0 in U, there is some number $r > 0$ such that every point with $\|\vec{x} - \vec{x}_0\| < r$ is within U.
Example. Any interval $(a, b) \subset \mathbb{R}$ is open.
Example.

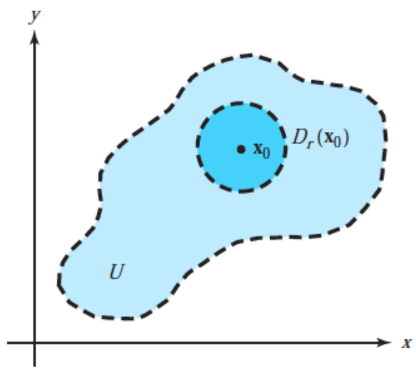

figure **2.2.2** An open set U is one that completely encloses some disk $D_r(\mathbf{x}_0)$ about each of its points $\mathbf{x}_0$.

Intuitively, U is open when the boundary points of U are not in U. (A point $z \in U$ is on the boundary of U if every neighborhood of z contains at least one pont in U and one not in U.)

Limits

Remember that in "standard" one-dimensional calculus, we need limits to study continuity, derivatives, improper integrals, etc. We would like to generalize this notion to functions of several variables.

Definition. Let $A \subset \mathbb{R}^n$ be an open set and let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Let $\vec{x}_0$ be in A or be on the boundary of A. We write

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}$$

when given any neighborhood N of $\vec{b}$ (i.e. an open set containing $\vec{b}$) f is eventually in N as $\vec{x}$ approaches $\vec{x}_0$. If f does not approach any vector as $\vec{x}$ approaches $\vec{x}_0$, we say the limit does not exist.

Properties of Limits

- If $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}_1$ and $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}_2$, then $\vec{b}_1 = \vec{b}_2$.
- If $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}_1$ and $\lim_{\vec{x} \to \vec{x}_0} g(\vec{x}) = \vec{b}_2$, then
  i) $\lim_{\vec{x} \to \vec{x}_0} cf(\vec{x}) = c\vec{b}_1$

  ii) $\lim_{\vec{x} \to \vec{x}_0} (f(\vec{x}) + g(\vec{x})) = \vec{b}_1 + \vec{b}_2$

  iii) when $m = 1$, i.e. b_1 and b_2 are scalars, $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x})g(\vec{x}) = b_1 b_2$

  iv) when $f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \vec{b}_1 = (b_1, b_2, \ldots, b_m)$ if and only if $\lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) = b_i$ for $i = 1, 2, \ldots, m$.

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + y^2$. Compute $\lim_{(x,y)\to(0,1)} f(x, y)$.
Solution.

$$\begin{aligned}\lim_{(x,y)\to(0,1)} f(x,y) &= \lim_{(x,y)\to(0,1)} (x^2+y^2) \\ &= 0^2 + 1^2 \\ &= 1.\end{aligned}$$

Example. Find the limit or show it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{7x^2}{x^2+y^2}.$$

Solution. If the limit exists, $\frac{7x^2}{x^2+y^2}$ approaches a definite value, say a, as (x, y) gets near $(0, 0)$. In particular, if (x, y) approaches $(0, 0)$ along any given path, then $\frac{7x^2}{x^2+y^2}$ should approach a.
If (x, y) approahces $(0, 0)$ along the line $y = 0$, the limiting value is

$$\lim_{x\to 0} \frac{7x^2}{x^2} = 7.$$

If (x, y) approaches $(0, 0)$ along the line $x = 0$, the limiting value is

$$\lim_{y\to 0} \frac{7(0)}{0+y^2} = 0 \neq 7.$$

Hence, $\lim_{(x,y)\to(0,0)} \frac{7x^2}{x^2+y^2}$ does not exist.

Continuous Functions

Definition. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function and let $\vec{x}_0 \in A$. We say that f is *continuous at* $\vec{x}_0$ if and only if

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0).$$

If f is continous at every point in A, we say that f is continuous.
Example. $f(x) = 2x^2 + 3x + 5$ is continuous.
Example. $f(x, y) = xy$ is continuous because

$$\lim_{(x,y)\to(x_0,y_0)} xy = \lim_{(x,y)\to(x_0,y_0)} x \lim_{(x,y)\to(x_0,y_0)} y = x_0y_0 = f(x_0, y_0)$$

for all points (x_0, y_0).
Example.

$$f(x,y) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

is not continuous. (Why? Think about $\lim_{(x,y)\to(0,0)} f(x, y)$)

Properties

Suppose f and g are continuous at $\vec{x}_0$. Then

i) cf is also continous at $\vec{x}_0$ where c is scalar.

ii) $f + g$ is also continuous at $\vec{x}_0$.

iii) When f and g are functions from $\mathbb{R}^n$ to $\mathbb{R}$ and are continuous at $\vec{x}_0$, fg is continuous at $\vec{x}_0$.

iv) Let $f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, then f is continuous if and only if the f_i's are all continuous.

v) Compositions of continuous functions are continuous. That is, if g is continuous at $\vec{x}_0$ and f is continuous at $y_0 = g(\vec{x}_0)$, then $f \circ g$ is continuous at $\vec{x}_0$. (Recall that $(f \circ g)(\vec{x}) = f(g(\vec{x}))$.)

Example. Evaluate or show the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

Solution. Let $f(u) = \begin{cases} \frac{sin(u)}{u}, & u \neq 0 \\ 1, & u = 0 \end{cases}$ and $g(x,y) = x^2 + y^2$. Then g is continuous at $(0,0)$ and f is continuous at $g(0,0) = 0$. Hence,

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} f(g(x,y)) = f(g(0,0)) = f(0) = 1.$$