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* 4.3 Vector Fields

A vector field \vec{F} is a map from \mathbb{R}^n to \mathbb{R}^n that assigns to each point $\vec{x} = (x_1, x_2, \dots, x_n)$ a vector $\vec{F}(\vec{x}) = \vec{F}(x_1, \dots, x_n)$.

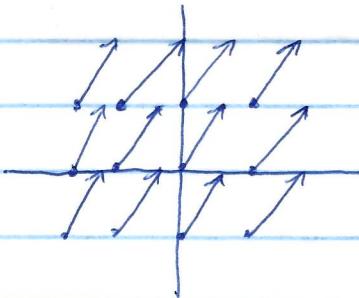
E.g. velocity fields (wind, fluids)

force fields (magnetic, gravitational).

$$\text{in 20: } \vec{F}(x,y) = M(x,y) \vec{i} + N(x,y) \vec{j}.$$

$$\text{in 3D: } \vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}.$$

E.g. $\vec{F} = \vec{i} + \vec{j}$



$$\vec{F} = y \vec{j}$$

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* Gradient vector fields:

Given a function $f(x, y, z)$, its gradient is

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}.$$

⇒ it assigns to each point a vector.

If $\vec{F} = \nabla f$ for some function f , we call \vec{F} a gradient vector field and f the potential of the vector field.

E.g. Gravitational force fields.

$$\vec{F} = -\frac{mMG}{\|\vec{r}\|^3} \vec{r} \quad \text{where } \vec{r}(x, y, z) = (x, y, z).$$

$$f = \frac{mMG}{\|\vec{r}\|}.$$

. $\vec{F} = y\vec{i} + x\vec{j}$ is a gradient vector field
with $f(x, y) = xy$.

{ Q: How to find f ?

A: Integrate.

$$\text{since } \frac{\partial f}{\partial x} = y. \rightarrow \int \frac{\partial f}{\partial x} dx = \int y dx = xy + C.$$

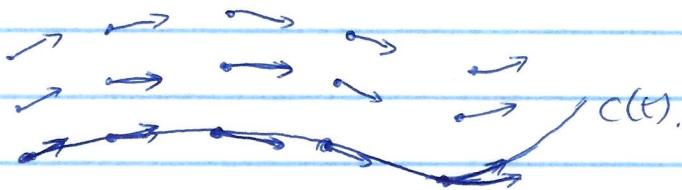
. $\vec{F} = y\vec{i} - x\vec{j}$ is not a gradient vector field.

$$\left(\begin{array}{l} \frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = -x. \\ \downarrow \\ \frac{\partial^2 f}{\partial y \partial x} = 1 \quad + \quad \frac{\partial^2 f}{\partial x \partial y} = -1 \end{array} \right).$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 + \frac{\partial^2 f}{\partial x \partial y} = -1$$

* Flow lines:

Given a vector field \vec{F} , a flow line is a path $\vec{c}(t)$ such that $\vec{F}(\vec{c}(t)) = \vec{c}'(t)$.



$$\text{E.g. } \vec{F}(x, y) = -y\vec{i} + x\vec{j}.$$

Find a flow line.

$$\vec{c}(t) = (x(t), y(t))$$

$$\vec{F}(\vec{c}(t)) = \vec{c}'(t).$$

$$-y(t)\vec{i} + x(t)\vec{j} = x'(t)\vec{i} + y'(t)\vec{j}.$$

$$\Rightarrow \begin{cases} -y(t) = x'(t) \\ x(t) = y'(t). \end{cases}$$

$x(t) = \cos(t)$ and $y(t) = \sin(t)$ work.

$$\Rightarrow \vec{c}(t) = (\cos(t), \sin(t))$$

$$\text{E.g. Let } \vec{F} = x\vec{i} + 2x\vec{j} + 3y\vec{k}.$$

Find a flow line.

$$\vec{F}(\vec{c}(t)) = \vec{c}'(t).$$

$$(x(t), 2x(t), 3y(t)) = (x'(t), y'(t), z'(t)).$$

$$\Rightarrow x(t) = x'(t) \Rightarrow x(t) = e^t.$$

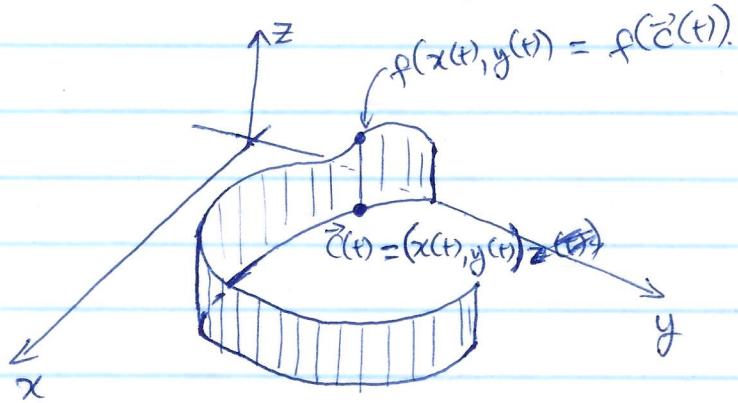
$$y'(t) = 2x(t) \Rightarrow 2e^t \Rightarrow y(t) = 2e^t.$$

$$z'(t) = 6e^t \Rightarrow z(t) = 6e^t.$$

$$\Rightarrow c(t) = (e^t, 2e^t, 6e^t).$$

Goal of the course: generalize the FTOC to several variables.

Need the concept of a path integral.



"area of the fence" whose base is the image of \vec{c} and height $f(x, y)$ at (x, y) .

* 7.1 The path integral.

(2 variables)

Def: The path integral, or the integral of $f(x, y)$ along the path \vec{c} , is defined by

$$\int_{\vec{c}} f \, ds := \int_a^b f(x(t), y(t)) \| \vec{c}'(t) \| \, dt.$$

where $\vec{c}: [a, b] \rightarrow \mathbb{R}^2$, \vec{c} is differentiable.

E.g. The base of the fence in the first quadrant

$$\vec{c}: [0, \frac{\pi}{2}] \rightarrow \mathbb{R}^2$$

$$\vec{c}(t) = (30 \cos^3 t, 30 \sin^3 t).$$

The height of the fence at (x, y) is

$$f(x, y) = 1 + \frac{y}{3}.$$

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$$\text{Area of fence} = \int_{\vec{C}} f \, ds \quad \text{scalar}$$

arc length $ds = \|\vec{c}'(t)\| dt.$

$$\vec{c}(t) = (-90\cos^3 t \sin t, 90\sin^3 t \cos t) = \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) \sqrt{90^2 \cos^4 t \sin^2 t + 90^2 \sin^4 t \cos^2 t} dt$$

$$= \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) 90 \cos t \sin t dt$$

$$= 90 \int_0^{\pi/2} (\sin t + 10 \sin^4 t) \cos t dt.$$

$$= 90 \int_0^1 (u + 10u^4) du.$$

$$= 90 \left[\frac{u^2}{2} + 2u^5 \right]_0^1$$

$$= 90 \left(\frac{1}{2} + 2 \right)$$

$$= 225.$$

3 variables:

$$\int_{\vec{C}} f \, ds := \int_a^b f(x(t), y(t), z(t)) \|\vec{c}'(t)\| dt.$$

E.g. $\vec{c}(t) = (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$.

$$f(x, y, z) = x^2 + y^2 + z.$$

Evaluate $\int_{\vec{C}} f \, ds$.

Sol: $\int_{\vec{C}} f \, ds = \int_0^{2\pi} f(\vec{c}(t)) dt \cdot \|\vec{c}'(t)\| dt.$

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$$\begin{aligned}
 f(\vec{c}(t)) &= f(x(t), y(t), z(t)) \\
 &= x^2(t) + y^2(t) + z(t) \\
 &= \cos^2 t + \sin^2 t + t \\
 &= 1 + t.
 \end{aligned}$$

$$\begin{aligned}
 \vec{c}'(t) &= (-\sin t, \cos t, 1) \\
 \|\vec{c}'(t)\| &= \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} \\
 &= \sqrt{2}.
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \int_{\vec{c}} f ds &= \int_0^{2\pi} (1+t) \cdot \sqrt{2} dt = \sqrt{2} \left(t + \frac{t^2}{2} \right) \Big|_0^{2\pi} \\
 &= \sqrt{2} \left(2\pi + \frac{(2\pi)^2}{2} \right) \\
 &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.
 \end{aligned}$$

7.2 Line Integrals

Motivation

$$W = (\text{force}) \cdot \cancel{\text{distance}} \quad (\text{displacement in the direction of force}).$$

over the short distance: $W \approx \vec{F} \cdot \vec{ds}$

total work over long distance along trajectory C :

$$W \approx \sum_i \vec{F}(t_i) \cdot (\Delta \vec{s})_i \rightarrow \int_C \vec{F} \cdot d\vec{s}.$$

$$\Rightarrow W = \int_C \vec{F} \cdot d\vec{s}.$$

Def: Let \vec{F} be a vector field in \mathbb{R}^3 , continuous on $\vec{c}: [a, b] \rightarrow \mathbb{R}^3$. The line integral of \vec{F} along \vec{c} is defined as

$$\int_C \vec{F} \cdot d\vec{s} = \int \vec{F}(x(t), y(t), z(t)) \cdot \vec{c}'(t) dt.$$

E.g. $\vec{c}(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$.

$$\vec{F}(x, y, z) = x^2 \vec{i} + y^2 \vec{j} + z \vec{k}.$$

Compute $\int_C \vec{F} \cdot d\vec{s}$.

Sol: Along \vec{c} ,

$$\begin{aligned} \vec{F}(x, y, z) &= \vec{F}(\cos t, \sin t, t) \\ &= \cos^2 t \vec{i} + \sin^2 t \vec{j} + t \vec{k}. \end{aligned}$$

then and

$$\vec{c}(t) = (-\sin t \vec{i} + \cos t \vec{j} + t \vec{k}).$$

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$$\begin{aligned}
 \int_{\vec{C}} \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} (\cos^2 t \vec{i} + \sin^2 t \vec{j} + t \vec{k}) \cdot (-\sin t \vec{i} + \cos t \vec{j} + \vec{k}) dt \\
 &= \int_0^{2\pi} -\cos^2 t \sin t + \sin^2 t \cos t + t dt \\
 &= \int_0^{2\pi} -\cos^2 t \sin t dt + \int_0^{2\pi} \sin^2 t \cos t dt + \int_0^{2\pi} t dt \\
 &= \left. \frac{\cos^3 t}{3} \right|_0^{2\pi} + \left. \frac{\sin^3 t}{3} \right|_0^{2\pi} + \left. \frac{t^2}{2} \right|_0^{2\pi} \\
 &= 0 + 0 + \frac{4\pi^2}{2} \\
 &= 2\pi^2.
 \end{aligned}$$

New notation: for the line integral

$$\vec{F} = (P, Q, R)$$

$$\text{and } d\vec{s} = (dx, dy, dz)$$

We write

$$\begin{aligned}
 \int_{\vec{C}} \vec{F} \cdot d\vec{s} &= \int_{\vec{C}} P dx + Q dy + R dz \\
 &= \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt
 \end{aligned}$$

Remark: This is not the sum of 3 integrals. It is just another way to write $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$. We still need to

parametrize and express everything in terms of the parameter.

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E.g. $\vec{c}(t) = (\cos t, \sin t, t) \quad 0 \leq t \leq 2\pi$

$$\vec{F}(x, y, z) = \underset{P}{\overset{x^2}{\uparrow}} \vec{i} + \underset{Q}{\overset{y^2}{\uparrow}} \vec{j} + \underset{R}{\overset{z}{\uparrow}} \vec{k}$$

$$\begin{aligned} \text{So } \int_{\vec{C}} \vec{F} \cdot d\vec{s} &= \int_C x^2 dx + \int_C y^2 dy + \int_C z dz \\ &= \int_a^{2\pi} \left(x^2(t) \frac{dx}{dt} + y^2(t) \frac{dy}{dt} + z^2(t) \frac{dz}{dt} \right) dt \\ &= \int_0^{2\pi} (\cos^2 t (-\sin t) + \sin^2 t \cos t + t) dt \\ &= 2\pi^2. \end{aligned}$$

E.g. Evaluate $\int_{\vec{C}} x^2 dx + xy dy$

where $\vec{c}(t) = (t, t^2), \quad 0 \leq t \leq 1.$

$$\begin{aligned} \int_{\vec{C}} x^2 dx + xy dy &= \int_0^1 \left(x^2(t) \frac{dx}{dt} + x(t) y(t) \frac{dy}{dt} \right) dt \\ &= \int_0^1 (t^2 + t \cdot t^2 \cdot 2t) dt \\ &= \int_0^1 (t^2 + 2t^4) dt \\ &= \left. \frac{t^3}{3} + \frac{2}{5} t^5 \right|_0^1 \\ &= \frac{1}{3} + \frac{2}{5}. \\ &= \frac{11}{15}. \end{aligned}$$

Remark: line integrals are independent of the parametrization as long as the parametrization is orientation preserving.

E.g. The curve $y = x^3$ from $(0,0)$ to $(1,1)$ can be parametrized as $\vec{c}(t) = (t, t^3)$ $0 \leq t \leq 1$ or as $\vec{p}(\theta) = (\sin \theta, \sin^3 \theta)$ $0 \leq \theta \leq \pi/2$.

$$\text{Let } \vec{F} = x\vec{i} + y\vec{j}.$$

$$\int \vec{F} \cdot d\vec{s} = \int_0^1 (\vec{t}\vec{i} + \vec{t}^3\vec{j}) \cdot \vec{c}'(t) dt$$

$$\text{or } = \int_0^{\pi/2} (\sin \theta \vec{i} + \sin^3 \theta \vec{j}) \cdot \vec{p}'(\theta) d\theta.$$

check!

* Line integral of gradient field:

$$\text{FTOC: } \int_a^b f'(t) dt = f(b) - f(a).$$

Thm: (Fundamental theorem of line integrals).

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$ differentiable and $c: [a, b] \rightarrow \mathbb{R}$.

(c is continuous or piecewise ~~cts~~ cts.)

$$\int_C (\nabla f) \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

Remark: If the field is a gradient field, only the end points matter.

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E.g. Evaluate $\int_{\vec{c}} \nabla f \cdot d\vec{s}$ where $f(x, y, z)$

$f(x, y, z) = \cos x + \sin y - xyz$
 and \vec{c} is a trajectory that starts at $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$
 and ends at $(\pi, 2\pi, 1)$.

Sol: Let $\vec{c}(t)$, $a \leq t \leq b$, be a path with
 $\vec{c}(a) = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$ & $\vec{c}(b) = (\pi, 2\pi, 1)$.

Then by ~~FROG~~ FTOLI.

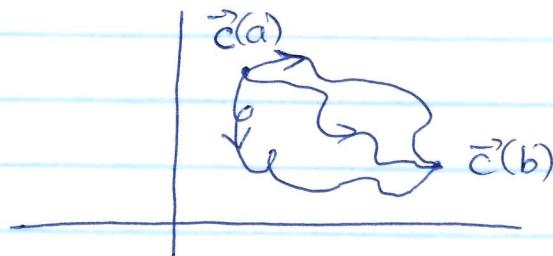
$$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

$$= f(\pi, 2\pi, 1) - f(\frac{\pi}{2}, \frac{\pi}{2}, 0)$$

$$= \cos \pi + \sin \frac{2\pi}{2} - \cancel{\frac{\pi^2}{2}} - \cos \frac{\pi}{2} - \sin \frac{\pi}{2}$$

$$= -1 + 0 - \frac{\pi^2}{2} - 0 - 1$$

$$= -2 - \frac{\pi^2}{2}.$$



- Integrals of scalar fields along curves
 \Rightarrow path integrals

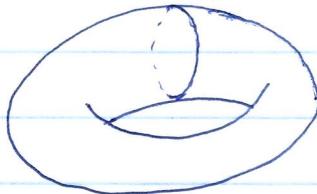
- Integrals of vector fields along curves
 \Rightarrow line integrals.

7.3 Parametrized Surfaces.

E.g. The graph of a function $f(x, y)$ is a surface.

But we still have surfaces that are not the graph of a function.

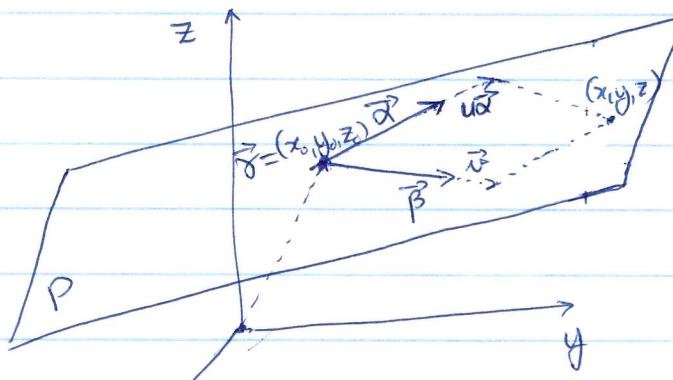
E.g. Torus (surface of a donut).



Def: A parametrization of a surface is a function $\Phi: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$. The surface is $S = \Phi(D)$ and $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$.

If $x(u, v)$, $y(u, v)$, $z(u, v)$ are differentiable, we call S a differentiable surface.

• Parametrization of a plane:



Let P be a plane that is parallel to \vec{u} and \vec{v} , and passes through \vec{r} .

For any $(x, y, z) \in P$, we can write (x, y, z) as

$$(x, y, z) = (x_0, y_0, z_0) + u\vec{u} + v\vec{v}$$

for some $u, v \in \mathbb{R}$

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$$\boxed{\text{S6 } \Phi(u, v) = \vec{x}u + \vec{y}v + \vec{z}}$$

E.g. Find a parametrization of the plane $x+y+z=1$

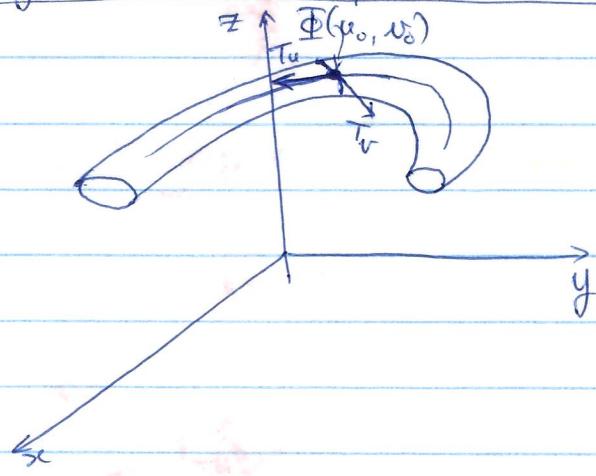
Sol: The point $(0,0,1)$ is on the plane.

Vectors $(1, -1, 0)$ and $(0, 1, -1)$ are parallel to the plane (why?).

Hence,

$$\Phi(u, v) = (1, -1, 0)u + (0, 1, -1)v + (0, 0, 1).$$

* Tangent vectors to parametrized surfaces:



Suppose that $\Phi(u, v)$ is diff. at (u_0, v_0) .

Fix v_0 and look at the map $t \mapsto \Phi(u_0, t, v_0)$
(in other words, we have a map $\mathbb{R} \rightarrow \mathbb{R}^3$) which identifies a ~~curve~~ curve on the surface

The vector tangent to this curve at (u_0, v_0) is given by

$$T_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0) \vec{i} + \frac{\partial y}{\partial u}(u_0, v_0) \vec{j} + \frac{\partial z}{\partial u}(u_0, v_0) \vec{k}.$$

It is also tangent to the surface.

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$$\text{Similarly, } T_w = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0) \vec{i} + \frac{\partial y}{\partial v}(u_0, v_0) \vec{j} + \frac{\partial z}{\partial v}(u_0, v_0) \vec{k}$$

Both T_u and T_v are tangent to the surface.
so $T_u \times T_v$ is normal to them (provided $T_u \times T_v \neq 0$).

To find the equation of the tangent plane at (u_0, v_0)
we calculate

$$\vec{n} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}_{(u_0, v_0)}$$

Tangent plane is

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

$$\Leftrightarrow n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \quad \vec{n} = (n_1, n_2, n_3)$$

E.g. Let $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$\Phi(u, v) = (u \cos v, u \sin v, u^2 + v^2)$$

Find the tangent plane at $\Phi(1, 0)$.

Sols: $T_u = (\cos v, \sin v, 2u)$

$$T_v = (-u \sin v, u \cos v, 2v)$$

$$\vec{n} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 2v \end{vmatrix}_{(1, 0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= -2\vec{i} - (0)\vec{j} + 1\vec{k}$$

$$= -2\vec{i} + \vec{k}.$$

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$$(x_0, y_0, z_0) = \Phi(u_0, v_0) = \Phi(1, 0) = (1, 0, 1).$$

The eq. of the tangent plane

$$-2(x-1) + 0(y-0) + 1(z-1) = 0$$

$$-2x + 2 + z - 1 = 0.$$

$$-2x + z = -1.$$

Remark: We say that a surface is regular or smooth, at $\Phi(u_0, v_0)$ if $T_u \times T_v \neq 0$ at (u_0, v_0) . We say that it is regular, if it is regular at all points $\Phi(u_0, v_0)$.

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7.4 Area of a surface.

Def: The surface area of a parametrized surface

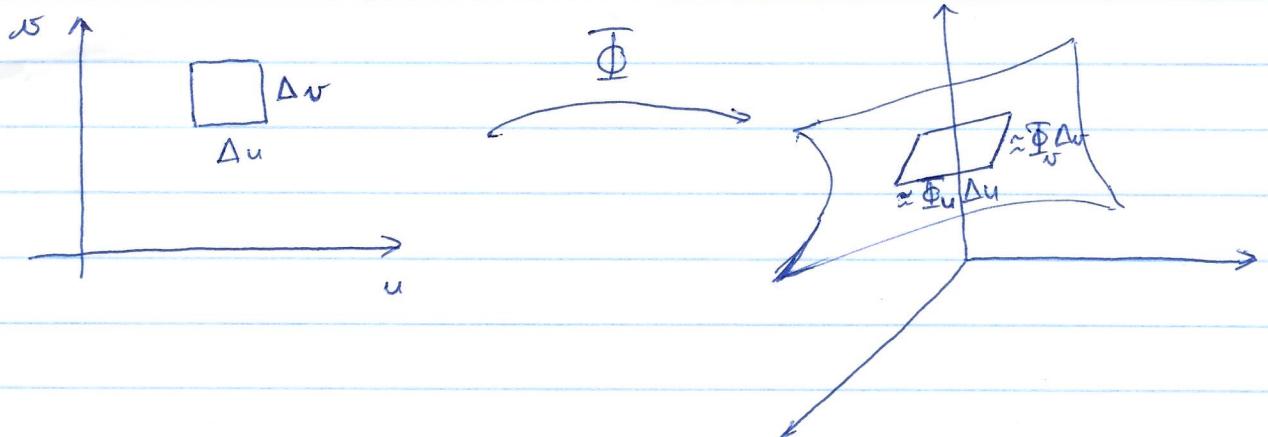
$$A(S) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| du dv.$$

 S is regular Φ one-to-one and differentiable.Recall: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, then

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}$$

So

$$A(S) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} du dv$$



The area of the image of the small rectangle is

$$\|(\Phi_u \Delta u) \times (\Phi_v \Delta v)\| = \|(\mathbf{T}_u \times \mathbf{T}_v)\| \Delta u \Delta v.$$

Summing the areas of all small rectangles as $\Delta u, \Delta v \rightarrow 0$ gives $A(S)$.

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E.g. Find the surface area of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 1.$$

Parametrize:

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

$$\text{Then } T_r = (\cos\theta, \sin\theta, 1)$$

$$T_\theta = (-r\sin\theta, r\cos\theta, 0).$$

$$T_r \times T_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta \vec{i} - r\sin\theta \vec{j} + r\vec{k}$$

$$\|T_r \times T_\theta\| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} \\ = \sqrt{2r^2}$$

$$A(\text{cone}) = \iint_0^{2\pi} \sqrt{2} r dr d\theta = 2\pi \sqrt{2} \frac{r^2}{2} \Big|_0^1 = \sqrt{2}\pi.$$

* Surface area of the graph of a function $f(x, y)$.We can parametrize S by $(x, y, f(x, y))$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} \quad \text{or} \quad (u, v, f(u, v)) \\ = -f_u \vec{i} - f_v \vec{j} + \vec{k}.$$

$$A(S) = \iint_D \sqrt{f_u^2 + f_v^2 + 1} \, du dv.$$

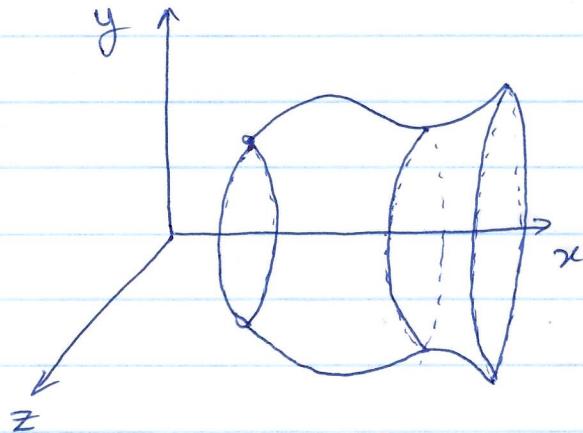
Exercise: use this to find the area of the cone

$$z = \sqrt{x^2 + y^2}$$

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* Surfaces of revolution.

Suppose S is obtained by rotating the graph of $y = f(x)$, $a \leq x \leq b$ around the x -axis.



We can parametrize S as $(u, f(u)\cos v, f(u)\sin v)$
 $a \leq u \leq b, 0 \leq v \leq 2\pi$.

$$\text{so } T_u = (1, f(u)\cos v, f(u)\sin v)$$

$$T_v = (0, -f(u)\sin v, f(u)\cos v)$$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f(u)\cos v & f(u)\sin v \\ 0 & -f(u)\sin v & f(u)\cos v \end{vmatrix}$$

$$= f'(u)f(u) \vec{i} - f(u)\cos v \vec{j} - f(u)\sin v \vec{k}.$$

$$\Rightarrow A(S) = \iint_{a}^{b} \sqrt{f'(u)^2 + \cos^2 v + \sin^2 v} |f(u)| du dv.$$

$$= \iint_{a}^{b} \sqrt{f'(u)^2 + 1} |f(u)| du.$$

$$= 2\pi \int_0^b \sqrt{f'(u)^2 + 1} |f(u)| du.$$

Exercise: use this to compute the area of the cone $z = \sqrt{x^2 + y^2}$. ($0 \leq x \leq 1$).

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7.5 Integrals of Scalar functions over surfaces.

Eg. The mass of a thin sheet of metal

$S = \Phi(D)$ where $D \subset \mathbb{R}^2$ and $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ where the density of the metal is given by $f(x, y, z)$.

Def: The integral of a scalar function over a surface

$$\iint_S f(x, y, z) \, ds = \iint_D f(\Phi(u, v)) \|T_u \times T_v\| \, du \, dv.$$

$$= \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} \, du \, dv$$

E.g. $f(x, y, z) = \sqrt{x^2 + y^2 + 1}$

$$S = (r \cos \theta, r \sin \theta, \theta)$$

$$D: 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$$

(helicoid

Example 2

Section 7.4.)

Compute $\iint_S f \, ds$.

$$\iint_S f \, ds = \iint_D f(r \cos \theta, r \sin \theta, \theta) \|T_r \times T_\theta\| \, dr \, d\theta$$

$$= \iint_D \sqrt{r^2 + 1} \|T_r \times T_\theta\| \, dr \, d\theta.$$

$$T_r = (\cos \theta, \sin \theta, 0)$$

$$T_\theta = (-r \sin \theta, r \cos \theta, 1)$$

$$T_r \times T_\theta = \sin \theta \vec{i} - \cos \theta \vec{j} + r \vec{k}.$$

$$\|T_r \times T_\theta\| = \sqrt{r^2 + 1}$$

$$\Rightarrow \iint_S f \, ds = \iint_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} \cdot \sqrt{r^2 + 1} \, dr \, d\theta.$$

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$$= 2\pi \left(r + \frac{r^3}{3} \right) \Big|_0^1 = \frac{8\pi}{3}.$$

* Surface integrals over graphs of functions:

Suppose S is the graph of a differentiable function

$$z = g(x, y)$$

\Rightarrow we can parametrize it as $(u, v, g(u, v))$

$$\text{Then } \|T_u \times T_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$$

$$\text{so } \iint_S f(x, y, z) ds = \iint_D f(u, v, g(u, v)) \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2} du dv$$

E.g. Compute $\iint_S \frac{x}{\sqrt{4x^2 + 4y^2 + 1}} ds$

where S is the hyperbolic paraboloid $z = y^2 - x^2$
over the region $-1 \leq y \leq 1, -1 \leq x \leq 1$.

we parametrize $(x, y, y^2 - x^2)$

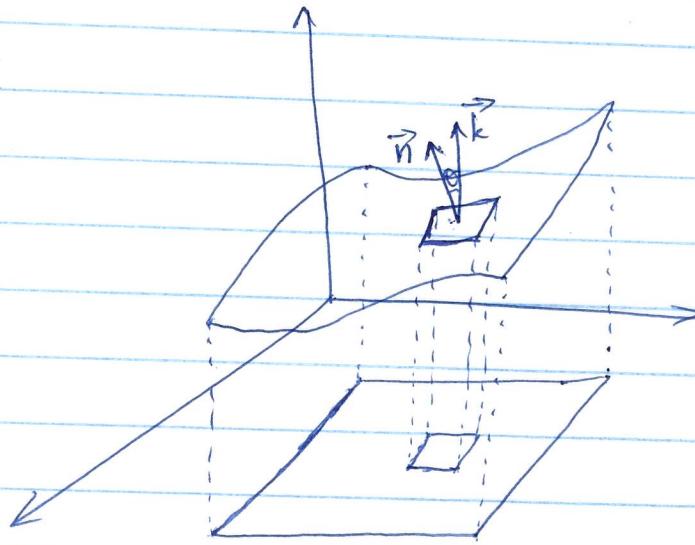
$$\begin{aligned} \iint_S f ds &= \iint_D \frac{x}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + (-2x)^2 + (2y)^2} dx dy \\ &= \iint_{-1}^1 \iint_{-1}^1 x dx dy = 0. \end{aligned}$$

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* Integrals over graphs:

$$S: (x, y, g(x, y))$$

D: simple region.



here $\vec{n} = \frac{\vec{N}}{\|\vec{N}\|}$ is the unit normal and

$\vec{N} = -\frac{\partial g}{\partial x} \vec{i} - \frac{\partial g}{\partial y} \vec{j} + \vec{k}$ is normal to the surface.

$$\text{Since } \vec{N} \cdot \vec{k} = \|\vec{N}\| \|\vec{k}\| \cos \theta = \|\vec{N}\| \cos \theta.$$

$$\begin{aligned} \text{then } \cos \theta &= \frac{\vec{N} \cdot \vec{k}}{\|\vec{N}\|} = \frac{\left(-\frac{\partial g}{\partial x} \vec{i} - \frac{\partial g}{\partial y} \vec{j} + \vec{k}\right) \cdot \vec{k}}{\|\vec{N}\|} \\ &= \frac{1}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \end{aligned}$$

$$\text{so } \iint_S f \, ds = \iint_D f(x, y, g(x, y)) \underbrace{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}_{\text{depends on } x \text{ & } y} \, dx \, dy$$

$$= \iint_D f(x, y, g(x, y)) \cdot \frac{1}{\cos \theta} \, dx \, dy.$$

$\uparrow \theta \text{ depends on } x \text{ & } y.$

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The ~~total~~

E.g. Compute the mass of the helicoid S : $(r \cos \theta, r \sin \theta, \theta)$,
where $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ if its mass density is
 $m(x, y, z) = \sqrt{x^2 + y^2}$.

The total mass of a surface with mass density m is given by

$$M(S) = \iint_S m(x, y, z) dS$$

$$\begin{aligned} &= \iint_S \sqrt{r^2} \|T_r \times T_\theta\| dr d\theta \\ &= \iint_0^{2\pi} \int_0^1 r \sqrt{r^2 + 1} dr d\theta \end{aligned}$$

$$\begin{aligned} &= 2\pi \cdot \left[\frac{1}{3} (r^2 + 1)^{3/2} \right]_0^1 \\ &= 2\pi \cdot \frac{1}{3} (2^{3/2} - 1) \end{aligned}$$

$$u = r^2 + 1$$

$$du = 2rdr$$

(53)

Surface integrals of vector fields.

Def: The surface integral of \vec{F} over Φ :

$$\iint_{\Phi} \vec{F} \cdot d\vec{S} := \iint_D \vec{F} \cdot (\vec{T}_u \times \vec{T}_v) du dv.$$

Example: $\vec{F}(x, y, z) = (x, y, z)$

S: sphere of radius 1.

Find $\iint_S \vec{F} \cdot d\vec{S}$.

Sol. Parametrize S by using spherical coordinates.

$$(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

Then $\vec{T}_\theta = (-\sin\theta \sin\phi, \cos\theta \sin\phi, 0)$

$\vec{T}_\phi = (\cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi)$

$$\begin{aligned}\vec{T}_\theta \times \vec{T}_\phi &= (-\cos\theta \sin^2\phi, -\sin\theta \sin^2\phi, -\sin^2\theta \sin\phi \cos\phi - \cos^2\theta \sin\phi \cos\phi) \\ &= (-\cos\theta \sin^2\phi, -\sin\theta \sin^2\phi, -\sin\phi \cos\phi).\end{aligned}$$

On the sphere, $\vec{F} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$.

$$\begin{aligned}\Rightarrow \nabla_\Phi \vec{F} \cdot (\vec{T}_\theta \times \vec{T}_\phi) &= -\cos^2\theta \sin^3\phi - \sin^2\theta \sin^3\phi - \sin\phi \cos^2\phi \\ &= -\sin^3\phi - \sin\phi \cos^2\phi \\ &= -\sin\phi.\end{aligned}$$

$$\Rightarrow \iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot (\vec{T}_\theta \times \vec{T}_\phi) d\theta d\phi$$

$$= \int_0^\pi \int_0^{2\pi} -\sin\phi d\theta d\phi$$

$$= -2\pi \int_0^\pi \sin\phi d\phi = +2\pi \cos\phi \Big|_0^\pi = -4\pi.$$

(54)

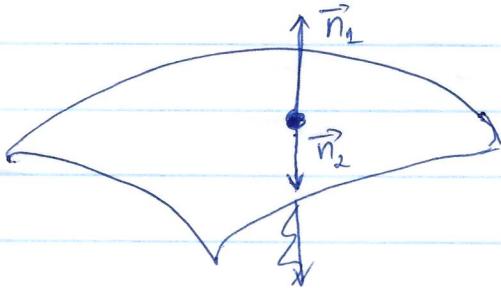
E.g. The volume of water per unit of time flowing "through a surface" S with velocity given by the field $\vec{F} = \iint_S \vec{F} \cdot d\vec{s}$

Interpret

Remark: We implicitly chose an orientation for the surface when we used $\vec{T}_\theta \times \vec{T}_\phi$ instead of $\vec{T}_\phi \times \vec{T}_\theta$.

⇒ orientation?

An oriented surface is one where at each $(x, y, z) \in S$ there are 2 unit normal vectors \vec{n}_1 and \vec{n}_2 with $\vec{n}_1 = -\vec{n}_2$ and each can be associated with a side of the surface.



Not every surface is orientable: e.g. Möbius strip is not.

E.g. The unit sphere can be given an oriented orientation by selecting $\vec{n} = \frac{\vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}}{\|\langle x, y, z \rangle\|}$.

This is an orientation preserving. (i.e. points outwards.)

But in the previous example with the sphere, our parametrization gave $\vec{T}_\theta \times \vec{T}_\phi = -\vec{n} \sin \phi$ ($0 \leq \phi \leq \pi \Rightarrow \sin \phi \geq 0$).

⇒ this parametrization was a orientation reversing.

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Thm: Surface integrals are independent of parametrization, provided they are orientation preserving.

$$\text{Thm: } \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \frac{dS}{\|\vec{T}_u \times \vec{T}_v\|} dudv$$

E.g. Heat flow

If $T(x, y, z)$ is the temperature at (x, y, z) , Then $\nabla T = \frac{\partial T}{\partial x} \vec{i} + \frac{\partial T}{\partial y} \vec{j} + \frac{\partial T}{\partial z} \vec{k}$ is the temperature gradient; $\vec{F} = -k \nabla T$ is a vector field associated with heat flow and $\iint_S \vec{F} \cdot d\vec{S}$ is the flux (or total rate of heat flow) across S .

Suppose $T(x, y, z) = x^2 + y^2 + z^2$. Find the heat flux across the unit sphere oriented with the outward normal.

(use $k=1$).

$$\vec{n} = x\vec{i} + y\vec{j} + z\vec{k} \text{. outward normal.}$$

$$\vec{F} = -\nabla T = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}.$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \frac{dS}{\|\vec{T}_u \times \vec{T}_v\|} dudv$$

$$= \iint_S (2x\vec{i} + 2y\vec{j} + 2z\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) \frac{dS}{\sqrt{x^2 + y^2 + z^2}}$$

$$= -2 \iint_S (x^2 + y^2 + z^2) dS$$

$$= -2 A(S).$$

$$= -2(4\pi)$$

$$= -8\pi.$$

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* Surface integrals over graphs:

Suppose S is the graph of a function, so

$$S: (x, y, g(x, y))$$

Additionally, suppose S is oriented with upward pointing normal (ie \vec{k} component)

$$\vec{n} = \frac{-\frac{\partial g}{\partial x} \vec{i} - \frac{\partial g}{\partial y} \vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

$$\text{since } T_x \times T_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix}$$

Then

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{s} &= \iint_D (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \cdot \left(-\frac{\partial g}{\partial x} \vec{i} - \frac{\partial g}{\partial y} \vec{j} + \vec{k}\right) dx dy \\ \Rightarrow \left\{ \iint_S \vec{F} \cdot d\vec{s} \right. &= \left. \iint_D \left(-F_1 \frac{\partial g}{\partial x} + -F_2 \frac{\partial g}{\partial y} + F_3\right) dx dy \right\}. \end{aligned}$$

$$\text{E.g. } z = x^2 + y^2, \quad x^2 + y^2 \leq 4$$

$$\text{Suppose } \vec{F} = -y \vec{i} + x \vec{j} + \vec{k}$$

$$\text{Compute } \iint_S \vec{F} \cdot d\vec{s}$$

S is the graph of a function.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{s} &= \iint_D (y \cdot 2x - x \cdot 2y + 1) dx dy \\ &= \iint_D dx dy \\ &= 4\pi. \end{aligned}$$