Lecture 21: Linear Differential Equations; Complex Matrices (Sections 5.4--5.5)

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Linear differential equations

- Goal. Solve the initial value problem

$$\boldsymbol{y}' = A\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0.$$

Linear differential equations

- Definition. Let A be $n \times n$. The **matrix exponential** is

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Theorem. Suppose $A = S\Lambda S^{-1}$. Then $e^A = Se^{\Lambda}S^{-1}$.

Linear differential equations

- Example. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then

$$e^{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \ldots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \ldots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

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Hence,

$$A = S\Lambda S^{-1}, \qquad \text{where } S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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Step 2: Compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

$$\boldsymbol{y} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix}.$$

Linear differential equations

- Example. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

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▶ Solution. Step 1: Diagonalize A

A has eigenvalues 2 and 4 (Why?).

- $\lambda = 2 \Longrightarrow$ eigenspace span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- $\lambda = 4 \Longrightarrow$ eigenspace span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

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The solution is

$$\Longrightarrow \boldsymbol{y} = e^{At}\boldsymbol{y}_0 = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}.$$

Complex Numbers and Their Conjugates

- Complex conjugate of $a + ib$ is $a - ib$ and is denoted by

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- Multiplying $a + ib$ by its conjugate produces $a^2 + b^2$:

$$(a + ib)(a - ib) = a^2 + b^2.$$

Lengths and Inner Products in the Complex Case

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- Example.

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ i \end{bmatrix} \Longrightarrow \|\boldsymbol{x}\| = \sqrt{|1|^2 + |i|^2} = 2.$$

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- Inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$

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- Example. Find inner products $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ and $\langle \boldsymbol{y}, \boldsymbol{x} \rangle$ of $\boldsymbol{x} = (1 + 3i, 3i)$ and $\boldsymbol{y} = (6 + 3i, 4 + i)$.

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$$\boxed{\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}}$$

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- **Example.**

$$\begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}.$$