Lecture 7: Solving $Ax = b$ and linear independence (Section 2.2-2.3)

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A basis of a vector space

- **Definition.** A set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ in V is a **basis** of V if
  - $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, and
  - the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ are linearly independent.

Theorem. Suppose that V has dimension d.

- A set of d vectors in V are a basis if they span V.
- A set of d vectors in V are a basis if they are linearly independent.

A basis of a vector space

▶ Example. Let $\mathscr{P}_2$ be the space of polynomials of degree at most 2.

a) What is the dimension of $\mathscr{P}_2$?

b) Is $\{t, 1-t, 1+t-t^2\}$ a basis of $\mathscr{P}_2$?

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$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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- **Example.** Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} \; : \; a, b, c, d \in \mathbb{R} \right\}.$$

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▶ Example. Consider

$$H = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

- What is the dimension of this subspace of $\mathbb{R}^3$?

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- **Example.** Consider

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  - What is the dimension of this subspace of $\mathbb{R}^3$?
  - Extend it to a basis of $\mathbb{R}^3$.

Bases for null spaces

To find a basis for $N(A)$:

Bases for null spaces

To find a basis for $N(A)$:

- find the parametric form of the solutions to $A\boldsymbol{x} = 0$,
- express solutions $\boldsymbol{x}$ as a linear combination of vectors with the free variables as coefficients,
- these vectors form a basis of $N(A)$.

Bases for null spaces

- Example. Find a basis for $N(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

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$$\left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & \boxed{1} & -6 & -15 & 0 \end{array}\right].$$

The solutions of $A\boldsymbol{x} = \boldsymbol{0}$ is of the form

$$\begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

Bases for null spaces

$$N(A) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-13\\0\\6\\1\\0\end{bmatrix}, \begin{bmatrix}-33\\0\\15\\0\\1\end{bmatrix}\right\}.$$

These above three vectors form a basis of $N(A)$. (Why?)

Bases for column spaces

Recall that the columns of A are independent
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$\boxed{\text{A basis for } C(A) \text{ is given by the pivot columns of } A.}$

Bases for column spaces

- Example. Find a basis for $C(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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- Example. Find a basis for $C(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

- Solution. Find row reduction form of A

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

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- Remark. For the basis of $C(A)$, you have to take the columns of A, *not the columns of an echelon form.*
- Remark. Row operations do not preserve *the column space.*
- Remark. Row operations do not preserve *the null space.*

Dimension of $C(A)$ and $N(A)$

Theorem. Let A be an $m \times n$ matrix. Then

- $\dim C(A)$ is the number of pivots of A.
- $\dim N(A)$ is the number of free variables of A.
- $\dim C(A) + \dim N(A) =$

- Proof.???

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- $\dim C(A) + \dim N(A) = n$

- Proof.???

Dimension of $C(A)$ and $N(A)$

- Example. Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find the dimension of $C(A)$ and $N(A)$.

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- Definition. The **rank** of a matrix A is the number of its pivots.

The four fundamental subspaces

- Example. Find a basis for $C(A)$ and $C(A^T)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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▶ Solution. To find $C(A)$, we can just use the echelon form of A. Likewise, we can also obtain $C(A^T)$ for an echelon form of A^T. But, it's not necessary!

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The rank of A is 2. And $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

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Recall that $C(A) \neq C(B)$. (We performed row operations).

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Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -5 \end{bmatrix} \right\}$ form a basis for $C(A)$.

The four fundamental subspaces

Theorem. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of rank r.

- $\dim C(A) = r$
- $\dim C(A^T) = r$
- $\dim N(A) = n - r$
- $\dim N(A^T) = m - r$