

Name: \_\_\_\_\_ PID: \_\_\_\_\_

TA: \_\_\_\_\_ Sec. No: \_\_\_\_\_ Sec. Time: \_\_\_\_\_

**Math 20A.**

**Midterm Exam 2**

**November 18, 2010**

*Turn off and put away your cell phone.*

*No calculators or any other electronic devices are allowed during this exam.*

*You may use one page of notes, but no books or other assistance during this exam.*

*Read each question carefully, and answer each question completely.*

*Show all of your work; no credit will be given for unsupported answers.*

*Write your solutions clearly and legibly; no credit will be given for illegible solutions.*

*If any question is not clear, ask for clarification.*

#	Points	Score
1	6	
2	6	
3	6	
4	6	
5	12	
$\Sigma$	36	

1. (6 points) Differentiate the following functions; you need not simplify.

(a)  $f(x) = \ln(\sin^3(x) + 1)$

$$f'(x) = \frac{1}{\sin^3(x) + 1} (\sin^3(x) + 1)' \quad \text{chain rule.}$$
$$= \frac{3\sin^2(x) \cos(x)}{\sin^3(x) + 1}$$

(b)  $g(x) = x^{2\sin(x)}$

First, take  $\ln$  both sides.

$$\ln g(x) = \ln x^{2\sin(x)}$$

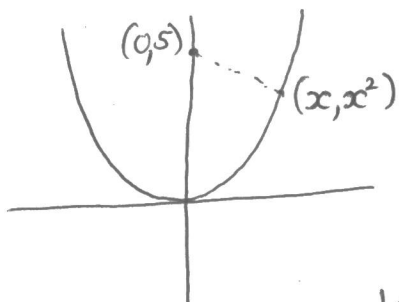
$$\ln g(x) = 2\sin(x) \ln(x)$$

Differentiate both sides with respect to  $x$

$$\frac{1}{g(x)} g'(x) = 2\cos(x) \ln(x) + 2\frac{\sin(x)}{x}$$

$$g'(x) = 2x^{2\sin(x)} \left( \cos(x) \ln(x) + \frac{\sin(x)}{x} \right)$$

2. (6 points) Find the points on the parabola  $y = x^2$  that are closest to the point  $(0, 5)$ .



The distance from  $(0, 5)$  to a point  $(x, x^2)$  on the parabola is

$$\sqrt{(0-x)^2 + (5-x^2)^2}$$

$$\begin{aligned} \text{Let } f(x) &= (0-x)^2 + (5-x^2)^2 \\ &= x^2 + 25 - 10x^2 + x^4. \end{aligned}$$

We want to find the <sup>absolute</sup> minimum of  $f(x)$ .

Method 1: Completing the square:

$$f(x) = \left(x^2 - \frac{9}{2}\right)^2 + \frac{19}{4}.$$

$$\Rightarrow f \text{ is minimized at } x^2 = \frac{9}{2} \Rightarrow x = \pm \sqrt{\frac{9}{2}}.$$

$\Rightarrow$  the points we want to find are  $\left(\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$  and  $\left(-\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$ .

Method 2:

$$f(x) = 4x^3 - 18x^2$$

$$f'(x) = 0 \Rightarrow 4x^3 - 18x = 0.$$

$$2x(2x^2 - 9) = 0.$$

$x = 0, \pm\sqrt{\frac{9}{2}}$  are critical points.

$$f''(x) = 12x^2 - 18.$$

$$f''(0) = -18 < 0 \Rightarrow f(0) \text{ is local max.}$$

$$f''\left(\sqrt{\frac{9}{2}}\right) = f''\left(-\sqrt{\frac{9}{2}}\right) = 12\left(\frac{9}{2}\right) - 18 > 0 \Rightarrow f\left(\sqrt{\frac{9}{2}}\right) \text{ and } f\left(-\sqrt{\frac{9}{2}}\right) \text{ are local } \cancel{\text{max}} \text{ min.}$$

$$f\left(\sqrt{\frac{9}{2}}\right) = f\left(-\sqrt{\frac{9}{2}}\right) = \frac{19}{4}. \text{ They are in fact absolute min.}$$

$\Rightarrow \left(\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$  and  $\left(-\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$  are the points we want to find.

3. (6 points) Find all points on the graph of  $y^2 + 2x^2 - xy = 14$  where the tangent line is horizontal.

Differentiate both sides w.r.t.  $x$ :

$$2y \frac{dy}{dx} + 4x - y - x \frac{dy}{dx} = 0$$

$$(2y - x) \frac{dy}{dx} = y - 4x.$$

$$\frac{dy}{dx} = \frac{y - 4x}{2y - x}.$$

The tangent line is horizontal if  $\frac{dy}{dx} = 0$ .

$$\Rightarrow \frac{y - 4x}{2y - x} = 0.$$

$$y = 4x.$$

Substitute  $y = 4x$  into  $y^2 + 2x^2 - xy = 14$ ,

$$(4x)^2 + 2x^2 - x(4x) = 14.$$

$$16x^2 + 2x^2 - 4x^2 = 14.$$

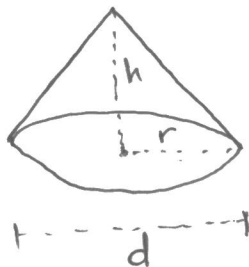
$$14x^2 = 14$$

$$x^2 = 1.$$

$$x = \pm 1.$$

$\Rightarrow$  The points are  $(1, 4)$  and  $(-1, -4)$ .

4. (6 points) Sand is pouring from a pipe at the rate of 16 cubic feet per second. The falling sand forms a conical pile whose height is always  $\frac{1}{4}$  the diameter of the base. How fast is the height of the sand pile increasing when the pile is 4 feet high?



$$h = \frac{1}{4} d.$$

$$\Rightarrow r = \frac{d}{2} = \frac{4h}{2} = 2h.$$

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (2h)^2 h = \frac{4}{3} \pi h^3.$$

$$V'(t) = 16.$$

We need to find  $h'(t)$  at the instant when  $h = 4$ .

Diff.  $V(t) = \frac{4}{3} \pi h^3(t)$  both sides w.r.t.  $t$ .

$$V'(t) = \frac{4}{3} \pi \underbrace{3h^2(t) h'(t)}_{\text{chain rule.}}$$

$$\Rightarrow h'(t) = \frac{V'(t)}{4\pi h^2(t)}.$$

$$= \frac{16}{4\pi 4^2}$$

$$= \frac{1}{4\pi}.$$

5. (12 points) Let  $h(x) = x^3 - 3x + 1$ .


(a) Find the intervals on which  $h$  is increasing and decreasing.

$$h'(x) = 3x^2 - 3$$

$$h'(x) = 0 \Rightarrow x = \pm 1.$$

$x$	Test value	sign of $h'$	$\Rightarrow h$ is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$
$(-\infty, -1)$	$h'(-2) = 9$	+	
$(-1, 1)$	$h'(0) = -3$	-	
$(1, \infty)$	$h'(2) = 9$	+	

(b) Find the local maxima and local minima of  $h$  and the points where they occur.


 local max  $h(-1) = -1 + 3 + 1 = 3$   
 at  $(-1, 3)$ .  
 local min  $h(1) = 1 - 3 + 1 = -1$   
 at  $(1, -1)$ .

(c) Find the absolute maximum and absolute minimum of  $h$  over the interval  $[-2, 2]$  and the points where they occur.

Since  $-1$  and  $1$  are both in  $[-2, 2]$ , we want to find

$$h(-1) = 3$$

$$h(1) = -1$$

$$h(-2) = -8 + 6 + 1 = -1$$

$$h(2) = 8 - 6 + 1 = 3$$

on  $[-2, 2]$ :

$$\Rightarrow \text{absolute max } h(-1) = 3 \text{ at } (-1, 3)$$

$$\text{max } h(2) = 3 \text{ at } (2, 3)$$

$$\text{absolute min } h(1) = -1 \text{ at } (1, -1)$$

$$h(-2) = -1 \text{ at } (-2, -1)$$

(d) Find the intervals on which the graph of  $h$  is concave up and concave down and find the inflection points.

$$h''(x) = 6x$$

$$h''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0.$$

Interval      test value      sign of  $h''$

$$(-\infty, 0) \quad h''(-1) = -6 \quad -$$

$$0 \quad h''(0) = 0 \quad 0$$

$$(0, \infty) \quad h''(1) = 6 \quad +$$

$\rightarrow h$  is concave down on  $(-\infty, 0)$  and concave up on  $(0, \infty)$   
 and  $(0, 1)$  is the inflection point.

Math 20A  
Midterm Exam 2  
November 20, 2012  
...  
Version A

Instructions

1. No calculators or other electronic devices are allowed during this exam.
2. You may use one page of notes, but no books or other assistance during this exam.
3. Write your *Name*, *PID*, and *Section* on the front of your Blue Book.
4. Write the *Version* of your exam at the top of the page on the front of your Blue Book.
5. Write your solutions clearly in your Blue Book
  - (a) Carefully indicate the number and letter of each question and question part.
  - (b) Present your answers in the same order they appear in the exam.
  - (c) Start each question on a new side of a page.
6. Read each question carefully, and answer each question completely.
7. Show all of your work; no credit will be given for unsupported answers.

0. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard.

1. (6 points) Find the derivatives of:

(a)  $f(x) = \ln(\sin(x) + 2)$

(b)  $f(x) = (\cos(x) + 2)^x$

2. (6 points) If  $g(x) + 2x \sin(g(x)) = x^2$  and  $g(1) = 0$ , find  $g'(1)$ .

3. (6 points) Let  $f(x) = x^3 + 6x^2 + 9x + 1$ .

(a) Find and classify all of the critical points of  $f(x)$ .

(b) Find the global maximum and global minimum of  $f(x)$  on the interval  $[-2, 0]$ .

4. (3 points) Compute  $\lim_{x \rightarrow \infty} x^3 e^{-2x^2}$ .

5. (6 points) The total volume of a Truffula Tree after  $t$  years is given by  $V(t) = \pi t h$ , where  $h$  is the height of the tree, which changes over time. The surface area of a Truffula Tree is given by  $A(t) = \frac{\pi}{2} h$ . Currently, for a ten-year old Truffula Tree, the measured rate of change of its volume is  $200 \text{ ft}^3/\text{yr}$ , and the measured rate of change of its surface area is  $1 \text{ ft}^2/\text{yr}$ . How high is the Truffula Tree?





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1) a)  $f(x) = \ln(\sin(x) + 2)$

$$f'(x) = \frac{1}{\sin(x) + 2} \cdot \cos(x).$$

b)  $f(x) = (\cos(x) + 2)^x$

$$\ln f(x) = x \ln(\cos(x) + 2).$$

Differentiate both sides w.r.t.  $x$

$$\frac{f'(x)}{f(x)} = \ln(\cos(x) + 2) + \frac{x}{\cos(x) + 2}.$$

$$f'(x) = (\cos(x) + 2)^x \left( \ln(\cos(x) + 2) + \frac{x}{\cos(x) + 2} \right).$$

2) Let  $y = g(x)$ . Then

$$y + 2x \sin(y) = x^2.$$

Implicitly differentiate both sides w.r.t.  $x$ :

$$\frac{dy}{dx} + 2 \sin(y) + 2x \cos(y) \frac{dy}{dx} = 2x.$$

$$(2x \cos(y) + 1) \frac{dy}{dx} = 2x - 2 \sin(y).$$

$$\frac{dy}{dx} = \frac{2x - 2 \sin(y)}{2x \cos(y) + 1}.$$

$$\Rightarrow g'(x) = \frac{2x - 2 \sin(g(x))}{2x \cos(g(x)) + 1}.$$

$$g'(1) = \frac{2(1) - 2 \sin(g(1))}{2(1) \cos(g(1)) + 1} = \frac{2 - 2 \sin(0)}{2 \cos(0) + 1} = \frac{2}{3}.$$

3)  $f(x) = x^3 + 6x^2 + 9x + 1$ .  $\text{Dom}(f) = \mathbb{R}$ .

a)  $f'(x) = 3x^2 + 12x + 9$ .

$f'(x) = 0 \Rightarrow 3x^2 + 12x + 9 = 0$ .

$x^2 + 4x + 3 = 0$ .

$(x+1)(x+3) = 0$ .

$x = -1$  and  $x = -3$ .

are critical points.

$x$	Test value	sign of $f'$
$(-\infty, -3)$	$f'(-4) = +9$	+
$-3$	$f'(-3) = 0$	0
$(-3, -1)$	$f'(-2) = -3$	-
$(-1, \infty)$	$f'(-1) = 0$	0
$-1$	$f'(0) = 9$	+

$\Rightarrow f(-3)$  is local max.

$f(-1)$  is local min.

b) Since  $-1 \in [-2, 0]$ ,

$f(-2) = -8 + 24 - 18 + 1 = -1$ .

$f(0) = \boxed{1} \rightarrow \text{absolute max}$

$f(-1) = -1 + 6 - 9 + 1 = \boxed{-3} \rightarrow \text{absolute min.}$

$$4) \quad \lim_{x \rightarrow \infty} x^3 e^{-2x^2} = \lim_{x \rightarrow \infty} \frac{x^3}{e^{2x^2}} \quad \frac{\infty}{\infty}.$$

$$\stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^{2x^2} (4x)}$$

$$= \frac{3}{4} \lim_{x \rightarrow \infty} \frac{x^2}{e^{2x^2}} \quad \frac{\infty}{\infty}.$$

$$\stackrel{\text{L'Hôpital}}{=} \frac{3}{4} \lim_{x \rightarrow \infty} \frac{1}{e^{2x^2} (4x)}$$

$$= \frac{3}{16} \lim_{x \rightarrow \infty} \frac{1}{x e^{2x^2}}$$

$$= \infty.$$

$$5) \quad V(t) = \pi t h. \quad A(t) = \frac{\pi}{2} h. \\ V'(10) = 200 \text{ ft}^3/\text{yr.} \quad \text{and} \quad A'(10) = 1 \text{ ft}^2/\text{yr.}$$

$$\text{Since } A(t) = \frac{\pi}{2} h(t), \quad A'(t) = \frac{\pi}{2} h'(t).$$

$$\Rightarrow \quad h'(t) = \frac{2}{\pi} A'(t)$$

$$h'(10) = \frac{2}{\pi} A'(10) = \frac{2}{\pi}.$$

$$\text{Since } V(t) = \pi t h, \quad V'(t) = \pi h(t) + \pi t h'(t).$$

$$\Rightarrow \quad h(t) = \frac{V'(t) - \pi t h'(t)}{\pi}.$$

$$h(10) = \frac{V'(10) - \pi \cdot 10 h'(10)}{\pi}$$

$$= \frac{200 - \pi \cdot 10 \left(\frac{2}{\pi}\right)}{\pi}$$

$$= \frac{180}{\pi} \text{ ft.}$$

