HW01 — Solution.

1) We can use Gaussian Elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

For part b, we consider the system

$$u + v + w = 2$$
$$u + 2v + 3w = 1$$
$$v + 2w = a,$$

and use GE to find a:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & a \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & a \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & a+1 \end{array}\right]$$

The system has solutions only when $a+1=0$ or $a=-1$.

Hence, the first two ~~soluti~~ equations become

$$u - w = 3 \quad \text{and} \quad v + 2v = -1.$$

To find one solution, just let $w = 0$ and solve for u and v.

$$\Rightarrow \quad u = 3, \quad v = -1, \quad w = 0.$$

2) Consider a system

$$A\vec{x} = \vec{b}.$$

Suppose that the system has two distinct solutions $\vec{u}$ and $\vec{v}$,

i.e. $A\vec{u} = \vec{b}$ and $A\vec{v} = \vec{b}$.

$$\Rightarrow \quad A\vec{u} = A\vec{v}$$
$$A(\vec{u} - \vec{v}) = 0.$$

There are many other solutions that we can find. For example, $\vec{w} = \frac{\vec{u}+\vec{v}}{2}$ is also another solution since $A\vec{w} = \frac{1}{2}(A\vec{u} + A\vec{v}) = \vec{b}$.

In fact, any point on the line between $\vec{u}$ and $\vec{v}$ is a solution. Indeed, they are of the form $t\vec{u} + (1-t)\vec{v}$ for $t \in [0,1]$.

$$\Rightarrow \quad A(t\vec{u} + (1-t)\vec{v}) = tA\vec{u} + (1-t)A\vec{v} = \vec{b}.$$

3) a) Conside the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$\Rightarrow$ we have to exchange the second and the third rows if

$$d - 10 = 0 \quad \text{or} \quad d = 10.$$

Provided that $d = 10$, the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

This is a triangular system.

b) If $d \neq 10$,

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - \frac{1}{d-10} R_2} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 0 & -1+\frac{1}{d-10} & 3 - \frac{2}{d-10} \end{array}\right]$$

The system will be singular if

$$\frac{1}{d-10} - 1 = 0 \quad \Rightarrow \quad d = 11.$$

4) a)
$$\begin{aligned} X = 2Y &\Rightarrow X - 2Y = 0 \\ X + Y = 39 &\Rightarrow X + Y = 39 \end{aligned} \quad \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 39 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 39 \end{array}\right] \Rightarrow \begin{aligned} 3Y = 39 &\Rightarrow Y = 13. \\ X = 2Y &= 26. \end{aligned}$$

b) The two points lie on the line, so we have.

$$\begin{aligned} 2m + c &= 5 \\ 3m + c &= 7. \end{aligned} \quad \Rightarrow \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 3 & 1 & 7 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{3}{2} R_1} \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \Rightarrow \begin{aligned} -\tfrac{1}{2} c = -\tfrac{1}{2} &\Rightarrow c = 1. \\ 2m + 1 = 5 &\Rightarrow m = 2. \end{aligned}$$

Therefore, the line is

$$y = 2x + 1.$$