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HW4.

Exercise 3.5.

a) $M = \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow M^TM = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$.

i) Find eigenvalues of M^TM:

$$\det\begin{bmatrix} 10-\lambda & 1 \\ 1 & 10-\lambda \end{bmatrix} = (10-\lambda)^2 - 1 = 0.$$

$$(9-\lambda)(11-\lambda) = 0.$$

$$\lambda_2 = 9 \text{ or } \lambda_1 = 11.$$

ii) Find eigenvectors:

For $\lambda_1 = 11$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \Rightarrow v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

For $\lambda_2 = 9$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

iii) Since $\lambda_1 = 11$, and $\lambda_2 = 9$, $\sigma_1 = \sqrt{11}$ and $\sigma_2 = 3$.

$$Mv_1 = \sigma_1 u_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sigma_1 u_1.$$

$$\Rightarrow \sigma_1 u_1 = \begin{bmatrix} 2/\sqrt{2} \\ 3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{11}}\begin{bmatrix} 2/\sqrt{2} \\ 3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{22}}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$Mv_2 = \sigma_2 u_2 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sigma_2 u_2.$$

$$\Rightarrow u_2 = \frac{1}{3}\begin{bmatrix} 0 \\ 3/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\Rightarrow M = \begin{bmatrix} 2/\sqrt{22} & 0 \\ 3/\sqrt{22} & 1/\sqrt{2} \\ 3/\sqrt{22} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{11} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

b) $M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow M^T M = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 14 \end{bmatrix}$

- Find eigenvalues:

$$\det \begin{bmatrix} 14-\lambda & 6 \\ 6 & 10-\lambda \end{bmatrix} = (14-\lambda)(14-\lambda) - 6^2 = 0.$$

$$\cancel{\lambda^2 - 24\lambda + 140 - 36} = 0.$$

$$\cancel{\lambda^2 - 24\lambda + 104 = 0}$$

$$(20-\lambda)(8-\lambda) = 0$$

$$\lambda_1 = 20 \quad \text{and} \quad \lambda_2 = 8$$

- Find eigenvectors:

For $\lambda_1 = 20$, $\begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

For $\lambda_2 = 8$, $\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

- $M v_1 = \sigma_1 u_1 \Rightarrow \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \sqrt{20}\, u_1 \Rightarrow u_1 = \begin{bmatrix} 2/\sqrt{40} \\ 2/\sqrt{40} \\ 4/\sqrt{40} \\ 4/\sqrt{40} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix}$

- $M v_2 = \sigma_2 u_2 \Rightarrow \begin{bmatrix} 2/\sqrt{2} \\ -2/\sqrt{2} \\ 2/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix} = \sqrt{8}\, u_2 \Rightarrow u_2 = \begin{bmatrix} 2/\sqrt{16} \\ -2/\sqrt{16} \\ 2/\sqrt{16} \\ -2/\sqrt{16} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$$M = \begin{bmatrix} 1/\sqrt{10} & 1/2 \\ 1/\sqrt{10} & -1/2 \\ 2/\sqrt{10} & 1/2 \\ 2/\sqrt{10} & -1/2 \end{bmatrix} \begin{bmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

3) $A = U\Sigma V^T = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

U: customer – concept (e.g. cuisine) similarity matrix.

V: restaurant – concept (eg. cuisine) similarity matrix.

1st cuisine (e.g. Western)

. $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ restaurant 1. restaurant 2

=> restaurant 2 is likely a Western one, but restaurant 1 may be not.

2nd cuisine (e.g. Asian)

. $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ restaurant 1. restaurant 2. => restaurant 1 is likely an Asian one, but restaurant 2 is not.

For u_1, it tells us how the customers like the 1st cuisine.

σ_i's are the strength of the concepts (e.g. Western and Asian cuisine).

$\sigma_1 - \sigma_2 = 4 - 2 = 2$ => the 1st concept is stronger then the second one.

Exercise 3.6. $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{bmatrix}$.

1) $B = A^T A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ +1 & -2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}$

$x^{(1)} = Bx / \|Bx\|_2 = \begin{bmatrix} 4 \\ 16 \end{bmatrix} \cdot \frac{1}{\sqrt{4^2+16^2}} = \cancel{4} \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$x^{(2)} = Bx^{(1)} / \|Bx^{(1)}\|_2 = \frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ 64 \end{bmatrix} \Big/ \frac{4}{\sqrt{17}} \sqrt{257} = \frac{1}{\sqrt{257}} \begin{bmatrix} \cancel{4}\ 1 \\ 16 \end{bmatrix}$

$x^{(3)} = Bx^{(2)} / \|Bx^{(2)}\|_2 = \frac{1}{\sqrt{257}} \begin{bmatrix} 4 \\ 256 \end{bmatrix} \Big/ \frac{1}{\sqrt{257}} \sqrt{4^2 + 256^2}$.

$\Rightarrow \quad x^{(3)} = \frac{1}{\sqrt{65552}} \begin{bmatrix} 4 \\ 256 \end{bmatrix}$ is an estimate of v_1.

2) $\lambda_1 = 16$ and $\lambda_2 = 4$ are eigenvalues of B.

$\Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Av_1 = \sigma_1 u_1 \Rightarrow \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = 4u_1 \Rightarrow u_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$.

$Av_2 = \sigma_2 u_2 \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 2u_2 \Rightarrow u_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

Note: the first component of $x^{(3)}$ is small. If we iterate more, the first component of that vector is getting smaller $\rightarrow$ closer to $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Exercise 3.10: Verify that

$$\sum_{i=1}^{r} c_i \vec{x}_i \vec{y}_i^T = \underbrace{\begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_r \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_r \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y_1^T \\ \vdots \\ y_r^T \end{bmatrix}}_{Y^T}.$$

Recall that Matrices A and B are identical if and only if for all vectors v, $Av = Bv$ (see Lemma. 3.3).
We will show that for any vector v,

$$\left(\sum c_i x_i y_i^T\right) v = X C Y^T v.$$

1) $$\text{LHS} = \sum_{i=1}^{r} c_i x_i y_i^T v = \sum_{i=1}^{r} c_i x_i \langle y_i, v\rangle.$$

$$= \sum_{i=1}^{r} \underbrace{c_i \langle y_i, v\rangle}_{\text{scalar}} x_i$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_r \end{bmatrix} \begin{bmatrix} c_1 \langle y_1, v\rangle \\ c_2 \langle y_2, v\rangle \\ \vdots \\ c_r \langle y_r, v\rangle \end{bmatrix}.$$

2) $$\text{RHS} = X C Y^T v = X C \begin{bmatrix} \langle y_1, v\rangle \\ \vdots \\ \langle y_r, v\rangle \end{bmatrix} = X \begin{bmatrix} c_1 \langle y_1, v\rangle \\ \vdots \\ c_r \langle y_r, v\rangle \end{bmatrix}.$$

$$\Rightarrow \quad \text{LHS} = X \begin{bmatrix} c_1 \langle y_1, v\rangle \\ \vdots \\ c_r \langle y_r, v\rangle \end{bmatrix} = \text{RHS}.$$

Exercise 3.12.

1) Let $u_1, \dots, u_r$ be the left singular vectors of A. Then

$$A = U \Sigma V^T.$$

$$\Rightarrow A^T A = (U\Sigma V^T)^T A U \Sigma V^T = V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T.$$

$$= V \Sigma \Sigma V^T.$$

$$= V \Sigma^2 V^T.$$

$$= \sum_{i=1}^{r} \sigma_i^2 v_i v_i^T.$$

2) For any $j \in \{1, \dots, r\}$:

$$A^T A v_j = \left(\sum_{i=1}^{r} \sigma_i^2 v_i v_i^T \right) v_j$$

$$= \sum_{i=1}^{r} \sigma_i^2 v_i \langle v_i, v_j \rangle.$$

$$\Rightarrow A^T A v_j = \sigma_j^2 v_j \qquad \text{since } \langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\Rightarrow$ v_j is an eigenvector of $A^T A$.

3) ~~Too~~ (Challenging see Trefethen & Bau textbook)

I will not ask this question on quiz & exam.

Exercise 3.13.

1) $\|A_k\|_F^2 = \sum_{i=1}^{k} \sigma_i^2$ (See Lemma 3.2)

2) $\|A_k\|_2^2 = \sigma_1^2$.

Since A_k has the largest singular value is σ_1.

3) $A - A_k = \sum_{i=k+1}^{r} \sigma_i u_i v_i^T$.

$\Rightarrow \|A - A_k\|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$.

4) $\|A - A_k\|_2^2 = \sigma_{k+1}^2$. since the largest singular value of $A - A_k$ is σ_{k+1}.

Exercise 3.16: (None) I was not able to go over all aspects of the power method!