7. Problem 5.1.14. Find the rank and all four eigenvalues for both the matrix of ones and the checkerboard matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Which eigenvectors correspond to nonzero eigenvalues?

Answer: Since all columns of A are equal, the rank of A is equal to 1; this implies that 0 is an eigenvalue of A, with eigenvectors given by the elements of the nullspace of A. Since elements of the nullspace of A are of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

we see that $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors corresponding to the eigenvalue 0.

To find the other eigenvalue of A (since A has rank 1, there can be at most one), we solve

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\
&\qquad + 1 \cdot \begin{vmatrix} 1 & 1-\lambda & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \\ 1 & 1 & 1 \end{vmatrix} \\
&= (1-\lambda) \left((1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} \right) \\
&\qquad - \left(1 \cdot \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} \right) \\
&\qquad + \left(1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} - (1-\lambda) \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right) \\
&\qquad - \left(1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} - (1-\lambda) \cdot \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right)
\end{aligned}$$

Then, after simplifying, we see that

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
&= (1-\lambda)^4 - 6(1-\lambda)^2 + 8(1-\lambda) - 3 \\
&= \lambda^4 - 4\lambda^3 \\
&= \lambda^3(\lambda - 4).
\end{aligned}$$

Therefore, the nonzero eigenvalue of A is 4; the corresponding eigenvector is in the nullspace of

$$A - 4I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

This matrix reduces to

$$\begin{bmatrix} -3 & 0 & 0 & 3 \\ 0 & -8/3 & 0 & 8/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the nullspace is the line containing the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, meaning that this is the eigenvector associated to the eigenvalue 4.

Turning to the checkerboard matrix C, notice that the first two columns are linearly independent, but that the third and fourth columns are repeats of the first two columns, so the matrix must have rank 2. Therefore, one eigenvalue is 0; the corresponding eigenvectors will be the elements of the nullspace of C. Since C reduces to

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for the nullspace of C, meaning that these two vectors are eigenvectors corresponding to the eigenvalue 0.

The other two eigenvalues will come from solving

$$
\begin{aligned}
0 &= \det(C - \lambda I) \\
&= \begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix} \\
&= -\lambda \cdot \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} + 0 - 1 \cdot \begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 1 & 0 & 1 \end{vmatrix} \\
&= -\lambda \left(-\lambda \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix} + 0 \right) - \left(1 \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix} + 0 \right) \\
&\quad + 0 - \left(1 \cdot \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} + \lambda \cdot \begin{vmatrix} 0 & -\lambda \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) \\
&= \lambda^4 - 4\lambda^2 \\
&= \lambda^2(\lambda^2 - 4) \\
&= \lambda^2(\lambda - 2)(\lambda + 2),
\end{aligned}
$$

so the nonzero eigenvalues of C are 2 and -2. For the eigenvalue 2, the corresponding eigenvector will be any vector in the nullspace for

$$
C - 2I = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}.
$$

You can easily check that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is such an eigenvector.

Likewise, the eigenvector corresponding to the eigenvalue -2 will be any vector in the nullspace for

$$
C - (-2)I = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.
$$

You can check that $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ is such an eigenvector.

8. Problem 5.1.24. What do you do to $A\vec{x} = \lambda\vec{x}$ in order to prove the following?

   (a) λ^2 is an eigenvalue of A^2.

Problem 2: Suppose the eigenvector matrix S has $S^T = S^{-1}$. Show that $A = S \Lambda S^{-1}$ is symmetric and has orthogonal eigenvectors.

Sol. Suppose $S = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$, where $\vec{v}_i$ are the eigenvectors of A. Since $S^T = S^{-1}$,

$$S S^T = S S^{-1} = I$$

and

$$S^T S = S^{-1} S = I.$$

Hence, S is an orthogonal matrix.

$$\Rightarrow A^T = (S \Lambda S^{-1})^T = (S^{-1})^T \Lambda^T S^T = S \Lambda^T S^T = S \Lambda S^{-1} = A.$$

since $\Lambda^T = \Lambda$ and $(S^{-1})^T = (S^T)^T = S$.

$\Rightarrow$ A is symmetric.

Problem 3: To diagonalize A, first find the eigenvalues

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 - 1^2$$

$$= (1-\lambda)(3-\lambda)$$

$$= (\lambda - 1)(\lambda - 3).$$

$\Rightarrow$ $\lambda_1 = 3$ and $\lambda_2 = 1$. are the eigenvalues.

For $\lambda_1 = 3$:

$$(A - 3I)\vec{x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ is the eigenvector.

For $\lambda_2 = 1$: $(A-I)\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow$ $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector.

$$\Rightarrow A = S\Lambda S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 1^k \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}.$$

<u>Problem 4</u>: First, we want to diagonalize A.

$\Rightarrow$ Find the eigenvalues of A.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) = 0.$$

$\Rightarrow$ $\lambda_1 = 1$ and $\lambda_2 = 0$ are the eigenvalues of A.

- For $\lambda_1 = 1$: $(A-I)\vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\Rightarrow$

  $\Rightarrow$ $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the eigenvector.

- For $\lambda_2 = 0$: $(A-0I)\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

  $\Rightarrow$ $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector.

Hence, the eigenvector matrix is

$$S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and the eigenvalue matrix is $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad A = S\Lambda S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

and $$e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{0} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{t}-1 \\ 0 & 1 \end{bmatrix}.$$