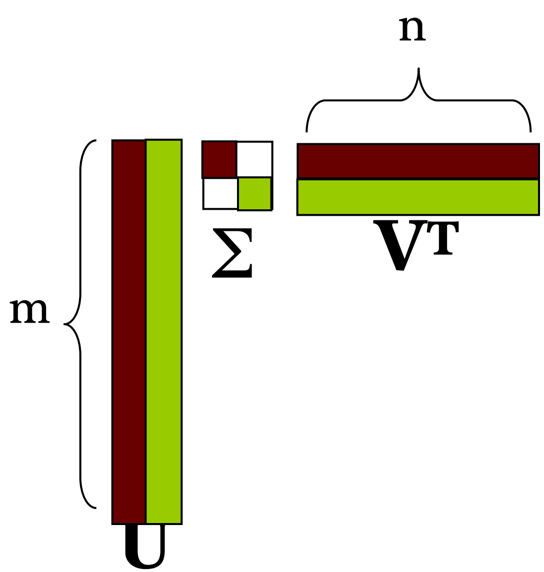
SVD – Example: Users-to-Movies

- **$A = U\,\Sigma\,V^T$ - example: Users to Movies**

$$
\begin{array}{c}
\\
\text{SciFi} \\
\\
\\
\text{Romnce} \\
\\
\end{array}
\begin{array}{ccccc}
\text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
\end{array}
$$

$$
\begin{array}{l|c}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\hline
\begin{array}{l} \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \\ \end{array} & \left[\begin{array}{ccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{array}\right]
\end{array}
=
$$



SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example: Users to Movies**

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- **A = U Σ V^T - example: Users to Movies**

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \\ \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{ccc} \text{SciFi-concept} & \text{Romance-concept} & \end{array} \\
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\end{array}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ **- example:**

U is "user-to-concept" similarity matrix

SciFi-concept

Romance-concept

$$
\begin{array}{c|c}
 & \text{Matrix} \quad \text{Alien} \quad \text{Serenity} \quad \text{Casablanca} \quad \text{Amelie} \\
\begin{array}{c} \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example:

SciFi-concept

“strength” of the SciFi-concept

$$
\begin{array}{cc}
 & \begin{array}{ccccc}\text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie}\end{array} \\
\begin{array}{c} \\ \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example:**

V is "movie-to-concept" similarity matrix

SciFi-concept

$$
\begin{array}{c|ccccc}
 & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
\hline
\text{SciFi} & 1 & 1 & 1 & 0 & 0 \\
\text{SciFi} & 3 & 3 & 3 & 0 & 0 \\
\text{SciFi} & 4 & 4 & 4 & 0 & 0 \\
\text{SciFi} & 5 & 5 & 5 & 0 & 0 \\
\text{Romnce} & 0 & 2 & 0 & 4 & 4 \\
\text{Romnce} & 0 & 0 & 0 & 5 & 5 \\
\text{Romnce} & 0 & 1 & 0 & 2 & 2
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SciFi-concept

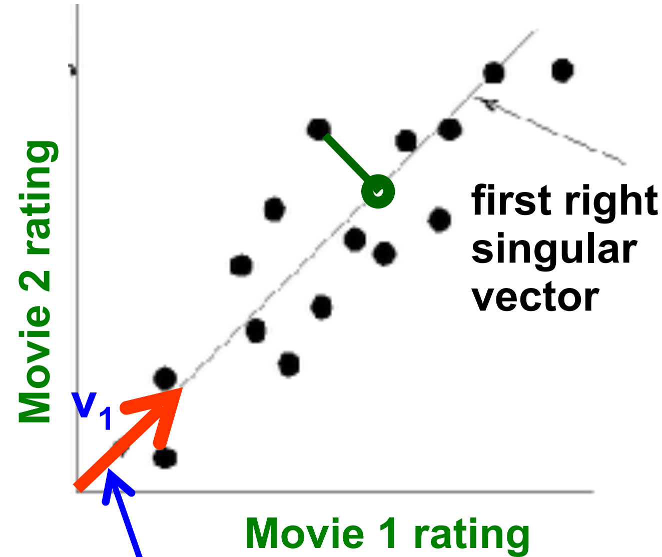
SVD - Interpretation #1

'movies', 'users' and 'concepts':

- **U: user-to-concept similarity matrix**
- **V: movie-to-concept similarity matrix**
- **Σ: its diagonal elements: 'strength' of each concept**

SVD - Interpretation #2

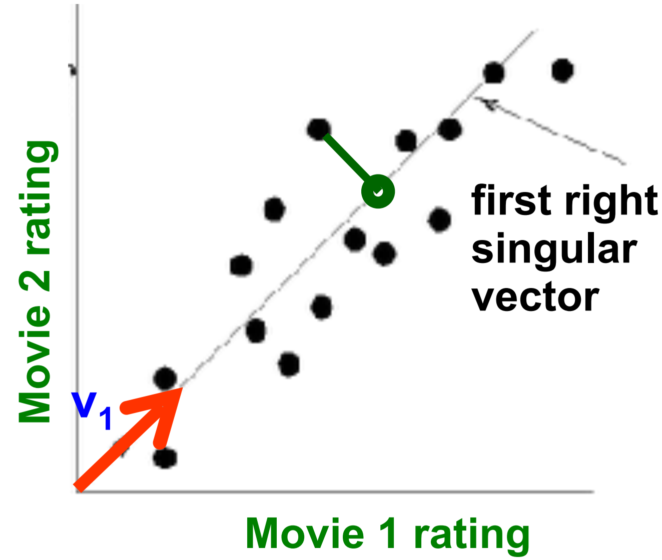
- **$A = U \Sigma V^T$ - example:**
  - V: "movie-to-concept" matrix
  - U: "user-to-concept" matrix



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix} \times \begin{bmatrix} \boxed{\mathbf{0.56} \quad \mathbf{0.59} \quad \mathbf{0.56} \quad 0.09 \quad 0.09} \\ 0.12 \quad -0.02 \quad 0.12 \quad \mathbf{-0.69} \quad \mathbf{-0.69} \\ 0.40 \quad \mathbf{-0.80} \quad 0.40 \quad 0.09 \quad 0.09 \end{bmatrix}$$

SVD - Interpretation #2

- $A = U \Sigma V^T$ - example:



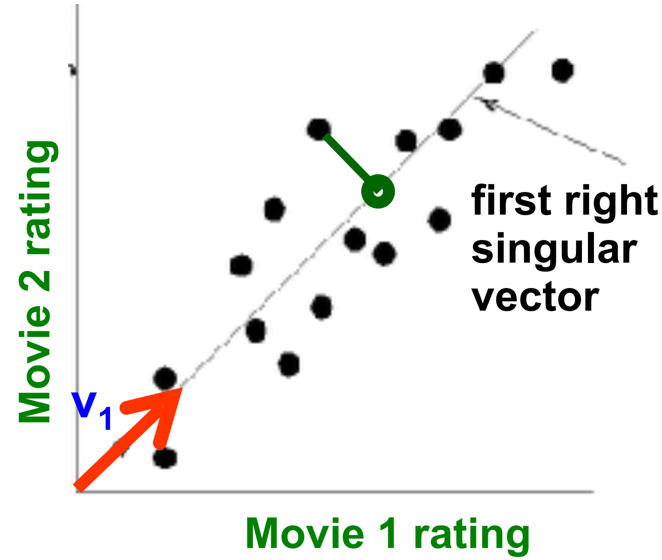
variance ('spread') on the v_1 axis

$$
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}
=
\begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix}
\times
\begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix}
\times
\begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}
$$

SVD - Interpretation #2

$A = U \Sigma V^T$ - example:

- $U \Sigma$: Gives the coordinates of the points in the projection axis



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Projection of users on the "Sci-Fi" axis $(U \Sigma)^T$:

$$\begin{bmatrix} \boxed{\begin{matrix} 1.61 \\ 5.08 \\ 6.82 \\ 8.43 \\ 1.86 \\ 0.86 \\ 0.86 \end{matrix}} & \begin{matrix} 0.19 \\ 0.66 \\ 0.85 \\ 1.04 \\ -5.60 \\ -6.93 \\ -2.75 \end{matrix} & \begin{matrix} -0.01 \\ -0.03 \\ -0.05 \\ -0.06 \\ 0.84 \\ -0.87 \\ 0.41 \end{matrix} \end{bmatrix}$$

SVD - Interpretation #2

More details

- **Q:** How exactly is dim. reduction done?

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SVD - Interpretation #2

More details

- **Q: How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \cancel{\mathbf{1.3}}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD - Interpretation #2

More details

- Q: How exactly is dim. reduction done?
- A: Set smallest singular values to zero

$$
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \cancel{\mathbf{1.3}} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}
$$

SVD - Interpretation #2

More details

- Q: **How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.13} & 0.02 & \cancel{-0.01} \\ \mathbf{0.41} & 0.07 & \cancel{-0.03} \\ \mathbf{0.55} & 0.09 & \cancel{-0.04} \\ \mathbf{0.68} & 0.11 & \cancel{-0.05} \\ 0.15 & \mathbf{-0.59} & \cancel{\mathbf{0.65}} \\ 0.07 & \mathbf{-0.73} & \cancel{\mathbf{-0.67}} \\ 0.07 & \mathbf{-0.29} & \cancel{\mathbf{0.32}} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \cancel{\mathbf{1.3}} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ \cancel{0.40} & \cancel{\mathbf{-0.80}} & \cancel{0.40} & \cancel{0.09} & \cancel{0.09} \end{bmatrix}$$

SVD - Interpretation #2

More details

- **Q:** How exactly is dim. reduction done?
- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 \\
\mathbf{0.41} & 0.07 \\
\mathbf{0.55} & 0.09 \\
\mathbf{0.68} & 0.11 \\
0.15 & \mathbf{-0.59} \\
0.07 & \mathbf{-0.73} \\
0.07 & \mathbf{-0.29}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 \\
0 & \mathbf{9.5}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69}
\end{bmatrix}
$$

SVD - Interpretation #2

More details

- **Q:** How exactly is dim. reduction done?
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.92} & \mathbf{0.95} & \mathbf{0.92} & 0.01 & 0.01 \\ \mathbf{2.91} & \mathbf{3.01} & \mathbf{2.91} & -0.01 & -0.01 \\ \mathbf{3.90} & \mathbf{4.04} & \mathbf{3.90} & 0.01 & 0.01 \\ \mathbf{4.82} & \mathbf{5.00} & \mathbf{4.82} & 0.03 & 0.03 \\ 0.70 & \mathbf{0.53} & 0.70 & \mathbf{4.11} & \mathbf{4.11} \\ -0.69 & 1.34 & -0.69 & \mathbf{4.78} & \mathbf{4.78} \\ 0.32 & \mathbf{0.23} & 0.32 & \mathbf{2.01} & \mathbf{2.01} \end{bmatrix}$$

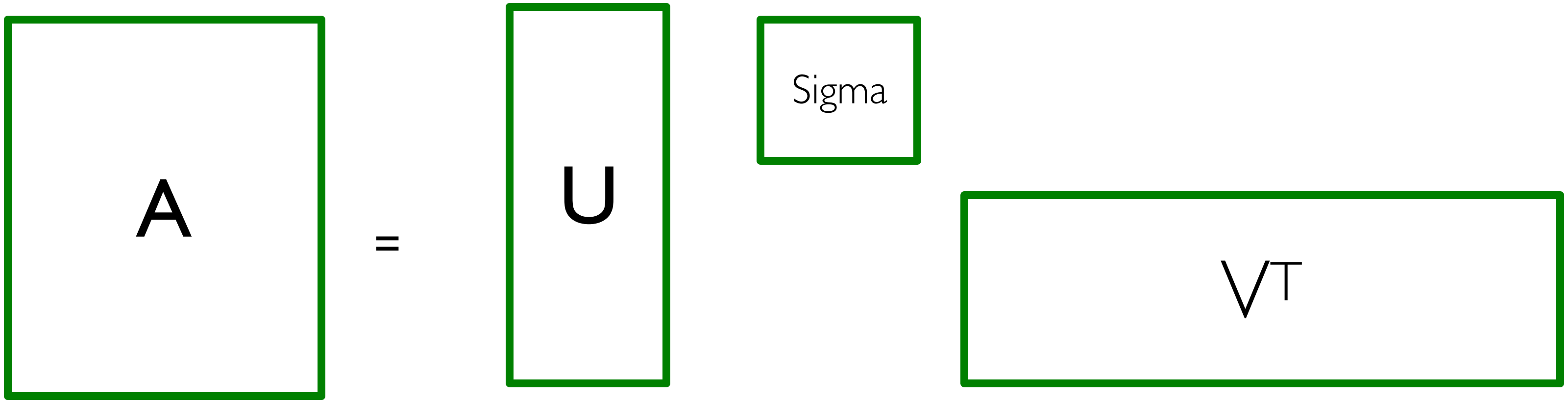
Frobenius norm:

$$\|M\|_F = \sqrt{\Sigma_{ij} M_{ij}^2}$$

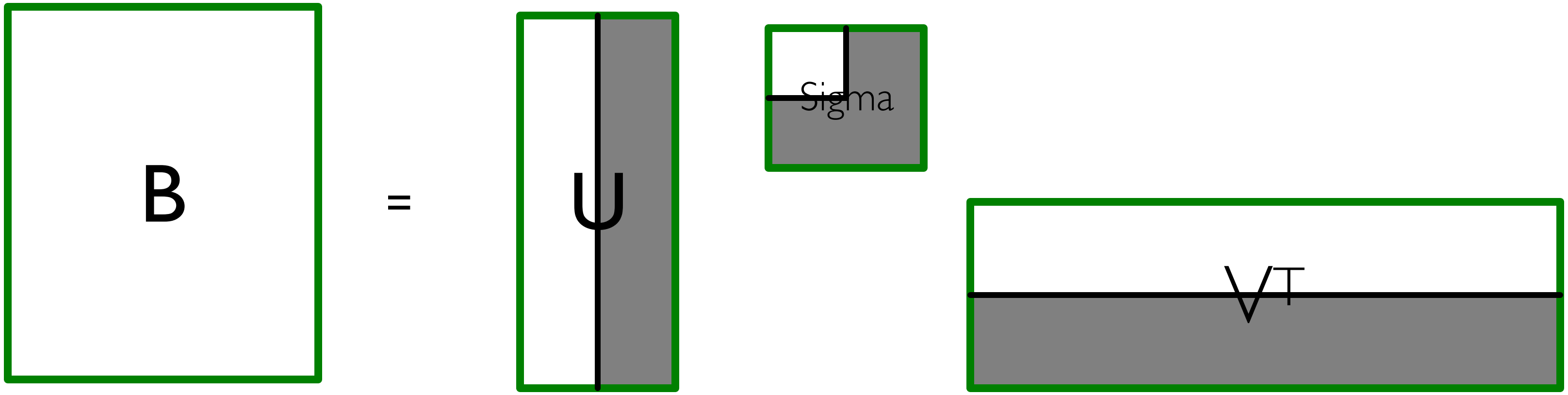
$$\|A-B\|_F = \sqrt{\Sigma_{ij} (A_{ij}-B_{ij})^2}$$

is "small"

SVD – Best Low Rank Approx.



B is best approximation of A

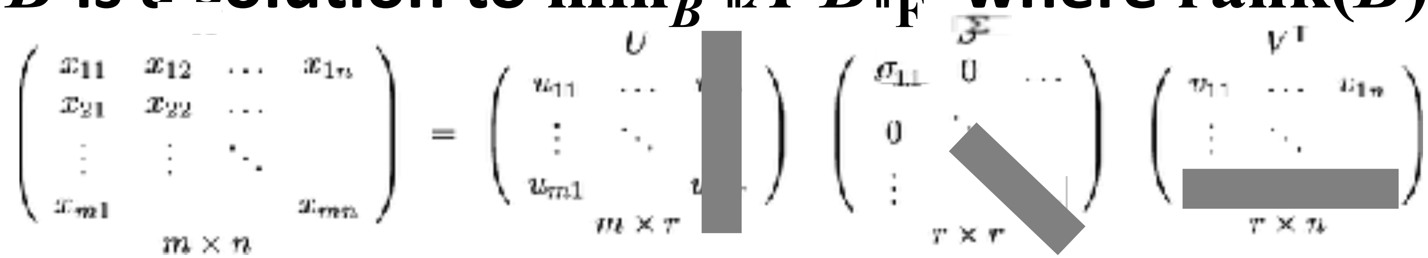


SVD – Best Low Rank Approx.

- **Theorem:**
  Let $A = U\,\Sigma\,V^T$ and $B = U\,S\,V^T$ where
  S = diagonal rxr matrix with $s_i=\sigma_i$ ($i=1\ldots k$) else $s_i=0$
  then B is a best rank(B)=k approx. to A

What do we mean by "best":

- B is a solution to $\min_B \|A-B\|_F$ where $\mathrm{rank}(B)=k$



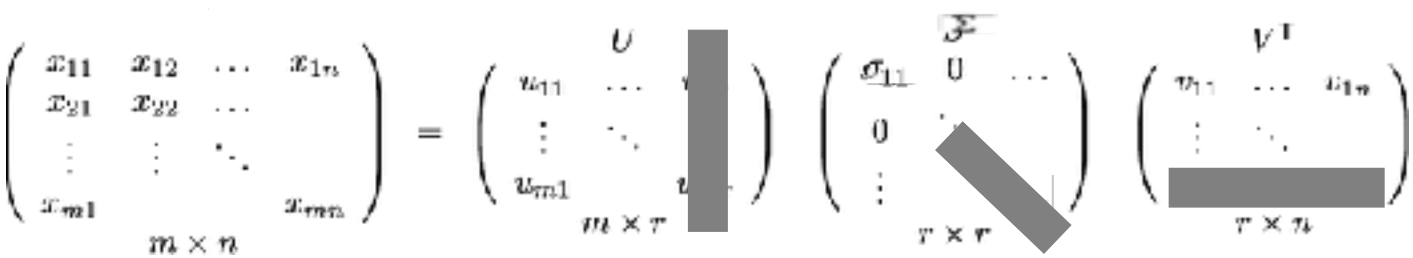
$$\|A-B\|_F = \sqrt{\sum_{ij}\left(A_{ij}-B_{ij}\right)^2}$$

SVD – Best Low Rank Approx.

Details!

- **Theorem:** Let $\mathbf{A} = \mathbf{U}\,\Sigma\,\mathbf{V}^T$ ($\sigma_1 \geq \sigma_2 \geq \ldots$, rank($\boldsymbol{A}$)=$\boldsymbol{r}$)
  then $\mathbf{B} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$
  - **S = diagonal rxr matrix** where $s_i = \sigma_i$ ($i=1\ldots k$) else $s_i = 0$

  is a best rank-$\boldsymbol{k}$ approximation to $\boldsymbol{A}$:
  - **B is a solution to $\min_B \|A-B\|_F$ where rank(B)=k**



- **We will need 2 facts:**
  - $\|M\|_F = \sum_i (q_{ii})^2$ where $\boldsymbol{M} = \boldsymbol{P}\,\boldsymbol{Q}\,\boldsymbol{R}$ is SVD of $\boldsymbol{M}$
  - $\mathbf{U}\,\Sigma\,\mathbf{V}^T - \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T = \mathbf{U}\,(\Sigma - \mathbf{S})\,\mathbf{V}^T$

SVD - Interpretation #2

Equivalent:

'spectral decomposition' of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \times \begin{bmatrix} \sigma_1 & \emptyset \\ \emptyset & \sigma_2 \end{bmatrix} \times \begin{bmatrix} — & v_1 & — \\ — & v_2 & — \end{bmatrix}$$

SVD - Interpretation #2

Equivalent:

'spectral decomposition' of the matrix

$$\begin{array}{cc} & m \\ n & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \end{array} = \overbrace{\sigma_1 \quad u_1 \quad v^T_1 \quad + \quad \sigma_2 \quad u_2 \quad v^T_2 \quad + \dots}^{k \text{ terms}}$$

$n \times 1$ $\quad$ $1 \times m$

Assume: $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq 0$

Why is setting small σ_i to 0 the right thing to do?

Vectors $\mathbf{u_i}$ and $\mathbf{v_i}$ are unit length, so σ_i scales them.

So, zeroing small σ_i introduces less error.

SVD - Interpretation #2

- **Q: How many σs to keep?**
  A: Rule-of-a thumb:
  keep 80-90% of 'energy' $= \sum_i \sigma_i^2$

$$
\begin{array}{c c}
 & \xleftarrow{\quad m \quad} \\
n \updownarrow & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}
\end{array}
= \sigma_1 \quad u_1 \quad v^T_1 \quad + \quad \sigma_2 \quad u_2 \quad v^T_2 \quad + \ldots
$$

Assume: $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots$

SVD - Complexity

- To compute SVD:
  - $O(nm^2)$ or $O(n^2m)$ (whichever is less)
- But:
  - Less work, if we just want singular values
  - or if we want first k singular vectors
  - or if the matrix is sparse
- Implemented in linear algebra packages like
  - Python, LINPACK, Matlab, SPlus, Mathematica ...

SVD - Conclusions so far

- SVD: A= U Σ V^T: unique
  - U: user-to-concept similarities
  - V: movie-to-concept similarities
  - Σ : strength of each concept
- Dimensionality reduction:
  - keep the few largest singular values (80-90% of 'energy')
  - SVD: picks up linear correlations

Examples of SVD

Case study 1: Recommend Movies

- **Q: Find users that like ‘Matrix’**
- **A: Map query into a ‘concept space’ – how?**

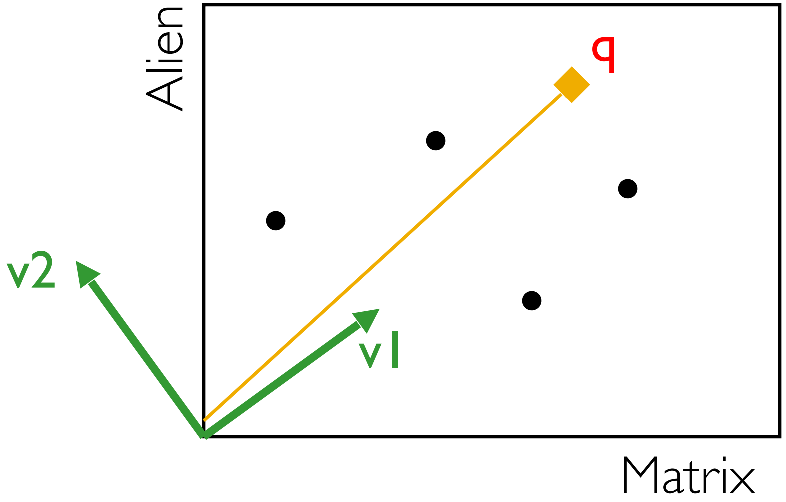
$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \\ \\ \text{SciFi} \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

Case study 1: Recommend Movies

- **Q: Find users that like 'Matrix'**
- **A: Map query into a 'concept space' – how?**

$$
\begin{array}{cccccc}
 & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
q = & \multicolumn{5}{c}{\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \end{bmatrix}}
\end{array}
$$

Project into concept space:
Inner product with each 'concept' vector $\mathbf{v_i}$



Case study 1: Recommend Movies

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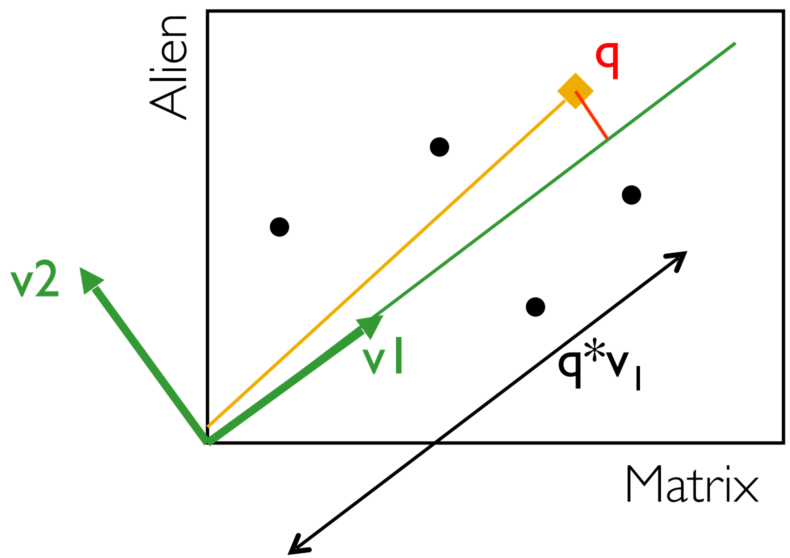
Project into concept space:
Inner product with each 'concept' vector $\mathbf{v_i}$



Case study: How to query?

Compactly, we have:

$$q_{concept} = q\ V$$

E.g.:

$$q = \begin{array}{c} \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\ \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \times \begin{bmatrix} 0.56 & 0.12 \\ 0.59 & -0.02 \\ 0.56 & 0.12 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix} = \begin{array}{c} \text{SciFi-concept} \\ \begin{bmatrix} 2.8 & 0.6 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 2.8 & 0.6 \end{bmatrix} \mathbf{V.T} = \begin{bmatrix} \mathbf{1.64} & \mathbf{1.64} & \mathbf{1.64} & -0.162 & -0.162 \end{bmatrix}$$

Case study: How to query?

- **How would the user d that rated ('Alien', 'Serenity') be handled?**

  $\mathbf{d_{concept} = d\ V}$

E.g.:

$$
d = \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\ \end{array}
$$

$$
d = \begin{bmatrix} 0 & 4 & 5 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.12 \\ 0.59 & -0.02 \\ 0.56 & 0.12 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix} = \begin{array}{cc} \text{SciFi-concept} & \\ \end{array}
$$

$$
= \begin{bmatrix} 5.2 & 0.4 \end{bmatrix}
$$

movie-to-concept similarities (V)

Case study: How to query?

- **Observation:** **User d that rated ('Alien', 'Serenity') will be similar to user q that rated ('Matrix'), although d and q have zero ratings in common!**

$$d = \begin{array}{c}\begin{array}{ccccc}\text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie}\end{array}\\ \begin{bmatrix}0 & 4 & 5 & 0 & 0\end{bmatrix}\end{array} \dashrightarrow \begin{array}{cc}\text{SciFi-concept} & \\ \downarrow & \\ \end{array}\begin{bmatrix}5.2 & 0.4\end{bmatrix}$$

$$q = \begin{bmatrix}5 & 0 & 0 & 0 & 0\end{bmatrix} \dashrightarrow \begin{bmatrix}2.8 & 0.6\end{bmatrix}$$

Zero ratings in common

Similarity ≠ 0

Case study 2: LSI

Latent Semantic Indexing (LSI) is a method for discovering hidden concepts in document data.

Case study 2: LSI

Suppose we have the following set of five documents

d1 : Romeo and Juliet.
d2 : Juliet: O happy dagger!
d3 : Romeo died by dagger.
d4 : "Live free or die", that's the New-Hampshire's motto.
d5 : Did you know, New-Hampshire is in New-England.

a search query: dies, dagger. ==> d3 > d2, d4.

Q. How about d1 and d5? Should they be returned as possibly interesting results to this query?

Case study 2: LSI

Term-Document Matrix

	d1	d2	d3	d4	d5
Romeo	1	0	1	0	0
Juliet	1	1	0	0	0
Happy	0	1	0	0	0
Dagger	0	1	1	0	0
Live	0	0	0	1	0
Die	0	0	1	1	0
Free	0	0	0	1	0
New Hampshire	0	0	0	1	1

Case study 2: LSI

$$\Sigma = \begin{bmatrix} 2.285 & 0 & 0 & 0 & 0 \\ 0 & 2.010 & 0 & 0 & 0 \\ 0 & 0 & 1.361 & 0 & 0 \\ 0 & 0 & 0 & 1.118 & 0 \\ 0 & 0 & 0 & 0 & 0.797 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -0.396 & 0.280 \\ -0.314 & 0.450 \\ -0.178 & 0.269 \\ -0.438 & 0.369 \\ -0.264 & -0.346 \\ -0.524 & -0.246 \\ -0.264 & -0.346 \\ -0.326 & -0.460 \end{bmatrix}}_{U_2} \underbrace{\begin{bmatrix} 2.285 & 0 \\ 0 & 2.010 \end{bmatrix}}_{\Sigma_2} \underbrace{\begin{bmatrix} -0.311 & -0.407 & -0.594 & -0.603 & -0.143 \\ 0.363 & 0.541 & 0.200 & -0.695 & -0.229 \end{bmatrix}}_{V_2^T}$$

Case study 2: LSI

$$U_2 \Sigma_2$$

$$romeo = \begin{bmatrix} -0.905 \\ 0.563 \end{bmatrix},\ juliet = \begin{bmatrix} -0.717 \\ 0.905 \end{bmatrix},\ happy = \begin{bmatrix} -0.407 \\ 0.541 \end{bmatrix},\ dagger = \begin{bmatrix} -1.001 \\ 0.742 \end{bmatrix},$$

$$live = \begin{bmatrix} -0.603 \\ -0.695 \end{bmatrix},\ die = \begin{bmatrix} -1.197 \\ -0.494 \end{bmatrix},\ free = \begin{bmatrix} -0.603 \\ -0.695 \end{bmatrix},\ new\text{-}hampshire = \begin{bmatrix} -0.745 \\ -0.925 \end{bmatrix},$$

$$\Sigma_2 V_2^T$$

$$d_1 = \begin{bmatrix} -0.711 \\ 0.730 \end{bmatrix},\ d_2 = \begin{bmatrix} -0.930 \\ 1.087 \end{bmatrix},\ d_3 = \begin{bmatrix} -1.357 \\ 0.402 \end{bmatrix},\ d_4 = \begin{bmatrix} -1.378 \\ -1.397 \end{bmatrix},\ d_5 = \begin{bmatrix} -0.327 \\ -0.460 \end{bmatrix}$$

Case study 2: LSI

The query is represented by a vector computed as the centroid of the vectors for its terms

$$q = \frac{\begin{bmatrix} -1.197 \\ -0.494 \end{bmatrix} + \begin{bmatrix} -1.001 \\ 0.742 \end{bmatrix}}{2} = \begin{bmatrix} -1.099 \\ 0.124 \end{bmatrix}$$

Cosine distance:

$$\frac{d_i \cdot q}{||d_i||_2 ||q||_2}$$

Case study 2: LSI