20A: PRACTICE FINAL #1

(1) [3 points] Find the equation of the line tangent to $x\cos(y) + 2x^2 + 2xy = 3$ at (1,0).
Implicit differentiation

(2) [4 points] Calculate the following limits or state that they do not exist (DNE).

$$(a)\ \lim_{x\to-\infty} xe^x \qquad (b)\ \lim_{x\to\infty} x^{1/x}$$

L'Hopital's rule — ln properties

(3) [6 points] Consider the function $f(x) = 2^x$ on $[0,6]$.

(a) Compute the right endpoint approximation R_3 to $\int_0^6 f(x)dx$. $\rightarrow R_N = \frac{6}{N}\sum_{j=1}^{N} f\left(j\frac{6}{N}\right)$

(b) Write down the most accurate phrase in your blue book:

"The correct answer to Part (a) is **less than** $\int_0^6 f(x)dx$."
"The correct answer to Part (a) is **greater than** $\int_0^6 f(x)dx$."
"The correct answer to Part (a) is **equal to** $\int_0^6 f(x)dx$."

~~(4) [6 points] A farmer has 24 feet of fencing and wishes to fence off a rectangular field that boarders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?~~

NO OPTIMIZATION

at $x=64$

(5) [6 points] (a) Use linear approximation and the fact that $\sqrt{64} = 8$ to estimate $\sqrt{65}$.

(b) Write down the most accurate phrase in your blue book:

"The correct answer to Part (a) is **less than** $\sqrt{65}$."
"The correct answer to Part (a) is **greater than** $\sqrt{65}$."
"The correct answer to Part (a) is **equal to** $\sqrt{65}$."

(6) [6 points] Calculate the following integrals.

$$(a)\ \int \frac{x^{2/3} + x^{1/2}}{x^{3/2}}dx \qquad (b)\ \int \frac{1}{x^2} + \sec(2x)\tan(2x)dx \qquad (c)\ \int_0^{\pi/2} \sin(2x)dx$$

↳ power rule (algebra first!) — ↳ build your own antiderivative... — ↳ FTC I

(7) [6 points] Calculate the derivatives of the following functions.

$$(a)\ f(x) = \sin(x)^{\cos(x)} \qquad (b)\ g(x) = \cos(\ln(x^2+1)) \qquad (c)\ h(x) = \int_1^{2/x} \tan(t^2)dt$$

properties of ln — chain rule — FTC II + chain rule

(8) [10 points] Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3$.

(a) Find the x-coordinates of all critical points of $f(x)$.

(b) Find the intervals of increase and decrease of $f(x)$.

(c) Classify all critical points of $f(x)$ as local maxima, local minima, or neither.

(d) Find the x-coordinates of all points of inflection of $f(x)$.

(e) Find the intervals of concavity of $f(x)$.

20A PRACTICE FINAL #1 SOL'N

(1) Find equ'n ~~of~~ tan line to $x\cos(y) + 2x^2 + 2xy = 3$ at the point $(1,0)$.

Recall the tangent line is the line through $(1,0)$ with slope $\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=0}}$, i.e. $y = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=0}}(x-1) + 0$

So we must find $\frac{dy}{dx}$; apply implicit differentiation:

$$\frac{d}{dx}\left(x\cos(y) + 2x^2 + 2xy\right) = \frac{d}{dx}(3)$$

$$\Rightarrow \frac{d}{dx}(x\cos(y)) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(2xy) = 0$$

$$\Rightarrow (1)\cos(y) + x\frac{d}{dx}(\cos(y)) + 4x + \left(2y + 2x\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \cos(y) - x\sin(y)\frac{dy}{dx} + 4x + 2y + 2x\frac{dy}{dx} = 0$$

$$\Rightarrow \cos(y) + 4x + 2y = x\sin(y)\frac{dy}{dx} - 2x\frac{dy}{dx}$$

$$\Rightarrow \frac{\cos(y) + 4x + 2y}{x\sin(y) - 2x} = \frac{dy}{dx}$$

then we evaluate at $(1,0)$:

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=0}} = \frac{\cos(0) + 4(1) + 2(0)}{(1)\sin(0) - 2(1)} = \frac{1+4+0}{-2} = \frac{-5}{2}$$

hence; the tangent line is given by:

$$\boxed{y = -\frac{5}{2}(x-1)}$$

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(2) Calculate the limits

(a) $\lim_{x\to-\infty} xe^{x} = \lim_{x\to-\infty} \dfrac{x}{e^{-x}}$ $\dfrac{\rightsquigarrow \infty}{\rightsquigarrow \infty}$

algebra

approaches indeterminate $\frac{\infty}{\infty}$ so apply L'Hôpital's rule.

$$\left[= \lim_{x\to-\infty} \frac{1}{-e^{-x}} \right]$$

$$= \lim_{x\to-\infty} -e^{x} = \boxed{0}$$

facts about exponential functions (recall limits @ infinity chapter!)

(b) $\lim_{x\to\infty} x^{1/x}$:

Set $L = \lim_{x\to\infty} x^{1/x}$

and apply ln to both sides; by continuity we get:

$$\ln(L) = \lim_{x\to\infty} \ln(x^{1/x})$$

property of ln

$$= \lim_{x\to\infty} \frac{1}{x}\ln(x)$$

L'Hopital (approaches $\frac{\infty}{\infty}$)

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\Rightarrow \ln(L) = 0$$

$$\Rightarrow e^{\ln(L)} = e^0$$

$$\Rightarrow \boxed{L = 1}$$

(3) Consider $f(x) = 2^x$ on $[0,6]$.

(a) find R_3 for an approx. of $\int_0^6 f(x)\,dx$

$$R_N = \frac{(b-a)}{N} \sum_{j=1}^{N} f\left(a + \frac{(b-a)}{N} j\right)$$

$$\Rightarrow R_3 = \frac{6}{3} \sum_{j=1}^{3} f\left(0 + \frac{6}{3} j\right)$$

$$= 2(f(2) + f(4) + f(6))$$

$$= \boxed{2(2^2 + 2^4 + 2^6)}$$

(good enough, just make it clear.)

$$= 2(4 + 16 + 64) = 2 \cdot 84$$

$$= \boxed{168}$$

(b) $\boxed{R_3 > \int_0^6 f(x)\,dx}$

because f is increasing on $[0,6]$.

(can check $f'(x)$ to verify if not clear!)

5 (a) use linear approx and $\sqrt{64}=8$ to estimate $\sqrt{65}$:

We want to use $f(x)=\sqrt{x}$; here $f(64)=\sqrt{64}=8$ and we want to estimate $f(65)$. Now: we approx. at $x=64$:

$$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$$

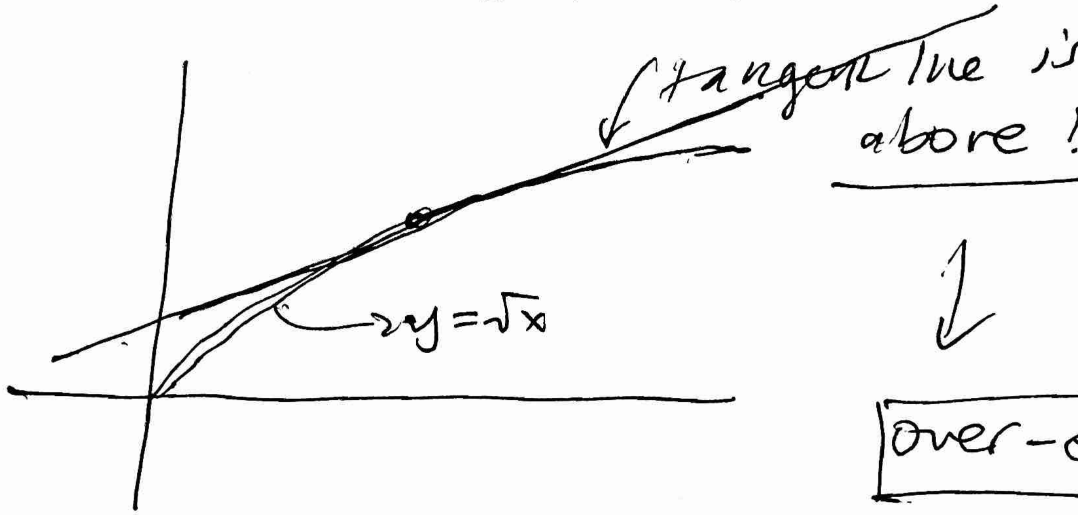
and $f'(64)=\dfrac{1}{2\sqrt{64}}=\dfrac{1}{16}$

and then

$$L(x)=f'(64)(x-64)+f(64)$$
$$=\frac{1}{16}(x-64)+8$$

then: $\sqrt{65}=f(65)\approx L(65)=\frac{1}{16}(65-64)+8$
$$=\boxed{\frac{1}{16}+8}$$

(b) consider the graph of $f(x) = y$



$\downarrow$

$\boxed{\text{over-estimate}}$

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(6) (a) Find the general antiderivative for

$$f(x) = \frac{x^{2/3} + x^{1/2}}{x^{3/2}} :$$

see first that $f(x) = \dfrac{x^{2/3}}{x^{3/2}} + \dfrac{x^{1/2}}{x^{3/2}} = x^{\frac{2}{3} - \frac{3}{2}} + x^{\frac{1}{2} - \frac{3}{2}}$

$$= x^{\frac{4}{6} - \frac{9}{6}} + x^{-1}$$

$$= x^{-\frac{5}{6}} + x^{-1}$$

then $F(x) = \dfrac{x^{\frac{1}{6}}}{\frac{1}{6}} + \ln|x| + C$

$$= \boxed{6x^{1/6} + \ln|x| + C}$$

(b) find the general antiderivative of $f(x) = \frac{1}{x^2} + \sec(2x)\tan(2x)$

First, see that $\frac{d}{dx}\left[\frac{-1}{x}\right] = \frac{1}{x^2}$ by the power rule.

(1)

Furthermore, see that

$$\frac{d}{dx}[\tan(2x)] = 2\sec(2x)\tan(2x)$$

chain rule

Hence: $\frac{1}{2}\frac{d}{dx}[\tan(2x)] = \sec(2x)\tan(2x)$

and thus: $\Rightarrow \frac{d}{dx}\left[\frac{1}{2}\tan(2x)\right] = \sec(2x)\tan(2x)$

(2)

So now

$$\boxed{F(x) = \frac{-1}{x} + \frac{1}{2}\tan(2x) + C}$$

(c) $\int_0^{\pi/2} \sin(2x)\,dx$

See first that $F(x) = \frac{-\cos(2x)}{2}$

is an antiderivative of $\sin(2x)$.

So by <u>FTC I</u>:

$$\int_0^{\pi/2} \sin(2x)\,dx = F\left(\tfrac{\pi}{2}\right) - F(0)$$

$$= \frac{-\cos\left(2\left(\frac{\pi}{2}\right)\right)}{2} + \frac{+\cos(0)}{2}$$

$$= \frac{-\cos(\pi)}{2} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

7 (a) $f(x) = \sin(x)^{\cos(x)}$

$f'(x) = ?$

first, take ln of both sides!

logarithmic differentiation

$\ln(f(x)) = \ln(\sin(x)^{\cos(x)})$

properties of ln

$\ln(f(x)) = \cos(x)\ln(\sin(x))$

$\xrightarrow{\frac{d}{dx}}$ $\frac{f'(x)}{f(x)} = -\sin(x)\ln(\sin(x)) + \cos(x)\frac{d}{dx}(\ln(\sin(x)))$

chain rule!

$= -\sin(x)\ln(\sin(x)) + \frac{\cos(x)}{\sin(x)} \cdot \cos(x)$

$\Rightarrow \frac{f'(x)}{f(x)} = -\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}$

multiply by $f(x)$

$\Rightarrow f'(x) = \left(-\sin(x)\ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)}\right)\underbrace{\sin(x)^{\cos(x)}}_{f(x)}$

$= \boxed{-\sin(x)^{\cos(x)+1}\ln(\sin(x)) + \sin(x)^{\cos(x)-1}\cos^2(x)}$

(b) $g(x) = \cos(\ln(x^2+1))$

$g'(x) = ?$

$$g'(x) = \frac{d}{dx}\left[\cos(\ln(x^2+1))\right]$$

$$= -\sin(\ln(x^2+1))\,\underbrace{\frac{d}{dx}\left[\ln(x^2+1)\right]}$$

chain rule

$$= -\sin(\ln(x^2+1))\overbrace{\left(\frac{1}{x^2+1}\right)\frac{d}{dx}(x^2+1)}$$

$$= -\sin(\ln(x^2+1))\frac{1}{x^2+1}(2x)$$

$$= \boxed{\frac{-2x\sin(\ln(x^2+1))}{x^2+1}}$$

(c) $h(x) = \int_1^{2/x} \tan(t^2)\,dt$

$h'(x) = ?$

Apply FTC II and the chain rule!

$$h'(x) = \frac{d}{dx}\int_1^{2/x} \tan(t^2)\,dt$$

outside by FTC II

chain rule

$$= \tan\left(\left(\frac{2}{x}\right)^2\right)\frac{d}{dx}\left(\frac{2}{x}\right)$$

inside with power rule

chain rule

$$= 2\tan\left(\frac{4}{x^2}\right)\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \boxed{\frac{-2\tan\left(\frac{4}{x^2}\right)}{x^2}}$$

(8) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3$

(a) Critical points:

$f'(x) = x^2 + x - 2$ (clearly defined everywhere)

$= (x-1)(x+2) = 0 \iff \boxed{x = 1, -2}$

(b) Intervals of increase/decrease

check the signs of f':

x	$f'(x)$	sign
-3	$(-)(-)$	$+$
-2	0	CP
0	$(-1)(2)$	$-$
1	0	CP
2	$(+)(+)$	$+$

so f increases on $(-\infty, -2) \cup (1, \infty)$
decreases on $(-2, 1)$

(c) determine local max/min

By the sign table, there is a $\boxed{\text{local max at } x=-2 \\ \text{local min at } x=1}$

(d) inflection points:

$f''(x) = 2x+1 = 0 \iff x = -\frac{1}{2}$

then we check the signs:

x	$f''(x)$	sign
-1	$-2+1=-1$	$-$
$-1/2$	0	PI
0	1	$+$

There is an $\boxed{\text{inflection point at } x=-\frac{1}{2}}$

(from concave ↓ to ↑)

(e) intervals of concavity

by the sign table; $\boxed{\text{concave down on } (-\infty,-\frac{1}{2}) \\ \text{concave up on } (-\frac{1}{2},\infty)}$