Section 3.1 Iterated Partial Derivatives

Recall. In single variable calculus we used the derivative $f'(x)$ to test for *critical points* ($f'(x_0) = 0$) and we checked $f''(x)$ to see if x_0 is a max ($f''(x_0) < 0$) or a min ($f''(x_0) > 0$).

Goal. Extend the methods to real valued functions of several variables ($f : \mathbb{R}^m \to \mathbb{R}$). In order to achieve this goal, we have to develop higher order derivatives and derive tests for maxima, minima, and saddle points.]

Iterated Partial Derivatives

Let $f : \mathbb{R}^3 \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$. We call such a function a C^1 function.

If each of these partials themselves have continuous partials, we say that f is a C^2.

Notation.

$$
\begin{gathered}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} \\
\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial z\, \partial y} \\
\text{etc.}
\end{gathered}
$$

If f is a function of only x and y and $\partial f/\partial x, \partial f/\partial y$ are continuously differentiable, then we get four second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\, \partial y}, \text{ and } \frac{\partial^2 f}{\partial y\, \partial x}.$$

All of these are called *iterated partial derivatives*, while $\frac{\partial^2 f}{\partial x\, \partial y}$, and $\frac{\partial^2 f}{\partial y\, \partial x}$ are called *mixed partial derivatives.*

Example. Find all the second partial derivatives of the function $f(x, y) = x^2y^3 + e^x$.
Solution. Since

$$\frac{\partial f}{\partial x} = 2xy^3 + e^x \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2,$$

we obtain

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(2xy^3 + e^x) = 2y^3 + e^x \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(3x^2y^2) = 6x^2y \\
\frac{\partial^2 f}{\partial x\, \partial y} &= \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2 \\
\frac{\partial^2 f}{\partial y\, \partial x} &= \frac{\partial}{\partial y}(2xy^3 + e^x) = 6xy^2.
\end{aligned}
$$

Example. Let $f(x, y) = \cos x \sin y$. Find all the second partial derivatives of $f(x, y)$.
Solution. Since

$$\frac{\partial f}{\partial x} = -\sin x \sin y \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos x \cos y,$$

we obtain

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(-\sin x \sin y) = -\cos x \sin y \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(\cos x \cos y) = -\cos x \sin y \\
\frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial x}(\cos x \cos y) = -\sin x \cos y \\
\frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}(-\sin x \sin y) = -\sin x \cos y.
\end{aligned}$$

Remark. In both examples, we had

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Theorem. If $f(x, y)$ is of class C^2, then the mixed partials are equal, i.e.

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Example. Let $w = f(x, y)$ where $x = u + v$ and $y = u - v$. Find $\frac{\partial^2 w}{\partial u\,\partial v}$.
Solution.

$$\begin{aligned}
\frac{\partial^2 w}{\partial u\,\partial v} &= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) \\
&= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\right) \\
&= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y}(-1)\right) \quad \text{(by the Chain Rule).}
\end{aligned}$$

Let $g(x, y) = \frac{\partial w}{\partial x}$ and $h(x, y) = \frac{\partial w}{\partial y}$. Then

$$\begin{aligned}
\frac{\partial^2 w}{\partial u\,\partial v} &= \frac{\partial}{\partial u}\left(g(x, y) - h(x, y)\right) \\
&= \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u} - \frac{\partial h}{\partial x}\frac{\partial x}{\partial u} - \frac{\partial h}{\partial y}\frac{\partial y}{\partial u} \\
&= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} - \frac{\partial h}{\partial y} \\
&= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x\,\partial y} - \frac{\partial^2 w}{\partial y\,\partial x} - \frac{\partial^2 w}{\partial y^2} \\
&= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$