Leture 1: Functions and Limits.

Calculus = study of functions of a real variable.

* What is a function?

A function is a rule (machine) that takes a real number $x \in \mathbb{R}$ as an input and produces a real number $f(x)$ as an output.

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x) \in \mathbb{R}$$

- Examples:

1) $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$.

A function that assigns to every real number its squares.

2) $f(x) = 4$ (constant)
3) $g(x) = 3x + 5$ (Linear: $g(x) = mx + b$)
4) $h(x) = 2x^2 + x + 2$ (quadratic: $h(x) = ax^2 + bx + c$).
5) $f(x) = \sin(x)$ (Trigonometric).
6) $f(x) = e^x$ (Exponential)

- Remark: 1) The set of numbers for which a function is defined is called its domain.

2) The set of all possible number $f(x)$ as x runs over the domain is called the range of the function.

- Example:

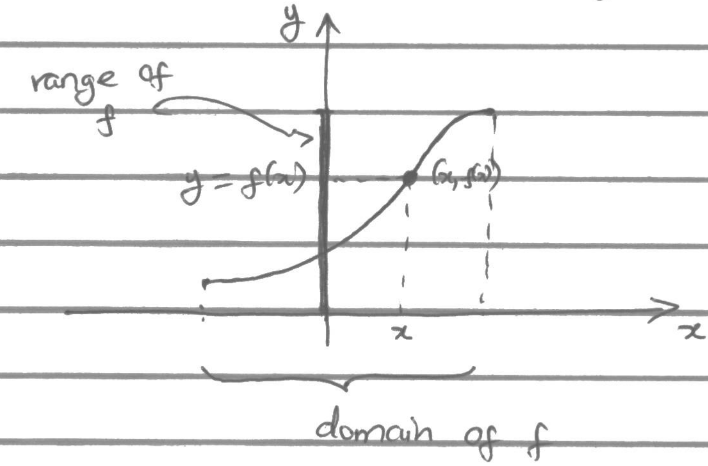
$f(x) = \sqrt{x}$

Domain: $x \geq 0$ and range: $[0, \infty)$.
$[0, +\infty)$

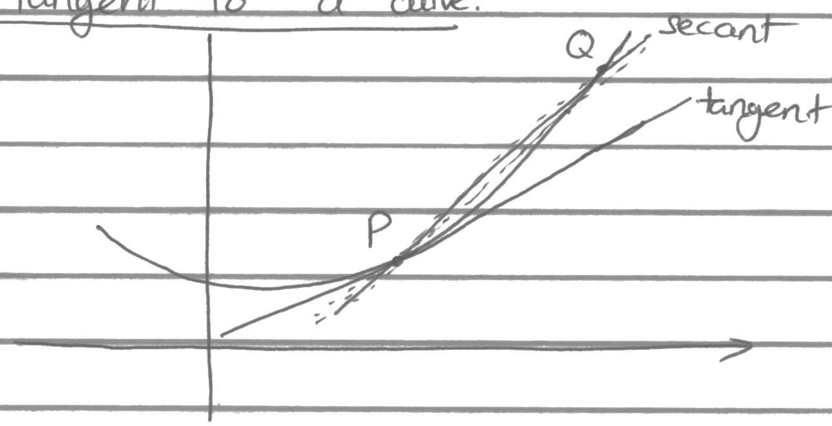
* Graph of a function:

We can get the graph of a function f by drawing all points whose coordinates are (x, y). Here $y = f(x)$.

E.g.

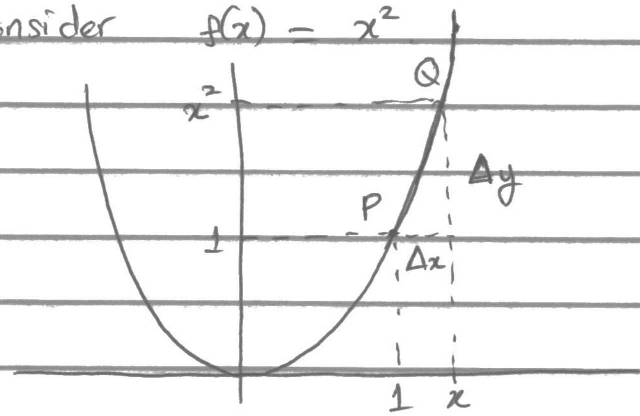


* The tangent to a curve:



. Example: Tangent to a parabola.

Consider $f(x) = x^2$



Recall that any line through $(1, 1)$ has equation

$y - 1 = m(x - 1)$, where m is the slope of the line.

Let Q be the other point on the parabola with coordinate (x, x^2). We want to find out how the line through P and Q changes if x is changed.
The slope of this line is

$$\frac{\text{rise}}{\text{run}} \qquad m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

As x gets closer to 1, m_{PQ} gets closer to the value $1+1=2$.
$\Rightarrow$ The tangent line to the parabola $y=x^2$ at $(1,1)$ has equation

$$y-1 = 2(x-1), \text{ i.e. } y = 2x-1.$$

* Instantaneous velocity.

$$\text{Average velocity} = \frac{\text{distance traveled}}{\text{time it took}}$$

When you are driving in your car, the speedometer tells you how fast your are going, i.e. what your velocity is. This ~~ve~~ is the velocity at the moment that you look at your speedometer.

Let t be the time (in hours) and let $s(t)$ be the distance we have covered.
Consider the average velocity during some short time interval beginning at time t. Denote Δt for the length of the time interval.

$\Rightarrow$ average velocity $= \dfrac{s(t+\Delta t) - s(t)}{\Delta t}$ miles per hour

$\Rightarrow$ The shorter you make the time, the closer this number should be to the instantaneous velocity at time t.

• Def: The velocity at time t.

$$\boxed{v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}}$$

* Rate of change:

Given a function $y=f(x)$, we want to know how much $f(x)$ changes if x changes. That is, if you change x to $x+\Delta x$, then y will change from $f(x)$ to $f(x+\Delta x)$.

$\Rightarrow$ Average rate of change $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

$\Rightarrow$ The rate of change of the function f at x:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Lecture 2: Limits.

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a.$$

ie, the limit of f as x approaches a is L.

That is, if we choose values of x close but not equal to a, $f(x)$ will be close to the value L.

E.g. 1) If $f(x) = x + 3$, then

$$\lim_{x \to 4} f(x) = 7.$$

2) Substituting numbers to guess a limit.

What is $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4}$?

$$f(x) = \frac{x^2 - 2x}{x^2 - 4}.$$

First, try to substitute $x = 2$:

$$\frac{2^2 - 2(2)}{2^2 - 4} = \frac{0}{0}.$$

which does not exist!

Now, try to substitute values of x close but not equal to 2.

x	$f(x)$
3.000	0.6
2.5	0.555556
2.01	0.501247
2.001	0.500125

E.g. Substituting number can suggest the wrong answer. Our first definition of limit is not very precise, because it says "x close to a," but how close is enough? Let's consider

$$g(x) = \frac{101000x}{100000x + 1}$$

What's $\lim_{x \to 0} g(x)$?

Substitution of some "small values of x" could lead us to believe that the limit is 1, Only when we substitute even smaller values, we find that limit is 0!

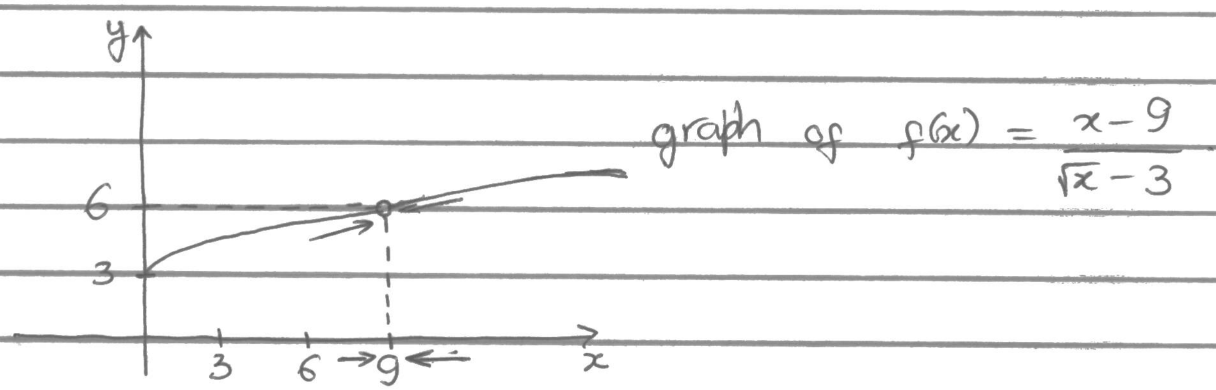
* Def: We say that L is the limit of $f(x)$ as $x \to a$, if

1) $f(x)$ need not be defined at $x = a$, but it must be defined for all x in some interval which contains a.

2) For every $\varepsilon > 0$, one can find a $\delta > 0$ such that for all x in the domain of f one has

$$|x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

E.g. Investigate $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$ graphically?



* Left and right limits:

- The value which $f(x)$ approaches as x approaches a through ~~for~~ values larger than a is defined by $\lim_{x \to a^+} f(x)$. (right limits)

- Similarly, the value which $f(x)$ approaches as x approaches a through values smaller than a is defined by $\lim_{x \to a^-} f(x)$. (left limits)

Thm: If both one-sided limits

$$\lim_{x \to a^+} f(x) = L_+ \quad \text{and} \quad \lim_{x \to a^-} f(x) = L_-$$

exist, then

$$\cancel{\text{limt}} \ \lim_{x \to a} f(x) \text{ exists} \iff L_+ = L_-$$

* Limits at infinity: What happens to $f(x)$ as x becomes larger and larger?

If there is a number L such that $f(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$$\lim_{x \to \infty} f(x) = L.$$

E.g. $\lim_{x \to +\infty} \frac{1}{x} = 0$.

Similarly, we write $\lim_{x \to -\infty} f(x) = M$ as x is negatively sufficient large.

Lecture 3: Properties of the limits and continuity.

* Limits of sums, products and quotients.

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ exist.

Then: 1) $\lim_{x\to a} (f(x) \pm g(x)) = L \pm M.$

2) $\lim_{x\to a} (f(x)\,g(x)) = LM.$

If $\lim_{x\to a} g(x) \neq 0$,

3) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}.$

4) $\lim_{x\to a} [f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n$

~~and~~

E.g. 1) $\lim_{x\to 2} \dfrac{x^3-1}{x^2-1} \overset{\text{why?}}{=} \dfrac{\lim_{x\to 2}(x^3-1)}{\lim_{x\to 2}(x^2-1)} = \dfrac{2^3-1}{2^2-1} = \dfrac{7}{3}.$

2) Compute $\lim_{x\to 2} \dfrac{x^2-2x}{x^2-4} = \lim_{x\to 2} \dfrac{x(x-2)}{(x-2)(x+2)}$

$= \lim_{x\to 2} \dfrac{x}{x+2}$

$= \dfrac{2}{4}$

$= \dfrac{1}{2}.$

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1) $f(a)$ defined
2) $\lim_{x\to a} f(x)$ exists
3) they are equal.

* **Definition:** A function f is continuous at a if
$$\lim_{x\to a} f(x) = f(a).$$

A ~~f~~ function is continuous if it is continuous at every ~~a~~ point in its domain.

- Polynomials are continuous.

E.g. Show that $P(x) = x^3 + 3x$ is continuous at, says, $x = 2$:

Need to show: $\lim_{x\to 2} P(x) = P(2)$.

$$P(2) = 2^3 + 3(2) = \lim_{x\to 2} P(x)$$

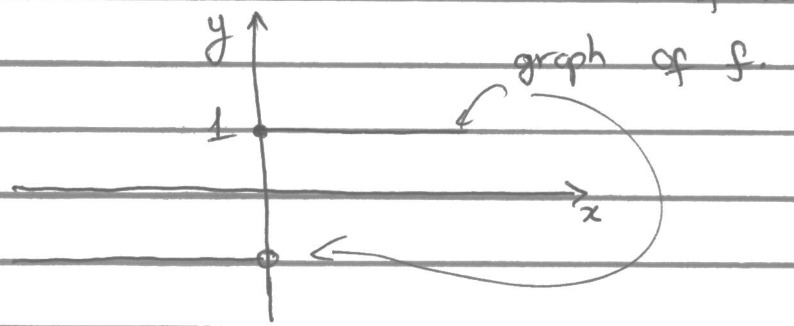
- Rational functions are continuous.

Proof: Consider a rational function $R(x) = \frac{P(x)}{Q(x)}$. Let a be any number in the domain of R, $Q(a) \neq 0$.

Then
$$\lim_{x\to a} R(x) = \lim_{x\to a} \frac{P(x)}{Q(x)}$$
$$= \frac{\lim_{x\to a} P(x)}{\lim_{x\to a} Q(x)}$$
$$= \frac{P(a)}{Q(a)} \quad \text{P and Q are continuous.}$$
$$= R(a).$$

$\Rightarrow$ R is continuous at a.

E.g. 1) $f(x) = \text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$.



f is discontinuous at 0.

$\lim\limits_{x \to 0^+} f(x) = 1$, $\lim\limits_{x \to 0^-} f(x) = -1$.

$\Rightarrow$ $\lim\limits_{x \to 0} f(x)$ does not exist since $\lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0^-} f(x)$

Other important basic continuous functions:

* Roots: $f(x) = \sqrt[n]{x} = x^{1/n}$ is continuous on $\underbrace{\text{the domain of } f}_{\text{(careful for even } n)}$ for any $n \in \mathbb{N}$.

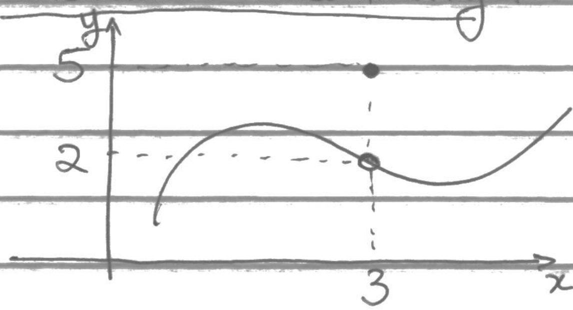
* Sine and cosine are continuous on all of $\mathbb{R}$.

* Exponentials: $f(x) = b^x$ continuous at all points of $\mathbb{R}$ for $b > 0$.

* Logarithms: $f(x) = \log_b x$ is continuous on $(0, \infty)$ for $b > 0$.

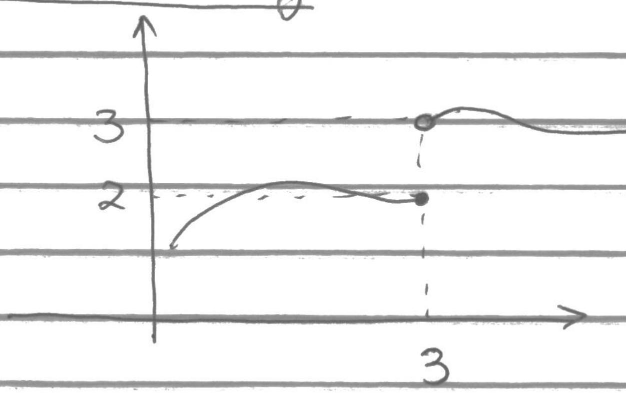
* Examples of discontinuities:

1) Removable discontinuity



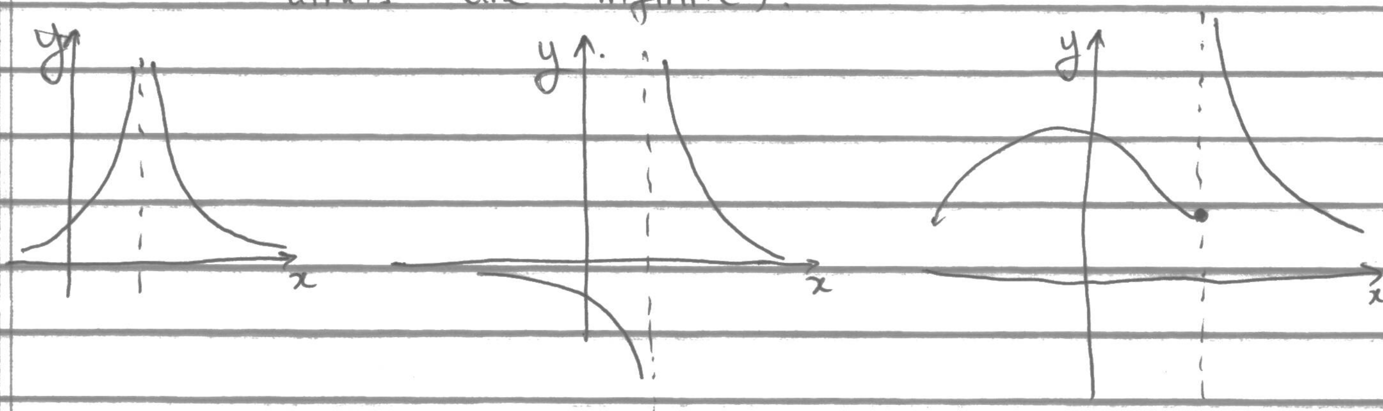
$$\lim_{x\to 3} f(x) = 2 \neq f(3) = 5.$$

2) Jump discontinuity: (one-sided limits exist but not equal).



$$\lim_{x\to 3^-} f(x) = 2 \neq \lim_{x\to 3^+} f(x) = 3.$$

3) Infinite discontinuity: (one or both of the one-sided limits are infinite).



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* Evaluating limits using continuity.

$$\lim_{x \to \frac{\pi}{2}} \cos(x) = ?$$

$$\lim_{x \to -1} \frac{3^x}{\sqrt{x+5}} = ?$$

Lecture 4: Evaluating limits (Sections: ~~2.5~~, 2.7)

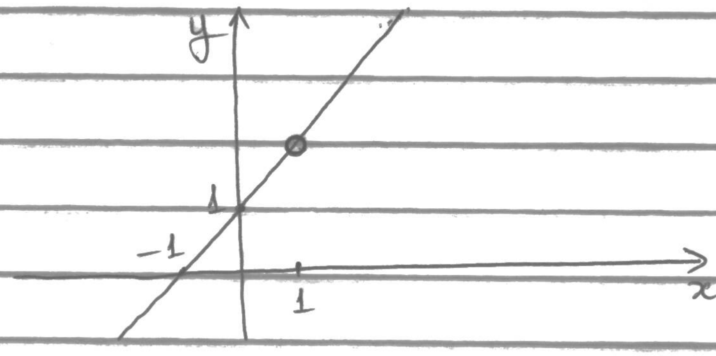
Recall that if ~~we know that~~ a function is continuous at c, to find $\lim_{x \to c} f(x)$, we can use substitution.

E.g. $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$ ← rational function

2 in domain.

But there are many limits $\lim_{x \to c} f(x)$ where $f(c)$ is not defined $\Rightarrow$ cannot use substitution.

E.g. $f(x) = \frac{x^2-1}{x-1}$. Domain: $x \neq 1$.



$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2$$

* Indeterminate form:

$f(x)$ has an indeterminate form at $x=2$ if $f(c) = \frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0,$ or $\infty - \infty$.

$\Rightarrow$ Strategy: transform it algebraically to functions which has the same limit.

E.g. 1) $\lim_{x\to -3} \frac{x^2+2x-3}{x^2+5x+6}$ $\qquad \frac{0}{0}$

$= \lim_{x\to -3} \frac{(x-1)\cancel{(x+3)}}{\cancel{(x+3)}(x+2)}$

$= \lim_{x\to -3} \frac{x-1}{x+2}$

$= \frac{-3-1}{-3+2}$

$= 4.$

2) Multiplying by the conjugate:

Evaluate $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $\qquad \frac{0}{0}$

$= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$ $\longrightarrow$ conjugate of $\sqrt{x}-2$.

$= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x\to 4} \frac{1}{\sqrt{x}+2}$

$= \frac{1}{\sqrt{4}+2}$

$= \frac{1}{4}$

Exer: Evaluate $\lim_{h\to 3} \frac{h-3}{\sqrt{h+1}-2}$

3) The form $\infty - \infty$.

Evaluate $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$

$$= \lim_{x \to 1} \frac{x^2 - 1 - 2(x-1)}{(x-1)(x^2-1)}$$

$$= \lim_{x \to 1} \frac{x^2 - 2x + 1}{(x-1)^2(x+1)}$$

$$= \lim_{x \to 1} \frac{(x-1)^2}{(x-1)^2(x+1)}$$

$$= \lim_{x \to 1} \frac{1}{x+1}$$

$$= \frac{1}{2}$$

4) Undefined limit.

$\lim_{x \to 3} \frac{x^2 + x + 6}{x - 3}$ $\qquad \frac{18}{0} \rightsquigarrow$ not indeterminate.

The limit does not limit (and the graph has asymptote).

* <u>Limits at infinity</u>:

Since ∞ and $-\infty$ are not in $\mathbb{R}$, they cannot be in the domain of f a function.

$\Rightarrow$ $f(\infty)$ is undefined.

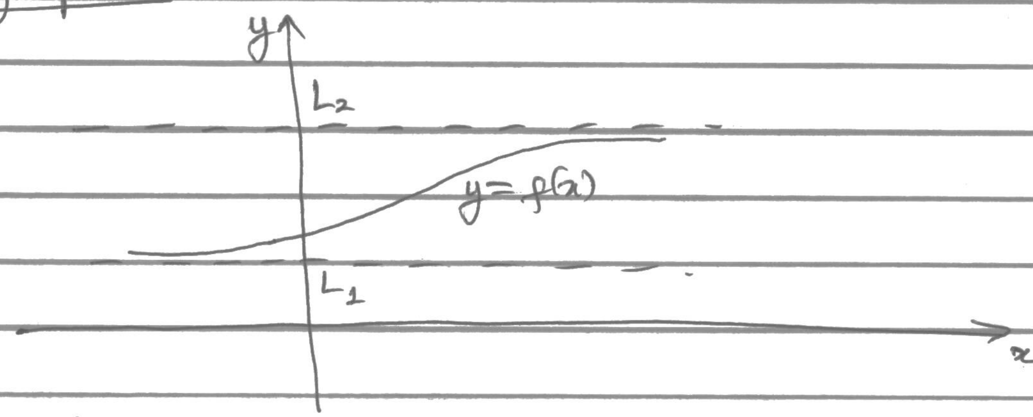
However, we may consider the limits of $f(x)$ as x approaches ∞ or $-\infty$:

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

can be read as : " the value $f(x) \in \mathbb{R}$, approaches $L \in \mathbb{R}$ as the value x increases without bound "

→ The limits at infinity describe horizontal asymptotes

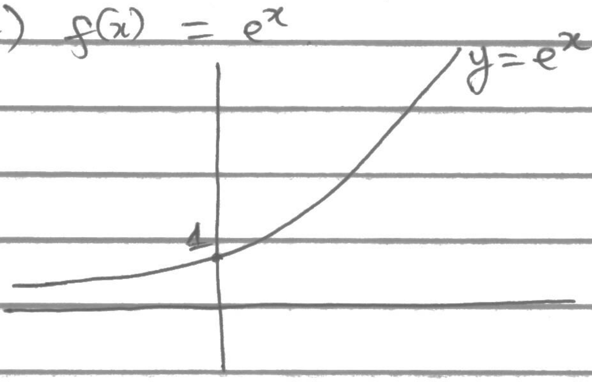


$$\lim_{x \to +\infty} f(x) = L_2$$

$$\lim_{x \to -\infty} f(x) = L_1$$

Important examples:

1) $f(x) = e^x$



$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^{+x} = 0$$

2) $f(x) = x^n$

- $n \in (0,\infty)$: $\lim_{x\to+\infty} x^n = +\infty$ $\qquad \lim_{x\to+\infty} \frac{1}{x^n} = 0$

- $n \in \mathbb{N}$:

$$\lim_{n\to-\infty} x^n = \begin{cases} +\infty & \text{if } n \text{ even} \\ -\infty & \text{if } n \text{ odd.} \end{cases}$$

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Thm: $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0}$ is a rational function.

$$\Rightarrow \lim_{x\to\pm\infty} f(x) = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}.$$

Examples:

1) $$\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9} = \frac{3}{4}\lim_{x\to\infty} x^0 = \frac{3}{4}$$

2) $$\lim_{x\to\infty} \frac{\sqrt{36x^4+7}}{9x^2+4} \underset{\text{algebra}}{=} \lim_{x\to\infty} \sqrt{\frac{36x^4+7}{(9x^2+4)^2}}$$

$$\underset{\text{limit laws}}{=} \sqrt{\lim_{x\to\infty} \frac{36x^4+7}{81x^4+72x^2+8}}$$

$$= \sqrt{\frac{36}{81} \lim_{x\to\infty} x^{4-4}}$$

$$= \sqrt{\frac{36}{81}}$$

$$= \frac{6}{9} = \frac{2}{3}.$$

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3) $\lim\limits_{t \to -\infty} \dfrac{4 + 6e^{2t}}{5 - 9e^{3t}} = \dfrac{\lim\limits_{t \to -\infty}(4 + 6e^{2t})}{\lim\limits_{t \to -\infty}(5 - 9e^{3t})}$

$= \dfrac{4 + 6(0)}{5 - 9(0)}$

$= \dfrac{4}{5}$