Lecture 6: Solving $Ax = b$ and linear independence (Section 2.2-2.3)

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Column spaces

▶ Definition. The **column space**, $C(A)$, of A is the span of the vector columns of A, i.e., if $A = [\ \boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_n\ ]$, then $C(A) = \text{span}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$.

Facts:

- $\boldsymbol{b}$ is in $C(A)$ if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. (Why?)
- If A is $m \times n$, then $C(A)$ is a subspace of $\mathbb{R}^m$. (Why?).

Column spaces

- Example. Find a matrix A such that $W = C(A)$ where

$$W = \left\{ \begin{bmatrix} 2x - y \\ 3y \\ 7x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

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- Solution.

$$\begin{bmatrix} 2x - y \\ 3y \\ 7x + y \end{bmatrix} = x \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} + y \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

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The matrix A whose $W = C(A)$ is

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 7 & 1 \end{bmatrix}.$$

$C(A)$ and solutions to $A\boldsymbol{x} = \boldsymbol{b}$

Theorem. Let $\boldsymbol{x}_p$ be a solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$. Then every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is a solution to the homogeneous equation $A\boldsymbol{x} = \boldsymbol{0}$.

$C(A)$ and solutions to $A\boldsymbol{x} = \boldsymbol{b}$

- Example. Find a parametric description of the solutions to

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

▶ Solution.

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array}\right] \to \ldots \to \left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 - 3x_2 + x_4 \\ x_2 \\ 1 - x_4 \\ x_4 \end{bmatrix}.$$

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$$= \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{elements of } N(A)}.$$

Linear independence

- span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$ is the set of all linear combinations

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \dots + c_m\boldsymbol{v}_m.$$

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- span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is a vector space.

Linear independence

- Example. Is $\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$ equal to $\mathbb{R}^3$

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- The span is equal to $\mathbb{R}^3$ if and only if the system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 1 & 2 & 1 & b_2 \\ 1 & 3 & 3 & b_3 \end{array} \right]$$

is consistent for all *values* of b_1, b_2, and b_3.

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But row reduction yields

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right].$$

That is, the system is inconsistent if $b_3 - 2b_2 + b_1 \neq 0$. Hence, the span is not equal to $\mathbb{R}^3$.

Linear independence

What went wrong?

Linear independence

What went wrong? The three vectors in the span satisfy

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The three vectors are **linearly dependent.**

Linear independence

- **Definition.** Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are said to be **linearly independent** if the equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m = 0$$

has only the trivial solution, namely, $x_1 = x_2 = \ldots = x_m = 0$.

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has only the trivial solution, namely, $x_1 = x_2 = \ldots = x_m = 0$.
- Definition. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are said to be **linearly dependent** if there exist coefficients $x_1, \ldots, x_m$ not all zero such that

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m = 0.$$

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- Example. The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ are linearly dependent.

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$$3\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\3\end{bmatrix} = 0,$$

or

$$\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix}\begin{bmatrix}3\\-2\\1\end{bmatrix} = \mathbf{0}.$$

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- The columns of A are linearly independent.
- $A\boldsymbol{x} = 0$ has only the solution $\boldsymbol{x} = 0$.
- $N(A) = \{0\}$.
- A has n pivots.

Linear independence

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b) $\left\{\begin{bmatrix}3\\2\\1\end{bmatrix}\right\}$

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b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

Yes! A set of a single nonzero vector $\{\boldsymbol{v}_1\}$ is always linearly independent. (Why?)

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d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

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No! A set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ containing the zero vector is linearly dependent. (Why?)

A basis of a vector space

▶ Definition. A set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ in V is a **basis** of V if

- $V = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$, and
- the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are linearly independent.

In other words, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ in V is a basis of V if and only if every vector $\boldsymbol{w}$ in V can be uniquely expressed as $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_m\boldsymbol{v}_m$.

A basis of a vector space

▶ Example. Let $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\mathbb{R}^3$.

It is called the **standard basis**.

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- Example. $\mathbb{R}^3$ has dimension 3. Likewise, $\mathbb{R}^n$ has dimension n.
- Example. Not all vector spaces have a finite basis. For instance, the vector space of all polynomials has *infinite dimension*.
  Its standard basis is $\{1, x, x^2, x^3, x^4, \ldots\}$

A basis of a vector space

Theorem. Suppose that V has dimension d.

- A set of d vectors in V are a basis if they span V.
- A set of d vectors in V are a basis if they are linearly independent.

▶ Example. Are the following sets a basis for $\mathbb{R}^3$?

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a) $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. No!

b) $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix}\right\}$.

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c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

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No!

▶ Example. Let $\mathscr{P}_2$ be the space of polynomials of degree at most 2.

a) What is the dimension of $\mathscr{P}_2$?

b) Is $\{t, 1-t, 1+t-t^2\}$ a basis of $\mathscr{P}_2$?

Shrinking and expanding sets of vectors

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$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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- Example. Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

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- Example. Consider

$$H = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

  - What is the dimension of this subspace of $\mathbb{R}^3$?

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- Example. Consider

$$H = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

  - What is the dimension of this subspace of $\mathbb{R}^3$?
  - Extend it to a basis of $\mathbb{R}^3$.