HW4.

1) $T(p(t)) = (2+3t)\,p(t)$.

a) For any polynomial $p, g \in \mathbb{P}_3$,

$$T(p(t)+g(t)) = (2+3t)(p(t)+g(t))$$
$$= (2+3t)\,p(t) + (2+3t)\,g(t)$$
$$= T(p(t)) + T(g(t))$$

For any $c \in \mathbb{R}$

$$T(c\,p(t)) = (2+3t)(c\,p(t))$$
$$= c\,(2+3t)\,p(t)$$
$$= c\,T(p(t))$$

Therefore, T is a linear transformation.

b) $\{1, t, t^2, t^3\}$ is standard basis for $\mathbb{P}_3$

$\{1, t, t^2, t^3, t^4\}$ is standard basis for $\mathbb{P}_4$.

$$T(1) = (2+3t)(1) = 2\cdot 1 + 3t + 0\cdot t^2 + 0\cdot t^3 + 0\cdot t^4$$
$$T(t) = (2+3t)(t) = 0\cdot 1 + 2t + 3\cdot t^2 + 0\cdot t^3 + 0\cdot t^4.$$
$$T(t^2) = (2+3t)(t^2) = 0\cdot 1 + 0t + 2\cdot t^2 + 3\cdot t^3 + 0\cdot t^4.$$
$$T(t^3) = (2+3t)(t^3) = 0\cdot 1 + 0\cdot t + 0\cdot t^2 + 2\cdot t^3 + 3\cdot t^4.$$

$$\Rightarrow \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

2) a) $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$ linear.

The representing matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b) $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$ linear.

c) $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ v_1 \end{bmatrix}$ linear.

d) $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ not linear.

Because if we take any vector $\vec{v}$ and $\vec{w} \in \mathbb{R}^2$,

$$T(\vec{v} + \vec{w}) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

But $T(\vec{v}) + T(\vec{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

$$\Rightarrow \quad T(\vec{v} + \vec{w}) \neq T(\vec{v}) + T(\vec{w}).$$

3) Note that I denote "if and only if" by $\Leftrightarrow$.

$\vec{x} - \vec{y}$ is orthogonal to $\vec{x} + \vec{y}$

$\Leftrightarrow \quad (\vec{x} - \vec{y}) \cdot (\vec{x} + \vec{y}) = 0$

$\Leftrightarrow \quad \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{y} - \vec{y} \cdot \vec{x} - \vec{y} \cdot \vec{y} = 0$

$\Leftrightarrow \quad \|\vec{x}\|^2 + \vec{x} \cdot \vec{y} - \vec{x} \cdot \vec{y} - \|\vec{y}\|^2 = 0$

since $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

$\Leftrightarrow \quad \|\vec{x}\|^2 = \|\vec{y}\|^2$

$\Leftrightarrow \quad \|\vec{x}\| = \|\vec{y}\|$.

4) $S = \text{span}\left\{\begin{bmatrix}1\\2\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\3\\2\end{bmatrix}\right\} \Rightarrow S = C\left(\begin{bmatrix}1 & 1\\2 & 3\\2 & 3\\3 & 2\end{bmatrix}\right)$

That is, S is the column space of $A = \begin{bmatrix}1 & 1\\2 & 3\\2 & 3\\3 & 2\end{bmatrix}$.

$\Rightarrow S^{\perp} = N(A^T)$.

Thus, we need to find the null space of A

$$\begin{bmatrix}1 & 2 & 2 & 3\\1 & 3 & 3 & 2\end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix}1 & 2 & 2 & 3\\0 & 1 & 1 & -1\end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix}1 & 0 & 0 & 5\\0 & 1 & 1 & -1\end{bmatrix}$$

$$\begin{array}{cccc} & & \uparrow & \uparrow \\ & & \text{free} & \end{array}$$

$\Rightarrow$ Any solution of $A^T\vec{x} = 0$ satisfies

$$\begin{aligned} x_1 + 5x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -5x_4 \\ x_2 &= -x_3 + x_4. \end{aligned}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-5x_4\\-x_3+x_4\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-5\\1\\0\\1\end{bmatrix}$$

Hence,

$$S^{\perp} = N(A^T) = \text{span}\left\{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\1\\0\\1\end{bmatrix}\right\}.$$