Why Reduce Dimensions?

Why reduce dimensions?

- **Discover hidden correlations/topics**
  - **Words that occur commonly together**
- **Remove redundant and noisy features**
  - **Not all words are useful**
- **Interpretation and visualization**
- **Easier storage and processing of the data**

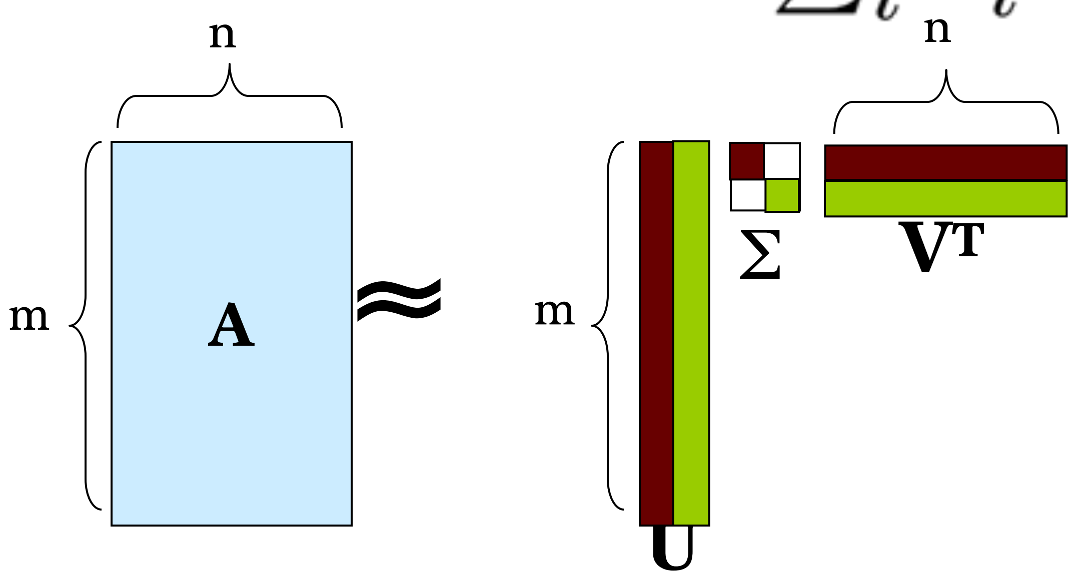
SVD - Definition

$$A_{[m \times n]} = U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$$

- **A: Input data matrix**
  - **m x n matrix (e.g., m documents, n terms)**
- **U: Left singular vectors**
  - **m x r matrix (m documents, r concepts)**
- **Σ: Singular values**
  - **r x r diagonal matrix (strength of each 'concept') (r : rank of the matrix A)**
- **V: Right singular vectors**
  - **n x r matrix (n terms, r concepts)**

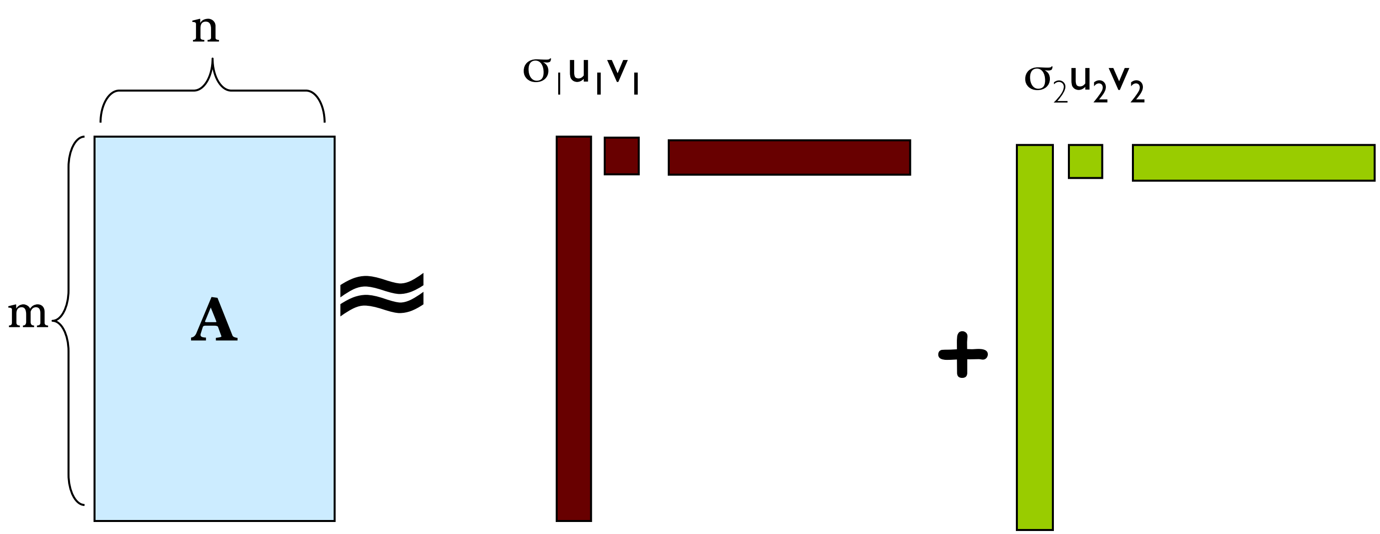
SVD

$$\mathbf{A} \approx \mathbf{U}\Sigma\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^{\mathrm{T}}$$



SVD

$$\mathbf{A} \approx \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^T$$



σ_i … scalar
u_i … vector
v_i … vector

SVD - Properties

It is always possible to decompose a real matrix A into $A = U \Sigma V^T$, where

- **U, Σ, V: unique**
- **U, V: column orthonormal**
  - **$U^T U = I$; $V^T V = I$ (I: identity matrix)**
  - **(Columns are orthogonal unit vectors)**
- **Σ: diagonal**
  - **Entries (singular values) are positive, and sorted in decreasing order ($\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$)**

Nice proof of uniqueness: http://www.mpi-inf.mpg.de/~bast/ir-seminar-ws04/lecture2.pdf

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example:** Users to Movies

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - **example:** Users to Movies

$$
\begin{array}{c}
\\
\\
\text{SciFi} \\
\\
\\
\text{Romnce} \\
\\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{array}\right]
\end{array}
=
\begin{array}{c}
\begin{array}{ccc} \text{SciFi-concept} & \text{Romance-concept} & \end{array} \\
\left[\begin{array}{ccc}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
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0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{array}\right]
\end{array}
\times
\left[\begin{array}{ccc}
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\end{array}\right]
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\left[\begin{array}{ccccc}
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0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{array}\right]
$$

SVD – Example: Users-to-Movies

- $A = U \, \Sigma \, V^T$ **- example:**

U is "user-to-concept" similarity matrix

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \\ \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
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\end{array}
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\end{bmatrix}
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\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- **$A = U \, \Sigma \, V^T$ - example:**

SciFi-concept

“strength” of the SciFi-concept

$$
\begin{array}{rc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{r} \\ \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
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\end{array}
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0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
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0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example:**

V is "movie-to-concept" similarity matrix

SciFi-concept

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \uparrow \\ \text{SciFi} \\ \downarrow \\ \uparrow \\ \text{Romnce} \\ \downarrow \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
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0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SciFi-concept

SVD - Interpretation #1

‘movies’, ‘users’ and ‘concepts’:

- **U: user-to-concept similarity matrix**
- **V: movie-to-concept similarity matrix**
- **Σ: its diagonal elements: ‘strength’ of each concept**