Lecture 18: Applications of Determinants; Eigenvectors and eigenvalues (Sections 4.3--5.1)

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Computation of A^{-1}

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \frac{1}{\det(A)} C^T.$$

Computation of A^{-1}

A^{-1} divides (the transpose) of the cofactors of A by $\det(A)$.

$$A^{-1} = \frac{C^T}{\det(A)} \quad \text{means} \quad (A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}.$$

(See the textbook for the proof.)

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$$A^{-1} = \frac{C^T}{\det A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The solution of $A\boldsymbol{x} = \boldsymbol{b}$

Suppose A is invertible.

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- Cramer's rule. The jth component of $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is the ratio

$$x_j = \frac{\det B_j}{\det A},$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{bmatrix}$$ has $\boldsymbol{b}$ in column j.

The solution of $A\boldsymbol{x} = \boldsymbol{b}$

- Example. Solve

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 4x_2 &= 6 \end{aligned}$$

- Solution.

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9, \quad x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3.$$

Volume of a box

$$\boxed{\text{The determinant equals the volume.}}$$

Eigenvectors and eigenvalues

A will be an $n \times n$ matrix.

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- **Example.** Use your geometric understanding to find the

eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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- For every vector $\boldsymbol{x}$ orthogonal to V, $P\boldsymbol{x} = 0$.

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The eigenspace of $\lambda = 1$ is V. The eigenspace of $\lambda = 0$ is $V^{\perp}$.

How to solve $Ax = \lambda x$

Key observation:

$$
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(A - \lambda I)\boldsymbol{x} &= 0
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This has a nonzero solution if and only if $\det(A - \lambda I) = 0$.

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▶ Recipe. To find eigenvectors and eigenvalues of A.

- First, find the eigenvalues λ using $\det(A - \lambda I) = 0$.
- Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = 0$.

How to solve $Ax = \lambda x$

- Example. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

Eigenvectors

Theorem. If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are eigenvectors A corresponding to different eigenvalues, then they are linearly independent.