Problem 1.

A colony $y(t)$ of yeast is growing in a bakery according to the differential equation

$$\frac{dy}{dt} = y^2(y^2 - 9), \; y(0) = y_0 > 0.$$

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) For what initial values $y_0 > 0$ will the yeast colony eventually die out?

`Solution:`

(i) *This is an autonomous differential equation. The critical points are found by setting*

$$f(y) = y^2(y^2 - 9) = 0 \implies y = 0 \text{ or } y = \pm 3.$$

We determine the sign of

$$f(y) = y^2(y^2 - 9)$$

as follows:

$$y < -3 \implies y^2(y^2 - 9) > 0$$
$$-3 < y < 0 \text{ or } 0 < y < 3 \implies y^2(y^2 - 9) < 0$$
$$y > 3 \implies y^2(y^2 - 9) > 0.$$

Thus -3 *is an asymptotically stable critical point,* 0 *is semistable, while* 3 *is unstable critical point.*

(ii) *We need*

$$\lim_{t \to \infty} y(t) = 0.$$

If $0 < y_0 < 3$*, solutions will converge to the critical point* 0 *(while for* $y_0 \geq 3$ *solutions will diverge to infinity).*

Problem 2.

Solve the initial value problem: $y' = 1 + 2xy, \;\; y(0) = 1$. (Your answer will require a definite integral.)

`Solution:` *We use integrating factors. We have*

$$y' - 2xy = 1,$$

and the integrating factor is

$$u = \exp^{\int -2x\,dx} = e^{-x^2}.$$

We multiply both sides by the integrating factor u *to find*

$$(e^{-x^2} y)' = e^{-x^2}.$$

Integrating we find

$$e^{-x^2} y = \int_0^x e^{-t^2}\,dt + C \implies y = e^{x^2} \int_0^x e^{-t^2}\,dt + Ce^{x^2}.$$

Using the initial condition

$$y(0) = 1 \implies C = 1.$$

Hence

$$y = e^{x^2} \int_0^x e^{-t^2}\,dt + e^{x^2}.$$

Problem 3.

Using undetermined coefficients, find the general solution of the differential equation

$$y'' - 2y' + 2y = 5\sin t.$$

`Solution:` *We solve the homogeneous equation*

$$y'' - 2y' + 2y = 0$$

by solving the characteristic equation

$$r^2 - 2r + 2 = 0 \implies r = 1 \pm i.$$

We find

$$y_1 = e^t \cos t,\ y_2 = e^t \sin t.$$

Next, we look for a particular solution in the form

$$y = A\cos t + B\sin t.$$

We calculate

$$y' = B\cos t - A\sin t$$
$$y'' = -A\cos t - B\sin t.$$

Hence

$$y'' - 2y' + 2y = (A - 2B)\cos t + (B + 2A)\sin t = 5\sin t.$$

Then

$$A = 2B,\ B + 2A = 5 \implies A = 2, B = 1.$$

Thus

$$y_p = 2\cos t + \sin t.$$

We find the general solution

$$y = 2\cos t + \sin t + C_1 e^t \cos t + C_2 e^t \sin t.$$

Problem 4.

Consider the differential equation

$$t^2y'' - 3ty' + 3y = 0, \text{ for } t > 0.$$

(i) Find the values of r such that $y = t^r$ is a solution to the differential equation.
(ii) Check that $y_1 = t$ and $y_2 = t^3$ form a fundamental pair of solutions.
(iii) Find the general solution of the differential equation

$$t^2y'' - 3ty' + 3y = t^3 \ln t.$$

`Solution:`

(i) *We have*

$$y = t^r \implies y' = rt^{r-1},\ y'' = r(r-1)t^{r-2}.$$

Substituting we find

$$t^2y'' - 3ty' + 3y = t^2 \cdot r(r-1)t^{r-2} - 3t \cdot rt^{r-1} + 3t^r = (r^2 - 4r + 3)t^r = 0$$
$$\implies r^2 - 4r + 3 = 0 \implies r = 1,\ r = 3.$$

(ii) *We calculate*

$$W(y_1, y_2) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 2t^3 \neq 0$$

hence y_1, y_2 form a fundamental pair of solutions for $t > 0$.

(iii) *The homogeneous solution is*

$$y_h = C_1t + C_2t^3.$$

We use variation of parameters to find the particular solution. We first write the equation in standard form

$$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = t \ln t.$$

Using integration by parts we find

$$u_1 = -\int \frac{t\ln t \cdot t^3}{2t^3}\, dt = -\frac{1}{2}\int t \ln t \, dt$$
$$= -\frac{1}{2}\left(\frac{1}{2}t^2 \ln t - \int \frac{1}{2}t^2 \, d\ln t\right) = -\frac{1}{4}t^2 \ln t + \frac{1}{4}\int t^2 \cdot \frac{1}{t}\, dt = -\frac{1}{4}t^2 \ln t + \frac{1}{4} \cdot \frac{t^2}{2}.$$

Next,

$$u_2 = \int \frac{t \ln t \cdot t}{2t^3}\, dt = \int \frac{\ln t}{2t}\, dt = \frac{1}{4}(\ln t)^2.$$

We conclude

$$y_p = u_1y_1 + u_2y_2 = \left(-\frac{1}{4}t^2 \ln t + \frac{1}{4} \cdot \frac{t^2}{2}\right) \cdot t + \frac{1}{4}(\ln t)^2 \cdot t^3 = -\frac{1}{4}t^3 \ln t + \frac{t^3}{8} + \frac{(\ln t)^2 \cdot t^3}{4}.$$

Therefore

$$y = y_h + y_p = C_1t + C_2t^3 - \frac{1}{4}t^3 \ln t + \frac{t^3}{8} + \frac{(\ln t)^2 \cdot t^3}{4}.$$

Rearranging constants, this can be rewritten as

$$y = c_1t + c_2t^3 - \frac{1}{4}t^3 \ln t + \frac{t^3(\ln t)^2}{4},$$

for $c_1 = C_1$ and $c_2 = C_2 - \frac{1}{8}$.

Problem 5.

Using the Laplace transform, solve the initial value problem

$$y'' - 2y' + y = t^{10}e^t, \;\; y(0) = 1, y'(0) = 1.$$

`Solution:` *Write $Y(s)$ for the Laplace transform of the solution y. We Laplace transform*

$$y'' - 2y' + y = t^{10}e^t$$

into

$$s^2Y(s) - s - 1 - 2(sY(s) - 1) + Y(s) = \frac{10!}{(s-1)^{11}}.$$

Rearranging terms, we obtain

$$(s-1)^2Y(s) - (s-1) = \frac{10!}{(s-1)^{11}} \implies Y(s) - \frac{1}{s-1} = \frac{10!}{(s-1)^{13}},$$

after dividing by $(s-1)^2$. We now use the inverse Laplace transform to find

$$y(t) - e^t = \frac{1}{12 \cdot 11} \cdot t^{12}e^t \implies y(t) = e^t + \frac{t^{12}e^t}{132}.$$

Problem 6.

The general solution of a certain first order system of differential equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{at} \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

where a is a *non-zero* real number.

(i) For what values of a is the origin a (proper) node? Will it be a source or a sink?
(ii) For what values of a is the origin a saddle equilibrium point? Carefully, draw the trajectories in this case.
(iii) For $a = 2$, find the matrix exponential e^{At}.

`Solution:`

(i) *A proper node corresponds to real distinct eigenvalues of the same sign. The eigenvalues are* 1 *and* a*. Thus we need* $a > 0$ *and* $a \neq 1$*. The origin will be a source.*
(ii) *A saddle corresponds to eigenvalues of opposite signs. Thus we need* $a < 0$*.*
(iii) *We have*

$$\Psi(t) = \begin{bmatrix} e^t & -2e^{2t} \\ 2e^t & e^{2t} \end{bmatrix}.$$

Thus

$$\Psi(0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

We calculate

$$e^{At} = \Psi(t) \cdot \Psi(0)^{-1} = \frac{1}{5} \begin{bmatrix} e^t & -2e^{2t} \\ 2e^t & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^t + 4e^{2t} & 2e^t - 2e^{2t} \\ 2e^t - 2e^{2t} & 4e^t + e^{2t} \end{bmatrix}.$$

Problem 7.

Find the general real-valued solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \mathbf{x}.$$

`Solution:` *We first find the eigenvalues and eigenvectors. We have*

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}$$

whose determinant equals

$$(1-\lambda)(3-\lambda) + 2 = 0 \implies \lambda^2 - 4\lambda + 5 = 0 \implies \lambda = 2 \pm i.$$

We use only one of the eigenvalues, say $\lambda = 2 + i$. *We find the eigenvector*

$$(A - (2+i)I)\vec{v} = 0 \implies \begin{bmatrix} -1+i & 1 \\ -2 & 1-i \end{bmatrix} \vec{v} = 0 \implies \vec{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}.$$

We calculate the complex valued solution

$$\vec{x}_c = e^{(2+i)t} \cdot \begin{bmatrix} 1-i \\ 2 \end{bmatrix} = e^{2t}(\cos t + i \sin t) \cdot \begin{bmatrix} 1-i \\ 2 \end{bmatrix} = e^{2t} \begin{bmatrix} \cos t + \sin t + i(\sin t - \cos t) \\ 2\cos t + 2i \sin t \end{bmatrix}.$$

Taking the real and imaginary parts, we find

$$\vec{x}_1 = e^{2t} \begin{bmatrix} \cos t + \sin t \\ 2\cos t \end{bmatrix}, \ \vec{x}_2 = e^{2t} \begin{bmatrix} \sin t - \cos t \\ 2\sin t \end{bmatrix}.$$

Then

$$x = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{2t} \begin{bmatrix} \cos t + \sin t \\ 2\cos t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin t - \cos t \\ 2\sin t \end{bmatrix}.$$

This is not the only possible form of the answer.

Problem 8.

Consider the differential equation
$$y'' + 2xy' + 2y = 0$$
whose solutions are power series in x centered at $x_0 = 0$.

(i) Find the recurrence relation between the coefficients of the power series y.
(ii) Write down the first three *non-zero* terms in each of the two linearly independent solutions.
(iii) What is the radius of convergence of the solutions which contains only even powers of x?

`Solution:`

(i) *We write*
$$y = \sum_{n=0}^{\infty} a_n x^n.$$
We calculate
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies 2xy' = \sum_{n=0}^{\infty} 2n a_n x^n,$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$
by shifting $n \to n+2$ in the last sum. We compute
$$y'' + 2xy' + 2y = \sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2na_n + 2a_n\right] x^n = 0.$$
Therefore
$$(n+1)(n+2)a_{n+2} + 2na_n + 2a_n = 0 \implies (n+1)(n+2)a_{n+2} + 2(n+1)a_n = 0$$
$$\implies (n+2)a_{n+2} + 2a_n = 0.$$

(ii) *For $n = 0$, we obtain*
$$2a_2 + 2a_0 = 0 \implies a_2 = -a_0.$$
For $n = 2$, we have
$$4a_4 + 2a_2 = 0 \implies a_4 = -\frac{a_2}{2} = \frac{a_0}{2}.$$
Similarly, for $n = 1$, we have
$$3a_3 + 2a_1 = 0 \implies a_3 = -\frac{2}{3}a_1.$$
For $n = 3$, we have
$$5a_5 + 2a_3 = 0 \implies a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1.$$
Clearly, a_0 determines all the even power coefficients, and a_1 determines the odd power coefficients. Then the general solution can be written in the form
$$y = a_0\left(1 - x^2 + \frac{x^4}{2} + \dots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \dots\right).$$
The two linearly independent solutions are
$$y_1 = 1 - x^2 + \frac{x^4}{2} + \dots$$

and

$$y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \dots.$$

(iii) *For the even power solution*

$$y_1 = \sum_k a_{2k} x^{2k}$$

we use the ratio test. We calculate

$$\rho = \lim_{k\to\infty} \left| \frac{a_{2k+2}x^{2k+2}}{a_{2k}x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{-2}{2k+2}x^2 \right| = 0 < 1.$$

To simplify the fraction we used the recurrence found in (i):

$$(2k+2)a_{2k+2} + 2a_{2k} = 0 \implies \frac{a_{2k+2}}{a_{2k}} = -\frac{2}{2k+2}.$$

Thus by the ratio test, we always have convergence or equivalently, the radius of convergence is infinite.

Problem 9.

(i) Find the inverse Laplace transform of the function
$$\frac{1}{(s+1)(s^2+4s+5)}.$$

(ii) Using the Laplace transform, solve the initial value problem
$$y'' + 4y' + 5y = e^{-t} + e^{-t+\pi}u_\pi(t), \quad y(0) = 0, y'(0) = 0.$$

`Solution:`

(i) *We use partial fractions to write*
$$\frac{1}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{B(s+2)+C}{s^2+4s+5}.$$
We solve
$$A(s^2+4s+5) + B(s+1)(s+2) + C(s+1) = 1.$$
From here
$$A = \frac{1}{2},\ B = -\frac{1}{2},\ C = -\frac{1}{2}.$$
Thus the fraction becomes
$$\frac{1}{(s+1)(s^2+4s+5)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\cdot\frac{(s+2)}{(s+2)^2+1} - \frac{1}{2}\cdot\frac{1}{(s+2)^2+1}.$$
The inverse Laplace transform is
$$\frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t.$$

(ii) *We Laplace transform the differential equation*
$$y'' + 4y' + 5y = e^{-t} + e^{-t+\pi}u_\pi(t)$$
into
$$s^2Y(s) + 4Y(s) + 5Y(s) = \frac{1}{s+1} + \frac{e^{-s\pi}}{s+1} \implies Y(s) = \frac{1}{(s+1)(s^2+4s+5)} + \frac{e^{-s+\pi}}{(s+1)(s^2+4s+5)}.$$
Using the previous part, we calculate
$$\begin{aligned} y(t) &= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t + u_\pi(t)\left(\frac{1}{2}e^{-t+\pi} - \frac{1}{2}e^{-2t+2\pi}\cos(t-\pi) - \frac{1}{2}e^{-2t+2\pi}\sin(t-\pi)\right) \\ &= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t + u_\pi(t)\left(\frac{1}{2}e^{-t+\pi} + \frac{1}{2}e^{-2t+2\pi}\cos t + \frac{1}{2}e^{-2t+2\pi}\sin t\right). \end{aligned}$$

Problem 10.

Two tanks A and B initially contain 2 gallons of fresh water. Water containing 2 lb salt/gallon flows into tank A at a rate of 3 gallons/minute. At the same time, water is drained from tank B at a rate of 3 gallon/minute.

The two tanks are connected by two pipes which allow water to flow in only one direction. Specifically, the first pipe allows water to flow from tank A into tank B at a rate of 4 gallons/minute. The second pipe allows water to flow from tank B into tank A at a rate of 1 gallon/minute.

(i) Let $Q_1(t)$ and $Q_2(t)$ be the quantities of salt (measured in pounds) in tanks A and B at time t. Show that

$$\mathbf{Q}' = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix} \mathbf{Q} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

(ii) Solve the system of differential equations (i) and determine the quantities $Q_1(t)$ and $Q_2(t)$ of salt present in each tank at time t. (Do not forget to take into account the initial conditions.) How much salt will each tank contain as time $t \to \infty$?

`Solution:`

(i) *Begin by drawing a picture. We use that*

$$dQ/dt = c_{in} \cdot rate_{in} - c_{out} \cdot rate_{out}.$$

Consider tank A:
- *there is inflow of salt contributing* $2lb/gal \cdot 3gal/min = 6$ *lb salt/minute,*
- *there is inflow of salt from tank B which contributes* $1 \cdot \frac{Q_2}{2}$ *lb salt/min,*
- *there is outflow of salt to tank B which contributes negatively* $4 \cdot Q_1/2 = 2Q_1$ *lb salt/min.*

Putting everything together

$$\frac{dQ_1}{dt} = 6 + \frac{Q_2}{2} - 2Q_1.$$

Now consider tank B:
- *there is inflow of water from tank A which contributes* $4 \cdot Q_1/2 = 2Q_1$ *lb/min salt,*
- *there is outflow of salt from tank B into tank A which contributes* $1 \cdot Q_2/2$ *lb salt/min,*
- *finally, salt is drained out of tank B, contributing negatively* $3 \cdot Q_2/2$ *lb salt/min.*

Putting things together

$$\frac{dQ_2}{dt} = \frac{1}{2}Q_2 + \frac{3}{2}Q_2 - 2Q_1 = 2Q_2 - 2Q_1.$$

The two equations above can be written in vector form

$$\vec{Q}' = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix} \vec{Q} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

The initial condition is

$$\vec{Q}(0) = 0.$$

(ii) *We find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix}.$$

Then

$$A - \lambda I = \begin{bmatrix} -2-\lambda & \frac{1}{2} \\ 2 & -2-\lambda \end{bmatrix}.$$

The determinant is $(-2-\lambda)^2 - 1 = 0$. *We find*

$$\lambda_1 = -1, \ \lambda_2 = -3.$$

We find the eigenvectors

$$(A+I)\vec{v}_1 = 0 \implies \begin{bmatrix} -1 & \frac{1}{2} \\ 2 & -1 \end{bmatrix} \vec{v}_1 = 0 \implies \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(A+3I)\vec{v}_2 = 0 \implies \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix} \vec{v}_2 = 0 \implies \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We find

$$\vec{Q}_h = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We look for a particular solution Q_p. *In fact, undetermined coefficients suggests that we look for* $\vec{Q}_p$ *as a constant solution which means*

$$\vec{Q}_p' = 0 \implies A\vec{Q}_p + \begin{bmatrix} 6 \\ 0 \end{bmatrix} = 0 \implies \vec{Q}_p = -A^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -2 & -\frac{1}{2} \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

The general solution is

$$\vec{Q} = \vec{Q}_p + \vec{Q}_h = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We now use the initial condition $\vec{Q}(0) = 0$ *to find the constants* c_1 *and* c_2. *We obtain*

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} 4 \\ 4 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

Therefore

$$\vec{Q} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - 3e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus

$$Q_1(t) = 4 - 3e^{-t} - e^{-3t}$$
$$Q_2(t) = 4 - 6e^{-t} + 2e^{-3t}.$$

Clearly

$$Q_1(t) \to 4, Q_2(t) \to 4$$

as $t \to \infty$.

Math 20D - Fall 2011 - Final Exam

Problem 1.

A population $y(t)$ of turtles is growing on an island according to the logistic equation with harvesting

$$\frac{dy}{dt} = y(600-y) - 50,000, \; y(0) = y_0 > 0.$$

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) Assume that at time $t=0$ there are 200 turtles on the island. How many turtles will there be on the island in the long run?

`Answer:`

(i) *We find the critical points*

$$\frac{dy}{dt} = y(600-y) - 50,000 = (-y+100)(y-500) = 0 \implies y = 100 \text{ and } y = 500.$$

The parabola $y(600-y) - 50,000$ *is concave, so the signs are negative for* $y < 100$*, positive for* $100 < y < 500$ *and negative for* $y > 500$*. In particular, the function* y *is decreasing for* $y < 100$*, increasing for* $100 < y < 500$ *and decreasing for* $y > 500$*. Drawing the phase line and sketching some of the solutions, we see that* $y = 100$ *repels solutions hence it is an unstable critical point. On the other hand* $y = 500$ *attracts solutions, hence* $y = 500$ *is a stable critical point.*

(ii) *Since* $y(0) = 200$ *which falls in the interval* $(100, 500)$*, it follows that the solution converges to the stable critical point*

$$\lim_{t\to\infty} y(t) = 500.$$

Problem 2.

Consider the inhomogeneous differential equation

$$(\star)\quad x^2y'' - xy' + y = x\ln x,\ \text{for } x > 0.$$

This problem has three main parts (A), (B), (C), all independent of each other.

(A.) Check that $y_1 = x$ is a solution to the homogeneous differential equation. We now proceed to find a second solution y_2 to the homogeneous equation.

(B.1) Show that for any fundamental pair of solutions (y_1, y_2) to the homogeneous equation we must have $W(y_1, y_2) = Cx$ for some constant $C \neq 0$.

(B.2) Set $y_1 = x$. Consider a second solution y_2 to the the homogeneous equation satisfying the initial values

$$y_2(1) = 0,\ y_2'(1) = 1.$$

Show that $W(y_1, y_2) = x$.

(B.3) Use part (B.2) to show that the solution y_2 must satisfy

$$xy_2' - y_2 = x.$$

(B.4) Use (B3) to find a second solution y_2.

(C) Using the solutions

$$y_1 = x \text{ and } y_2 = x\ln x$$

to the homogeneous equation, find the general solution to the inhomogeneous equation $(\star)$ by variation of parameters.

`Answer:`

(A) *We verify that $y_1 = x$ is a solution by computing $y_1' = 1, y_1'' = 0$. Direct computation then shows that the differential equation is verified*

$$x^2y_1'' - xy_1' + y_1 = 0.$$

(B1) *This follows by Abel's theorem. We first bring the equation in standard form*

$$y'' - \frac{1}{x}y + \frac{1}{x^2}y = 0.$$

Abel's theorem states that

$$W(y_1, y_2) = C\exp\left(\int \frac{1}{x}\,dx\right) = C\exp(\ln x) = Cx$$

as needed.

(B2) *We compute*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & y_2 \\ 1 & y_2' \end{vmatrix} = xy_2' - y_2.$$

Evaluating at $x = 1$ we find

$$W(y_1, y_2)(1) = 1 \cdot y_2'(1) - y_2(1) = 1$$

using the initial conditions $y_2(1) = 0, y_2'(1) = 1$. *Since we already showed in (B1) that* $W(y_1, y_2) = Cx$ *it follows*

$$W(y_1, y_2)(1) = C \cdot 1 = C$$

from where $C = 1$ *by comparing with the preceding equation. Thus* $W(y_1, y_2) = x$.

(B3) *We showed in part (B2) that*

$$W(y_1, y_2) = xy_2' - y_2 \text{ and } W(y_1, y_2) = x$$

from where the conclusion follows.

(B4) *To find* y_2 *we use integrating factors. We first write the equation* $xy_2' - y_2 = x$ *in standard form*

$$y_2' - \frac{1}{x} y_2 = 1.$$

The integrating factor is

$$\mu = \exp\left(-\int \frac{1}{x}\right) = \exp(-\ln x) = \frac{1}{x}.$$

Multiplying both sides by the integrating factor we find

$$\left(\frac{1}{x} y_2\right)' = \frac{1}{x} \implies \frac{1}{x} y_2 = \ln x + K \implies y_2 = x \ln x + Kx.$$

To find the constant K *we use the initial value* $y_2(1) = 0$ *which yields* $K = 0$ *so that*

$$y_2 = x \ln x.$$

(C) *We bring the equation to be solved into standard form*

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{\ln x}{x}.$$

We have computed $W(y_1, y_2) = x$ *above. By variation of parameters a particular solution is*

$$y_p = u_1 y_1 + u_2 y_2.$$

We have

$$u_1 = -\int \frac{\ln x}{x} \cdot \frac{y_2}{W}\, dx = -\int \frac{\ln x}{x} \cdot \frac{x \ln x}{x}\, dx = -\int \frac{(\ln x)^2}{x} = -\int (\ln x)^2 \cdot (\ln x)'\, dx = -\frac{1}{3}(\ln x)^3.$$

Similarly,

$$u_2 = \int \frac{\ln x}{x} \cdot \frac{y_1}{W}\, dx = \int \frac{\ln x}{x} \cdot \frac{x}{x}\, dx = \int \frac{\ln x}{x}\, dx = \int (\ln x) \cdot (\ln x)'\, dx = \frac{1}{2}(\ln x)^2.$$

A particular solution is found by substituting into the above expression

$$y_p = -\frac{1}{3}(\ln x)^3 \cdot x + \frac{1}{2}(\ln x)^2 \cdot x \ln x = \frac{1}{6} x (\ln x)^3.$$

The general solution takes the form

$$y = y_p + y_h = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{6} x (\ln x)^3 + c_1 x + c_2 x \ln x.$$

Problem 3.

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$. (You do not need to check this fact.)

(i) Find a fundamental pair of solutions to the system.
(ii) Draw the trajectories of the general solution. What is the type of the phase portrait you obtained?
(iii) Calculate the matrix exponential e^{At}.
(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
(v) Use variation of parameters to find a particular solution the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} 12t \\ 0 \end{bmatrix}.$$

`Answer:`

(i) *We find eigenvectors for the two eigenvalues. Letting* $A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}$ *we compute for the first eigenvalue*

$$A + 4I = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} \implies \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

For the second eigenvalue, we compute

$$A + 6I = \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \implies \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

We form the two fundamental solutions

$$\vec{x}_1 = e^{-4t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{x}_2 = e^{-6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(ii) *The general solution is*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{-4t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

When $t \to -\infty$, the solutions are of large magnitude and follow the dominant term e^{-6t} in the direction of the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. When $t ø \infty$, the solutions approach zero, and they follow the dominant term e^{-4t} in the direction $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. The origin is a node sink.

(iii) *We have*

$$e^{At} = \Phi(t) = \Psi(t) \cdot \Psi(0)^{-1}.$$

We find the fundamental matrix

$$\Psi(t) = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix}.$$

Thus

$$\Psi(0) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}.$$

Substituting we find

$$e^{At} = \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4e^{-4t} - 2e^{-6t} & -8e^{-4t} + 8e^{-6t} \\ e^{-4t} - e^{-6t} & -2e^{-4t} + 4e^{-6t} \end{bmatrix}.$$

(iv) *We have*

$$\vec{x} = e^{At} \cdot \vec{x}_0 = \frac{1}{2} \begin{bmatrix} 4e^{-4t} - 2e^{-6t} & -8e^{-4t} + 8e^{-6t} \\ e^{-4t} - e^{-6t} & -2e^{-4t} + 4e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4e^{-4t} + 6e^{-6t} \\ -e^{-4t} + 3e^{-6t} \end{bmatrix}.$$

(v) *We compute*

$$\vec{x} = \Psi(t) \int \Psi(t)^{-1} \begin{bmatrix} 12t \\ 0 \end{bmatrix} dt.$$

We have

$$\Psi(t)^{-1} = \frac{1}{2e^{-10t}} \begin{bmatrix} e^{-6t} & -2e^{-6t} \\ -e^{-4t} & 4e^{-4t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{4t} & -2e^{4t} \\ -e^{6t} & 4e^{6t} \end{bmatrix}.$$

Thus

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \int \frac{1}{2} \begin{bmatrix} e^{4t} & -2e^{4t} \\ -e^{6t} & 4e^{6t} \end{bmatrix} \cdot \begin{bmatrix} 12t \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \int \begin{bmatrix} 6te^{4t} \\ -6te^{6t} \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-4t} & 2e^{-6t} \\ e^{-4t} & e^{-6t} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2}\left(t - \frac{1}{4}\right) e^{4t} \\ -\left(t - \frac{1}{6}\right) e^{6t} \end{bmatrix} = \begin{bmatrix} 4t - \frac{7}{6} \\ \frac{1}{2}t - \frac{5}{24} \end{bmatrix}. \end{aligned}$$

The integrals were computed via integration by parts. For instance

$$\int 6te^{6t}\, dt = \int t(e^{6t})'\, dt = te^{6t} - \int e^{6t}\, dt = te^{6t} - \frac{1}{6}e^{6t} = \left(t - \frac{1}{6}\right) e^{6t}.$$

The second integral is similar

$$\int 6te^{4t}\, dt = \int \frac{3}{2} t(e^{4t})'\, dt = \frac{3}{2}\left(te^{4t} - \int e^{4t}\, dt\right) = \frac{3}{2}\left(t - \frac{1}{4}\right) e^{4t}.$$

Problem 4.

Find two independent real valued solutions of the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \vec{x}.$$

`Answer:` *We write* $A = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$. *We compute Tr* $A = 4, \det A = 8$ *so the characteristic polynomial is*

$$\lambda^2 - 4\lambda + 8 = 0 \implies (\lambda - 2)^2 + 4 = 0 \implies \lambda - 2 = \pm 2i \implies \lambda = 2 \pm 2i.$$

We use only one of the eigenvalues below, say $\lambda = 2 + 2i$. *We find an eigenvector by computing*

$$A - (2+2i)I = A = \begin{bmatrix} 1-(2+2i) & 1 \\ -5 & 3-(2+2i) \end{bmatrix} = A = \begin{bmatrix} -1-2i & 1 \\ -5 & 1-2i \end{bmatrix} \implies \vec{v} = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}.$$

Thus a complex valued solution is given by

$$\vec{x}_1 = e^{(2+2i)t} \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} = e^{2t}(\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} \cos 2t + i \sin 2t \\ (1+2i)(\cos 2t + i \sin 2t) \end{bmatrix} = e^{2t} \begin{bmatrix} \cos 2t + i \sin 2t \\ \cos 2t - 2 \sin 2t + i(2 \cos 2t + \sin 2t) \end{bmatrix}.$$

We find the real valued solutions by taking the real and imaginary part of the complex valued solution. We have

$$u_1 = e^{2t} \begin{bmatrix} \cos 2t \\ \cos 2t - 2 \sin 2t \end{bmatrix}, v_1 = e^{2t} \begin{bmatrix} \sin 2t \\ 2 \cos 2t + \sin 2t \end{bmatrix}.$$

are the real valued solutions. There are other possible answers here as well.

Problem 5.

Consider the differential equation

$$y'' - xy' - y = 0$$

whose solutions are power series in x centered at $x_0 = 0$.

(i) Find the recurrence relation between the coefficients of the power series y.
(ii) Write down the first three *non-zero* terms in each of the two linearly independent solutions.
(iii) Express the solution involving only even powers of x in closed form. The final answer should be a familiar exponential. You may need to recall the series expansion

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots + \frac{y^n}{n!} + \ldots.$$

`Answer:`

(i) *We write*

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We compute

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

where in the above we used that the term corresponding to $n = 0$ is in fact zero $na_n = 0$ for $n = 0$. In addition,

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n,$$

where the shift $n \to n+2$ was done in the last step. Thus

$$\begin{aligned} y'' - xy' - y &= \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[a_{n+2}(n+1)(n+2) - n a_n - a_n\right] \cdot x^n \\ &= \sum_{n=0}^{\infty} \left[a_{n+2}(n+1)(n+2) - a_n(n+1)\right] \cdot x^n. \end{aligned}$$

Since $y'' - xy' - y = 0$ we conclude

$$a_{n+2}(n+1)(n+2) - a_n(n+1) = 0 \implies a_{n+2}(n+2) - a_n = 0$$

for all n.

(ii) *We write down the first coefficients of the even solution by using $n = 0, n = 2$. We find*

$$2a_2 - a_0 = 0 \implies a_2 = \frac{a_0}{2}$$

$$4a_4 - a_2 = 0 \implies a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4}.$$

The even solution is

$$y^{even} = a_0 + a_2x^2 + a_4x^4 + \ldots = a_0 + \frac{a_0}{2}x^2 + \frac{a_0}{2\cdot 4}x^4 + \ldots$$
$$= a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \ldots\right).$$

Here we can even set $a_0 = 1$ *if we wish to find an answer without any undetermined constants.*

For the odd solution we use $n = 1$ *and* $n = 3$ *to find*

$$3a_3 - a_1 = 0 \implies a_3 = \frac{a_1}{3}$$
$$5a_5 - a_3 = 0 \implies a_5 = \frac{a_3}{5} = \frac{a_1}{3\cdot 5}.$$

This yields

$$y^{odd} = a_1x + a_3x^3 + a_5x^5 + \ldots = a_1\left(x + \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 + \ldots\right).$$

Again, we could use $a_1 = 1$ *if we wish to find an answer without any undetermined constants.*

(iii) *We wish to first the pattern for the even solution. If we continue further with* $n = 4$ *we find*

$$6a_6 - a_4 = 0 \implies a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot 4\cdot 6}$$

while $n = 6$ *yields*

$$8a_8 - a_6 = 0 \implies a_8 = \frac{a_6}{8} = \frac{a_0}{2\cdot 4\cdot 6\cdot 8}.$$

The pattern is now clear

$$a_{2k} = \frac{a_0}{2\cdot 4\cdot 6\cdot \ldots \cdot (2k)} = \frac{a_0}{2^k\cdot(1\cdot 2\cdot\cdots\ldots\cdot k)} = \frac{a_0}{2^k k!}.$$

Let us set $a_0 = 1$ *since we wish to speak about a specific even solution (which is only unique up to scaling). Then*

$$a_{2k} = \frac{1}{2^k k!}$$

and

$$y^{even} = \sum_{k=0}^{\infty} a_{2k}x^{2k} = \sum_{k=0}^{\infty}\frac{1}{2^k k!}x^{2k} = \sum_{k=0}^{\infty}\frac{1}{k!}\cdot\left(\frac{x^2}{2}\right)^k = e^{\frac{x^2}{2}}.$$

Problem 6.

Consider the function

$$h(t) = \begin{cases} 0 & t < 1 \\ t^2 & 1 \le t < 2 \\ t^2 + t - 2 & t \ge 2. \end{cases}$$

(i) Express h in terms of unit step functions.

(ii) Find the Laplace transform of h. You may leave your answer as a sum of fractions.

`Answer:`

(i) *We have* $h(t) = t^2 u_1(t) + (t-2)u_2(t)$.

(ii) *We use that*

$$f(t-c)u_c(t) \mapsto e^{-cs}F(s).$$

In our case, the second term is a direct application (taking $c = 2$ *and* $f(t) = t$ *so that* $F(s) = \frac{1}{s^2}$*), so*

$$(t-2)u_2(t) \mapsto \frac{e^{-2s}}{s^2}.$$

For the first term, we wish to write

$$t^2 u_1(t) = f(t-1)u_1(t)$$

for some suitable function f *in order to apply the formula. This means*

$$f(t-1) = t^2 \implies f(t) = (t+1)^2 = t^2 + 2t + 1 \implies F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}.$$

We have

$$t^2 u_1(t) = f(t-1)u_1(t) \mapsto e^{-s}F(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s}.$$

Therefore

$$H(s) = \frac{2e^{-s}}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2}.$$

Problem 7.

Use Laplace transforms to solve the initial value problem

$$y'' + 2y' + 5y = e^{-2t},\ y(0) = 0, y'(0) = 1.$$

`Answer:` *We have*

$$y'' \mapsto s^2Y - sy(0) - y'(0) = s^2Y - 1,$$
$$y' \mapsto sY - y(0) = sY.$$

The equation to be solved becomes after applying Laplace transform

$$s^2Y - 1 + 2sY + 5Y = \frac{1}{s+2} \implies (s^2 + 2s + 5)Y = 1 + \frac{1}{s+2}$$
$$\implies Y = \frac{1}{s^2 + 2s + 5} + \frac{1}{(s+2)(s^2 + 2s + 5)}.$$

We need to compute the inverse Laplace transforms of the above expression. The first term

$$\frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 4} \text{ has inverse Laplace equal to } \frac{1}{2}\sin 2te^{-t}.$$

The second term is more difficult. We use partial fractions to write

$$\frac{1}{(s+2)(s^2 + 2s + 5)} = \frac{A}{s+2} + \frac{Bs + C}{s^2 + 2s + 5}.$$

Direct computation yields

$$A(s^2 + 2s + 5) + (s+2)(Bs + C) = 1 \iff s^2(A + B) + s(2A + 2B + C) + 5A + 2C = 1$$
$$\iff A + B = 0, 2A + 2B + C = 0, 5A + 2C = 0 \iff A = \frac{1}{5}, B = -\frac{1}{5}, C = 0.$$

Thus

$$\frac{1}{(s+2)(s^2 + 2s + 5)} = \frac{1}{5}\left(\frac{1}{s+2} - \frac{s}{s^2 + 2s + 5}\right) = \frac{1}{5}\left(\frac{1}{s+2} - \frac{s+1}{(s+1)^2 + 4} + \frac{1}{(s+1)^2 + 4}\right).$$

The Laplace inverse equals

$$\frac{1}{5}\left(e^{-2t} - e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right).$$

Collecting all terms

$$y(t) = \frac{1}{2}\sin 2te^{-t} + \frac{1}{5}\left(e^{-2t} - e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right)$$

or simplifying

$$y(t) = \frac{e^{-2t}}{5} + \frac{3}{5}e^{-t}\sin 2t - \frac{1}{5}e^{-t}\cos 2t.$$

Problem 8.

Consider the forcing function

$$h(t) = u_\pi(t) - u_{4\pi}(t).$$

(i) Solve the following initial value problem using Laplace transform

$$y'' + y = h(t),\ y(0) = y'(0) = 0.$$

(ii) Write your solution $y(t)$ explicitly over each of the three intervals

$$0 \le t < \pi, \quad \pi \le t < 4\pi, \quad 4\pi \le t < \infty.$$

(iii) Draw the graph of the solution you found in (i).

`Answer:`

(i) *Using the Laplace of* $u_c(t) \mapsto \frac{e^{-cs}}{s}$, *we compute*

$$H(s) = \frac{e^{-s\pi}}{s} - \frac{e^{-4\pi s}}{s}.$$

The Laplace transform of the differential equation becomes

$$s^2Y + Y = H(s) \implies Y = \frac{H(s)}{s^2+1} = \frac{e^{-\pi s} - e^{-4\pi s}}{s(s^2+1)}.$$

We need to find the inverse Laplace transform of this last expression. We first decompose into partial fractions

$$F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

This is the Laplace transform of the function

$$f(t) = 1 - \cos t.$$

Using that $e^{cs}F(s)$ *has Laplace inverse* $u_c(t)f(t-c)$ *we have*

$$Y = e^{-\pi s}F(s) - e^{-4\pi s}F(s) \implies y = u_\pi(t)f(t-\pi) - u_{4\pi}(t)f(t-4\pi)$$

$$\implies y = u_\pi(t)(1-\cos(t-\pi)) - u_{4\pi}(t)(1-\cos(t-4\pi)).$$

Using periodicity this can be further simplified to

$$y = u_\pi(t)(1+\cos t) - u_{4\pi}(1-\cos t).$$

(ii)
- *For* $t < \pi$ *we have* $u_\pi(t) = u_{4\pi}(t) = 0$ *so* $y = 0$
- *For* $\pi \le t < 4\pi$ *we have* $u_\pi(t) = 1$ *but* $u_{4\pi}(t) = 0$ *so* $y = 1 + \cos t$
- *Finally for* $t > 4\pi$ *we have* $u_\pi(t) = u_{4\pi}(t) = 1$ *so* $y = 1 + \cos t - (1 - \cos t) = 2\cos t$.

Thus

$$y(t) = \begin{cases} 0 & \text{if } t < \pi \\ 1 + \cos t & \text{if } \pi \le t < 4\pi \\ 2\cos t & \text{if } t \ge 4\pi \end{cases}.$$

Fall 2016 – Math 20D – Solution.

1) Solve $t^3 y' + 4t^2 y = e^{t^2}$, $y(1) = e$, $t > 0$.

Devide both sides by t^3

$$y' + \frac{4}{t} y = t^{-3} e^{t^2}$$

integrating factor.

$$u(t) = e^{\int \frac{4}{t} dt} = e^{4 \ln t} = t^4$$

$$\Rightarrow \quad u(t) y = \int t^4 (t^{-3} e^{t^2}) dt.$$

$$= \int t e^{t^2} dt.$$

$$\therefore \quad u(t) y = \frac{e^{t^2}}{2} + C.$$

$$y = \frac{\frac{e^{t^2}}{2} + C}{t^4} = \frac{t^{-4} e^{t^2}}{2} + C t^{-4}$$

Since $y(1) = e$,

$$\frac{1^{-4} e^{1^2}}{2} + C \cdot 1^{-4} = e.$$

$$\frac{e}{2} + C = e$$

$$C = \frac{e}{2}.$$

$$\Rightarrow \quad y = \frac{t^{-4} e^{t^2}}{2} + \frac{e}{2} t^{-4}$$

2) $y' = xy^2e^x$, $y(0) = 3$.

This is a separable eq.

$$\frac{dy}{dx} = xy^2e^x.$$

$$\int \frac{dy}{y^2} = \int xe^x\,dx.$$

$$-\frac{1}{y} = \int xe^x\,dx$$ $\longrightarrow$ integration by parts.

$u = x$. $dv = e^x dx$

$du = dx$ $v = e^x$.

$$-\frac{1}{y} = xe^x - \int e^x dx$$

$$-\frac{1}{y} = xe^x - e^x + C$$

$$y = -\frac{1}{xe^x - e^x + C}.$$

Since $y(0) = 3$,

$$3 = -\frac{1}{0 - e^0 + C}$$

$$3 = -\frac{1}{-1+C}$$

$$-3+3C = -1.$$

$$C = \frac{2}{3}.$$

$$\Rightarrow y = -\frac{1}{xe^x - e^x + \frac{2}{3}}.$$

3) $y'' + 3y' + 6y = 2t.$

use undetermined coefficients method:

Find $y = y_h + y_p$.

i) Find y_h:

characteristic eq: $r^2 + 3r + 6 = 0$.

$$r = \frac{-3 \pm \sqrt{9-24}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{15}}{2}.$$

$$\Rightarrow \quad y_h = C_1 e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{15}}{2}t\right) + C_2 e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{15}}{2}t\right).$$

ii) Find particular solution:

$y = At + B$.

$\Rightarrow \quad y' = A$

$y'' = 0$.

$0 + 3A + 6(At + B) = 2t$.

$6At + (3A + B) = 2t$.

$$\Rightarrow \begin{cases} 6A = 2 \\ 3A + B = 0 \end{cases} \Rightarrow \quad A = \frac{1}{3}, \quad B = -1$$

$$\Rightarrow \quad y_p = \frac{1}{3}t - 1.$$

General solution:

$$y = e^{-\frac{3}{2}t}\left(c_1 \cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{15}}{2}t\right)\right) + \frac{1}{3}t - 1.$$

4) $\vec{x}' = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} \vec{x}$. $\quad \vec{x}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

- Find eigenvalues:

$$\det \begin{bmatrix} 3-\lambda & 5 \\ 0 & -1-\lambda \end{bmatrix} = 0 \Rightarrow (3-\lambda)(-1-\lambda) = 0.$$

$\lambda = 3$ and $\lambda = -1$.

- Find eigenvectors:

For $\lambda = 3 \Rightarrow \begin{bmatrix} 0 & 5 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 5\xi_2 = 0. \\ -4\xi_2 = 0. \end{matrix} \Rightarrow \xi_2 = 0.$

Take $\xi_1 = 1$, and $\xi_2 = 0$.

$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda = -1$, $\begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4\xi_1 + 5\xi_2 = 0$

Take $\xi_1 = 1$ and $\xi_2 = -\frac{4}{5}(1) = -\frac{4}{5}$.

$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix}$.

General sol: $\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} e^{-t}$.

$\vec{x}(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 4 \\ -\frac{4}{5} c_2 = 8 \end{cases} \Rightarrow \begin{cases} c_1 = 14 \\ c_2 = -10 \end{cases}$

$$\boxed{\vec{x} = 14 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} - 10 \begin{bmatrix} 1 \\ -\frac{4}{5} \end{bmatrix} e^{-t}}$$

5) $\vec{x}^{(1)} = \begin{bmatrix} t^2 \\ 3t \end{bmatrix}$ and $\vec{x}^{(2)} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$.

check to see if they form a fundamental set of solutions.

$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \begin{bmatrix} t^2 & e^{2t} \\ 3t & 0 \end{bmatrix} = 3te^{2t} \neq 0$ for $t>0$.

$\Rightarrow$ they form a fundamental set of solutions.

We need to find matrix A such that

$$\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}.$$

$\Rightarrow \begin{bmatrix} 2t \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t^2 \\ 3t \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$

$\begin{cases} 2t = at^2 + 3bt \\ 3 = ct^2 + 3dt \end{cases}$ $\qquad$ $\left.\begin{matrix} 2e^{2t} = a e^{2t} \\ 0 = c e^{2t} \end{matrix}\right\} \Rightarrow \begin{matrix} a=2 \\ c=0 \end{matrix}$

$\Rightarrow \begin{cases} 2t = 2t^2 + 3bt. \\ 3 = 0 + 3dt \end{cases}$

$\begin{cases} 2 = 2t + 3b. \\ 1 = dt \end{cases}$

$b = \dfrac{2-2t}{3}$

$d = \dfrac{1}{t}$.

$$A = \begin{bmatrix} 2 & \frac{2-2t}{3} \\ 0 & \frac{1}{t} \end{bmatrix}.$$

6) I will not ask this question.

7) $$f(t) = \begin{cases} 0, & t<2 \\ (t-2)^2, & 2 \le t < 5 \\ 4, & t \geq 5 \end{cases} = \begin{cases} 0 & t<2 \\ (t-2)^2 & 2\le t<5 \\ 0 & t \geq 5 \end{cases} + \begin{cases} 0 & t<2 \\ 0 & 2 \le t< \\ 4 & t\geq 5 \end{cases}$$

$$= (t-2)^2 \cdot \begin{cases} 0 & t<2 \\ 1 & 2\le t<5 \\ 0 & t \geq 5 \end{cases} + 4 \cdot \begin{cases} 0 & t<5 \\ 1 & t\geq 5 \end{cases}$$

$$f(t) = (t-2)^2 \left(u_2(t) - u_5(t)\right) + 4\, u_5(t).$$

$$\mathcal{L}\{f\} = \mathcal{L}\{u_2(t)(t-2)^2\} - \mathcal{L}\{u_5(t)(t-2)^2\} + 4\,\mathcal{L}\{u_5(t)\}.$$

$$= e^{-2s}\,\mathcal{L}\{t^2\}. - e^{-5s}\,\mathcal{L}\{(t+3)^2\} + 4\,\frac{e^{-5s}}{s}.$$

$$= e^{-2s} \cdot \frac{2!}{s^3} - e^{-5s}\,\mathcal{L}\{t^2+6t+9\} + \frac{4e^{-5s}}{s}.$$

$$= \frac{2e^{-2s}}{s^3} - e^{-5s}\left(\mathcal{L}\{t^2\} + 6\,\mathcal{L}\{t\} + 9\,\mathcal{L}\{1\}\right) + \frac{4e^{-5s}}{s}.$$

$$= \frac{2e^{-2s}}{s^3} - e^{-5s}\left(\frac{\cancel{t}\,2!}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right) + \frac{4e^{-5s}}{s}.$$

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8) $t^2 y'' - 2y = 0, \quad y_1 = t^2, \quad t > 0.$

let $y_2 = t^2 v(t).$

$y' = 2tv + t^2 v'$

$y'' = 2v + 2tv' + 2tv' + t^2 v'' = t^2 v'' + 4tv' + 2v.$

Suppose $y = t^2 v$ is a solution,

$t^2(t^2 v'' + 4tv' + 2v) - 2t^2 v = 0.$

$t^2 v'' + 4tv' + 2v - 2v = 0.$

$t^2 v'' + 4tv' = 0.$

$tv'' + 4v' = 0.$

let $u = v'$, then

$tu' + 4u = 0.$

$tu' = -4u. \rightsquigarrow$ separable.

$\int \frac{du}{u} = \int -\frac{4}{t}\, dt.$

$\ln u = -4\ln t \ \cancel{+C}.$

$u = e^{-4\ln t} = t^{-4}.$

$\Rightarrow v' = t^{-4}.$

$v = \int t^{-4}\, dt = -\frac{t^{-3}}{3}.$

$\Rightarrow y_2 = t^2\left(-\frac{t^{-3}}{3}\right) = -\frac{t^{-1}}{3}.$

$W[y_1, y_2] = \det\begin{bmatrix} t^2 & -\frac{t^{-1}}{3} \\ 2t & \frac{t^{-2}}{3} \end{bmatrix} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3} \neq 0.$

$\Rightarrow$ General solution: $\boxed{y = c_1 t^2 - c_2 \frac{t^{-3}}{3}}$

Math 20D - Spring 2017 - Final Exam

Problem 1.

Using undetermined coefficients, find the general solution of the differential equation

$$y'' - 2y' = e^{2t} - 4t.$$

`Solution:` *The homogeneous equation* $y'' - 2y' = 0$ *has characteristic equation* $r^2 - 2r = 0$ *which gives* $r = 0$ *and* $r = 2$. *Thus, a fundamental pair of solutions for the homogeneous equation is given by*

$$y_1 = e^{0\cdot t} = 1, \;\; y_2 = e^{2t}.$$

The homogeneous solution is

$$y_h = c_1 + c_2 e^{2t}.$$

For the particular solution, we seek

$$y_p = Ate^{2t} + (Bt^2 + Ct + D).$$

The presence of te^{2t} *is motivated by the fact that we need not replicate any of the homogeneous solutions. Similarly, the degree of the polynomial part of the solution is seen to be* 2 *because of the presence of* y' *and the term* t *in the answer. We have*

$$y_p' = A(2t+1)e^{2t} + (2Bt + C)$$
$$y_p'' = A(4t+4)e^{2t} + 2B.$$

Therefore

$$y_p'' - 2y_p' = \big(A(4t+4)e^{2t} + 2B\big) - 2\left((2t+1)e^{2t} + (2Bt + C)\right) = 2Ae^{2t} + (-4Bt + 2B - C) = e^{2t} - 4t.$$

This gives

$$2A = 1, -4B = -4, 2B - 2C = 0.$$

Thus

$$A = \frac{1}{2}, B = 1, C = 1 \implies y_p = \frac{1}{2}te^{2t} + t^2 + t.$$

We chose here $D = 0$ *since we need only one particular solution. The general solution is found by superimposing*

$$y = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{2}te^{2t} + t^2 + t + c_1 + c_2 e^{2t}.$$

Problem 2.

Using integrating factors, find the general solution of the differential equation

$$ty' = t\cos t^4 - 3y.$$

`Solution:` *We first write the equation in standard form*

$$ty' = t\cos t^4 - 3y \implies ty' + 3y = t\cos t^4 \implies y' + \frac{3}{t}y = \cos t^4.$$

The integrating factor is

$$u = \exp\left(\int \frac{3}{t}\,dt\right) = \exp(3\ln t) = t^3.$$

Multiplying by the integrating factor throughout we find

$$(t^3y)' = t^3\cos t^4 \implies t^3y = \int t^3\cos t^4\,dt = \frac{1}{4}\sin t^4 + C.$$

This gives

$$y = \frac{1}{4t^3}\sin t^4 + \frac{C}{t^3}.$$

Problem 3.

Consider the differential equation

$$x^2y'' - 2xy' + (2 - x^2)y = x^3e^x.$$

(i) Find the values of r for which $y = xe^{rx}$ is a solution to the *homogeneous* equation.

(ii) Using variation of parameters, find a particular solution to the *inhomogeneous* equation.

`Solution:`

(i) *If $y = xe^{rx}$ then direct computation shows*

$$y' = (rx + 1)e^{rx}, \ \ y'' = (r^2x + 2r)e^{rx}.$$

Thus

$$x^2y'' - 2xy' + (2 - x^2)y = e^{rx} \cdot \left(x^2(r^2x + 2r) - 2x(rx + 1) + (2 - x^2)x\right) = e^{rx}(r^2x^3 - x^3) = e^{rx}x^3(r^2 - 1).$$

For the homogeneous equation, the last expression should be 0 *for all* x*, hence* $r^2 - 1 = 0$ *so* $r = \pm 1$*. The two solutions are*

$$y_1 = xe^x, y_2 = xe^{-x}.$$

(ii) *We look for a particular solution*

$$y_p = u_1y_1 + u_2y_2.$$

First, we bring the equation into standard form

$$y'' - \frac{2}{x} \cdot y' + \frac{2 - x^2}{x^2} \cdot y = xe^x.$$

We have

$$W(y_1, y_2) = \begin{vmatrix} xe^x & xe^{-x} \\ (x + 1)e^x & (-x + 1)e^{-x} \end{vmatrix} = xe^x \cdot (-x + 1)e^{-x} - xe^x \cdot (x + 1)e^{-x} = -2x^2.$$

By variation of parameters, we have

$$u_1 = -\int \frac{xe^x}{-2x^2} \cdot (xe^{-x})\, dx = \int \frac{1}{2}\, dx = \frac{x}{2},$$

$$u_2 = \int \frac{xe^x}{-2x^2} \cdot (xe^x)\, dx = \int \frac{-1}{2}e^{2x}\, dx = -\frac{1}{4}e^{2x}.$$

Then

$$y_p = \frac{x}{2} \cdot (xe^x) - \frac{1}{4}e^{2x} \cdot (xe^{-x}) = \frac{x^2}{2}e^x - \frac{1}{4}xe^x.$$

Problem 4.

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}.$$

(i) Find a fundamental pair of solutions to the system.
(ii) Draw the trajectories of the general solution. What is the type of the phase portrait you obtained?
(iii) Calculate the normalized fundamental matrix $\Phi(t)$ with $\Phi(0) = I$.
(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
(v) Use variation of parameters to find a particular solution the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} te^{3t} \\ 0 \end{bmatrix}.$$

`Solution:`

(i) *We find Tr* $A = 6, \det A = 9$. *The eigenvalues are roots of the characteristic polynomial*

$$\lambda^2 - 6\lambda + 9 = 0 \implies \lambda = 3.$$

This is a repeated eigenvalue and the matrix is defective. We find the eigenvector by computing

$$A - 3I = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}.$$

Thus

$$(A - 3I)\vec{v} = 0 \implies \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus

$$\vec{x}_1 = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We find a generalized eigenvector by solving

$$(A - 3I)\vec{w} = \vec{v} \implies \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \vec{w} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}.$$

Other choices for $\vec{v}, \vec{w}$ *are possible here. We have*

$$\vec{x}_2 = e^{3t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right).$$

(ii) *The general solution is found by superimposing the two solutions found above*

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right).$$

The trajectory is an improper node source. The dominant term is $e^{3t} t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and solutions follow the direction* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *both when* $t \to -\infty$ *and when* $t \to \infty$. *To determine the direction of the trajectory, we need to compute the velocity vector at one point. For instance, we can pick*

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \vec{x}'(0) = A\vec{x}(0) = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

This vector points down. Since the trajectories diverge away from the origin, in order to match the direction of the velocity vector, the trajectories must move clockwise.

(iii) *We have*

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}.$$

Note that

$$\Psi(0) = \begin{bmatrix} 1 & -1/2 \\ 1 & 0 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{1/2}\begin{bmatrix} 0 & 1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}.$$

Thus

$$\Phi(t) = \Psi(t)\cdot\Psi(0)^{-1} = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} (1-2t)e^{3t} & 2te^{3t} \\ -2te^{3t} & (1+2t)e^{3t} \end{bmatrix}.$$

(iv) *We have*

$$\vec{x} = \Phi(t)\vec{x}(0) = \begin{bmatrix} (1-2t)e^{3t} & 2te^{3t} \\ -2te^{3t} & (1+2t)e^{3t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1-2t)e^{3t} \\ -2te^{3t} \end{bmatrix}.$$

(v) *We have*

$$\vec{x}_p = \Psi(t)\int \Psi(t)^{-1}\begin{bmatrix} te^{3t} \\ 0 \end{bmatrix} dt.$$

We compute $\det\Psi(t) = e^{6t}/2$ *so that*

$$\Psi(t)^{-1} = \frac{1}{e^{6t}/2}\begin{bmatrix} te^{3t} & -e^{3t}(t-1/2) \\ -e^{3t} & e^{3t} \end{bmatrix}.$$

Then

$$\Psi(t)^{-1}\begin{bmatrix} t^2e^{3t} \\ 0 \end{bmatrix} = \frac{1}{e^{6t}/2}\begin{bmatrix} te^{3t} & -e^{3t}(t-1/2) \\ -e^{3t} & e^{3t} \end{bmatrix}\begin{bmatrix} te^{3t} \\ 0 \end{bmatrix} = \frac{2}{e^{6t}}\begin{bmatrix} t^2e^{6t} \\ -te^{6t} \end{bmatrix} = \begin{bmatrix} 2t^2 \\ -2t \end{bmatrix}.$$

Substituting, we obtain

$$\vec{x}_p = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}\int\begin{bmatrix} 2t^2 \\ -2t \end{bmatrix} dt = \begin{bmatrix} e^{3t} & e^{3t}(t-1/2) \\ e^{3t} & e^{3t}t \end{bmatrix}\begin{bmatrix} 2t^3/3 \\ -t^2 \end{bmatrix} = e^{3t}\begin{bmatrix} 2t^3/3 - t^2(t-1/2) \\ 2t^3/3 - t^2\cdot t \end{bmatrix}.$$

Thus

$$\vec{x}_p = e^{3t}\begin{bmatrix} -t^3/3 + t^2/2 \\ -t^3/3 \end{bmatrix}.$$

Problem 5.

Consider the differential equation

$$y'' - 3xy' - 3y = 0 \text{ with initial conditions } y(0) = 1, y'(0) = 0$$

whose solution is written as a power series

$$y = a_0 + a_1 x + a_2 x^2 + \dots.$$

(i) Using the initial conditions, calculate the coefficients a_0 and a_1.

(ii) Find the recurrence relation between the coefficients of the power series y.

(iii) Write down the first four *non-zero* terms of the solution. Is the solution even or odd?

(iv) Write down the general expression for the non-zero coefficients. Express the solution y in closed form. The final answer should be a familiar function. You may need to recall the series expansion

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \dots + \frac{w^n}{n!} + \dots.$$

`Solution:`

(i) *Substituting $x = 0$ we obtain*

$$y(0) = a_0 = 1$$

and computing derivatives we find

$$y'(0) = a_1 = 0.$$

Thus $a_0 = 1, a_1 = 0$.

(ii) *We have $y = \sum_{n=0}^{\infty} a_n x^n$ which gives*

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \implies xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

where we reinserted the term $n = 0$ since the expression above covers this case as well. Next,

$$y'' = \sum_{n=2} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n.$$

Thus

$$y'' - 3xy' - 3y = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 3 \sum_{n=0}^{\infty} n a_n x^n - 3 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left((n+1)(n+2) a_{n+2} - (3n+3) a_n \right) x^n = 0.$$

Therefore

$$(n+1)(n+2) a_{n+2} - (3n+3) a_n = 0 \implies a_{n+2} = \frac{3(n+1)}{(n+1)(n+2)} a_n \implies a_{n+2} = \frac{3}{n+2} a_n.$$

(iii) *We have $a_1 = 0$. The above recursions works in steps of 2, so $a_n = 0$ for all n odd. Thus the solution only has even terms, hence y is even.*

We use the recurrence for $n = 0, 2, 4, 6$ to find

$$a_0 = 1, \;\; a_2 = \frac{3}{2} a_0 = \frac{3}{2}$$

$$a_4 = \frac{3}{4} a_2 \implies a_4 = \frac{3}{4} \cdot \frac{3}{2}$$
$$a_6 = \frac{3}{6} \cdot a_4 \implies a_6 = \frac{3}{6} \cdot \frac{3}{4} \cdot \frac{3}{2}.$$

Thus

$$y = 1 + \frac{3}{2} x^2 + \frac{3}{4} \cdot \frac{3}{2} x^4 + \frac{3}{6} \cdot \frac{3}{4} \cdot \frac{3}{2} x^6 + \dots.$$

(iv) *The general even term is*

$$a_{2n} = \frac{3}{2n} \cdot \frac{3}{(2n-2)} \cdot \dots \cdot \frac{3}{2} = \frac{3^n}{(2n)(2n-2) \cdot \dots \cdot 2} = \frac{3^n}{2^n n!}.$$

Thus

$$y = \sum_{n=0}^{\infty} \frac{3^n}{2^n n!} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3x^2}{2} \right)^n = \exp\left(\frac{3x^2}{2} \right).$$

Problem 6.

Consider the function
$$h(t) = \begin{cases} 2t + t^3 e^t & 0 \le t < 2 \\ t^2 + t^3 e^t & 2 \le t. \end{cases}$$

(i) Express h in terms of unit step functions.

(ii) Find the Laplace transform of h. You may leave your answer as a sum of fractions.

`Solution:`

(i) *We have*
$$h(t) = (2t + t^3 e^t) + (t^2 - 2t)u_2(t).$$

(ii) *The first term* $2t + t^3 e^t$ *has Laplace transform*
$$\frac{2}{s^2} + \frac{6}{(s-1)^4},$$
where the exponential shift formula was used above. For the second term, we write $(t^2 - 2t)u_2(t) = f(t-2)u_2(t)$ *where*
$$f(t-2) = t^2 - 2t \implies f(t) = (t+2)^2 - 2(t+2) = t^2 + 2t \implies F(s) = \frac{2}{s^3} + \frac{2}{s^2}.$$
Thus $(t^2 - 2t)u_2(t) = f(t-2)u_2(t)$ *has Laplace transform*
$$e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right).$$
Thus
$$H(s) = \left(\frac{2}{s^2} + \frac{6}{(s-1)^4}\right) + e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right).$$

Problem 7.

Use Laplace transforms to solve the initial value problem

$$y'' + 4y' + 5y = 10e^t, \;\; y(0) = 3, y'(0) = -2.$$

`Solution:` *Using Laplace transform, we find*

$$s^2 Y - 3s + 2 + 4(sY - 3) + 5Y = \frac{10}{s-1}.$$

We solve

$$(s^2 + 4s + 5)Y = (3s + 10) + \frac{10}{s-1} = \frac{(3s+10)(s-1) + 10}{s-1} = \frac{3s^2 + 7s}{s-1}.$$

Thus

$$Y(s) = \frac{3s^2 + 7s}{(s-1)(s^2 + 4s + 5)}.$$

We write this into a sum of partial fractions

$$\frac{3s^2 + 7s}{(s-1)(s^2 + 4s + 5)} = \frac{A}{s-1} + \frac{B(s+2)}{(s+2)^2 + 1} + \frac{C}{(s+2)^2 + 1}.$$

We solve for the undetermined coefficients

$$3s^2 + 7s = A(s^2 + 4s + 5) + B(s+2)(s-1) + C(s-1) = (A+B)s^2 + (4A + B + C)s + (5A - 2B - C)$$

$$\implies A + B = 3, 4A + B + C = 7, 5A - 2B - C = 0 \implies A = 1, B = 2, C = 1.$$

Thus

$$Y(s) = \frac{1}{s-1} + \frac{2(s+2)}{(s+2)^2 + 1} + \frac{1}{(s+2)^2 + 1}$$

which yields

$$y(t) = e^t + 2e^{-2t}\cos t + e^{-2t}\sin t.$$

Problem 8.

Consider the forcing function

$$h(t) = u_1(t) + u_2(t).$$

(i) Solve the following initial value problem using Laplace transform

$$y'' - y = h(t),\ y(0) = y'(0) = 0.$$

(ii) Write your solution $y(t)$ explicitly over each of the three intervals

$$0 \le t < 1, \quad 1 \le t < 2, \quad 2 \le t < \infty.$$

`Solution:`

(i) *Using Laplace transform we obtain*

$$s^2Y - Y = \frac{e^s}{s} + \frac{e^{2s}}{s} \implies Y(s) = \frac{e^{-s}}{s(s^2-1)} + \frac{e^{-2s}}{s(s^2-1)}.$$

We have

$$\frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}.$$

This gives

$$1 = A(s^2-1) + Bs(s+1) + Cs(s-1) = (A+B+C)s^2 + (B-C)s - A$$
$$\implies A+B+C = 0, B-C = 0, -A = 1 \implies A = -1, B = C = \frac{1}{2}.$$

Thus

$$\frac{1}{s(s^2-1)} = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1},$$

which comes via Laplace transform from the function

$$-1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$

Thus

$$y(t) = u_1(t)\left(-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}\right) + u_2(t)\left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-t+2}\right).$$

(ii) *For $t < 1$ we have $u_1(t) = u_2(t) = 0$ so*

$$y(t) = 0.$$

For $1 \le t < 2$ we have $u_1(t) = 1$ and $u_2(t) = 0$ so

$$y(t) = -1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}.$$

For $t \ge 2$ we have $u_1(t) = u_2(t) = 1$ so

$$y = \left(-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1}\right) + \left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-t+2}\right) = -2 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-t+1} + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-t+2}.$$