Final Exam Topics

Math 20D, Lecture D00, Fall 2017

1. **First order equations.**

   (a) Linear equations
   $$y' + p(t)y = q(t).$$

   i. Solve by **integrating factors**. Bring the equation in standard linear form;

   ii. Integrating factor
   $$u(t) = \exp\left(\int p(t)\,dt\right)$$

   iii. Multiply by u, rewrite the equation as
   $$(uy)' = uq$$
   and solve from here.

   (b) Nonlinear equations

   i. **Separable**
   $$\frac{dy}{dx} = f(x)g(y).$$
   Separate variables, then integrate.

   ii. **Autonomous equations**
   $$\frac{dy}{dx} = f(y).$$
   Equilibrium solutions/critical points are found by $f(y) = 0$. Type of critical points: stable, unstable. Phase line.

   iii. **Exact**
   $$M(x, y) + N(x, y)y' = 0.$$
   Check exactness:
   $$M_y = N_x.$$
   Find a function ϕ such that $\phi_x = M, \phi_y = N$. Set $\phi = \text{constant}$.

2. **Second order homogeneous** equations
   $$y'' + p(t)y' + q(t)y = 0.$$

   (a) General facts:

   i. **Superposition**: if y_1, y_2 are solutions, $c_1y_1 + c_2y_2$ is also a solution.

   ii. **Fundamental pair of solutions**: the Wronskian
   $$W(y_1, y_2)(t) = \det\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \neq 0.$$
   For a fundamental pair, the general solution is
   $$y = c_1y_1 + c_2y_2.$$

iii. **Abel's theorem**

$$W(y_1, y_2) = C \exp\left(-\int p(t)\,dt\right).$$

(b) **Constant coefficient equation**: $p(t) = a, q(t) = b$.

i. Characteristic equation $r^2 + br + c = 0$.

ii. Distinct real roots r_1 and r_2, then the fundamental solutions are

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}.$$

iii. Complex roots: fundamental solutions are the real and imaginary part of $e^{r_1 t}$. If $r_1 = \alpha + i\beta$, then

$$y_1 = e^{\alpha t} \cos \beta t, y_2 = e^{\alpha t} \sin \beta t.$$

iv. Repeated roots $r_1 = r_2 = a$: fundamental solutions are

$$y_1 = e^{at}, y_2 = te^{at}.$$

(c) **Inhomogeneous second order** equations.

i. General solution

$$y = y_h + y_p,$$

where y_h is the homogenous solution, and y_p is the particular solution.

ii. Find a *particular solution* by **undetermined coefficients**

$$y'' + py' + qy = g(t).$$

There are three cases: $g(t)$ can be polynomial, trigonometric, or exponential.

A. For $g(t)$ polynomial: look for y_p as a polynomial with undetermined coefficients. (Try to guess its degree first.)

B. For $g(t)$ trigonometric: look for $y_p = A \cos t + B \sin t$.

C. For exponential case $g(t) = e^{\lambda t}$, use

$$y_p = Ce^{\lambda t}$$

unless λ is a root of the characteristic equation $r^2 + br + c = 0$. In this case, look for $y_p = e^{\lambda t}(At + B)$ for undetermined A, B. If λ is a double root for the characteristic equation, you will need to work with $y_p = e^{\lambda t}(At^2 + Bt + C)$.

D. For a term $g(t) = e^{\lambda t} \times$ polynomial or trigonometric function, substitute $y = e^{\lambda t} u$, find the differential equation for u, then solve for u by undetermined coefficients.

iii. Alternatively, you may use **variation of parameters**

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)}\,dt, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)}\,dt.$$

3. **First order systems** of equations $\vec{x}' = A\vec{x}$.

   (a) Find **eigenvalues** λ of A:

   $$\det(A - \lambda I) = 0.$$

   Eigenvectors are found by solving the system

   $$(A - \lambda I)\vec{\xi} = 0.$$

   (b) Finding solutions: find eigenvalues λ_1, λ_2 with eigenvectors $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$.

   - If real eigenvalues, general solution

   $$\vec{x} = c_1 e^{\lambda_1 t}\,\vec{\xi}^{(1)} + c_2 e^{\lambda_2 t}\,\vec{\xi}^{(2)}.$$

   - If complex eigenvalues, take real and imaginary part of $e^{\lambda_1 t}\vec{\xi}^{(1)}$ to obtain the fundamental solutions, then superimpose.

   (c) Repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$.
   Defective case: one eigenvector $\vec{\xi}$. Solutions

   $$\vec{x}^{(1)} = e^{\lambda t}\vec{\xi}^{(1)}, \quad \vec{x}^{(2)} = e^{\lambda t}(\vec{\xi}^{(2)} + t\vec{\xi}^{(1)})$$

   where $\vec{\xi}^{(2)}$ is a generalized eigenvector

   $$(A - \lambda I)\vec{\xi}^{(2)} = \vec{\xi}^{(1)}.$$

   (d) **Phase portraits**.

   - Saddles: real eigenvalues of opposite sign.
   - nodes: real distinct eigenvalues of same sign.
   - spiral: complex eigenvalues. To find the direction of spirals, compute the velocity vector at a point on the trajectory.
   - (Not on the exam) improper nodes: repeated eigenvalues.

4. **Series solutions.**

   - Power series, summation notation, shift in the indices, differentiation of power series.
   - Applications to ODEs: find the recurrence relations between the coefficients, assembling the solutions.

5. **Laplace transform.**

   (a) Laplace transforms of the standard functions: $1, t^n, \sin(at), \cos(at), e^{at}, e^{at}f(t)$.

   (b) Solve differential equations with Laplace transform. Recall that

   $$\mathcal{L}\{f'\} = sF(s) - f(0), \quad \mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0).$$

   (c) Discontinous functions (Heaviside function).

   (d) Solving differential equations with discontinuous response functions via Laplace transforms.