Lecture 3: Matrix Operations and LU Decomposition

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Basic notation

Consider an $m \times n$ matrix A (m rows, n columns).

$$A = [\ \boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots\ \boldsymbol{a_n}\ ] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$a_{ij} =$ entry in ith row, jth column.

Row times column/Inner product

$$\begin{bmatrix} 2 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Multiplication of a matrix and a vector

- $A\boldsymbol{v}$ by rows

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$$

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- $A\boldsymbol{v}$ by rows

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 5 + 6 \cdot 0 \\ 3 \cdot 2 + 0 \cdot 5 + 3 \cdot 0 \\ 1 \cdot 2 + 1 \cdot 5 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

- $A\boldsymbol{v}$ by column

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- $A\boldsymbol{v}$ by column

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}.$$

In general,

$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{a}_n.$$

The product of a matrix A with a vector $\boldsymbol{x}$ is a linear combination of the columns of A with weights given by the entries of $\boldsymbol{x}$.

▶ Example.

$$
\begin{array}{rcrcrcrcl}
5x_1 & & & + & x_3 & + & 10x_4 & = & 3 \\
x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 4 \\
-3x_1 & + & 4x_2 & & & + & 5x_4 & = & 2 \\
7x_1 & + & x_2 & - & x_3 & - & x_4 & = & 5
\end{array}
$$

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$$\Longrightarrow \begin{bmatrix} 5 & 0 & 1 & 10 \\ 1 & 2 & 3 & 1 \\ -3 & 4 & 0 & 5 \\ 7 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 5 \end{bmatrix}.$$

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▶ Example. Suppose A is $m \times n$ and $\boldsymbol{x}$ is in $\mathbb{R}^p$. Under which condition does $A\boldsymbol{x}$ make sense?

Matrix multiplication

The product of two matrices is given by

$$AB = [\ A\boldsymbol{b}_1\ A\boldsymbol{b}_2\ \cdots\ A\boldsymbol{b}_p\ ], \quad \text{where } B = [\ \boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots\ \boldsymbol{b}_p\ ].$$

▶Example.

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = ?$$

▶Example. Suppose A is $m \times n$ and B is $p \times q$. Under which condition does AB make sense? How about BA?

Another way to look at matrix multiplication

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

- **Example.** Use row-column rule to compute

$$\begin{bmatrix} 2 & 6 & 5 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 0 \end{bmatrix} = ?$$

Basic properties

- Example.

a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is 3×3 **identity matrix.**

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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b)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

Basic properties

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$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

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c)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$$

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Basic properties

Theorem. Let A, B, C be matrices of appropriate size. Then

- Associative: $A(BC) = (AB)C$.
- Left-distribute: $A(B + C) = AB + AC$.
- Right-distributive: $(A + B)C = AC + BC$.

▶ Matrix multiplication is not commutative!

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$$\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} =$$

Transpose of a matrix

- Definition. The **transpose** A^T of a matrix A is the matrix whose columns are formed from the corresponding rows of A.
- Example.

a)

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \\ -5 & 1 \end{bmatrix}^T =$$

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a)

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b)

$$[\, x_1 \ x_2 \ x_3 \,]^T =$$

LU Decomposition

Elementary matrices

- **Definition.** An **elementary matrix** is one that obtained by performing a single elementary row operation on an identity matrix.
- **Example.**

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Elementary matrices

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b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$

Elementary matrices

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$

Elementary matrices

c) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

d) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

Elementary matrices

▶ Example. Elementary matrices are invertible because row operations are reversible.

a) $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} =$$

b) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} =$$

c) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} =$$

Gaussian elimination revisited

▶ Example. Keeping track of the elementary matrices during Gaussian elimination on A:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$$

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Note that

$$A = E^{-1} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

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We factored A as the product of a lower and upper triangular matrix! We say that A has *triangular factorization*.

Gaussian elimination revisited

$A = LU$ is known as the **LU decomposition** of A.

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- Definition.

lower triangular

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

LU decompostion

- Example. Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ as $A = LU$.
- Solution.

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- Solution.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$$E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

LU decompostion

$$E_3E_2E_1A = U \Longrightarrow A = (E_3E_2E_1)^{-1}U = E_1^{-1}E_2^{-1}E_3^{-1}U$$

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The factor L is given by

$$\begin{aligned} L &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

LU decompostion

We found the following LU decomposition of A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Why LU decomposition?

Once we have $A = LU$, it is simple to solve $A\boldsymbol{x} = \boldsymbol{b}$.

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$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ L(U\boldsymbol{x}) &= \boldsymbol{b} \\ L\boldsymbol{c} &= \boldsymbol{b} \text{ and } U\boldsymbol{x} = \boldsymbol{c}. \end{aligned}$$

Both of the final systems are triangular and hence easily solved:

- $L\boldsymbol{c} = \boldsymbol{b}$ by forward substitution to find $\boldsymbol{c}$, and then
- $U\boldsymbol{x} = \boldsymbol{c}$ by backward substitution to find $\boldsymbol{x}$.

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- $L\boldsymbol{c} = \boldsymbol{b}$ by forward substitution to find $\boldsymbol{c}$, and then
- $U\boldsymbol{x} = \boldsymbol{c}$ by backward substitution to find $\boldsymbol{x}$.

▶ Example. Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}.$$