Lecture 11: Orthogonality (Section 3.1 - 3.2)

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The fundamental theorem, part II

- Definition. Let W be a subspace of $\mathbb{R}^n$, and $\boldsymbol{v}$ in $\mathbb{R}^n$.
  - $\boldsymbol{v}$ is **orthogonal** to W, if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ for all $\boldsymbol{w}$ in W.

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  - Another subspace V is **orthogonal** to W, if every vector in V is orthogonal to W.
  - The **orthogonal complement** of W is the space $W^\perp$ of all vectors in $\mathbb{R}^n$ that are orthogonal to W. (Is $W^\perp$ a vector space?)
- **Example.** Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$, then $N(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ and $C(A^T) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ are orthogonal subspaces.

The fundamental theorem, part II

Theorem. (Fundamental Theorem of Linear Algebra, Part II)

- $N(A)$ is orthogonal to $C(A^T)$. (The two spaces are orthogonal complements)
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▶ Proof.

Theorem. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of rank r.

- $\dim C(A) = \dim C(A^T) = r$.
- $\dim N(A) = n - r$
- $\dim N(A^T) = m - r$.

- **Example.** Find all vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

▶ Example. Find all vectors orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

▶ Example. Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \; : \; a + b = 2c \right\}$. Find a basis for the orthogonal complement of V.

A new perspective on $A\boldsymbol{x} = \boldsymbol{b}$

$A\boldsymbol{x} = \boldsymbol{b}$ is solvable

$\iff \boldsymbol{b}$ is in $C(A)$

$\iff \boldsymbol{b}$ is orthogonal to $N(A^T)$

A new perspective on $Ax = b$

$$
\begin{aligned}
&A\boldsymbol{x} = \boldsymbol{b} \text{ is solvable} \\
&\iff \boldsymbol{b} \text{ is in } C(A) \\
&\iff \boldsymbol{b} \text{ is orthogonal to } N(A^T)
\end{aligned}
$$

- Example. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

Orthogonal bases

- Recall that if $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal, then $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent.

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- Example. The standard basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. (Why?)

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- Example. Are the vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal basis for $\mathbb{R}^3$?

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Orthogonal bases

- Example. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V. Find $c_1, \ldots, c_n$ such that

$$\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n.$$

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▶ Solution. Take the dot product of $\boldsymbol{v}_1$ with both sides

$$\begin{aligned}\boldsymbol{v}_1 \cdot \boldsymbol{w} &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1.\end{aligned}$$

Hence, $c_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$.

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\boldsymbol{v}_1 \cdot \boldsymbol{w} &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\
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&= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1.
\end{aligned}$$

Hence, $c_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$. In general, $c_j = \frac{\boldsymbol{v}_j \cdot \boldsymbol{w}}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}$.

Orthogonal bases

- Example. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \quad \text{with} \quad c_j = \boldsymbol{v}_j \cdot \boldsymbol{w}.$$

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- **Example.** Is the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ orthonormal?

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▶ Example. Is the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ orthonormal? If not, normalize the vectors to produce an orthonormal basis.

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▶ Example. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Orthogonal projections

▶ Definition. The **orthogonal projection** of vector $\boldsymbol{x}$ onto vector $\boldsymbol{y}$ is

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  It follows that $c = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}}$.
- $\boldsymbol{x}^{\perp}$ is also called the component of $\boldsymbol{x}$ orthogonal to $\boldsymbol{y}$.

Orthogonal projections

- Example. What is the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\boldsymbol{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

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- Example. What is the orthogonal projection of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?