Math 102 - Winter 2013 - Midterm I

Name: ______________________________

Student ID: ______________________________

Section time: ______________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones. You may use a "cheat sheet" of notes which should be at most half a page, front and back.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 50 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		7
2		22
3		10
4		10
Total		50

Problem 1.

Find the LU-decomposition of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}.$$

`Solution:` *We start with the row operations*

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$$

yielding the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix}.$$

Next, we look at

$$R_3 \to R_3 + 5R_2$$

yielding the upper triangular matrix

$$U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

The lower triangular matrix equals

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}.$$

Problem 2.

Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 & 1 \\ -4 & 5 & -9 & 1 \\ 2 & -5 & 12 & 2 \end{bmatrix}.$$

(i) Give a basis for the column space$C(A)$.

`Solution:` *We begin by row reducing the matrix*

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ -4 & 5 & -9 & 1 \\ 2 & -5 & 12 & 2 \end{bmatrix}.$$

We have the row operations

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$$

yielding

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & 9 & 3 \end{bmatrix}.$$

We next perform the row operation

$$R_3 \to R_3 + 3R_2$$

yielding the matrix

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for $C(A)$ is given by the pivot columns of A.. The pivot columns are the first and second column, hence

$$C(A) = span\left\{\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}\right\}.$$

The rank of A equals 2..

(iii) Give a basis for the null space of A. What is the nullity of A?

`Solution:` *We need to row reduce further in order to find the null space of A. In part (i) we arrived at the echelon form*

$$\begin{bmatrix} -2 & 2 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can further calculate

$$R_1 \to R_1 - 2R_2$$

to obtain the rref

$$\begin{bmatrix} -2 & 0 & 3 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_3 and x_4 while the pivot variables are x_1, x_2. We have

$$-2x_1 + 3x_3 + 3x_4 = 0 \implies x_1 = \frac{3}{2}x_3 + \frac{3}{2}x_4$$

$$x_2 - 3x_3 - x_4 = 0 \implies x_2 = 3x_3 + x_4.$$

We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \cdot \begin{bmatrix} \frac{3}{2} \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$N(A) = span \left\{ \begin{bmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(iv) Show that the columns $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}$ of A are linearly dependent by exhibiting explicit relations between them.

`Solution:` *Vectors in the null space give relations between the columns of A. We have*

$$\begin{bmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \in N(A) \implies \frac{3}{2}\mathbf{c_1} + 3\mathbf{c_2} + \mathbf{c_3} = 0$$

$$\begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in N(A) \implies \frac{3}{2}\mathbf{c_1} + \mathbf{c_2} + \mathbf{c_4} = 0.$$

(v) Does A admit a left inverse, a right inverse, either or neither?

`Solution:` *Neither since the rank does not match the number of columns nor the number of rows.*

(vi) What is the dimension of the left null space of A? What is the dimension of the row space of A?

`Solution:` *The left null space has dimension equal to number of rows minus the rank, in our case $3 - 2 = 1$. The row space has dimension equal to the rank, namely 2.*

(vii) Write down the general solution to the following system of equations

$$Ax = \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}.$$

`Solution:` *The particular solution is* $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. *The general solution is of the form*

$$x = x_p + x_h = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 3.

Are the following vector spaces or not?

(i) The set of upper triangular $n \times n$ matrices.

`Solution:` *Vector space: sum of upper triangular matrices is upper triangular. Multiplication of upper triangular matrices by scalars is upper triangular.*

(ii) The set $\{(x_1, x_2, x_3) \text{ in } \mathbb{R}^3 : x_1x_2x_3 \geq 0\}$.

`Solution:` *Not a vector space. For instance* $(-1, -1, 1)$ *is in the set, but its scalar multiple* $(1, 1, -1)$ *is not.*

(iii) The set $\{(x, y, z, w) \text{ in } \mathbb{R}^4 : 2x - 3y + z - 2w = 0 \text{ and } x - 2y - z + 3w = 0\}$.

`Solution:` *Vector space. The set in question is the null space of the matrix*

$$A = \begin{bmatrix} 2 & -3 & 1 & -2 \\ 1 & -2 & -1 & 3 \end{bmatrix}.$$

Null spaces are always vector spaces.

(iv) The set of polynomials $P(x, y)$ of degree at most 3 in two variables such that

$$P(0,0) = \frac{\partial P}{\partial x}(0,0) = 0.$$

`Solution:` *Vector space. Sum of two polynomials* P *and* Q *with this property and scalar multiples still satisfy the equation in the question.*

(v) The set of $n \times n$ matrices $\texttt{rref} A = I$.

`Solution:` *Not a vector space. The zero vector is not part of the set.*

(vi) The set of $n \times n$ matrices A that commute with a fixed permutation matrix P that is, $PA = AP$.

`Solution:` *Vector space. If* $PA = AP$ *and* $PB = BP$ *then* $P(A + B) = (A + B)P$. *Similarly, if* $PA = AP$ *then* $P(cA) = cPA = cAP = (cA)P$.

Problem 4.

Consider the vector space $\mathcal{P}$ of polynomials $f(x)$ of degree less or equal to 2. Let

$$T : \mathcal{P} \to \mathcal{P}$$

be the transformation

$$f \to f'' + xf'.$$

(i) Show that $\{1, x, x^2 - x - 1\}$ is a basis for $\mathcal{P}$.

`Solution:` *We know that* $\{1, x, x^2\}$ *is a basis of* $\mathcal{P}$ *hence* $\dim \mathcal{P} = 3$. *We need to show that* $\{1, x, x^2 - x - 1\}$ *is also a basis, hence we need to prove that the three polynomials are linearly independent. Assume otherwise. Then*

$$a \cdot 1 + b \cdot x + c \cdot (x^2 - x - 1) = 0$$

and we need to show that $a = b = c = 0$. *Indeed, rewrite the above equation as*

$$(a - c) + (b - c)x + cx^2 = 0 \implies a - c = b - c = c = 0 \implies a = b = c = 0$$

proving linear independence, as needed.

(ii) Explain why T is a linear transformation.

`Solution:` *To check* T *is a linear transformation we need to show*

$$T(f + g) = T(f) + T(g),\ T(cf) = cT(f).$$

The first equality rewrites as

$$(f + g)'' + x(f + g)' = (f'' + xf') + (g'' + xg')$$

which is clearly true. The second equality is

$$(cf)'' + x(cf)' = c(f'' + xf')$$

which is also true.

(iii) Find the matrix of the transformation T in the above basis.

`Solution:` *We have*

$$T(1) = 0$$

$$T(x) = x$$

$$T(x^2 - x - 1) = 2 + x(2x - 1) = 2(x^2 - x - 1) + 1 \cdot x + 4 \cdot 1.$$

The matrix of the transformation is

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$