Lecture 7: Linear independence and Four Fundamental Subspaces (Section 2.3-2.4)

Thang Huynh, UC San Diego

1/24/2018

A basis of a vector space

- **Definition.** A set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ in V is a **basis** of V if
  - $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, and
  - the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ are linearly independent.

Theorem. Suppose that V has dimension d.

- A set of d vectors in V are a basis if they span V.
- A set of d vectors in V are a basis if they are linearly independent.

A basis of a vector space

▶ Example. Let $\mathscr{P}_2$ be the space of polynomials of degree at most 2.

a) What is the dimension of $\mathscr{P}_2$?

b) Is $\{t, 1 - t, 1 + t - t^2\}$ a basis of $\mathscr{P}_2$?

Shrinking and expanding sets of vectors

We can find a basis for $V = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ by discarding, if necessary, some of the vectors in the spanning set.

Shrinking and expanding sets of vectors

We can find a basis for $V = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ by discarding, if necessary, some of the vectors in the spanning set.

- Example. Produce a basis of $\mathbb{R}^2$ from the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Shrinking and expanding sets of vectors

We can find a basis for $V = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ by discarding, if necessary, some of the vectors in the spanning set.

- **Example.** Produce a basis of $\mathbb{R}^2$ from the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- **Example.** Find a basis and the dimension of

$$W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Shrinking and expanding sets of vectors

Every set of linearly independent vectors can be extended to a basis.

Shrinking and expanding sets of vectors

Every set of linearly independent vectors can be extended to a basis.

▶ Example. Consider

$$H = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- What is the dimension of this subspace of $\mathbb{R}^3$?

Shrinking and expanding sets of vectors

Every set of linearly independent vectors can be extended to a basis.

- Example. Consider

$$H = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

  - What is the dimension of this subspace of $\mathbb{R}^3$?
  - Extend it to a basis of $\mathbb{R}^3$.

Bases for null spaces

To find a basis for $N(A)$:

Bases for null spaces

To find a basis for $N(A)$:

- find the parametric form of the solutions to $A\boldsymbol{x} = 0$,
- express solutions $\boldsymbol{x}$ as a linear combination of vectors with the free variables as coefficients,
- these vectors form a basis of $N(A)$.

Bases for null spaces

- Example. Find a basis for $N(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

Bases for null spaces

- Example. Find a basis for $N(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

- Solution. First solve $A\boldsymbol{x} = 0$.

Bases for null spaces

- Example. Find a basis for $N(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

- Solution. First solve $A\boldsymbol{x} = 0$.

$$\left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & \boxed{1} & -6 & -15 & 0 \end{array}\right].$$

The solutions of $A\boldsymbol{x} = \mathbf{0}$ is of the form

$$\begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}.$$

Bases for null spaces

$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

These above three vectors form a basis of $N(A)$. (Why?)

Bases for column spaces

Recall that the columns of A are independent

$\iff A\boldsymbol{x} = 0$ has only the trivial solution (namely, $\boldsymbol{x} = 0$),

$\iff A$ has no free variables.

Bases for column spaces

Recall that the columns of A are independent

$\iff A\boldsymbol{x} = 0$ has only the trivial solution (namely, $\boldsymbol{x} = 0$),

$\iff A$ has no free variables.

A basis for $C(A)$ is given by the pivot columns of A.

Bases for column spaces

- Example. Find a basis for $C(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Bases for column spaces

- Example. Find a basis for $C(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

- Solution. Find row reduction form of A

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

Bases for column spaces

- Remark. For the basis of $C(A)$, you have to take the columns of A, *not the columns of an echelon form.*

Bases for column spaces

- Remark. For the basis of $C(A)$, you have to take the columns of A, *not the columns of an echelon form.*
- Remark. Row operations do not preserve *the column space.*

Bases for column spaces

- Remark. For the basis of $C(A)$, you have to take the columns of A, *not the columns of an echelon form.*
- Remark. Row operations do not preserve *the column space.*
- Remark. Row operations do not preserve *the null space.*

Dimension of $C(A)$ and $N(A)$

Theorem. Let A be an $m \times n$ matrix. Then

- $\dim C(A)$ is the number of pivots of A.
- $\dim N(A)$ is the number of free variables of A.
- $\dim C(A) + \dim N(A) =$

▶ Proof.???

Dimension of $C(A)$ and $N(A)$

Theorem. Let A be an $m \times n$ matrix. Then

- $\dim C(A)$ is the number of pivots of A.
- $\dim N(A)$ is the number of free variables of A.
- $\dim C(A) + \dim N(A) = n$

- Proof.???

Dimension of $C(A)$ and $N(A)$

▶ Example. Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find the dimension of $C(A)$ and $N(A)$.

The four fundamental subspaces

- Definition. The **row space** of A is the column space of A^T.

The four fundamental subspaces

- Definition. The **row space** of A is the column space of A^T.
- Definition. The **left null space** of A is the null space of A^T.

Why ”left”?

The four fundamental subspaces

- Definition. The **row space** of A is the column space of A^T.
- Definition. The **left null space** of A is the null space of A^T.

Why ”left”? $\boldsymbol{y}$ is in $N(A^T)$ if and only if $y^T A = 0$.

The four fundamental subspaces

- Definition. The **row space** of A is the column space of A^T.
- Definition. The **left null space** of A is the null space of A^T.

Why "left"? $\boldsymbol{y}$ is in $N(A^T)$ if and only if $y^T A = 0$.

- Definition. The **rank** of a matrix A is the number of its pivots.

The four fundamental subspaces

- Example. Find a basis for $C(A)$ and $C(A^T)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

The four fundamental subspaces

- Example. Find a basis for $C(A)$ and $C(A^T)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

- Solution. To find $C(A)$, we can just use the echelon form of A. Likewise, we can also obtain $C(A^T)$ for an echelon form of A^T. But, it's not necessary!

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The rank of A is 2. And $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The rank of A is 2. And $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

Recall that $C(A) \neq C(B)$. (We performed row operations).
However, $C(A^T) = C(B^T)$.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The rank of A is 2. And $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

Recall that $C(A) \neq C(B)$. (We performed row operations).

However, $C(A^T) = C(B^T)$.

$\boxed{\text{Row space is preserved by elementary row operations.}}$

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The rank of A is 2. And $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

Recall that $C(A) \neq C(B)$. (We performed row operations).
However, $C(A^T) = C(B^T)$.

Row space is preserved by elementary row operations.

Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -5 \end{bmatrix} \right\}$ form a basis for $C(A)$.

The four fundamental subspaces

Theorem. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of rank r.

- $\dim C(A) = r$
- $\dim C(A^T) = r$
- $\dim N(A) = n - r$
- $\dim N(A^T) = m - r$