Name: **PID:**

TA: **Sec. Time:**

Math 170A
Midterm #2
May 13th, 2015

Turn off and put away your cell phone.
You may not use any notes or calculators during this exam.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

ANSWERS

#	Points	Score
1	25	
2	25	
3	25	
4	25	
$\sum$	100	

1a. (10 points) $Ax = b$ with $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 12 \\ 15 \\ 17 \end{bmatrix}$. Starting with guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, calculate two steps of Jacobi's Method.

$$X^{(1)} = \begin{matrix} \frac{1}{5}(12-1-1) \\ \frac{1}{4}(15-1-1) \\ \frac{1}{3}(17-1-1) \end{matrix} = \begin{bmatrix} 2 \\ \frac{13}{4} \\ 5 \end{bmatrix}$$

$$X^{(2)} = \begin{matrix} \frac{1}{5}(12-\frac{13}{4}-5) \\ \frac{1}{4}(15-2-5) \\ \frac{1}{3}(17-2-\frac{13}{4}) \end{matrix} = \begin{matrix} \frac{15}{20} \\ \frac{8}{4} \\ \frac{47}{12} \end{matrix} = \begin{bmatrix} \frac{3}{4} \\ 2 \\ \frac{47}{12} \end{bmatrix}$$

1b. (10 points) Using $A, b, x^{(0)}$ from Question 1a, calculate two steps of Gauss-Seidel Method.

$$X^{(1)} = \begin{matrix} \frac{1}{5}(12-1-1) \\ \frac{1}{4}(15-2-1) \\ \frac{1}{3}(17-2-3) \end{matrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$X^{(2)} = \begin{matrix} \frac{1}{5}(12-3-4) \\ \frac{1}{4}(15-1-4) \\ \frac{1}{3}(17-1-\frac{5}{2}) \end{matrix} = \begin{matrix} 1 \\ \frac{10}{4} \\ \frac{27}{6} \end{matrix} = \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{9}{2} \end{bmatrix}$$

1c. (5 points) Using $A, b, x^{(0)}$ from Question 1a, calculate the initial error, $e^{(0)}$, and the initial residual, $r^{(0)}$ knowing that the true solution is $x = \begin{bmatrix} 1.02 & 2.36 & 4.54 \end{bmatrix}^T$.

$$e^{(0)} = X - X^{(0)} = \begin{bmatrix} 0.02 \\ 1.36 \\ 3.54 \end{bmatrix}$$

$$r^{(0)} = b - Ax^{(0)} = \begin{matrix} 12-7 \\ 15-6 \\ 7-5 \end{matrix} = \begin{bmatrix} 5 \\ 9 \\ 12 \end{bmatrix}$$

2a. (15 points) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Calculate the following six norms; $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, $\|b\|_1$, $\|b\|_2$, $\|b\|_\infty$. Hint: $\|A\|_2 = \sqrt{\rho(A^T A)}$.

$$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 20 \end{bmatrix}$$

$\det(A^T A - \lambda I) = 0$

$\lambda^2 - 30\lambda + 100 = 0$

$\lambda = \dfrac{30 \pm \sqrt{900 - 400}}{2}$

$$\boxed{\begin{array}{ll} \|A\|_1 = 6 & \|b\|_1 = 3 \\ \|A\|_2 = \sqrt{15 + 5\sqrt{5}} & \|b\|_2 = \sqrt{5} \\ \|A\|_\infty = 7 & \|b\|_\infty = 2 \end{array}}$$

2b. (10 points) Let $v, w \in R^n$, and let $A = vw^T$. Show that $\|A\|_2 = \|v\|_2 \|w\|_2$. Hint: show $\|A\|_2 \le \|v\|_2 \|w\|_2$ and show $\|A\|_2 \ge \|v\|_2 \|w\|_2$.

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|vw^T x\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{|w^T x| \, \|v\|_2}{\|x\|_2}$$

by Cauchy-Schwarz $\le \max\limits_{x \neq 0} \dfrac{\|w\|_2 \|x\|_2 \|v\|_2}{\|x\|_2} = \|w\|_2 \|v\|_2$

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \ge \underset{\text{choice}}{\text{specific}} \frac{\|Aw\|_2}{\|w\|_2} = \frac{\|vw^T w\|_2}{\|w\|_2}$$

$$= \frac{|w^T w| \, \|v\|_2}{\|w\|_2} = \frac{\|w\|_2^2 \, \|v\|_2}{\|w\|_2} = \|w\|_2 \|v\|_2$$

Therefore $\|A\|_2 = \|v\|_2 \|w\|_2$

3a. (20 points) Write a function in Matlab that takes as input an $n \times n$ matrix A, and the number n, and returns as output the matrix infinity norm of A, and the number of flops used. Use only programming basics. Hints: $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{i,j}|$ and the command "abs(c)" returns the absolute value of c in Matlab.

```
function [max flops] = infnorm(A, n)

    max = 0 ;
    flops = 0 ;
    for i = 1 : n
        sum = 0 ;
        for j = 1 : n
            sum = sum + abs(A(i,j)) ;
            flops = flops + 1 ;
        end
        if sum > max
            max = sum ;
        end
    end
end
```

3b. (5 points) Prove $\|A\|_1 \geq \max_j \sum_{i=1}^{n} |a_{i,j}|$. Hint: this involves defining a specific vector $y \in R^n$.

Let k be such that $\sum_{i=1}^{n} |a_{i,k}| = \max_j \sum_{i=1}^{n} |a_{i,j}|$

Let $y \in \mathbb{R}^n$ such that $y_i = \begin{Bmatrix} 0 & i \neq k \\ 1 & i = k \end{Bmatrix}$

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \geq \frac{\|Ay\|_1}{\|y\|_1} = \max_j \sum_{i=1}^{n} |a_{i,j}|$$

4a. (20 point) Calculate the convergence rate of Jacobi's Method applied to $Ax = b$ with $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$?

When solving this problem, use two steps of the Power Method starting with an initial guess $q_0 = [1 \quad 1 \quad 1]^T$ to approximate any eigenvalues you may need.

Jacobi convergence rate $= r = \rho(G) = \max_k |\lambda_k|$

$$G = D^{-1}(E+F) = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/5 & -1/5 \\ -1/4 & 0 & -1/4 \\ -1/3 & -1/3 & 0 \end{bmatrix}$$

Use two steps of the Power Method to find $\max_k |\lambda_k|$

$$Gq_0 = \begin{bmatrix} -2/5 \\ -2/4 \\ -2/3 \end{bmatrix} \quad q_1 = \begin{bmatrix} 3/5 \\ 3/4 \\ 1 \end{bmatrix} \quad Gq_1 = \begin{bmatrix} -7/20 \\ -8/20 \\ -9/20 \end{bmatrix} \quad q_2 = \begin{bmatrix} 7/9 \\ 8/9 \\ 1 \end{bmatrix}$$

Therefore $\boxed{r \approx \frac{9}{20}}$

4b. (5 points). Assume that a specific iteration method defined by the formula $Mx^{(k+1)} = Nx^{(k)} + b$ has $\rho(G) = \rho(M^{-1}N) = \frac{1}{2}$. Approximately how many iterations of this method are needed to reduce the error by a factor of 1000?

Need $\left(\frac{1}{2}\right)^k \leq \frac{1}{1000}$

$k \log\left(\frac{1}{2}\right) \leq \log\left(\frac{1}{1000}\right)$

$k \geq \log_2 1000$

$\boxed{k = 10}$