①

Math 152 – HW01 – Solution.

1) a) Let $M = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix}$. Then

$$MM^T = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix}.$$

and $$M^TM = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix}$$

b) We need to show that $A^TA = (A^TA)^T$ and $AA^T = (AA^T)^T$. These are based on properties of matrix transposes.

(i.e., $(BC)^T = C^TB^T$ and $(B^T)^T = B$).

$$\Rightarrow (A^TA)^T = A^T(A^T)^T = A^TA.$$

and $$(AA^T)^T = (A^T)^T A^T = AA^T.$$

2) Observe that

$$d_1 = c_1 f_1(1) + c_2 f_2(1) + \dots + c_8 f_8(1)$$
$$d_2 = c_1 f_1(2) + c_2 f_2(2) + \dots + c_8 f_8(2)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$d_8 = c_1 f_1(8) + c_2 f_2(8) + \dots + c_8 f_8(8)$$

$$\Rightarrow \vec{d} = F\vec{c}, \text{ where}$$

(2)

$$\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix} \qquad \cancel{F = \begin{matrix} f_1(1) & f_1(2) & \cdots & f_1(8) \\ f_1(2) & f_2(2) & \cdots & f_2(8) \end{matrix}}$$

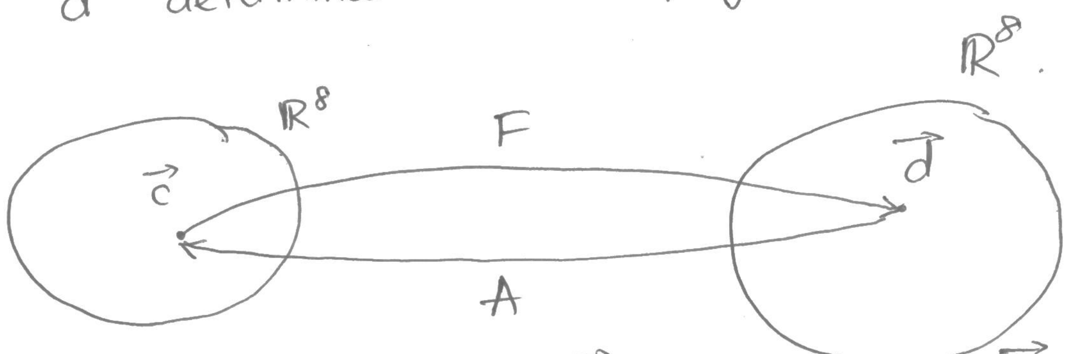
$$F = \begin{bmatrix} f_1(1) & f_2(1) & \cdots & f_8(1) \\ f_1(2) & f_2(2) & \cdots & f_8(2) \\ \vdots & \vdots & & \vdots \\ f_1(8) & f_2(8) & \cdots & f_8(8) \end{bmatrix} \quad \text{and} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}.$$

The problem says that given any $\vec{d} \in \mathbb{R}^8$, one can find $\vec{c} \in \mathbb{R}^8$ such that $F\vec{c} = \vec{d}$. That is, F is of full rank, or $C(F) = \mathbb{R}^8$.

$\Rightarrow$ F is invertible.

$\Rightarrow$ $\vec{d}$ determines $\vec{c}$ uniquely.

b)



We know that ~~$F\vec{c} = \vec{d}$~~ given any $\vec{d} \in \mathbb{R}^8$, there exists $\vec{c} \in \mathbb{R}^8$ such that $\vec{d} = F\vec{c}$

$\Rightarrow$ $\vec{c} = F^{-1}\vec{d}$.

Hence, $F^{-1} = A$ and $A^{-1} = F$.

$\Rightarrow$ $(A^{-1})_{ij} = F_{ij} = f_j(i)$.

③.

3) Let $R \in \mathbb{R}^{m\times m}$ be a nonsingular upper-triangular matrix. Show that R^{-1} is also upper-triangular.

Since R is invertible, there exists a matrix $A \in \mathbb{R}^{m\times m}$ such that $AR = I_{m\times m}$. (That is, $A = R^{-1}$.)

Let $\vec{r}_j$ be the jth column of R and $\vec{a}_j$ be the jth column of A. Then

$$(*)\quad \begin{cases} \vec{a}_1 r_{11} = \vec{e}_1 \\ \vec{a}_1 r_{12} + \vec{a}_2 r_{22} = \vec{e}_2. \\ \cdots \\ \vec{a}_1 r_{1j} + \vec{a}_2 r_{2j} + \cdots + \vec{a}_j r_{jj} = \vec{e}_j. \\ \cdots \\ \vec{a}_1 r_{1m} + \vec{a}_2 r_{2m} + \cdots + \vec{a}_m r_{mm} = \vec{e}_m. \end{cases}$$

where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$ is the standard basis of $\mathbb{R}^m$.

Solving the system $(*)$, we obtain

$$\vec{a}_1 = \vec{e}_1 / r_{11}.$$

$$\vec{a}_2 = (\vec{e}_2 - \vec{a}_1 r_{12}) / r_{22}.$$

$$\vdots$$

$$\vec{a}_j = \left(\vec{e}_j - \sum_{k=1}^{j-1} \vec{a}_k r_{kj}\right) / r_{jj}$$

for $j = 1, \ldots, m$.

So, we see that for each column vector $\vec{a}_j$, it has zeros on the components that have indexes larger than j,

$\Rightarrow$ A is an upper triangular matrix.

④.

4) Let $A \in \mathbb{R}^{m \times n}$, $m \geqslant n$.

Show that A has full rank if and only if given any $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that $\vec{x} \neq \vec{y}$, then $A\vec{x} \neq A\vec{y}$.

Proof.

($\Rightarrow$) Suppose A is of full rank. Then

$$\mathrm{Null}(A) = \{0\}.$$

Then take any vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ such that $\vec{x} \neq \vec{y}$, i.e., $\vec{x} - \vec{y} \neq 0$.

$$A(\vec{x} - \vec{y}) \neq 0 \quad \text{as} \quad \vec{x} - \vec{y} \notin \mathrm{Null}(A).$$

$$\Rightarrow \quad A\vec{x} \neq A\vec{y}.$$

($\Leftarrow$) Take any $\vec{x} \in \mathbb{R}^n$ such that $\vec{x} \neq \vec{0}$. Then

$$A\vec{x} \neq A\vec{0}.$$

$$\Rightarrow \quad A\vec{x} \neq \vec{0}.$$

$$\Rightarrow \quad \vec{x} \notin \mathrm{Null}(A).$$

Therefore, $\mathrm{Null}(A) = \{0\}$.

$\therefore$ A is of full rank.