Math 170A Introduction to Numerical Analysis – Fall 2016

Final Exam Review

Topics:

1.2, 8.1 Convert ODEs, spring carts into $Ax = b$.

1.3 Perform forward and backward substitution.

1.4 Calculate Cholesky Decomposition given a spd A. Symmetric positive definite matrix theory.

1.5 Banded and sparse matrix theory.

8.2 Perform Jacobi and Gauss-Seidel.

2.1 Calculate vector and matrix norms. Vector and matrix norm theory.

8.3 Calculate convergence rates for iterative methods given $Ax = b$. Iterative method theory.

5.2 Calculate eigenvalues and eigenvectors directly.

5.3 Perform the power method.

1.7 Calculate LU without pivoting given A such that $A = LU$.

1.8 Calculate LUP with pivoting given A such that $A = P^T LU$.

2.2 Calculate condition numbers.

2.3, 2.4 Use perturbation error bounds to bound relative error.

3.1 Calculate the best fit line and/or parabola given data.

3.2 Calculate reflector matrices. Calculate QR decomposition given simples matrices A.

3.3 Use QR decomposition to find least squares solution.

Sample questions:

1.3 Given $L = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, and $b = \begin{bmatrix} 14 \\ 23 \\ 18 \end{bmatrix}$, solve $Ly = b$ for y.

1.4a Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 8 \\ 3 & 8 & 11 \end{bmatrix}$. Calculate R such that $A = R^T R$.

1.4b Prove that an $n \times n$ symmetric matrix A is spd if A has n positive real eigenvalues and n orthonormal real eigenvectors.

8.2 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Starting with $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, calculate two iteration of Jacobi's method and Gauss-Seidel method.

2.1a Let $A = \begin{bmatrix} 2 & -4 \\ 8 & 1 \end{bmatrix}$, $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Calculate $\|A\|_1, \|A\|_\infty, \|x\|_1, \|x\|_2, \|x\|_\infty$. How would you calculate $\|A\|_2$?

2.1b Prove $\|x\|_2 \leq \|x\|_1$. and $\|x\|_1 \leq \sqrt{n}\|x\|_2$ for all vectors $x \in \mathbb{R}^n$.

8.3a Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, calculate the convergence rate of Gauss-Seidel method and Jacobi's method applied to $Ax = b$.

8.3b Prove that $e^{(k+1)} = M^{-1}Ne^{(k)}$ for any iterative method defined by $Mx^{(k+1)} = Nx^{(k)} + b$ to solve $Ax = b$ with $A = M - N$.

5.2a Calculate the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1.8 & -0.4 \\ -0.4 & 1.2 \end{bmatrix}$ by direct calculation.

5.2b Approximate the largest eigenvalue and corresponding eigenvector of $A = \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix}$ with two iterations of the power method starting with an initial guess of ones.

1.7 Calculate LU without pivoting such that $LU = A$ with $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 6 \\ 6 & 6 & 12 \end{bmatrix}$.

1.8 Calculate $P^T LU$ with pivoting such that $LU = PA$ with $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 6 \\ 6 & 6 & 12 \end{bmatrix}$.

2.2a Calculate $\kappa_1(A)$ for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

2.2b Prove that $\kappa_1(A) \geq 1$ for all matrices A.

2.3a We know that $\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}$. Bound the relative error $\frac{\|x-\hat{x}\|_\infty}{\|x\|_\infty}$ of the approximate solution $\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $Ax = b$ with $A = \begin{bmatrix} 49 & 50 \\ 50 & 50 \end{bmatrix}$ and $b = \begin{bmatrix} 99.005 \\ 100 \end{bmatrix}$.

2.3b Prove that $\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}$.

3.1 Given the points $(1, 1), (2, 3), (3, 4)$ and $(4, 4)$, calculate the equation of the line that best approximates this data, i.e., find the least squares line that minimizes the residual 2-norm.

3.3 Let $A = \begin{bmatrix} 0 & 3 \\ 0.6 & 1.2 \\ 0.8 & 1.6 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0.6 & 0 & 0.8 \\ 0.8 & 0 & -0.6 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 15 \\ 10 \\ 10 \end{bmatrix}$ with $A = QR$. Calculate the least squares solution of $Ax = b$.

3.2 Calculate the QR decomposition of $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.