Midterm 2A

MATH 102, WINTER 2018

NAME: Key.

PID:

- Print your *NAME* on every page and write your PID in the space provided above.
- Show all of your work in the spaces provided. No credit will be given for unsupported answers, even if correct.
- Supporting work for a problem must be on the page containing that problem. No scratch paper will be accepted.
- No calculators, tables, phones, or other electronic devices are allowed during this exam. You may use your double-sided handwritten notes, but no books or other assistance.

DO NOT TURN PAGE UNTIL INSTRUCTED TO DO SO

(This exam is worth 40 points)

Problem 0.(1 points.) Follows the instructions on this exam and any additional instructions given during the exam.

NAME:

Problem 1.(9 points.) Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{bmatrix}.$$

$$\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix} \overset{R_3-R_1}{=} \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -4 & -4 \\ 0 & -4 & -3 & 2 \end{vmatrix}$$

$$\underset{R_4+R_2}{\overset{R_3+R_2}{=}} \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= 2(4)(-1)(-1)$$

$$= 8$$

Note: Cofactor expansion is also allowed!

NAME:

Problem 2.(10 points.) This problem finds the curve $y = C + D \cdot 2^x$ which gives the best least squares fit to the points $(x, y) = (0,6), (1,4), (2,0)$.

a) (5 points) Write down the 3 equations that would be satisfied if the curve went through all 3 points.

b) (5 points) Find the coefficients C and D of the best curve

$$y = C + D \cdot 2^x.$$

a)
$$C + D2^0 = 6 \Rightarrow C + D = 6$$
$$C + D2^1 = 4 \Rightarrow C + 2D = 4$$
$$C + D2^2 = 0 \Rightarrow C + 4D = 0.$$

b) The matrix equation is

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}}_{\vec{b}}.$$

$\Rightarrow$ least squares solutions:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

and
$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \frac{1}{63-49} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$
$$= \frac{1}{14} \begin{bmatrix} 210 - 98 \\ -70 + 42 \end{bmatrix}$$
$$= \frac{1}{14} \begin{bmatrix} 112 \\ -28 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$\Rightarrow y = 8 - 2 \cdot 2^x$$

NAME:

Problem 3.(10 points.) Let U be the *orthogonal complement* to span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right\}$.

a) (4 points) Find a basis of U.

b) (4 points) Find an orthonormal basis of U.

a) Since U is the orthogonal complement to span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right\}$,

$$U = N\left(\begin{bmatrix} 1 & 2 & -5 \end{bmatrix}\right)$$

$\Rightarrow$ Find null space of the matrix $\begin{bmatrix} 1 & 2 & -5 \end{bmatrix}$.

The vector in this null space is of the form:

$$\begin{bmatrix} -2x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $U = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of U.

b) Use Gram-Schmidt process:

$\vec{a}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ $\vec{b}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \vec{q}_1 = \dfrac{\vec{b}_1}{\|\vec{b}_1\|} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

$$\vec{b}_2 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}}(-10) \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\vec{q}_2 = \frac{\vec{b}_2}{\|\vec{b}_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(check: $\langle \vec{q}_1, \vec{q}_2 \rangle = \dfrac{1}{\sqrt{5}} \times \dfrac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0$ ✓)

$\Rightarrow$ $\{\vec{q}_1, \vec{q}_2\} = \left\{ \dfrac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis of U.

NAME:

c) (4 points) Find the distance between $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ and U.

First, we need to find $\text{proj}_U \vec{v}$:

$$\hat{v} = \text{proj}_U \vec{v} = \langle v, q_1 \rangle q_1 + \langle v, q_2 \rangle q_2$$

$$= \left\langle \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\rangle \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}(-6+1) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{6}(3+2+7) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= -1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

the distance between v and U is

$$\| v^{\perp} \| = \| v - \hat{v} \|$$

$$= \left\| \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \right\|$$

$$= \sqrt{1^2 + 2^2 + 5^2}$$

$$= \sqrt{1 + 4 + 25}$$

$$= \sqrt{30}.$$

NAME:

Problem 4.(10 points.) Consider the space $\mathcal{P}_2$ of polynomials of degree up to 2, together with the inner product

$$\langle p(t), g(t)\rangle = \int_0^1 p(t)g(t)\,dt.$$

a) (4 points) Show that the standard basis $\{1, t, t^2\}$ is not an orthogonal basis.

b) (4 points) Apply Gram-Schmidt to $\{1, t, t^2\}$ to obtain an orthonormal basis of $\mathcal{P}_2$.

a) Check with 1 and t:

$$\langle 1, t\rangle = \int_0^1 1\cdot t\,dt = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}.$$

$\Rightarrow$ 1 and t are not orthogonal

thus, $\{1, t, t^2\}$ is not an orthogonal basis.

b) let $a_1 = 1$, $a_2 = t$ and $a_3 = t^2$.

$b_1 = a_1 = 1 \quad \Rightarrow \quad q_1 = \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{\langle b_1, b_1\rangle}} = \frac{1}{\sqrt{\int_0^1 1\,dt}} = 1.$

$$b_2 = \underbrace{t}_{a_2} - \underbrace{\left(\int_0^1 t\cdot 1\,dt\right)}_{\langle a_2, q_1\rangle}\cdot\underbrace{1}_{q_1} = t - \left(\frac{t^2}{2}\Big|_0^1\right) = t - \frac{1}{2}$$

$$q_2 = \frac{b_2}{\|b_2\|} = \frac{t - 1/2}{\sqrt{\langle t-\frac{1}{2}, t-1/2\rangle}} = \frac{t-1/2}{\sqrt{\int_0^1 (t-1/2)^2\,dt}}$$

let calculate $\int_0^1 (t-\frac{1}{2})^2\,dt = \int_0^1 t^2 - t + \frac{1}{4}\,dt = \frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{4}t\Big|_0^1$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{4-6+3}{12} = \frac{1}{12}.$$

$$q_2 = \frac{t - 1/2}{\sqrt{\frac{1}{12}}} = \sqrt{12}\,t - \sqrt{3}.$$

$\Rightarrow$ ~~$q_2 = 12t - 6$~~

To find q_3, first we need to calculate

$$\langle a_3, q_1\rangle = \int_0^1 t^2\cdot 1\,dt = \frac{1}{3}.$$

$$\langle a_3, q_2\rangle = \int_0^1 t^2\cdot(\sqrt{12}t - \sqrt{3})\,dt = \cancel{\sqrt{6}}\ \frac{\sqrt{3}}{6}.$$

$$\Rightarrow b_3 = t^2 - \frac{1}{3}\cdot 1 - \frac{\sqrt{3}}{6}\cdot(\sqrt{12}t - \sqrt{3}) = t^2 - t + \frac{1}{6}.$$

$$\Rightarrow q_3 = \frac{t^2 - t + \frac{1}{6}}{\|t^2 - t + \frac{1}{6}\|} = \frac{t^2 - t + \frac{1}{6}}{\sqrt{\int_0^1 (t^2 - t + \frac{1}{6})^2\,dt}} = \frac{t^2 - t + 1/6}{\sqrt{\frac{1}{180}}} = 6\sqrt{5}(t^2 - t) + \sqrt{5}$$

NAME:

c) (2 points) What is the orthogonal projection of t^2 onto span$\{1, t\}$?

$$\operatorname{span}\{1,t\} = \operatorname{span}\{\underbrace{1}_{q_1}, \underbrace{\sqrt{12}\,t - \sqrt{3}}_{q_2}\}.$$

$\Rightarrow$ projection of t^2 onto span$\{1,t\}$ is

$$\langle t^2, 1\rangle 1 + \langle t^2, \sqrt{12}\,t - \sqrt{3}\rangle(\sqrt{12}\,t - \sqrt{3})$$

$$= \left(\int_0^1 t^2\,dt\right) + \left(\int_0^1 t^2(\sqrt{12}\,t - \sqrt{3})\,dt\right)(\sqrt{12}\,t - \sqrt{3})$$

$$= \frac{1}{3} + \left(\sqrt{12}\,\frac{t^4}{4} - \sqrt{3}\,\frac{t^3}{3}\Big|_0^1\right)(\sqrt{12}\,t - \sqrt{3})$$

$$= \frac{1}{3} + \left(\frac{\sqrt{12}}{4} - \frac{\sqrt{3}}{3}\right)(\sqrt{12}\,t - \sqrt{3}).$$

Midterm 2B

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NAME:

Problem 1.(9 points.) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -2 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -2 \end{bmatrix}.$$

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -2 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -2 \end{vmatrix} \overset{R_3-R_1}{=\!=} \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -2 & -2 \end{vmatrix}$$

$$\underset{R_4-R_2}{\overset{R_3+R_2}{=}} \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$$= (1)(1)(-3)(-4).$$

$$= 12.$$

NAME:

Problem 2.(10 points.) This problem finds the curve $y = C + D \cdot 2^x$ which gives the best least squares fit to the points $(x, y) = (0, 8), (1, -4), (2, 0)$.

a) (5 points) Write down the 3 equations that would be satisfied if the curve went through all 3 points.

b) (5 points) Find the coefficients C and D of the best curve

$$y = C + D \cdot 2^x.$$

a) $C + D = 8$

$C + 2D = -4$

$C + 4D = 0$

b) The matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$

$$\Rightarrow \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

$$= \begin{bmatrix} \cancel{48}\,84/14 \\ -28/14 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

$$\Rightarrow \quad y = 6 \cancel{-1} - 2 \cdot 2^x.$$

NAME:

Problem 3.(10 points.) Let U be the *orthogonal complement* to span $\left\{ \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \right\}$.

a) (4 points) Find a basis of U.

b) (4 points) Find an orthonormal basis of U.

a) $\begin{bmatrix} -5x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

$\Rightarrow \left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the basis of $U = N([1\ 5\ -2])$.

b) $b_1 = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} \Rightarrow q_1 = \frac{1}{\sqrt{26}} \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$.

$b_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{26}}(-10) \frac{1}{\sqrt{26}} \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \frac{10}{26} \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 - \frac{25}{13} \\ 5/13 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/13 \\ 5/13 \\ 1 \end{bmatrix}$.

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$\cancel{q_2 =}\ \|b_2\| = \sqrt{\frac{1}{169} + \frac{25}{169} + \frac{169}{169}} = \sqrt{\frac{175}{169}} = \frac{\sqrt{175}}{13}$

$q_2 = \frac{b_2}{\|b_2\|} = \frac{1}{\sqrt{175}} \begin{bmatrix} 1 \\ 5 \\ 13 \end{bmatrix}$

$\Rightarrow \left\{ \frac{1}{\sqrt{26}} \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{175}} \begin{bmatrix} 1 \\ 5 \\ 13 \end{bmatrix} \right\}$ is an orthonormal basis of U.

NAME:

c) (4 points) Find the distance between $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ and U.

$$\langle v, q_1 \rangle = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \cdot \frac{1}{\sqrt{26}} \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{26}}(-15+1) = -\frac{14}{\sqrt{26}},$$

$$\langle v, q_2 \rangle = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \cdot \frac{1}{\sqrt{175}} \begin{bmatrix} 1 \\ 5 \\ 13 \end{bmatrix} = \frac{1}{\sqrt{175}} [$$

the distance between v and U is the norm of the projection of v onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \right\} = U^{\perp}$

$$\Rightarrow \quad \operatorname{proj}_{U^\perp} v = \frac{\left\langle \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}.$$

$$= \frac{(3+5-14)}{1+25+4} \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}.$$

$$= \frac{-6}{30} \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}.$$

$$= -\frac{1}{5} \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}.$$

$$\Rightarrow \|\operatorname{proj}_{U^\perp} v\| = \frac{1}{5}\sqrt{1+25+4} = \frac{\sqrt{30}}{5}.$$

NAME:

Problem 4.(10 points.) Consider the space $\mathcal{P}_2$ of polynomials of degree up to 2, together with the inner product

$$\langle p(t), g(t) \rangle = \int_0^1 p(t)g(t)\,dt.$$

a) (4 points) Show that the standard basis $\{1, t, t^2\}$ is not an orthogonal basis.

b) (4 points) Apply Gram-Schmidt to $\{1, t, t^2\}$ to obtain an orthonormal basis of $\mathcal{P}_2$.

Same as Ver A.

c) (4 points) What is the orthogonal projection of t onto $\operatorname{span}\{1, t^2\}$?

$\operatorname{span}\{1, t^2\}$

Find orthonormal basis:

$q_1 = 1$.

$b_2 = t^2 - \left(\int_0^1 t^2\,dt\right) = t^2 - \frac{1}{3}$.

$$\|b_2\|^2 = \int_0^1 \left(t^2 - \frac{1}{3}\right)^2 dt = \int_0^1 t^4 - \frac{2}{3}t^2 + \frac{1}{9}\,dt.$$

$$= \frac{t^5}{5} - \frac{2}{9}t^3 + \frac{1}{9}t\Big|_0^1$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$= \frac{4}{45}$$

$$\Rightarrow q_2 = \frac{\sqrt{45}}{2}\left(t^2 - \frac{1}{3}\right).$$

projection of t onto $\operatorname{span}\{1, t^2\}$ is

$$\langle t, 1\rangle 1 + \left\langle t, \frac{\sqrt{45}}{2}\left(t^2 - \frac{1}{3}\right)\right\rangle \frac{\sqrt{45}}{2}\left(t^2 - \frac{1}{3}\right).$$

$$= \left(\int_0^1 t\,dt\right) + \left(\int_0^1 t\left(t^2 - \frac{1}{3}\right)dt\right)\frac{45}{4}\left(t^2 - \frac{1}{3}\right)$$

$$= \frac{1}{2} + \int_0^1 \left(t^3 - \frac{1}{3}t\right)dt \quad \frac{45}{4}\left(t^2 - \frac{1}{3}\right)$$

$$= \frac{1}{2} + \left(\frac{t^4}{4} - \frac{t^2}{6}\right)\Big|_0^1 \quad \frac{45}{4}\left(t^2 - \frac{1}{3}\right)$$

$$= \frac{1}{2} + \left(\frac{1}{4} - \frac{1}{6}\right)\frac{45}{4}\left(t^2 - \frac{1}{3}\right).$$