* Implicit Differentiation.

An implicitly defined function is a function $y = f(x)$ which is defined by an equation of the form

$$F(x, y) = 0.$$

($\Rightarrow$ To find $y = f(x)$ for a given value of x, we must solve the equation $F(x, y) = 0$ for y)

E.g. $y^2 + x = xy$.

or $y^2 + x - xy = 0$.

E.g. Consider $f(x) = \sqrt[4]{1-x^4}$, $-1 \leq x \leq 1$.

Find $f'(x)$.

- 1st method: chain rule.

Note that $f(x) = (1-x^4)^{1/4}$

$$f'(x) = \frac{1}{4}(1-x^4)^{-3/4} \cdot (1\cancel{+} - x^4)'$$

$$= \frac{1}{4}(1-x^4)^{-3/4} \quad (-4x^3).$$

$$= \frac{-x^3}{(1-x^4)^{3/4}}$$

- 2nd method:

Let $y = \sqrt[4]{1-x^4} \quad \Rightarrow \quad f'(x) = \frac{dy}{dx}$.

Then $y^4 = 1 - x^4$

$$y^4 + x^4 - 1 = 0.$$

Differentiate both sides w.r.t. x:

$$\frac{d(y^4)}{dx} + \frac{d(x^4)}{dx} - \frac{d(1)}{dx} = 0.$$

$$\frac{d(y^4)}{dy} \cdot \frac{dy}{dx} + 4x^3 - 0 = 0.$$

$$4y^3 \frac{dy}{dx} + 4x^3 = 0.$$

$$\frac{dy}{dx} = -\frac{4x^3}{4y^3}$$

$$\frac{dy}{dx} = -\frac{x^3}{(1-x^4)^{3/4}},$$

E.g. Let $y=f(x)$ be a function defined by

$$2y + \sin(y) = x.$$

i.e. $$2y + \sin(y) - x = 0 \qquad (*)$$

For example, if $x = 2\pi$ then $y = \pi$.

To find $f'(x) = \frac{dy}{dx}$, we differentiate both sides of $(*)$.

$$\frac{d(2y)}{dx} + \frac{d\sin(y)}{dx} - \frac{dx}{dx} = 0.$$

$$\frac{d(2y)}{dy} \cdot \frac{dy}{dx} + \frac{d(\sin(y))}{dy} \cdot \frac{dy}{dx} - 1 = 0.$$

$$2\frac{dy}{dx} + \cos(y)\frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{2+\cos(y)}.$$

* Leibniz's notation:

Suppose that $y = g(x)$ and $z = f(y) = (f \circ g)(x)$.

$\Rightarrow$ The derivative of z with respect to x is the derivative of the function $f \circ g$.

The derivative of z w.r.t. y is the derivative of f, and the derivative of y w.r.t. x is the derivative of g.

$$\frac{dz}{dx} = (f \circ g)'(x), \quad \frac{dz}{dy} = f'(y), \quad \frac{dy}{dx} = g'(x).$$

The chain rule:

$$\boxed{\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}}$$

E.g. Find $f'(x)$ of $f(x) = \cancel{\sin(2x)}\ \sin(x^2 + x)$.

Let $z = \sin(x^2 + x)$ and $y = x^2 + x$.

Then $z = \sin(y)$.

$$\begin{array}{c} z \\ \downarrow \\ y \\ \downarrow \\ x \end{array} \qquad \Rightarrow f'(x) = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \cos(y) \cdot (2x+1)$$

$$= \cos(x^2+x) \cdot (2x+1).$$

- Derivatives of Arc Sine and Arc tangent:

Recall that

$$y = \arcsin(x) \iff x = \sin(y), \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

and

$$y = \arctan(x) \iff x = \tan(y) \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Thm: a) $\dfrac{d(\arcsin(x))}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

b) $\dfrac{d(\arctan(x))}{dx} = \dfrac{1}{1+x^2}.$

pf: a) If $y = \arcsin(x)$, then

$$x = \sin(y).$$

or $\quad \sin(y) - x = 0.$

differentiate both sides w.r.t. x,

$$\frac{d(\sin(y))}{dx} - \frac{dx}{dx} = 0.$$

$$\cos(y)\frac{dy}{dx} - 1 = 0.$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$= \frac{1}{\sqrt{1-\sin^2(y)}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

b) Exercise!.

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Derivatives of exponential and logarithmic functions.

. Def (the natural logarithm) $\ln(x)$ is the function with **domain** $(0, \infty)$ that gives an inverse to e^x:

$$e^{\ln(x)} = x$$

$$\ln(e^x) = x$$

[It's a logarithm with base $e \Rightarrow \ln(x) = \log_e(x)$.

. Let $f(x) = a^x$ for $a>0$.

How to find $f'(x)$?

$\Rightarrow$ use the chain rule

First, let rewrite $f(x)$ as:

$$f(x) = a^x = e^{\ln(a)\cdot x}$$

$$\Rightarrow f'(x) = e^{\ln(a)x} \cdot \ln(a)$$

$$= a^x \ln(a)$$

$$\Rightarrow \boxed{\frac{d}{dx}(a^x) = a^x \ln(a)}$$

. How to find $\frac{d}{dx}(\ln(x))$?

$\Rightarrow$ use implicit differentiation.

Let $y = \ln(x) \Rightarrow e^y = x$.

$$\Rightarrow e^y - x = 0.$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(e^y) - \frac{d(x)}{dx} = 0.$$

$$e^y \frac{dy}{dx} - 1 = 0$$

$$\rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\therefore \quad \boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}} \quad \text{and} \quad \frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}$$

* Properties of logarithms:

$\log_b(x)$ is the inverse of b^x for $b>0$ and $x>0$

That is, $\log_b(b^a) = a$

and $b^{\log_b a} = a$.

1) $\log_b(a) = \frac{\ln(a)}{\ln(b)}$ $(b>1)$.

2) $\log_b(AB) = \log_b(A) + \log_b(B)$ $(A, B > 0)$.

3) $\log_b\left(\frac{A}{B}\right) = \log_b(A) - \log_b(B)$ $(A, B > 0)$.

4) $\log_b(A^x) = x \log_b(A)$. $(A>0, x \in \mathbb{R})$.

E.g. 1) Let $y = 6^{2x-x^2}$. Find $\frac{dy}{dx}$?

Sol. $\frac{dy}{dx} = \frac{d}{dx}\left(6^{2x-x^2}\right)$.

$$\overset{\text{chain rule}}{=} 6^{2x-x^2}(\ln 6) \cdot (2-2x)$$

$$= (\ln 6)(2-2x)\, 6^{2x-x^2}.$$

2) Let $y = \log_2(1 + 4x^{-1})$. Find the equation of the tangent line at $x = 4$.

Sol. $\frac{d}{dx}[\log_2(1 + 4x^{-1})] = ?$

$$2^y = 1 + 4x^{-1}$$

Differentiate implicitly

$$\frac{d}{dx}(2^y) = \frac{d}{dx}(1 + 4x^{-1})$$

$$(\ln 2)\, 2^y \frac{dy}{dx} = -4x^{-2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4}{x^2 (2^y)(\ln 2)}$$

$$\frac{dy}{dx} = \frac{-4}{x^2(1 + 4x^{-1}) \ln 2} = \frac{-4}{(x^2 + 4x)\ln 2}$$

At $x = 4$:

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{-4}{(4^2 + 4 \cdot 4)\ln 2} = -\frac{1}{8\ln(2)}$$

The eq. of the tangent line at $x = 4$:

$$y - f(4) = f'(4)(x - 4).$$

$$y - 1 = -\frac{1}{8\ln(2)}(x - 4).$$

- Logarithmic differentiations:

$$\boxed{\frac{d}{dx}\left[\ln(f(x))\right] = \frac{f'(x)}{f(x)}}$$

E.g. $f(x) = x^x$. Find $f'(x)$?

Let $y = x^x$. Then $\ln(y) = \ln x^x = x\ln(x)$.

Implicitly differentiate:

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[x\ln(x)].$$

$$\frac{y'}{y} = \ln(x) + x \cdot \frac{1}{x}.$$

$$\frac{y'}{y} = \ln(x) + 1.$$

$$y' = y(\ln(x) + 1)$$

$$y' = x^x(\ln x + 1).$$

* Derivatives of Hyperbolic functions:

~~st~~ $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

1) $\frac{d}{dx}(\sinh x) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2}$.

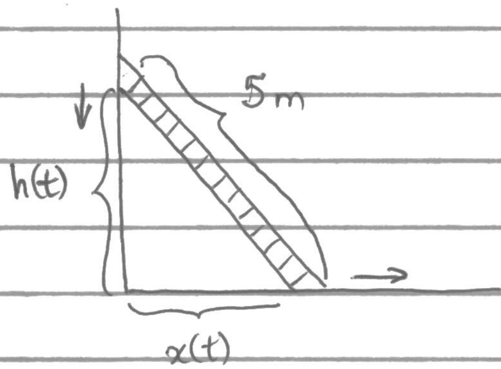
$\Rightarrow$ $\frac{d}{dx}(\sinh x) = \cosh(x)$.

2) $\frac{d}{dx}(\cosh x) = \sinh(x)$.

Eg. Find $\frac{d}{dx}(\tanh x) = \ldots = \operatorname{sech}^2 x$.

Applications of derivatives : Related Rates

Example: "the ladder problem".



Suppose a 5 m ladder slides down a wall, where the bottom is 1.5 m from the wall at time $t=0$.

Let $x(t)$ = distance from the bottom of the ladder to the wall.

$h(t)$ = height of the ladder's top.

Then $x(0) = 1.5$ m.

Q1: If the bottom moves away from the wall at 0.8 m/s, how fast will the top fall at 1 seconds?

$\Rightarrow$ $x'(t) = 0.8$ m/s.

want $h'(1)$?

First, we need to find the relationship between $x(t)$ and $h(t)$:

$\Rightarrow$ Pythagorean theorem:

$$h(t)^2 + x(t)^2 = 5^2$$

Differentiate both sides w.r.t. t:

$$\frac{d}{dt}\left[h(t)^2 + x(t)^2\right] = 0$$

chain rule $\longrightarrow$ $2h(t)\,h'(t) + 2x(t)\,x'(t) = 0.$

$$\Rightarrow \quad h(t)\,h'(t) + x(t)\,x'(t) = 0$$

substitute $t = 1$:

$$h(1)\,h'(1) + x(1)\,x'(1) = 0$$

$$\Rightarrow \quad h'(1) = -\frac{x(1)\,x'(1)}{h(1)}$$

Since the derivative of $x(t)$ is a constant,

$$x'(1) = \frac{x(1) - x(0)}{1 - 0}$$

$$0.8 = x(1) - 1.5$$

$$x(1) = 2.3$$

$$\Rightarrow \quad h^2(1) + x^2(1) = 25$$

$$h(1) = \sqrt{25 - 2.3^2}$$

$$\therefore \quad h'(1) = -\frac{2.3\,(0.8)}{\sqrt{25 - 2.3^2}} \approx -0.41 \text{ m/s}$$

negative because it is moving down the wall!

Q2: If the top slides down the wall at a rate of 1 m/s. Find the velocity of the bottom of the ladder ~~at $t = 2$~~ at the point where the top is 2 m ~~aw~~ from the floor.

$\Rightarrow$ Given: $h'(t) = -1$ m/s.

Find $x'(a)$ at $t = a$ where $h(a) = 2$ m

$$\Rightarrow \quad h(t)\, h'(t) + x(t)\, x'(t) = 25.$$

$$x'(t) = -\frac{h(t)\, h'(t)}{x(t)}$$

substitute $t = a$:

$$x'(a) = -\frac{h(a)\, h'(a)}{x(a)} = -\frac{2(-1)}{x(a)}.$$

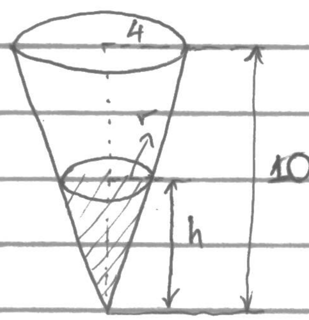
By the Pythagorean theorem:

$$x(a) = \sqrt{25 - h^2(a)} = \sqrt{25-4} = \sqrt{21}$$

$$\Rightarrow \quad x'(a) = +\frac{2}{\sqrt{21}}$$

<u>Example:</u> Filling a conical tank.

Water pours into a conical tank at the rate $6\ m^3/min$.



a) Find the rate of the water level rising when the level is 3m.

similar triangles,

$$\frac{r}{h} = \frac{4}{10}.$$

$\Rightarrow$ Let $V(t)$ = volume of the tank.
$h(t)$ = height of the water

$$\boxed{V = \frac{1}{3}\pi h r^2}$$

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$$V = \frac{1}{3}\pi h(0.4h)^2 = \frac{0.16}{3}\pi h^3.$$

$$\Rightarrow \quad V = \frac{0.16}{3}\pi h^3.$$

$$\frac{dV}{dt} = \frac{0.16}{\cancel{3}}\pi \underbrace{h^2 \frac{dh}{dt}}_{\text{chain rule.}}.$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{6}{(0.16)\pi h^2}.$$

when $h = 3$, $\quad \frac{dh}{dt} = \frac{6}{(0.16)\pi 5^2} \approx 0.48$ m/min.

Application of derivatives:
Linear Approximation.

Suppose you want to compute $\sqrt{101}$ without using a calculator. What can you do?

You notice that 101 is close to 100

and $\sqrt{100} = 10$.

$\Rightarrow$ $\sqrt{101}$ is close to 10.

But how to find an (approximate) value of $\sqrt{101}$?

If $f(x)$ is differentiable at $x = a$, then the tangent line provides an approximation for $f(x)$ in some interval a.

I.e.,

$$\boxed{\begin{cases} f(x) \approx f'(a)(x-a) + f(a) =: L(x). \\ \text{for } x \text{ near } a. \end{cases}}$$

The approximation error is given by $|f(x) - L(x)|$.
and the % error is:

$$\left(\left| \frac{f(x) - L(x)}{f(x)} \right| \cdot 100 \right) \%.$$

E.g. 1) Estimate $\sqrt{101}$.

Let $f(x) = \sqrt{x}$. We will estimate $\sqrt{101}$ with the tangent line at $x = \cancel{25}: 100$.

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(100) = \frac{1}{2}\,100^{-1/2} = \frac{1}{2\sqrt{100}} = \frac{1}{20}.$$

Then
$$L(x) = f'(100)(x-100) + f(100)$$
$$= \frac{1}{20}(x-100) + 10.$$

Thus, $\sqrt{101} \approx \frac{1}{20}(101-100) + 10 = \frac{1}{20} + 10 = \frac{201}{20}.$

2) Estimate $\sqrt{26} - \sqrt{25}$

We first estimate $\sqrt{26}$ with the tangent line at $x = 25$.

Let $f(x) = \sqrt{x}$

$$\Rightarrow f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(25) = \frac{1}{2}\,25^{-1/2} = \frac{1}{2\sqrt{25}} = \frac{1}{10}.$$

$$\Rightarrow L(x) = f'(25)(x-25) + f(25)$$
$$= \frac{1}{10}(x-25) + 5.$$

$$\Rightarrow \sqrt{26} - \sqrt{25} = f(26) - f(25)$$
$$\approx L(26) - 5 = \frac{1}{10}(26-25) = \frac{1}{10}.$$

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3) Estimate $\arctan(1.05)$. ~~Find the approximation error.~~

Note that $\arctan(1) = \frac{\pi}{4}$.

Let $f(x) = \arctan(x)$.

$f'(x) = \frac{1}{1+x^2}$.

$f'(1) = \frac{1}{2}$.

$\Rightarrow \quad L(x) = f'(1)(x-1) + f(1)$

$= \frac{1}{2}(x-1) + \frac{\pi}{4}$.

$\Rightarrow \quad f(1.05) \approx \frac{1}{2}(1.05-1) + \frac{\pi}{4} = \frac{\pi}{4} + \frac{0.05}{2}$.

~~Approximation Error:~~

$\cancel{|f(1.05) - L(1.05)| =}$

4) How much larger is $\sqrt[4]{16.1}$ is larger than $\sqrt[4]{16} = 2$.

Let $f(x) = \sqrt[4]{x} = x^{1/4}$

Find the linear approximation at $x = 16$.

$f'(x) = \frac{1}{4} x^{-3/4}$.

$\cancel{f'(2) = 1}$ $\quad f'(16) = \frac{1}{4} 16^{-3/4} = \frac{1}{4} \frac{1}{16^{3/4}} = \frac{1}{4 \cdot 2^3}$

$= \frac{1}{32}$.

$$L(x) = f'(16)(x-16) + f(16)$$
$$= \frac{1}{32}(x-16) + \sqrt[4]{16}.$$

$$\Rightarrow \sqrt[4]{16.1} - \sqrt[4]{16} \simeq \frac{1}{32}(16.1-16) + \sqrt[4]{16} - \sqrt[4]{16}$$

$$\Rightarrow \sqrt[4]{16.1} - \sqrt[4]{16} \simeq \frac{0.1}{32}.$$

5) when solving problems in geometric optics, engineers and physicists often use the simplifying assumption that, for small angle θ, $\sin(\theta)$ is approximately equal to θ. Find a linear approximation for $\sin(\theta)$ that shows why this is a reasonable assumption.

Sol. Let $f(x) = \cancel{x}\ \sin(x)$.

since θ is a small angle, the linear approximation will be about $x = 0$.

$$f'(x) = \cos(x).$$

$$\Rightarrow L(x) = \cos(0)(x-0) + \sin(0)$$
$$= 1\ (x) + 0$$
$$= x.$$

$\Rightarrow \theta$ near 0, $\sin(\theta) \approx \theta$.