# Practice Problems for Midterm II

Math 102, Winter 2018

1. Consider the vector subspace of $\mathbb{R}^4$ spanned by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) Find a basis for the orthogonal complement $V^\perp$.

   (b) Using Gramm-Schmidt, find an orthonormal basis for $V^\perp$.

   (c) Find the matrix of the orthogonal projection onto $V^\perp$ using the basis you found.

   (d) Find the projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto $V^\perp$. Derive from here the projection of the same vector onto $V$.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -4 \\ 1 & 1 \\ 1 & -4 \end{bmatrix}, \text{ and the vector } \boldsymbol{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}.$$

   (a) Find the left inverse of the matrix $A$.

   (b) Using the calculation in part (a), find the matrix of the orthogonal projection onto the column space of $A$.

   (c) Find the least squares solution to the system

$$A\boldsymbol{x} = \boldsymbol{b}$$

   using the left inverse you calculated in part (a).

   (d) Find the QR decomposition of $A$.

   (e) Now redo part (c). That is, find the least squares solution to the system

$$A\boldsymbol{x} = \boldsymbol{b}$$

   using the QR decomposition you found in part (d).

3. Calculate the determinant of the matrix

$$\begin{bmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{bmatrix}$$

   (a) using either row or column operations;

   (b) ~~using the method of cofactors.~~

4. ~~Find the inverse of the matrix~~

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

~~using the method of cofactors.~~

5. The Laguerre polynomials are important in quantum mechanics, in writing down the solution of the Schrodinger equation for the hydrogen atom.

   (a) Show that
   $$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx$$
   is an inner product on the space $\mathcal{P}$ of polynomials of degree at most 2.

   (b) Starting with the basis $\{1, x, x^2\}$, obtain an orthogonal basis for $\mathcal{P}$ using the Gram-Schmidt method. The resulting polynomials are the Laguerre polynomials.

   For this problem you may use the values of the integrals (called the gramma function):
   $$\int_0^\infty x^n e^{-x}\,dx = n!$$

6. (a) Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & 0 \\ -5 & 0 & 6 \\ 0 & 8 & 9 \end{bmatrix}$.

   (b) Find the area of the triangle with vertices at points $(1, 1), (2, 3), (-1, 5)$.

7. Use the Gram-Schmidt process to find 2 orthonormal vectors forming a basis for the column space of the matrix $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

8. True-False. Tell whether the following statements are true or false. If true, give a brief explanation and if false, give a counterexample.

   (a) Every matrix $A$ is diagonalizable (i.e., $A$ is of the form $A = PDP^{-1}$ with $D$ diagonal).

   (b) $\det(AB) = \det(BA)$.

# 1 Solution of Problem 1

(i) Let $A = \begin{bmatrix} -2 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. We have $V = C(A)$ hence

$$V^{\perp} = N(A^T).$$

Row reducing $A^T$ we find the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

whose null space is spanned by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

(ii) We apply Gram-Schmid to the basis we found above. We nornalize the first basis vector

$$u_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and compute an orthogonal vector to it

$$y_2 = v_2 - (v_2 \cdot u_1)u_1 = v_2 + \sqrt{3}u_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Normalizing the answer again, we obtain

$$u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

(iii) We let

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

whose columns are the vectors $u_1, u_2$ we found in (ii). The projection has matrix

$$AA^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix}.$$

(iv) The projection of the vector onto $V^{\perp}$ equals

$$AA^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \\ 4/3 \\ 1/3 \end{bmatrix}.$$

The projection onto $V$ and that onto $V^{\perp}$ add up to the original vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. The projection onto $V$ becomes

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2/3 \\ 4/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

## 2 Solution of Problem 2

(i) We have $A^+ = (A^TA)^{-1}A^T$. We find

$$A^TA = \begin{bmatrix} 4 & -6 \\ -6 & 34 \end{bmatrix} \implies (A^TA)^{-1} = \frac{1}{100}\begin{bmatrix} 34 & 6 \\ 6 & 4 \end{bmatrix}.$$

This yields

$$A^+ = \frac{1}{10}\begin{bmatrix} 4 & 1 & 4 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

(ii) The matrix of the orthogonal projection is

$$AA^+ = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

(iii) The least squares solution is

$$x^\star = A^+b = \frac{1}{10}\begin{bmatrix} 4 & 1 & 4 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 29/10 \\ -9/10 \end{bmatrix}$$

(iv) We run the Gramm-Schmid process for the vectors

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -4 \\ 1 \\ -4 \end{bmatrix}.$$

For the first step we have

$$q_1 = \frac{u_1}{||u_1||}$$

where $||u_1|| = 2$ hence

$$q_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

This step yields already the first row of $Q$, namely the vector $q_1$.

The second step yields the second rows of $Q$ and $R$. We first orthogonalize

$$y_2 = u_2 - (u_2 \cdot q_1)q_1.$$

We have

$$u_2 \cdot q_1 = -3$$

yielding

$$y_2 = u_2 + 3q_1 = \begin{bmatrix} 5/2 \\ -5/2 \\ 5/2 \\ -5/2 \end{bmatrix}.$$

Next, we have

$$||y_2|| = 5$$

yielding the second normalized vector

$$q_2 = \frac{y_2}{||y_2||} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

We create the matrix $Q$ from the vectors $q_1, q_2$ and the matrix $R$ from the dot products we computed during Gramm-Schmidt. We have

$$Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \quad R = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

(v) We explained in class that the least squares solution is found by solving the system

$$Rx^\star = Q^Tb$$

which in our case becomes

$$\begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix} x^\star = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/2 \\ -9/2 \end{bmatrix}.$$

This can be solved by back substitution yielding

$$5x_2^\star = -9/2 \implies x_2^\star = -9/10$$

$$2x_1^\star - 3x_2^\star = 17/2 \implies x_1^\star = 29/10.$$

## 3 Solution of Problem 3

There are many ways to do it. Try your favorite. The answer is 5.

## 4 Solution of Problem 4

There will be no such questions on the exam (no cofactor questions).

# 5 Solution of Problem 5

(i) We need to verify the axioms of inner products. There are 4 such axioms:

- $(f, g+h) = (f, g) + (f, h)$;
- $(f, g) = (g, f)$;
- $c(f, g) = (f, cg) = (cf, g)$;
- $(f, f) \geq 0$ with equality if and only if $f = 0$.

The first three axioms follow from definitions. Indeed, the first axiom reads

$$(f, g+h) = \int_0^\infty f(x)(g(x)+h(x))e^{-x}\,dx = \int_0^\infty f(x)g(x)e^{-x}\,dx + \int_0^\infty f(x)h(x)e^{-x}\,dx = (f, g)+(f, h)$$

which is clearly satisfied. The second is verified the same way:

$$(f, g) = \int_0^\infty f(x)g(x)e^{-x}\,dx = \int_0 g(x)f(x)e^{-x}\,dx = (g, f).$$

and the third is entirely similar (and left to the reader). For the last axiom, we calculate

$$(f, f) = \int_0^\infty f(x)^2 e^{-x}\,dx \geq 0$$

since we are integrating a nonnegative function $(f(x))^2 e^{-x} \geq 0$. Equality happens if and only if $(f(x))^2 e^{-x} = 0 \implies f = 0$.

(ii) Using the orthogonalization procedure for the polynomials

$$P_1 = 1, P_2 = x, P_3 = x^2$$

we find:

Step 1: $Q_1 = P_1 = 1$;

Step 2:

$$Q_2 = P_2 - \frac{(P_2, Q_1)}{(Q_1, Q_1)} Q_1.$$

We have

$$(P_2, Q_1) = \int_0^\infty x \cdot 1 \cdot e^{-x}\,dx = 1$$

and

$$(Q_1, Q_1) = \int_0^\infty 1 \cdot 1 \cdot e^{-x}\,dx = 1.$$

Here we used the values of the integral supplied by the problem. This yields

$$Q_2 = x - 1.$$

Step 3:

$$Q_3 = P_3 - \frac{(P_3, Q_1)}{(Q_1, Q_1)} Q_1 - \frac{(P_3, Q_2)}{(Q_2, Q_2)} Q_2.$$

We have

$$(P_3, Q_1) = \int_0^\infty x^2 \cdot 1 \cdot e^{-x}\,dx = 2! = 2$$

and

$$(Q_1, Q_1) = \int_0^\infty 1 \cdot 1 \cdot e^{-x}\,dx = 1.$$

Again we used the integrals provided by the text of the problem. Similarly,

$$(P_3, Q_2) = \int_0^\infty x^2 \cdot (x-1) \cdot e^{-x}\,dx = \int_0^\infty x^3 e^{-x}\,dx - \int_0^\infty x^2 e^{-x}\,dx = 3! - 2! = 4.$$

Also

$$(Q_2, Q_2) = \int_0^\infty (x-1) \cdot (x-1) \cdot e^{-x}\,dx = \int_0^\infty (x^2 - 2x + 1)e^{-x}\,dx$$
$$= \int_0^\infty x^2 e^{-x}\,dx - 2\int_0^\infty x e^{-x}\,dx + \int_0^\infty e^{-x}\,dx = 2! - 2 \cdot 1 + 1 = 1.$$

This yields

$$Q_3 = x^2 - \frac{2}{1} \cdot 1 - \frac{4}{1}(x-1) = x^2 - 4x + 2.$$

The basis of Laguerre polynomials for $\mathcal{P}$ is $\{1, x-1, x^2 - 4x + 2\}$.

## 6 Solution of Problem 6, 7, 8

See Solution of the Old 2007 Exam. There will be no eigevenvalues/eigenvectors and diagonalizable questions.