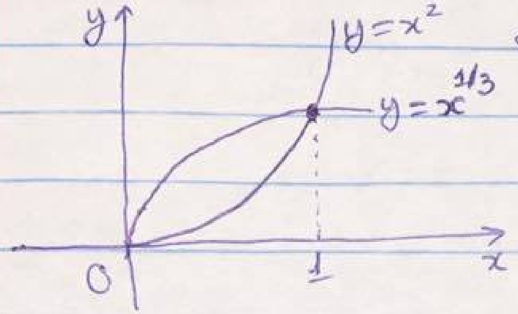
Fall '14

9) Evaluate $\iint_R 3y^2\,dA$. R first quadrant $y=x^2$ & $y=x^{1/3}$.



$y=x^2 \iff x=\sqrt{y}$.

$y=x^{1/3}$. $x=y^3$.

Find the intersection :

$$x^2 = x^{1/3}$$
$$x^6 = x$$
$$x^6 - x = 0$$
$$x(x^5-1)=0$$
$$x=0 \quad \text{or} \quad x^5-1=0$$
$$x=0 \quad \text{or} \quad x=1$$

$$\iint_R 3y^2\,dA = \int_0^1\int_{x^2}^{x^{1/3}} 3y^2\,dy\,dx$$
$$= \int_0^1 y^3\Big|_{y=x^2}^{x^{1/3}}\,dx$$
$$= \int_0^1 x - x^6\,dx$$
$$= \frac{x^2}{2} - \frac{x^7}{7}\Big|_0^1$$
$$= \frac{1}{2} - \frac{1}{7}$$
$$= \frac{5}{14}$$

Extra Q. Interchange the order:

$$\int_0^1\int_{y^3}^{\sqrt{y}} 3y^2\,dx\,dy = \int_0^1 3y^2x\Big|_{x=y^3}^{\sqrt{y}}\,dy$$

$$= \int_0^1 3y^2\sqrt{y} - 3y^5\,dy$$

$$= \int_0^1 3y^{5/2} - 3y^5\,dy$$

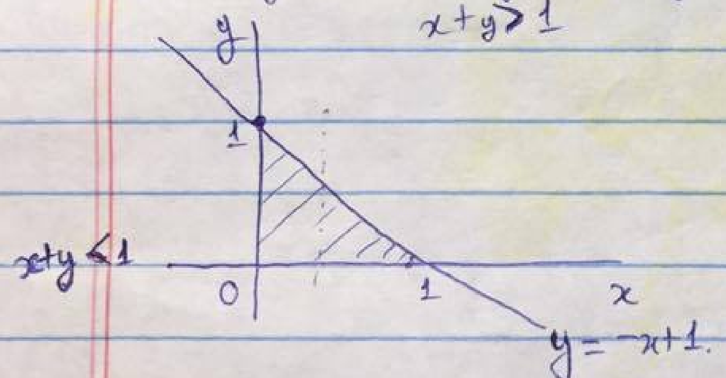
$$= 3\left(\frac{2}{7}\right)y^{7/2} - 3\,\frac{y^6}{6}\Big|_{y=0}^{1}$$

$$= \frac{6}{7} - \frac{1}{2}$$

$$= \frac{5}{14}.$$

8) Integrate $f(x,y) = 2xy + 1$ $\quad R = \{(x,y): x \geq 0,\ y \geq 0,\ x+y \leq 1\}.$

$x+y > 1$



$x + y = 1$

$y = \cancel{1} - x + 1.$

$$\iint_R f(x,y)\,dA = \int_0^1\int_0^{-x+1} (2xy+1)\,dy\,dx$$

$$= \int_0^1 xy^2 + y\Big|_{y=0}^{-x+1}\,dx$$

$$= \int_0^1 x(-x+1)^2 - x + 1\,dx.$$

$$= \int_0^1 x\left[(-x)^2 - 2x + 1\right] - x + 1\,dx$$

$$= \int_0^1 x^3 - 2x^2 + 1 \, dx$$

$$= \frac{x^4}{4} - \frac{2x^3}{3} + x \Big|_0^1$$

$$= \frac{1}{4} - \frac{2}{3} + 1$$

$$= \frac{7}{12}.$$

7) Let $f(x,y) = 2xy + 1$. Find critical points on $x^2+y^2=4$
Compute max & min. values.

<u>Sol.</u> Let $g(x,y) = x^2 + y^2$.

$$\nabla f(x,y) = (2y, 2x)$$
$$\nabla g(x,y) = (2x, 2y).$$

$$\nabla f = \lambda \nabla g.$$
$$(2y, 2x) = \lambda(2x, 2y).$$
$$\begin{cases} 2y = \lambda 2x. \\ 2x = \lambda 2y. \end{cases}$$
$$\begin{cases} y = \lambda x & (1) \\ x = \lambda y & (2) \\ x^2+y^2=4 & \end{cases}$$

Sub. (1) into (2),

$$x = \lambda(\lambda x)$$
$$x = \lambda^2 x.$$
$$x = 0 \quad \text{or} \quad \lambda = \pm 1.$$

When $x = 0$, $y = 0$. Not a sol. since $x^2 + y^2 = 4$.

When $\lambda = 1$, $\Rightarrow x = y$.

$\Rightarrow$ sub into (3): $2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x \pm \pm\sqrt{2}$

$\Rightarrow (\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

when $\lambda = -1$, $y = -x$

$\Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$.

$(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

$$\left.\begin{aligned} f(\sqrt{2}, \sqrt{2}) &= 2(\sqrt{2})(\sqrt{2}) + 1 = 5 \\ f(-\sqrt{2}, -\sqrt{2}) &= 5 \end{aligned}\right\} \text{max.}$$

$$\left.\begin{aligned} f(\sqrt{2}, -\sqrt{2}) &= -4 + 1 = -3 \\ f(-\sqrt{2}, \sqrt{2}) &= -4 + 1 = -3 \end{aligned}\right\} \text{min.}$$

<u>Extra bonus:</u> Find abs min & abs max of $f(x,y) = 2xy + 1$ on $x^2 + y^2 \leq 4$. ?

6) Let $f(x,y) = x^2 + y^3 - 12x - 3y + 15$.

$\nabla f = (3x^2 - 12, 3y^2 - 3)$.

$f_x = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$f_y = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 = 1 \qquad y = \pm 1$.

$\Rightarrow$ 4 critical points: $(2,1)$, $(2,-1)$, $(-2,1)$, $(-2,-1)$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = 0$

At $(2,1)$, $D(2,1) = 12 \cdot 6 > 0$, $f_{xx}(2,1) = 12 > 0$ $\} \Rightarrow$ local min.

At $(2,-1)$, $D(2,-1) = 12(-6) < 0 \Rightarrow$ saddle point.

At $(-2,1)$, $D(-2,1) = -12(6) < 0 \Rightarrow$ saddle point.

At $(-2,-1)$, $D(-2,-1) = 12(6) > 0$, $f_{xx} < 0 \Rightarrow$ local max.

5) $\vec{c}(t) = (5t^2, \sqrt{t}, t + \ln t) \Rightarrow \vec{c}'(t) = (10t, \frac{1}{2}t^{-1/2}, 1 + \frac{1}{t})$

$\vec{r}(t), \quad \vec{r}(1) = (2, 4, 3) \quad \vec{r}'(1) = (1, -2, -1).$

$$\frac{d}{dt}\left(\vec{c}(t)\cdot\vec{r}(t)\right)\Big|_{t=1}$$

$$\frac{d}{dt}\left(\vec{c}(t)\cdot\vec{r}(t)\right)\Big|_{t=1} = \vec{c}'(t)\cdot\vec{r}(t)\Big|_{t=1} + \vec{c}(t)\cdot\vec{r}'(t)\Big|_{t=1}$$

$$= \vec{c}'(1)\cdot\vec{r}(1) + \vec{c}(1)\cdot\vec{r}'(1)$$

$$= (10, \tfrac{1}{2}, 2)\cdot(2,4,3) + (5,1,1)\cdot(1,-2,-1)$$

$$= 20 + 2 + 6 + 5 - 2 - 1.$$

$$= \ldots$$

4) $\underbrace{xy + z^2}_{f(x,y,z)} = 7. \quad$ at $\quad (-2, 1, 3).$

$\nabla f = (y, x, 2z).$

$\nabla f(-2,1,3) = (1, -2, 6).$

$1(x+2) - 2(y-1) + 6(z-3) = 0.$

3) Let $f(x,y,z) = xy + \dfrac{x^2}{y} - e^{z^3-1}. \qquad P = (2,2,1)$

a)

2) $\vec{c}(t) = (\cos(2t), 9, e^{-2t}).$

Find eq. of tangent line at $t = 0$

Sol. $\vec{c}'(t) = (-2\sin(2t), 0, -2e^{-2t}).$

$\vec{c}'(0) = (0, 0, -2).$

$\vec{c}(0) = (1, 9, 1).$

$\vec{l}(t) = (1, 9, 1) + (t-0)\cdot(0, 0, -2).$