Lecture 12: Cosines and Projections onto Lines (Section 3.2)

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Orthogonal bases

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- Example. The standard basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.

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- Example. The set of the vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$? (Do we need to check that the three vectors are independent?)

Orthogonal bases

- Example. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V. Find $c_1, \ldots, c_n$ such that

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- Solution. Take the dot product of $\boldsymbol{v}_1$ with both sides

$$\begin{aligned} \boldsymbol{v}_1 \cdot \boldsymbol{w} &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1. \end{aligned}$$

Hence, $c_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$.

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Hence, $c_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$. In general, $c_j = \frac{\boldsymbol{v}_j \cdot \boldsymbol{w}}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}$.

Orthogonal bases

- Example. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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- **Example.** Is the basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ orthonormal?

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▶ Example. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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  It follows that $c = \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}$.
- $\boldsymbol{b}^{\perp}$ is also called the component of $\boldsymbol{b}$ orthogonal to $\boldsymbol{a}$.

Orthogonal projections

- Example. What is the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\boldsymbol{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

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- Example. What is the orthogonal projection of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Projection matrix of rank 1

Recall that the projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is

$$\widehat{\boldsymbol{b}} = \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\, \boldsymbol{a} = \underbrace{\frac{\boldsymbol{a}\,\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}}}_{P}\, \boldsymbol{b}.$$

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▶ Example. The matrix that projects onto the line through $\boldsymbol{a} = (1, 1, 1)$ is

$$P = \frac{\boldsymbol{a}\,\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Properties of projection matrices

- P is a symmetric matrix.
- Its square is itself: $P^2 = P$.
- The rank of P is 1.

Orthogonal projection onto subspaces

Theorem. Let W be a subspace of $\mathbb{R}^n$. Then, each $\boldsymbol{x}$ in $\mathbb{R}^n$ can be uniquely written as

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- Once $\hat{\boldsymbol{x}}$ is determined, $\boldsymbol{x}^{\perp} = \boldsymbol{x} - \hat{\boldsymbol{x}}$.

Orthogonal projection onto subspaces

▶ Example. Let $W = \text{span}\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, and $\boldsymbol{x} = \begin{bmatrix}0\\3\\10\end{bmatrix}$.

- Find the orthogonal projection of $\boldsymbol{x}$ onto W.
- Write $\boldsymbol{x}$ as a vector in W plus a vector orthogonal to W.

Orthogonal projection onto subspaces

- **Definition.** Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be an orthogonal basis of W, a subspace of $\mathbb{R}^n$. The projection map $\pi_W : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\pi_W(\boldsymbol{x}) = \left(\frac{\boldsymbol{x} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \right) \boldsymbol{v}_1 + \ldots + \left(\frac{\boldsymbol{x} \cdot \boldsymbol{v}_m}{\boldsymbol{v}_m \cdot \boldsymbol{v}_m} \right) \boldsymbol{v}_m$$

is linear (why?). The matrix P representing π_W with respect to the standard basis is the corresponding **projection matrix.**

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- Example. Find the projection matrix P which corresponds to orthogonal projection onto $W = \text{span}\left\{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ in $\mathbb{R}^3$. Then find the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$ onto W.