Section 7.3. Systems of Linear Equation.

Matrices $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Find A^{-1} using Gaussian Elimination.

For this class, we only focus on 2×2 (and 3×3) matrices, and 2×1 vectors.

1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ if $\det A \neq 0$ $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2) $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. (recall Wronskian).

3) if $\det(A) \neq 0$, A^{-1} exist.

4) if $\det(A) = 0$, A^{-1} does not exist.

Now let's look at linear equations:

$$A\vec{x} = \vec{b}.$$

where A is 2×2 matrix, $\vec{x}$ and $\vec{b}$ are 2×1 vectors.

i) If $\det(A) \neq 0$,

sol. exists ~~equr~~ uniquely, $x = A^{-1}b$.

ii) If $\det(A) = 0$,

there may be ~~infite~~ infinitely many sols. or there is no solution.

To find the solution, one can use Gaussian Elimination.

E.g. 1 $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

Find x such that $A\vec{x} = \vec{b}$.

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -4 & -4 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

So we have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ $x_1 = 1$ and $x_2 = 1$.

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

E.g. 2: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\vec{x}$ such that $A\vec{x} = \vec{b}$.

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\Rightarrow$ no solutions.

E.g. 3: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find $\vec{x}$ such that $A\vec{x} = \vec{b}$.

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 1.$$

Let $x_1 = c$ and $x_2 = 1-c$.

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c \\ 1-c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - c\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

infinitely many solutions.

---

* Linear Dependence v. Linear Independence

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly dependent</u>

if and only if there exists scalars $c_1, c_2, \ldots, c_n$ at least one of them nonzero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}.$$

Otherwise, they are <u>linearly independent</u>.

<u>E.g.</u>: 1) $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Since $2\vec{x} - \vec{y} = 0$, $\vec{x}$ and $\vec{y}$ are linearly dependent.

2) $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

are <u>linearly independent</u>. (why?)

Given two vectors in $\mathbb{R}^2$.

$$\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}.$$

then $\vec{v}_1, \vec{v}_2$ are linearly independent if and only if $\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$, i.e., $ad - cb \neq 0$.

$\vec{v}_1, \vec{v}_2$ are linearly dependent if $ad - cb = 0$.

* **Eigenvalues and Eigenvectors:**

If $A\vec{x} = \lambda\vec{x}$ with $\vec{x} \neq \vec{0}$,

we say $\vec{x}$ is an eigenvector of A with eigenvalue λ.

To find eigenvalue, we have $A\vec{x} = \lambda I\vec{x}$.

$\Rightarrow (A-\lambda I)\vec{x} = \vec{0}$.

$\Rightarrow \det(A-\lambda I) = \vec{0}$.

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\det(A-\lambda I) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc.$$

$$= \lambda^2 - (a+d)\lambda + ad-bc.$$

$\Rightarrow \boxed{\lambda^2 - (a+d)\lambda + ad-bc = 0.}$

Characteristic equation.

E.g. $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

Find eigenvalues and eigenvectors of A.

$$\det(A-\lambda I) = (3-\lambda)(3-\lambda) - 4.$$

$$= \lambda^2 - 6\lambda + 5$$

$$= (\lambda-5)(\lambda-1) = 0$$

$\Rightarrow \lambda_1 = 1$ and $\lambda_2 = 5$

The eigenvector corresponding to $\lambda_1 = 1$ satisfies.

$(A-\lambda_1 I)\vec{v_1} = \vec{0}$.

$\therefore \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\vec{v_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v_1} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The eigenvector corresponding to λ_2 satisfies

$$(A-\lambda_2 I)\,\vec{v}_2 = \vec{0}.$$

$$\therefore \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

we can take c_1, c_2 to be any number,
here we take $c_1 = c_2 = 1$.

So the eigenvalues of A are $1, 5$, and the corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

---

normalized vector: $\dfrac{\vec{v}}{\sqrt{\vec{v}\cdot\vec{v}}}$.

E.g. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\vec{v}\cdot\vec{v} = 2$.

$$\frac{\vec{v}}{\sqrt{\vec{v}\cdot\vec{v}}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

* Algebraic multiplicity and geometric multiplicity:

A is an $n\times n$ matrix, it has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, some of them may be repeated.

If one of the λ_i appears m times as a root to $\det(A-\lambda I)=0$, then λ_i as an eigenvalue of A has algebraic multiplicity m.

If λ_i has q linearly independent eigenvectors, we say λ_i has geometric multiplicity q.

62

E.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$.
is of algebraic multiplicity of 2.

$(A - \lambda I)\vec{v} = \vec{0}$.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{v} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two independent eigenvectors of A correspond to $\lambda = 1$.

So, $\lambda = 1$ has geometric multiplicity 2.

Section 7.4: Basic Theory of Systems of ODEs.

Example:
$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2. \end{cases} \quad (*)$$

Then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Recall that $x_1(t)$ and $x_2(t)$ are functions of t.

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

$(*) \Rightarrow \vec{x}' = A\vec{x}$.

Verify that $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

are solutions to $(*)$.

* Principle of Superposition:

If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ solutions to $\vec{x}' = A\vec{x}$, then $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a sol. to $\vec{x}' = A\vec{x}$, for arbitrary c_1 and c_2.

Q: What is the general sol. to $\vec{x}' = A\vec{x}$?

. Def: Wronskian

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \det[\vec{x}^{(1)}, \vec{x}^{(2)}].$$

If $W[\vec{x}^{(1)}, \vec{x}^{(2)}] \neq 0$ for $\alpha < t < \beta$, $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ form a fundamental set of solutions.

from our example.

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \det\begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$$

$$= e^{3t}(-2e^{-t}) - (2e^{3t})(e^{-t})$$

$$= -2e^{2t} - 2e^{2t}$$

$$= -4e^{2t} \neq 0.$$

So $\vec{x} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$ is a general solution to $(*)$.

<u>Thm:</u> If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are sol to $\vec{x}' = A\vec{x}$, on the interval $\alpha < t < \beta$, then in this interval $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}]$ either is identically zero or else never vanishes.

65

7.4

#7 $\vec{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ $\vec{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

a) Compute the Wronskian of $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$

$$W = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = t^2e^t - 2te^t = te^t(t-2)$$

b) These solutions are linearly independent on

$\boxed{t<0, \quad 0<t<2, \quad 2<t}$

since at $t=0$ and at $t=2$ the Wronskian is zero

c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$?

One or more coefficients must be discontinuous at $t=0$ and at $t=2$.

d) Need to find A so that $\vec{x}' = A\vec{x}$ for all $\vec{x} = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$.

$$\begin{bmatrix} 2t & e^t \\ 2 & e^t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t^2 & e^t \\ 2t & e^t \end{bmatrix}$$

$at^2 + b(2t) = 2t$ $\qquad (a+b)e^t = e^t$

$ct^2 + d(2t) = 2$ $\qquad (c+d)e^t = e^t$

$at + 2b = 2$ $\qquad a+b=1$

$ct + 2d = 2/t$ $\qquad c+d=1$

(66)

#7 (continued)

$at + 2b = 2 \qquad a + b = 1$

$b = 1 - a$

$at + 2(1-a) = 2$

$at + 2 - 2a = 2$

$a(t-2) = 0 \qquad a = 0, \ b = 1$

$ct + 2d = 2/t \qquad c + d = 1$

$d = 1 - c$

$ct + 2(1-c) = 2/t$

$ct + 2 - 2c = 2/t$

$c(t-2) = 2/t - 2$

$c = \frac{2-2t}{t(t-2)} \qquad c = 2\frac{1-t}{t(t-2)}$

$d = 1 - 2\frac{1-t}{t(t-2)}$

$= \frac{t(t-2) - 2 + 2t}{t(t-2)}$

$= \frac{t^2-2}{t(t-2)}$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{2-2t}{t(t-2)} & \frac{t^2-2}{t(t-2)} \end{bmatrix}$$

Section 7.5: Homogeneous linear Systems with constant coefficients.

Consider $\vec{x}' = A\vec{x}$. (*)

Recall: if $\vec{x}$ is a 1-dimensional function:

$$\frac{dx}{dt} = ax$$

$$\Rightarrow x(t) = Ce^{at}. \text{ is a sol.}$$

How to find the solution to (*)?

Try $\vec{x} = \vec{\xi} e^{rt}$, where $\vec{\xi}$ is a vector.

$\Rightarrow \vec{x}' = r\vec{\xi} e^{rt}$

and $A\vec{x} = A\vec{\xi} e^{rt}$.

So $\vec{x}' = A\vec{x}$ implies that

$r\vec{\xi} e^{rt} = A\vec{\xi} e^{rt}$

$\therefore A\vec{\xi} = r\vec{\xi}$.

$\Rightarrow r$ is an eigenvalue of A

and $\vec{\xi}$ is its corresponding eigenvector!

E.g. Solve $\vec{x}' = A\vec{x}$ with $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

Step 1: Find eigenvalues of A:

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^2 - 3^2$$

$$\Rightarrow \quad \det(A - \lambda I) = 0$$

$$(2-\lambda)^2 - 3^2 = 0$$

$$(-1-\lambda)(5-\lambda) = 0.$$

$$\lambda_1 = 5 \text{ and } \lambda_2 = -1.$$

Step 2: Find eigenvectors.

- For $\lambda_1 = 5$,

$$A - \lambda_1 I = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

$$\Rightarrow (A - \lambda_1 I)\vec{\xi} = \vec{0}.$$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

$$-3\xi_1 + 3\xi_2 = 0.$$

Take $\xi_1 = 1 \Rightarrow \xi_2 = 1$.

$$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- For $\lambda_2 = -1$,

$$A - \lambda_2 I = \begin{bmatrix} 2+1 & 3 \\ 3 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}.$$

$$(A - \lambda_2 I)\vec{\xi} = 0.$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0.$$

$$3\xi_1 + 3\xi_2 = 0.$$

Take $\xi_1 = 1$ and $\xi_2 = -1$,

$$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\Rightarrow \vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} \quad \text{and} \quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-\cancel{4}t}.$$

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \det\begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} = -2e^{4t} \neq 0.$$

$$\Rightarrow \vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$ is the general solution.

69

E.g. $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$.

Find eigenvalues:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} = 0.$$

$$(1-\lambda)(-2-\lambda) - 4 = 0.$$

$$\lambda^2 + \lambda - 6 = 0.$$

$$(\lambda-2)(\lambda+3) = 0.$$

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = -3.$$

Find eigenvectors:

$\lambda_1 = 2 \Rightarrow \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow \xi_1 = \xi_2 \Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = -3 \Rightarrow \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow 4\xi_1 + \xi_2 = 0$

take $\xi_1 = 1$ and $\xi_2 = -4$.

$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

$\Rightarrow \vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$ and $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$.

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = \det \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix} = -4e^{-t} - e^{-t} = -5e^{-t} \neq 0$$

$\Rightarrow$ General sol.

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}.$$

* Sketch a few trajectories:

* Sketch a few trajectories:

- If $c_2 = 0$, then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$.

  or $x_1 = c_1 e^{2t}$ and $x_2 = c_1 e^{2t}$.

  $\Rightarrow \quad x_1 = x_2$.

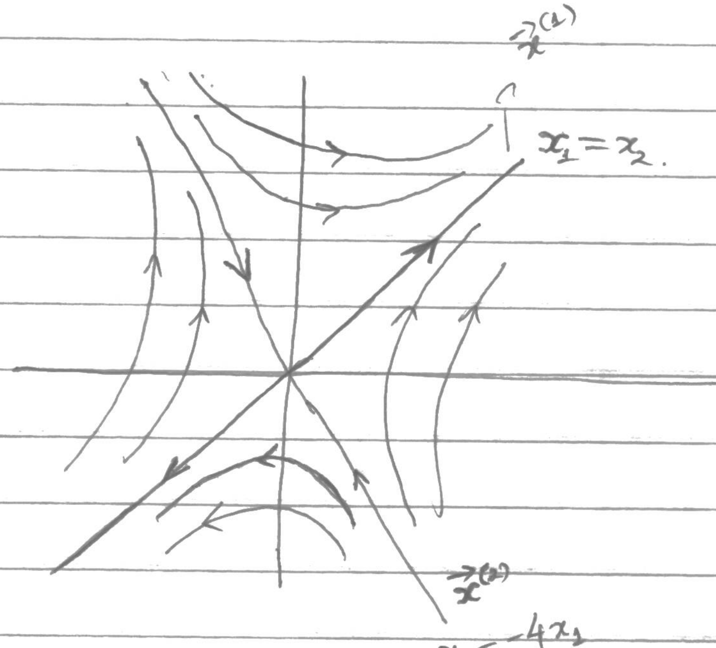
- If $c_1 = 0$, then

  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$

  or $x_1 = c_2 e^{-3t}$

  $x_2 = -c_2 4 e^{-3t}$

  $\Rightarrow x_2 = -4x_1$.



arrows shows direction of motion as t increases.

Discuss solutions as $t \to \infty$.

If $c_1 = 0$, then $\vec{x} \to \vec{0}$ as $t \to \infty$.

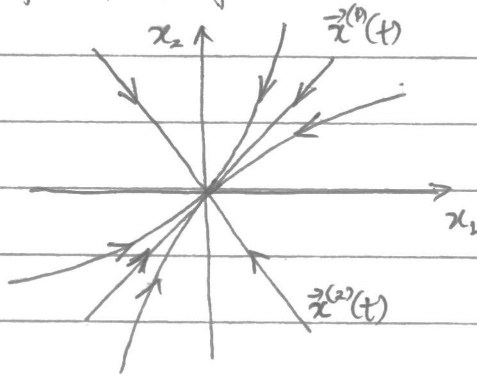
otherwise $x_1, x_2 \to +\infty$ if $c_1 > 0$

$x_1, x_2 \to -\infty$ if $c_1 < 0$.

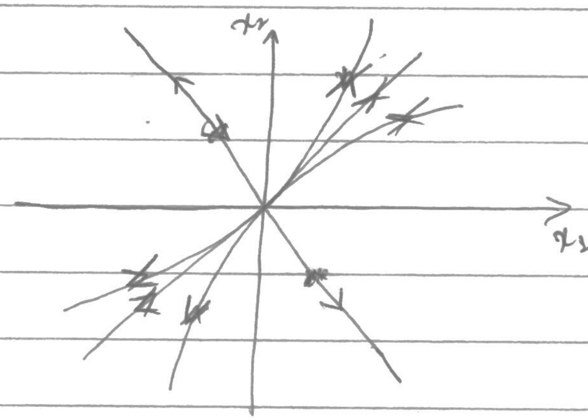
The origin is a <u>saddle point</u>.

Phase portrait of sol. to $\vec{x}' = A\vec{x}$.

* if λ_1, λ_2 of the same sign.



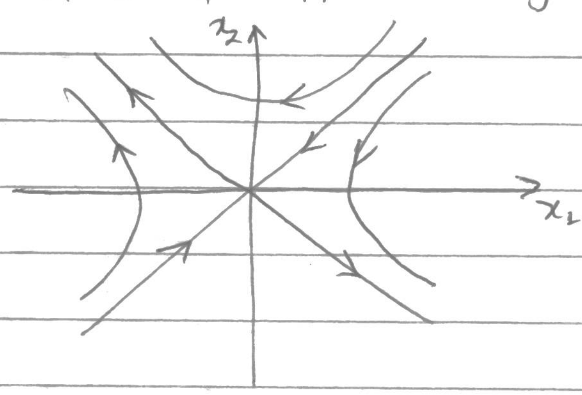
stable



unstable.

origin is called a node.

* If λ_1, λ_2 of opposite sign.



origin is called a saddle point.

Section 7.6 Complex Eigenvalues.

Characteristic Equation:

$$\det(A - \lambda I) = 0.$$

How about when roots are complex?

E.g. $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \vec{x}$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 2^2$$

$$= \lambda^2 - 2\lambda + 5$$

Characteristic Eq:

$$\lambda^2 - 2\lambda + 5 = 0.$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

For $\lambda_1 = 1 + 2i$,

$$A - \lambda_1 I = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - (1+2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(1+2i) & -2 \\ 2 & 1-(1+2i) \end{bmatrix}$$

$$= \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2i\,\xi_1 - 2\xi_2 = 0$$

Take $\xi_1 = 1$, $\xi_2 = -i$

$$\rightarrow \vec{x}^{(1)} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{t} \left(\cos(2t) + i \sin(2t) \right).$$

$$= \begin{bmatrix} \cos(2t) + i\sin(2t) \\ \sin(2t) - i\cos(2t) \end{bmatrix} e^t$$

$$= \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} e^t + \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix} e^t \, i .$$

Now let $\vec{u} = Re(\vec{x}^{(1)}) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} e^t$

and $\vec{v} = Im(\vec{x}^{(1)}) = \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix} e^t$

we can verify that $\vec{u}$ and $\vec{v}$ solves

$$\vec{x}' = A\vec{x} .$$

and

$$W[\vec{u}, \vec{v}] = \det \begin{bmatrix} \cos(2t)e^t & \sin(2t)e^t \\ \sin(2t)e^t & -\cos(2t)e^t \end{bmatrix}$$

$$= -\cos^2(2t)e^{2t} - \sin^2(2t)e^{2t}$$

$$= -e^{2t} \neq 0 .$$

$\Rightarrow$ General solution:

$$\vec{x}(t) = c_1 \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix} e^t .$$

* General procedure for complex roots:

$$\vec{x}' = A\vec{x}.$$

$$\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda = \alpha \pm \beta i.$$

Find eigenvector corresponding to $\lambda_1 = \alpha + \beta i$.

$$(A - (\alpha + \beta i)I)\vec{\xi}^{(1)} = 0$$

$$\Rightarrow \quad \vec{\xi}^{(1)} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + i\begin{bmatrix} \xi_3 \\ \xi_4 \end{bmatrix}.$$

Let $$\vec{u} = \left[\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}\cos(\beta t) - \begin{bmatrix} \xi_3 \\ \xi_4 \end{bmatrix}\sin(\beta t)\right] e^{\alpha t}.$$

$$\vec{v} = \left[\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}\overset{\sin(\beta t)}{\cancel{\cos(\beta t)}} + \begin{bmatrix} \xi_3 \\ \xi_4 \end{bmatrix}\overset{\cos(\beta t)}{\cancel{\sin(\beta t)}}\right] e^{\alpha t}.$$

→ General solution

$$\vec{x} = c_1\vec{u} + c_2\vec{v}.$$