Lecture 19: Eigenvectors and eigenvalues; Diagonalization (Sections 5.1--5.2)

Thang Huynh, UC San Diego

2/28/2018

Eigenvectors and eigenvalues

A will be an $n \times n$ matrix.

- Definition. An eigenvector of A is a nonzero $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{for some scalar } \lambda.$$

The scalar λ is the corresponding eigenvalue.

Eigenvectors and eigenvalues

A will be an $n \times n$ matrix.

- Definition. An eigenvector of A is a nonzero $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda \boldsymbol{x} \quad \text{for some scalar } \lambda.$$

The scalar λ is the corresponding eigenvalue.

- Definition. Given λ, the set of all eigenvectors with eigenvalue λ is called the eigenspace of A corresponding to λ.

How to solve $Ax = \lambda x$

Key observation:

$$\begin{aligned} A\boldsymbol{x} &= \lambda \boldsymbol{x} \\ A\boldsymbol{x} - \lambda \boldsymbol{x} &= 0 \\ (A - \lambda I)\boldsymbol{x} &= 0 \end{aligned}$$

This has a nonzero solution if and only if $\det(A - \lambda I) = 0$.

How to solve $Ax = \lambda x$

Key observation:

$$\begin{aligned} A\boldsymbol{x} &= \lambda \boldsymbol{x} \\ A\boldsymbol{x} - \lambda \boldsymbol{x} &= 0 \\ (A - \lambda I)\boldsymbol{x} &= 0 \end{aligned}$$

This has a nonzero solution if and only if $\det(A - \lambda I) = 0$.

- Recipe. To find eigenvectors and eigenvalues of A.
  - First, find the eigenvalues λ using $\det(A - \lambda I) = 0$.
  - Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = 0$.

How to solve $Ax = \lambda x$

- Example. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

How to solve $Ax = \lambda x$

- Example. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Solution. The characteristic polynomial is

$\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda) \implies A$ has eigenvalues $2, 3, 6.$

How to solve $Ax = \lambda x$

- Example. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Solution. The characteristic polynomial is

$\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda) \Longrightarrow A$ has eigenvalues $2, 3, 6$.

  - For $\lambda_1 = 2$, $(A - 2I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_1 = (2, -5/2, 1)^T$.

How to solve $Ax = \lambda x$

- **Example.** Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Solution. The characteristic polynomial is

$$\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda) \Longrightarrow A \text{ has eigenvalues } 2, 3, 6.$$

  - For $\lambda_1 = 2$, $(A - 2I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_1 = (2, -5/2, 1)^T$.
  - For $\lambda_2 = 3$, $(A - 3I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_2 = (1, 0, 0)^T$.

How to solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$

- **Example.** Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Solution. The characteristic polynomial is

$\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda) \Longrightarrow A$ has eigenvalues $2, 3, 6$.

  - For $\lambda_1 = 2$, $(A - 2I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_1 = (2, -5/2, 1)^T$.
  - For $\lambda_2 = 3$, $(A - 3I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_2 = (1, 0, 0)^T$.
  - For $\lambda_3 = 6$, $(A - 6I)\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_3 = (2, 3, 0)^T$.

(Note that these three eigenvectors are linearly independent.)

Eigenvectors

> Theorem. If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are eigenvectors A corresponding to *different eigenvalues*, then they are linearly independent.

Algebraic Multiplicity

- Example. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Algebraic Multiplicity

- Example. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

- Solution. The characteristic polynomial is

$\det(A - \lambda I) = (2 - \lambda)(\lambda - 2)(\lambda - 4)$. A has eigenvalues $2, 2, 4$. Since $\lambda = 2$ is a double root, it has algebraic multiplicity 2. (It's an exercise to find eigenvectors.)

Eigenvalues and Eigenvectors

An $n \times n$ matrix A has up to n different eigenvalues.

- For each eigenvalue λ, A has at least one eigenvector.
- If λ has multiplicity m, then A has up to m (independent) eigenvectors for λ.

Eigenvalues and Eigenvectors

Ideally, we would like to find a total of n (independent) eigenvectors of A.

Eigenvalues and Eigenvectors

Ideally, we would like to find a total of n (independent) eigenvectors of A. But there are two sources of trouble: eigenvalues can be

- complex numbers or
- repeated roots of characteristic polynomial.

Eigenvalues and Eigenvectors

- Example. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Geometrically, what is the trouble?)

Eigenvalues and Eigenvectors

- Example. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Geometrically, what is the trouble?)
- Solution. The characteristic polynomial $\det(A - \lambda I) = \lambda^2 + 1 \Longrightarrow \lambda_1 = i$ and $\lambda_2 = -i$.

Eigenvalues and Eigenvectors

- **Example.** Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Geometrically, what is the trouble?)
- Solution. The characteristic polynomial $\det(A - \lambda I) = \lambda^2 + 1 \implies \lambda_1 = i$ and $\lambda_2 = -i$.
  - For $\lambda_1 = i$, $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

- Example. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Geometrically, what is the trouble?)
- Solution. The characteristic polynomial $\det(A - \lambda I) = \lambda^2 + 1 \implies \lambda_1 = i$ and $\lambda_2 = -i$.
  - For $\lambda_1 = i, \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.
  - For $\lambda_2 = -i, \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

- Example. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

▶ Example. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

▶ Solution. The characteristic polynomial $\det(A - \lambda I) = (1 - \lambda)^2$. So $\lambda = 1$ is the only eigenvalue (of multiplicity 2).

$$(A - \lambda I)\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} = 0 \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So the eigenspace is span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. Only dimension 1!

Diagonalization

Diagonal matrices are very easy to work with.

- Example. For instance, it is easy to compute their powers.

Let's consider $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then

Diagonalization

Diagonal matrices are very easy to work with.

- Example. For instance, it is easy to compute their powers.

Let's consider $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 2^2 & & \\ & 3^2 & \\ & & 4^2 \end{bmatrix} \quad \text{and} \quad A^{100} = \begin{bmatrix} 2^{100} & & \\ & 3^{100} & \\ & & 4^{100} \end{bmatrix}.$$

Diagonalization

- Example. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$

Diagonalization

▶ Example. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$ Let's find eigenvalues and eigenvectors of A:

- $\lambda_1 = 4 \Longrightarrow$ eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- $\lambda_2 = 5 \Longrightarrow$ eigenvector $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Diagonalization

▶ Example. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$ Let's find eigenvalues and eigenvectors of A:

- $\lambda_1 = 4 \Longrightarrow$ eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- $\lambda_2 = 5 \Longrightarrow$ eigenvector $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Key observation: if $\boldsymbol{v}$ is an eigenvector of A corresponding to an eigenvalue λ,

$$A^m \boldsymbol{v} = \lambda^m \boldsymbol{v}.$$

Diagonalization

Let $B = A^{100} = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Then $\boldsymbol{b}_1 = A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{b}_2 = A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Diagonalization

Let $B = A^{100} = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Then $\boldsymbol{b}_1 = A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{b}_2 = A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Therefore, finding A^{100} is equivalent to finding $A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Diagonalization

Let $B = A^{100} = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Then $\boldsymbol{b}_1 = A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{b}_2 = A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Therefore, finding A^{100} is equivalent to finding $A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Observe that

$$\begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix} = -\boldsymbol{v}_1 + 2\boldsymbol{v}_2.$$

$$A^{100}\begin{bmatrix}1\\0\end{bmatrix} = -4^{100}\begin{bmatrix}1\\2\end{bmatrix} + 2\cdot 5^{100}\begin{bmatrix}1\\1\end{bmatrix}.$$

Diagonalization

Let $B = A^{100} = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Then $\boldsymbol{b}_1 = A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{b}_2 = A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Therefore, finding A^{100} is equivalent to finding $A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Observe that

$$\begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix} = -\boldsymbol{v}_1 + 2\boldsymbol{v}_2.$$

$$A^{100}\begin{bmatrix}1\\0\end{bmatrix} = -4^{100}\begin{bmatrix}1\\2\end{bmatrix} + 2\cdot 5^{100}\begin{bmatrix}1\\1\end{bmatrix}.$$

Exercise: find $A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.