Name: ______________________ PID: ____________

TA: ________________ Sec. No: _____ Sec. Time: _____

**Math 20A.**
**Midterm Exam 2**
**November 18, 2010**

*Turn off and put away your cell phone.*
*No calculators or any other electronic devices are allowed during this exam.*
*You may use one page of notes, but no books or other assistance during this exam.*
*Read each question carefully, and answer each question completely.*
*Show all of your work; no credit will be given for unsupported answers.*
*Write your solutions clearly and legibly; no credit will be given for illegible solutions.*
*If any question is not clear, ask for clarification.*

| # | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 6 | |
| 3 | 6 | |
| 4 | 6 | |
| 5 | 12 | |
| $\Sigma$ | 36 | |

1. (6 points) Differentiate the following functions; you need not simplify.

(a) $f(x) = \ln\left(\sin^3(x) + 1\right)$

$$f'(x) = \frac{1}{\sin^3(x)+1} \left(\sin^3(x)+1\right)'$$

Chain rule.

$$= \frac{3\sin^2(x)\cos(x)}{\sin^3(x)+1}$$

(b) $g(x) = x^{2\sin(x)}$

First, take ~~log~~ ln both sides.

$$\ln g(x) = \ln x^{2\sin(x)}$$

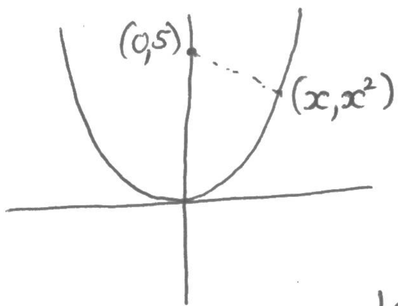
$$\ln g(x) = 2\sin(x)\ln(x).$$

Differentiate both sides with respect to $x$

$$\frac{1}{g(x)} g'(x) = 2\cos(x)\ln(x) + \frac{2\sin(x)}{x}.$$

$$g'(x) = 2x^{2\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right).$$

2. (6 points) Find the points on the parabola $y = x^2$ that are closest to the point $(0, 5)$.




The distance from $(0,5)$ to a point $(x, x^2)$ on the parabola is

$$\sqrt{(0-x)^2 + (5-x^2)^2}.$$

Let
$$f(x) = (0-x)^2 + (5-x^2)^2$$
$$= x^2 + 25 - 10x^2 + x^4.$$
$$= x^4 - 9x^2 + 25.$$

We want to ~~mai~~ find the absolute minimum of $f(x)$.

- Method 1: Completing the square:

$$f(x) = \left(x^2 - \frac{9}{2}\right)^2 + \frac{19}{4}.$$

$\Rightarrow$ $f$ is minimized at $x^2 = \frac{9}{2}$ $\Rightarrow$ $x = \pm\sqrt{\frac{9}{2}}$.

$\Rightarrow$ the points we want to find are $\left(\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$ and $\left(-\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$.

- Method 2:

$$f'(x) = 4x^3 - 18x$$

$f'(x) = 0 \Rightarrow$
$$4x^3 - 18x = 0.$$
$$2x(2x^2 - 9) = 0.$$
$x = 0, \pm\sqrt{\frac{9}{2}}$ are critical points.

$$f''(x) = 12x^2 - 18.$$

$f''(0) = -18 < 0 \Rightarrow f(0)$ is local max.

$f''\left(\sqrt{\frac{9}{2}}\right) = f''\left(-\sqrt{\frac{9}{2}}\right) = 12\left(\frac{9}{2}\right) - 18 > 0 \Rightarrow f\left(\sqrt{\frac{9}{2}}\right)$ and $f\left(-\sqrt{\frac{9}{2}}\right)$ are local ~~max~~. min.

$f\left(\sqrt{\frac{9}{2}}\right) = f\left(-\sqrt{\frac{9}{2}}\right) = \frac{19}{4}$. They are in fact absolute min.

$\Rightarrow$ $\left(\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$ and $\left(-\sqrt{\frac{9}{2}}, \frac{9}{2}\right)$ are the points we want to find.

3. (6 points) Find all points on the graph of $y^2 + 2x^2 - xy = 14$ where the tangent line is horizontal.

Differentiate both sides w.r.t. $x$:

$$2y \frac{dy}{dx} + 4x - y - x\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 4x.$$

$$\frac{dy}{dx} = \frac{y-4x}{2y-x}.$$

The tangent line is horizontal if $\frac{dy}{dx} = 0$.

$$\Rightarrow \quad \frac{y-4x}{2y-x} = 0.$$

$$y = 4x.$$

Substitute $y = 4x$ into $y^2 + 2x^2 - xy = 14$,

$$(4x)^2 + 2x^2 - x(4x) = 14.$$

$$16x^2 + 2x^2 - 4x^2 = 14.$$

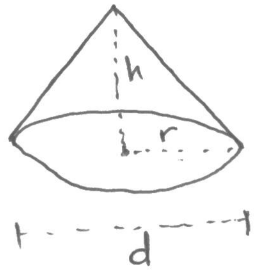
$$14x^2 = 14$$

$$x^2 = 1.$$

$$x = \pm 1.$$

$\Rightarrow$ The points are $(1, 4)$ and $(-1, -4)$.

4. (6 points) Sand is pouring from a pipe at the rate of 16 cubic feet per second. The falling sand forms a conical pile whose height is always $\frac{1}{4}$ the diameter of the base. How fast is the height of the sand pile increasing when the pile is 4 feet high?




$$h = \frac{1}{4} d.$$

$$\Rightarrow \quad r = \frac{d}{2} = \frac{4h}{2} = 2h.$$

$$V = \frac{1}{3}\pi r^2 h. = \frac{1}{3}\pi (2h)^2 h = \frac{4}{3}\pi h^3.$$

$$V'(t) = 16.$$

We need to find $h'(t)$ at the instant when $h = 4$.

Diff. $\quad V(t) = \frac{4}{3}\pi h^3(t) \quad$ both sides. w.r.t. $t$.

$$V'(t) = \frac{4}{3}\pi \underbrace{3h^2(t)\, h'(t)}_{\text{chain rule.}}$$

$$\Rightarrow \quad h'(t) = \frac{V'(t)}{4\pi h^2(t)}.$$

$$= \frac{16}{4\pi\, 4^2}$$

$$= \frac{1}{4\pi}.$$

5. (12 points) Let $h(x) = x^3 - 3x + 1$.

(a) Find the intervals on which $h$ is increasing and decreasing.

$h'(x) = 3x^2 - 3$

$h'(x) = 0 \Rightarrow x = \pm 1.$

| $x$ | Test value | sign of $h'$ |
|---|---|---|
| $(-\infty, -1)$ | $h'(-2) = 9$ | + |
| $(-1, 1)$ | $h'(0) = -3$ | − |
| $(1, \infty)$ | $h'(2) = 9$ | + |

$\Rightarrow$ $h$ is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$

(b) Find the local maxima and local minima of $h$ and the points where they occur.

$\nearrow \searrow \nearrow$

local max $h(-1) = -1 + 3 + 1 = 3.$
at $(-1, 3).$

local $\overset{\text{min}}{\cancel{\text{max}}}$ $h(1) = 1 - 3 + 1 = -1.$
at $(1, -1).$

(c) Find the absolute maximum and absolute minimum of $h$ over the interval $[-2, 2]$ and the points where they occur.

Since $-1$ and $1$ (critical points) are both in $[-2, 2]$, we want to find

$h(-1) = 3$

$h(1) = -1$

$h(-2) = -8 + 6 + 1 = -1.$

$h(2) = 8 - 6 + 1 = 3$

on $[-2, 2]$:
$\Rightarrow$ absolute max $h(-1) = 3$ at $(-1, 3)$
max $h(2) = 3$ at $(2, 3)$
absolute min $h(1) = -1$ at $(1, -1)$
$h(-2) = -1$ at $(-2, -1)$

(d) Find the intervals on which the graph of $h$ is concave up and concave down and find the inflection points.

$h''(x) = 6x.$

$h''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0.$

| Interval | test value | sign of $h''$. |
|---|---|---|
| $(-\infty, 0)$ | $h''(-1) = -6$ | − |
| 0 | $h''(0) = 0$ | 0 |
| $(0, \infty)$ | $h''(1) = 6$ | + |

$\Rightarrow$ $h$ is concave $\overset{\text{down}}{\cancel{\text{up}}}$ on $(-\infty, 0)$ and concave up on $(0, \infty)$ and $(0, 1)$ is the inflection point.

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Midterm Exam 2
November 20, 2012
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# Version A

**Instructions**

1. **No calculators or other electronic devices are allowed during this exam.**
2. **You may use one page of notes, but no books or other assistance during this exam.**
3. **Write your *Name*, *PID*, and *Section* on the front of your Blue Book.**
4. **Write the *Version* of your exam at the top of the page on the front of your Blue Book.**
5. **Write your solutions clearly in your Blue Book**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order they appear in the exam.**
   (c) **Start each question on a new side of a page.**
6. **Read each question carefully, and answer each question completely.**
7. **Show all of your work; no credit will be given for unsupported answers.**

0. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard.

1. (6 points) Find the derivatives of:
   (a) $f(x) = \ln(\sin(x) + 2)$
   (b) $f(x) = (\cos(x) + 2)^x$

2. (6 points) If $g(x) + 2x\sin(g(x)) = x^2$ and $g(1) = 0$, find $g'(1)$.

3. (6 points) Let $f(x) = x^3 + 6x^2 + 9x + 1$.
   (a) Find and classify all of the critical points of $f(x)$.
   (b) Find the global maximum and global minimum of $f(x)$ on the interval $[-2, 0]$.

4. (3 points) Compute $\lim_{x\to\infty} x^3 e^{-2x^2}$.

5. (6 points) The total volume of a Truffula Tree after $t$ years is given by $V(t) = \pi\, t\, h$, where $h$ is the height of the tree, which changes over time. The surface area of a Truffula Tree is given by $A(t) = \frac{\pi}{2} h$. Currently, for a ten-year old Truffula Tree, the measured rate of change of its volume is 200 $\text{ft}^3/\text{yr}$, and the measured rate of change of its surface area is 1 $\text{ft}^2/\text{yr}$. How high is the Truffula Tree?

Midterm 2 — 2012.

1) a) $f(x) = \ln(\sin(x) + 2)$.

$$f'(x) = \frac{1}{\sin(x)+2} \cdot \cos(x).$$

b) $f(x) = (\cos(x) + 2)^x$

$$\ln f(x) = x \ln(\cos(x) + 2).$$

Differentiate both sides w.r.t. $x$

$$\frac{f'(x)}{f(x)} = \ln(\cos(x) + 2) - \frac{x\sin(x)}{\cos(x)+2}.$$

$$f'(x) = (\cos(x) + 2)^x \left( \ln(\cos(x)+2) - \frac{x\sin(x)}{\cos(x)+2} \right).$$

2) Let $y = g(x)$. Then

$$y + 2x\sin(y) = x^2.$$

Implicitly differentiate both sides w.r.t. $x$:

$$\frac{dy}{dx} + 2\sin(y) + 2x\cos(y)\frac{dy}{dx} = 2x.$$

$$(2x\cos(y) + 1)\frac{dy}{dx} = 2x - 2\sin(y).$$

$$\frac{dy}{dx} = \frac{2x - 2\sin(y)}{2x\cos(y) + 1}.$$

$$\Rightarrow \quad g'(x) = \frac{2x - 2\sin(g(x))}{2x\cos(g(x)) + 1}.$$

$$g'(1) = \frac{2(1) - 2\sin(g(1))}{2(1)\cos(g(1)) + 1} = \frac{2 - 2\sin(0)}{2\cos(0) + 1} = \frac{2}{3}.$$

3) $f(x) = x^3 + 6x^2 + 9x + 1$. $\quad Dom(f) = \mathbb{R}$.

a) $f'(x) = 3x^2 + 12x + 9$.

$f'(x) = 0 \Rightarrow 3x^2 + 12x + 9 = 0$.

$x^2 + 4x + 3 = 0$.

$(x+1)(x+3) = 0$.

$x = -1$ and $x = -3$.

are critical points.

| $x$ | Test value | sign of $f'$ |
|---|---|---|
| $(-\infty, -3)$ | $f'(-4) = +9$ | $+$ |
| $-3$ | $f'(-3) = 0$ | $0$ |
| $(-3, -1)$ | $f'(-2) = -3$ | $-$ |
| ~~$(-1, \infty)$~~ | | |
| $-1$ | $f'(-1) = 0$ | $0$ |
| $(-1, \infty)$ | $f'(0) = 9$ | $+$ |

$\Rightarrow f(-3)$ is local max.

$f(-1)$ is local min.

b) Since $-1 \in [-2, 0]$,

$f(-2) = -8 + 24 - 18 + 1 = -1$.

$f(0) = \boxed{1} \rightarrow$ absolute max

$f(-1) = -1 + 6 - 9 + 1 = \boxed{-3} \rightarrow$ absolute min.

4) $\lim\limits_{x\to\infty} x^3 e^{-2x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{2x^2}}$ $\qquad \dfrac{\infty}{\infty}$.

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{3x^2}{e^{2x^2}(4x)}$$

$$= \frac{3}{4} \lim_{x\to\infty} \frac{x}{e^{2x^2}} \qquad \frac{\infty}{\infty}.$$

$$\overset{\text{L'Hôpital}}{=} \frac{3}{4} \lim_{x\to\infty} \frac{1}{e^{2x^2}(4x)}$$

$$= \frac{3}{16} \lim_{x\to\infty} \frac{1}{x e^{2x^2}}$$

$$= \infty.$$

5) $V(t) = \pi t h$. $\qquad A(t) = \frac{\pi}{2} h$.

$V'(10) = 200 \text{ ft}^3/\text{yr}$. $\quad$ and $\quad A'(10) = 1 \text{ ft}^2/\text{yr}$.

Since $A(t) = \frac{\pi}{2} h(t)$, $A'(t) = \frac{\pi}{2} h'(t)$.

$$\Rightarrow \cancel{A} \; h'(t) = \frac{2}{\pi} A'(t)$$

$$h'(10) = \frac{2}{\pi} A'(10) = \frac{2}{\pi}.$$

Since $V(t) = \pi t h$, $V'(t) = \pi h(t) + \pi t h'(t)$.

$$\Rightarrow h(t) = \frac{V'(t) - \pi t h'(t)}{\pi}.$$

$$h(10) = \frac{V'(10) - \pi\, 10\, h'(10)}{\pi}$$

$$= \frac{200 - \pi\, 10 \left(\frac{2}{\pi}\right)}{\pi}$$

$$= \frac{180}{\pi} \text{ ft}.$$