HW01

Math 102, Winter 2018

Due Week 2

Reading: Review vector norms, matrix norms, orthogonality, projections.

1. Sketch the unit circle $\{\boldsymbol{x}, \|\boldsymbol{x}\|_p = 1\}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ for $p = 1, 2$, and ∞.

2. (a) Write the definition of the vector norm $\|\boldsymbol{x}\|_2$.

   (b) Show that if Q is an orthogonal matrix, then $\|Q\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$.

   (c) Let $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$ and

$$Q = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad \text{(this is a Hadamard matrix).}$$

   Without calculating $Q\boldsymbol{x}$ directly, what is the value of $\|Q\boldsymbol{x}\|_2$?

3. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^m$, the matrix $A = I + \boldsymbol{u}\boldsymbol{v}^T$ is know as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\boldsymbol{u}\boldsymbol{v}^T$ for some scalar α, and give an expression for α. For what $\boldsymbol{u}$ and $\boldsymbol{v}$ is A singular? If it is singular, what is $\mathrm{Null}(A)$?

4. Given $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$, show that if $E = \boldsymbol{u}\boldsymbol{v}^T$, then $\|E\|_2 = \|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2$. Is the same true for the Frobenius norm, i.e., $\|E\|_F = \|\boldsymbol{u}\|_F\|\boldsymbol{v}\|_F$? Prove it or give a counterexample.

5. Consider the matrix
$$A = \begin{bmatrix} -2 & 3 & 2 \\ -4 & 5 & 1 \\ 1 & -2 & 4 \end{bmatrix}.$$
   What are the ℓ^1, ℓ^2, ℓ^∞-, and Frobenius norms of A?

6. Given $A \in \mathbb{R}^{m\times n}$ with $m \geq n$, show that A^TA is nonsingular if and only if A has full rank.

7. Consider the matrices
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   (a) What is the orthogonal projector P onto $\mathrm{range}(A)$, and what is the image under P of the vector $(1, 2, 3)^T$?

   (b) Same questions for B.