HW03 — Solution.

1) $C(A) = \text{span}\left\{\begin{bmatrix}1\\1\\5\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}\right\}$ and $N(A) \ni \begin{bmatrix}1\\1\\2\end{bmatrix}$

$\Rightarrow \dim C(A) = 2$ $\qquad\qquad \dim N(A) \geq 1.$

since the above two vectors are linearly independent

therefore, A has at least 3 colums. because

$\dim C(A) + \dim N(A) =$ the number of columns of A.

Consider

$$A = \begin{bmatrix}1 & 0 & a_1\\1 & 3 & a_2\\5 & 1 & a_3\end{bmatrix}.$$

Since $\begin{bmatrix}1\\1\\2\end{bmatrix} \in N(A)$, $\begin{bmatrix}1 & 0 & a_1\\1 & 3 & a_2\\5 & 1 & a_3\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$

$$\Rightarrow \begin{cases} 2a_1 = 0 \\ 1 + 3 + 2a_2 = 0 \\ 5 + 1 + 2a_3 = 0 \end{cases} \Rightarrow \quad \begin{aligned} a_1 &= 0 \\ a_2 &= -2 \\ a_3 &= -3 \end{aligned}$$

$$\therefore \quad A = \begin{bmatrix}1 & 0 & 0\\1 & 3 & -2\\5 & 1 & -3\end{bmatrix}.$$

2) $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

In augmented form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 4 \\ 2 & 5 & 7 & 6 & 3 \\ 2 & 3 & 5 & 2 & 5 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 4 \\ 0 & 3 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 \to 3R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 4 \\ 0 & 3 & 3 & 2 & -5 \\ 0 & 0 & 0 & -8 & -4 \end{array}\right] \xrightarrow[\substack{R_2 \to \frac{1}{3}R_2 \\ R_3 \to -\frac{1}{8}R_3}]{R_1 \to R_1 - \frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & 1 & 1 & 2/3 & -5/3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \quad (*)$$

a) $\Rightarrow$ pivot columns of A are the first, ~~third~~ second, and fourth.

$\Rightarrow$ $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \right\}$.

Use echelon form, the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in $N(A)$ if.

$$\cancel{\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 0 \\ 3x_2 + 3x_3 + 2x_4 &= 0 \\ -8x_4 &= 0. \end{aligned}} \Rightarrow \begin{cases} x_1 + x_3 + \frac{4}{3}x_4 = 0 \\ x_2 + x_3 + \frac{2}{3}x_4 = 0 \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_4 = 0. \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

$\Rightarrow$ $N(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

b) To solve $A\vec{x} = \vec{b}$, from $(*)$ we have

$$\begin{cases} x_1 + x_3 + \frac{4}{3}x_4 = \frac{17}{3} \\ x_2 + x_3 + \frac{2}{3}x_4 = -\frac{5}{3} \\ x_4 = \frac{1}{2} \end{cases} \Rightarrow \begin{aligned} x_1 &= \frac{17}{3} - \frac{4}{3} \cdot \frac{1}{2} - x_3 = 5 - x_3 \\ x_2 &= -\frac{5}{3} - \frac{2}{3} \cdot \frac{1}{2} - x_3 = -2 - x_3 \\ x_4 &= \frac{1}{2} \end{aligned}$$

Solution: $\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 0 \\ 1/2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ for any $x_3 \in \mathbb{R}$.

3) $x + 2y - 3z - t = 0$.

The plane contains all points (x, y, z, t) which are solutions of the equation

$$\begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0.$$

Hence, the plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$

We can find three linearly independent vectors on the plane by finding these vectors of the nullspace of A.

Recall that A only has 1 pivot and 3 free variables.

$\Rightarrow$ any vector $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in N(A)$ satisfies

$$x = -2y + 3z + t.$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2y+3z+t \\ y \\ z \\ t \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and these three vectors are linearly independent.

The plane cannot contain 4 linearly independent vectors. because if it does. The plane is exactly the space $\mathbb{R}^4$. That is, any vector in $\mathbb{R}^4$ has to be on the plane too. But we can find many points which are in $\mathbb{R}^4$ but are not on the plane. For example, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, etc.

4) a)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A is already in the reduced echelon form.

A has two pivot columns (the second and third).

$\Rightarrow \quad C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$

$C(A^T) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$ the row space of A.

To find $N(A)$, we need to solve $Ax = 0$

$\Rightarrow \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ is free variable.} \end{cases}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$

The left nullspace of A, i.e., $N(A^T)$, consists of those vectors $\vec{y}$ such that $A^T\vec{y} = 0$. Since

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ y_3 is free variable. and $y_1 = y_2 = 0$.

$\Rightarrow$ solution of $A^T\vec{y} = 0$ is of the form.

$$\begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$\Rightarrow \quad N(A^T) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

b) $I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$C(I+A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ since $I+A$ has 3 pivot columns.

$C(I^T + A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\dim N(I+A) = 0 \Rightarrow N(I+A) = \{\vec{0}\}$.

$\dim N(I^T + A^T) = 0 \Rightarrow N(I^T + A^T) = \{\vec{0}\}$.