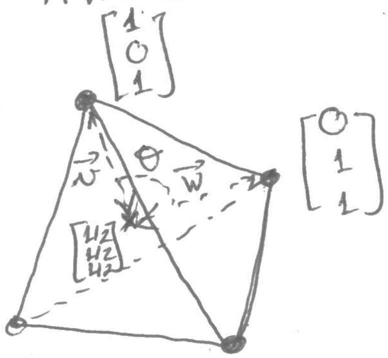
HW05 – Solution.

1)



$$\vec{v} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1/2\end{bmatrix}$$

$$\vec{w} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1/2\end{bmatrix}$$

We know that

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

$$\Rightarrow \quad \cos\theta = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \, \|\vec{w}\|}$$

$$= \frac{\frac{1}{2}\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)}{\sqrt{(1/2)^2 + (-1/2)^2 + (1/2)^2}\ \sqrt{(-1/2)^2 + (1/2)^2 + (1/2)^2}}$$

$$= \frac{-1/4}{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= -\frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

2) First, we need to find a vector on the line of intersection. Any such vector is necessarily a solution of the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using Reduced Row Reduction, we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

$$\Rightarrow \quad \cancel{x - y =} \quad \begin{aligned} x - t &= 0 \\ y + 2t &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= t \\ y &= -2t. \end{aligned}$$

Therefore, solutions to the matrix equation are of the form

$$t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The vector $\vec{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is the direction vector of the intersection line.

$\Rightarrow$ The projection matrix P is

$$P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

3) $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

a) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\Rightarrow$ the row space of A is equal to V.

That is, $V = C(A^T)$.

By FTLA, $C(A^T)$ is the orthogonal complement of $N(A)$.

$$\Rightarrow \quad V^\perp = N(A).$$

Hence, we need to find $N(A)$. by solving

$$A\vec{x} = \vec{0}.$$

$$\Rightarrow \quad x_1 = -x_2 - x_4$$
$$x_3 = 0.$$

Therefore, the solution of the homogeneous equation are of the form

$$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \quad N(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

b) The projection matrix P onto V.

From part a, we have that V is the row space of A or, equivalently, V is the column space of

$$B = A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow$ The projection matrix P onto $V = C(B)$ is

$$P = B(B^T B)^{-1} B^T =$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

c) $\vec{b} = (0, 1, 0, -1)$.

The closest vector to $\vec{b}$ in V will necessarily be the projection of $\vec{b}$ onto V. Since $\vec{b}$ is perpendicular to V, we know this will be a zero vector.

Check:

$$P\vec{b} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4) If the data actually lay on a straight line $y = C + Dt$, we would have

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

But this system is not solvable.

$\Rightarrow$ we want to find $\hat{x}$ such that $A\hat{x}$ is as close as possible to $\vec{b}$. This happens when

$$\hat{x} = (A^TA)^{-1} A^T\vec{b}.$$

Now,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

$$\Rightarrow (A^TA)^{-1} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix}$$

$$\rightarrow \quad \hat{x} = (A^TA)^{-1}A^Tb$$

$$= \begin{bmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -3/10 \\ -12/5 \end{bmatrix}.$$

Therefore, the best fit line for the data is

$$y = -\frac{3}{10} - \frac{12}{5}t.$$