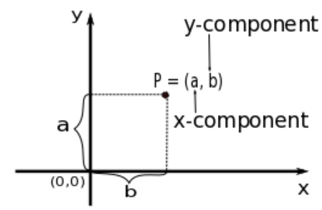
1.1 Vectors in Two- and Three-Dimensional Space

Spring 17, UCSD

- We have already learned about *differentiation* (computing rates of change, tangents to curves, velocities, ...), *integration* (areas under curves, ...), and the relationship between them (Fundamental Theorem of Calculus).
- But we were working with functions of *one variable* only.
- We live in a 3-dimensional world (4, if you count time) and many interesting functions depend not just on one variable but *many*.
- We would ultimately like to generalize differentiation and integration, and understand their relationship and their applications when we have **many variables**.



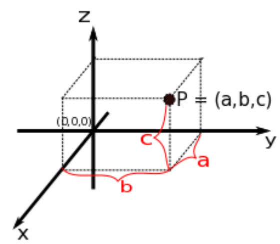
Vector Addition and Scalar Multiplication

Define: $(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$.

So $(1, 1, 2) + (-1, 7, \pi) = (0, 8, 2 + \pi)$.

Define: $\underbrace{\alpha}_{\text{scalar}} \underbrace{(a, b, c)}_{\text{vector}} = (\alpha a, \alpha b, \alpha c)$.

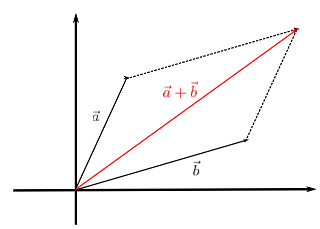
So $2(1, -7, 5) = (2, -14, 10)$.



Properties:

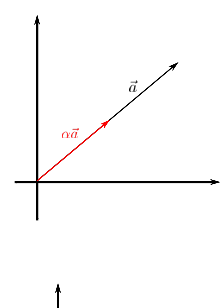
- $\alpha(0, 0, 0) = (0, 0, 0)$
- $0(a, b, c) = (0, 0, 0)$
- $1(a, b, c) = (a, b, c)$
- $(\alpha\beta)(a, b, c) = \alpha(\beta(a, b, c))$ (associativity)
- $(\alpha + \beta)(a, b, c) = \alpha(a, b, c) + \beta(a, b, c)$ (distribitivity)
- $\alpha[(a_1, a_2, a_3) + (b_1, b_2, b_3)] = \alpha(a_1, a_2, a_3) + \alpha(b_1, b_2, b_3)$

(Proofs follow "easily" from the definitions)



Geometry of Vector Operations

Vectors: Directed line segments, with a beginning (tail) and an ending (head). *Translating a vector gives you the same vector.*



Standard Basis Vectors $\vec{i}, \vec{j}$, and $\vec{k}$

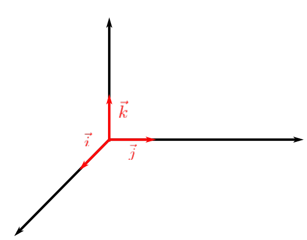
It will be convenient to give names to the following vectors

- $\vec{i} = (1, 0, 0)$
- $\vec{j} = (0, 1, 0)$
- $\vec{k} = (0, 0, 1)$

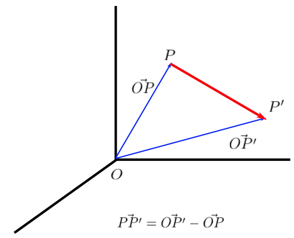
So

$$\begin{aligned}(3, 4, 5) &= 3(1, 0, 0) + 4(0, 1, 0) + 5(0, 0, 1)\\ &= 3\vec{i} + 4\vec{j} + 5\vec{k}.\end{aligned}$$

In general, $(a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$.



The vector joining 2 points: If the point P has coordinates (x, y, z) and the point P' has coordinates (x', y', z'). Then the vector $\vec{PP'}$ has components $(x' - x, y' - y, z' - z)$.

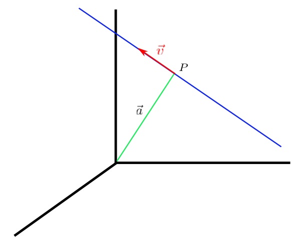


Example: Add the vector from $(1, 1, 2)$ to $(2, 2, 3)$ to the vector from $(0, 0, 1)$ to $(-7, -5, -3)$.

Solution. The first vector is $(2, 2, 3) - (1, 1, 2) = (1, 1, 1)$, and the second vector is $(-7, -5, -3) - (0, 0, 1) = (-7, -5, -4)$. Hence, their sum is $(1, 1, 1) + (-7, -5, -4) = (-6, -4, -3) = -6\vec{i} - 4\vec{j} - 3\vec{k}$.

Equations of Lines:

Equations of a line ℓ passing through a point P with the direction of a vector $\vec{v}$.



We can see that the point $\vec{a} + \vec{v}$ is on the line ℓ, $\vec{a} + 2\vec{v}$ is also on the line, $\vec{a} - 0.5\vec{v}$ is also on ℓ. In fact, any point on ℓ is of the form $\vec{a} + t\vec{v}$ for some scalar t.

Point-Direction in Form of a Line:

The equation of the line ℓ passing through the tip of $\vec{a} = (a_1, a_2, a_3)$ in the direction of $\vec{v} = (v_1, v_2, v_3)$ is

$$\ell(t) = \vec{a} + t\vec{v}.$$

In coordinate form, the equations are

$$\begin{cases} x = a_1 + v_1 t \\ y = a_2 + v_2 t \\ z = a_3 + v_3 t \end{cases}$$

Remark. When working in 2D we simply drop the z-coordinate.

Example. Find the equation of the line passing through the point $(1, 1, 2)$ in the direction $2\vec{i} - 3\vec{j} + 4\vec{k}$.

Solution. $\ell(t) = \vec{a} + t\vec{v}$ where $\vec{a} = (1, 1, 2)$ and $\vec{v} = (2, -3, 4)$, so

$$\begin{cases} x = 1 + 2t \\ y = 1 - 3t \\ z = 2 + 4t \end{cases}$$

Example. In what direction does the line

$$\begin{cases} x = -3t + 2 \\ y = -2t + 2 \\ z = 8t + 2 \end{cases}$$

point? Answer. It points in the direction of $\vec{v} = (-3, -2, 8)$.

Remark. Equation of a line is *not* unique. (Why?)

Point-point form of a line

Want an equation of a line passing through the endpoints of 2 vectors $\vec{a}$ and $\vec{b}$. To do this, note that the vector $\vec{b} - \vec{a}$ is in the direction of the line. So

$$\ell(t) = \vec{a} + (\vec{b} - \vec{a})t \text{ or } \ell(t) = (1 - t)\vec{a} + t\vec{b}.$$

Rewriting in coordinate form:

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

(Parametric equation of a line in point-point form)

Remark. We can eliminate t to get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(provided the denominators are not zero).

Example. Find the equation of the line passing through $(4, 5, 6)$ and $(2, 1, 2)$.

Solution.

$$\begin{aligned} x &= 4 + (2 - 4)t \\ y &= 5 + (1 - 5)t \\ z &= 6 + (2 - 6)t \end{aligned}$$

or

$$\ell(t) = (4, 5, 6) + t(-2, -4, -4).$$

As two different lines through the origin determine a plane through the origin, so do two nonparalell vectors. Given two vectors $\vec{v}$ and $\vec{w}$, the equation of the plane spanned by $\vec{v}$ and $\vec{w}$ is

$$\ell(s, t) = s\vec{v} + t\vec{w}.$$

Example. What is the equation of the plane spanned by $(1, 0, 0)$ and $(0, 1, 0)$?

Solution. The equation is

$$\ell(s, t) = s(1, 0, 0) + t(0, 1, 0) = (s, t, 0).$$

This is the xy-plane.

Example. What is the equation of the plane spanned by $(1, -1, 2)$ and $(0, 1, 3)$?

Solution. The equation is

$$\ell(s, t) = s(1, -1, 2) + t(0, 1, 3) = (s, -s + t, 2s + 3t).$$