## * Complex Numbers (S.1)

Q: How do we solve equations like

$$x^2 + 1 = 0 \ ?$$

A: Introduce a new number!

$$\boxed{\begin{array}{l} i \text{ is the number with the property} \\ \qquad i^2 = -1 \iff i = \sqrt{-1} \end{array}}$$

"imaginary number"

Complex ~~m~~ number: (e.g. $3 + i5$, or $-7 + i3$)
is a number of the form $a + ib$ where

real numbers
$\underset{\sim}{a}$ is called the real part
$\underset{\sim}{b}$ is called the imaginary part.

Complex numbers have many important applications (signal processing, quantum mechanics, etc.)

## * Properties:

- Addition: $(a + ib\cancel{i}) \pm (c + id\cancel{i}) = (a + c) \pm (b + d)i$
  (& subtraction)

  e.g. $(4 + 3i) - (2 - 5i) = 2 + 8i$.

- Multiplication: (like polynomials)

$$\begin{aligned} (a + ib)(c + id) &= ac + iad + ibc + i^2 bd. \\ &= ac + i(ad + bc) - 1 \cdot bd. \end{aligned}$$

$$\Rightarrow (a + ib)(c + id) = ac - bd + i(ad + bc).$$

- Division:

$$\begin{aligned} \frac{a + ib}{c + id} &= \frac{(a + ib)(c - id)}{(c + id)(c - id)} \\ &= \frac{(a + ib)(c - id)}{c^2 + d^2} \end{aligned}$$

* **Def:** Complex conjugate.

Given a complex number $\alpha = a + ib$, it's complex conjugate is $\boxed{\bar{\alpha} = a - ib}$

* Properties of the complex conjugate:

- $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$ ~~eg.~~

  e.g. $\overline{(3+i4) + (2+i5)} = (3-i4) + (2-i5)$.

- $\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}$

  e.g. $\overline{(1+i2)(2+i3)} = (1-i2)(2-i3)$.

  $= 2 - 6 - i7$

  $= -4 - i7$.

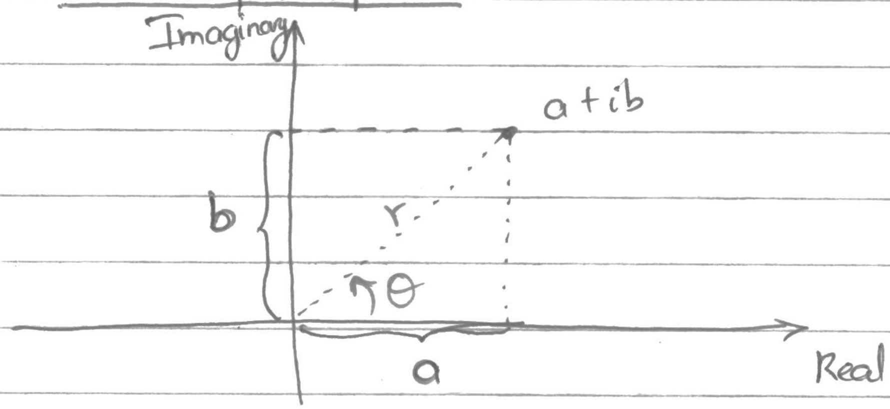
- $\overline{\left(\frac{\alpha}{\beta}\right)} = \frac{\bar{\alpha}}{\bar{\beta}}$.

* **Def:** The magnitude of a complex number.

$$|\alpha| = |a + ib| = \sqrt{a^2 + b^2}.$$

e.g. $|3 + i4| = \sqrt{3^2 + 4^2} = 5$.

* The complex plane:




Sometimes, it's convenient to represent complex numbers in "polar coordinates".

That is, $r = \sqrt{a^2 + b^2} = |a + ib|$

$\theta$ is called the argument of $a + ib$.

E.g. Write $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ in polar form.

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

$$\Rightarrow \quad \frac{1}{2} + i\frac{\sqrt{3}}{2} = 1\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right].$$

Polar form is useful for multiplying complex number. Suppose we want to multiply

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then $z_1 z_2 = r_1 r_2 \left[\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + i\left(\underbrace{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}_{\sin(\theta_1+\theta_2)}\right)\right]$

$$\Rightarrow \boxed{z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)}$$

Similarly, $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$.

Can also show

$$z = r[\cos\theta + i\sin\theta] \Rightarrow \boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

De Moivre's theorem.

E.g. We have $\sqrt{3} + i \overset{\text{check!}}{=} 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

$$\Rightarrow (\sqrt{3}+i)^{17} = 2^{17}\left(\cos\frac{17\pi}{6} + i\sin\frac{17\pi}{6}\right)$$

$$= 2^{17}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= 2^{17}\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$$

$$= 2^{16}(-\sqrt{3} + i).$$

* <u>Roots of a complex number.</u>

The <u>nth root</u> of $z$ is a complex number $w$ such that

$\boxed{w^n = z}$

So if $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$ we have

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta).$$

$$\Rightarrow \quad s = r^{1/n} \qquad \phi = \frac{\theta + 2\pi k}{n} \quad \text{where } \boxed{k = 0, 1, \dots, n-1}$$

So there are n-roots:

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

nth root of $r(\cos\theta + i\sin\theta)$.

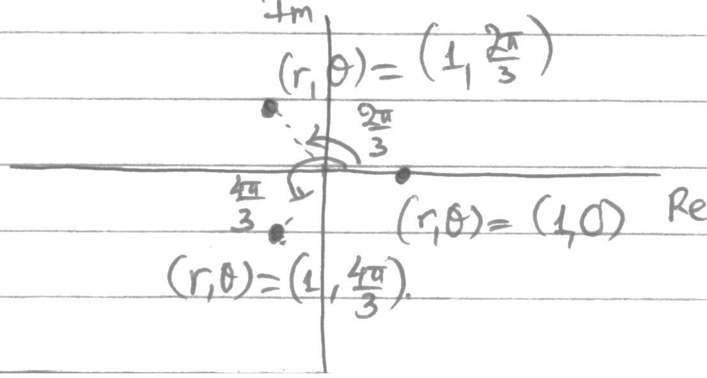
<u>E.g.</u> Find the $3^{rd}$ roots of 1. (called the $3^{rd}$ roots of unity).

<u>Sol:</u> $1 = 1(\cos 0 + i\sin 0)$.

$$\Rightarrow \quad w_k = 1^{1/3}\left[\cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right)\right] \quad \text{for } k = 0, 1, 2.$$

So $w_0 = 1$, $w_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$w_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.



* Complex Exponentials:

Def: $$e^{a+\theta i} = e^{a}(\cos\theta + i\sin\theta)$$
where $a$ and $\theta$ are real ~~[illegible]~~ numbers.

With this definition:
$$e^{a_1+\theta_1 i}\, e^{a_2+\theta_2 i} \underset{\uparrow\,(\text{why?})}{=} e^{(a_1+a_2)+(\theta_1+\theta_2)i}.$$

so $e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$ where $z_1$ and $z_2$ are complex.

Euler's formula:
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$$

Consequences of Euler's formula:

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \text{and } \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned}\right.$$

very helpful for deriving trig. identities.

E.g. $$\cos^2\theta = \cos\theta\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$= \frac{e^{i2\theta}+2+e^{-i2\theta}}{4}$$
$$= \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$
$$= \frac{1+\cos(2\theta)}{2}.$$

Exercise: show that $e^{i\pi}+1=0$.

* Trigonometric Integrals (7.2)
and Integrals involving hyperbolic function (7.4)

- E.g. Evaluate $\int \cos(3x)\sin(x)\,dx$.

1) Method 1: Use trig. identities and sub / int. by parts (see textbook)
2) Method 2: Convert $\cos(3x)$ and $\sin(x)$ to complex exponentials and integrate.

$$\int \cos(3x)\sin(x)\,dx = \int \left(\frac{e^{i3x}+e^{-i3x}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx$$

$$= \int \frac{e^{i4x}-e^{i2x}+e^{-i2x}-e^{-i4x}}{4i}\,dx$$

$$= \int \frac{(e^{i4x}-e^{-i4x})-(e^{i2x}-e^{-i2x})}{4i}\,dx$$

integrate normally

$$= \frac{1}{4i}\left[\left(\frac{e^{i4x}}{i4}-\frac{e^{-i4x}}{-4i}\right)-\left(\frac{e^{i2x}}{2i}-\frac{e^{-i2x}}{-i2}\right)\right]+C$$

$$= \frac{1}{4i}\left[\left(\frac{e^{i4x}+e^{-i4x}}{4i}\right)-\left(\frac{e^{i2x}+e^{-i2x}}{2i}\right)\right]+C$$

$$= \frac{1}{4i}\left(\frac{\cos(4x)}{2i}-\frac{\cos(2x)}{i}\right)+C.$$

$$= \frac{\cos 4x}{-8}+\frac{\cos 2x}{4}+C.$$

- E.g. Evaluate $\int e^{2x}\sin x\,dx$

Note that you can do it by parts.

$$\int e^{2x}\sin x\,dx = \int e^{2x}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx$$

$$= \frac{1}{2i}\int\left(e^{(2+i)x} - e^{(2-i)x}\right)dx.$$

$$= \frac{1}{2i}\left[\frac{e^{(2+i)x}}{2+i} - \frac{e^{(2-i)x}}{2-i}\right] + C.$$

$$= \frac{1}{2i}\left[\frac{e^{2x}e^{ix}}{2+i} - \frac{e^{2x}e^{-ix}}{2-i}\right] + C$$

$$= \frac{e^{2x}}{2}\left[\frac{e^{ix}}{-1+2i} - \frac{e^{-ix}}{1+2i}\right] + C$$

$$= \frac{e^{2x}}{2}\left[\frac{e^{ix}(-1-2i)}{5} - \frac{e^{-ix}(1-2i)}{5}\right] + C$$

$$= \frac{e^{2x}}{2}\left[\frac{-e^{ix} - 2ie^{ix} - e^{-ix} + 2ie^{-ix}}{5}\right] + C$$

$$= \frac{e^{2x}}{2}\left[\frac{-(e^{ix}+e^{-ix}) - 2i(e^{ix}-e^{-ix})}{5}\right] + C$$

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\sin x = \frac{e^{ix}-e^{-ix}}{2}$

$$= \frac{e^{2x}}{2}\left[\frac{-2\cos x + 4\sin x}{5}\right] + C.$$

* Hyperbolic sine and cosine:

$$\boxed{\cosh(x) := \frac{e^{x}+e^{-x}}{2}} \quad \text{and} \quad \boxed{\sinh(x) := \frac{e^{x}-e^{-x}}{2}}$$

E.g. Evaluate $I = \int \cos^2(x)\sinh^2(x)\,dx$.

Sol. $\cos^2(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 = \frac{e^{i2x}+e^{-i2x}+2}{4}$

$$= \frac{1}{2}\left(\frac{e^{i2x}+e^{-i2x}}{2} + 1\right).$$

Hyperbolic sine and cosine:

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \quad \text{etc.}$$

$$\Rightarrow \quad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = \sinh(x).$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x). \qquad \text{etc.}$$

and $\int \sinh(x)\,dx = \cosh(x) + C$, $\int \cosh(x)\,dx = \sinh(x) + C$

$$\int \operatorname{sech}^2(x)\,dx = \tanh^2(x) + C.$$

E.g. Calculate $\int x^4 \sinh(x^5 + 2)\,dx$.

Sol. Use sub.

let $u = x^5 + 2$, $du = 5x^4\,dx$.

$$\int x^4 \sinh(x^5 + 2)\,dx = \int \frac{\sinh(u)}{5}\,du = \frac{1}{5}\cosh(u) + C$$

$$= \frac{1}{5}\cosh(x^5+2) + C$$

Helpful identity: $\cosh^2(x) = 1 + \sinh^2(x)$.

or $\cosh^2(x) - \sinh^2(x) = 1$.

E.g. $\int \sinh^3(x)\cosh(x)\,dx$

Sol: Let $u = \sinh(x)$, $du = \cosh(x)\,dx$.

$$\int \sinh^3(x)\cosh(x)\,dx = \int u^3\,du = \frac{u^4}{4} + C$$

$$= \frac{\sinh^4(x)}{4} + C$$

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E.g. Calculate $\int \sin(x)\cosh(x)\,dx$.

either use int. by parts (exercise).

or Use exponentials.

$$\int \sin(x)\cosh(x)\,dx = \int \left(\frac{e^{ix}-e^{-ix}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)dx$$

$$= \frac{1}{4}\int e^{(1+i)x} + e^{(-1+i)x} - e^{(1-i)x} - e^{-(1+i)x}\,dx.$$

$$= \frac{1}{4}\left[\frac{e^{(1+i)x}}{1+i} + \frac{e^{(-1+i)x}}{-1+i} - \frac{e^{(1-i)x}}{1-i} - \frac{e^{-(1+i)x}}{-1-i}\right] + C$$

$= \ldots$ continue!

## Section 7.5. The method of Partial Fractions.

Use to integrate rational functions

$$f(x) = \frac{P(x)}{Q(x)} \longrightarrow \text{Polynomials.}$$

* Some useful facts:

**Fundamental theorem of Algebra:**

Any nonconstant polynomial

$n$ = degree of $Q(x)$

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^{n}$$

can be factored as

$$Q(x) = \underbrace{c}_{\text{complex constant}}(x-\beta_1)^{n_1}(x-\beta_2)^{n_2}\dots(x-\beta_k)^{n_k}.$$

complex

- The $\beta_j$'s are called the zeros of $Q(x)$.
- The $n_j$'s are called the multiplicity of the $\beta_j$.

Examples:

- $2x^2 - 8x + 6 = 2(x-3)(x-1)$
- $x^3 - 8 = (x-2)(x^2 + 2x + 4)$
  $= (x-2)(x+1+i\sqrt{3})(x+1-i\sqrt{3}).$
- $(x^2+1)^3 = (x-i)^3(x+i)^3$

  $-i$ is a zero of multiplicity 3.

**Fundamental theorem of Algebra: real factors.**

Any nonconstant polynomial with real coefficient's

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

can be factored as

$$Q(x) = c(x-\underset{\text{real}}{r_1})^{m_1}\dots(x-\underset{\text{real}}{r_k})^{m_k}\underbrace{(x^2+b_1x+c_1)^{n_1}\dots(x^2+b_lx+c_l)^{n_l}}_{\text{real with no real roots}}$$

E.g. Evaluate $\int \frac{dx}{x^2 - 3x + 2}$.

Sol: • Step 1: Factor.

$$I = \int \frac{dx}{x^2 - 3x + 2} = \int \frac{1}{(x-2)(x-1)} dx$$

• Step 2: write the partial fraction expansion.

$$I = \int \frac{A}{x-2} + \frac{B}{x-1} dx.$$

and find A & B:

$$\frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x-1) + B(x-2)}{(x-2)(x-1)} = \frac{(A+B)x - A - 2B}{(x-2)(x-1)} \underset{\text{we want this}}{=} \frac{1}{(x-2)(x-1)}$$

$$\Rightarrow (A+B)x + (-A-2B) = 0x + 1.$$

$$\Rightarrow \left.\begin{array}{l} A+B = 0 \\ \text{and } -A-2B = 1 \end{array}\right\} \begin{array}{l} A = -B = 1 \\ B = -1 \end{array}$$

• Step 3: Substitute and integrate:

$$I = \int \frac{1}{x-2} + \frac{-1}{x-1} dx$$

$$= \int \frac{1}{x-2} dx - \int \frac{1}{x-1} dx.$$

$$= \ln|x-2| - \ln|x-1| + C.$$

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Evaluate: $I = \int \frac{x^2+2}{(x-1)(2x-8)(x+2)}\,dx$.

Sol: Since the bottom is already factored, we go straight to the partial fraction expansion.

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2}.$$

Find A, B, and C:

$$x^2+2 = A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8)$$

To find A, set $x=1$:

$$3 = A(-6)(3) + 0 + 0.$$

$$\Rightarrow A = -\frac{1}{6}.$$

To find B, set $x=4$:

$$18 = B(3)(6)$$

$$\Rightarrow B = 1.$$

To find C, set $x=-2$:

$$6 = C(-3)(-12)$$

$$\Rightarrow C = \frac{1}{6}.$$

So $I = \int \frac{-1/6}{x-1} + \frac{1}{2x-8} + \frac{1/6}{x+2}\,dx.$

$$= -\frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|x-4| + \frac{1}{6}\ln|x+2| + C.$$

Remark: In both these examples, $\frac{P(x)}{Q(x)}$ was proper that $\deg(P) < \deg(Q)$.

* Example (Here $\deg(P) \geq \deg(Q)$).

Evaluate $\quad I = \displaystyle\int \frac{x^3+1}{x^2-4}\,dx.$

Here, since $\deg(P) \geq \deg(Q)$ we have to do long division

$$\begin{array}{r|l} & \quad x \\ \hline x^2-4 & x^3+1 \\ & x^3-4x \\ \hline & \quad 4x+1. \end{array}$$

$$\Rightarrow \quad \frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4}.$$

$$\text{So } I = \int \left(x + \frac{4x+1}{x^2-4}\right) dx$$

$$= \int x\,dx + \int \frac{4x+1}{(x-2)(x+2)}\,dx.$$

$$= \frac{x^2}{2} + \int \frac{A}{x-2} + \frac{B}{x+2}\,dx.$$

$$= \frac{x^2}{2} + \int \frac{9/4}{x-2} + \frac{7/4}{x+2}\,dx.$$

$$= \frac{x^2}{2} + \frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + C.$$

* Example: Evaluate $\quad I = \displaystyle\int \frac{3x-9}{(x-1)\underbrace{(x+2)^2}_{\text{repeated linear factor}}}\,dx.$

1) $\displaystyle\frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$

note these two terms.

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To find A, B, C:

$$3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1).$$

Plug in $x=-2$: $\Rightarrow$ $C=5$

Plug in $x=1$: $\Rightarrow$ $A=-2/3$.

To find B, plug in some other number

$$x=0 \Rightarrow -9 = 4A + -2B + -C.$$

$$B = \frac{+9+4A-C}{2} = \frac{2}{3}.$$

$$\Rightarrow I = \int -\frac{2/3}{x-1}\,dx + \int \frac{2/3}{x+2}\,dx + \int \frac{5}{(x+2)^2}\,dx$$

$$= -\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + \frac{-5}{x+2} + C.$$

Example:

$$I = \int \frac{18}{(x+3)\underbrace{(x^2+19)}_{\text{irreducible quadratic function.}}}\,dx$$

$$\frac{18}{(x+3)(x^2+19)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9}.$$

$$\Rightarrow 18 = A(x^2+9) + (Bx+C)(x+3).$$

~~= A + B)~~

Set $x=-3$, then $A=1$.

$$\Rightarrow 18 = (B+1)x^2 + (3B+C)x + (3C+9)$$

so $B=-1$ because $B+1=0$

and $C=3$. $3B+C=0$

$$\Rightarrow I = \int \frac{1}{x+3}\,dx + \int \frac{-x+3}{x^2+9}\,dx = \ln|x+3| - \int \frac{x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

$$= \ln|x+3| - \frac{1}{2}\ln|x^2+9| + 3\tan^{-1}\left(\frac{x}{3}\right) + C$$

- Summary: $\frac{P(x)}{Q(x)}$.

($a_i$'s are distinct)

1) $Q(x) = (x-a_1)(x-a_2)\dots(x-a_n)$, then

$$\frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}.$$

To calculate the constants, clear denominators and substitute, in turn, the values $x = a_1, a_2, \dots, a_n$.

E.g. Evaluate $I = \int \frac{4x-9}{(x-2)(x-3)}\,dx$

$$\frac{4x-9}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

~~Set $x=2$,~~

$$4x - 9 = A(x-3) + B(x-2)$$

set $x = 2$, then $-1 = -A \Rightarrow A = 1$.

$x = 3$, then $3 = B \Rightarrow B = 3$.

$$I = \int \frac{1}{x-2}\,dx + \int \frac{3}{x-3}\,dx = \ln|x-2| + 3\ln|x-3| + C.$$

2) If $Q(x)$ has $(x-a)^M$ where $M > 2$,

the partial fraction has:

$$\frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \dots + \frac{B_M}{(x-a)^M}.$$

If $Q(x)$ has $(x^2+b)^N$ where $N \geq 1$,

the partial fraction has

$$\frac{A_1x+B_1}{x^2+b} + \frac{A_2x+B_2}{(x^2+b)^2} + \dots + \frac{A_Nx+B_N}{(x^2+b)^2}.$$

E.g. $I = \displaystyle\int \frac{6x^2+9x+1}{(x-1)(x+1)^2}\,dx.$

$$\frac{6x^2+9x+1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

$\Rightarrow \quad 6x^2+9x+1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1).$

Take $x = 1$, then

$$16 = 4A \Rightarrow A = 4.$$

Take $x = -1$, then

$$-2 = -2C \Rightarrow C = 1.$$

Take $x = 0$,

$$1 = A - B \cancel{+} - C$$

$$\Rightarrow B = A - C - 1 = 4 - 1 - 1 = 2.$$

$$\Rightarrow I = \int \frac{4}{x-1}\,dx + \int \frac{2}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx$$

$$= 4\ln|x-1| + 2\ln|x+1| \cancel{+} - \frac{1}{x+1} + \text{Constant}$$

E.g. $I = \displaystyle\int \frac{\cancel{10}\;25}{x(x^2+25)}\,dx.$

$$\frac{25}{x(x^2+25)} = \frac{A}{x} + \frac{Bx+C}{x^2+25}.$$

$\Rightarrow \quad \cancel{1}\;25 = A(x^2+25) + (Bx+C)x.$

Take $x = 0$,

$$25 = 25A \Rightarrow A = \cancel{1/25}\;1.$$

And

$$\cancel{1}\;25 = (\cancel{\tfrac{1}{25}+}\,(B+1)x^2 + Cx + 25A$$

$\Rightarrow \quad B + 1 = 0 \quad \Rightarrow \quad B = -1$

and $\quad C = 0$

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$$I = \int \frac{1}{x}\,dx + \int \frac{-x}{x^2+25}\,dx.$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+25| + \text{Constant}.$$

3) If $\deg(P) \geq \deg(Q)$, use long division.

## Section 7.7 Improper Integrals

$$\int_{-\infty}^{a} f(x)\,dx, \quad \int_{a}^{\infty} f(x)\,dx, \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\,dx$$

But what do they mean?

We define $\displaystyle\int_{-\infty}^{a} f(x)\,dx = \lim_{c\to-\infty}\int_{c}^{a} f(x)\,dx$

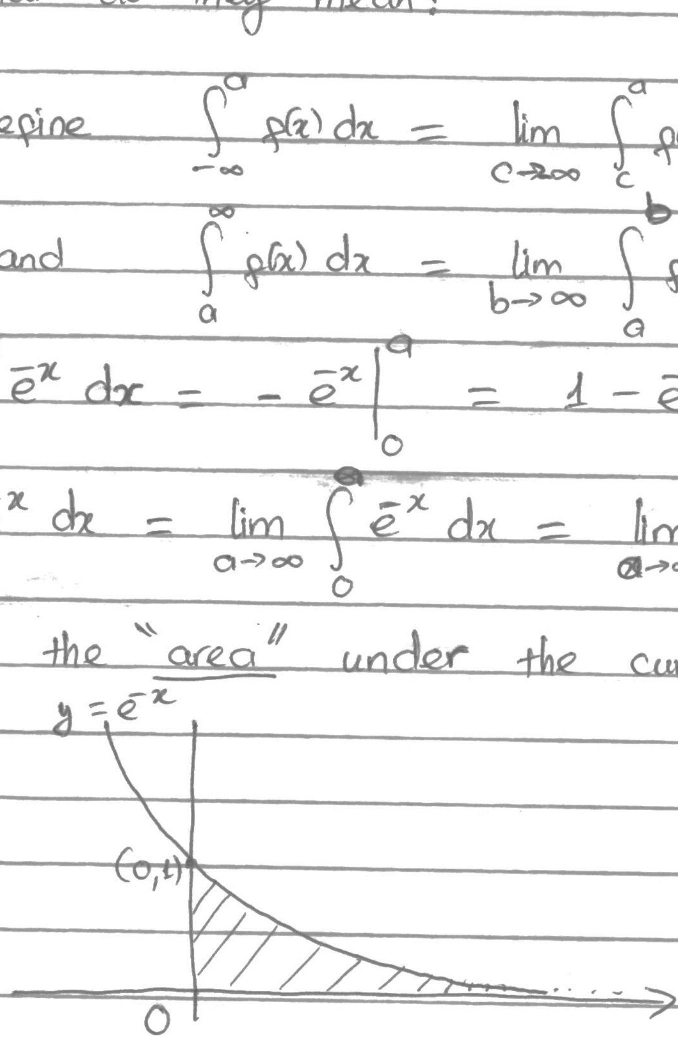
and $\displaystyle\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx$

E.g. $\displaystyle\int_{0}^{a} e^{-x}\,dx = -e^{-x}\Big|_{0}^{a} = 1-e^{-a}$.

$\Rightarrow \displaystyle\int_{0}^{\infty} e^{-x}\,dx = \lim_{a\to\infty}\int_{0}^{a} e^{-x}\,dx = \lim_{a\to\infty}\left(1-e^{-a}\right) = 1-0$

$= 1$.

This is the "area" under the curve $e^{-x}$ from $0$ onward.




**Def:** The improper integral of $f(x)$ over $[a,\infty)$ is

$$\int_{a}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{a}^{R} f(x)\,dx$$

(if the limit exists).

If the limit exists, we say that the improper integral

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converges. Otherwise, we say it diverges

E.g. $\int_3^{\infty} \frac{1}{x^2}\, dx = \lim_{R\to\infty} \int_3^{R} x^{-2}\, dx$

$= \lim_{R\to\infty} - x^{-1} \Big|_3^{R}$

$= \lim_{R\to\infty} -R^{-1} + 3^{-1}$

$= 3^{-1}.$

E.g. Does $\int_{-\infty}^{-1} \frac{1}{x}\, dx$ diverge?

Sol. $\int_{-\infty}^{-1} \frac{1}{x}\, dx = \lim_{R\to -\infty} \int_R^{-1} \frac{1}{x}\, dx$

$= \lim_{R\to -\infty} \ln|x| \Big|_R^{-1}$

$= \lim_{R\to -\infty} \ln|-1| - \ln|R|$

$= -\infty$

Thm: For $a > 0$,

$$\int_a^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Exer: Check this?

Example. $\displaystyle\int_0^\infty x e^{-x}\,dx = \lim_{R\to\infty}\int_0^R x e^{-x}\,dx$

Int. by parts

$u = x \quad v' = e^{-x}$

$u' = 1 \quad v = -e^{-x}$

$$= \lim_{R\to\infty} -(x+1)e^{-x}\Big|_0^R$$

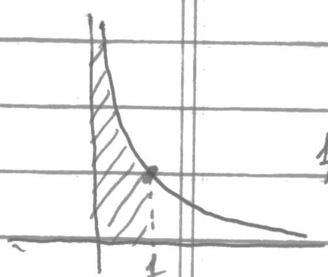
$$= \lim_{R\to\infty} -\frac{R+1}{e^{R}} + 1.$$

$$= 1 - \lim_{R\to\infty}\frac{R+1}{e^{R}}$$

$$= 1 - 0 \qquad \text{(L'Hopital's rule)}$$

Example: (Integrands with infinite discontinuities)

1)



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+}\int_R^1 \frac{dx}{\sqrt{x}} = \lim_{R\to 0^+} 2\sqrt{x}\,\Big|_R^1$$

$1/\sqrt{x}$ is disc. at $x=0$

$$= \lim_{R\to 0^+} 2 - 2\sqrt{R}$$

$$= 2.$$

2) $$\int_0^7 \frac{1}{x}\,dx = \lim_{R\to 0^+}\int_R^7 \frac{1}{x}\,dx = \lim_{R\to 0^+} \ln|x|\,\Big|_R^7$$

$1/x$ is discont. at $x=0$

$$= \lim_{R\to 0^+} \ln 7 - \ln|R|$$

$$= +\infty.$$

$\Rightarrow$ the integral diverges.

- **Comparing Integrals:**

Sometimes we can't (don't know how) evaluate an integral directly but want to know whether or not it converges.

E.g. $I = \int_1^\infty \frac{e^{-x}}{x}\, dx$.

when $x \geq 1$,

$$0 < \frac{1}{x} \leq 1.$$

$$\Rightarrow \quad 0 < \frac{e^{-x}}{x} \leq e^{-x}.$$

$$\Rightarrow \quad 0 < \int_1^\infty \frac{e^{-x}}{x}\, dx \leq \int_1^\infty e^{-x}\, dx = e^{-1}.$$

so $\int_1^\infty \frac{e^{-x}}{x}\, dx$ converges!

Thm: (Comparison test)
If $0 \leq f(x) \leq g(x)$ for $x \geq a$,
then:
- If $\int_a^\infty g(x)\, dx$ converges, $\int_a^\infty f(x)\, dx$ converges.
- If $\int_a^\infty f(x)\, dx$ diverges, $\int_a^\infty g(x)\, dx$ diverges.

E.g. Does $\int_1^\infty \frac{1}{\sqrt{x} + x^2}$ converges?

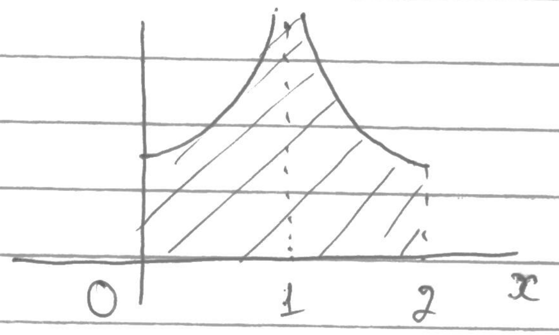
Sol: When $x \geq 1$, $\sqrt{x} + x^2 \geq x^2$

$$\Rightarrow \quad 0 < \frac{1}{\sqrt{x} + x^2} \leq \frac{1}{x^2}$$

and $\int_1^\infty \frac{1}{x^2}\, dx = \lim_{R \to +\infty} -\frac{1}{x}\Big|_1^R = 1.$

$\Rightarrow \int_1^\infty \frac{1}{\sqrt{x} + x^2}\, dx$ converges.

E.g. Calculate $\int_0^2 \frac{dx}{(x-1)^{2/3}}$.




infinite discontinuity at $x = 1$.

$$\Rightarrow \int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^2 \frac{dx}{(x-1)^{2/3}}.$$

We consider each integral individually.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{R\to 1^-} \int_0^R \frac{dx}{(x-1)^{2/3}} = \lim_{R\to 1^-} 3(x-1)^{1/3}\Big|_0^R$$

$$= \lim_{R\to 1^-} 3(R-1)^{1/3} - 3(-1)^{1/3}$$

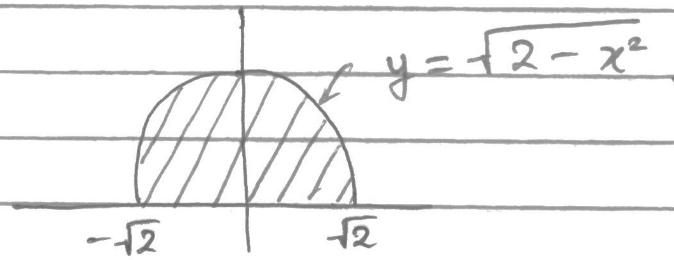
$$= 3.$$

~~Similarly,~~

Continue!!

## Section 7.3 Trigonometric Substitution

Suppose we want to find the area of half of a circle.




$$\text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\, dx$$

At first, we might try the substitution $u = 2-x^2$, but this will actually make the integral more complicated!
$\Rightarrow$ need a different approach.

Sol. Let $x = \sqrt{2}\sin\theta$, then $dx = \sqrt{2}\cos\theta\, d\theta$
and ~~then~~ $\sqrt{2-x^2} = \sqrt{2-2\sin^2\theta} = \sqrt{2(1-\sin^2\theta)}$

$$= \sqrt{2\cos^2\theta}$$
$$= \sqrt{2}\,|\cos\theta|.$$

How about the limits of integration?
when $x = -\sqrt{2}$, $\sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$

$x = \sqrt{2}$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\, dx = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta|\, \sqrt{2}\cos\theta\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2\theta))\,d\theta$$

$$= \theta + \frac{\sin(2\theta)}{2} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{\sin(\pi)}{2} - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right)$$

$$= \pi.$$

Table of Trigonometric Substitutions:

| Expression | Sub. | identity |
|---|---|---|
| $\sqrt{a^2-x^2}$ | $x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ | $1-\sin^2\theta = \cos^2\theta$ |
| $\sqrt{a^2+x^2}$ | $x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ | $1+\tan^2\theta = \sec^2\theta$ |
| $\sqrt{x^2-a^2}$ | $x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ | $\sec^2\theta - 1 = \tan^2\theta.$ |

E.g. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Sol: . step 1: Sub. to eliminate the square root.

let $x = 3\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Then $\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta}$

$= 3\cos\theta.$

and $dx = 3\cos\theta\, d\theta.$

. step 2: Evaluate the integral

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta.$$

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$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta \, d\theta$$

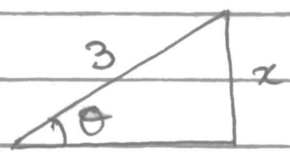
$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C.$$

Step 3: Convert back to the original variable.

$\Rightarrow$ use right triangle method:

Recall $x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$




$\sqrt{9-x^2} \Leftarrow$ Pythagorean theorem.

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}.$$

$$\Rightarrow \int \frac{\sqrt{9-x^2}}{x^2}\, dx = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$

E.g. Calculate $\int \frac{1}{x^2\sqrt{x^2+4}}\, dx$.

Step 1: let $x = 2\tan\theta$. for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

then $dx = 2\sec^2\theta \, d\theta$

and $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4}$

$= 2\sqrt{\tan^2\theta + 1}$

$= 2|\sec\theta|$

$= 2\sec\theta.$

Step 2: $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta}\, 2\sec^2\theta\, d\theta.$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta.$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\, d\theta.$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta.$$

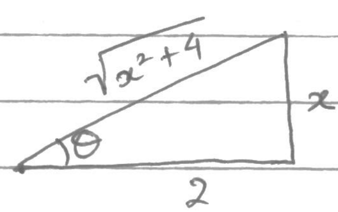
$u = \sin\theta$ ~~$\cos\theta$~~

$$= \frac{1}{4}\int \frac{1}{u^2}\, du.$$

$$= -\frac{1}{4}\frac{1}{u} + C$$

$$= -\frac{1}{4\sin\theta} + C.$$

Step 3: Convert back to $x$.




$$\tan\theta = \frac{x}{2}$$

$$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}.$$

$$\Rightarrow \int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{1}{4\frac{x}{\sqrt{x^2+4}}} + C$$

$$= -\frac{\sqrt{x^2+4}}{4x} + C.$$

Exercise: Try $\int \frac{x}{\sqrt{x^2+4}}\,dx.$

E.g. Evaluate $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}\,dx$.

Step 0: $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}\,dx = \int \frac{1}{\sqrt{4(x^2-\frac{1}{4})}}\,dx.$

$$= \int \frac{1}{2\sqrt{x^2-\frac{1}{4}}}\,dx.$$

Step 1: Let $x = \frac{1}{2}\sec\theta$.

then $\displaystyle\sqrt{x^2-\frac{1}{4}} = \sqrt{\frac{1}{4}\sec^2\theta - \frac{1}{4}} = \sqrt{\frac{1}{4}(\sec^2\theta - 1)}$

$$= \frac{1}{2}|\tan\theta|$$

$$= \frac{1}{2}\tan\theta.$$

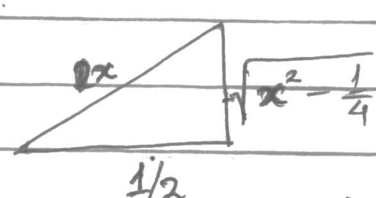
and $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

Step 2: $\displaystyle\frac{1}{2}\int \frac{1}{\sqrt{x^2-\frac{1}{4}}}\,dx = \frac{1}{2}\int \frac{1}{\frac{1}{2}\tan\theta}\cdot\frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$= \frac{1}{2}\int \sec\theta\,d\theta.$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

Step 3: $\sec\theta = 2x = \dfrac{x}{1/2}$




$$\Rightarrow \int \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\ln\left|2x + \frac{\sqrt{x^2-1/4}}{x}\right| + C.$$

- Completing the square:

E.g. $I = \int \frac{dx}{\sqrt{x^2+4x+13}}$

Step 1: complete the square: Note that $x^2 + 4x + 13 = x^2 + 4x + 4 + 9$

$= (x+2)^2 + 9$

Step 2: u-sub.

$$I = \int \frac{dx}{\sqrt{(x+2)^2+9}}$$

let $u = x+2$. Then $du = dx$.

$$I = \int \frac{du}{\sqrt{u^2+9}}.$$

Step 3: Trig. sub.

let $u = 3\tan\theta$.

then $u^2 + 9 = 9\sec^2\theta$.

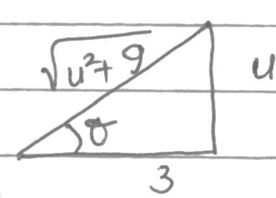
$du = 3\sec^2\theta \, d\theta$.

$$\Rightarrow I = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9\sec^2\theta}}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Step 4: Convert back.




$$I = \ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| + C$$

$$= \ln\left|\frac{\sqrt{(x+2)^2+9}}{3} + \frac{x+2}{3}\right| + C.$$

## Section 10.1 Sequences

A sequence is a function $f(n)$ on a set of integers n.

E.g.

$$\underset{a_1}{1}, \underset{a_2}{2}, \underset{a_3}{3}, \underset{a_4}{4}, \dots \quad : \quad a_n = n \quad \text{for } n = 1, 2, 3, \dots$$

$$\underset{a_1}{\frac{1}{1}}, \underset{a_2}{\frac{1}{2}}, \underset{a_3}{\frac{1}{3}}, \underset{a_4}{\frac{1}{4}}, \dots \qquad a_n = \frac{1}{n} \quad \text{for } n = 1, 2, 3, \dots$$

$$\underset{a_0}{\frac{1}{1}}, \underset{a_1}{\frac{1}{2}}, \underset{a_2}{\frac{1}{4}}, \underset{a_3}{\frac{1}{8}}, \dots \qquad \underset{\text{index}}{a_n} = 2^{-n} \quad \text{for } n = 0, 1, 2, \dots$$

Example: (Recursive sequence)

$$a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right).$$

$$\Rightarrow a_1 = 1, \quad a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}.$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \approx 1.4167$$

$$a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{2} + \frac{2}{17/2}\right) \approx 1.414216.$$

This sequence looks like its getting closer and closer to $\sqrt{2}$!

- **Limit of a sequence:**

We say that a sequence $\{a_n\}$ converges to a limit $L$, and we write

$$\lim_{n\to\infty} a_n = L \quad (\text{or } a_n \to L \text{ as } n \to \infty)$$

If for every $\varepsilon > 0$, there exists an integer $N$ such that $|a_n - L| < \varepsilon$ for every $n > N$.

In words, by going far enough in the sequence we can get as close as we like to $L$.

. If no limit exists, we say $\{a_n\}$ diverges.

. If the terms increase w/out bound, we say $\{a_n\}$ diverges to infinity.

E.g. Let $a_n = \frac{1}{n}$, for $n = 1, 2, \ldots$

What is $\lim_{n\to\infty} a_n$?

Sol: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

Thm: If $\lim_{x\to\infty} f(x) = L$, then $a_n = f(n)$ converges to $L$ as well.

E.g. Find the limit of the sequence

$$a_n = \frac{n^2 - 7}{n^2}, \quad n = 1, 2, \ldots$$

Sol. $\lim_{n\to\infty} \frac{n^2-7}{n^2} = \lim_{n\to\infty} 1 - \frac{7}{n^2} = 1.$

* Limit laws for sequences:

Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.

Then

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M.$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \quad \text{if} \quad M \neq 0.$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL \quad (\text{for any constant } c)$$

* Squeeze theorem:

If $b_n \leq a_n \leq c_n$, and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} a_n = L$.

E.g. Find $\lim_{n\to\infty} \frac{\sin(n)}{n}$

Sol. $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$.

and $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0$

by the squeeze theorem.

Limit inside a continuous function.

If $f(x)$ is continuous and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

E.g. Find $\lim_{n\to\infty}\left(\frac{3n}{n+1}\right)^7$

Sol: $\lim_{n\to\infty}\left(\frac{3n}{n+1}\right)^7 = \left(\lim_{n\to\infty}\frac{3n}{n+1}\right)^7 = 3^7.$

E.g. Find $\lim_{n\to\infty}\cos\left(\frac{n}{n^2+1}\right)$

Sol. $\lim_{n\to\infty}\cos\left(\frac{n}{n^2+1}\right) = \cos\left(\lim_{n\to\infty}\frac{n}{n^2+1}\right)$

$= \cos(0)$

$= 1.$

* Section 10.2 Infinite Series

Q: How does a calculator compute $\sin(x)$ or $\cos(x)$?

A: Roughly speaking, use infinite series.

It turns out that for example

$$\sin(2) = \underbrace{2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots}_{\text{infinite series}}$$

$$\pi = \overbrace{4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots}$$

Q: But how do we compute these infinite sums?

A: We use limits.

Partial sums: Let $\{a_n\}$ be a sequence. Its Nth partial sum is

$$\boxed{S_N = a_1 + a_2 + \cdots + a_N}$$

E.g. For the $\sin(2)$ sequence

$S_1 = 2$

$S_2 = 2 - \frac{2^3}{6} \approx 0.667$

$S_3 = 2 - \frac{2^3}{6} + \frac{2^5}{120} \approx 0.9333$

$S_4 = 2 - \frac{2^3}{6} + \frac{2^5}{120} - \frac{2^7}{7!} \approx 0.9079$

$S_5 = \cdots \approx 0.909347$

$\vdots$

$\lim_{N\to\infty} S_N = \sin(2) \approx 0.909297$.

In general, we say an infinite series $\sum_{n=k}^{\infty} a_n$ converges to $S$ if

$$\lim_{N\to\infty} S_N = S.$$

In this case we write $\sum_{n=k}^{\infty} a_n = S$.

If the limit doesn't exist, we say the series diverges.

Example. (Telescoping series)

Using the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Sol. We want to find $S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)}_{S_N}$$

Let's write out a few terms.

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$

$$S_4 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$$

In general, $S_N = 1 - \frac{1}{N+1} \Rightarrow \lim_{N\to\infty} S_N = 1$.

Remark: • A sequence is a "list of numbers" $\{a_n\}$
• A series is their sum.

* Properties of Infinite Series:

If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ converge,

then $\sum a_n \pm \sum b_n = \sum (a_n \pm b_n)$

and $\sum c a_n = c \sum a_n$.

$c$ ← any constant.

• Geometric Series:

Example: $\sum_{n=0}^{\infty} (0.5)^n = 0.5^0 + 0.5^1 + 0.5^2 + \dots$

$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$= 2$

In general,

$$\boxed{\underbrace{\sum_{n=0}^{\infty} cr^n}_{c \text{ any constant},\ |r|<1} = c + cr + cr^2 + cr^3 + \dots = \frac{c}{1-r}}$$

and

$$\boxed{S_N = \sum_{n=0}^{N} cr^n = \frac{c(1-r^{N+1})}{1-r}}$$

Example: Evaluate $\sum_{n=0}^{\infty} 3^n$

Sol. $S = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{3}\right)^n}_{r^n} = \frac{1}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

Example: Evaluate $\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n$

Sol. $S = \sum_{n=3}^{\infty} \underset{\uparrow\ c}{7} \underbrace{\left(-\frac{3}{4}\right)^n}_{r^n}$

careful! The sum is from $n=3$

How do we fix it?

Let $m = n-3 \Rightarrow n = m+3$

$$S = \sum_{m+3=3}^{\infty} 7\left(-\frac{3}{4}\right)^{m+3}$$

$$= \sum_{m=0}^{\infty} \underbrace{7\left(-\frac{3}{4}\right)^3}_{c} \underbrace{\left(-\frac{3}{4}\right)^m}_{r^m}$$

$$= 7\left(-\frac{3}{4}\right)^3 \frac{1}{1-\left(-\frac{3}{4}\right)}$$

In general,

$$\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}$$