Section 1.3 Matrices, Determinants, and Cross-Product

Define the cross product $\vec{a} \times \vec{b} = \vec{c}$ where $\vec{c}$ is orthogonal to the plane spanned by $\vec{a}$ and $\vec{b}$.

First, we need to introduce a few things.

Matrices

A 2×2 (2 rows and 2 columns) matrix is an array

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where a_{11}, a_{12}, a_{21}, and a_{22} are scalars.
Similarly a 3×3 (3 rows and 3 columns) matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Example. $\begin{bmatrix} 1 & 5 \\ -3 & 7 \end{bmatrix}$, and $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 4 & 5 & -1 \end{bmatrix}$.

The *determinant* of a 2×2 matrix is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Example. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1(1) - 0(0) = 1$, and $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = -2$.

The determinant of a 3×3 matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example.

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 7 \cdot 5) \\ &= -3 + 12 - 9 \\ &= 0. \end{aligned}$$

Properties of Determinants

- Interchanging two rows or two columns results in a change of sign.

  Example. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$ (interchanging rows), and $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 6 & 8 & 7 \end{vmatrix}$ (interchanging columns).

- We can factor a scalar out of any row or column, i.e.

$$\begin{vmatrix} \alpha a_{11} & a_{12} \\ \alpha a_{21} & a_{22} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & \alpha a_{12} \\ a_{21} & \alpha a_{22} \end{vmatrix} = \begin{vmatrix} \alpha a_{11} & \alpha a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{vmatrix}.$$

  (same for 3×3 matrices).
- Adding a row (column) from the matrix to *another* row (or column) from the matrix does not change the determinant.
  Example. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+c & b+d \\ c & d \end{vmatrix} = \begin{vmatrix} a+b & b \\ c+d & d \end{vmatrix}$.

Cross Products

Suppose that $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = (a_1, a_2, a_3)$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = (b_1, b_2, b_3)$. The *cross product* (or vector product) of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k},$$

or symbolically

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example.

$$\begin{aligned} (3\vec{i} - \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - \vec{k}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \vec{k} \\ &= (1-2)\vec{i} - (-3-1)\vec{j} + (6+1)\vec{k} \\ &= -\vec{i} + 4\vec{j} + 7\vec{k}. \end{aligned}$$

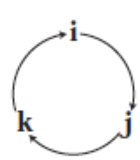
Properties of Cross Products

i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
   In particular, $\vec{a} \times \vec{a} = -\vec{a} \times \vec{a}$, so $\vec{a} \times \vec{a} = 0$.

ii) $\vec{a} \times (\beta\vec{b} + \gamma\vec{c}) = \beta\vec{a} \times \vec{b} + \gamma\vec{a} \times \vec{c}$
   $(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha\vec{a} \times \vec{c} + \beta\vec{b} \times \vec{c}$

In particular, by Property i) we have $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$.
Also,

$$\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j}. \end{aligned}$$



Example. Without using the determinant of a matrix, find the cross product of $(3\vec{i} - \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - \vec{k})$.

Solution.

$$\begin{aligned}(3\vec{i}-\vec{j}+\vec{k})\times(\vec{i}+2\vec{j}-\vec{k}) &= 3\vec{i}\times\vec{i}+3\vec{i}\times 2\vec{j}-3\vec{i}\times\vec{k}-\vec{j}\times\vec{i}-2\vec{j}\times\vec{j}+\vec{j}\times\vec{k}+\vec{k}\times\vec{i}+2\vec{k}\times\vec{j}-\vec{k}\times\vec{k}\\ &= 0+6\vec{k}-3(-\vec{j})-(-\vec{k})-0+\vec{i}+\vec{j}+2(-\vec{i})-0\\ &= -\vec{i}+4\vec{j}+7\vec{k}.\end{aligned}$$

From the definitions of cross product and scalar product, we have

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\vec{c} &= \left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}\right)\cdot(c_1\vec{i}+c_2\vec{j}+c_3\vec{k})\\ &= \begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}c_1-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}c_2+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}c_3\\ \vec{a}\times\vec{b} &= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned}$$

Suppose that $\vec{v}$ lies in the plane spanned by $\vec{a}$ and $\vec{b}$, then $\vec{v}=\alpha\vec{a}+\beta\vec{b}$ for some scalars α,β.

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\vec{v} &= (\vec{a}\times\vec{b})\cdot(\alpha\vec{a}+\beta\vec{b})\\ &= (\vec{a}\times\vec{b})\cdot\alpha\vec{a}+(\vec{a}\times\vec{b})\cdot\beta\vec{b}\\ &= \alpha(\vec{a}\times\vec{b})\cdot\vec{a}+\beta(\vec{a}\times\vec{b})\cdot\vec{b}\\ &= 0\ \text{(Why?)}.\end{aligned}$$

That is, $(\vec{a}\times\vec{b})\cdot\vec{v}=0$. So $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a},\vec{b}$, and to all vectors $\vec{v}$ spanned by $\vec{a}$ and $\vec{b}$. The following figure is taken from Marsden and Tromba's figure 1.3.2

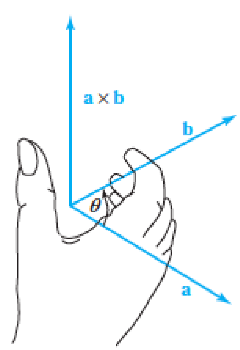


figure **1.3.2** The right-hand rule for determining in which of the two possible directions **a** × **b** points.

We have figured out the direction of $\vec{a}\times\vec{b}$. Let us now figure out its length. Since

$$\vec{a}\times\vec{b}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k},$$

we obtain

$$
\begin{aligned}
\|\vec{a}\times\vec{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\
&= \ldots \text{(try to calculate it by yourself)} \\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2 \\
&= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2 \\
&= \|\vec{a}\|^2\|\vec{b}\|^2 - (\|\vec{a}\|^2\|\vec{b}\|\cos\theta)^2 \\
&= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta) \\
&= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta.
\end{aligned}
$$

Therefore, $\boxed{\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta.}$

Consequently, $\|\vec{a}\times\vec{b}\|^2 + (\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2\|\vec{b}\|^2$.

Fact: length of $\vec{a}\times\vec{b}$ = area of parallelogram formed by $\vec{a}$ and $\vec{b}$. (See Figure 1.3.3 from Marsden and Tromba)

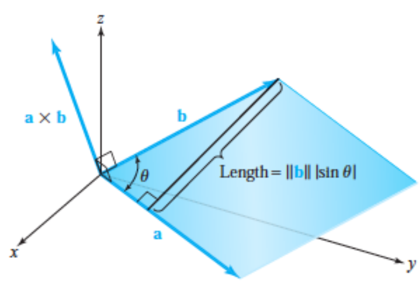


figure 1.3.3 The length of **a** × **b** is the area of the parallelogram formed by **a** and **b**.

Example. Find the area of the parallelogram spanned by $\vec{a} = (1,2,3)$ and $\vec{b} = (1,1,1)$.

Solution. Since the area of the parallelogram $= \|\vec{a}\times\vec{b}\|$, we first need to find

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} + 2\vec{j} - \vec{k}.$$

Hence, Area $= \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$.

Example. Find a unit vector orthogonal to $\vec{a} = (1,2,3)$ and $\vec{b} = (1,1,1)$.

Solution. The vector $\frac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|}$ is orthogonal to $\vec{a}$ and $\vec{b}$, and is of unit length. By the previous example $\vec{a}\times\vec{b} = (-1,2,-1)$, so

$$\frac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|} = -\frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k}.$$

Remark. $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ = volume of the parallelepipe spanned by $\vec{a},\vec{b}$, and $\vec{c}$.

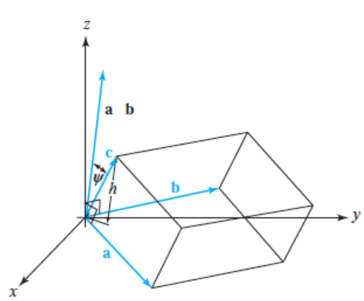


figure 1.3.5 The volume of the parallelepiped spanned by **a**, **b**, **c** is the absolute value of the determinant of the 3 × 3 matrix having **a**, **b**, **c** as its rows.

Example. Find the volume of the parallelepiped spanned by $\vec{a} = \vec{i} + 3\vec{k}, \vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}, \vec{c} = 5\vec{i} + 4\vec{k}$.
Solution. Since

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 5 & 0 & 4 \end{vmatrix} = \ldots \text{(check!)} = -11,$$

the volume $= |-11| = 11$.

Equations of Planes

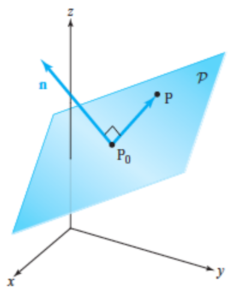


figure **1.3.6** The points P of the plane through P_0 and perpendicular to **n** satisfy the equation $\overrightarrow{P_0P} \cdot \mathbf{n} = 0$.

Let $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ be normal to the plane, so $\vec{n}$ is orthogonal to all vectors in the plane. So, if $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$ are two points on the plane,

$$\vec{n} \cdot \overrightarrow{P_0P} = 0, \text{ i.e. } (A\vec{i} + B\vec{j} + C\vec{k}) \cdot ((x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}) = 0.$$

Therefore, the equation of the plane normal to $\vec{n} = (A, B, C)$, and passing through $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$, is

$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0},$$

i.e. $Ax + By + Cz + D = 0$ where $D = -Ax_0 - By_0 - Cz_0$.

Example. Find the equation of the plane containing three points $P = (1, 1, 1), Q = (2, 0, 0)$, and $R = (1, 1, 0)$.
Solution. We need to find a normal vector to the plane. To do that, we first find $\overrightarrow{QP} = (-1, 1, 1)$ and $\overrightarrow{RP} = (0, 0, 1)$. Then

$$\vec{n} = \overrightarrow{QP} \times \overrightarrow{RP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j}.$$

So the equation of the plane is

$$1(x - 1) + 1(y - 1) + 0(z - 1) = 0.$$

Simplifying it, we obtain

$$x + y = 2.$$

Two planes are *parallel* when their normal vectors are parallel. So

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0$$

are parallel if

$$(A_1, B_1, C_1) = \alpha(A_2, B_2, C_2) \qquad \text{for some } \alpha \neq 0.$$

Example. The planes given by

$$2x + y - 3z + 5 = 0 \text{ and } 4x + 2y - 6z + 2017 = 0$$

are parallel.

Distance from a point to a plane

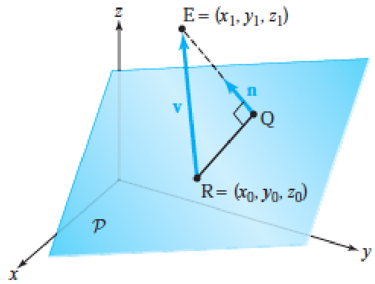


figure **1.3.7** The geometry for determining the distance from the point E to plane $\mathcal{P}$.

Let $\vec{n}$ be the unit normal to the plane $\mathcal{P}$ given by $Ax + By + Cz + D = 0$, so

$$\vec{n} = \frac{A\vec{i} + B\vec{j} + C\vec{k}}{\sqrt{A^2 + B^2 + C^2}}.$$

From the picture: distance = length of projection of $\overrightarrow{RE}$ onto $\vec{n}$. Hence,

$$\boxed{\text{distance } = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.}$$

Example. Find the distance of the point $(1, 1, 1)$ to the plane $2x + 3y - 4z + 1 = 0$.
Solution. distance $= \frac{|2(1)+3(1)-4(1)+1|}{\sqrt{2^2+3^2+4^2}} = \frac{2}{\sqrt{29}}$.