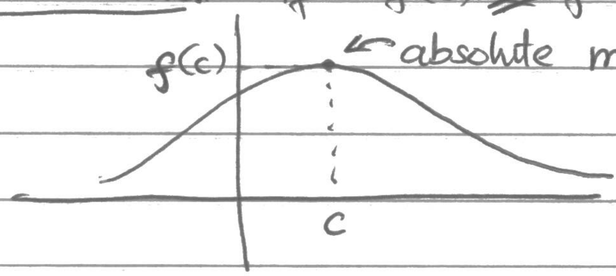
Extreme Values.

Given a function f, what are the maximum and minimum value of $f(x)$?

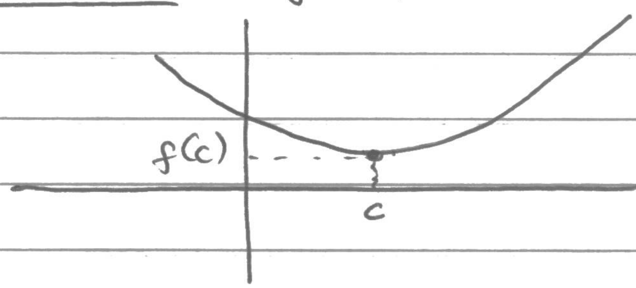
Def: Let f be a function defined on an interval I and let $c \in I$.

Then $f(c)$ is

- absolute max if $f(c) \geq f(x)$ for all $x \in I$.



- absolute min: if $f(c) \leq f(x)$ for all $x \in I$.



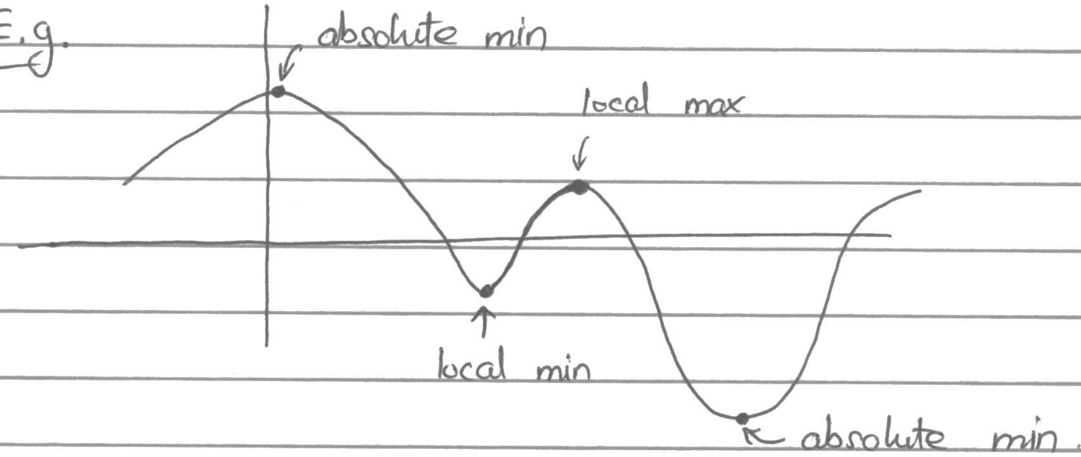
Remark: absolute extrema do not always exists.

Thm: (Existence of absolute extrema)
A continuous function f on a closed and bounded interval $[a, b]$ takes on an absolute max and min values.

Def: $f(c)$ is a

- local min: if $f(c)$ is a min value on some open interval
- local max: if $f(c)$ is a max value on some open interval.

E.g.



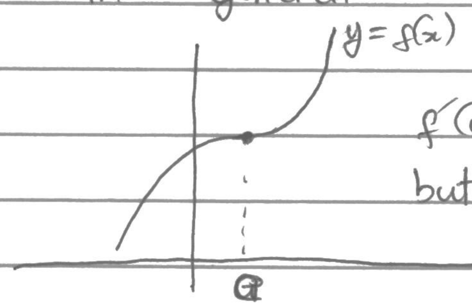
Def: A number $c \in \mathrm{Dom}(f)$ is called a critical point if either:

1) $f'(c) = 0$ or
2) $f'(c)$ DNE.

Thm: (Fermat's theorem for local extrema).

$f(c)$ local min or max $\Longrightarrow$ c is a critical point.

Remark: $\not\Leftarrow$ the other direction is not true in general



$f'(a) = 0$
but $x = a$ is not local min/max.

E.g. 1) $f(x) = \dfrac{x^2}{x^2-4x+8}$. Find all critical points.

$$f'(x) = \frac{2x(x^2-4x+8) - x^2(2x-4)}{(x^2-4x+8)^2}$$

$$= \frac{2x^3 - 8x^2 + 16x - 2x^3 + 4x^2}{(x^2-4x+8)^2}$$

$$= \frac{-4x^2+16x}{(x^2-4x+8)^2}$$

Find critical points by solving $f'(x) = 0$. and note that the domain of f is all real number (why?)

$$\frac{-4x^2+16x}{(x^2-4x+8)^2} = 0$$

$$-4x^2 + 16x = 0.$$

$$-4x(x-4) = 0$$

$$x = 0 \quad \text{or} \quad x = 4.$$

$\Rightarrow$ $x=0$ and 4 are critical points.

2) $f(t) = 4t - \sqrt{t^2+1}$. Find all critical points.

$$f'(t) = 4 - \frac{1}{2}(t^2+1)^{-1/2}(2t)$$

$$= 4 - \frac{t}{\sqrt{t^2+1}}$$

First, note that $t^2+1>0$ for all t, f' is defined everywhere.

Now solve $f'(t) = 0$.

$$4 - \frac{t}{\sqrt{t^2+1}} = 0$$

$$4\sqrt{t^2+1} = t.$$

$$16(t^2+1) = t^2.$$

$$15t^2 = -16$$

$$t^2 = -\frac{16}{15}.$$

$\Rightarrow$ no solution.

$\Rightarrow$ f has no critical points.

(Exer: $f(x) = x + |2x+1|$

Thm: (Extreme Values on a closed interval) Assume that f is continuous on $[a,b]$ and let $f(c)$ be the minimum or maximum value on $[a,b]$. Then c is either a critical point or one of the endpoints a or b.

Procedure to find extrema of continuous function f on a closed interval $[a,b]$.

1) Find the critical points of f in the interval $[a,b]$.

2) Compute $f(x)$ for the critical points x and the endpoints $x=a$ and $x=b$.

3) The largest (or smallest) value among those in step 2) is the absolute max (or min) of $f(x)$ on $[a,b]$.

E.g. Find absolute min and max values of
$f(x) = x^3 - 24\ln(x)$ on $[\frac{1}{2}, 3]$.

1) $f'(x) = 3x^2 - \frac{24}{x}$.

$f'(x) = 0 \Rightarrow 3x^2 - \frac{24}{x} = 0$.

$3x^2 = \frac{24}{x}$.

$3x^3 = 24$.

$x^3 = 8$

$x = 2$

The critical point is 2, and the endpoints are $\frac{1}{2}$ and 3.

2) $f(2) = \boxed{8 - 24\ln(2)}$ min

$f(\frac{1}{2}) = \boxed{\frac{1}{8} + 24\ln 2}$. max.

$f(3) = 27 - 24\ln 3$.

Thm: (Rolle) f continuous on $[a,b]$.
f differentiable on (a,b).

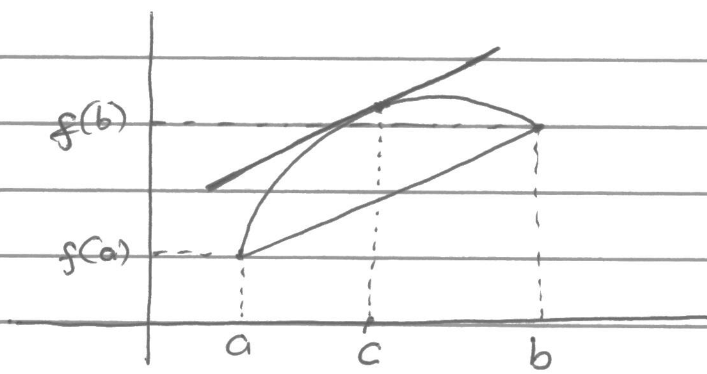
If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that

$$f'(c) = 0.$$

Mean Value Theorem (MVT)

Thm: f continuous on $[a,b]$.
f differentiable on (a,b).
Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



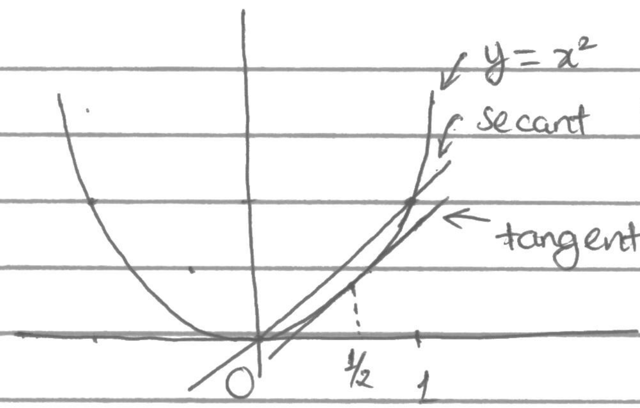
The secant line for the points $(a, f(a))$ and $(b, f(b))$ has the same slope as some tangent line to the graph $y = f(x)$ for some point between a and b. (this point is c).

E.g. Given $f(x) = x^2$. Find a point $x = c$ in $[0,1]$ satisfying the ~~c/o~~ conclusion of the MVT and graph the resulting tangent and secant line.

Sol. By MVT, $\exists c \in (0,1)$ s.t.

$$f'(c) = \frac{f(1)-f(0)}{1-0} = 1.$$

$$\Rightarrow \quad 2c = 1. \Rightarrow \quad c = \frac{1}{2}.$$



$y = x^2$

secant line ~~p~~ through $(0,0)$, $(1,1)$.

tangent line at $x = \frac{1}{2}$.

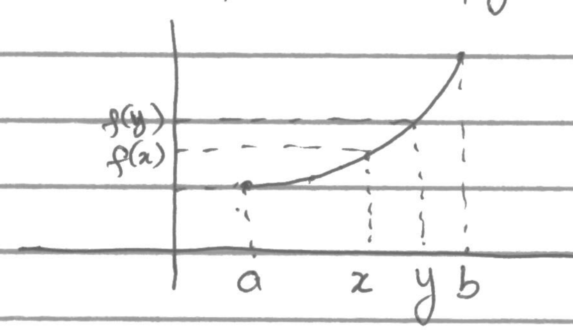
These lines are parallel.

<u>Corollary</u>: If f is differentiable and $f'(x) = 0$ for all $x \in (a,b)$, then f is <u>constant</u> on (a,b).

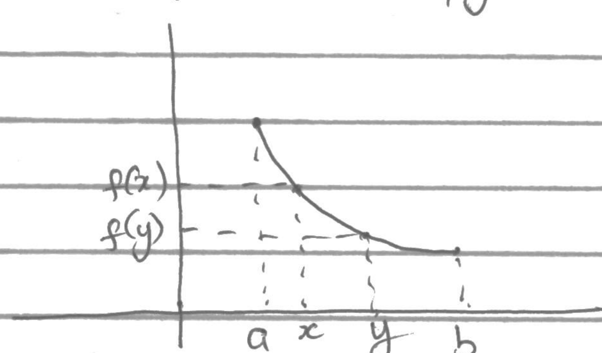
* <u>Monotone functions + 1st derivative test</u>.

A function f is

- <u>increasing</u> on (a,b) if $f(x) < f(y)$ for all $x, y \in (a,b)$ and $x < y$.



- <u>decreasing</u> on (a,b) if $f(x) > f(y)$ for all $x, y \in (a,b)$ and $x < y$.



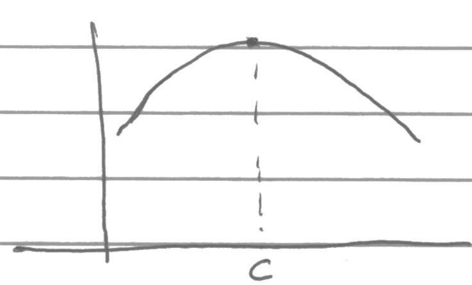
Thm: (Sign of the derivative).
Suppose f differentiable on (a,b).
1) $f'(x) > 0$ for all $x \in (a,b)$
$\Rightarrow$ f increasing on (a,b).

2) $f'(x) < 0$ for all $x \in (a,b)$
$\Rightarrow$ f decreasing on (a,b).

$\Rightarrow$ We can use the theorem to determine when a critical point gives a local max or min.

Thm: (First derivative test). Let f be continuous and differentiable on an interval containing critical point c, then for $a < c < b$,
1) If $f'(a) > 0$ and $f'(b) < 0$
then $f(c)$ is local max.
2) If $f'(a) < 0$ and $f'(b) > 0$,
then $f(c)$ is local min.

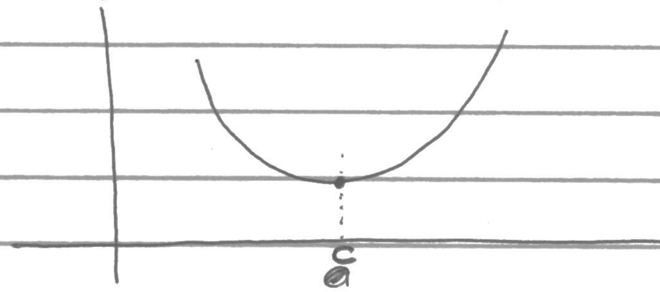
1) $f'(x)$ changes from ⊕ to ⊖
$\Rightarrow$ $f(x)$ increases, then decreases.



$\Rightarrow$ $f(c)$ local max.

2) $f'(x)$ changes from ⊖ to ⊕.

$\Rightarrow$ $f(x)$ ~~changes~~ decrease, then increases.



$\Rightarrow$ $f(c)$ local min.

E.g. 1) Let $f(x) = \dfrac{2x+1}{x^2+1}$. Find the critical points, the intervals of increase/ decrease, and the local extrema.

Sol. $$f'(x) = \frac{2(x^2+1) - (2x+1)(2x)}{(x^2+1)^2}$$
$$= \frac{2(1-x-x^2)}{(x^2+1)^2}.$$

Since ~~f~~ $(x^2+1)^2 > 0$, $f'(x)$ is defined everywhere.

. Find critical points:

$$f'(x) = 0$$
$$\Rightarrow \frac{2(1-x-x^2)}{(x^2+1)^2} = 0.$$
$$1 - x - x^2 = 0.$$
$$x^2 + x - 1 = 0.$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

So the critical points are $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$.

Now consider the sign changes:

x		$\frac{-1-\sqrt{5}}{2}$		$\frac{-1+\sqrt{5}}{2}$	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	decreases		increases		decreases

$\Rightarrow$ f is decreasing on $\left(-\infty, \frac{-1-\sqrt{5}}{2}\right)$

increasing on $\left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$

decreasing on $\left(\frac{-1+\sqrt{5}}{2}, \infty\right)$.

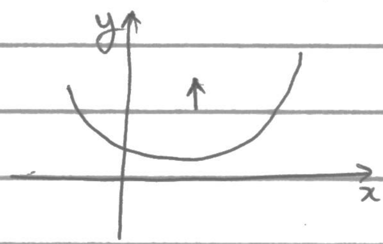
$\Rightarrow$ f has <u>local min</u> at $x = \frac{-1-\sqrt{5}}{2}$

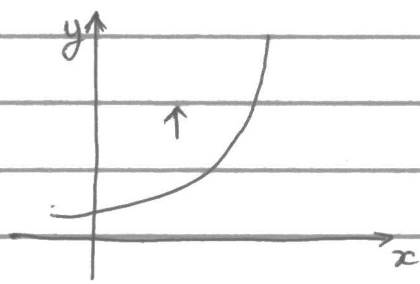
has <u>local max</u> at $x = \frac{-1+\sqrt{5}}{2}$.

[<u>Exer</u>, $f(x) = (x^2 - 2x)e^x$.

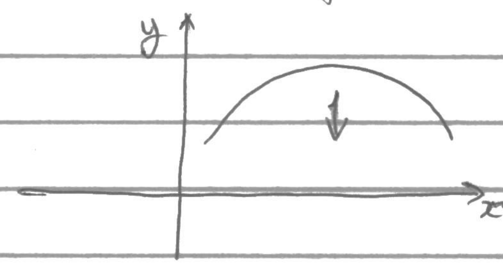
Concavity. "up like a cup, down like a frown"

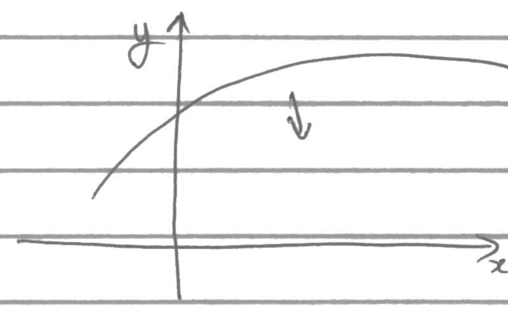
Concave up:
"Like a cup"





Concave down
"like a frown".





* Def: For f differentiable on (a, b) we have:
  - f is concave up on (a,b) if f' is increasing.
  - f is concave down on (a,b) if f' is decreasing.

Thm: (Test for concavity)
Suppose f' is differentiable on (a,b). Then
- $f''(x) > 0$ for $x \in (a,b) \Rightarrow f$ is concave up on (a,b)
- $f''(x) < 0$ for $x \in (a,b) \Rightarrow f$ is concave down on (a,b).

Thm: (Test for inflection).
If $f''(c) = 0$ or $f''(c)$ DNE. and the sign of $f''(x)$ changes at c. Then f has an inflection point at $x = c$.
(change of convavity)

E.g. 1) $\overset{f(x)}{\cancel{y}} = x + \sin^2(x)$ on $[0, \pi]$.

Find the intervals of concavity and the inflection points.

Sol. $f'(x) = 1 + 2\sin(x)\cos(x)$.

$$f''(x) = 2\cos(x)\cos(x) - 2\sin(x)\sin(x)$$
$$= 2\cos^2(x) - 2\sin^2(x).$$

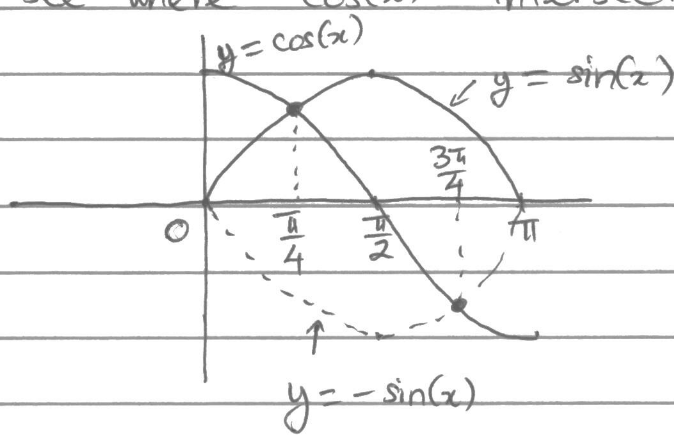
$\Rightarrow$ $f''(x)$ is defined everywhere.

~~Solve~~ $\Rightarrow$ to find the inflection points,

solve $f''(x) = 0$.

$$2\cos^2(x) - 2\sin^2(x) = 0$$
$$\Rightarrow \cos^2(x) = \sin^2(x)$$
$$\cos(x) = \pm \sin(x).$$

Let's see where $\cos(x)$ intersects $\sin(x)$ and $-\sin(x)$.



$\Rightarrow$ $\cos(x) = \sin(x)$ at $x = \frac{\pi}{4}$

and $\cos(x) = -\sin(x)$ at $x = \frac{3\pi}{4}$.

Now, consider the sign changes for $f''(x)$:

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x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f''(x)$	2	0	-2	0	2
sign of f''	$+$	0	$-$	0	$+$

$\Rightarrow$ $f(x)$ is concave up on $[0, \frac{\pi}{4})$
concave down on $(\frac{\pi}{4}, \frac{3\pi}{4})$

and concave up on $(\frac{3\pi}{4}, \pi]$.

Additionally, f has inflection points at
$x = \frac{\pi}{4}$ (from concave up to down)
and $x = \frac{3\pi}{4}$ (from concave down to up).

Thm: (2^{nd} derivative test)

Let c be a critical point for f; ~~and such that~~
If $f''(c)$ exists, then:

1) $f''(c) > 0 \Rightarrow f(c)$ local min.
2) $f''(c) < 0 \Rightarrow f(c)$ local max.
3) $f''(c) = 0 \Rightarrow$ inconclusive ($f(c)$ may be a local min, or max, or neither)

the 2^{nd} derivative test is useful for testing for extrema.

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E.g. $f(x) = \frac{1}{x^4+1}$.

Find critical points, local extrema
interval of concavity, points of inflection.

Sol. $f'(x) = -(x^4+1)^{-2}(4x^3) = \frac{-4x^3}{(x^4+1)^2}$

$f'(x)$ is defined everywhere.

Set $f'(x) = 0$.

$\frac{-4x^3}{(x^4+1)^2} = 0 \Rightarrow -4x^3 = 0 \Rightarrow \boxed{x = 0. \text{ is a critical point}}$

Now find the second derivative:

$$f''(x) = \frac{(-12x^2)(x^4+1)^2 - (-4x^3)\,2(x^4+1)(4x^3)}{(x^4+1)^4}$$

$$= \frac{-12x^2(x^4+1)^2 + 32x^6(x^4+1)}{(x^4+1)^4}$$

$$= \frac{-12x^2(x^4+1) + 32x^6}{(x^4+1)^3}$$

$$= \frac{-12x^6 - 12x^2 + 32x^6}{(x^4+1)^3}$$

$$= \frac{20x^6 - 12x^2}{(x^4+1)^3}$$

Set $f''(x) = 0$.

$\frac{20x^6 - 12x^2}{(x^4+1)^3} = 0$

$20x^6 - 12x^2 = 0$

$\Rightarrow 4x^2(5x^4 - 3) = 0 \Rightarrow x^2 = 0$ or $x^4 = \frac{3}{5}$

$x = 0$ or $x = \pm\sqrt[4]{3/5}$.

(75)

x	-1	0 ↙ critical point	1
$f'(x)$	$\cancel{4}$ $+$	0	$-\cancel{1}$
$f''(x)$		0	

By the 2nd derivative test,

→ since $f''(0) = 0$, ⇒ ~~inclu~~ inconclusive.

But ~~by~~ the sign ~~chart~~ change of $f'(x)$ at $x = 0$, tells us that

f has a <u>local max</u> at $x = 0$.

Now, consider the following chart for concavity.

x	-1	$-(3/5)^{1/4}$	$-1/2$	0	$1/2$	$(3/5)^{1/4}$	1
$f''(x)$	1	0	$\frac{20}{2^6} - \frac{12}{2^2} < 0$	0	$\frac{20}{2^6} - \frac{12}{2^2}$	0	1
sign of $f''(x)$	$+$	0	$-$	0	$-$ ~~+~~	0	$+$

Hence,

f is concave up on $\left(-\infty, -\sqrt[4]{\frac{3}{5}}\right)$

concave down on $\left(-\sqrt[4]{\frac{3}{5}}, \sqrt[4]{\frac{3}{5}}\right)$

concave up on $\left(\sqrt[4]{\frac{3}{5}}, +\infty\right)$

and has inflection points at

$x = -\sqrt[4]{\frac{3}{5}}$ and $x = \sqrt[4]{\frac{3}{5}}$.

Our next goal is to study how to sketch the graph of a function.

But before we do that, we first learn L'Hôpital Rule which is a helpful tool for computing certain limits and also for determining asymptotic behavior.

Thm: (L'Hôpital Rule)

f and g are differentiable on an open interval containing a such that

$$f(a) = 0 = g(a).$$

Suppose also that $g(x) \neq 0$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Infinite version:

The theorem is also true for $a = \pm\infty$ provided that, instead,

- $\lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x)$

or

- $\lim_{x \to \infty} f(x) = \pm\infty$ and $\lim_{x \to \infty} g(x) = \pm\infty$.

E.g. 1) $\lim_{x\to 0} \frac{\sin(x)}{x}$ $\qquad \sin(0) = 0$.

$\overset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = \lim_{x\to 0} \cos(x) = \cos(0) = 1.$

2) $\lim_{x\to 2} \frac{x^2 - 4}{x^3 + 2x - 12}$ $\qquad \left(\frac{0}{0}\text{ indeterminate}\right).$

$= \lim_{x\to 2} \frac{2x}{3x^2 + 2}$

$= \frac{4}{3(2^2) + 2}$

$= \frac{4}{14} = \frac{2}{7}.$

Remark: To apply L'Hôpital rule, we need an indeterminate form.

E.g. Evaluate $\lim_{x\to 1} \frac{x^2+1}{2x+1} \underset{\uparrow \text{ direct sub.}}{=} \frac{1+1}{2(1)+1} = \frac{2}{3}.$

But if you use L'Hôpital Rule:

$\lim_{x\to 1} \frac{x^2+1}{2x+1} \underset{\uparrow \text{ False}}{=} \lim_{x\to 1} \frac{2x}{2} = \lim_{x\to 1} x = 1.$

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3) $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x$

<u>Trick:</u> $\left(\frac{x}{x+1}\right)^x = e^{\ln\left(\frac{x}{x+1}\right)^x} = e^{x\ln\left(\frac{x}{x+1}\right)} = e^{\frac{\ln\left(\frac{x}{x+1}\right)}{1/x}}$

$\Rightarrow \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\frac{\ln\left(x/x+1\right)}{1/x}}$

$= e^{\lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{1/x}}$

$\Rightarrow$ We find

$$\lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{1/x} \qquad \frac{0}{0} \text{ form.}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\frac{1}{x/(x+1)}\cdot\left(\frac{x}{x+1}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{x+1}{x}\cdot\frac{1(x+1)-x(1)}{(x+1)^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{-x}{x+1}$$

$$= \lim_{x\to\infty}\frac{-x/x}{\frac{x}{x}+\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{-1}{1+\frac{1}{x}}$$

$$= \frac{-1}{1+0} = -1.$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = e^{-1}.$$

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\text{: (Growth of } f(x) = e^x) \\ \displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} = \infty \quad \text{for every exponent } n.\end{array}}$$

In other words, exponential functions <u>increase more rapidly</u> than the power functions.

$$x^n \ll e^x.$$

<u>Pf</u>: For $n = 1, 2, 3$.

$$\lim_{x\to\infty} \frac{e^x}{x} \underset{\frac{0}{0}}{\overset{\text{L'Hôpital's}}{=}} \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \infty.$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \frac{1}{2} \lim_{x\to\infty} \frac{e^x}{x} = \infty.$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \frac{1}{3} \lim_{x\to\infty} \frac{e^x}{x^2} = \infty.$$

Graph Sketching and Asymptotes.

1) Four basic shapes

f' \ f''	+ Concave up	− Concave down
+ increasing	++	+−
− decreasing	−+	−−

2) Transition points:
   - local min or max (sign change of f')
   - point of inflections (sign change of f'').

3) Asymptotic behavior.
   The behavior of $f(x)$ as x approaches either $\pm\infty$ or a vertical asymptote.

Examples.

1) $f(x) = \frac{1}{3}x^3 + x^2 + 3x$.

Domain : $\mathbb{R}$.

$f'(x) = x^2 + 2x + 3$. $\Rightarrow$ Domain of f' : $\mathbb{R}$.

. Find critical points:

$f'(x) = 0$

$x^2 + 2x + 3 = 0$.

$(x+1)^2 + 2 = 0$.

$\Rightarrow$ no roots. $\Rightarrow$ no critical points.

Additionally,

$f'(x) = (x+1)^2 + 2 > 0$ for all values x.

$\Rightarrow$ $f(x)$ is increasing everywhere.

$f''(x) = 2x + 2$ Domain of f'': $\mathbb{R}$.

$f''(x) = 0$.

$2x + 2 = 0$.

$x = -1$.

Interval	Test Value	Sign of f''
$(-\infty, -1)$	$f''(-2) = -2$	$-$
~~$(-1, \infty)$~~	~~$f''(0) = 2$~~	~~$+$~~
-1	$f''(-1) = 0$	0
$(-1, \infty)$	$f''(0) = 2$	$+$

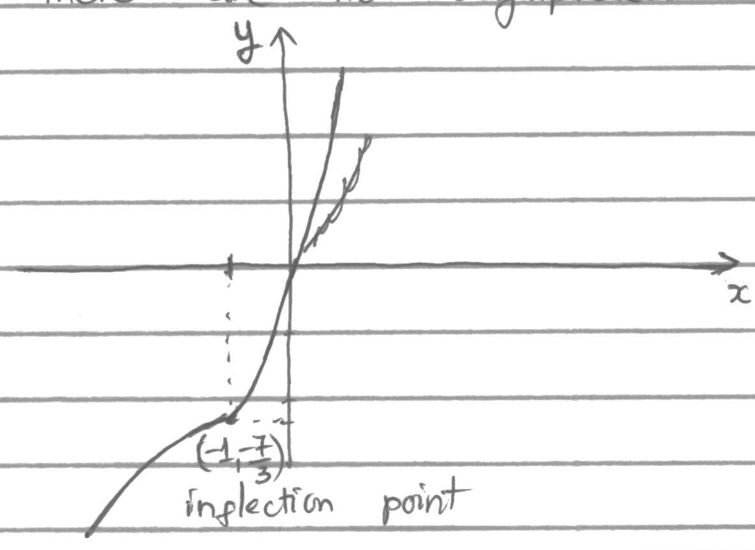
$\Rightarrow$ -1 inflects from concave down to concave up.

$\Rightarrow$ $\left(-1, -\frac{7}{3}\right)$

Find asymptotes:

$\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

$\Rightarrow$ there are no asymptotes.



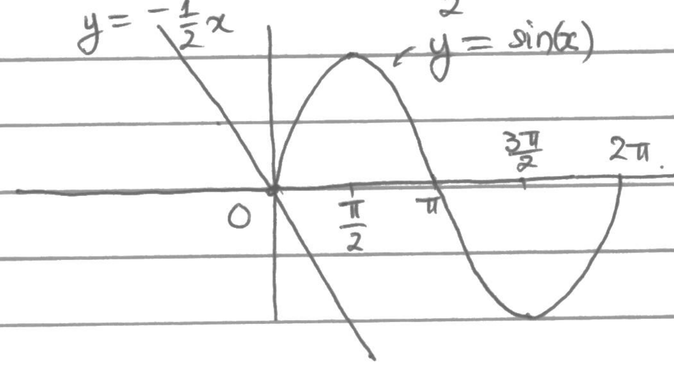
2) $f(x) = \sin(x) + \frac{1}{2}x$ over $[0, 2\pi]$.

Domain: $[0, 2\pi]$

. Find x-intercepts:

$$\sin(x) + \frac{1}{2}x = 0.$$

$$\sin(x) = -\frac{1}{2}x.$$



$\Rightarrow$ x-intercept at $(0,0)$.

. Find critical points and sign of f':

$f'(x) = \cos(x) + \frac{1}{2}$ $\Rightarrow$ ~~doo~~ domain: $[0, 2\pi]$

Now if $f'(x) = 0$

$$\cos(x) + \frac{1}{2} = 0$$

$$\cos(x) = -\frac{1}{2}.$$

$\Rightarrow$ $x = \frac{4\pi}{3}$ and $x = \frac{2\pi}{3}$ in $[0, 2\pi]$.

So critical points at $x = \frac{4\pi}{3}$ and $\frac{2\pi}{3}$

Interval	Test Value	sign of f'
$[0, \frac{2\pi}{3})$	$f'(0) = \frac{3}{2}$	$+$
$\frac{2\pi}{3}$	$f'(\frac{2\pi}{3}) = 0$	0.
$(\frac{2\pi}{3}, \frac{4\pi}{3})$	$f'(\pi) = -\frac{1}{2}$	$-$
$\frac{4\pi}{3}$	$f'(\frac{4\pi}{3}) = 0$	0.
$(\frac{4\pi}{3}, 2\pi]$	$f'(2\pi) = \frac{3}{2}$	$+$

$f''(x) = -\sin(x)$. $\Rightarrow$ domain: $[0, 2\pi]$.

$f''(x) = 0$

$-\sin(x) = 0$

$x = 0, \pi, 2\pi$.

x	Test Value	sign of f''
$(0, \pi)$	$f'(\frac{\pi}{2}) = -1$	$-$
π	$f'(\pi) = 0$	0.
$(\pi, 2\pi)$	$f'(\frac{3\pi}{2}) = +1$	$+$

Note that we didn't check the endpoints since inflections can only occur across a point which is contained in an open interval also contained in the domain.

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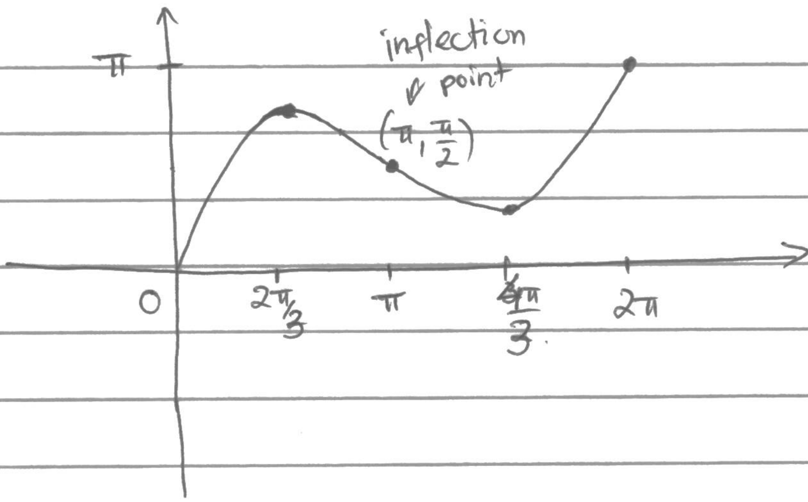

3) $f(x) = \dfrac{x-2}{x-3}$

Domain: $x \neq 3$.

x-intercept: $x = 2$.

$$f'(x) = \frac{1(x-3) - (x-2)(1)}{(x-3)^2} = \frac{-1}{(x-3)^2} = -(x-3)^{-2}$$

$\Rightarrow$ no critical points.

and $f'(x) < 0$ for all $x \neq 3$.

$\Rightarrow$ f is decreasing.

$$f''(x) = 2(x-3)^{-3}$$

x	Test value	sign of $f''(x)$
$(-\infty, 3)$	$f''(0) = -\frac{2}{3^3}$	$-$
3	undefined	
$(3, \infty)$	$f''(4) = \frac{2}{1}$	$+$

Although $f''(x)$ changes sign at $x=3$, we do not call $x=3$ an inflection point because it is not in the domain of f.

Asymptotes:

$$\lim_{x\to 3^+} f(x) = +\infty \qquad \lim_{x\to\infty} f(x) = 1$$

$$\lim_{x\to 3^-} f(x) = -\infty \qquad \lim_{x\to -\infty} f(x) = 1.$$

$\Rightarrow$ $x=3$ is the vertical asymptote.

$y=1$ is the horizontal asymptote