MATH 102 - SOLUTIONS FOR FINAL PRACTICE PROBLEMS

1.

(i) The row operations $R_2 \to R_2 - 2R_1$ followed by $R_3 \to R_3 + 2R_2$ yield the upper triangular matrix

$$U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

The lower triangular matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

We have

$$A = LU.$$

For the LDU decomposition, we divide the rows of U by the diagonal entries of U to achieve 1's on the diagonal. We find the LDU decomposition

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) Clearly, from the LU decomposition L has determinant 1 and U has determinant -2, thus $\det A = \det L \det U = -2$. You could also use the LDU decomposition to find the same answer.

(iii) From the LU decomposition, we solve the two systems

$$Ly = \begin{bmatrix} 2 \\ 9 \\ 8 \end{bmatrix}, \ Ux = y.$$

The solution of the first system is found by back substitution yielding $y = \begin{bmatrix} 2 \\ 5 \\ 18 \end{bmatrix}$. The second system

$$Ux = y = \begin{bmatrix} 2 \\ 5 \\ 18 \end{bmatrix}$$

is also solved by back substitution yielding

$$x = \begin{bmatrix} 73 \\ 49 \\ 9 \end{bmatrix}.$$

2.A.

(i) We have $x + 3y + 3z + 2w = 0$, $5z - 7w = 0$, $5w = 0$. Thus $z = w = 0$ and $x = -3y$. Thus the null space is spanned by the vector $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The nullity is 1. Clearly, y is a free variable.

(ii) The vector $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ gives the relation $-3c_1 + c_2 = 0$. Thus the columns are dependent.

(iii) Since y is free, the pivot variables are the remaining variables x, z, w. A basis for $C(A)$ is thus given by the $1^{st}, 3^{rd}, 4^{th}$ columns of A:

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \\ 14 \\ -12 \end{bmatrix}.$$

The rank of A is 3.

(iv) The first 3 rows of the reduced matrix span the row space of A. The basis is $[1 \;\; 3 \;\; 3 \;\; 2], [0 \;\; 0 \;\; 5 \;\; -7], [0 \;\; 0 \;\; 0 \;\; 1]$.

(v) The left null space has dimension $5 - 3 = 2$.

(vi) A doesn't admit either inverse, because the rank does not equal the number of rows or the number of columns.

2.B.

(i) Solutions are of the form

$$x = x_p + x_h, \text{ where } x_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \; x_h \in N(A).$$

From the row-reduced form we find the null space of A:

$$x_h = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \implies x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} + .y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

This is not a subspace since it does not contain the vector 0. Indeed, 0 is not a solution to the system since $A \cdot 0 \neq \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$.

(ii) Let v_1, v_2, v_3, v_4, v_5 be the columns of A. Vectors in the nullspace of A give relations between the columns.

The first vector $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in N(A)$ gives $-2v_1 + v_2 = 0 \implies v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The second vector $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \in N(A)$ gives $-v_1 - v_3 + v_4 = 0 \implies v_3 = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$.

Finally, the last vector $\begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \in N(A)$ gives $-2v_1 - 2v_3 + v_5 = 0 \implies v_5 = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$.

Therefore

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & -2 & -1 & -2 \\ 1 & 2 & -2 & -1 & -2 \end{bmatrix}.$$

3. Note that

$$\mathrm{Rot}(e_2) = e_2$$

$$\mathrm{Rot}(e_1) = e_1 \cos\theta + e_3 \sin\theta$$

$$\mathrm{Rot}(e_3) = -e_1 \sin\theta + e_3 \cos\theta.$$

The matrix equals

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

4.

(i) If $f = a_0 + a_1 x + \ldots + a_n x^n$ then

$$Tf = \int_{-x}^{0} (a_0 + a_1 t + \ldots + a_n t^n)\, dt = a_0 x - \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 - \ldots + \frac{(-1)^n a_n}{n+1} x^{n+1}.$$

Thus Tf is a polynomial of degree at most equal to $n+1$.

(ii) To T is a linear transformation, we need

$$T(f+g) = T(f) + T(g), T(cf) = cT(f).$$

For instance, the first property follows since

$$\int_{-x}^{0} (f+g)(t)\, dt = \int_{-x}^{0} f(t)\, dt + \int_{-x}^{0} g(t)\, dt$$

while the second is equivalent to

$$\int_{-x}^{0} (cf)(t)\,dt = c \int_{-x}^{0} f(t)\,dt.$$

(iii) We have

$$T(1) = x, T(x) = -\frac{x^2}{2}, T(x^2) = \frac{x^3}{3}, \dots, T(x^n) = \frac{(-1)^n}{n+1} x^{n+1}.$$

Thus the matrix of T is

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{3} & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \frac{(-1)^n}{n+1} \end{bmatrix}.$$

5. 6.

(i) We have $V = C(A)$ for the matrix $A = \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 0 & 4 \\ 1 & 1 \end{bmatrix}$. Therefore $V^\perp = N(A^T)$. Row-reducing A^T we find the matrix

$$\text{rref } A^T = \begin{bmatrix} 1 & 0 & 4/3 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

which gives the null space spanned by the vectors

$$V^\perp = \text{span} \left\{ \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(ii) Let $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ be the basis of V. We normalize v_1: $y_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and let

$$w_2 = v_2 - (v_2 \cdot y_1) y_1 = v_2 - \sqrt{2} y_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}.$$

Normalizing w_2 we find the vector $y_2 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$. The basis is $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\}$.

(iii) We have

$$u \cdot y_1 = \sqrt{2},\ u \cdot y_2 = 5$$

which shows

$$\mathrm{Proj}_V(u) = (u \cdot y_1)y_1 + (u \cdot y_2)y_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}.$$

(iv) We have

$$\mathrm{Proj}_{V^\perp}(u) = u - \mathrm{Proj}_V(u) = \begin{bmatrix} -4 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

(v) The matrix of the projection is AA^T where $A = \begin{bmatrix} 0 & 3/5 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 4/5 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ is the matrix whose columns are found in (ii).

7. Let v_1, v_2, v_3 be the columns of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We begin by normalizing the first column of A:

$$\boxed{||v_1|| = 2} \implies q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

This is the first column of Q. The first column of R simply contains the entry 2 in the upper left corner and zeros elsewhere.

Next, we produce a vector y_2 perpendicular to q_1. We have

$$\boxed{q_1 \cdot v_2 = \frac{3}{2}}$$

hence

$$y_2 = v_2 - \frac{3}{2}q_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

We have

$$\boxed{||y_2|| = \sqrt{12}/4}.$$

Thus

$$q_2 = \frac{y_2}{||y_2||} = \begin{bmatrix} -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

This is the second column of Q. The second column of R contains the entries $3/2$ and $\sqrt{12}/4$.

Finally, we produce a vector y_3 perpendicular to q_1, q_2. We have

$$\boxed{v_3 \cdot q_1 = 1}, \boxed{v_3 \cdot q_2 = 2/\sqrt{12}}$$

hence

$$y_3 = v_3 - q_1 - \frac{2}{\sqrt{12}} q_2 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Finally,

$$\boxed{||y_3|| = \sqrt{6}/3}$$

and

$$q_3 = \frac{y_3}{||y_3||} = \begin{bmatrix} 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

We have

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

and

$$R = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & \sqrt{12}/4 & 2/\sqrt{12} \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}$$

8. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -3 \\ -1 & -2 \\ 1 & 3 \end{bmatrix}.$$

(i) We have

$$A^+ = (A^T A)^{-1} A^T$$

which can be calculated directly to be

$$A^+ = \frac{1}{2} \begin{bmatrix} 3 & 2 & -3 & -2 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

(ii) The projection onto the column space of A is

$$AA^+ = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

(iii) The left null space of A is orthogonal to the column space of A. Thus the two projections onto $C(A)$ and $N(A^T)$ add up to the identity. This is simply the decomposition of a vector into components parallel and perpendicular to a subspace. Thus the matrix of the projection onto the left null space is

$$I - \text{matrix in } (ii) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(iv) The least squares solution is found by multiplying by A^+, so

$$x = A^+ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}.$$

9. Let $\mathcal{P}$ be the space degree at most equal to 2 polynomials with real coefficients.

(i) We need to verify the axioms of inner products. There are 4 such axioms:

- $(f, g+h) = (f, g) + (f, h)$;
- $(f, g) = (g, f)$;
- $c(f, g) = (f, cg) = (cf, g)$;
- $(f, f) \geq 0$ with equality if and only if $f = 0$.

The first three axioms follow from definitions. Indeed, the first axiom reads

$$(f, g+h) = \int_{-\infty}^{\infty} f(x)(g(x)+h(x))e^{-x}\, dx = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\, dx + \int_{-\infty}^{\infty} f(x)h(x)e^{-x^2}\, dx = (f, g)+(f, h)$$

which is clearly satisfied. The second is verified the same way:

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\, dx = \int_{-\infty} g(x)f(x)e^{-x^2}\, dx = (g, f).$$

and the third is entirely similar (and left to the reader). For the last axiom, we calculate

$$(f, f) = \int_{-\infty}^{\infty} f(x)^2 e^{-x^2}\, dx \geq 0$$

since we are integrating a nonnegative function $(f(x))^2 e^{-x^2} \geq 0$. Equality happens if and only if $(f(x))^2 e^{-x} = 0 \implies f = 0$.

(ii) Using the orthogonalization procedure for the polynomials

$$P_1 = 1, P_2 = x, P_3 = x^2$$

we find:

Step 1: $Q_1 = P_1 = 1$;

Step 2:

$$Q_2 = P_2 - \frac{(P_2, Q_1)}{(Q_1, Q_1)} Q_1.$$

We have

$$(P_2, Q_1) = \int_{-\infty}^{\infty} x \cdot 1 \cdot e^{-x^2}\, dx = 0$$

because the function we are integrating is odd. This yields $Q_2 = x$.

Step 3:

$$Q_3 = P_3 - \frac{(P_3, Q_1)}{(Q_1, Q_1)} Q_1 - \frac{(P_3, Q_2)}{(Q_2, Q_2)} Q_2.$$

First,

$$(Q_1, Q_1) = \int_{-\infty}^{\infty} 1 \cdot 1 \cdot e^{-x^2}\, dx = \sqrt{\pi}.$$

Here we used the integrals provided by the text of the problem. Next, we have

$$(P_3, Q_1) = \int_{-\infty}^{\infty} x^2 \cdot 1 \cdot e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

This integral is calculated using integration by parts for $u = \frac{-x}{2}, v = e^{-x^2}$. Thus the integral is

$$\int_{-\infty}^{\infty} u\, dv = -\int_{\infty}^{\infty} v\, du = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi},$$

where the last integral was provided by the problem. Similarly,

$$(P_3, Q_2) = \int_{-\infty}^{\infty} x^2 \cdot x \cdot e^{-x^2}\, dx = \int_{-\infty}^{\infty} x^3 e^{-x^2}\, dx = 0$$

again since we are integrating an odd function.

This yields

$$Q_3 = x^2 - \frac{1}{2}.$$

The basis of Hermite polynomials for $\mathcal{P}$ is $\{1, x, x^2 - \frac{1}{2}\}$.

(iii) We set

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \overline{f(x)} g(x) e^{-x^2}\, dx$$

where $\overline{f(x)}$ is the polynomial whose coefficients are the complex conjugates of the coefficients of f.

10.

(i) We easily find $\det A = -2$ by expanding along any row or column. The inverse of A can be found using the matrix of cofactors $C_{ij} = (-1)^{i+j} M_{ij}$, and

$$A^{-1} = \frac{C^T}{\det A} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

(ii) If $AB = -BA$ then $\det(AB) = \det(-BA) = -\det(BA)$ hence $\det A \cdot \det B = -\det B \cdot \det A$. Since $\det A = -2$, this yields $\det B = 0$ hence B cannot be invertible.

11.A. Similar matrices have the characteristic polynomials. The characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^3 - b\lambda - a.$$

Since the eigenvalues of B are -1 and 2 we must have that -1 and 2 are roots of the above polynomial. Hence

$$-1 + b - a = 0, 8 - 2b - a = 0$$

yielding

$$a = 2, b = 3.$$

We don't stop here though. These values are just potential candidates, but they may not work, since A and B may in fact just have the same characteristic polynomial without being similar (the implication is only in one direction). Since B is diagonal, we actually need to investigate if A is diagonalizable. For that let us look at the eigenvalue $\lambda = -1$ which has multiplicity 2 for B, hence it should have multiplicity 2 for A as well. But the eigenspace is the null space of

$$A + I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}.$$

This matrix row reduces to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has a one dimensional null space spanned by $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$. Thus the values of A and B we found are not actual solutions. The answer is that A and B cannot be similar.

11.B. We first diagonalize A. The trace of A is 4, the determinant is 3, hence the characteristic polynomial is $\lambda^2 - 4\lambda + 3$ with roots

$$\lambda = 1, \lambda = 3.$$

The eigenvalue for $\lambda_1 = 1$ is $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, while for $\lambda_2 = 3$, we have $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \implies C^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have

$$e^{tA} = Ce^{tD}C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 \\ -e^t + e^{3t} & e^t \end{bmatrix}.$$

12.A. Write $x_n = \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}$. We have

$$\begin{bmatrix} G_{n+2} \\ G_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix}.$$

Thus

$$x_{n+1} = Ax_n, \text{ for } A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

Thus

$$x_n = A^n x_0 = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To calculate A^n, we need to first diagonalize A. Indeed, the eigenvalues of A are 1 and 2. The eigenvalue $\lambda_1 = 1$ has eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

while for $\lambda_2 = 2$ we find

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus, for

$$C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \implies C^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

We have

$$A = C \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} C^{-}1 \implies A^n = C \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} C^{-1}.$$

Thus

$$x_n = C \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} C^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2^{n+1} - 1 \\ 2^n - 1 \end{bmatrix}$$

which yields

$$G_n = 2^n - 1.$$

12.B. We have $\mathrm{Tr} A = 2a, \det A = a^2 - 1$. The characteristic polynomial of A is

$$\lambda^2 - 2a\lambda + (a^2 - 1)$$

which has roots

$$\lambda_1 = a - 1, \lambda_2 = a + 1.$$

These eigenvalues are real. When $a < -1$, the eigenvalues are negative and the equation is stable. When $a = -1$, the equation is neutrally stable since a root is 0 while the other one is -2. When $a > -1$, the equation is unstable because the eigenvalue $a + 1$ is positive.

12.C. Observe that A is a Markov matrix. The limit $\lim Y_n$ is always a multiple of the $\lambda = 1$ eigenvector. The $\lambda = 1$ eigenvector is found by computing $N(I - A)$. In fact, an eigenvector equals

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This can be seen without any computation because the rows of A add up to 1 as well. To find the limit, recall that the sum of entries of Y_n is preserved (recall the California population model from lecture). The sum of entries of Y_0 is 8. The same should happen in the limit. Now, the limit is a multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Thus

$$\lim_{n\to\infty} Y_n = \frac{8}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

13.A. We have

$$A = UDU^{-1} = UDU^H$$

where $U^{-1} = U^H$ because U is unitary. We compute using the rules of Hermitian transpose

$$A^H = (U^H)^H D^H U^H = UD^H U^H.$$

Thus

$$AA^H = (UDU^H)(UD^H U^H) = UDD^H U^H$$

using that $U^H U = I$ for the middle terms. Similarly,

$$A^H A = (UD^H U^H)(UDU^H) = UD^H DU^H.$$

Since D is diagonal, $DD^H = D^H D$ as it can be readily checked. Thus $AA^H = A^H A$.

13.B.

(i) $A = \frac{1}{2}\begin{bmatrix}1+i & 1-i\\1-i & 1+i\end{bmatrix}$ is unitary. Indeed

$$A^H = \frac{1}{2}\begin{bmatrix}1-i & 1+i\\1+i & 1-i\end{bmatrix}.$$

One checks that

$$AA^H = A^H A = I.$$

This matrix is also normal because any unitary matrix is normal. Clearly A is not Hermitian or skew Hermitian.

(ii) $A = \begin{bmatrix}3 & 2-i & -3i\\2+i & 0 & 1-i\\3i & 1+i & 0\end{bmatrix}$ is Hermitian since

$$A^H = \begin{bmatrix}3 & 2-i & -3i\\2+i & 0 & 1-i\\3i & 1+i & 0\end{bmatrix} = A.$$

This matrix is automatically normal. The matrix cannot be skew Hermitian, or unitary (just look at the lengths of any column).

(iii) $A = \begin{bmatrix}i & 2+i\\-2+i & 4i\end{bmatrix}$ is skew Hermitian since

$$A^H = \begin{bmatrix}-i & -2-i\\2-i & -4i\end{bmatrix} = -A.$$

This matrix is therefore also normal. The matrix is not unitary (just look at the length of columns).

(iv) $A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This matrix is not unitary (look at the length of columns), Hermitian or skew-Hermitian. It is however normal, since $AA^H = A^H A$ as one can easily check.

(v) $A = \frac{1}{\sqrt{2}} \begin{bmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{bmatrix}$ where z is a complex number of modulus 1. We have

$$A^H = \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{z} & -i\bar{z} \\ z & iz \end{bmatrix}.$$

If A is Hermitian then comparing upper right corners we must have $\bar{z} = -i\bar{z} \implies \bar{z} = 0 \implies z = 0$ which is not allowed. Similarly A cannot be skew Hermitian. However, A is unitary since

$$AA^H = \frac{1}{2} \begin{bmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{bmatrix} \cdot \begin{bmatrix} \bar{z} & -i\bar{z} \\ z & iz \end{bmatrix} = \begin{bmatrix} z\bar{z} & 0 \\ 0 & z\bar{z} \end{bmatrix} = I$$

since $z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 = 1$ as $|z| = 1$. Clearly A is also normal.

13.C. We have

$$\text{Tr}A = 10, \det A = 24 - (2-2i)(2+2i) = 16.$$

The characteristic polynomial is $\lambda^2 - 10\lambda + 16$ with roots

$$\lambda_1 = 2, \lambda_2 = 8.$$

The eigenspace for $\lambda_1 = 2$ is the null space of

$$A - 2I = \begin{bmatrix} 2 & 2+2i \\ 2-2i & 4. \end{bmatrix}.$$

which is spanned by $v_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$. This vector doesn't have length 1 as required for unitary matrices, hence we renormalize it by its length. We have

$$||v_1||^2 = |(-1-i)|^2 + 1^2 = 3$$

hence

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1-i \\ 1 \end{bmatrix}.$$

Similarly $\lambda_2 = 8$ yields the eigenvector

$$v_2 = \begin{bmatrix} 1 \\ -i+1 \end{bmatrix} \implies ||v_2|| = \sqrt{3} \implies u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -i+1 \end{bmatrix}.$$

We have

$$A = U \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} U^{-1}$$

where

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} -1-i & 1 \\ 1 & -i+1 \end{bmatrix}.$$

Other answers may also be correct.

13.D. Clearly

$$A^H = I^H - 2(\mathbf{v}^H)^H \cdot \mathbf{v}^H = I - 2\mathbf{v} \cdot \mathbf{v}^H = A$$

hence A is Hermitian. Now we need to check $AA^H = I$ or equivalently $A^2 = I$. We compute

$$A^2 = (I - 2\mathbf{v} \cdot \mathbf{v}^H)^2 = I - 4\mathbf{v} \cdot \mathbf{v}^H + 4\mathbf{v} \cdot \mathbf{v}^H \cdot \mathbf{v} \cdot \mathbf{v}^H = I - 4\mathbf{v} \cdot \mathbf{v}^H + 4\mathbf{v} \cdot \mathbf{v}^H = I.$$

In the above calcualtion, we used

$$\mathbf{v}^H \mathbf{v} = 1$$

which is true because the product calculates the length square of v which is 1.

13.E.

(i) False. We have seen many diagonalizable non-symmetric matrices. Their diagonal form is symmetric and similar to the original non-symmetric matrix.

(ii) True. Similar matrices must have the same trace. The trace of $A+I$ equals Tr $A+n$, while the trace of $A-I$ equals Tr $A-n$ for a matrix of size n. These traces are clearly not equal, hence the matrices can't be similar.

(iii) True. Determinants are preserved by similarity. An invertible matrix has nonzero determinant so it can't be similar to a singular matrix.

(iv) False. If $A = PD_1P^{-1}, B = QD_2Q^{-1}$ then $AB = PD_1P^{-1}QD_2Q^{-1}$ which may not be diagonalizable unless $P = Q$. Product of simultaneously diagonalizable matrices is however diagonalizable.

(v) False. If $A^H = A, B^H = B$ then $(AB)^H = B^H A^H = BA$. It could be that A and B don't commute.

(vi) True. AA^H is Hermitian hence normal. To see AA^H is Hermitian, we compute $(AA^H)^H = (A^H)^H A^H = AA^H$.

(vii) True. The determinant is product of the 4 eigenvalues, all of which are purely imaginary. Product of 2 purely imaginary numbers is real because $i^2 = -1$, so product of 4 purely imaginary numbers is real as well.

(viii) False. A unitary matrix has 4 eigenvalues of absolute value 1. Their sum has absolute value at most equal to 4. Since the sum of eigenvalues is the trace, and $3+4i$ has absolute value 5, we conclude the trace can't be $3+4i$.

(ix) True. This was stated in class.

14.

(i) Since the trace is 9 and is the sum of eigenvalues, and 2 is a repeated eigenvalue, we conclude that the eigenvalues of A are $2, 2, 5$. The eigenspace for $\lambda_1 = \lambda_2 = 2$ is the null space of

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

A basis for this eigenspace is

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

These vectors are not orthogonal so we need to run Gram-Schmidt for them. First

$$q_1 = \frac{1}{\sqrt{2}} u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

To find q_2, we first calculate the vector

$$y_2 = u_2 - (u_2 \cdot q_1) q_1 = u_2 - \frac{1}{\sqrt{2}} q_1 = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}.$$

We have $||y_2|| = \frac{\sqrt{6}}{2}$ hence

$$q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The eigenvector for $\lambda_3 = 5$ is found from the null space of

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

An eigenvector is

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We need to normalize this vector to have length 1:

$$q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

There are other correct possible answers.

(ii) The eigenvalues of A are $2, 2, 5$ hence A is positive definite. We can pick $R = \sqrt{D}Q^T$ as a possible answer. In this case

$$R = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ \sqrt{1/3} & -2/\sqrt{3} & 1/\sqrt{3} \\ \sqrt{5/3} & \sqrt{5/3} & \sqrt{5/3} \end{bmatrix}.$$

There are other possible answers.

(iii) The three squares

$$f = f_1^2 + f_2^2 + f^3$$

are found by computing

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

This yields $f_1 = x - z$, $f_2 = \frac{1}{\sqrt{3}}(x - 2y + z)$, $f_3 = \sqrt{5/3}(x + y + z)$. There are other possible answers.

15.

(i) The corresponding symmetric matrix is

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$$

which has

$$\mathrm{Trace}(A) = 5 > 0, \det A = 2 > 0$$

Therefore Q is positive definite.

(ii) The corresponding matrix is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}.$$

The characteristic polynomial of A equals

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -3 & -3 \\ -3 & \lambda - 1 & -3 \\ -3 & -3 & \lambda - 1 \end{bmatrix} = (\lambda - 7)(\lambda + 2)^2.$$

The eigenvalues 7 and -2 have opposite sign, so Q is indefinite.

16.

(i) We have

$$A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

with eigenvalues 18 and 0. The eigenvectors are

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

This gives already the matrix

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It remains to find U. First

$$AA^T = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

with eigenvalues $18, 0, 0$. The $\lambda = 18$ unit eigenvector is

$$u_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}.$$

The $\lambda = 0$ eigenspace is the null space of the above matrix, that is

$$2x - 4y + 4z = 0.$$

A possible basis is

$$w_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, w_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

This basis is not orthonormal, so we apply Gram-Schmidt. We first normalize w_2 to the vector

$$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{bmatrix}.$$

Then we compute

$$y_3 = w_3 - (u_2 \cdot w_3)u_2 = w_3 - \frac{4}{\sqrt{5}}u_2 = \frac{1}{5}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

which we normalize to the vector

$$u_3 = \begin{bmatrix} 2/\sqrt{45} \\ 5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}.$$

We find

$$U = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \\ 2/3 & -1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix}.$$

(ii) The pseudoinverse is

$$A^+ = V\Sigma^+ U^T,$$

where

$$\Sigma^+ = \begin{bmatrix} \frac{1}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find

$$A^+ = \frac{1}{18}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}.$$

(iii) The matrix of the projection is

$$AA^+ = \frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

(iv) The least squares of minimum length is $A^+b = \frac{1}{18}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(v) The rank of A is 1. The first column of U is spans the column space of A, the second and third columns of U span the left null space. The first column of V spans the row space of A, the second column of V spans the null space of A.