MATH 102 - SOLUTIONS TO PRACTICE PROBLEMS - MIDTERM I

1. We carry out row reduction. We begin with the row operations

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$$

yielding the matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

This is already upper triangular hence

$$U = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

The lower triangular matrix equals

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

2. For further reference, we begin by row-reducing the augumented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & b_1 \\ 2 & 3 & 5 & 7 & b_2 \\ -1 & 0 & -1 & -2 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & b_1 \\ 0 & -1 & -1 & -1 & b_2 - 2b_1 \\ 0 & 2 & 2 & 2 & b_1 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 & 2b_2 - 3b_1 \\ 0 & 1 & 1 & 1 & -b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & 2b_2 + b_3 - 3b_1 \end{bmatrix}.$$

(i) The null space of A is also the null space of the row-reduced matrix

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two variables x, y are pivots, the last two variables z, w are free. We obtain the system

$$x + z + 2w = 0 \implies x = -z - 2w$$

$$y + z + w = 0 \implies y = -z - w.$$

We conclude

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $N(A)$ has the basis

$$N(A) = \mathrm{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The nullity of A equals 2.

(ii) Each vector in the null space gives a relation between the columns of A. For instance, the vector

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \in N(A) \implies -\mathbf{c_1} - \mathbf{c_2} + \mathbf{c_3} = 0$$

and

$$\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in N(A) \implies -2\mathbf{c_1} - \mathbf{c_2} + \mathbf{c_4} = 0.$$

(iii) The last row of zeros in the row-reduced augumented matrix gives the equation

$$2b_2 + b_3 - 3b_1 = 0$$

that must be satisfied by all vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the column space.

(iv) The pivot columns, namely the first and second columns of A, give a basis for $C(A)$:

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

The rank of A equals 2.

(v) The pivots are in the first and second row, so a basis for the row space of A is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(vi) The dimension of the left-null space of A is the number of rows minus the rank which is $3 - 2 = 1$.

(vii) We can easily find a particular solution

$$x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The general solution is $x = x_p + x_h$ hence

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

(viii) The rank of A equals 2 which not equal to the number of rows or columns of A. Hence A does not admit a left inverse nor a right inverse.

3. Assume first that the vectors $\{\mathbf{u}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - 2\mathbf{w}\}$ are linearly dependent. There exist constants a, b, c not all zero such that

$$a\mathbf{u} + b(\mathbf{v} - \mathbf{w}) + c(\mathbf{u} + \mathbf{v} - 2\mathbf{w}) = 0.$$

Rearranging, we obtain

$$(a + c)\mathbf{u} + (b + c)\mathbf{v} + (b - 2c)\mathbf{w} = 0.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis, the vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ must be independent. This implies that

$$a + c = 0,\ b + c = 0,\ b - 2c = 0.$$

Solving for a, b, c we find

$$a = b = c = 0$$

which is impossible by assumption that not all (a, b, c) are zero.

It follows that $\{\mathbf{u}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - 2\mathbf{w}\}$ are three linearly independent vectors in $\mathbb{R}^3$ hence they must form a basis of $\mathbb{R}^3$.

4.

(i) Not a vector space. For instance $(1, 0, \ldots, 0)$ is in the set, but twice the vector $2(1, 0, \ldots, 0)$ is not in the set.

(ii) Vector space. This is the null space of the matrix

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Null spaces are always vector spaces.

(iii) Vector space. This is a bit harder to see. First, we determine the set a bit more explicitly. The square of the distance to $(3, -4, 0, 0)$ equals

$$(x - 3)^2 + (y + 4)^2 + z^2 + w^2$$

while the square distance to $(0, -3, 4, 0)$ equals

$$x^2 + (y + 3)^2 + (z - 4)^2 + w^2.$$

Since the two distances must be equal, we have

$$(x - 3)^2 + (y + 4)^2 + z^2 + w^2 = x^2 + (y + 3)^2 + (z - 4)^2 + w^2.$$

Expanding out and canceling, we obtain the linear equation

$$-3x + y + 4z = 0.$$

The requirement that the distances from $(0, -3, 4, 0)$ and $(0, 4, 3, 0)$ be equal can be worked out in a similar fashion. We obtain the equation

$$-7y + z = 0.$$

Thus the set in question can be described by the two linear equations

$$-3x + y + 4z = 0, \;\; -7y + z = 0.$$

This is a null space of the matrix

$$\begin{bmatrix} -3 & 1 & 4 & 0 \\ 0 & -7 & 1 & 0 \end{bmatrix}$$

hence it is a subspace.

(iv) Vector space. Indeed, if P and Q are two polynomials with

$$P(0) = P'(0) = P''(0) = 0, Q(0) = Q'(0) = Q''(0) = 0$$

then their sum $P + Q$ also satisfies

$$(P + Q)(0) = (P + Q)'(0) = (P + Q)''(0) = 0.$$

Similarly, scalar multiplication is preserved since cP satisfies

$$(cP)(0) = (cP)'(0) = (cP)''(0) = 0.$$

(v) Vector space. This is the set of matrices

$$A^T = -A.$$

This is a subspace since if A is skew symmetric so is cA since

$$(cA)^T = cA^T = -cA.$$

Similarly, if A, B are skew symmetric so is their sum $A + B$ since

$$(A + B)^T = A^T + B^T = -A - B = -(A + B).$$

(vi) Not a vector space. If y is a solution of the equation, then $y'' + 4y = \sin t$. But $2y$ is not a solution since

$$(2y)'' + 4(2y) = 2 \sin t.$$

Since the equation changes, the solution set is not a vector space.

5. We calculate the null space of rref (A). This is the space spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. Elements in the null space of A give relations between the columns of A. Writing v_1, v_2, v_3, v_4 for the columns of A we must have

$$v_2 = v_4 - v_1 - v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

6. Note that

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$T\left(\begin{bmatrix}0\\2\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right) = 2\cdot\begin{bmatrix}2\\-1\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix}.$$

7.

(i) Clearly, $\{1, x, x^2\}$ is a basis of $\mathcal{P}_2$ hence

$$\dim \mathcal{P}_2 = 3.$$

To show that $\{1, x-1, x^2\}$ is also a basis, it suffices to prove that the three vectors are linearly dependent. Assume otherwise, so that

$$a + b(x-1) + cx^2 = 0$$

for some constants a, b, c. This rewrites as

$$(a-b) + bx + cx^2 = 0 \implies a - b = 0, b = 0, c = 0 \implies a = b = c = 0$$

proving the independence.

(ii) We calculate

$$T(1) = 1, T(x-1) = (x-1), T(x^2) = 2x + x^2 = x^2 + 2(x-1) + 2\cdot 1.$$

For the last equality, we needed to express $T(x^2) = 2x + x^2$ in terms of the basis elements $1, x-1, x^2$. The matrix of T in this basis is given by collecting the coefficients of the basis elements $1, x-1, x^2$, yielding the matrix

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}.$$

8. True. To explain this fact, note that C has fewer rows than columns. Then C must have free variables. This in turn means that the null space $N(C)$ contains a non-zero vector $\mathbf{x}$:

$$C\mathbf{x} = 0.$$

We claim that $\mathbf{x}$ is also in the null space of A. Indeed,

$$A\mathbf{x} = BC\mathbf{x} = B\cdot 0 = 0.$$

This shows that $N(A)$ contains a nonzero vector.

9. Consider Mat_2 the space of 2×2 matrices. This is a 4-dimensional vector space. The 5 matrices I, A, A^2, A^3, A^4 are vectors in Mat_2. Since 5 vectors in a 4-dimensional vector space must be linearly dependent, we can therefore find constants c_0, c_1, c_2, c_3, c_4 not all zero such that

$$c_0 I + c_1 A + c_2 A^2 + c_3 A^3 + c_4 A^4 = 0.$$