Math 102 - Winter 2013 - Midterm II

Name: ______________________________

Student ID: ___________________________

Section time: __________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones.

Read each question carefully, and show all your work.

There are 5 questions which are worth 50 points. You have 50 minutes to complete the test.

Question	Score	Maximum
1		11
2		15
3		8
4		8
5		8
Total		50

Problem 1. [*11 points.*]

Consider the subspace V of $\mathbb{R}^3$ spanned by the **orthonormal** vectors

$$\vec{v}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}.$$

(i) [4] Find the projection of the vector $\vec{v} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$ onto V.

`Solution:` *We have* $v \cdot v_1 = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = 3$. *Similarly* $v \cdot v_2 = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} = 3$.

Thus

$$Proj_V(v) = (v \cdot v_1)v_1 + (v \cdot v_2)v_2 = 3v_1 + 3v_2 = 3 \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} + 3 \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.$$

(ii) [4] Find a basis for the orthogonal complement $V^{\perp}$.

`Solution:` *For the matrix*

$$A = \begin{bmatrix} 2/3 & 2/3 \\ 2/3 & -1/3 \\ 1/3 & -2/3 \end{bmatrix},$$

we have $V = C(A)$. *Hence*

$$V^{\perp} = N(A^T) = N\left(\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}\right) = N\left(\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \mathit{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right\}.$$

(iii) [3] Find the matrix of the projection onto $V^{\perp}$.

`Solution:` *We know that the matrix of the projection is obtained as*

$$\frac{w \cdot w^T}{w^T \cdot w} = \frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & 4 \\ 2 & -4 & -4 \end{bmatrix}.$$

Problem 2. [*15 points.*]

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & -2 \\ -1 & -1 \end{bmatrix} \text{ and the vector } b = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(i) [4] Find the left inverse of A.

`Solution:` *We know the left inverse $A^+ = (A^TA)^{-1}A^T$. We compute*

$$A^TA = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$$

hence

$$(A^TA)^{-1} = \frac{1}{4} \begin{bmatrix} 10 & -6 \\ -6 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}.$$

Finally,

$$A^+ = \frac{1}{2} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & -2 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 2 & 1 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(ii) [4] Find the matrix of the orthogonal projection onto the column space of A.

`Solution:` *The matrix equals*

$$AA^+ = \frac{1}{2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

(iii) [4] Find the QR decomposition of the matrix A.

`Solution:` *We find the first column of Q. We have*

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \implies \boxed{||v_1|| = 2} \implies q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

To find the second column of Q we use Gram-Schmidt on the vector $v_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$. *We have*

$y_2 = v_2 - (v_2 \cdot q_1)q_1$. *Note that* $\boxed{v_2 \cdot q_1 = 3}$ *hence*

$$y_2 = v_2 - 3q_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

Note that $\boxed{||y_2|| = 1}$ *hence* $q_2 = y_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$. *Thus*

$$Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}.$$

(iv) [3] Find the least squares solution of the system $Ax = b$ using any method you like.

`Solution:` *The least squares solution has the form*

$$x^\star = A^+ b = \frac{1}{2} \begin{bmatrix} -1 & 2 & 1 & -2 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Problem 3. [*8 points.*]

Consider the two matrices

$$A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(i) [3] Find the value of a for which the matrices A and B have the same determinant.

`Solution:` *We calculate*

$$\det A = a, \det B = -2$$

by expanding along the rows or columns. We must therefore have $a = -2$.

(ii) [5] Are there any values of a for which the matrices A and B are similar?

`Solution:` *Note that similar matrices have the same determinant, hence $a = -2$. Now, the question is asking if A is similar to the diagonal matrix B, hence diagonalizable. The matrix B has eigenvalues $1, 1, -2$. The same must be true about A as well. We just need to verify that the eigenvalue 1 has a 2 dimensional eigenspace. For this we compute*

$$E_1 = N(A - I) = N\left(\begin{bmatrix} -1 & 0 & -2 \\ 1 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Since this is one dimensional, A and B can never be similar.

Problem 4. [*8 points.*]

Let $\mathcal{P}$ be the vector space of polynomials of degree less or equal to 2, endowed with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)x^2\,dx.$$

Find an orthogonal basis for $\mathcal{P}$ using Gram-Schmidt on the standard basis $\{1, x, x^2\}$.

`Solution:` *We have $P_1 = 1, P_2 = x, P_3 = x^2$. Running Gram-Schmidt, we first set $Q_1 = P_1 = 1$. To find Q_2 we have*

$$Q_2 = P_2 - \frac{\langle P_2, Q_1\rangle}{\langle Q_1, Q_1\rangle}\cdot Q_1 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}\cdot 1 = x - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x\,dx}\cdot 1 = x.$$

Finally, we have

$$Q_3 = P_3 - \frac{\langle P_3, Q_1\rangle}{\langle Q_1, Q_1\rangle}\cdot Q_1 - \frac{\langle P_3, Q_2\rangle}{\langle Q_2, Q_2\rangle}\cdot Q_2.$$

Note that

$$\begin{aligned}\langle Q_1, Q_1\rangle &= \langle 1, 1\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}\\ \langle P_3, Q_1\rangle &= \langle x^2, 1\rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5}\\ \langle P_3, Q_2\rangle &= \langle x^2, x\rangle = \int_{-1}^{1} x^5\,dx = 0\end{aligned}$$

hence substituting we find

$$Q_3 = x^2 - \frac{2/5}{2/3}\cdot 1 = x^2 - \frac{3}{5}.$$

The orthogonal basis is $\{1, x, x^2 - \frac{3}{5}\}$.

Problem 5. [*8 points.*]

Let V be subspace of $\mathbb{R}^n$ of dimension k for $0 < k < n$, and let T denote the orthogonal projection $\mathrm{Proj} : \mathbb{R}^n \to \mathbb{R}^n$ onto V.

(i) [1] What is the dimension of $V^\perp$?

`Solution:` *We know that* $\dim V + \dim V^\perp = n$ *hence*

$$\dim V^\perp = n - k.$$

(ii) [2] Explain that the eigenvalues of T equal either 0 and 1, and write down the two corresponding eigenspaces.

`Solution:` *For $v \in V$, we have $T(v) = v$ hence 1 is an eigenvalue for T and V is the eigenspace. For $v \in V^\perp$, we have $T(v) = 0$ hence 0 is an eigenvalue for T and $V^\perp$ is the eigenspace.*

(iii) [3] Is T diagonalizable? Why or why not?

`Solution:` *T is diagonalizable since the eigenspaces V and $V^\perp$ span $\mathbb{R}^n$.*

(iv) [2] What is the trace of T?

`Solution:` *We know that the trace of T equals the sum of eigenvalues, in this case $1 + \ldots + 1 + 0 + \ldots + 0 = k$ since the eigenvalue 1 is counted with multiplicity k.*