## * 4.3 Vector Fields:

A vector field $\vec{F}$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that assigns to each point $\vec{x} = (x_1, x_2, \ldots, x_n)$ a vector $\vec{F}(\vec{x}) = \vec{F}(x_1, \ldots, x_n)$.

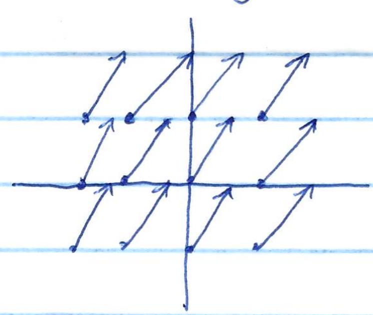
E.g. velocity fields (wind, fluids)

force fields (magnetic, gravitational).

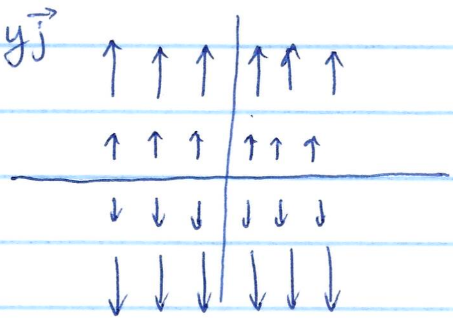
in 2D: $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$.

in 3D: $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$.

E.g. $\vec{F} = \vec{i} + \vec{j}$



$\vec{F} = y\vec{j}$



* Gradient vector fields:

Given a function $f(x,y,z)$, its gradient is

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

$\Rightarrow$ it assigns to each point a vector.

If $\vec{F} = \nabla f$ for some function $f$, we call $\vec{F}$ a gradient vector field and $f$ the potential of the vector field.

E.g. • Gravitational force fields.

$$\vec{F} = -\frac{mMG}{\|\vec{r}\|^3}\vec{r} \qquad \text{where } \vec{r}(x,y,z) = (x,y,z).$$

$$f = \frac{mMG}{\|\vec{r}\|}.$$

• $\vec{F} = y\vec{i} + x\vec{j}$ is a gradient vector field with $f(x,y) = xy$.

Q: How to find $f$?

A: Integrate.

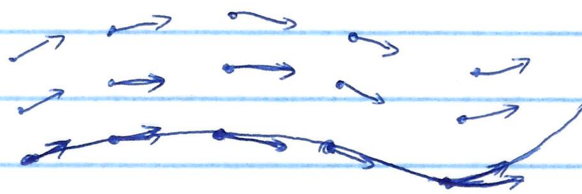
since $\frac{\partial f}{\partial x} = y. \Rightarrow \int \frac{\partial f}{\partial x}\,dx = \int y\,dx = xy + c$

• $\vec{F} = y\vec{i} - x\vec{j}$ is not a gradient vector field.

$$\left( \begin{array}{ccc} \frac{\partial f}{\partial x} = y & \text{and} & \frac{\partial f}{\partial y} = -x. \\ \Downarrow & & \\ \frac{\partial^2 f}{\partial y \partial x} = 1 & \neq & \frac{\partial^2 f}{\partial x \partial y} = -1 \end{array} \right).$$

* Flow lines:

Given a vector field $\vec{F}$, a flow line is a path $\vec{c}(t)$ such that $\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$.




E.g. $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$.

Find a flow line.

$$\vec{c}(t) = (x(t), y(t))$$
$$\vec{F}(\vec{c}(t)) = \vec{c}\,'(t).$$
$$-y(t)\vec{i} + x(t)\vec{j} = x'(t)\vec{i} + y'(t)\vec{j}.$$

$$\Rightarrow \begin{cases} -y(t) = x'(t) \\ x(t) = y'(t). \end{cases}$$

$x(t) = \cos(t)$ and $y(t) = \sin(t)$ work.

$\Rightarrow \vec{c}(t) = (\cos(t), \sin(t))$

E.g. Let $\vec{F} = x\vec{i} + 2x\vec{j} + 3y\vec{k}$.

Find a flow line.

$$\vec{F}(\vec{c}(t)) = \vec{c}\,'(t).$$
$$(x(t), 2x(t), 3y(t)) = (x'(t), y'(t), z'(t)).$$

$\Rightarrow$ $x(t) = x'(t) \Rightarrow x(t) = e^t$.

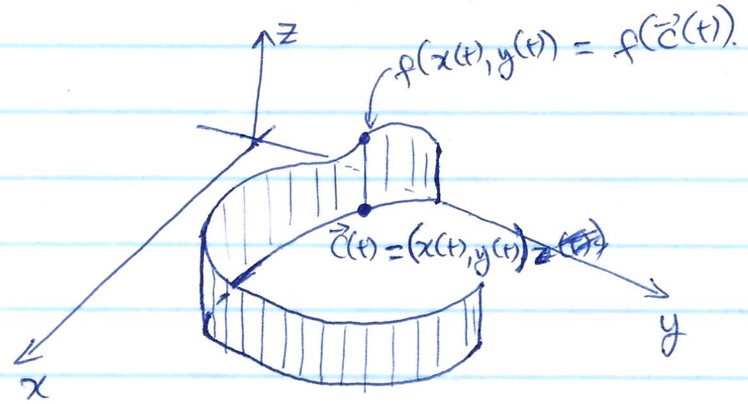
$y'(t) = 2x(t) \Rightarrow 2e^t \Rightarrow y(t) = 2e^t$.

$z'(t) = 6e^t \Rightarrow z(t) = 6e^t$.

$\Rightarrow c(t) = (e^t, 2e^t, 6e^t)$.

Goal of the course: generalize the FTOC to several variables.

Need the concept of a path integral.




"area of the fence" whose base is the image of $\vec{c}$ and height $f(x,y)$ at $(x,y)$.

* 7.1 The path integral.

Def: (2 variables) The path integral, or the integral of $f(x,y)$ along the path $\vec{c}$, is defined by

$$\int_{\vec{c}} f\,ds := \int_a^b f(x(t), y(t))\, \|c'(t)\|\, dt.$$

where $\vec{c}: [a,b] \to \mathbb{R}^2$, $\vec{c}$ is differentiable.

E.g. The base of the fence in the first quadrant

$$\vec{c}: [0, \tfrac{\pi}{2}] \to \mathbb{R}^2$$

$$\vec{c}(t) = (30\cos^3 t,\ 30\sin^3 t).$$

the height of the fence at $(x,y)$ is

$$f(x,y) = 1 + \frac{y}{3}.$$

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scalar

Area of fence $= \int_{\vec{c}} f\, ds$

arc length $\quad ds = \|\vec{c}\,'(t)\|\, dt.$

$$\boxed{c'(t) = (-90\cos^2 t \sin t,\ 90\sin^2 t \cos t)} = \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right)\sqrt{90^2\cos^4 t \sin^2 t + 90^2 \sin^4 t \cos^2 t}\ dt$$

$$= \int_0^{\pi/2} \left(1 + \frac{30\sin^3 t}{3}\right) 90 \cos t \sin t\ dt$$

$$= 90\int_0^{\pi/2} (\sin t + 10 \sin^4 t) \cos t\ dt.$$

$u = \sin t$

$du = \cos t\, dt$

$$= 90 \int_0^1 (u + 10u^4)\, du.$$

$$= 90 \left[\frac{u^2}{2} + 2u^5\right]_0^1$$

$$= 90\left(\frac{1}{2} + 2\right)$$

$$= 225.$$

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3 variables:

$$\int_{\vec{c}} f\, ds := \int_a^b f(x(t), y(t), z(t))\, \|\vec{c}\,'(t)\|\, dt.$$

E.g. $\vec{c}(t) = (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$.

$f(x, y, z) = x^2 + y^2 + z.$

Evaluate $\int_{\vec{c}} f\, ds.$

Sol: $\int_{\vec{c}} f\, ds = \int_0^{2\pi} f(\vec{c}(t))\, \cancel{dt}\, \|\vec{c}\,'(t)\|\, dt.$

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$$
\begin{aligned}
f(\vec{c}(t)) &= f(x(t), y(t), z(t)) \\
&= x^2(t) + y^2(t) + z(t) \\
&= \cos^2 t + \sin^2 t + t \\
&= 1 + t.
\end{aligned}
$$

$$\vec{c}\,'(t) = (-\sin t, \cos t, 1)$$

$$
\begin{aligned}
\|\vec{c}\,'(t)\| &= \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} \\
&= \sqrt{2}.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \int_{\vec{c}} f\,ds &= \int_0^{2\pi} (1+t)\sqrt{2}\,dt = \sqrt{2}\left(t + \frac{t^2}{2}\right)_0^{2\pi} \\
&= \sqrt{2}\left(2\pi + \frac{(2\pi)^2}{2}\right) \\
&= 2\sqrt{2}\,\pi + 2\sqrt{2}\,\pi^2.
\end{aligned}
$$

## 7.2 Line Integrals.

Motivation

$$W = (\text{force}) \cdot \cancel{\text{distance}} \ (\text{displacement in the direction of force}).$$

over the short distance: $W \approx \vec{F} \cdot \vec{\Delta s}$

total work over long distance along trajectory $c$:

$$W \approx \sum_i \vec{F}(t_i) \cdot (\Delta \vec{s})_i \longrightarrow \int_{\vec{c}} \vec{F} \cdot d\vec{s}.$$

$$\Rightarrow \quad W = \int_{\vec{c}} \vec{F} \cdot d\vec{s}.$$

**Def:** Let $\vec{F}$ be a vector field in $\mathbb{R}^3$, continuous on $\vec{c}: [a, b] \to \mathbb{R}^3$. The <u>line integral</u> of $\vec{F}$ along $\vec{c}$ is defined as

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int \vec{F}(x(t), y(t), z(t)) \cdot \vec{c}'(t)\, dt.$$

E.g. $\vec{c}(t) = (\cos t, \sin t, t), \quad 0 \le t \le 2\pi.$

$\vec{F}(x, y, z) = x^2 \vec{i} + y^2 \vec{j} + z\vec{k}.$

Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{s}.$

Sol: Along $\vec{c}$,

$$\vec{F}(x, y, z) = \vec{F}(\cos t, \sin t, t) = \cos^2 t\, \vec{i} + \sin^2 t\, \vec{j} + t\vec{k}.$$

~~then~~ and

$$\vec{c}'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + 1\vec{k}.$$

$$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (\cos^2 t\, \vec{i} + \sin^2 t\, \vec{j} + t\vec{k}) \cdot (-\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k})\, dt$$

$$= \int_0^{2\pi} - \cos^2 t \sin t + \sin^2 t \cos t + t\, dt.$$

$$= \int_0^{2\pi} -\cos^2 t \sin t\, dt + \int_0^{2\pi} \sin^2 t \cos t\, dt + \int_0^{2\pi} t\, dt$$

$$= \left.\frac{\cos^3 t}{3}\right|_0^{2\pi} + \left.\frac{\sin^3 t}{3}\right|_0^{2\pi} + \left.\frac{t^2}{2}\right|_0^{2\pi}$$

$$= \cancel{\cos^3}\ 0 + 0 + \frac{4\pi^2}{2}$$

$$= 2\pi^2.$$

New notation: for the line integral

$\vec{F} = (P, Q, R)$

and $d\vec{s} = (dx, dy, dz)$

We write

$$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} P dx + Q dy + R dz$$

$$= \int_a^b \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

Remark: this is not the sum of 3 integrals. It is just another way to write $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$. We still need to parametrize and express everything in terms of the parameter.

E.g. $\vec{c}(t) = (\cos t, \sin t, t) \qquad 0 \le t \le 2\pi$

$$\vec{F}(x,y,z) = \underset{\uparrow \atop P}{x^2}\vec{i} + \underset{\uparrow \atop Q}{y^2}\vec{j} + \underset{\uparrow \atop R}{z}\vec{k}.$$

so $\displaystyle\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_{\vec{c}} x^2 dx + \oint y^2 dy + z dz.$

$$= \int_a^{2\pi} \left( x^2(t)\frac{dx}{dt} + y^2(t)\frac{dy}{dt} + z(t)\frac{dz}{dt} \right) dt$$

$$= \int_0^{2\pi} \left( \cos^2 t\,(-\sin t) + \sin^2 t\,\cos t + t \right) dt.$$

$$= 2\pi^2.$$

E.g. Evaluate $\displaystyle\int_{\vec{c}} x^2 dx + xy\,dy$

where $\vec{c}(t) = (t, t^2), \qquad 0 \le t \le 1.$

$$\int_{\vec{c}} x^2 dx + xy\,dy = \int_0^1 \left( x^2(t)\frac{dx}{dt} + x(t)\,y(t)\frac{dy}{dt} \right) dt.$$

$$= \int_0^1 \left( t^2 + t\cdot t^2\cdot 2t \right) dt$$

$$= \int_0^1 \left( t^2 + 2t^4 \right) dt$$

$$= \frac{t^3}{3} + \frac{2}{5}t^5 \Big|_0^1$$

$$= \frac{1}{3} + \frac{2}{5}.$$

$$= \frac{11}{15}.$$

Remark: line integrals are independent of the parametrizatio[n] as long as the parametrization is orientation preserving.

E.g. The curve $y = x^3$ from $(0,0)$ to $(1,1)$ can be parametrized as $\vec{c}(t) = (t, t^3)$ $\quad 0 \le t \le 1$

or as $\vec{p}(\theta) = (\sin\theta, \sin^3\theta)$ $\quad 0 \le \theta \le \pi/2$.

Let $\vec{F} = x\vec{i} + y\vec{j}$.

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^1 (t\vec{i} + t^3\vec{j}) \cdot \vec{c}\,'(t)\, dt$$

$$\text{or} \quad = \int_0^{\pi/2} (\sin\theta\, \vec{i} + \sin^3\theta\, \vec{j}) \cdot \vec{p}\,'(\theta)\, d\theta .$$

Check!

* Line integral of gradient field:

FTOC: $\displaystyle\int_a^b f'(t)\, dt = f(b) - f(a)$.

$$\boxed{\begin{array}{l}\text{Thm: (Fundamental theorem of line integrals).} \\ f: \mathbb{R}^3 \to \mathbb{R} \text{ differentiable and } c: [a,b] \to \mathbb{R}. \\ (c \text{ is continuous or piecewise cts.}) \\ \displaystyle\int_{\vec{c}} (\nabla f) \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))\end{array}}$$

Remark: If the field is a gradient field, only the end points matter.

E.g. Evaluate $\int_{\vec{c}} \nabla f \cdot d\vec{s}$ where ~~$f(x,y,z)$~~

$$f(x,y,z) = \cos x + \sin y - xyz$$

and $\vec{c}$ is a trajectory that starts at $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ and ends at $(\pi, 2\pi, 1)$.

Sol: Let $\vec{c}(t)$, $a \le t \le b$, be a path with $\vec{c}(a) = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ & $\vec{c}(b) = \left(\pi, \frac{2\pi}{\cancel{2}}, 1\right)$.

Then by ~~FTOQL~~ FTOLI.

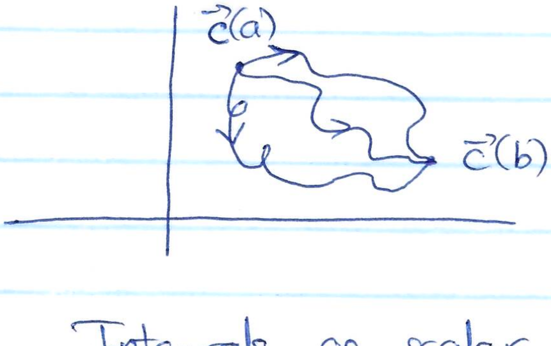
$$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

$$= f\left(\pi, \frac{2\pi}{\cancel{2}}, 1\right) - f\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$$

$$= \cos\pi + \sin\frac{2\pi}{\cancel{2}} - \frac{\cancel{\pi}\, 2\pi^2}{\cancel{2}} - \cos\frac{\pi}{2} - \sin\frac{\pi}{2}$$

$$= -1 + 0 - \frac{\pi^2}{2} - 0 - 1$$

$$= -2 - \frac{\pi^2}{2}.$$




- Integrals of <u>scalar fields</u> along curves
  $\Rightarrow$ path integrals

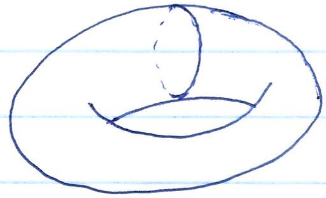
- Integrals of <u>vector fields</u> along curves
  $\Rightarrow$ line integrals.

## 7.3 Parametrized Surfaces.

E.g. The graph of a function $f(x,y)$ is a surface.

But we still have surfaces that are not the graph of a function.

E.g. Torus (surface of a donut).

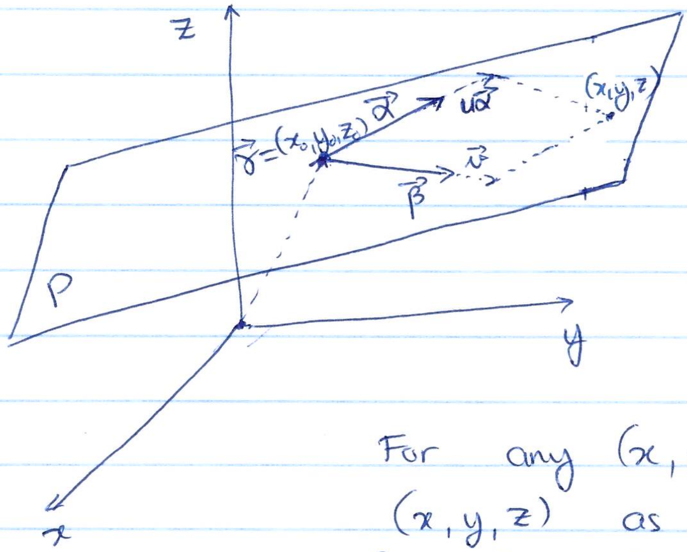


**Def:** A parametrization of a surface is a function $\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$. The surface is $S = \Phi(D)$ and

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v)).$$

If $x(u,v)$, $y(u,v)$, $z(u,v)$ are differentiable, we call $S$ a differentiable surface.

- Parametrization of a plane:




Let $P$ be a plane that is parallel to $\vec{\alpha}$ and $\vec{\beta}$, and passes through $\vec{\gamma}$.

For any $(x,y,z) \in P$, we can write $(x,y,z)$ as

$$(x,y,z) = (x_0, y_0, z_0) + u\vec{\alpha} + v\vec{\beta}$$

for some $u, v \in \mathbb{R}$

$$\boxed{\text{So } \quad \Phi(u,v) = \vec{\alpha} u + \vec{\beta} v + \vec{\gamma}}$$

E.g. Find a parametrization of the plane $x+y+z=1$

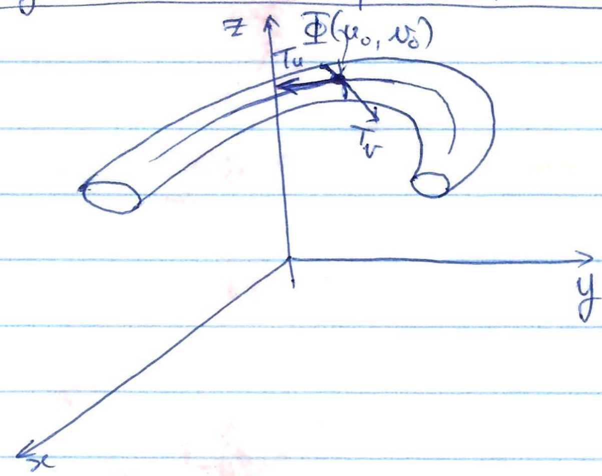
Sol: The point $(0,0,1)$ is on the plane.

Vectors $(1,-1,0)$ and $(0,1,-1)$ are parallel to the plane (why?)

Hence,

$$\Phi(u,v) = (1,-1,0)u + (0,1,-1)v + (0,0,1).$$

* Tangent vectors to parametrized surfaces:




Suppose that $\Phi(u,v)$ is diff. at $(u_0, v_0)$.

Fix $v_0$ and look at the map $t \mapsto \Phi(u_0+t, v_0)$

(in other words, we have a map $\mathbb{R} \to \mathbb{R}^3$) which identifies a curve on the surface

The vector tangent to this curve at $(u_0, v_0)$ is given by

$$T_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)\,\vec{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\vec{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\vec{k}.$$

It is also tangent to the surface.

Similarly, $T_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0, v_0)\vec{k}$

Both $T_u$ and $T_v$ are tangent to the surface.
so $T_u \times T_v$ is normal to them (provided $T_u \times T_v \neq 0$).

To find the equation of the tangent plane at $(u_0, v_0)$
we calculate

$$\vec{n} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}_{(u_0, v_0)}$$

Tangent plane is

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

$$\Leftrightarrow \quad n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \qquad \vec{n} = (n_1, n_2, n_3)$$

E.g. Let $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$\Phi(u, v) = (u\cos v, u\sin v, u^2 + v^2)$
Find the tangent plane at $\Phi(1,0)$.

Sol: $T_u = (\cos v, \sin v, 2u)$
$T_v = (-u\sin v, u\cos v, 2v)$

$$\vec{n} = T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 2v \end{vmatrix}_{(1,0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= -2\vec{i} - (0)\vec{j} + 1 \cdot \vec{k}$$

$$= -2\vec{i} + \vec{k}.$$

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$$(x_0, y_0, z_0) = \Phi(u_0, v_0) = \Phi(1,0) = (1, 0, 1).$$

The eq. of the tangent plane

$$-2(x-1) + 0(y-0) + 1(z-1) = 0$$

$$-2x + 2 + z - 1 = 0.$$

$$-2x + z = -1.$$

Remark: We say that a surface is regular or smooth, at $\Phi(u_0, v_0)$ if $T_u \times T_v \neq 0$ at $(u_0, v_0)$. We say that it is regular if it is regular at all points $\Phi(u_0, v_0) \in S$.

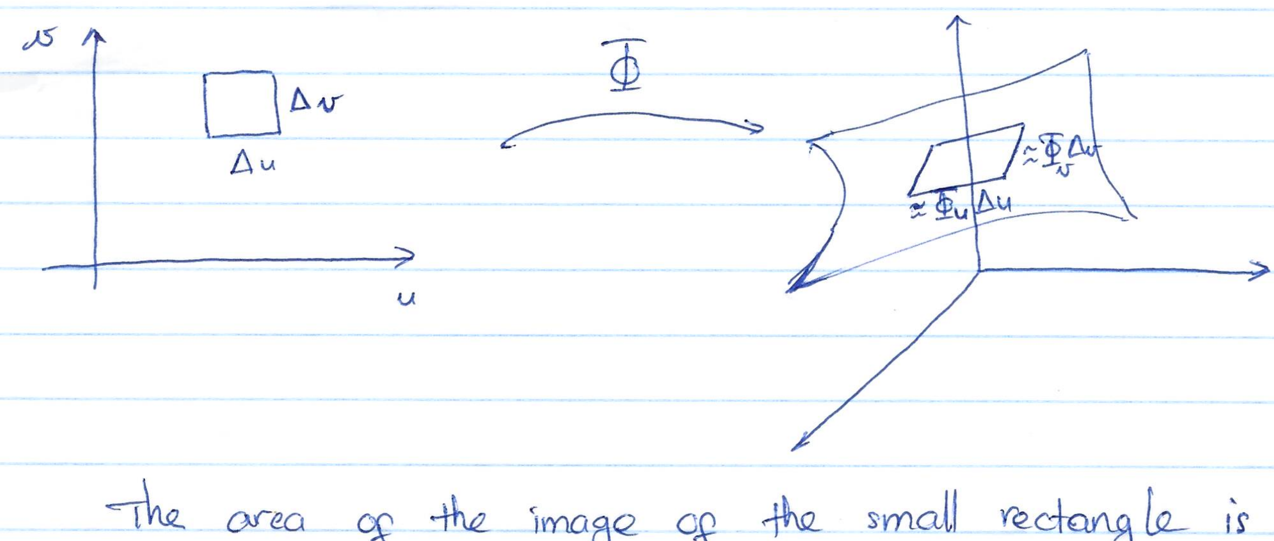
## 7.4 Area of a surface

$\boxed{\begin{array}{l}\text{Def: the surface area of a parametrized surface} \\ \quad A(S) = \displaystyle\iint_D \|T_u \times T_v\| \, du\, dv \\ \quad S \text{ is regular} \\ \quad \Phi \quad \text{one-to-one and differentiable.}\end{array}}$

Recall: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, then

$$\|T_u \times T_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}$$

So $\boxed{A(S) = \displaystyle\iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2} \, du\, dv}$




The area of the image of the small rectangle is

$$\|(\Phi_u \Delta u) \times (\Phi_v \Delta v)\| = \|\Phi_u \times \Phi_v\| \, \Delta u \, \Delta v.$$

Summing the areas of all small rectangles as $\Delta u, \Delta v \to 0$ gives $A(S)$.

E.g. Find the surface area of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1.$$

Parametrize:

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le r \le 1, \; 0 \le \theta \le 2\pi.$$

Then $T_r = (\cos\theta, \sin\theta, 1)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$

$$T_r \times T_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$$

$$\|T_r \times T_\theta\| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = \sqrt{2r^2} = \sqrt{2}\, r$$

$$A(\text{cone}) = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\,dr\,d\theta = 2\pi\sqrt{2}\,\frac{r^2}{2}\Big|_0^1 = \sqrt{2}\,\pi.$$

* Surface area of the graph of a function $f(x,y)$.

We can parametrize $S$ by $(x, y, f(x,y))$

or $(u, v, f(u,v))$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = -f_u\,\vec{i} - f_v\,\vec{j} + 1\vec{k}.$$

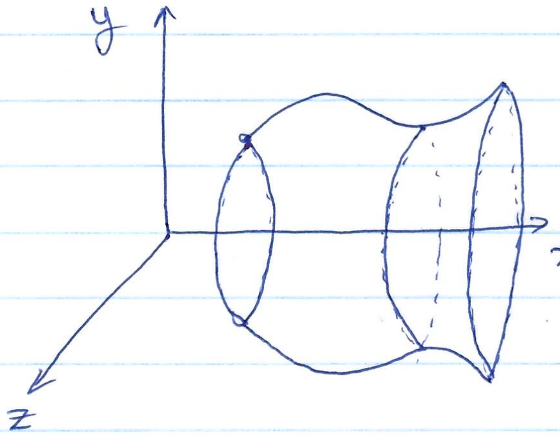
$$A(S) = \iint_D \sqrt{f_u^2 + f_v^2 + 1}\; du\,dv.$$

Exercise: use this to find the area of the cone

$$z = \sqrt{x^2 + y^2}$$

* Surfaces of revolution.

Suppose $S$ is obtained by rotating the graph of $y = f(x)$, $a \le x \le b$ around the $x$-axis.




We can parametrized $S$ as $(u, f(u)\cos v, f(u)\sin v)$

$a \le u \le b$, $0 \le v \le 2\pi$.

So $T_u = (1, f'(u)\cos v, f'(u)\sin v)$

$T_v = (0, -f(u)\sin v, f(u)\cos v)$

$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(u)\cos v & f'(u)\sin v \\ 0 & -f(u)\sin v & f(u)\cos v \end{vmatrix}$$

$$= f'(u) f(u)\,\vec{i} - f(u)\cos v\,\vec{j} - f(u)\sin v\,\vec{k}.$$

$$\Rightarrow A(S) = \int_0^{2\pi}\int_a^b \sqrt{f'(u)^2 + \cos^2 v + \sin^2 v}\,|f(u)|\,du\,dv.$$

$$= \int_a^b\int_0^{2\pi} \sqrt{f'(u)^2 + 1}\,|f(u)|\,dv\,du.$$

$$= 2\pi \int_a^b \sqrt{f'(u)^2 + 1}\,|f(u)|\,du.$$

Exercise: use this to compute the area of the cone $z = \sqrt{x^2+y^2}$. $(0 \le x \le 1)$.

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7.5 Integrals of Scalar functions over surfaces.

**E.g.** The mass of a thin ~~set~~ sheet of metal $S = \Phi(D)$ where $D \subset \mathbb{R}^2$ and $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$ where the density of the metal is given by $f(x,y,z)$.

Def: The integral of a scalar function over a surface

$$\iint_S f(x,y,z)\, dS = \iint_D f(\Phi(u,v)) \|T_u \times T_v\| \,du\,dv.$$

$$= \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2}\, du\,dv$$

E.g. $f(x,y,z) = \sqrt{x^2+y^2+1}$ (helicoid Example 2 Section 7.4.)

$S:$ $(r\cos\theta, r\sin\theta, \theta)$

$D:$ $0 \le \theta \le 2\pi$, $0 \le r \le 1$

Compute $\iint_S f\, dS$.

$$\iint_S f\, dS = \iint_D f(r\cos\theta, r\sin\theta, \theta) \|T_r \times T_\theta\|\, dr\, d\theta$$

$$= \iint_D \sqrt{r^2+1}\, \|T_r \times T_\theta\|\, dr\, d\theta.$$

$T_r = (\cos\theta, \sin\theta, 0)$

$T_\theta = (-r\sin\theta, r\cos\theta, 1)$

$T_r \times T_\theta = \sin\theta\, \vec{i} - \cos\theta\, \vec{j} + r\vec{k}.$

$\|T_r \times T_\theta\| = \sqrt{r^2+1}$.

$$\Rightarrow \iint_S f\, dS = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1} \cdot \sqrt{r^2+1}\, dr\, d\theta.$$

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$$= 2\pi \left(r + \frac{r^3}{3}\right)\Big|_0^1 = \frac{8\pi}{3}.$$

* Surface integrals over graphs of functions:

Suppose $S$ is the graph of a differentiable function $z = g(x,y)$

$\Rightarrow$ we can parametrize it as $(u, v, g(u,v))$

Then $\|T_u \times T_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$

so $$\iint_S f(x,y,z)\, dS = \iint_D f(u, v, g(u,v)) \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}\, du\, dv$$

E.g. Compute $\displaystyle\iint_S \frac{x}{\sqrt{4x^2 + 4y^2 + 1}}\, dS$

where $S$ is the hyperbolic paraboloid $z = y^2 - x^2$ over the region $-1 \leq y \leq 1$, $-1 \leq x \leq 1$.

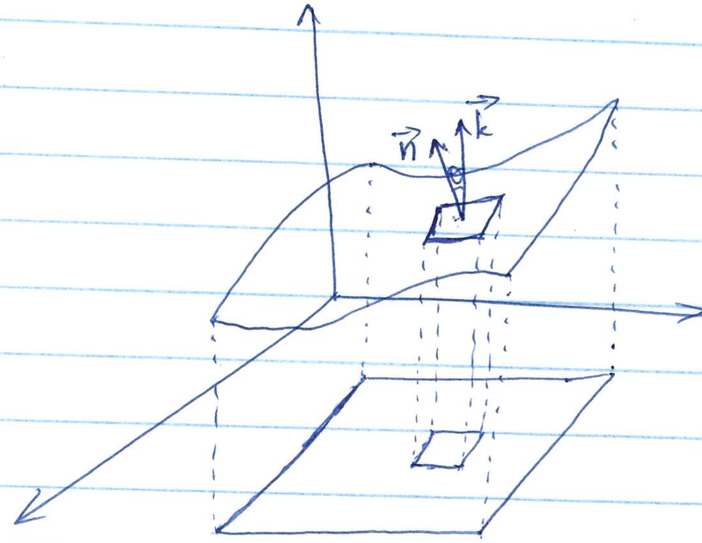
we parametrize $(x, y, y^2 - x^2)$

$$\iint_S f\, dS = \iint_D \frac{x}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + (-2x)^2 + (2y)^2}\, dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\, dx\, dy = 0.$$

## * Integrals over graphs:

$S: (x, y, g(x,y))$ D: simple region.




here $\vec{n} = \frac{\vec{N}}{\|\vec{N}\|}$ is the unit normal and

$\vec{N} = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}$ is normal to the surface.

Since $\vec{N}\cdot\vec{k} = \|\vec{N}\|\,\|\vec{k}\|\cos\theta = \|\vec{N}\|\cos\theta$.

then $\cos\theta = \frac{\vec{N}}{\|\vec{N}\|}\cdot\vec{k} = \frac{\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right)\cdot\vec{k}}{\|\vec{N}\|}$

$$= \frac{1}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

So $$\iint_S f\,ds = \iint_D f(x,y,g(x,y))\underbrace{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}\,dxdy$$

$$= \iint_D f(x,y,g(x,y))\,\frac{1}{\cos\theta}\,dxdy.$$

$\theta$ depends on $x$ & $y$.

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~~The total~~

E.g. Compute the mass of the helicoid $S: (r\cos\theta, r\sin\theta, \theta)$ where $0 \le r \le 1$, $0 \le \theta \le 2\pi$ if its mass density is $m(x,y,z) = \sqrt{x^2+y^2}$.

The total mass of ~~a~~ a surface with mass density $m$ is given by

$$M(S) = \iint_S m(x,y,z)\,dS$$

$$= \iint_D \sqrt{r^2}\, \|T_r \times T_\theta\|\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\sqrt{r^2+1}\; dr\,d\theta$$

$u = r^2+1$

$du = 2r\,dr$

$$= 2\pi \cdot \left.\tfrac{1}{3}(r^2+1)^{3/2}\right|_0^1$$

$$= 2\pi \cdot \tfrac{1}{3}\left(2^{3/2}-1\right)$$

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# Surface integrals of vector fields.

Def: The surface integral of $\vec{F}$ over $\Phi$:

$$\iint_{\Phi} \vec{F} \cdot d\vec{S} := \iint_{D} \vec{F} \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv.$$

Example. $\vec{F}(x, y, z) = (x, y, z)$

$S$: sphere of radius $1$.

Find $\iint_S \vec{F} \cdot d\vec{S}$.

Sol. Parametrize $S$ by using spherical coordinates.

$(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$.

Then $\vec{T}_\theta = (-\sin\theta \sin\phi, \cos\theta \sin\phi, 0)$

$\vec{T}_\phi = (\cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi)$.

$$\vec{T}_\theta \times \vec{T}_\phi = (-\cos\theta \sin^2\phi, \; -\sin\theta \sin^2\phi, \; -\sin^2\theta \sin\phi \cos\phi - \cos^2\theta \sin\phi \cos\phi)$$

$$= (-\cos\theta \sin^2\phi, -\sin\theta \sin^2\phi, -\sin\phi \cos\phi).$$

On the sphere, $\vec{F} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$.

$$\Rightarrow \cancel{T} \; \vec{F} \cdot (\vec{T}_\theta \times \vec{T}_\phi) = -\cos^2\theta \sin^3\phi - \sin^2\theta \sin^3\phi - \sin\phi \cos^2\phi.$$

$$= -\sin^3\phi - \sin\phi \cos^2\phi.$$

$$= -\sin\phi.$$

$$\Rightarrow \iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot (T_\theta \times T_\phi)\, d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} -\sin\phi \, d\theta\, d\phi$$

$$= -2\pi \int_0^{\pi} \sin\phi \, d\phi = +2\pi \cos\phi \Big|_0^{\pi} = -4\pi.$$

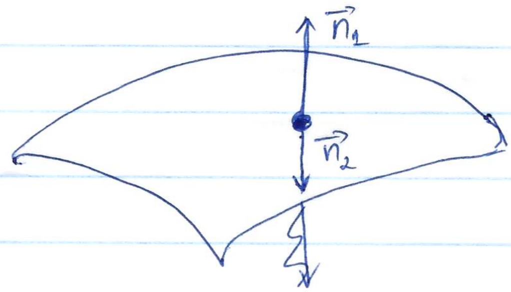
E.g. The volume of water per unit of time flowing "through a surface" $S$ with velocity given by the field $\vec{F} = \iint_{\Phi} \vec{F} \cdot d\vec{S}$

~~Interpret~~

Remark: We implicitly chose an orientation for the surface when we used $\vec{T}_\theta \times \vec{T}_\phi$ instead of $\vec{T}_\phi \times \vec{T}_\theta$.

$\Rightarrow$ orientation?

An oriented surface is one where at each $(x, y, z) \in S$ there are 2 unit normal vectors $\vec{n}_1$ and $\vec{n}_2$ with $\vec{n}_1 = -\vec{n}_2$. and each can be associated with a side of the surface.




Not every surface is orientable: e.g. Mobius strip is not.

E.g. The unit sphere can be given an ~~oriented~~ orientation by selecting $\vec{n} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\cancel{\|(x, y, z)\|}}$.

This is an orientation preserving. (i.e. points outwards.)

But in the previous example with the sphere, our parametrization gave $\vec{T}_\theta \times \vec{T}_\phi = -\vec{n} \sin\phi$ $\quad (0 \le \phi \le \pi \Rightarrow \sin\phi \ge 0)$.

$\Rightarrow$ this parametrization was a orientation reversing.

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Thm: Surface integrals are independent of parametrization, provided they are orientation preserving.

$$\boxed{\text{Thm: } \iint_S \vec{F}\cdot \underbrace{d\vec{S}}_{(\vec{T}_u\times\vec{T}_v)\,du\,dv} = \iint_S \vec{F}\cdot \underbrace{\vec{n}}_{\frac{\vec{T}_u\times\vec{T}_v}{\|T_u\times T_v\|}}\, \underbrace{dS}_{\|T_u\times T_v\|\,du\,dv}}$$

E.g. Heat flow

If $T(x,y,z)$ is the temperature at $(x,y,z)$,
Then $\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$ is the temperature gradient; $\vec{F} = -k\nabla T$ is a vector field associated with heat flow and $\iint \vec{F}\cdot d\vec{S}$ is the flux (or total rate of heat flow) accross $S$.

Suppose $T(x,y,z) = x^2 + y^2 + z^2$. Find the heat flux accross the unit sphere oriented with the outward normal. (use $k = 1$).

$\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$. outward normal.

$\vec{F} = -\nabla T = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= -\iint_S (2x\vec{i} + 2y\vec{j} + 2z\vec{k})\cdot(x\vec{i} + y\vec{j} + z\vec{k})\, dS$$

$$= -2\iint_S (x^2 + y^2 + z^2)\, dS$$

$$= -2\,A(S).$$

$$= -2(4\pi)$$

$$= -8\pi.$$

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* Surface integrals over graphs:

Suppose $S$ is the graph of a function, so

$S: (x, y, g(x,y))$

Additionally, suppose $S$ is oriented with upward pointing normal ~~(i.e. $\vec{k}$ component~~

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \qquad \text{since } T_x \times T_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix}$$

Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) \cdot \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) dx\,dy$$

$$\Rightarrow \boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-F_1\frac{\partial g}{\partial x} + -F_2\frac{\partial g}{\partial y} + F_3\right) dx\,dy.}$$

E.g. $z = x^2 + y^2$, $x^2 + y^2 \le 4$

Suppose $F = -y\vec{i} + x\vec{j} + \vec{k}$.

Compute $\iint_S \vec{F} \cdot d\vec{S}$.

$S$ is the graph of a function.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (+y \cdot 2x - x\,2y + 1)\, dx\,dy.$$

$$= \iint_D dx\,dy.$$

$$= 4\pi.$$