Lecture 20: Diagonalization; Linear Differential Equations (Sections 5.2--5.4)

Thang Huynh, UC San Diego

3/2/2018

Diagonalization

Diagonal matrices are very easy to work with.

- Example. For instance, it is easy to compute their powers.

Let's consider $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 2^2 & & \\ & 3^2 & \\ & & 4^2 \end{bmatrix} \quad \text{and} \quad A^{100} = \begin{bmatrix} 2^{100} & & \\ & 3^{100} & \\ & & 4^{100} \end{bmatrix}.$$

Diagonalization

▶ Example. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$ Let's find eigenvalues and eigenvectors of A:

- $\lambda_1 = 4 \Longrightarrow$ eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- $\lambda_2 = 5 \Longrightarrow$ eigenvector $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Key observation: if $\boldsymbol{v}$ is an eigenvector of A corresponding to an eigenvalue λ,

$$A^m \boldsymbol{v} = \lambda^m \boldsymbol{v}.$$

Diagonalization

Let $B = A^{100} = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Then $\boldsymbol{b}_1 = A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{b}_2 = A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Diagonalization

Let $B = A^{100} = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$. Then $\boldsymbol{b}_1 = A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $\boldsymbol{b}_2 = A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Therefore, finding A^{100} is equivalent to finding $A^{100}\begin{bmatrix}1\\0\end{bmatrix}$ and $A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Observe that

$$\begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix} = -\boldsymbol{v}_1 + 2\boldsymbol{v}_2.$$

$$A^{100}\begin{bmatrix}1\\0\end{bmatrix} = -4^{100}\begin{bmatrix}1\\2\end{bmatrix} + 2\cdot 5^{100}\begin{bmatrix}1\\1\end{bmatrix}.$$

Exercise: find $A^{100}\begin{bmatrix}0\\1\end{bmatrix}$.

Diagonalization

The key idea of the previous example was to work with respect to a basis given by the eigenvectors.

Diagonalization

The key idea of the previous example was to work with respect to a basis given by the eigenvectors.
Put the eigenvectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ as columns into a matrix S.

$$A \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 \boldsymbol{x}_1 & \cdots & \lambda_n \boldsymbol{x}_n \\ | & & | \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$

Diagonalization

The key idea of the previous example was to work with respect to a basis given by the eigenvectors.

Put the eigenvectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ as columns into a matrix S.

$$A \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 \boldsymbol{x}_1 & \cdots & \lambda_n \boldsymbol{x}_n \\ | & & | \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$AS = S\Lambda$$

Diagonalization

Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Then A can be diagonalized as $A = S\Lambda S^{-1}$.

- the columns of S are the eigenvectors
- the diagonal matrix Λ has the eigenvalues on the diagonal.

(Such a diagonalization is possible if and only if A has enough eigenvectors.)

Diagonalization

- Example. Consider the rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Diagonalization

- Example. Consider the rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Eigenvalues and eigenvectors:

$$\lambda_1 = i \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \lambda_2 = i \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Diagonalization

- Example. Consider the rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Eigenvalues and eigenvectors:

$$\lambda_1 = i \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \lambda_2 = i \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Then

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}.$$

Powers A^k

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Longrightarrow A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

Powers A^k

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Longrightarrow A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

If λ is an eigenvalue of A with $\boldsymbol{v}$ as its corresponding eigenvector, λ^2 is an eigenvalue of A^2 with corresponding eigenvector $\boldsymbol{v}$.

Powers A^k

$$Ax = \lambda x \Longrightarrow A^2 x = \lambda^2 x.$$

If λ is an eigenvalue of A with v as its corresponding eigenvector, λ^2 is an eigenvalue of A^2 with corresponding eigenvector v.

$$A = S\Lambda S^{-1} \quad \Longrightarrow \quad A^2 = S\Lambda^2 S^{-1}.$$

Powers A^k

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Longrightarrow A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

If λ is an eigenvalue of A with $\boldsymbol{v}$ as its corresponding eigenvector, λ^2 is an eigenvalue of A^2 with corresponding eigenvector $\boldsymbol{v}$.

$$A = S\Lambda S^{-1} \quad \Longrightarrow \quad A^2 = S\Lambda^2 S^{-1}.$$

In general,

$$A^k = S\Lambda^k S^{-1} \quad \text{or} \quad S^{-1}A^k S = \Lambda^k.$$

Powers A^k

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Longrightarrow A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

If λ is an eigenvalue of A with $\boldsymbol{v}$ as its corresponding eigenvector, λ^2 is an eigenvalue of A^2 with corresponding eigenvector $\boldsymbol{v}$.

$$A = S\Lambda S^{-1} \quad \Longrightarrow \quad A^2 = S\Lambda^2 S^{-1}.$$

In general,

$$A^k = S\Lambda^k S^{-1} \quad \text{or} \quad S^{-1}A^k S = \Lambda^k.$$

Additionally, if A is invertible,

$$A^{-1} = S\Lambda^{-1}S^{-1}.$$

Linear differential equations

- **Example.** The **differential equation** $y' = ay$ with **initial condition** $y(0) = C$ is solved by $y(t) = Ce^{at}$.
- **Example.** Our goal is to solve systems of differential equations:

$$\begin{aligned} y_1' &= 2y_1 & y_1(0) &= 1 \\ y_2' &= -y_1 + 3y_2 + y_3 & y_2(0) &= 2 \\ y_3' &= -y_1 + y_2 + 3y_3 & y_3(0) &= 1. \end{aligned}$$

Linear differential equations

- Example. The **differential equation** $y' = ay$ with **initial condition** $y(0) = C$ is solved by $y(t) = Ce^{at}$.
- Example. Our goal is to solve systems of differential equations:

$$\begin{aligned} y_1' &= 2y_1 & y_1(0) &= 1 \\ y_2' &= -y_1 + 3y_2 + y_3 & y_2(0) &= 2 \\ y_3' &= -y_1 + y_2 + 3y_3 & y_3(0) &= 1. \end{aligned}$$

In matrix form:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Linear differential equations

- Definition. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$$

Linear differential equations

- Definition. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

Then

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Linear differential equations

- Definition. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

Then

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

The solution to $\boldsymbol{y}' = A\boldsymbol{y}, \boldsymbol{y}(0) = \boldsymbol{y}_0$ is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0.$$

Linear differential equations

- Example. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then

$$e^{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \ldots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \ldots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

Linear differential equations

- Example. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then

$$e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \ldots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \ldots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

Theorem. Suppose $A = S\Lambda S^{-1}$. Then $e^A = Se^{\Lambda}S^{-1}$.

Linear differential equations

- Example. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Linear differential equations

- Example. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- Solution. The solution is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Linear differential equations

- Example. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- Solution. The solution is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Step 1: Diagonalize A

Linear differential equations

- **Example.** Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- Solution. The solution is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Step 1: Diagonalize A

- $\lambda_1 = 1 \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $\lambda_2 = -1 \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Linear differential equations

▶ Example. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

▶ Solution. The solution is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Step 1: Diagonalize A

- $\lambda_1 = 1 \Longrightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $\lambda_2 = -1 \Longrightarrow \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence,

$$A = S\Lambda S^{-1}, \qquad \text{where } S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Step 2: Compute the solution $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

$$\boldsymbol{y} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix}.$$