Section 3.4

Constrained extrema & Lagrange multipliers

Suppose that you want to minimize the surface area of a can, subject to keeping the volume fixed (for example, to minimize the cost of the materials).

Or, suppose that a particle moves along a curve in a container where the force (or temp, or pressure) is given by some function and we want to find the max/min ~~of~~ force that the particle experiences.

In other words, we want to solve

(maximize)
minimize $f(\vec{x})$
subject to $g(\vec{x}) = c$

e.g. minimize $2\pi r h + 2\pi r^2$

subject to $\pi r^2 h = 1$

Theorem (the method of Lagrange multipliers)

- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ & $g: U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1 functions.

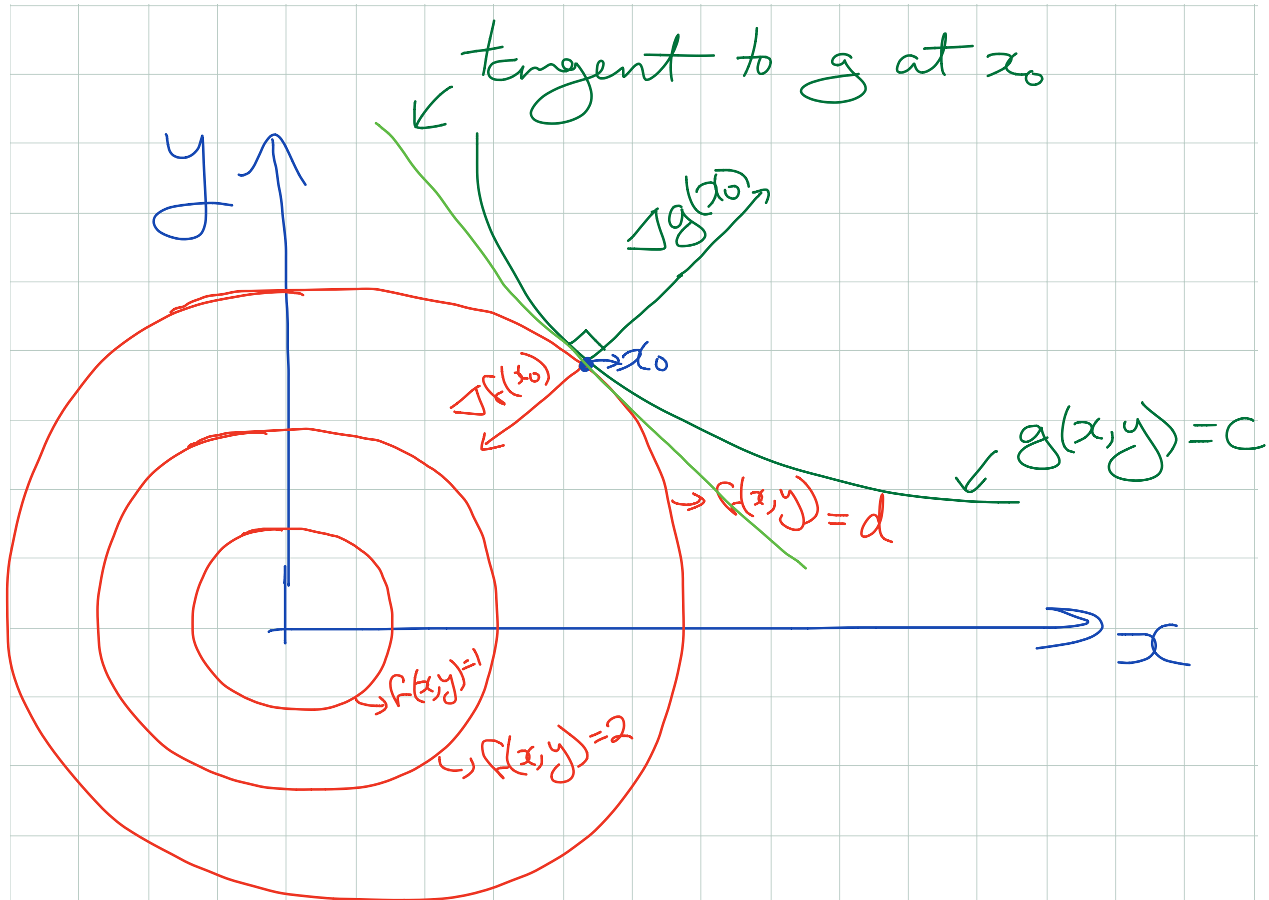
- Let $\vec{x}_0 \in U$ & $g(\vec{x}_0) = c$, $\nabla g(\vec{x}_0) \neq 0$.
- Let S = level set for g with value c

$$= \{ \vec{x} \in \mathbb{R}^n ,\ g(\vec{x}) = c \}$$

If f restricted to S (denoted $f|_S$) has a local max or min, then there is a number λ such that
(possibly $\lambda = 0$)

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$$

We call x_0 a critical pt of $f|_S$.



But what does this mean for us?

Suppose that f is continuous and the constraint is closed and bounded.

From the first theorem, we have

$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$ at local max or min x_0 for some λ

we just need to find the points that satisfy

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x})$$ — lagrange equations (or lagrange condition)

and check to see if they are minima or maxima or neither

So: Step 1: Write the Lagrange Equations

$$\nabla f = \lambda \nabla g$$

Step 2: Solve for λ, x, y, z

step 3: Compute f at the critical pts. and select the min & max.

Remark: If the constraint is not closed & bounded the min or max may not exist. eg. $f(x,y) = x^2+y^2$ s.t. $x+y=1$

Example: Find the extrema of $f(x,y) = x^2-y^2$ on the circle $x^2+y^2=1$

Sol'n. • $g(x,y) = x^2+y^2$

Step 1 $\nabla f(x,y) = (2x, -2y)$

$\nabla g(x,y) = (2x, 2y)$

Step 2

So we must find λ and x and y s.t.

$$(2x, -2y) = \lambda(2x, 2y) \quad \& \quad x^2+y^2=1$$

$\Rightarrow x = \lambda x \implies x = 0$ or $\lambda = 1$

<u>If $x = 0$</u>

$x^2 + y^2 = 1 \implies y = \pm 1$

$y = -\lambda y \implies \lambda = -1$

<u>If $\lambda = 1$</u>

$y = -\lambda y \Rightarrow y = 0$

& $x = \pm 1$

So we get the pts $(0,1)$, $(0,-1)$, $(1,0)$ & $(-1,0)$

<u>Step 3</u>

we can now check them to see if they are maxima or minima

$\Rightarrow$ max is $f(1,0) = f(-1,0) = 1$
min is $f(0,1) = f(0,-1) = -1$

—x—

<u>Example</u>

Find the minimum & maximum of $f(x,y,z) = x+y+z$ subject to $x^2+2y^2+3z^2=6$

<u>Sol'n</u> <u>Step 1</u>

$\nabla f(x,y,z) = (1, 1, 1)$

$\nabla g(x,y,z) = (2x, 4y, 6z)$

Want $\nabla f = \lambda \nabla g$

So we want $\begin{cases} 1 = 2\lambda x \\ 1 = 4\lambda y \\ 1 = 6\lambda z \\ x^2+2y^2+3z^2=6 \end{cases}$

4 eq'ns
4 unknowkn

Step 2 (Solve for x, y, z, λ)

So $\lambda x = 2\lambda y = 3\lambda z$

and $\lambda \neq 0$ (otherwise $1 = 2\lambda x$ is not satisfied

thus $\boxed{x = 2y = 3z}$ — (*)

$\Rightarrow x^2 = 4y^2 = 9z^2$

$\Rightarrow y^2 = x^2/4$ & $z^2 = x^2/9$

So $x^2 + \frac{2x^2}{4} + \frac{3x^2}{9} = 6 \Leftrightarrow \left(3/2 + \frac{1}{3}\right)x^2 = 6$

$\Leftrightarrow \frac{11}{6}x^2 = 6 \Rightarrow x^2 = 36/11$

$x_1 = \frac{6}{\sqrt{11}}$, $\quad x_2 = \frac{-6}{\sqrt{11}}$

$\Downarrow \qquad \Downarrow$

by (*) $y_1 = \frac{x_1}{2} = 3/\sqrt{11}$ $\qquad y_2 = \frac{x_2}{2} = -3/\sqrt{11}$

$z_1 = \frac{x_1}{3} = \frac{2}{\sqrt{11}}$ $\qquad z_2 = \frac{x_2}{3} = \frac{-2}{\sqrt{11}}$

$P_1 = \frac{1}{\sqrt{11}}(6, 3, 2)$ & $P_2 = \frac{1}{\sqrt{11}}(-6, -3, -2)$

Step 3

Finally, checking P_1 & P_2, we see that $f(P_1) \geq f(P_2)$. Moreover, our constraint was closed & bounded

So $f(P_1)$ is max

$f(P_2)$ is min.

Chapter 4

Sec 4.1 Acceleration

Recall: Given a path $\vec{c}(t) = (x(t), y(t), z(t))$

we can compute $\underbrace{\vec{c}'(t)}_{\vec{v}(t)} = (x'(t), y'(t), z'(t))$

velocity vector

Recall: $\vec{c}'(t_0)$ is the tangent vector to the path at the point $\vec{c}(t_0)$.

Recall: $\|\vec{c}'(t)\|$ is the speed.

Differentiation rules: $\vec{b}: \mathbb{R} \to \mathbb{R}^3$, $\vec{c}: \mathbb{R} \to \mathbb{R}^3$

- $\dfrac{d}{dt}\left(\vec{b}(t) + \vec{c}(t)\right) = \vec{b}'(t) + \vec{c}'(t).$

- If $f(t): \mathbb{R} \to \mathbb{R}$

$$\left(f(t)\,\vec{c}(t)\right)' = f'(t)\,\vec{c}(t) + f(t)\,\vec{c}'(t)$$

e.g.

$$\left.\begin{aligned} f(t) &= t \\ \vec{c}(t) &= (t, t^2, t^3) \end{aligned}\right\} \Rightarrow \left(f(t)\,\vec{c}(t)\right)' = 1(t, t^2, t^3) + t(1, 2t, 3t^2$$

$$= (2t, 3t^2, 4t^3)$$

- $\left(\vec{b}(t) \cdot \vec{c}(t)\right)' = \vec{b}'(t) \cdot \vec{c}(t) + \vec{b}(t) \cdot \vec{c}'(t)$

exercise: prove it

- $\left(\vec{b}(t) \times \vec{c}(t)\right)' = \vec{b}'(t) \times \vec{c}(t) + \vec{b}(t) \times \vec{c}'(t)$

- $\left(\vec{c}(f(t))\right)' = \vec{c}'(f(t))\, f'(t)$

<u>Example</u>: If $\|\vec{c}(t)\| = \text{const}$

then $\|\vec{c}(t)\|^2 = \vec{c}(t)\cdot\vec{c}(t) = \text{const}.$

$$\Rightarrow (\vec{c}(t)\cdot\vec{c}(t))' = 0$$

$$\Rightarrow \underbrace{\vec{c}\,'(t)\cdot\vec{c}(t) + \vec{c}(t)\cdot\vec{c}\,'(t)}_{2\vec{c}(t)\cdot\vec{c}\,'(t)} = 0$$

$$2\vec{c}(t)\cdot\vec{c}\,'(t) = 0$$

so $\vec{c}\,'(t)$ is orth. to $\vec{c}(t)$ for all t.

<u>Definition</u>: $\vec{a}(t) = \vec{V}'(t) = \vec{c}\,''(t)$ is the acceleration of the curve.

$$\Rightarrow \vec{a}(t) = (x''(t), y''(t), z''(t))$$

<u>Example</u>: Suppose that the acceleration of a particle is $\vec{a}(t) = -\vec{k}\ \forall t$ (constant acc.)
Suppose that $\vec{c}(0) = (0,0,1)$ & $\vec{V}(0) = (1,1,0)$
Find the time & spatial coordinates when the particle reaches $z = 0$.

Sol'n : want to find t & $\vec{c}(t) = (x(t), y(t), z(t))$

where $z(t) = 0$.

we know that $\vec{a}(t) = (0, 0, -1) \quad \forall t$

& $\vec{a}(t) = (x''(t), y''(t), z''(t))$

but $\vec{v}(t) = (x'(t), y'(t), z'(t))$

So $x'(t) = \text{const}$

$y'(t) = \text{const}$

$z'(t) = -t + \text{const}$

But $\vec{x}'(0) = 1 \implies x'(t) = 1$

$y'(0) = 1 \implies y'(t) = 1$

$z'(0) = 0 \implies z'(t) = -t$

$\Downarrow$

Integrating again

$x(t) = t + \text{const}$ and $x(0) = 0 \implies \boxed{x(t) = t}$

$y(t) = t + \text{const}$ and $y(0) = 0 \implies \boxed{y(t) = t}$

$z(t) = -t^2/2 + \text{const}$ & $z(0) = 1 \implies \boxed{z(t) = 1 - t^2/2}$

We want t : $z(t) = 0$ so we solve

$1-t^2/2=0 \Rightarrow t=\sqrt{2}$ (bec $t\geq 0$)

So $\vec{c}(\sqrt{2})=(\sqrt{2},\sqrt{2},0)$ is the position of the particle when it crosses the $z=0$ plane

Physics example (How long is a planet's year, knowing only its radius)

Suppose $\vec{c}(t)=\left(r\cos\frac{st}{r}, r\sin\frac{st}{r}\right)$ ← circular orbit

is the orbit of a planet.

Then, the period is $\boxed{T=\frac{2\pi r}{s}}$ ← time to complete revolution & $\boxed{\|\vec{c}(t)\|=r}$ ← radius of orbit

The planet's velocity is $\boxed{\vec{v}(t)=\left(-s\sin\frac{st}{r}, s\cos\frac{st}{r}\right)}$

and its speed is $\boxed{\|\vec{v}(t)\|=\sqrt{(\quad)^2+(\quad)^2}=s}$

The acceleration is $\vec{a}(t)=\vec{v}\,'(t)=\left(-\frac{s^2}{r}\cos\frac{st}{r}, -\frac{s^2}{r}\sin\frac{st}{r}\right)$

So $\vec{a}(t)=\underbrace{-\frac{s^2}{r^2}}_{-\omega^2}\underbrace{\left(r\cos\frac{st}{r}, r\sin\frac{st}{r}\right)}_{\vec{c}(t)}$

(call $s^2/r^2=\omega^2$) (ω is the frequency),

so $\boxed{\vec{a}(t)=-\omega^2\vec{c}(t)}$

But Physics tells us that force $\vec{F} = \underset{\substack{\uparrow \\ \text{mass}}}{m}\vec{a}$

and that $\vec{F} = -\dfrac{\overset{\text{const.}}{G}mM}{r^3}\vec{r}$

mass of planet

Mass of sun

newton's law of gravity

So now $-m\omega^2\vec{r}(t) = -\dfrac{GmM}{r^3}\vec{r}(t)$

taking norms on both sides

~~so~~ $-m\dfrac{s^2}{r^{\cancel{2}}}\,\cancel{r} = -\dfrac{GmM\,\cancel{r}}{r^{3^2}}$

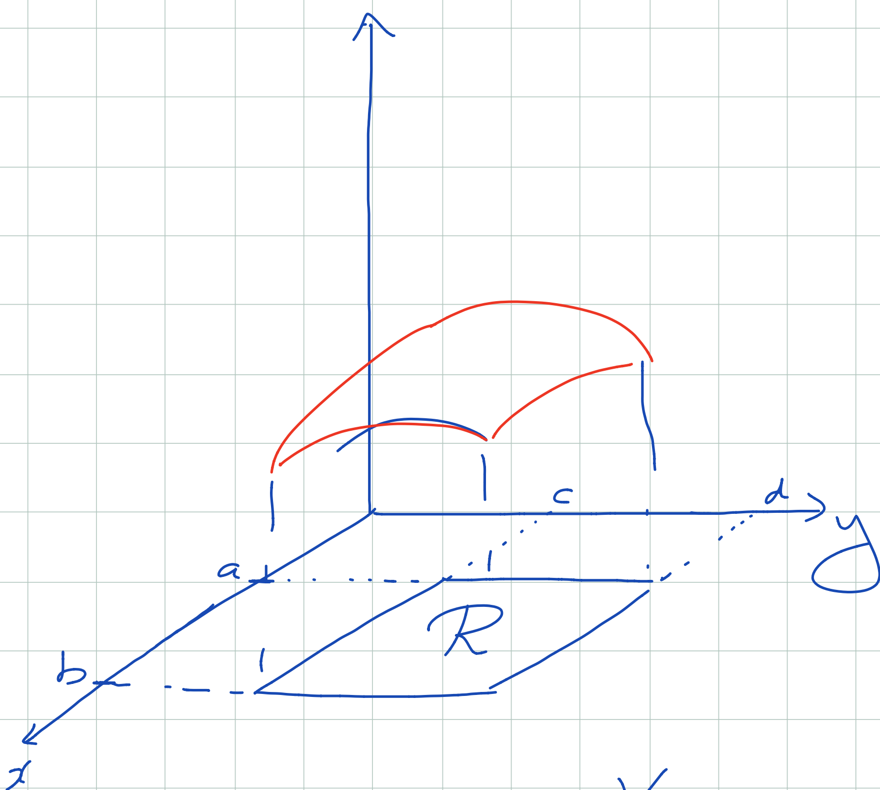
so $\underbrace{s^2}_{\left(\frac{2\pi r}{T}\right)^2} = \dfrac{GM}{r}$

$$\Rightarrow \boxed{T^2 = r^3\frac{(2\pi)^2}{GM}}$$

Kepler's law

Ch 5 Double & Triple Integrals

Sec 5.1



Want to find the volume V under the graph of the function

$$z = f(x,y) \quad (f(x,y) \geq 0)$$

and above the rectangle $[a,b] \times [c,d]$

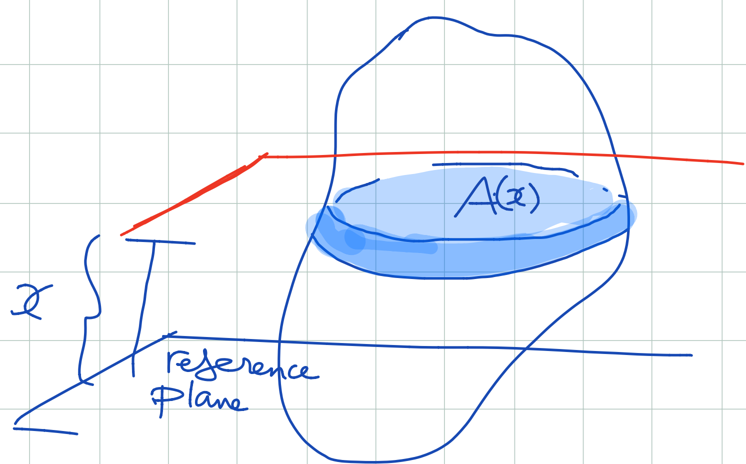
Double Integral

The Volume above the region R and under the graph of $f(x,y)$ $(f(x,y) \geq 0)$

is called the double integral of f over R and we denote it by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

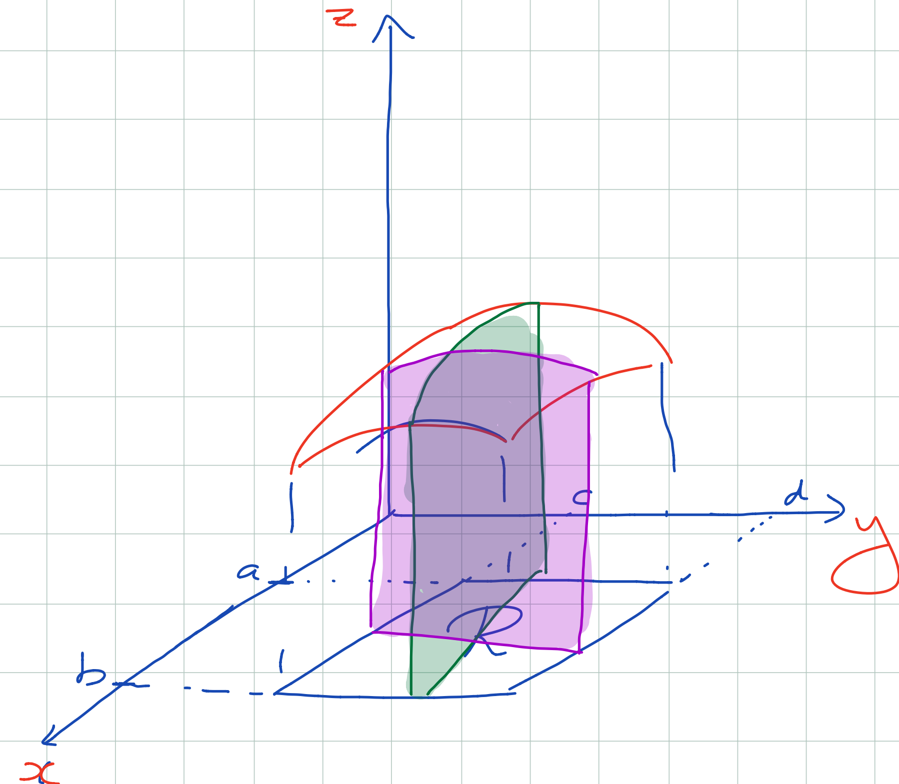
The Slice Method for computing Volumes



Idea: The volume of the object is the sum of the volumes of thin slices of it

$$\Rightarrow \text{Volume} = \int_a^b A(x)\,dx$$

Let's now use the slice method to get something nicer:



Slicing parallel to the yz-plane

$$V = \int_a^b A(x)\,dx = \underbrace{\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx}$$

slice principle

Slicing parallel to the xz-plane

$$V = \underbrace{\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy}$$

Iterated integrals

Examples:

Evaluate the integral

$$\iint_R x^2+y^2 \, dA \quad \text{where } R = [-1,1] \times [0,1]$$

Sol'n

Using iterated integrals:

$$\iint_R (x^2+y^2) \, dA = \int_{-1}^{1} \left[\int_0^1 x^2+y^2 \, dy \right] dx$$

$$= \int_{-1}^{1} \left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=0}^{y=1} dx$$

$$= \int_{-1}^{1} \left(x^2 + \frac{1}{3} \right) dx$$

$$= \frac{x^3}{3} + \frac{x}{3} \Big|_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$$

Let's do it again in the other order:

$$\iint_R x^2+y^2 \, dA = \int_0^1 \left[\int_{-1}^{1} x^2+y^2 \, dx \right] dy$$

$$= \int_0^1 \frac{x^3}{3} + xy^2 \Big|_{x=-1}^{x=1} dy$$

$$= \int_0^1 \frac{2}{3} + 2y^2 \, dy = \frac{2}{3}y + \frac{2}{3}y^3 \Big|_0^1$$

$$= \frac{4}{3}$$

We just found the volume under the paraboloid $z = x^2 + y^2$ and above the rectangle $[-1,1] \times [0,1]$!

Example

Find $\int_0^1 \int_0^1 xy e^{x+y} \, dx \, dy$

Sol'n

$$\int_0^1 \int_0^1 xy e^{x+y} \, dx \, dy = \int_0^1 \int_0^1 (xe^x)(ye^y) \, dx \, dy$$

But $\int_0^1 xe^x \, dx = xe^x - e^x$

So $\int_0^1\int_0^1 xe^x ye^y \, dx\,dy = \int_0^1 (xe^x - e^x) ye^y \Big|_{x=0}^{1} dy$

$= \int_0^1 (1) ye^y \, dy = (1) (ye^y - e^y) \Big|_0^1$

$= 1$

<u>Example</u>

Evaluate

$$V = \int_0^2 \int_{-1}^{0} xe^x \sin \frac{\pi}{2} y \, dy \, dx$$

<u>Sol'n</u>

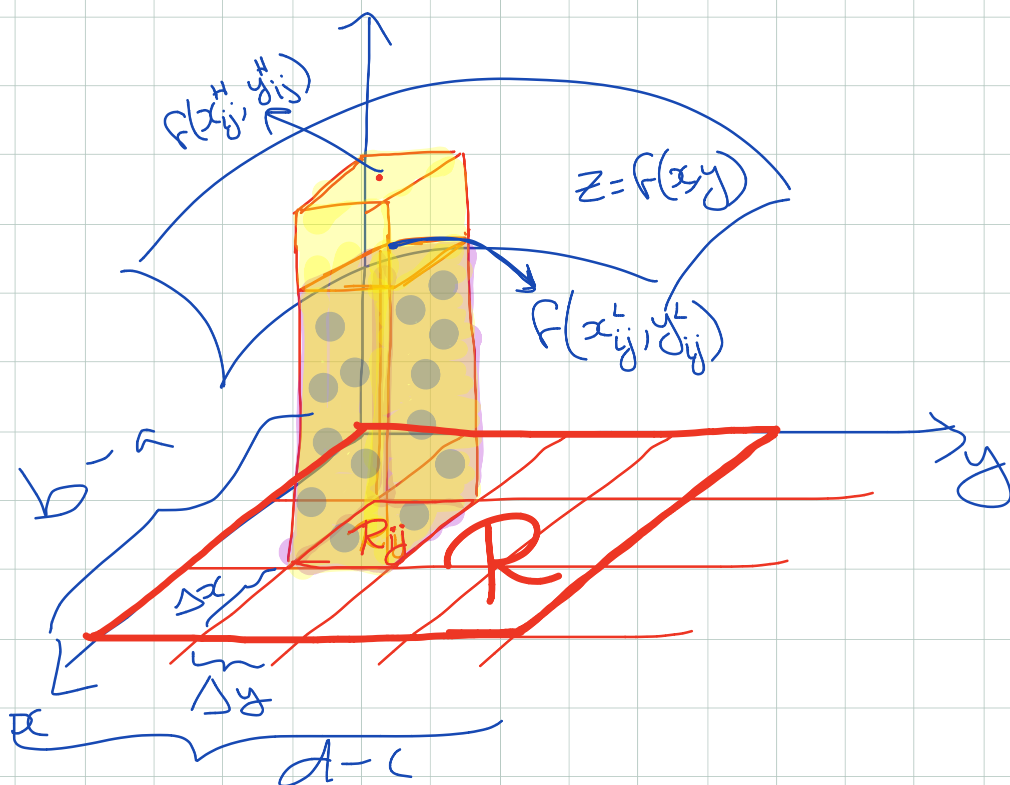
$$V = \int_0^2 -\frac{2}{\pi} xe^x \cos \frac{\pi}{2} y \Big|_{y=-1}^{0} dx$$

$= \int_0^2 \frac{2}{\pi} xe^x (1-0) \, dx = \frac{2}{\pi} \int_0^2 xe^x \, dx$

$= \frac{2}{\pi} (xe^x - e^x) \Big|_{x=0}^{x=2} = \frac{2}{\pi} (2e^2 - e^2 - 1)$

$= \frac{2}{\pi} (e^2 - 1)$

5.2 Double integrals over a rectangle



The volume above the rectangle R_{ij} and below the graph of $z=f(x,y)$ is

1) Less than the volume of the yellow rectang. box

2) Greater than the volume of the dotted box.

Taking limits as the number of rectangles R_{ij} increases and summing all the volumes:

If

$$\lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(x_{ij}^{L}, y_{ij}^{L}) \Delta x \Delta y$$

$$= \lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(x_{ij}^{H}, y_{ij}^{H}) \Delta x \Delta y$$

$$= \lim_{n\to\infty} \sum_{i,j=0}^{n-1} f(x_{ij}, y_{ij}) \Delta x \Delta y = S$$

(Converges to a limit S)

We say f is integrable over R

The integral is written

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

$$\text{or} \quad \iint_R f\,dA$$

Theorem: A continuous function on a closed rectangle is integrable

Properties: Same as single variable integration

- $$\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$$
- $$\iint_R cf\,dA = c\iint_R f\,dA$$
- $$\iint_{R_1\cup R_2} f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA$$ when $R_1 \cap R_2 = \phi$

Theorem: (Fubini) Let $R = [a,b]\times[c,d]$ and let f be continuous on R.

then $$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

you can change the order of integration!

Example: Compute $\iint_R (x^2+y)\,dA$ $R=[0,1]\times[0,1]$

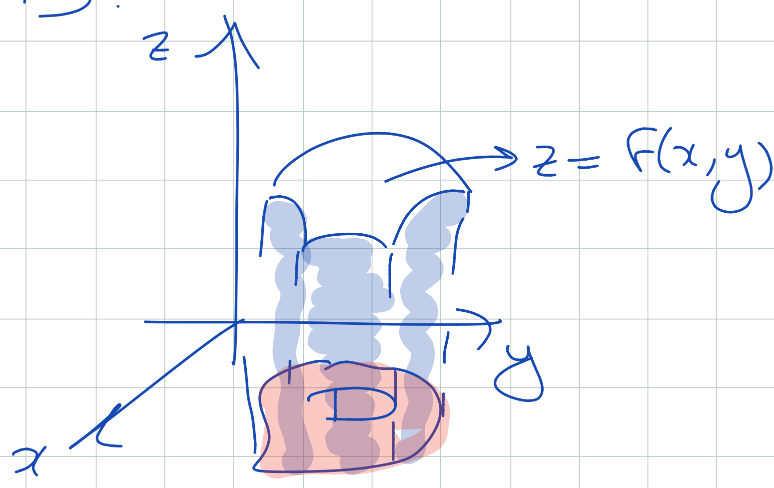
Sol'n: $\iint_R (x^2+y)\,dA = \int_0^1\int_0^1 (x^2+y)\,dx\,dy$

$$= \int_0^1 \left[\frac{x^3}{3} + xy\right]_{x=0}^{x=1} dy$$

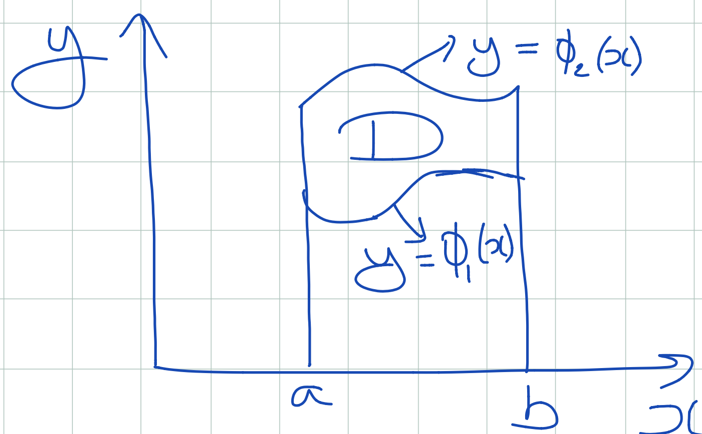
$$= \int_0^1 \frac{1}{3} + y\,dy = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

5.3 Double integrals over more general Regions:

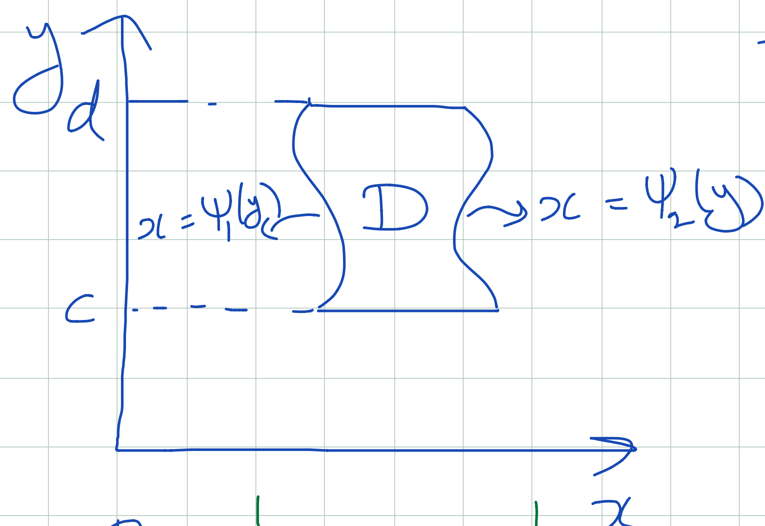
Want the volume "under" the graph of a function and above A general (non-rectangular) region D.



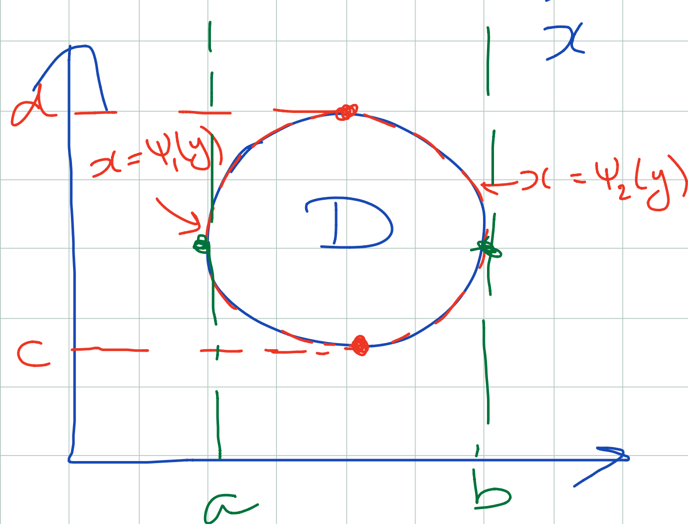
Elementary Regions:



y-simple region



x-simple region



simple region (both x-simple & y-simple)

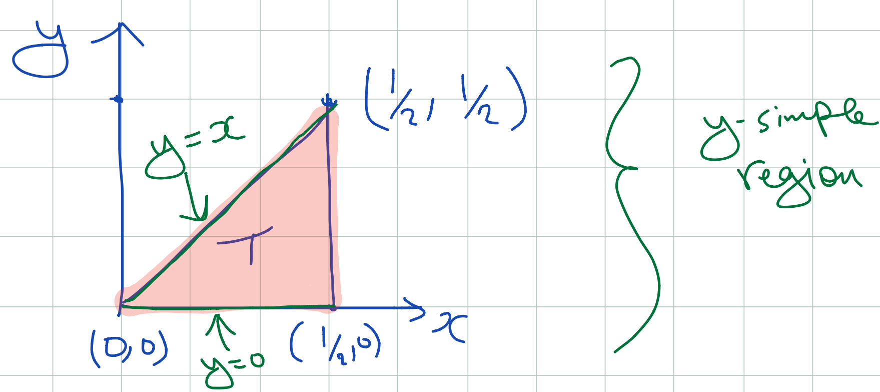
Example:

Find $\iint_T (y + x^2)\, dA$

where T is the triangle with vertices

$(0,0), (\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$.

Sol'n Step 1: Sketch the region



Step 2

$$\iint_T f(x,y)\,dA = \int_0^{\frac{1}{2}} \int_0^x y + x^2 \; dy\,dx$$

$$= \int_0^{1/2} \frac{y^2}{2} + yx^2 \Big|_{y=0}^{y=x} dx$$

$$= \int_0^{1/2} \frac{x^2}{2} + x^3 \; dx = \frac{x^3}{6} + \frac{x^4}{4} \Big|_0^{1/2}$$

In general: If D is a simple y-region

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

In general: If D is a simple x-region

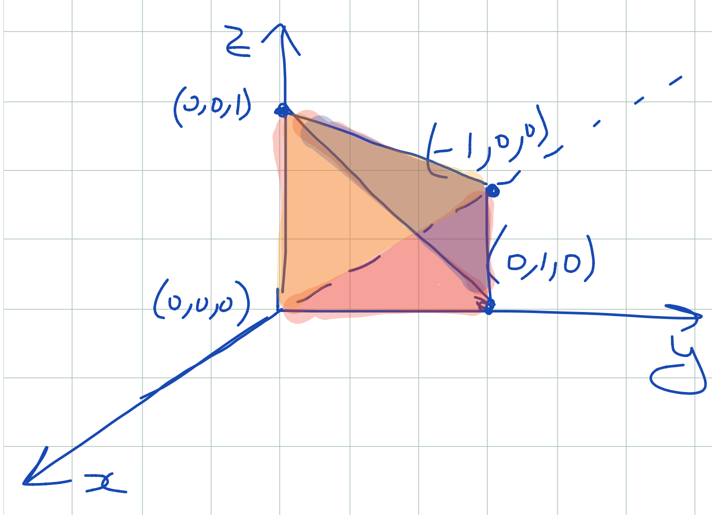
$$\iint_D f(x,y)\,dA = \int_c^d \int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\,dx\,dy$$

Example:

Find the Volume of the tetrahedron bounded by the planes $y=0$, $z=0$, $x=0$ &

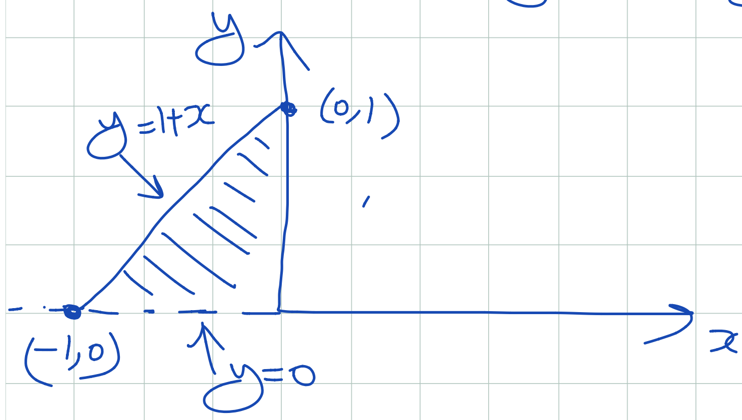
$$-x+y+z=1 \iff z = 1+x-y$$

Sol'n: Step 1: Sketch!



So on the xy-plane $(z=0)$

$$x=0,\ y=0,\ -x+y=1$$



$$\Rightarrow V(\text{tetrahedron}) = \int_{x=-1}^{x=0}\int_{y=0}^{y=1+x} 1+x-y \, dy\, dx$$

$$= \int_{-1}^{0} y + xy - y^2/2 \Big|_{y=0}^{1+x} dx$$

$$= \int_{-1}^{0} \underbrace{(1+x)+(1+x)x-(1+x)^2/2 - 0}_{1+x+x+x^2-\frac{(1+x)^2}{2}} \, dx$$

$$= \int_{-1}^{0} \frac{1}{2}+x+\frac{x^2}{2}\, dx = \frac{x}{2}+\frac{x^2}{2}+\frac{x^3}{6}\Big|_{-1}^{0}$$

$$= 0-\left(-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{6}\right) = \tfrac{1}{6}$$

Sec 5.4:

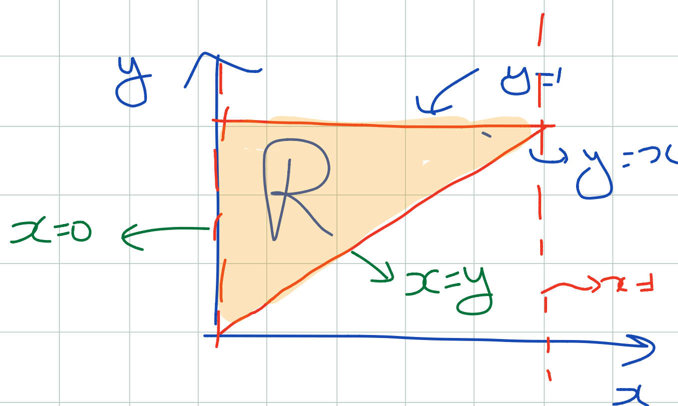
Sometimes evaluating an iterated integral can be hard so we may need to change the order of integration

e.g. $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$

Here, e^{y^2} doesn't have an antiderivative that we know

So how do we change the order of integration?

We draw



R is both x-simple & y-simple

so we can also define this region by $\underset{\psi_2(y)}{0} \leq x \leq \underset{\psi_1(y)}{y}$, $0 \leq y \leq 1$

$$\text{So } \int_0^1 \int_x^1 e^{y^2}\,dy\,dx = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

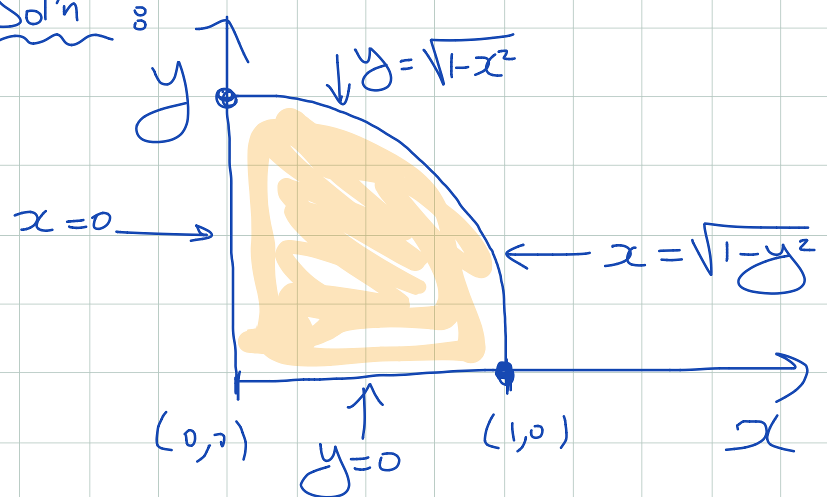
$$= \int_0^1 x e^{y^2} \Big|_{x=0}^{x=y} dx\,dy = \int_0^1 y e^{y^2}\,dy$$

$$= \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{e-1}{2}$$

Example: Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx$$

Sol'n:



The region is both x & y-simple, so

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx\, dy$$

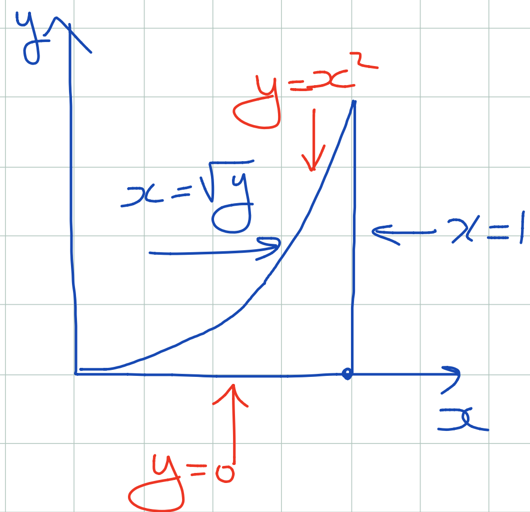
$$= \int_0^1 x\sqrt{1-y^2} \Big|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 1-y^2\, dy$$

$$= y - y^3/3 \Big|_0^1 = 2/3$$

Example: $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$

Sol'n



$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy = \int_0^1 \int_{y=0}^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_0^1 y e^{x^3} \Big|_{y=0}^{y=x^2} dx$$

$$= \int_{x=0}^{x=1} x^2 e^{x^3}\,dx, \quad \text{Let } u = x^3, \; du = 3x^2 dx$$

$$= \int_{u=0}^{u=1} \frac{1}{3} e^u\,du$$

$$= e/3$$

Exercise: do $\int_0^4 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$

5.5 The triple integral

$$\iiint_R f(x,y,z)\, \underbrace{dV}_{dA\,dz \,=\, dx\,dy\,dz}$$

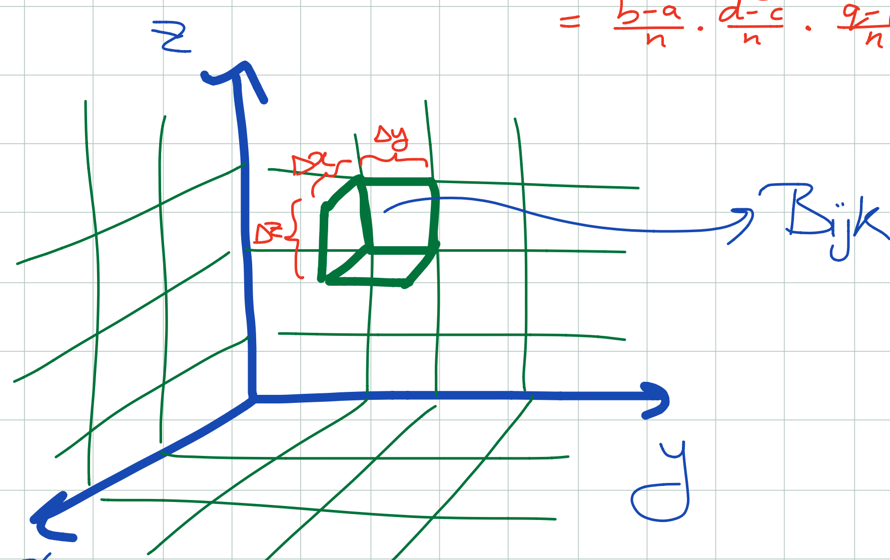
$R \rightarrow$ solid in space

Like Before, we start with integrals over a box (rectangular parallelpiped) $B = [a,b]\times[c,d]\times[p,q]$

Similar to double and single integrals:

$\vec{c}_{ijk} \in B_{ijk}$, the ijk'th box

Define $$S_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(\vec{c}_{ijk})\, \underbrace{\Delta V}_{=\,\Delta x\,\Delta y\,\Delta z \,=\, \frac{b-a}{n}\cdot\frac{d-c}{n}\cdot\frac{q-p}{n}}$$



Def'n: If f is a bounded function of 3 variables on B and if $S = \lim_{n \to \infty} S_n$ exists (and is independent of the choice of $\vec{c}_{ijk} \in B_{ijk}$) then: f is integrable and $S =: \iiint_B f(x,y,z)\, dV$ is the triple integral of f over B.

Properties: If f is integrable over B, then the triple integral can be evaluated as an iterated integral

example: Let B be the box given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $-1 \leq z \leq 0$

evaluate $\iiint_B x^2 + xy + z^2 y \, dV$

$$= \int_0^1 \int_0^2 \int_{-1}^0 x^2 + xy + z^2 y \, dz\, dy\, dx \quad \text{(iterated integral)}$$

$$= \int_0^1 \int_0^2 x^2 z + xyz + \frac{z^3}{3} y \Big|_{z=-1}^{z=0} dy\, dx$$

$$= \int_0^1 \int_0^2 x^2 + xy + \frac{1}{3} y \, dy\, dx$$

$$= \int_0^1 x^2 y + xy^2/2 + y^2/6 \Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 2x^2 + 2x + 2/3 \, dx = 2/3 + 1 + 2/3 = 7/3$$

Exercise: Evaluate the triple integral in a different order.

Next Step: Elementary Regions:

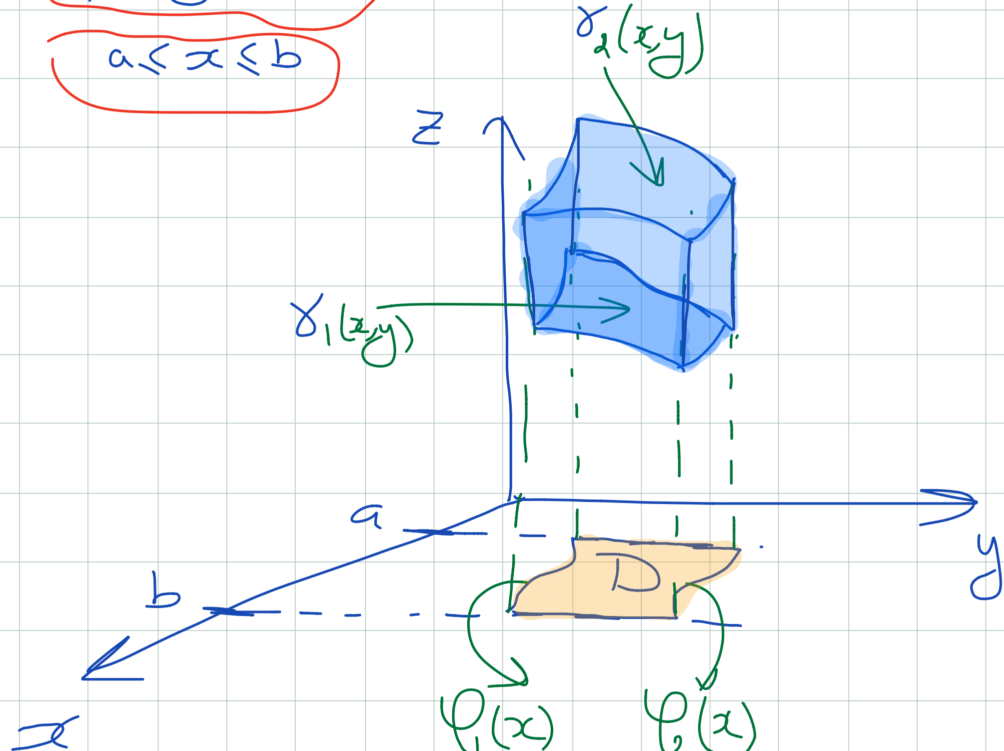
Example: We call a region elementary if it can be described as

$\gamma_1(x,y) \leq z \leq \gamma_2(x,y)$

$(x,y) \in D$ where D is simple.

so $\varphi_1(x) \leq y \leq \varphi_2(x)$

$a \leq x \leq b$



Example: The unit ball can be described as an elementary region: bec. $x^2+y^2+z^2 \leq 1$ can be written as

$$\begin{cases} -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2} \\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ -1 \leq x \leq 1 \end{cases}$$

} D: y-simple

Physical interpretation of triple integrals:

$$\iiint_W 1\, dV = \text{Volume}(W)$$

$$\iiint_W f(x,y,z)\, dV = \text{mass of an object } W \text{ with non-homog. density given by } f(x,y,z).$$

Triple Integral by iterated integration

Example: Find the Volume of a ball of radius 1

Sol'n: We want $\iiint_{Ball} 1\, dV$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dy\, dx$$

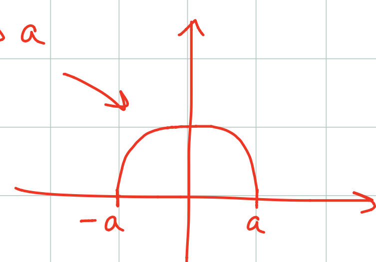
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy\, dx$$

Let $a = \sqrt{1-x^2}$

$\Rightarrow 2\sqrt{1-x^2-y^2} = 2(a^2-y^2)^{1/2}$

(only to make our calculation easier

$$= 2\int_{-1}^{1}\underbrace{\int_{-a}^{a}\sqrt{a^2-y^2}\, dy}_{\text{area of semicircle of radius } a}\, dx$$



$$= 2\int_{-1}^{1}\frac{\pi a^2}{2}\, dx \qquad a^2 = 1-x^2$$

$$= \pi\int_{-1}^{1} 1-x^2\, dx = \pi\left(x - x^3/3\right)\Big|_{-1}^{1} = \pi\left(1-\tfrac{1}{3}-(-1+\tfrac{1}{3})\right)$$

$$= 4/3\,\pi$$