# Lecture 8: Four Fundamental Subspaces and Linear Transformations (Section 2.4 and 2.6)

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- Definition. The **rank** of a matrix $A$ is the number of its pivots.

# The four fundamental subspaces

- Example. Find a basis for $C(A)$ and $C(A^T)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

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▶ Solution. To find $C(A)$, we can just use the echelon form of $A$. Likewise, we can also obtain $C(A^T)$ for an echelon form of $A^T$. But, it's not necessary!

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The rank of $A$ is $2$. And $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ form a basis for $C(A)$.

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Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -5 \end{bmatrix} \right\}$ form a basis for $C(A)$.

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**Theorem.** (Fundamental Theorem of Linear Algebra, Part I)
Let $A$ be an $m \times n$ matrix of rank $r$.

- $\dim C(A) = r$
- $\dim C(A^T) = r$
- $\dim N(A) = n - r$
- $\dim N(A^T) = m - r$

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- Example.

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▶ Example. Suppose $A$ is a $5 \times 5$ matrix, and that $\boldsymbol{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of $A$. What can you say about the number of solutions to $A\boldsymbol{x} = 0$.

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## Linear transformations

Consider vector spaces $V$ and $W$.

- Definition. A map $T : V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

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- Example. Let $A$ be an $m \times n$ matrix. Then the map $T(\boldsymbol{x}) = A\boldsymbol{x}$ is a liearn transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Why?
- Example. Let $\mathscr{P}_n$ be the vector space of all polynomials of degree at most $n$. Consider the map $T : \mathscr{P}_n \to \mathscr{P}_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

This map is a linear transformation! Why?

## Representing linear maps by matrices

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for $V$. A linear map $T : V \to W$ is determined by the values $T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)$.

## Representing linear maps by matrices

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▶ Definition. (From linear maps to matrices)

Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ be a basis for $V$, and $\boldsymbol{y}_1, \dots, \boldsymbol{y}_m$ a basis for $W$. The **matrix representing** $T$ with respect to these bases

- has $n$ columns (one for each of the $\boldsymbol{x}_j$),
- the $j$-th column has $m$ entries $a_{1j}, \dots, a_{mj}$ determined by

$$T(\boldsymbol{x}_j) = a_{1j}\boldsymbol{y}_1 + \dots + a_{mj}\boldsymbol{y}_m.$$

## Representing linear maps by matrices

- Example. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let $T$ be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}.$$

What is the matrix $A$ representing $T$ with respect to the standard bases?

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What is the matrix $B$ representing $T$ with respect to the following bases?

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix} \text{ for } \mathbb{R}^2, \quad \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ for } \mathbb{R}^3.$$

▶ Example. Let $T : \mathscr{P}_3 \to \mathscr{P}_2$ be the linear map given by

$$T(p(t)) = \frac{d}{dt}p(t).$$

What is the matrix $A$ representing $T$ with respect to the standard bases?