HW02 - Math 102

1) The first two elimination steps are to subtract 2 times row 1 from row 2 and to subtract 3 times row 1 from row 3, yielding

$$(*) \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & -1 \end{bmatrix}.$$

these two steps correspond to the matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Now looking at (*), the final elimination is to subtract twice row 2 from row 3, yielding

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

this elimination step corresponds to the matrix

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Hence, we obtain

$$E_{32} E_{31} E_{21} A = U.$$

(Note that there may be many answer of E_{21}, E_{31}, E_{32}).

The action of E_{21} is undone by adding twice row 1 to row 2,

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, $E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

$$\Rightarrow L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

2) a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

We write the augmented matrix $[A|I]$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

In the augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Hence,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

3) a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$$

Hence, the equation is solvable only if

$b_2 - 2b_1 = 0$ and $b_3 + b_1 = 0$.

In other words, the righ-hand sides of the equation must be a vector of the form.

$$\begin{bmatrix} b_1 \\ 2b_1 \\ -b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

for any real number b_1. That is, the column space of the given matrix is the line containing the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

b) The augmeted matrix is

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{array}\right]$$

The given equation is solvable only if

$b_3 + b_1 = 0 \iff b_3 = -b_1$.

Hence, the possible right-hand sides are vectors of the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The column space of the given matrix is the plane containing the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

4 a) Show that $A^2 = 0$ is possible.

For example, we can consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ Then } A \neq 0, \text{ but}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

b) Show that $A^T A = 0$ is not possible unless $A = 0$.

<u>Solution.</u> Suppose A is $m \times n$ and $A \neq 0$.

$$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_m] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$ where $\vec{a}_i$ is the i-th column vector of A.

Then,

$$A^T A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix} [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$$

$$= \begin{bmatrix} \langle \vec{a}_1, \vec{a}_1 \rangle & \langle \vec{a}_1, \vec{a}_2 \rangle & \cdots & \langle \vec{a}_1, \vec{a}_n \rangle \\ \langle \vec{a}_2, \vec{a}_1 \rangle & \langle \vec{a}_2, \vec{a}_2 \rangle & \cdots & \langle \vec{a}_2, \vec{a}_n \rangle \\ \vdots & \vdots & & \\ \langle \vec{a}_n, \vec{a}_1 \rangle & \langle \vec{a}_n, \vec{a}_2 \rangle & \cdots & \langle \vec{a}_n, \vec{a}_n \rangle \end{bmatrix}.$$

Note that the diagonal entries are of the form

$$\langle \vec{a}_i, \vec{a}_i \rangle = \|\vec{a}_i\|_2^2 = \sum_{k=1}^{m} a_{ik}^2 \geqslant 0.$$

If $A^T A = 0$, all its diagonal entries are 0. Hence, all a_{ik}'s are zero. This is not possible as we assume $A \neq 0$.

Therefore, $A^T A \neq 0$.