**Math 170A - Lecture B00**
**Fall 2016**
**Midterm #1, version B**
**10/21/2016**
**Time Limit: 50 Minutes**

**Name:** Key

PID: ____________________

Section Time: ____________________

This exam contains 6 pages (including this cover page) and 5 questions.
Total of points is 100.

Turn off and put away your cell phone.
You may not use any notes (except your cheat sheet) or calculators during this exam.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

Grade Table (for teacher use only)

| Question | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 10 | |
| 5 | 15 | |
| Total: | 100 | |

1. (25 points) Find the *two iterations* of the Jacobi method for the linear system

$$\begin{aligned} 10x_1 - \quad x_2 \qquad\quad &= 9 \\ -x_1 + 10x_2 - \ \ 2x_3 &= 7 \\ - \ \ 2x_2 - 10x_3 &= 6 \end{aligned}$$

using $x^{(0)} = (0,0,0)$.

**Solution.** From the system, we have that

$$\begin{aligned} x_1 &= \frac{9 + x_2}{10}, \\ x_2 &= \frac{7 + x_1 + 2x_3}{10}, \\ x_3 &= \frac{6 + 2x_2}{-10}. \end{aligned}$$

For the 1sth iteration, using $x^{(0)} = (0,0,0)$, we obtain

$$\begin{aligned} x_1^{(1)} &= \frac{9 + x_2^{(0)}}{10} = \frac{9}{10}, \\ x_2^{(1)} &= \frac{7 + x_1^{(0)} + 2x_3^{(0)}}{10} = \frac{7}{10}, \\ x_3^{(1)} &= \frac{6 + 2x_2^{(0)}}{-10} = \frac{-6}{10} = -\frac{3}{5}. \end{aligned}$$

For the 2nd iteration, using $x^{(1)} = (9/10, 7/10, -3/5)$, we obtain

$$\begin{aligned} x_1^{(2)} &= \frac{9 + x_2^{(1)}}{10} = \frac{9 + \frac{7}{10}}{10} = \frac{97}{100}, \\ x_2^{(1)} &= \frac{7 + x_1^{(1)} + 2x_3^{(1)}}{10} = \frac{7 + \frac{9}{10} - \frac{6}{5}}{10} = \frac{67}{100}, \\ x_3^{(1)} &= \frac{6 + 2x_2^{(1)}}{-10} = \frac{6 + \frac{14}{10}}{-10} = -\frac{74}{100}. \end{aligned}$$

2. (25 points) Let

$$A = \begin{bmatrix} 25 & 10 & 5 \\ 10 & 8 & 4 \\ 5 & 4 & 6 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}.$$

Matrix $A$ is symmetric positive definite.

a. (10 points) Compute the Cholesky factor $R$ of $A$.

b. (5 points) Check that $R^T R = A$.

c. (10 points) Solve $Ax = b$ for $x$ using the forward substitution and the backward substitution. (*Hint:* Notice that $R^T Rx = b$ and let $Rx = y$. Solve the first equation for $y$, then solve the second one for $x$.)

**Solution.**

a. $r_{11} = \sqrt{25} = 5$.
$r_{1j} = \frac{a_{1j}}{r_{11}}$ for $j = 2, 3$ so $r_{12} = \frac{10}{5} = 2$ and $r_{13} = \frac{5}{5} = 1$.
$r_{22} = \sqrt{a_{22} - r_{12}^2} = \sqrt{8 - 4} = 2$.
$r_{23} = \frac{a_{23} - r_{12} r_{13}}{r_{22}} = \frac{4 - 2(1)}{2} = 1$.
$r_{33} = \sqrt{a_{33} - r_{13}^2 - r_{23}^2} = \sqrt{6 - (1)^2 - 1^2} = 2$. Hence,

$$R = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

b. $R^T R = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 \times 5 & 5 \times 2 & 5 \times (1) \\ 2 \times 5 & 2 \times 2 + 2 \times 2 & 2 \times (1) + 2 \times (1) \\ 1 \times 5 & 1 \times 2 + 1 \times 2 & (1) \times (1) + 1 \times 1 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 25 & 10 & 5 \\ 10 & 8 & 4 \\ 5 & 4 & 6 \end{bmatrix} = A$.

c. First, let $y = Rx$. And solve $R^T y = b$ for $y$. That is,

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}.$$

Using the forward substitution method, we obtain

$$\begin{aligned} 5y_1 = 10 &\Rightarrow y_1 = 2 \\ 2y_1 + 2y_2 = -2 &\Rightarrow y_2 = -1 - y_1 = -3 \\ y_1 + y_2 + 2y_3 = -5 &\Rightarrow y_3 = \frac{-5 - y_1 - y_2}{2} = \frac{-5 - 2 + 3}{2} = -2. \end{aligned}$$

Now, solve the system $Rx = y$ for $x$. That is,

$$\begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}.$$

Using the backward method, we get

$$
\begin{aligned}
2x_3 = -2 &\Rightarrow x_3 = -1 \\
2x_2 + x_3 = -3 &\Rightarrow x_2 = \frac{-3 - x_3}{2} = -1 \\
5x_1 + 2x_2 + x_3 = 2 &\Rightarrow x_1 = \frac{2 - 2x_2 - x_3}{5} = \frac{2 - 3(-1) + 1}{5} = 1.
\end{aligned}
$$

3. (25 points) Consider the differential equation $-u''(x) = \pi^2\cos(\pi x)$ for $0 < x < 1$ with boundary condition $u(0) = 1$ and $u(1) = -1$. We wish to solve it approximately by the finite difference method. We subdivide the interval $[0, 1]$ into 4 equal subintervals of length $h = \frac{1}{4}$. The subdivision points of the intervals are $x_i = \frac{1}{4}i$ for $i = 0, \ldots, 4$. Write the system of equations in the form $Au = b$ that we would use to solve for the approximations $u(x_1), u(x_2), u(x_3)$. DON'T solve the system.
(*Hint:* $u''(x_i) \approx \frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}$.)

**Solution.** Observe that $x_0 = 0, x_1 = 1/4, x_2 = 2/4, x_3 = 3/4$, and $x_4 = 1$. Since $u_0 = u(x_0) = u(0) = 1$ and $u_4 = u(x_4) = u(1) = -1$, there are three unknowns we need to find. They are $u_1 = u(x_1), u_2 = u(x_2)$, and $u_3 = u(x_3)$.

Substituting $u''(x_i) \approx \frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}$ into the equation $-u''(x) = \pi^2\cos(\pi x)$ , we obtain that, for $i = 1, 2, 3$,

$$-\frac{u_{i+1} - 2u(x_i) + u(x_{i-1})}{h^2} = \pi^2\cos(\pi x_i).$$

For $i = 1$, we have

$$\begin{aligned}-16(u_2 - 2u_1 + u_0) &= \pi^2\cos(\pi x_1)\\ 32u_1 - 16u_2 &= \pi^2\cos(\frac{\pi}{4}) + 16.\end{aligned}$$

Similarly for $i = 2$ and 3, we have

$$-16u_1 + 32u_2 - 16u_3 = \pi^2\cos(\frac{2\pi}{4}),$$

and

$$-16u_2 + 32u_3 = \pi^2\cos(\frac{3\pi}{4}) - 16.$$

The system $Au = b$ is

$$\begin{bmatrix} 32 & -16 & 0 \\ -16 & 32 & -16 \\ 0 & -16 & 32 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \pi^2\cos(\frac{\pi}{4}) + 16 \\ \pi^2\cos(\frac{2\pi}{4}) \\ \pi^2\cos(\frac{3\pi}{4}) - 16 \end{bmatrix}.$$

4. (10 points) Let
$$A = \begin{bmatrix} -6 & 1 & 2 \\ 1 & 0 & -2 \\ 6 & 1 & 4 \end{bmatrix}.$$
Find $\|A\|_\infty$ and $\|A\|_1$.

**Solution.** Since $\sum_{j=1}^3 |a_{1j}| = 9, \sum_{j=1}^3 |a_{2j}| = 3$, and $\sum_{j=1}^3 |a_{3j}| = 11$,
$$\|A\|_\infty = \max_{1\le i\le 3} \sum_{j=1}^3 |a_{ij}| = 11.$$
Since $\sum_{i=1}^3 |a_{i1}| = 13, \sum_{i=1}^3 |a_{i2}| = 2$, and $\sum_{i=1}^3 |a_{i3}| = 8$,
$$\|A\|_1 = \max_{1\le j\le 3} \sum_{i=1}^3 |a_{ij}| = 13.$$

5. (15 points) Given any $p$-norm on $\mathbb{R}^2$, the *unit circle* with respect to that norm is the set $\{x \in \mathbb{R}^2 \mid \|x\|_p = 1\}$. On the single set of coordinate axes, sketch the unit circle with respect to the $p$-norm for $p = 1, 2$, and $\infty$.

**Solution.**