Section 2.4 Paths and Curves

Definition. A function $\vec{c} : [a, b] \to \mathbb{R}^n$ is called a *path* (in the plane when $n = 2$ and in the space when $n = 3$). Here, $\vec{c}(a)$ and $\vec{c}(b)$ are its endpoints. C is the collection of points $\vec{c}(t)$ as t varies in $[a, b]$. C is called a *curve*. The path $\vec{c}$ is said to *parametrize* the curve C.

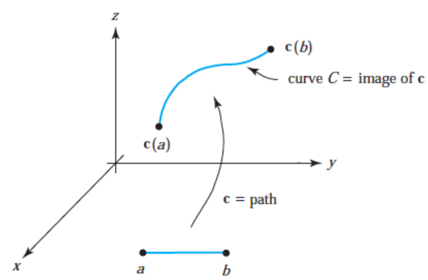


figure **2.4.1** The map **c** is the path; its image C is the curve we "see."

Example. The path $\vec{c}(t) = (t, t^2)$ traces out a paraboloic arc. Here $x(t) = t$ and $y(t) = t^2$. This curve coincides with the graph $f(x) = x^2$.

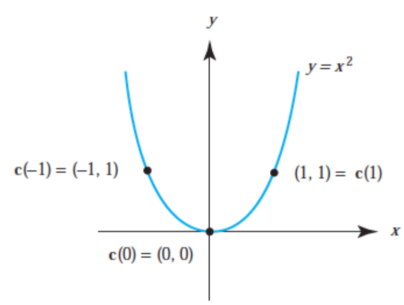


figure **2.4.4** The image of $\mathbf{c}(t) = (t, t^2)$ is the parabola $y = x^2$.

Example. The path $\vec{c}(t) = (a_1, a_2, a_3) + t(v_1, v_2, v_3)$ traces out a line.

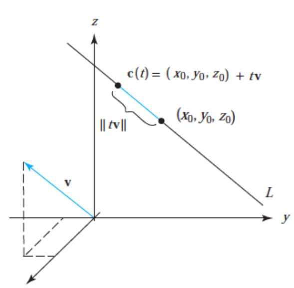


figure **2.4.2** L is the straight line in space through (x_0, y_0, z_0) and in direction $\mathbf{v}$; its equation is $\mathbf{c}(t) = (x_0, y_0, z_0) + t\mathbf{v}$.

Example. The unit circle $C : x^2 + y^2 = 1$ in the plane is the image of the path

$$\vec{c} : [0, 2\pi] \to \mathbb{R}^2, \quad \vec{c}(t) = (\cos t, \sin t)$$

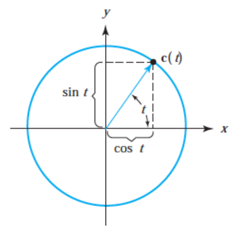


figure **2.4.3** $\mathbf{c}(t) = (\cos t, \sin t)$ is a path whose image C is the unit circle.

Note that different paths may parametrize the same curve. For example, $\vec{d}(t) = (\cos 2t, \sin 2t)$ for $0 \leq t \leq \pi$ also parametrizes the unit circle.

We usually think of t as time so $\vec{c}(t)$ is position at time t. Then we can talk about the *velocity* $\vec{v}(t)$ at time t. If $\vec{c}$ is a path and it's differentiable, we say $\vec{c}$ is a *differentiable path*.

Definition. The *velocity* of $\vec{c}$ at time t is given by

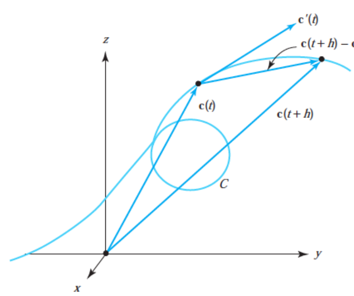
$$\vec{c}(t)' = \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}.$$

And the *speed* is $\|\vec{c}(t)'\|$, the length of the velocity vector.

Note that if $\vec{c}(t) = (x(t), y(t), z(t))$, then $\vec{c}(t)' = (x'(t), y'(t), z'(t)) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$.

Remark. $\vec{c}(t)'$ is tangent to the path $\vec{c}(t)$ at time t.

figure **2.4.8** The vector $\mathbf{c}'(t)$ is tangent to the path $\mathbf{c}(t)$.



Example. Compute the tangent vector to the path $\vec{c}(t) = (t, t + t^2, e^{2t})$ at $t = 1$.
Solution. Since $\vec{c}(t)' = (1, 1 + 2t, 2e^{2t})$, $\vec{c}(1)' = (1, 2, 2e^2)$.

Example. (A helix) $\vec{c}(t) = (\cos t, \sin t, t)$, so $\vec{c}(t)' = (-\sin t, \cos t, 1)$.

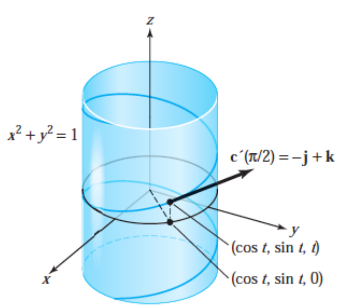


figure **2.4.9** The helix $\mathbf{c}(t) = (\cos t, \sin t, t)$ wraps around the cylinder $x^2 + y^2 = 1$.

Tangent Line to a Path

Suppose $\vec{c}(t)$ is a path and $\vec{c}(t_0)' \neq 0$ at t_0. The equation of the *tangent line* at $\vec{c}(t_0)$ is

$$\vec{\ell}(t) = \vec{c}(t_0) + (t - t_0)\vec{c}(t_0)'.$$

Example. Suppose that a particle follows the path $\vec{c}(t) = (\cos t, \sin t, t)$ and flies off at a tangent at $t = \frac{\pi}{2}$. Where is it at $t = \frac{\pi}{2} + 1$?
Solution. We need to find the equation of the line tangent to the curve at $\vec{c}(\frac{\pi}{2}) = (0, 1, \frac{\pi}{2})$.
Since $\vec{c}(t)' = (-\sin t, \cos t, 1), \vec{c}(\frac{\pi}{2})' = (-1, 0, 1)$. The equation of the tangent line to the curve at $(0, 1, \frac{\pi}{2})$ is

$$\vec{\ell}(t) = (0, 1, \frac{\pi}{2}) + (t - \frac{\pi}{2})(-1, 0, 1)$$

. Hence, at $t = \frac{\pi}{2} + 1$,

$$\ell(\frac{\pi}{2} + 1) = (0, 1, \frac{\pi}{2}) + (1)(-1, 0, 1) = (-1, 1, \frac{\pi}{2} + 1).$$