Section 2.3 Differentiation

Recall that for a one-variable function $f : \mathbb{R} \to \mathbb{R}$, we say that f is differentiable at x if the following limit exists

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Partial Derivatives

So, for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ we need a definition of what we mean by the phrase "$f(x_1, \ldots, x_n)$ is *differentiable* at $\vec{x} = (x_1, \ldots, x_n)$". This definition is not quite as simple as one might think. Let us introduce the notion of *partial derivative*. We start first with a function $f : \mathbb{R}^3 \to \mathbb{R}$. The partial derivatives of such f are defined by

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y, z) &= \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}, \\
\frac{\partial f}{\partial y}(x, y, z) &= \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}, \\
\frac{\partial f}{\partial z}(x, y, z) &= \text{ what should it be?}
\end{aligned}$$

In other words, $\frac{\partial f}{\partial x}$ is just the derivative of f with respect to the variable x, with the other variables held fixed; $\frac{\partial f}{\partial y}$ is just the derivative of f with respect to the variable y, with the other variables held fixed; and $\frac{\partial f}{\partial z}$ is just the derivative of f with respect to the variable z, with the other variables held fixed.

Notice that the partial derivatives are themselves functions of x, y, and z.
Example. Let $f(x, y, z) = x^2y^3 + \sin(xy) + z^2y$. Find $\frac{\partial f}{\partial y}(0, 1, 1)$ and $\frac{\partial f}{\partial z}(1, 0, 1)$.
Solution.

$$\begin{aligned}
&\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(x^2y^3 + \sin(xy) + z^2y) = 3x^2y^2 + x\cos(xy) + z^2 \Rightarrow \frac{\partial f}{\partial y}(0, 1, 1) = 1. \\
&\frac{\partial f}{\partial z}(x, y, z) = ?????
\end{aligned}$$

Definition. (Partial Derivatives) Let $U \subset \mathbb{R}^n$ be an open set and suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function. Then $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$, the **partial derivatives** of f with respect to the first, second, ..., nth variable, are the real-valued functions of n variables, which, at the point $(x_1, \ldots, x_n) = \vec{x}$, are defined by

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n) &= \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h} \\
&= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{e_j}) - f(\vec{x})}{h}
\end{aligned}$$

if the limits exist, where $1 \le j \le n$ and $\vec{e_j}$ is the jth standard basis vector defined by $\vec{e_j} = (0, \ldots, 1, \ldots, 0)$, with 1 in the j slot.

In other words, $\frac{\partial f}{\partial x_j}$ is just the derivative of f with respect to the variable x_j, with the other variables held fixed.

Notation. To indicate that a partial derivative is to be evaluated at a particular point, for example, at (x_0, y_0), we write

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0} \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}.$$

Differentiability for Functions of Two Variables

We start to define the notion of differentiability for two variables.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is *differentiable* at (x_0, y_0)

- if $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist
- and if

(1)
$$\frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\|(x,y)-(x_0,y_0)\|} \longrightarrow 0$$

as $(x, y) \to (x_0, y_0)$.

But what does (1) mean? If f is a one-variable function, then (1) is equivalent to

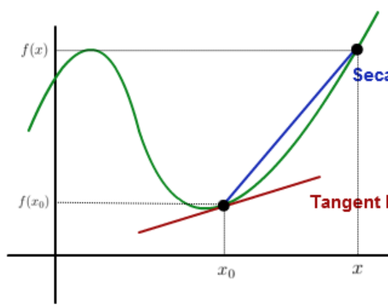
$$\frac{f(x) - f(x_0) - \left[\frac{\partial f}{\partial x}(x_0)\right](x-x_0)}{x - x_0} \longrightarrow 0 \quad \text{as } x \to x_0,$$

which is equivalent to

$$\underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\text{slope of secant}} \longrightarrow \underbrace{\frac{\partial f}{\partial x}(x_0)}_{\text{slope of tangent}} \qquad \text{as } x \to x_0$$

and to

$$f(x) \longrightarrow \underbrace{\frac{\partial f}{\partial x}(x_0)(x-x_0) + f(x_0)}_{\text{linear approximation}} \quad \text{as } x \to x_0.$$



Now let us get back to (1). If f is differentiable, when we are close to (x_0, y_0), the graph of the tangent plane is close to the graph of f.

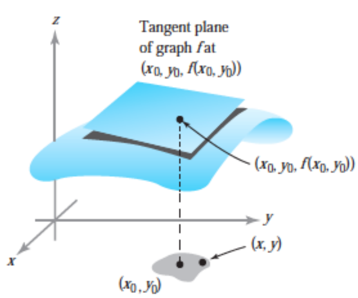


figure 2.3.3 For points (x, y) near (x_0, y_0), the graph of the tangent plane is close to the graph of f.

That is, when (x, y) is close to (x_0, y_0),

$$f(x,y) \approx f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0).$$

We call $L(x,y) = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$ the *linear approximation* of $f(x,y)$ near (x_0,y_0).

Definition. (Tangent Plane) Let $f:\mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0,y_0). The plane in $\mathbb{R}^3$ given by

$$z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$$

is called the tangent plane to the graph of f at $(x_0,y_0,f(x_0,y_0))$.

Example. Let $f(x,y) = xe^{xy}$. Find

a) its tangent plane at $(1,0)$, and

b) use it to approximate $f(1.1,-0.1)$.

Solution. a) We calculate the following

$$\begin{aligned} f(1,0) &= 1e^0 = 1 \\ \frac{\partial f}{\partial x}(x,y) &= e^{xy} + xye^{xy} \implies \frac{\partial f}{\partial x}(1,0) = 1 \\ \frac{\partial f}{\partial y}(x,y) &= x^2e^{xy} \implies \frac{\partial f}{\partial y}(1,0) = 1. \end{aligned}$$

The equation of the tangent plane at $(1,0)$ is

$$\begin{aligned} z &= f(1,0) + \frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)(y-0) \\ &= 1 + 1(x-1) + 1(y) \\ &= x + y. \end{aligned}$$

b) For (x,y) near $(1,0)$, the linear approximation of $f(x,y)$ is $L(x,y) = x+y$. Thus,

$$f(1.1,-0.1) \approx L(1.1,-0.1) = 1.1 - 0.1 = 1.$$

Example. Find the equation of the tangent plane to the graph of $f(x,y) = x + y^2 + \cos(xy)$ at $(0,1)$.
Solution. We first evaluate the following

$$\begin{aligned} f(0,1) &= 0 + 1 + \cos(0) = 2 \\ \frac{\partial f}{\partial x}(x,y) &= 1 - y\sin(xy) \implies \frac{\partial f}{\partial x}(0,1) = 1 - 1\sin(0) = 1 \\ \frac{\partial f}{\partial y}(x,y) &= 0 + 2y - x\sin(xy) \implies \frac{\partial f}{\partial y}(0,1) = 2. \end{aligned}$$

So the equation of the tangent plane is

$$\begin{aligned} z &= f(0,1) + \frac{\partial f}{\partial x}(0,1)(x-0) + \frac{\partial f}{\partial y}(0,1)(y-1) \\ &= 2 + 1(x) + 2(y-1) \\ &= x + 2y. \end{aligned}$$

Differentiability (The general case)

The derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $\vec{x}_0$ denoted by $Df(\vec{x}_0)$ is an $m \times n$ *matrix* T whose elements are $t_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_{\vec{x}_0}$.
So, if $f(\vec{x}_0) = (f_1(\vec{x}_0), f_2(\vec{x}_0), \dots, f_m(\vec{x}_0))$,

$$T = Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{\vec{x}_0} & \frac{\partial f_1}{\partial x_2}\Big|_{\vec{x}_0} & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_{\vec{x}_0} \\ \frac{\partial f_2}{\partial x_1}\Big|_{\vec{x}_0} & \frac{\partial f_2}{\partial x_2}\Big|_{\vec{x}_0} & \cdots & \frac{\partial f_2}{\partial x_n}\Big|_{\vec{x}_0} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_{\vec{x}_0} & \frac{\partial f_m}{\partial x_2}\Big|_{\vec{x}_0} & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_{\vec{x}_0} \end{bmatrix}$$

Definition. Let U be an open set in $\mathbb{R}^n$ and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. Then f is *differentiable* at $\vec{x}_0 \in U$ if the partial derivatives of f exist at $\vec{x}_0$ and if

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\|f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0)\|}{\|\vec{x} - \vec{x}_0\|} = 0.$$

Example. Let $f(x, y, z) = (x^2 + \cos y, -ye^z)$. Find $Df(x, y, z)$.
Solution.

$$\begin{aligned} Df(x,y,z) &= \begin{bmatrix} \frac{\partial(x^2+\cos y)}{\partial x} & \frac{\partial(x^2+\cos y)}{\partial y} & \frac{\partial(x^2+\cos y)}{\partial z} \\ \frac{\partial(-ye^z)}{\partial x} & \frac{\partial(-ye^z)}{\partial y} & \frac{\partial(-ye^z)}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} 2x & -\sin y & 0 \\ 0 & -e^z & -ye^z \end{bmatrix}. \end{aligned}$$

Remark. If $f : U \subset \mathbb{R}^n \to \mathbb{R}$, then $Df(\vec{x}_0)$ is a $1 \times n$ matrix. The corresponding derivative matrix is the vector $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n})$ called the *gradient* and denoted by ∇f.

Example. Let $f(x, y, z) = 2x + y^2 + ze^x$, then

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (2 + ze^x, 2y, e^x) = (2 + ze^x)\vec{i} + 2y\vec{j} + e^x\vec{k}.$$

Theorem. If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0 \in U$, then f is continuous at $\vec{x}_0$.

Theorem. If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is such that the partials $\frac{\partial f_i}{\partial x_j}$

- all exist and

- are continuous in a neighborhood of $\vec{x} \in U$,
  then f is differentiable at $\vec{x}$.

Example. $f(x, y, z) = (x^2 + \cos y, -ye^z)$ from the previous example is differentiable because the partials exist and are continuous.
Example. $f(x, y) = \frac{\cos x + e^{xy}}{x^2+y^2}$ is differentiable at all points $(x, y) \neq 0$.

Example. But $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ is continuous, has partial derivatives at $(0, 0)$, yet is *not differentiable* there because its partial derivatives cannot be continuous at $(0, 0)$.