Math 20A. Midterm Exam 1. 2012.

1) a) $$\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x+3} - \frac{1}{3}\right) = \lim_{x\to 0} \frac{1}{x}\cdot\left(\frac{3-(x+3)}{(x+3)3}\right)$$

$$= \lim_{x\to 0} \frac{1}{x}\ \frac{-x}{3(x+3)}$$

$$= \lim_{x\to 0} \frac{-1}{3(x+3)}$$

$$= \frac{-1}{3(0+3)}$$

$$= -\frac{1}{9}.$$

b) $$\lim_{x\to\infty}\left(\sqrt{x+2} - 2\sqrt{x}\right) \qquad (\infty - \infty).$$

$$= \lim_{x\to\infty}\left(\sqrt{x+2} - 2\sqrt{x}\right).\ \frac{\sqrt{x+2}+2\sqrt{x}}{\sqrt{x+2}+2\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{x+2-4x}{\sqrt{x+2}+2\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{2-3x}{\sqrt{x+2}+2\sqrt{x}}$$

$$\cancel{= \lim_{x\to\infty} \frac{\frac{2}{x}-3}{\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\frac{2}{\sqrt{x}}}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}} - \frac{3x}{\sqrt{x}}}{\frac{\sqrt{x+2}}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}} - 3\sqrt{x}}{\sqrt{1+\frac{2}{x}} + 2}$$

$$= -\infty$$

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2) Let f be differentiable function.

$f(1) = 3$, $f'(1) = -2$, $f(2) = -3$. $f'(2) = 1$, $f(3) = 1$, $f'(3) = 0$

a) The equation for the tangent line at $x = 2$ is

$$y - f(2) = f'(2)(x-2).$$

$$y - (-3) = 1(x-2).$$

$$y + 3 = x - 2$$

$$\boxed{y = x - 5.}$$

b) f is differentiable $\Rightarrow$ f is continuous

Since $f(1) = 3 > 0 > -3 = f(2)$, by the Intermediate Value theorem, there exists $c_1 \in (1, 2)$ such that $f(c_1) = 0$.

Similarly, since $f(2) = -3 < 0 < f(3) = 1$, there exists $c_2 \in (2, 3)$ such that $f(c_2) = 0$.

3) a) $f(x) = x^2 e^x \Rightarrow f'(x) = 2xe^x + x^2 e^x.$

b) $f(x) = \dfrac{x^5 - e^x}{5x + 1} \Rightarrow f'(x) = \dfrac{(5x^4 - e^x)(5x+1) - (x^5 - e^x)(5)}{(5x+1)^2}$

$$= \frac{25x^5 + 5x^4 - 5xe^x - e^x - 5x^5 + 5e^x}{(5x+1)^2}$$

$$= \frac{20x^5 + 5x^4 - 5xe^x + 4e^x}{(5x+1)^2}$$

c) $f(x) = \pi^3(x+1) \Rightarrow f'(x) = \pi^3$.

4) $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0. \end{cases}$

a) $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{f(x)}{x} = \lim_{x \to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{x}$

$= \lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

~~b)~~

b) For $x \neq 0$,

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$.

$\Rightarrow \quad -|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$.

Since $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$,

By the squeeze theorem,

$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.