Lecture 2: Gaussian Elimination and Matrix Operations

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How to solve systems of linear equations

- Strategry: replace system with an equivalent system which is *easier to solve*.
- Definition. Linear systems are **equivalent** if they have the same set of solutions.
- Example.

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0 \end{aligned} \xrightarrow{R_2 \rightsquigarrow R_1 + R_2} \begin{aligned} x_1 + x_2 &= 1 \\ 2x_2 &= 1. \end{aligned}$$

In a *triangular form* $\Longrightarrow$ Use *back-substitution*:

$$x_2 = \frac{1}{2}, \quad x_1 = ?$$

▶ Example.

$$\begin{array}{rcr} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \end{array} \xrightarrow{R_3 \rightsquigarrow R_3 + 4R_1} \begin{array}{rcr} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -3x_2 + 13x_3 & = & -9 \end{array}$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 + \frac{3}{2}R_2} \begin{array}{rcr} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ x_3 & = & 3 \end{array}$$

In triangular form $\Longrightarrow$ use back-substitution:

$$x_3 = 3, \quad x_2 = ..., \quad x_1 = ...$$

Matrix notation: $A\boldsymbol{x} = \boldsymbol{b}$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \longrightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

- **Definition.** An *elementary row operation* is one of the following:
  - Replacement: Add one row to a multiple of another row.
  - Interchange: Interchange two rows.
  - Scaling: Multiply all entries in a row by a nonzero constant.
- **Definition.** Two matrices are *row equivalent*, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.
- **Definition.** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \Big\downarrow R_3 \longrightarrow R_3+4R_1$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \Big\downarrow R_3 \longrightarrow R_3 + \frac{3}{2}R_2$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Instead of back-substitution, we can continue with row operations:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & 2x_2 & & & = & 32 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{ccccr} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Row reduction (Gaussian elimination) and echelon forms

- **Definition.** A matrix is in echelon form (or row echelon form) if:
  - Each leading entry (i.e. leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
  - All entries in a column below a leading entry are zero.
  - All nonzero rows are above any rows of all zeros.
- **Example.**

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($*$ stands for any value, and $\blacksquare$ for any nonzero value.)

- **Example.** Are the following matrices in echelon form?

a)

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * \\ \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} \blacksquare & 0 & 0 \\ * & \blacksquare & 0 \\ * & 0 & \blacksquare \\ * & 0 & 0 \end{bmatrix}$$

▶ Definition. A leading entry in an echelon form is called a *pivot*.

▶ Definition. A matrix is in *reduced echelon form* if, in addition to being in echelon form, it also satisfies:

- Each pivot is 1.
- Each pivot is the only nonzero entry in its column.

▶ Example. Our initial matrix in echelon form put into reduced echelon form:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dashrightarrow \begin{bmatrix} 0 & \blacksquare & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution of linear systems via row reduction

- Example. Find a parametric description of the solution set of:

$$\begin{array}{rrrrrcr} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & 12x_3 & -9x_4 & +6x_5 & = & 15. \end{array}$$

- Solution. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

We find its reduced echelon form as (exercise)

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

The pivot variables are ? The free variables are ?

Existence and uniqueness

Theorem.(Existence and uniqueness theorem)
A linear system is consistent if and only if an echelon form of the augmented matrix has no row of the form

$$[\ 0\ \ldots\ 0 \mid a]$$

where a is nonzero.
If a linear system is consistent, then the solution contains either

- a unique solution (when there are no free variables) or
- infinitely many solutions (when there is at least one free variable).

Example. For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h. \end{aligned}$$

Basic notation

Consider an $m \times n$ matrix A (m rows, n columns).

$$A = [\ \boldsymbol{a}_1\ \boldsymbol{a}_2\ \cdots\ \boldsymbol{a}_{\boldsymbol{n}}\ ] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix times vector

$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_n\boldsymbol{a}_n, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The product of a matrix A with a vector $\boldsymbol{x}$ is a linear combination of the columns of A with weights given by the entries of $\boldsymbol{x}$.

▶ Example.

a)

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}.$$

b)

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 + x_2 \end{bmatrix}.$$

▶ Example. Suppose A is $m \times n$ and $\boldsymbol{x}$ is in $\mathbb{R}^p$. Under which condition does $A\boldsymbol{x}$ make sense?

Matrix times matrix

The product of two matrices is given by

$$AB = [\, A\boldsymbol{b}_1 \; A\boldsymbol{b}_2 \; \cdots \; A\boldsymbol{b}_p \,], \quad \text{where } B = [\, \boldsymbol{b}_1 \; \boldsymbol{b}_2 \; \cdots \; \boldsymbol{b}_p \,].$$

▶Example.

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = ?$$

▶Example. Suppose A is $m \times n$ and B is $p \times q$. Under which condition does AB make sense? How about BA?