Lecture 4: LU Decomposition and Matrix Inverse

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Gaussian elimination revisited

▶ Example. Keeping track of the elementary matrices during Gaussian elimination on A:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$$

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Note that

$$A = E^{-1} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

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We factored A as the product of a lower and upper triangular matrix! We say that A has *triangular factorization*.

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- Definition.

lower triangular

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

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- Example. Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ as $A = LU$.
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$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

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$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

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$$E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

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The factor L is given by

$$
\begin{aligned}
L &= E_1^{-1}E_2^{-1}E_3^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

LU decompostion

We found the following LU decomposition of A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

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$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ L(U\boldsymbol{x}) &= \boldsymbol{b} \\ L\boldsymbol{c} &= \boldsymbol{b} \text{ and } U\boldsymbol{x} = \boldsymbol{c}. \end{aligned}$$

Both of the final systems are triangular and hence easily solved:

- $L\boldsymbol{c} = \boldsymbol{b}$ by forward substitution to find $\boldsymbol{c}$, and then
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▶ Example. Solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}.$$

The inverse of a matrix

- Definition. An $n \times n$ matrix A is **invertible** if there is a matrix B such that

$$AB = BA = I_{n \times n}.$$

In that case, B is the **inverse** of A and is denoted by A^{-1}.

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▶ Remark.

- The inverse of a matrix is unique. (Why?)

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- **Remark.**
  - The inverse of a matrix is unique. (Why?)
  - Do not write $\frac{A}{B}$.

The inverse of a matrix

- Example. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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- Example. The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible.
- Example. A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible **if and only if** $ad - bc \neq 0$.

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- A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. (Why?)

Solving systems using matrix inverse

Theorem. Let A be invertible. Then the system $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

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- To solve $A\boldsymbol{x} = \boldsymbol{b}$, we do row reduction on $[\, A \,|\, b \,]$.

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- To solve $AX = I$, we do row reduction on $[\,A \,|\, I\,]$.
- To compute A^{-1} (The Gauss-Jordan Method):
  - Form the augmented matrix $[\,A \,|\, I\,]$.
  - Compute the reduced echelon form.
  - If A is invertible, the result is of the form $[\,I \,|\, A^{-1}\,]$.

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- Solution.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

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Computing the inverse

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & \frac{-5}{16} & \frac{-6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & \frac{-3}{8} & \frac{-2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \left[\begin{array}{ccc} \frac{12}{16} & \frac{-5}{16} & \frac{-6}{16} \\ \frac{4}{8} & \frac{-3}{8} & \frac{-2}{8} \\ -1 & 1 & 1 \end{array}\right]$$

Why does it work?

- Each row reduction corresponds to multiplying with an elementary matrix E:

$$[\,A|I\,] \rightarrow [\,E_1A|E_1I\,] \rightarrow [\,E_2E_1A|E_2E_1\,] \rightarrow \ldots$$

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$$\ldots \to [\,FA\,|\,F\,] \quad \text{where } F = E_r \ldots E_2E_1.$$

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$$\ldots \to [\,FA\,|\,F\,] \quad \text{where } F = E_r \ldots E_2E_1.$$

- If we manage to reduce $[\,A|I\,]$ to $[\,I|F\,]$, this means

$$FA = I \Longrightarrow A^{-1} = F.$$