Section 2.1 The Geometry of Real-Valued Functions

In previous calculus courses, you have learned functions of one variable and scalar-valued functions. Here you will study functions of several variables and vector-valued functions. In this section, there are four keypoints: functions, graph, level sets, and sections.

Functions

We first introduce some terminology. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function/mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. In other words, f is a function that takes n inputs and give m outputs.
- When $m = 1$, f is *scalar-valued* since $f(x_1, \ldots, x_n) \in \mathbb{R}$.
- When $m > 1$, f is *vector-valued* since $f(x_1, \ldots, x_n) \in \mathbb{R}^m$. The subset $A \subset \mathbb{R}^n$ is called the *domain* of $f(\vec{x})$.
In particular, if $n = 1$, f is the function of *one variable*; and if $n > 1$, f is the function of *several variables*.

Example. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = x^2 + y^2 + z^2$$

is scalar-valued. (other notation $f : (x, y, z) \mapsto x^2 + y^2 + z^2$.)
But the function $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by

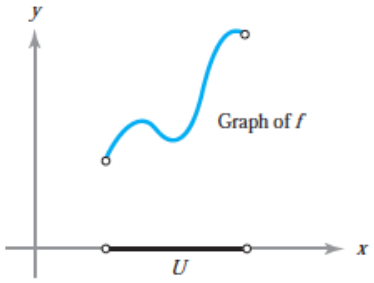
$$g : (x, y, z) \mapsto (x^2 + y^2 + z^2, x + y + z)$$

is vector-valued.

Graph

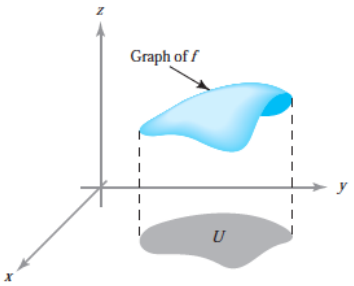
Recall that for a one-variable scalar function $f : U \subset \mathbb{R} \to \mathbb{R}$, its graph is the set of all points $(x, f(x))$ in $\mathbb{R}^2$ such that $x \in U$. That is,

$$\text{graph of } f = \{(x, y) | x \in U, y = f(x)\} \subset \mathbb{R}^2.$$



How about several-variable functions?
For example, let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, then

$$\text{graph of } f = \{(x_1, x_2, \ldots, x_n, z) \mid (x_1, \ldots, x_n) \in U, \text{ and } z = f(x_1, \ldots, x_n)\} \subset \mathbb{R}^{n+1}.$$



Level Sets

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The *level set* of value c is the set $\{\vec{x} \in U \mid f(\vec{x}) = c\} \subset \mathbb{R}$.
Remark. When $n = 2$, the level set is called the level curve; and when $n = 3$, it is called the level surface. (Why?)

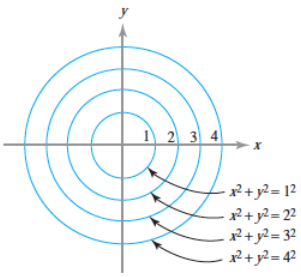


figure **2.1.6** Some level curves for the function $f(x, y) = x^2 + y^2$.

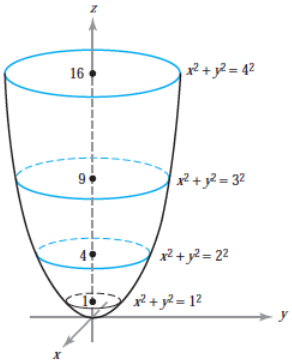


figure **2.1.7** Level curves in Figure 2.1.6 raised to the graph.

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