

The code distance of Floquet codes

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For fault-tolerant quantum memory defined by periodic Pauli measurements, called *Floquet codes*, we prove that every correctable, undetectable spacetime error occurring during the steady stage is a product of (i) measurement operators inserted at the time of the measurement and (ii) pairs of identical Pauli operators sandwiching a measurement that commutes with the operator. We call such errors *benign*; they define a binary vector subspace of spacetime errors which properly generalize stabilizers of static Pauli stabilizer codes. Hence, the code distance of a Floquet code is the minimal weight of an undetectable spacetime Pauli error that is not benign. Our results apply more generally to families of dynamical codes for which every instantaneous stabilizer is inferred from measurements in a time interval of bounded length.

1 Introduction

In a conventional construction of fault-tolerant quantum memory, one starts with a Pauli stabilizer code before designing a Clifford circuit to measure the stabilizers. The latter is as important as the former, and it is well known that the error correction capability of a Pauli stabilizer code as indicated by its code distance is only a performance target, *not* a guarantee. The appreciation of circuit level noise has grown over time, and realizable quantum error correction must be centered around circuits rather than a subspace of the Hilbert space of a many-qubit system [Got22, MBG23, DP23].

Prior work [HH21] establishes that carefully constructed measurement dynamics can function as a fault-tolerant quantum memory despite not stabilizing a subspace sufficient to encode even a single logical qubit. Perhaps the simplest class of such quantum memory is defined by periodic sequences of Pauli measurements, called *Floquet codes*. Of course, there is no fundamental reason to require measurement periodicity, and one can also consider more general *dynamical codes* [Got22] defined by Clifford circuits with Pauli measurements.

The code distance of Floquet or dynamical codes must be the minimal weight of an undetected spacetime error that causes a logical failure. While the detectability of such an error is relatively straightforward [MBG23, DP23] (but see also [FG24]), the question of whether the error causes a logical failure is often subtle, being highly sensitive to time boundary conditions.

In some concrete examples of Floquet codes (*e.g.*, the honeycomb code without [HH21, FDB⁺23] and with [HH22, PKD⁺23, GNM22] boundaries, the CSS honeycomb code [DTB23], the X-cube Floquet code [ZAV23], the coupled spin chain construction [YCC24], and Floquet Bacon–Shor code [AR24]), the code distance is estimated by analyzing the map from errors to syndromes [MBG23].

These analyses happen to rely on decoding graphs, meaning that there are exactly two flipped syndrome bits for each elementary error, which is not always true in general Floquet codes.

It is not always straightforward to identify logical failures caused by spacetime error configurations. [Vui21] constructs a Floquet code on the honeycomb graph for which all instantaneous stabilizer groups have $O(\sqrt{n})$ code distance, but whose time dynamics allow certain $O(1)$ weight spacetime errors to induce logical failures while remaining undetected. Moreover, some errors may force the physical state out of the instantaneous code space; nonetheless, under certain measurement dynamics, subsequent measurements can be not only unaffected — they may even *absorb* the error. Prior works have called this phenomenon *self-correction* [AR24].

This subtlety is rooted in the status that there is no criterion to tell whether an undetectable spacetime error will eventually have implemented a nontrivial logical operation on the encoded qubit. Such uncertainty is in contrast with conventional Pauli stabilizer codes, where any undetectable error must fall in precisely one of two classes: the trivial logical class consisting of all stabilizers, and the non-trivial logical class consisting of logical operators. The code distance of a Pauli stabilizer code is therefore the minimum weight of an undetectable error that is not a stabilizer. Classifying a given undetectable error within this dichotomy is efficient and amounts to checking its commutativity with any basis of the logical operators.

Here, we give such a dichotomy for undetectable spacetime errors in Floquet codes. We identify a class of errors which manifestly preserve logical states, generated by two sets of errors: (i) a Pauli measurement operator inserted at the time of the measurement, and (ii) a pair of identical Pauli operators ¹ inserted immediately before and after a measurement which commutes with the operator. We prove that any undetectable spacetime error inserted during the steady stage of a Floquet code that is not generated by the aforementioned errors implements a nontrivial logical operation on the encoded qubits. This conclusion follows from the result that, in any Floquet code, every undetectable spacetime error is equivalent to a logical operator of an instantaneous Pauli stabilizer code at some future time. We show that, given an undetectable spacetime error, computation of the equivalent logical operator is efficient; hence, as in the stabilizer code case, such a dichotomy can be checked efficiently. Our results hold in any dynamical code (and in particular, Floquet codes) for which every instantaneous stabilizer is inferred by measurements from a constant-width time window, which we call the *bounded-inference property*.

1.1 Prior work

Delfosse and Paetznick [DP23] considered arbitrary, finite Clifford circuits that include Pauli measurements. They defined two closely related codes: the *outcome code* is a binary vector space (a classical code) consisting of all possible outcomes of the measurements in a circuit in the absence of noise, and the *spacetime code* is a Pauli stabilizer code which maps spacetime Pauli errors to deviations of the measurement outcomes from the outcome code. In other words, their work focuses on *detection* of spacetime errors. This would provide sufficient machinery to study finite Clifford circuits whose operational meaning is self-contained, since one could encode the final logical outcome into some Pauli measurements. However, it is not always clear how to adapt this machinery if the finite circuit is treated as a fault-tolerant component (*e.g.*, a quantum memory) which maintains its own performance guarantees in modular applications. For Floquet codes, one might consider a sequence of Clifford circuits, starting with some logical encoding and ending with logical measure-

¹One is the inverse of the other in case of prime qudit codes.

ments with increasing memory times. The problem remains as to how to ascribe the error correction performance of the resulting circuit to the time boundary conditions or the underlying Floquet code; see also [Lemma 4.3](#).

Fu and Gottesman [\[FG24\]](#) study dynamical codes specified by an initial Pauli stabilizer stabilizer code S_0 and a finite sequence of Pauli measurements \mathbf{M} . A code state \mathbf{c} of S_0 is prepared, after which \mathbf{M} is measured through *once*; their fault model considers a one-time error inserted immediately following the preparation of \mathbf{c} . Our setting is considerably different: we do not fix S_0 a priori, instead considering a periodic measurement schedule of length P . The instantaneous stabilizer codes are induced entirely by the recurrent measurement dynamics, which persist indefinitely. Most importantly, whereas [\[FG24\]](#) considers a one-time error insertion, we consider general spacetime errors which may be inserted anywhere in spacetime after the code enters *steady stage* (when the instantaneous codes reaches a constant dimension).

There exists a graphical representation of quantum circuits, called ZX calculus, to study Clifford circuits with Pauli measurements [\[vdW20\]](#). Bombín *et al.* [\[BLN⁺24\]](#) used this approach on several examples, illustrating that detectors and logical failures can be probed using the commutation relations between errors and “Pauli webs” corresponding to detectors and logical operators. [\[BLN⁺24\]](#) focused on quantum circuits measuring Pauli observables supported in X, Z . Magdalena de la Fuente *et al.* [\[dLFOTT⁺25\]](#) extended ZX calculus to a tri-color ZXY calculus which more naturally characterizes circuits with arbitrary Pauli measurements, and showed how Pauli webs can be constructed in this more general setting. Xu and Dua [\[XD25\]](#) applied this approach to dynamical codes obeying certain spacetime-locality conditions, arguing for their fault-tolerance. In both [\[dLFOTT⁺25\]](#) and [\[XD25\]](#), it is important that there are input and output Pauli stabilizer codes that are finitely separated in time, and the same limitation as [\[DP23\]](#) applies. See also [Lemma 2.9](#).

Given the subtlety of probing logical failures without future time boundaries, one may wonder how numerical study of Floquet codes is even possible. This can indeed be subtle in general as one must often impose temporal boundaries to make a simulated circuit finite, and the temporal boundary may need to be tailored to the exact construction under consideration. On topological codes, the issue is less severe since the correlation between error correction operators is short ranged in time [\[DKLP02\]](#). Typically, error-free time steps and noiseless measurements are inserted in last stages of the circuit [\[GNFB21, GNM22, PKD⁺23, MBG23\]](#), and simulation software directly gives what remains after decoding. A justification for these noiseless final rounds is that any logical qubit will be measured out destructively in practice, and all these simulated quantum codes have a property that single-qubit measurements allow for reconstruction of all logical and syndrome bits, which is a property enjoyed by all CSS Pauli stabilizer codes. It however remains true that these time boundary conditions make it difficult to compare Floquet codes.

1.2 Settings

We recall the evolution of the stabilizer group under a Pauli measurement schedule. Suppose that the system is the maximally mixed state in a common eigenspace of a Pauli stabilizer group, called an **instantaneous stabilizer group** or $\text{ISG}(t)$ at time t , and that we measure a Pauli operator M in the subsequent time step. If $M \in \text{ISG}(t)$, then the physical state is an eigenstate of M . The measurement reveals this eigenvalue, and the physical state is undisturbed; hence, $\text{ISG}(t+1) = \text{ISG}(t)$. If on the other hand $M \notin \text{ISG}(t)$, we must consider two cases:

- (i) M commutes with every element of $\text{ISG}(t)$, or

(ii) M fails commute with some element $S \in \text{ISG}(t)$.

In case of (i), M is a nontrivial logical operator of the instantaneous code, and the post-measurement physical state is projected to an eigenstate of M . Depending on the outcome, M or $-M$ becomes a new stabilizer, and $\text{ISG}(t+1) = \langle \pm M, \text{ISG}(t) \rangle$; the rank of the stabilizer group increases by 1. In case of (ii), since the post-measurement physical state must be an eigenstate of M , the anticommuting operator S cannot be in $\text{ISG}(t+1)$. On the other hand, any element of $\text{ISG}(t)$ that commutes with M is still a stabilizer. The new stabilizer group $\text{ISG}(t+1)$ is generated by this M -commutant subgroup of $\text{ISG}(t)$ and $\pm M$; the rank of the stabilizer group stays the same.

Definition 1.1. A **dynamical code** is defined by a time sequence $\mathbf{M}_1, \mathbf{M}_2, \dots$ of sets $\mathbf{M}_t = \{M_{t,i}\}_i$ of Pauli operators $M_{t,i}$, where $M_{t,i}$ and $M_{t,j}$ must commute for all i, j . There is no requirement that operators $M_{t,i}$ and $M_{t',j}$ at different time steps $t \neq t'$ need to commute. If the sequence \mathbf{M}_t is periodic in t , then we call the dynamical code a **Floquet code**.

The implementation of a dynamical code is nothing but measuring $M_{t,i}$ for all i at time t . Since all the measurements at one time step is assumed to commute with each other, they can be implemented simultaneously.

Remark 1.2. One can include Clifford unitaries in the definition of a dynamical or Floquet code, but we do not consider them in this paper for simplicity. Note that all our results can be adapted in the unitary-allowed settings by conjugating Pauli errors by the unitary gates in the code.

Our dynamics always start with the maximally mixed state and the trivial instantaneous stabilizer group $\text{ISG}(0) = \{\mathbf{1}\}$. Later instantaneous stabilizer groups $\text{ISG}(t)$ are not determined before one actually implements the schedule. However, it is standard to calculate which measurements will have deterministic or nondeterministic outcomes, and, in this paper, $\text{ISG}(t)$ will mean an instantaneous stabilizer group determined by any valid set of measurement outcomes assuming that no errors occur during the dynamics.

Although the measured signs of Pauli operators are central to *performing* Pauli error correction, much of our derivation will not require detailed sign information. Thus, we will use

$$\overline{\text{ISG}} = \langle -\mathbf{1}, \text{ISG} \rangle \tag{1}$$

to denote the extension of the instantaneous stabilizer group by signs. This eases notation; for example, $M \in \overline{\text{ISG}}$ means that either $M \in \text{ISG}$ or $-M \in \text{ISG}$. Additionally, we will refer to “logical operators of ISG ” as shorthand for the more precise phrase “logical operators of the Pauli stabilizer code defined by ISG .”

Definition 1.3. Let \mathcal{P} be the Pauli group on n qubits, and let \mathcal{P}_t for any integer $t \geq 0$ be a copy of \mathcal{P} . A **spacetime Pauli error** $E = (\dots, E_t, \dots)$, or just an **error** for short, is an element of the infinite direct sum $\mathcal{E} = \bigoplus_{t \geq 0} \mathcal{P}_t$. If I is a set of time steps such that $E_t \in \mathbb{C}\mathbf{1}$ for all $t \notin I$, then we say E is **supported** on I . We write $[O]_{t'}$ for any Pauli operator O to mean the Pauli operator O inserted at time t' ; that is, the bracket $[\cdot]$ strips off the time coordinate, if any, and resets it by the subscript.

If we mod out all phase factors, the set \mathcal{E} becomes a \mathbb{F}_2 -linear space where the addition of vectors $E = (\dots, E_t, E_{t+1}, \dots)$ and $E' = (\dots, E'_t, E'_{t+1}, \dots)$ is inherited from an obvious multiplication rule of operators: $EE' = (\dots, E_t E'_t, E_{t+1} E'_{t+1}, \dots)$.

Definition 1.4 (Error timing convention). Consider a dynamical code with measurement schedule $\mathbf{M}_1, \mathbf{M}_2, \dots$ where \mathbf{M}_t denotes the set of measurements performed at time t . For any error $E = (\dots, E_t, \dots) \in \mathcal{E}$, the error component $E_t \in \mathcal{P}_t^n$ is inserted immediately *following* all measurements in \mathbf{M}_t have been performed.

1.3 Action of errors

We can regard any measurement as a quantum channel that projects the system and adjoins measurement outcomes. For clarity, let us momentarily assume (without loss of generality) that there is one measurement at a time. Then, the measurement channel will be

$$\mathcal{M}_{t+1} : \rho_t \mapsto \rho_{t+1} = \Pi \rho_t \Pi \otimes |0\rangle\langle 0| + (\mathbf{1} - \Pi) \rho_t (\mathbf{1} - \Pi) \otimes |1\rangle\langle 1| \quad (2)$$

If a measurement is deterministic, one of the two terms is zero; otherwise, they both have trace 1/2 in the steady stage. The measurement result register gets enlarged by one bit every measurement. At any given execution of a dynamical code, we will have a pure state on the measurement outcome register, but here we consider the probabilistic ensemble over all measurement outcomes. A space-time error E gives additional insertions of Pauli channels in the sequence of measurement channels, and modifies the chain of outcome-recorded density matrices:

$$E : \begin{pmatrix} \rho_0 \\ \rho_1 = \mathcal{M}_1(\rho_0) \\ \vdots \\ \rho_t = \mathcal{M}_t(\rho_{t-1}) \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \rho'_0 = E_0 \rho_0 E_0^\dagger \\ \rho'_1 = E_1 \mathcal{M}_1(E_0 \rho_0 E_0^\dagger) E_1^\dagger \\ \vdots \\ \rho'_t = E_t \mathcal{M}_t(\rho'_{t-1}) E_t^\dagger \\ \vdots \end{pmatrix} \quad (3)$$

Proposition 1.5. *We have a group action of the group \mathcal{E} of all spacetime errors on the set of all chains of outcome-recorded density matrices.*

Proof. Clearly, the empty error acts by the identity. We have to show the associativity. Let $E, F \in \mathcal{E}$. Applying E and then F , we have

$$\begin{pmatrix} \vdots \\ \rho_t = \mathcal{M}_t(\rho_{t-1}) \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \rho'_t = E_t \mathcal{M}_t(\rho'_{t-1}) E_t^\dagger \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \rho''_t = F_t E_t \mathcal{M}_t(\rho''_{t-1}) E_t^\dagger F_t^\dagger \\ \vdots \end{pmatrix}, \quad (4)$$

which is the same as applying $FE \in \mathcal{E}$ at once:

$$\begin{pmatrix} \vdots \\ \rho_t = \mathcal{M}_t(\rho_{t-1}) \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} \vdots \\ \rho''_t = F_t E_t \mathcal{M}_t(\rho''_{t-1}) E_t^\dagger F_t^\dagger \\ \vdots \end{pmatrix}. \quad (5) \quad \square$$

2 Ancestries of stabilizers and detectors

In this section, we will understand how error detection works in dynamical codes. This has been discussed previously in various forms [MBG23, BLN⁺24], but our exposition clarifies how instantaneous stabilizer groups are affected by errors. We will first examine how a current stabilizer is

constructed from past measurements in the absence of any errors, and then explain why certain errors leave detectable signatures in measurement outcomes. By considering reference systems, this knowledge will prove useful to understand logical failures. We will identify for each detector an element of \mathcal{E} whose commutation relation with a given error determines whether the detector will be triggered.

To avoid unnecessary complication, we assume in this section that exactly one measurement is taken at a time even if many measurements could be implemented simultaneously. Under this assumption, every element $S \in \text{ISG}(t)$ is derived from a unique ancestry, defined recursively as follows.

Definition 2.1. Let $0 \leq \tau \leq t$ be time steps, $S_t \in \text{ISG}(t)$ a stabilizer, and M_t the measurement operator at time t with outcome $m_t \in \{\pm 1\}$ so that $m_t M_t \in \text{ISG}(t)$. We define the **ancestry** of S_t , denoted $\text{ance}_\tau^t(S_t) \in \mathcal{E}$, by

$$\text{ance}_\tau^t(S_t) = \begin{cases} \mathbf{1}_t \cdot \text{ance}_\tau^{t-1}([S_t]_{t-1}) & \text{if } [S_t]_{t-1} \in \text{ISG}(t-1) \text{ and } t > \tau, \\ (m_t M_t) \cdot \text{ance}_\tau^{t-1}([(m_t M_t)^\dagger S_t]_{t-1}) & \text{if } [S_t]_{t-1} \notin \text{ISG}(t-1) \text{ and } t > \tau, \\ S_t & \text{if } t = \tau. \end{cases} \quad (6)$$

Lemma 2.2. In the second case of (6), we always have $[(m_t M_t)^\dagger S_t]_{t-1} \in \text{ISG}(t-1)$ which commutes with $[M_t]_{t-1}$. Therefore, the ancestry is well-defined.

Proof. The condition that $[S_t]_{t-1} \notin \text{ISG}(t-1)$ implies that S_t is a new member introduced to $\text{ISG}(t)$ because of the measurement of M_t . (The measurement of M_t cannot just change the sign of S_t , and hence $-[S_t]_{t-1} \notin \text{ISG}(t-1)$.) That is, $S_t \in (m_t M_t) \cdot \mathcal{S} \subseteq (m_t M_t) \cdot [\text{ISG}(t-1)]_t$ where $\mathcal{S} \subseteq [\text{ISG}(t-1)]_t$ consists of all those that commute with M_t . \square

The ancestry of a stabilizer tells us which prior measurement outcomes are used to infer the eigenvalue of a present stabilizer. The recursive definition gives an algorithm to find the ISG elements in the past upon which the eigenvalue of a current stabilizer depends.

Corollary 2.3. Let $S_t \in \text{ISG}(t)$ and let $A_t A_{t-1} \cdots A_{\tau+1} B_\tau = \text{ance}_\tau^t(S_t)$ with $B_\tau \in \text{ISG}(\tau)$ where $A_j \neq \mathbf{1}_j$ only if $A_j = m_j M_j \in \text{ISG}(j)$ (with the outcome $m_j = \pm 1$ of M_j). Define for each $j \in [\tau+1, t]$

$$P_j = [A_j]_j [A_{j-1}]_j \cdots [A_{\tau+1}]_j [B_\tau]_j \in \text{ISG}(j). \quad (7)$$

Then, for each $j \in [\tau+1, t]$, the operator A_j commutes with P_j and $[A_j^\dagger P_j]_{j-1} \in \text{ISG}(j-1)$. We also have that $S_t = P_t$.

Proof. The second claim that $S_t = P_t$ is clear from the definition of the ancestry. If $A_j = \mathbf{1}_j$, the first claim is trivial, which appears in the first case of (6). If A_j is the measurement operator at time j , the second case of (6) must have happened. By Lemma 2.2, we know that $[A_j^\dagger P_j]_{j-1} \in \text{ISG}(j-1)$ and it must commute with $[A_j]_{j-1}$. \square

Example 2.4. Suppose that we measure XI at time 1, IX at time 2, XX at time 3, and ZZ at time 4. Then, $YY \in \overline{\text{ISG}}(4)$ has the ancestry $[ZZ]_4[II]_3[IX]_2[XI]_1$ up to signs.

From Lemma 2.3 we have an expression for any current stabilizer S_t as a product of measurement operators, which is correct including all the signs. Note that every nonidentity operator A_j for

$j \in [\tau + 1, t]$ is associated with a *nondeterministic* measurement; if it were deterministic, the definition of ancestry must have skipped it. Hence, there cannot be any information about errors in the outcomes of M_j . The only useful information is obtained when S_t gets measured in a subsequent step.

Suppose an error E_τ is inserted in one time step τ . This is after M_τ is measured, but before $M_{\tau+1}$ is measured. Before the error, the physical state is stabilized by B_τ , but after the error it is stabilized by δB_τ where $\delta = 1$ if E_τ commutes with B_τ and $\delta = -1$ otherwise. At each subsequent time step j , we have

$$P'_j = [m_j M_j]_j [m_{j-1} M_{j-1}]_j \cdots [m_{\tau+1} M_{\tau+1}]_j [\delta B_\tau]_j \in \text{ISG}'(j) \quad (8)$$

where m_j is the outcome of the measurement by M_j . This differs by δ from what is predicted from measurement outcomes:

$$P_j = [m_j M_j]_j [m_{j-1} M_{j-1}]_{j-1} \cdots [m_{\tau+1} M_{\tau+1}]_j [B_\tau]_j \in \text{ISG}(j). \quad (9)$$

Suppose that an additional one-time-slice error $F_{\tau'}$ is inserted at time $\tau' \geq \tau$. Before $F_{\tau'}$ is inserted but after $M_{\tau'}$ is measured, the physical state is stabilized by $P'_{\tau'} = \delta P_{\tau'}$. Inserting $F_{\tau'}$, the physical state is stabilized by $\eta P'_{\tau'} = \eta \delta P_{\tau'}$ where $\eta = \pm 1$ is determined by the commutation relation between $P_{\tau'}$ and $F_{\tau'}$. Arriving at time $t \geq \tau' \geq \tau$, we see that the state with $F_{\tau'} E_\tau$ inserted is stabilized by $\eta \delta S_t$. Observe that P_τ that was responsible for the commutation relation is the time- τ component of $\text{ance}_\tau^t(S_t)$. The same is true for $P_{\tau'}$ with τ' replacing τ .

More generally, we arrive at

Proposition 2.5. *Let $\pi_\tau : (\mathcal{E} = \bigoplus_t \mathcal{P}_t) \rightarrow \mathcal{P}_\tau$ be the canonical projection of the direct sum. Let $S_t \in \text{ISG}(t)$ be an instantaneous stabilizer and $E = (\dots, E_j, \dots)$ be an error supported on a time interval $[a, b]$. The physical state at time $t \geq b$ after the insertion of E is stabilized by $(-1)^\sigma S_t$ where*

$$\begin{aligned} \sigma &= \sum_{j=a}^b \lambda(E_j, \pi_j \circ \text{ance}_j^t(S_t)) \mod 2, \\ \lambda(P, Q) &= \begin{cases} 0 & \text{if } P \text{ and } Q \text{ commute,} \\ 1 & \text{otherwise} \end{cases} \end{aligned} \quad (10)$$

Hence, the measurement of an existing stabilizer is a **detector** [GNFB21, MBG23]. Conversely, since we make one measurement at a time in this section (even though many measurements can be implemented simultaneously), any detector is a measurement at time step t of an element of $\text{ISG}(t-1)$.² We define the **ancestry of a detector** at time t to be the ancestry of the stabilizer at time $t-1$ that the detector measures. **Lemma 2.5** says that we can test whether a given detector at time t is triggered by a spacetime error E from the commutation relation between E and

$$D = \left(\pi_j \circ \text{ance}_j^{t-1}(\pm[M_t]_{t-1}) \right)_{j=0}^{t-1} \in \mathcal{E}. \quad (11)$$

Definition 2.6. An **undetectable** error is one that does not trigger any detector any time.

²In general, a detector is a linear constraint on measurement outcomes at time t given all the outcomes from the earlier time steps [GNFB21, MBG23, DP23].

The syndrome bit from a detector is the *difference* between the inferred eigenvalue of a stabilizer from its ancestry and the current measured value.³ In [BLN⁺24, dLFOTT⁺25, XD25], detectors are described by certain “Pauli webs” or “Pauli flows” that are contained in the interior of a ZX diagram, which is the same as D in (11). Our exposition emphasizes operational meaning of D .

For detectors associated with persistent stabilizers that are used for the syndrome bits of the honeycomb code [HH21], the differences of the measured eigenvalues between nearer time steps are taken as a basis for the syndrome. More generally, we can understand such a basis as follows.

Proposition 2.7. *For any $t \geq \tau \geq 0$, the map $\text{ance}_\tau^t : \text{ISG}(t) \rightarrow \mathcal{E}$ with all signs ignored is \mathbb{F}_2 -linear.*

Proof. Since we ignore all signs for Pauli operators, we will suppress signs in this proof. We use induction in t . If $t = \tau$, the map is an inclusion map. Assume the claim for $\text{ance}_\tau^{t-1} : \text{ISG}(t-1) \rightarrow \mathcal{E}$. Let $A_t, B_t \in \text{ISG}(t)$ and M_t the measurement operator at time t .

- (i) Suppose that $[A_t]_{t-1}, [B_t]_{t-1} \in \text{ISG}(t-1)$. Then, it follows that $\text{ance}_\tau^t(A_t) = \text{ance}_\tau^{t-1}([A_t]_{t-1})$ and $\text{ance}_\tau^t(B_t) = \text{ance}_\tau^{t-1}([B_t]_{t-1})$. By the induction hypothesis, their product is given by $\text{ance}_\tau^{t-1}([A_t B_t]_{t-1})$, which equals $\text{ance}_\tau^t(A_t B_t)$ since $[A_t B_t]_{t-1} \in \text{ISG}(t-1)$.
- (ii) Suppose that $[A_t]_{t-1} \in \text{ISG}(t-1)$ and $[B_t]_{t-1} \notin \text{ISG}(t-1)$. Then, $[A_t B_t]_{t-1} \notin \text{ISG}(t-1)$. The three relevant images of ance_τ are $\text{ance}_\tau^t(A_t) = \text{ance}_\tau^{t-1}([A_t]_{t-1})$, $\text{ance}_\tau^t(B_t) = M_t \cdot \text{ance}_\tau^{t-1}([M_t^\dagger B_t]_{t-1})$, $\text{ance}_\tau^t(A_t B_t) = M_t \cdot \text{ance}_\tau^{t-1}([M_t^\dagger A_t B_t]_{t-1})$, which are consistent with linearity.
- (iii) Suppose that $[A_t]_{t-1} \notin \text{ISG}(t-1)$ and $[B_t]_{t-1} \notin \text{ISG}(t-1)$. Then, $\text{ISG}(t) = \langle M_t \rangle \oplus W$ for some $W \subseteq [\text{ISG}(t-1)]_t$. It follows that $[A_t B_t]_{t-1} \in \text{ISG}(t-1)$. Now the three relevant images are $\text{ance}_\tau^t(A_t) = M_t \cdot \text{ance}_\tau^{t-1}([M_t^\dagger A_t]_{t-1})$, $\text{ance}_\tau^t(B_t) = M_t \cdot \text{ance}_\tau^{t-1}([M_t^\dagger B_t]_{t-1})$, and $\text{ance}_\tau^t(A_t B_t) = \text{ance}_\tau^{t-1}([A_t B_t]_{t-1})$, which are consistent with linearity.⁴

□

Lemma 2.7 says that $\text{ance}_\tau : \bigoplus_{t \geq \tau} \text{ISG}(t) \rightarrow \mathcal{E}$ with signs ignored is \mathbb{F}_2 -linear. The kernel of this linear map reveals interdependence among detectors. For example, the ancestry of a persistent stabilizer S ($[S]_r \in \text{ISG}(r)$ for all $r \geq \tau$) is the stabilizer $[S]_\tau$ itself. Hence, two detectors measuring the same persistent stabilizer at different time steps have the empty ancestor $\mathbf{1} \in \text{ISG}(\tau)$ at τ . In this case, the two measurements must reveal the same eigenvalue in the absence of any errors. A spacetime-local generating set of the kernel, if exists, is useful since the spacetime location of a syndrome bit is more directly associated with the spacetime location of a component of an error. For known Floquet codes [HH21, HH22, DTB23] that are based on topological codes, there exists such a spacetime-local generating set for the kernel.

³In [HH21] the calculation for syndrome bits for a given error was performed by looking at measurements that are “flipped” by an error and the criterion for a flipped measurement is whether the measurement operator anticommutes with a given error ignoring all the time coordinates. While this rule gave correct results, it is not entirely sound reasoning to say that “the measurement outcome is flipped because the error anticommutes with the measurement operator” especially if the measurement is nondeterministic regardless of the error. After all, any detector probes the flip of some element of $\text{ISG}(\tau)$.

⁴This part might appear to use something special about \mathbb{F}_2 , but does not. Over p -dimensional qudits, one must take the commutation value $c \in \mathbb{Z}_p$ into account and consider $M^{\pm c} A$.

Remark 2.8. To probe logical failures, we bring an independent reference system R that is at time 0 maximally entangled with a code system C . The measurements are made on qubits of C only. Some entanglement will break down by measurements on C , but if the dynamical code encodes k logical qubits, there must be k Bell pairs between C and R . At any moment t , the entire system CR is a pure Pauli stabilizer state, and every logical operator $L = L^C$ of $\text{ISG}(t)$ is a tensor factor⁵ of a stabilizer $L^C U^R$ of CR . By considering the ancestry of $L^C U^R$, we generate an element $D(L)^{CR} \in \mathcal{E}^{CR}$ by (11), and the commutation relation between an error and $D(L)^{CR}$ will tell us if the insertion of the error flips the eigenvalue of $L^C U^R$. Since the error of interest is on C but not on R , we may truncate $D(L)^{CR}$ to have an element $D(L)$ on the spacetime of C , and use it as a probe for the failure of the logical operator L .

Remark 2.9. Note that it is not clear how to extend $D(L)$ beyond the time t that was used to construct $D(L)$ in Lemma 2.8. At minimum, one has to assume that the logical operator L is not going to be measured at a future time. In [BLN⁺24, dLFOTT⁺25, XD25], this problem does not exist because there is always a future temporal boundary, which leads to a set of open legs in a corresponding ZX-diagram. In our case, we do not have any future boundary and a logical failure will be determined by finding an instantaneous logical operator at a future time step that is equivalent to an undetectable error.

We record a simple observation that the “CSS-ness” of the measurement operators implies the “CSS-ness” of the detectability.

Proposition 2.10. *Suppose that every measurement operator of a dynamical code is either an X-type Pauli operator or Z-type. Then, a spacetime error $E = E_X E_Z$ is undetectable if and only if both E_X , the X-part of E , and E_Z , the Z-part of E , are undetectable.*

Proof. It suffices to show that the element D of (11) is either X-type or Z-type. Let M_t be a deterministic measurement and suppose it is Z-type. If D had a nontrivial X-component, there must be the greatest time step $\tau < t$ and some Z-type element $S \in \text{ISG}(\tau)$ such that M_τ is X-type, and $[S]_{\tau-1} \notin \text{ISG}(\tau-1)$, but $[M_\tau^\dagger S]_{\tau-1} \in \text{ISG}(\tau-1)$. Since all measurements are either X- or Z-type, $\text{ISG}(\tau-1)$ is a direct sum of its X-type and Z-type subgroups. The Z-part of $[M_\tau^\dagger S]_{\tau-1}$ is $[S]_{\tau-1} \in \text{ISG}(\tau-1)$, which contradicts the hypothesis that τ existed. \square

3 Initial and steady stages of dynamical codes

We will later consider errors that happens during the steady stages of dynamical codes. Here we give definitions for initial and steady stages of dynamical codes, and remark on how long initialization step can take for Floquet codes. Fu and Gottesman [FG24, §5] have shown optimal bounds on the initialization time; our exposition is perhaps easier to understand. A simple result (Lemma 3.3) in this section will be importantly used in a later argument that every Floquet code is bounded-inference.

Every Pauli measurement does not decrease the rank of ISG . Therefore, given a measurement schedule defined on n qubits, there exists a limit

$$n - k = \lim_{t \rightarrow \infty} \text{rank } \text{ISG}(t) \leq n, \quad (12)$$

⁵This observation was also stated in [dLFOTT⁺25], where they note that a logical flow from an input stabilizer code to an output stabilizer code defines a logically entangled state between the two.

from which we read off the number k of the encoded qubits in the dynamical code.

Definition 3.1. A time step t is said to be in the **steady** stage if $\text{rank } \text{ISG}(t) = \text{rank } \text{ISG}(t')$ for all $t' > t$. Otherwise, t is in the **initial** stage. The **initialization time** is the least $t \geq 0$ in the steady stage. We will often speak of *an error in the steady stage* to mean that the error is supported on a time interval $[t_0, t_1]$ where t_0 is in the steady stage.

Proposition 3.2. *If t is in the steady stage and if a measurement operator M at time $t + 1$ is not in $\overline{\text{ISG}}(t)$, then there exists $S \in \text{ISG}(t)$ that anticommutes with M .*

Proof. Otherwise, $\text{ISG}(t + 1) \supseteq \langle \pm M, \text{ISG}(t) \rangle$ whose rank is higher. \square

Proposition 3.3. *Let P be the period and T the initialization time of a Floquet code. Then, $[\overline{\text{ISG}}(t)]_{t+P} \subseteq \overline{\text{ISG}}(t + P)$ for any $t \geq 0$ where the equality holds if and only if $t \geq T$.*

Proof. If we show that $\overline{\text{ISG}}(t) \subseteq \overline{\text{ISG}}(t + P)$ for any $t \geq 0$, then the equality condition follows by the definition of the initialization time.

For the set inclusion, we may assume that there is one measurement at a time without loss of generality. If $t = 0$, there is nothing to show. Let $S \in \text{ISG}(t)$ with $t \geq 1$. [Lemma 2.3](#) gives a time sequence of operators $P_1, \dots, P_t = S$ where $A_j^\dagger P_j = [P_{j-1}]_j$ commutes with the measurement operator M_j at time j for all $j \in [1, t]$. Due to the periodicity, we have that $[A_j^\dagger P_j]_{j+P}$ commutes with M_{j+P} and hence $[P_t]_{t+P}$ is a member of $\overline{\text{ISG}}(t + P)$. Therefore, $[S]_{t+P} \in \overline{\text{ISG}}(t + P)$. \square

Corollary 3.4. *For a Floquet code on n qubits with period P , the initialization time T satisfies*

$$T \leq nP. \quad (13)$$

Proof. By [Lemma 3.3](#), if $t < T$ is in the initial stage, then $[\overline{\text{ISG}}(t)]_{t+P} \subsetneq \overline{\text{ISG}}(t + P)$, which implies $\text{rank } \text{ISG}(t) < \text{rank } \text{ISG}(t + P)$. Since $\text{rank } \text{ISG}(t) \leq n$ for all $t \geq 0$, the rank can increase at most n times. Therefore, the initialization time satisfies $T \leq nP$. \square

Proposition 3.5. *For each even number n there exists a Floquet code of period 4 on n qubits whose initialization time is $T = 2n$.*

While this particular code is not useful for quantum error correction since there will not be any logical qubit, this demonstrates that the initialization time can scale linearly with system size, establishing that $T = \Theta(nP)$ in the worst case. This bound was shown in [\[FG24, §5\]](#).

Proof. Index the qubits by $j \in \{1, 2, \dots, n\}$ and define the measurement schedule:

- At $t = 1 \bmod 4$: measure $Z_j Z_{j+1}$ for $j \in \{1, 3, 5, \dots, n-1\}$.
- At $t = 2 \bmod 4$: measure X_j for $j \in \{2, 4, 6, \dots, n\}$.
- At $t = 3 \bmod 4$: measure $Z_j Z_{j+1}$ for $j \in \{2, 4, 6, \dots, n-2\}$.
- At $t = 4 \bmod 4$: measure X_j for $j \in \{1, 3, 5, \dots, n-1\}$.

After the first measurement at $t = 1$, we have $\text{rank } \text{ISG}(1) = n/2$ from the ZZ measurements. The subsequent X measurements at $t = 2$ anticommute with these stabilizers. The Z measurements at $t = 3$ leave X_{n-1} in the ISG, maintaining $\text{rank } \text{ISG}(1) = \text{rank } \text{ISG}(2) = n/2$. However, at $t = 4$, the ISG consists of X_j on all odd j , plus X_n , so $\text{rank } \text{ISG}(4) = (n/2) + 1$. Continuing this accounting, one can verify that in each subsequent period, $\text{rank } \text{ISG}(4m) = n/2 + m$ for $m \geq 1$, reaching rank n after $n/2$ periods. \square

4 Benign errors and equivalent errors

In this section, we elaborate on a seemingly small subset of spacetime errors, that we call benign, whose action on the history of density matrices is trivial. We will see that benign errors and only benign errors may define a meaningful equivalence relation on undetectable errors in view of their logical action.

4.1 Benign errors

Definition 4.1. A **sandwiching** error $E_{t+1}E_t$ is one that is supported on two consecutive time steps t and $t + 1$ such that $[E_{t+1}]_t = E_t^\dagger$ and E_{t+1} commutes with every measurement operator at time $t + 1$. A **vacuous** error is one that equals the measurement operator at a time t . A **benign** error is any finite product of vacuous and sandwiching errors. We will refer to either a vacuous or a sandwiching error as a **benign generator**.

Note that while a vacuous error E_t could also be interpreted to be a measurement operator $[M_{t+1}]_t$, such an error is the product of benign generators:

$$[M_{t+1}]_t = ([M_{t+1}]_t[M_{t+1}^\dagger]_{t+1})M_{t+1}. \quad (14)$$

The action by a vacuous error is the identity since it is a multiplication by an operator on its eigenstate and density matrices are invariant under scalar conjugations. The action by a sandwiching error is not necessarily the identity action on the chain of all density matrices ([Lemma 4.4](#)), but the second of the sandwiching error cancels the first, and no measurement outcome is affected. By [Lemma 1.5](#), we have

Proposition 4.2. *Every benign error is undetectable.*

Remark 4.3. At the end of §5 of [\[DP23\]](#), essentially the same set of errors as our benign errors is mentioned. The authors of [\[DP23\]](#) assert that these exhaust all undetectable errors “with trivial effect.” Note that the set of all benign errors supported on a time interval I can be properly larger than the set generated by benign generators on I . (See also [Lemma 5.13](#).) For example, an instantaneous stabilizer at time t may have deep ancestry, in which case the stabilizer cannot be written as a product of benign generators supported on that one time slice t ; a benign error that is a product of benign generators supported in the future may not commute with some elements of $\text{ISG}(t)$, in which case it cannot be a product of benign generators supported in the past and present.

Remark 4.4. A benign error can make the physical state at some point in time different from that predicted by measurement outcomes. For example, consider a period-two measurement dynamics on one qubit where we measure X at time 1 and then Z at time 2. The insertion of Z_2 immediately after the Z measurement is benign (where the subscript is the time), but a sandwiching error Z_1Z_2 makes a benign error Z_1 where the underlying state is an eigenstate of X .

Note that the ancestry of a stabilizer is always a finite product of vacuous errors. When we recursively find ancestors of a stabilizer, the placement of $M_t^\dagger S_t$ to the preceding time step $([M_t^\dagger S_t]_{t-1})$ is the multiplication by a sandwiching error. The commutation requirement is fulfilled by [Lemma 2.3](#).

Hence, an ancestry $\text{ance}(S_t)$ can always be multiplied by a finite product of sandwiching errors to result in S_t . Explicitly, if $\text{ance}_\tau^t(S_t) = \prod_{i=\tau}^t M_i$, then we have a telescopic product

$$S_t = \text{ance}_\tau^t(S_t) \cdot \prod_{j=\tau+1}^t \left(\left(\prod_{i=\tau}^j [M_i]_j \right) \left(\prod_{i=\tau}^j [M_i]_{j-1}^\dagger \right) \right). \quad (15)$$

As an example, if $\text{ance}_\tau^t(S_t) = M_{t-2} M_{t-1} M_t$, then

$$S_t = M_t M_{t-1} M_{t-2} \cdot \left([M_{t-2}]_{t-1} M_{t-2}^\dagger \right) \left([M_{t-1} M_{t-2}]_t [M_{t-1} M_{t-2}]_{t-1}^\dagger \right). \quad (16)$$

This implies

Proposition 4.5. *For any $t \geq 0$, every element of $\text{ISG}(t)$ is benign. In fact, it is a product of vacuous errors and sandwiching errors, each supported on $[0, t]$. Conversely, if a product E of benign generators, each supported on $[0, t]$, is supported on one time step t , then $E \in \overline{\text{ISG}}(t)$.*

Lemma 4.6. *Any benign error $(\dots, \mathbf{1}_{\tau+1}, E_\tau, \mathbf{1}_{\tau-1}, \dots)$ supported on one time step τ in the steady stage is an element of $\overline{\text{ISG}}(\tau)$, or else, E_τ anticommutes with some element of $\text{ISG}(\tau)$.*

This is roughly the converse of [Lemma 4.5](#). This is however false without the assumption of τ being in the steady stage. Suppose that a measurement of an operator $M_{\tau+1}$ at time $\tau + 1$ makes $\text{rank } \text{ISG}(\tau + 1) > \text{rank } \text{ISG}(\tau)$. This means that $M_{\tau+1}$ commutes with all elements in $\text{ISG}(\tau)$ but lies outside $\text{ISG}(\tau)$. The vacuous error $M_{\tau+1}$ at $\tau + 1$ can be commuted back to time step τ , which is a multiplication by one vacuous error and one sandwiching error, and becomes a nontrivial logical operator of $\text{ISG}(\tau)$. Nonetheless, we will see that the residual impact of a benign error always disappears as the dynamics evolves.

Proof. We may serialize all measurements $\{M_t\}$.

Suppose that E_τ commutes with every elements of $\text{ISG}(\tau)$, as otherwise the claim is trivially true. The error E_τ is a product of vacuous and sandwiching errors from the past, present, and future. Those from the past and the present can only generate an element of $\text{ISG}(\tau)$ by [Lemma 4.5](#), so we may assume that E_τ is a product of vacuous errors $\{V_t \in \mathcal{E} : \tau < t \leq T\}$ and sandwiching errors $\{W_{t,t-1} \in \mathcal{E} : \tau < t \leq T\}$ from the future. Put $\mathcal{B} = \{V_t, W_{t,t-1} : \tau < t \leq T\}$.

We use induction in $T - \tau$. If $T = \tau$, then $\mathcal{B} = \emptyset$ and we have proven the claim. Suppose $T > \tau$. The only error in \mathcal{B} that has support on τ is the sandwiching error $W_{\tau+1,\tau}$. The product $W_{\tau+1,\tau} E_\tau$ is then supported on $[\tau + 1, T]$. This means that $W_{\tau+1,\tau} = E_{\tau+1} E_\tau^\dagger$ where $E_{\tau+1} = [E_\tau]_{\tau+1}$ and $[M_{\tau+1}]$ commutes with $[E_\tau]$. The next group $\text{ISG}(\tau + 1)$ is generated by a subset of $\text{ISG}(\tau)$ and $\{M_{\tau+1}\}$, both of which element-wise commute with $E_{\tau+1}$. Since $E_{\tau+1}$ is now a product of benign generators from $[\tau + 1, T]$, the induction hypothesis implies that $\pm E_{\tau+1} \in \text{ISG}(\tau + 1)$. If $\pm [M_{\tau+1}]_\tau \in \text{ISG}(\tau)$, then $\text{ISG}(\tau) = [\text{ISG}(\tau + 1)]_\tau \ni E_\tau$. If $\pm [M_{\tau+1}]_\tau \notin \text{ISG}(\tau)$, then [Lemma 3.2](#) supplies $S_\tau \in \text{ISG}(\tau)$ that anticommutes with $[M_{\tau+1}]_\tau$. Now, $E = [E_\tau]$ commutes with $M = [M_{\tau+1}]$, $S = [S_\tau]$, and all the logical operators (trivial or not) of $[\text{ISG}(\tau + 1)]$. It follows that E belongs to the intersection of $[\text{ISG}(\tau)]$ and $[\text{ISG}(\tau + 1)]$. \square

4.2 Equivalent errors

Definition 4.7. Two spacetime Pauli errors are **equivalent** if their difference is benign.

Tautologically, every benign error is equivalent to no error. [Lemma 4.5](#) says that every element of $\text{ISG}(t)$ for any t is equivalent to no error.

Corollary 4.8. *If τ is in the steady stage, equivalent logical operators of $\text{ISG}(\tau)$ differ by $\text{ISG}(\tau)$.*

Proof. The difference between two logical operators is assumed to be benign and commutes with every element of $\text{ISG}(t)$. [Lemma 4.6](#) proves the claim. \square

In the context of static Pauli stabilizer codes, equivalent logical operators are *defined* in this way. Here, we have defined the equivalence by our own notion of benign spacetime errors, and hence the statement is more than a definition. [Lemma 4.8](#) justifies that our definition of equivalence is a generalization of that of static Pauli stabilizer codes. Note that τ being in the steady stage is an essential assumption as remarked below [Lemma 4.6](#), which is always satisfied for any static Pauli stabilizer code, for which we measure every stabilizer every time.

Proposition 4.9. *Equivalent errors E, E' induce the same physical state eventually. Specifically, if $E'E^\dagger$ is a product of benign generators, each supported on $[0, t]$, then the physical states at time t obtained by inserting E and E' are the same for any valid history of measurement outcomes.*

Proof. [Lemma 1.5](#) says that the chain of outcome-recorded density matrices upon insertion of E' can be obtained from that of E by inserting a benign error $E'E^\dagger$. Clearly, the action of a benign error changes the chain of density matrices at only finitely many time steps, and the possible changes are only possible in the support of benign generators that constitute $E'E^\dagger$. \square

Moreover, the action of an undetectable error supported on the past can always be reproduced by an error at the present:

Proposition 4.10. *Let E be an undetectable error supported on a time interval $I = [t_0, t_1]$ in the steady stage. Then, for any time interval $I' = [t'_0, t'_1]$ with $t'_0 \geq t_0$ and $t'_1 \geq t_1$, there exists an undetectable error E' supported on I' equivalent to E . Moreover, $E'E^\dagger$ is a product of benign generators supported on $[t_0, t'_0]$ and some element of $\text{ISG}(t_0)$. In particular, $E'E^\dagger$ is a product of benign generators supported on $[0, t'_0]$.*

Proof. We may assume that all measurements are serialized. It suffices to show the claim for $t'_0 = t_0 + 1$ and $t'_1 = t_1$. Let M_{t_0+1} be a measurement operator at time $t_0 + 1$.

(i) Suppose that $[M_{t_0+1}]_{t_0} \in \overline{\text{ISG}}(t_0)$. Then, the measurement of M_{t_0+1} is a detector, and since E is undetectable, the error component E_{t_0} and the measurement operator M_{t_0+1} must commute with each other. By a sandwiching error $[E_{t_0}]_{t_0+1}E_{t_0}^\dagger$, we can push E_{t_0} to the next time step.

(ii) Suppose that $[M_{t_0+1}]_{t_0} \notin \overline{\text{ISG}}(t_0)$. By [Lemma 3.2](#), there exists $S_{t_0} \in \text{ISG}(t_0)$ that anticommutes with $[M_{t_0+1}]_{t_0}$. If $[M_{t_0+1}]_{t_0}$ commutes with E_{t_0} we use a sandwiching error $[E_{t_0}]_{t_0+1}E_{t_0}^\dagger$ to push E_{t_0} to the next time step. Otherwise, we know that $S_{t_0}E_{t_0}$ can be pushed to the next time step. By [Lemma 4.5](#), S_{t_0} is benign and is a product of benign generators, each of which is supported on $[0, t_0]$ and hence the push-up is done by a benign error from the past. \square

5 The code distance

In this section, we finally obtain a full characterization of undetectable errors in what we call bounded-inference dynamical codes, and will arrive at a well-motivated definition of the code distance of dynamical code. The class of bounded-inference dynamical codes includes all Floquet

codes, and is defined by a mild property that we believe must be satisfied by all useful dynamical codes. We will conclude with some facts on Floquet codes, with which one can write an algorithm for computing the code distance of Floquet codes. Our definition of the code distance involves the infinite time axis, and the results at the end of this section guarantee that some finite calculation suffices.

Definition 5.1. The **inference window width** at time t for a dynamical code is the least number $\mu(t) \in [0, t - 1]$ such that every element $S \in \text{ISG}(t)$ is determined by measurements supported on time interval $[t - \mu(t), t]$; that is, S is a product of benign generators supported on $[t - \mu(t), t]$. A **bounded-inference** dynamical code is one in which $\mu = \sup_t \mu(t) < \infty$.

Now we use the periodicity of a Floquet code.

Proposition 5.2. *Every Floquet code is bounded-inference with an inference window width $\mu \leq T + P - 2$ where $T \geq 1$ is the initialization time and $P \geq 1$ is the period.*

A static Pauli stabilizer code may be promoted to a Floquet code by measuring all stabilizer generators in parallel at every step, so that $P = 1$. If the $n - k$ generators are measured at $t = 1$, then $\text{rank } \text{ISG}(1) = n - k$ and $T = 1$. The bound $\mu \leq T + P - 2 = 0$ is therefore saturated: each $S \in \text{ISG}(t)$ is inferred from the single-round window $[t, t]$, reflecting that all stabilizers are remeasured every step. In contrast, the bound can be loose — for the honeycomb code [HH21], $T = 4$ and $P = 3$ while $\mu = 3$.

Proof. Let t be a time, and let $f \in [T, T + P - 1]$ be an integer such that $f = t \bmod P$. Every element of $\text{ISG}(f)$ has ancestry supported on $[1, f]$ and therefore is inferred by the measurements on the same time interval. Since $[\overline{\text{ISG}}(f)]_t = \overline{\text{ISG}}(t)$ by Lemma 3.3, every element of $\text{ISG}(t)$ is also inferred by the measurements in the time window $[t - f + 1, t]$. \square

The bounded-inference property will be important because of

Lemma 5.3. *Let E_τ be an undetectable error supported on one time step τ in a bounded-inference dynamical code with a uniform inference window width μ . The physical state of the system at time $\tau + \mu + 1$ with E_τ inserted is in the code space of $\text{ISG}(\tau + \mu + 1)$.*

Proof. All the eigenvalues of the stabilizers in $\text{ISG}(\tau + \mu + 1)$ and those in later time steps are determined by the outcomes measured after the insertion of E_τ . These eigenvalues are valid and independent of whether E_τ is inserted since E_τ is undetectable. This implies that the state is in the correct code space predicted by the measurement outcomes. \square

Theorem 5.4. *In any bounded-inference dynamical code (e.g., a Floquet code), an undetectable error in the steady stage is always equivalent to an error L_t supported on one time step t such that L_t commutes with every element of $\text{ISG}(t)$.*

In fact, as we will see in the proof, if E is an undetectable error supported on a time interval $[a, b]$, then the operator L_t can be found by “pushing” E into the time step $b + \mu + 1$ by applying Lemma 4.10.

Proof. Let μ be an inference window width. Lemma 5.2 says that a Floquet code has a bounded inference window. By Lemma 4.10 an undetectable error is equivalent to another undetectable error E_τ supported on one time step τ . Lemma 5.3 says that the physical state at time $t = \tau + \mu + 1$ with E_τ inserted is in the instantaneous code space. By Lemma 4.10 again, we multiply E_τ by

a benign error consisting of benign generators in the past of t to obtain an equivalent error L_t . The action of L_t on a pristine state gives the same physical state as the one with E_τ inserted by [Lemma 4.9](#), and hence the action of a Pauli operator L_t results in the code space. Therefore, L_t must commute with all the stabilizers at time t . \square

Corollary 5.5. *Let E be an undetectable error on a bounded-inference dynamical code (e.g., a Floquet code) in the steady stage. The error E is correctable if and only if E is benign.*

Put verbosely, if E is benign, then the physical state eventually becomes pristine with the code's measurement dynamics; if E is not benign, no decoder may correct E and the encoded qubits are unrecoverably damaged.

Proof. [Lemma 5.4](#) says that E is equivalent to a logical operator L , trivial or not, of $\text{ISG}(t)$ at some time t . If L is a stabilizer, then L is benign, and so is E . In this case there is no damage to the logical qubit. If L is not a stabilizer, then L cannot be corrected, and [Lemma 4.6](#) says that L is not benign. \square

Remark 5.6. [Lemmas 5.4](#) and [5.5](#) remain true for slightly more general dynamical codes where the inference window width $\mu(t)$ satisfies

$$\lim_{t \rightarrow \infty} t - \mu(t) = \infty. \quad (17)$$

We finally arrive at a desired characterization of the code distance of Floquet codes. Let a **weight** be a function $\mathcal{E} \rightarrow \mathbb{Z}_{\geq 0}$. One must use a well motivated function for a weight, reflecting the underlying noise of qubits. If every location in spacetime suffers from independent stochastic noise, it makes sense to define the weight to be the number of nonidentity factors of a spacetime error, which is usually assumed and we, too, use below.

Definition 5.7. The **code distance** of a bounded-inference dynamical code is the minimal weight of a nonbenign undetectable error $E \in \mathcal{E}$ inserted into the steady stage.

This is to be compared with a conventional notion: the code distance of a Pauli stabilizer code is the minimal weight of a nonstabilizer undetectable error.

Proposition 5.8. *Promote a static Pauli stabilizer code to a Floquet code by measuring all stabilizer generators every time step. Then, the code distance of this Floquet code is equal to the conventional code distance of the Pauli stabilizer code.*

Proof. Since the code distance of the Floquet code cannot exceed the code distance d of the instantaneous Pauli stabilizer code, it suffices to consider undetectable spacetime errors of weight at most d . Since we infer all stabilizers every time step, every time slice of an undetectable spacetime error must be undetectable on its own; otherwise, the earliest nonlogical slice must flip some of the succeeding stabilizer measurements. Since the total weight is at most d , the weight in each slice is at most d and hence each slice is a stabilizer unless the weight is d . Every stabilizer is benign by [Lemma 4.5](#). \square

Corollary 5.9. *It is no easier to compute the code distance of a Floquet code than to compute the conventional code distance of a Pauli stabilizer code.*

Theorem 5.10. *If d is the code distance of a Floquet code, any spacetime error of weight less than $d/2$ occurred during the steady stage can be corrected.*

Proof. Such an error E of weight less than $d/2$ is either detectable or benign. If it is benign, no correction is needed; once we go sufficiently far into the future past the error, the physical state is the same as if no error has occurred. If it is detectable, then we have to find a finite time window in which the error can be supported. Given such a time window, we can solve an inhomogeneous \mathbb{F}_2 -linear equation to find some error C that reproduces the same syndrome and has weight less than $d/2$. The combination EC^\dagger has weight less than d and is undetectable, and hence is benign by Lemma 5.5.

It remains to determine a sufficiently large but finite time window that supports E (or its relevant factors). To this end, we redefine detectors as follows. Bring μ from Lemma 5.2. Our detector at t is redefined to be the difference between the outcome of the detector at t and the prediction made by the outcomes in the time window $[t - \mu, t]$.⁶

Under this redefinition of detectors, there must exist a nonidentity component of E within the μ -neighborhood of an unhappy detector along the time axis; otherwise, the detector cannot see the error and hence cannot be unhappy. If E stretches farther along the time axis than $\mu d/2$ from unhappy detectors of the greatest or least time coordinate, then there must be a time interval of length $> \mu$ that lacks any error factor. In that case, the factors of E that sits far away from the unhappy detectors must be undetectable on their own, and hence benign. In conclusion, it suffices to look for C in the time window $[t_0 - \mu d/2, t_1 + \mu d/2]$ where t_0 is the time coordinate of the earliest unhappy detector and t_1 is that of the latest. \square

Remark 5.11. We have not discussed any measurement errors so far, but they can be accounted for by a conjugating Pauli error that anticommutes with the measurement operator if all measurement operators at a time step have nonoverlapping support. When the weight means the number of nonidentity Pauli factors in an error, this means that a measurement error corresponds to a Pauli error of weight 2. If a measurement dynamics contains overlapping measurement operators at a time step, the representation of measurement errors by conjugating Pauli may not be appropriate since the conjugating Pauli error may correspond to two or more measurement outcome flips. In that case, one can introduce fictitious time steps to make the measurement operators nonoverlapping.

Remark 5.12. Computing the code distance of a Pauli stabilizer code is in general inefficient [IP15], and for Floquet code it is only more complex.⁷ A new problem is that the time axis is infinite. However, thanks to periodicity, some finite computation is enough. Suppose that an undetectable error E is a product AB where A is supported on a time interval $[a_0, a_1]$ and B on $[b_0, b_1]$. If $a_1 < b_0 - \mu$ where μ is a uniform bound on inference time window (Lemma 5.2), then each of A and B must be undetectable. Thus, in the computation of the code distance of a Floquet code, it suffices to consider errors E that do not allow such decomposition, implying that the time support of E must be “connected.” Let d_0 be the minimum of the code distances of all ISG’s in the steady stage. We see that it suffices to consider all undetected errors supported on $[T, T+d_0\mu]$ to determine the code distance of a Floquet code. In practice, one must further restrict the time window to ease the search by, for example, (i) considering subsystem codes defined by two neighboring ISG’s to

⁶We have discussed a similar choice of a basis for syndrome above in the context of ancestors of stabilizers.

⁷This is a necessary consequence since a Floquet code can express all syndrome measurement gadgets along with data qubits and furthermore erases the distinction between data and ancilla qubits.

reduce d_0 , (ii) optimizing the inference window of detectors, and (iii) lower bounding the weight on each time slice for an error to be undetectable from immediately following detectors. To test if an undetectable error is benign, one can use either [Lemma 2.8](#) or [Lemma 5.13](#).

Proposition 5.13. *For any bounded-inference dynamical code with inference window width μ , all benign errors supported on a time window $[a, b]$ in the steady stage are generated by benign generators supported on $[a - \mu, b + \mu]$.*

See [Section 6.1](#) for an example.

Proof. Let E be a benign error supported on $[a, b]$. We know that E is undetectable by [Lemma 4.2](#), and hence is equivalent to an error F_b supported on one time step b such that $F_b E^\dagger$ is a product of an element of $\overline{\text{ISG}}(a)$ and some benign generators supported on $[a, b]$ by [Lemma 4.10](#). Since the dynamical code is assumed to be bounded-inference, $F_b E^\dagger$ is a product of benign generators supported on $[a - \mu, b]$. We may push F_b further onto the time step $b + \mu$ to obtain $G_{b+\mu}$ that commutes with every element of $\text{ISG}(b+\mu)$ by [Lemma 5.4](#). $G_{b+\mu} F_b^\dagger$ is a product of benign generators on $[b - \mu, b + \mu]$. Since $G_{b+\mu}$ is benign, $G_{b+\mu}$ is an element of $\overline{\text{ISG}}(b+\mu)$ by [Lemma 4.6](#) and hence is a product of benign generators on $[b, b + \mu]$. We have found a decomposition $E = (E F_b^\dagger)(F_b G_{b+\mu}^\dagger) G_{b+\mu}$ of E in terms of benign generators on $[a - \mu, b + \mu]$. \square

6 Examples

Here, we examine three Floquet codes. The first is a toy code from [\[HH21\]](#) that is called a ladder code. We include this mainly for pedagogical purposes.

The second example is a planar version of the honeycomb code from [\[Vui21\]](#). This example was noted to have a constant-weight spacetime error that causes logical failure, but an explicit calculation seemed complicated. In view of previous results [\[HH21, dFOTT⁺25\]](#), it would seem necessary that one has to first calculate all detectors, find a suitable undetectable spacetime error, and then show that the error induces nontrivial logical action. Using our results, the calculation for this example becomes simple: we just have to find an equivalent spacetime error that is a logical (trivial or not) operator of some instantaneous stabilizer group by “pushing” the error towards the future after multiplying by suitable benign errors. By [Lemmas 4.8](#) and [5.4](#), this pushing procedure will reveal a unique logical operator (trivial or not) if and only if the original error was undetectable. This way, we can bypass the calculation of any detectors and the logical action of the error becomes evident.

The third example is the Floquet Bacon–Shor code of [\[AR24\]](#). An upper bound on the spacetime code distance was shown, and numerical evidence was given that the upper bound was sharp. By examining the requirement for an undetectable error to be equivalent to another error supported arbitrarily far in the future, we will give a simple proof that the reported upper bound is indeed the spacetime code distance of the Floquet Bacon–Shor code.

6.1 Ladder code

There are $n = 4m$ physical qubits placed on vertices with the periodic boundary conditions as shown in [Fig. 1](#); the ends of the legs are connected. The faces are labeled in an alternating 0 and 1 pattern. The period $P = 4$ measurement schedule is as follows.

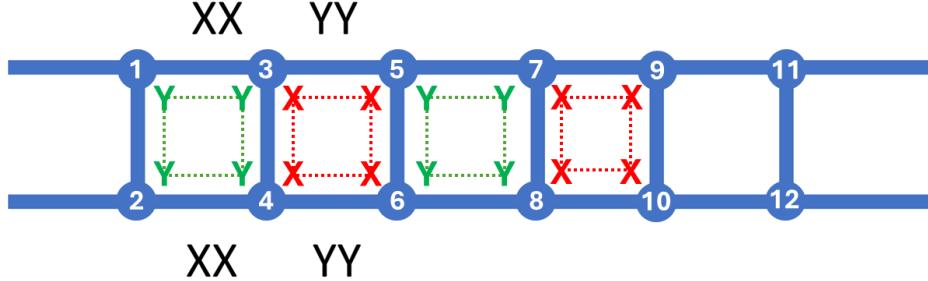


Figure 1: Ladder code for $m = 3$. Qubits are associated to vertices, enumerated as shown. Measurement operators on vertical legs are ZZ on the two qubits. Measurement operators on horizontal legs are alternately XX or YY ; a couple are shown. Persistent stabilizers of the parent subsystem code are weight-4 plaquettes of type Y (green) or X (red) which are not measured directly, but rather inferred through the measurement schedule.

- ($t = 1 \bmod 4$) Measure two-qubit ZZ on the rungs.
- ($t = 2 \bmod 4$) Measure two-qubit XX on horizontal leg segments straddling 0-labeled faces.
- ($t = 3 \bmod 4$) Measure two-qubit ZZ on the rungs.
- ($t = 4 \bmod 4$) Measure two-qubit YY on horizontal leg segments straddling 1-labeled faces.

The code has persistent stabilizers shown in Fig. 1; we let S_X and S_Y denote the persistent X - and Y -type stabilizers, respectively. With the exception of $i = 1$, S_X is inferred following the measurement of \mathbf{M}_i for $i = 1, 4 \bmod 4$ and S_Y is inferred following the measurement of \mathbf{M}_i for $i = 2, 3 \bmod 4$. The inference window is sliding with constant width $\mu = 2$, meaning persistent stabilizers are inferred every round (except $i = 1$) and that the inference is based on measurements taken at time steps i and $i - 1$ only. When $n = 4m$, we have $|S_Y| = |S_X| = m$.

[Lemma 5.13](#) says that the set of benign errors supported on a time interval $[a, b]$ is generated by benign generators supported on a larger time window $[a - \mu, b + \mu]$. Let us see why this larger time window is necessary. Let $a = 3$ and $b = 4$. Consider a spacetime error

$$E = E_a E_b = [X^1 X^2 X^3 X^4]_a [X^1 X^2 X^3 X^4]_b. \quad (18)$$

Note that E is *not* a sandwiching error since $X^1 X^2 X^3 X^4$ does not commute with $Y^3 Y^5 \in \mathbf{M}_4$. Still, E is benign; observe

- (i) $E_a \in \text{ISG}(a)$; indeed, it is the product of a persistent stabilizer $Y^1 Y^2 Y^3 Y^4$ and the measurement operators $Z^1 Z^2$ and $Z^3 Z^4$ measured at time $t = a$; and
- (ii) $E_b \notin \text{ISG}(b)$, but it commutes past \mathbf{M}_{b+1} . Since $[E_b]_{b+1} \in \overline{\text{ISG}}(b+1)$, it is then reabsorbed and has no lasting effect on the history of physical states.

By [Lemma 4.5](#), the error E decomposes into a product of benign generators. Ignoring signs, we have

$$\begin{aligned} E_{a=3} &= [Z^1 Z^2]_a [Z^3 Z^4]_a [Y^1 Y^2 Y^3 Y^4]_a, \\ \text{ance}_{a=3}^{b+1=5}([E_b]_{b+1}) &= [Z^1 Z^2]_{b+1} [Z^3 Z^4]_{b+1} [Y^1 Y^2 Y^3 Y^4]_a. \end{aligned} \quad (19)$$

Hence, although E is a benign error supported on $[a, b]$, it is a product of vacuous errors at $t = b + 1$ and $t = a$ with sandwiching errors in $[a, b + 1]$.

We next demonstrate the “pushing” algorithm that appears in the proof of [Lemma 4.10](#). Let $\tau = 4j$ for some $j \geq 1$ corresponds to a time step for which the YY measurements \mathbf{M}_4 are implemented. Consider a spacetime error of weight 7 given by

$$E_{[\tau, \tau+3]} = [Z^6 X^7 Y^8]_{\tau+2} [Z^4 Z^8]_{\tau+1} [X^3 X^6]_\tau \quad (20)$$

It may not be obvious how to go about determining whether the error is undetectable, and if so, whether it induces a nontrivial logical action. Let us try to push $E_{[\tau, \tau+3]}$ into the future.

- Between τ and $\tau + 1$: $X^3 X^6$ anticommutes with $Z^3 Z^4, Z^5 Z^6 \in \mathbf{M}_1$; accordingly, we insert the vacuous error $V_\tau = [Y^3 Y^5]_\tau$ so that

$$V_\tau \cdot [X^3 X^6]_\tau = [Y^3 Y^5]_\tau [X^3 X^6]_\tau = [Z^3 Y^5 X^6]_\tau$$

now commutes with all measurement operators in M_1 . Since $V_t \cdot [X^3 X^6]_\tau$ commutes with M_1 , we may insert the sandwiching error $W_{\tau+1} = [Z^3 Y^5 X^6]_\tau [Z^3 Y^5 X^6]_{\tau+1}$ to cancel the time- τ component.

- Between $\tau + 1$ and $\tau + 2$: the original component $[Z^4 Z^8]_{\tau+1}$ is modified to $[Z^3 Z^4 Y^5 X^6 Z^8]_{\tau+1}$. We insert a vacuous error $V_{\tau+1} = [Z^3 Z^4]_{\tau+1} [Z^7 Z^8]_{\tau+1}$ to ensure commutativity with \mathbf{M}_2 :

$$V_{\tau+1} \cdot W_{\tau+1} \cdot V_t \cdot [Z^4 Z^8]_{\tau+1} [X^3 X^6]_\tau = [Y^5 X^6 Z^7]_{\tau+1}.$$

We insert a sandwiching error $W_{\tau+2} = [Y^5 X^6 Z^7]_{\tau+1} [Y^5 X^6 Z^7]_{\tau+2}$.

- At $\tau + 2$: finally, we obtain

$$E'_{[\tau+2]} = W_{\tau+2} \cdot V_{\tau+1} \cdot W_{\tau+1} \cdot V_t \cdot E_{[\tau, \tau+2]} = [Y^5 Y^6 Y^7 Y^8]_{\tau+2} \in [S_Y]_{\tau+2}.$$

Thus, we have shown that the spacetime error $E_{[\tau, \tau+2]}$ is equivalent to a persistent stabilizer at time $\tau + 2$. Therefore, $E_{[\tau, \tau+2]}$ is benign and hence is an undetectable (correctable) error.

6.2 Planar honeycomb code by Vuillot

Vuillot’s code [\[Vui21\]](#) is shown in [Fig. 2](#), with physical qubits associated with the vertices. It is a planar variant of the honeycomb code [\[HH21\]](#). The measurement schedule has period $P = 3$, cycling through measurements of XX, ZZ, YY , in order. At each time step, the bulk protocol is identical to [\[HH21\]](#) while the boundary introduces a pair of defects, denoted by circles on opposite ends of the hexagonal patch. These defects, called *corners*, are single-qubit measurements in the same Pauli basis associated with the given time step. As in [\[HH21\]](#), the instantaneous codes remain Clifford-equivalent to a surface code, with distance $\Theta(\sqrt{n})$. [Figure 2](#) shows the Clifford-equivalent surface code at each time step; in the figure, the distance of each instantaneous code is 5. Hence, the spacetime distance δ is bounded as $2 \leq \delta \leq \Theta(\sqrt{n})$.

We are going to prove that a spacetime error

$$E = [X^1]_1 [Y^1]_3$$

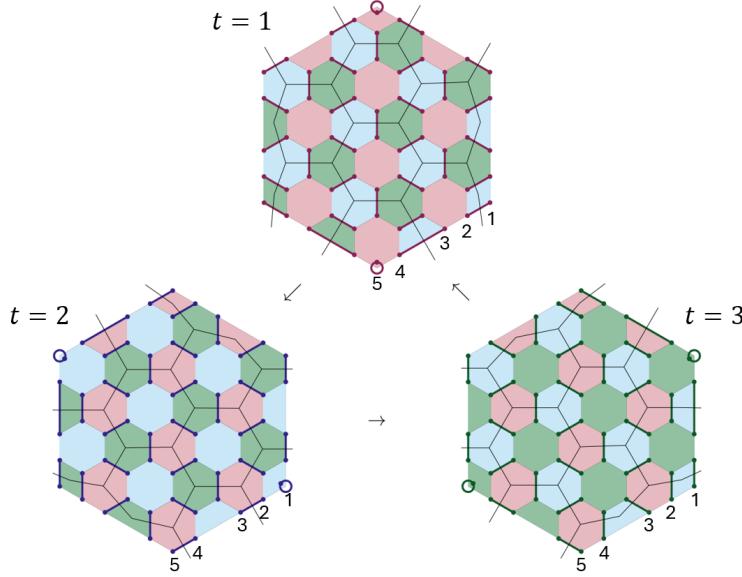


Figure 2: Measurement schedule of planar-boundary honeycomb variant. Measurement operators at each time step are two-body measurements, with a pair of single-qubit measurements at opposite corners. Red, blue, and green measurement operators are supported in X , Z , and Y , respectively. Thin black lines represent the equivalent hexagonal surface code describing the state at each time step. Illustration provided by C. Vuillot and used with permission.

is undetectable and induces logical failure, so $\delta = 2$, underscoring that large instantaneous code distances are insufficient for a large spacetime code distance. Here, the qubits on the hexagonal patch are labeled as shown in Fig. 2. The proof of the claim follows a similar calculation as in the ladder code. We will find equivalent errors following the procedure in the proof of Lemma 4.10, until we see that an equivalent error is a logical operator L of some $\text{ISG}(t)$. Then, since a logical operator of an ISG is always undetectable, and by Lemma 4.2 benign errors are undetectable, it follows that E itself is undetectable. Lemma 4.9 implies that the action on the physical state by E is the same as that by L , and hence E causes a logical failure.

Below, we will write shorthand $X^{1,2,3,5}$ and $Y^{2,3}$, etc., in place of $X^1X^2X^3X^5$ and Y^2Y^3 .

- **t = 1:** the error $[X^1]_1$ is introduced to the system. X^1 does not commute with the Z -measurement at time $t = 2$; following Lemma 4.10, we multiply $[X^1]_1$ by a vacuous error $[X^{1,2,3,4,5}]_1 \in \text{ISG}(1)$, which is a product of measurement operators $X^{1,2}, X^{3,4}, X^5$ at $t = 1$. The resulting error is given by

$$[X^1]_1 [X^{1,2,3,4,5}]_1 = [X^{2,3,4,5}]_1$$

which commutes with the Z -type measurement operators at time $t = 2$.

- **t = 2:** the error $[X^{2,3,4,5}]_1$ commutes with the Z -measurements and commutes forward to become $[X^{2,3,4,5}]_2$ via a sandwiching error W ; explicitly,

$$W_2 = [X^{2,3,4,5}]_1 [X^{2,3,4,5}]_2.$$

The error $[X^{2,3,4,5}]_2$ does not commute with the Y -measurements at $t = 3$, so we again apply [Lemma 4.10](#) and multiply by a vacuous error $[Z^{2,3,4,5}]_2$, the product of measurement operators $Z^{2,3}, Z^{4,5}$ at $t = 2$. The resulting error is given by

$$[X^{2,3,4,5}]_2 [Z^{2,3,4,5}]_2 = [Y^{2,3,4,5}]_2.$$

- **$t = 2$:** the error $[Y^{2,3,4,5}]_2$ commutes with the Y -measurements at time $t = 3$, so we may introduce a sandwiching error $W_3 = [Y^{2,3,4,5}]_2 [Y^{2,3,4,5}]_3$ to commute the error forward to become $[Y^{2,3,4,5}]_3$. At this point, the error $[Y^1]_3$ is introduced, resulting in $[Y^{1,2,3,4,5}]_3$.

Thus, $E = [X^1]_1 [Y^1]_3$ is equivalent to $E' = [Y^{1,2,3,4,5}]_3$, which is a nontrivial logical operator of $\text{ISG}(3)$. This completes the proof of the claim that $\delta = 2$.

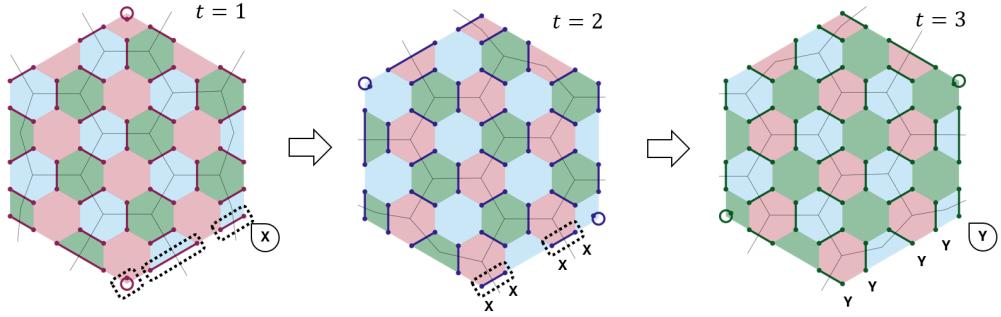


Figure 3: Compression of weight-2 spacetime error supported on time window $\{1, 2, 3\}$ to a single-time error at time $t = 3$, at which point the error is a logical of $\text{ISG}(3)$. Past supports of the spacetime error are commuted forward by multiplying the error by measurement operators in the dashed boxes, following the procedure of [Lemma 4.10](#). Teardrop arrows indicate the introduction of spacetime error. Illustration provided by C. Vuillot and used with permission.

6.3 Floquet Bacon–Shor code

Alam and Rieffel [\[AR24\]](#) construct a Floquet variant of the Bacon–Shor code [\[Bac06\]](#), a prototype of subsystem codes. While stabilizer codes directly measure the stabilizer elements, subsystem codes decompose the stabilizer elements over a basis of so-called *gauge operators*, which span a generally nonabelian subgroup of the Pauli group. It is shown that the Floquet variant of the Bacon–Shor code can support k dynamical logical qubits (in addition to one static logical qubit) through a 4-periodic measurement schedule with k “gauge defects”. Here we only discuss the case with $k = 1$. The code is defined on a $d \times d$ grid for odd $d = 2c + 1$; physical qubits are associated to vertices. The measurement schedule is shown in [Fig. 4](#).

[\[AR24\]](#) proved that the Floquet Bacon–Shor code has spacetime distance bounded above by $2c$, which was validated in simulation. Moreover, they conjecture the bound is tight. We prove this conjecture by applying [Lemma 5.4](#), demonstrating its use in the distance analysis of Floquet codes.

Proposition 6.1. *Let $d \geq 3$ be odd. The spacetime distance of the Floquet Bacon–Shor code encoding two logical qubits (one static and one dynamic) on a $d \times d$ square lattice is $\delta = d - 1$.*

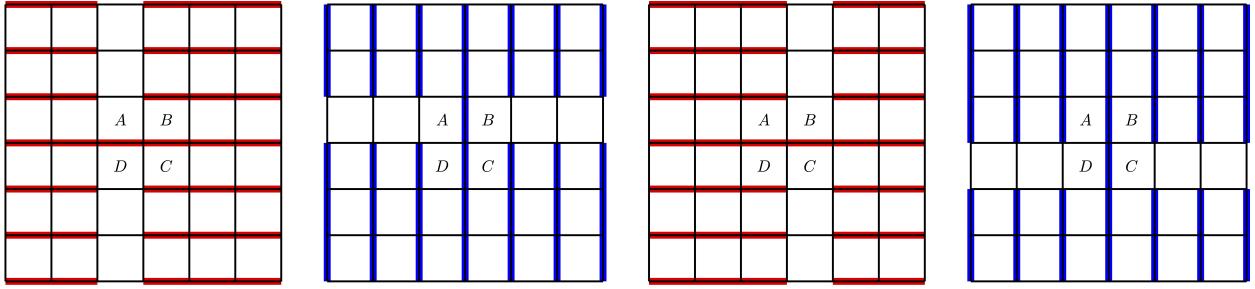


Figure 4: 4-periodic measurement schedule of the Floquet Bacon–Shor code. At times $t = 1, 3 \bmod 4$, XX is measured on the horizontal line segments. When $t = 2, 4 \bmod 4$, ZZ is measured on the vertical line segments. We denote each set of measurements by M_i , $i \in [4]$. Reproduced from Figure 3 of Alam and Rieffel [AR24], licensed under CC-BY 4.0.

We first show the following lemma. Let $d = 2c + 1$ and index the vertices of the $d \times d$ square lattice by $[-c, c]^2$. We say that an error $O = (O_{t_0}, \dots, O_{t_1})$ is of X -type if $O_i \in \{I, X\}^{\otimes d^2}$ for all $i \in [t_0, t_1]$.

Lemma 6.2. *Let $\tilde{E} = (\tilde{E}_i)_{i \in [\alpha, \beta]}$ be an X -type spacetime error supported on odd time steps such that each $\tilde{E}_i \neq I$ is neither a static logical operator nor a persistent stabilizer of the Bacon–Shor code on the $d \times d$ grid. Then, there exists an equivalent X -type spacetime error $E = (E_i)_{i \in [a, b]}$ of weight at most that of \tilde{E} such that*

- (i) *every nonidentity E_i anticommutes with some element of \mathbf{M}_{i+1} ; and*
- (ii) *E_i is supported on the set \mathcal{V} of far left, right vertices of the grid, except for the midpoint of the right edge (see Fig. 5).*

Proof. For the first claim, observe that any \tilde{E}_i commuting with M_{i+1} may be commuted forward by the sandwiching error $W_{i+1} = [\tilde{E}_i]_{i+1} \tilde{E}_i$, which can only lower the spacetime weight of \tilde{E} . After $\tau \leq 4$ such steps, $\tilde{E}_{i+\tau}$ must anticommute with $\mathbf{M}_{\tau+1}$; otherwise, \tilde{E}_i is a static logical/stabilizer. The second claim follows from observing that multiplication by X -type vacuous errors can translate the support of \tilde{E}_i to the far left, right vertices of the lattice without increasing the spacetime weight. \square

Lemma 6.3. *Let $t \geq 5$ be an odd time step. Let E_t be a one-time X -type error supported on \mathcal{V} of weight $< 2c$ such that E_t anticommutes with some element of \mathbf{M}_{t+1} . Then, SE_t anticommutes with some element of M_{t+1} for all $S \in \text{ISG}(t)$.*

Proof. Assume otherwise. Without loss of generality, we may fix $t = 1 \bmod 4$. [AR24] shows that any X -type element $S \in \text{ISG}(t)$ is a product of persistent Bacon–Shor stabilizers and some elements of \mathbf{M}_1 . Since persistent Bacon–Shor stabilizers commute with every measurement, we may assume that $S \in \langle \mathbf{M}_1 \rangle$. On the other hand, the only operator that commutes with every element of \mathbf{M}_2 is a product of “vertical strings” depicted in the right pane of Fig. 5. So, SE_t must be some product of these vertical strings. It is clear by inspection that we can further modify SE_t by \mathbf{M}_1 such that $S'E_t$ is commuting with all of \mathbf{M}_2 and $S'E_t$ is supported on the left, right edges of the grid. Then, since E_t is supported on the left, right edges, $S' = (S'E_t)(E_t)$ is supported on the left, right edges,

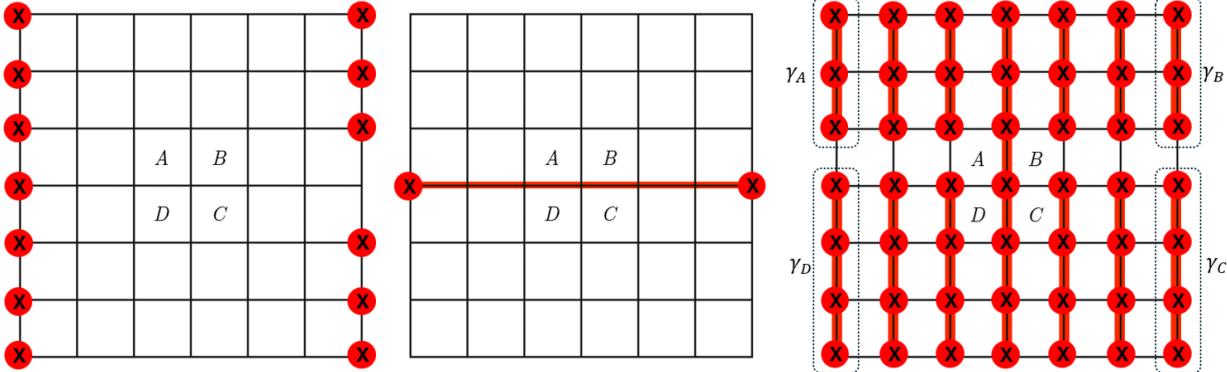


Figure 5: **Left:** up to vacuous errors from M_1 , X -type errors are generated by single-qubit X operators supported on the far left, right vertices of the lattice. Any error component lying on the x -axis is translated to the left. **Center:** the only element of $\langle M_1 \rangle$ supported on the far left, right vertices. **Right:** the basis of all X -type errors commuting with M_2 ; the far left, right elements are labeled as shown. Modified from Figure 3 of Alam and Rieffel [AR24], licensed under CC-BY 4.0.

but there is only one possible choice of S' as shown in the middle pane of Fig. 5. It follows that E_t must contain a factor $S'\gamma_C\gamma_D$ of weight $2c$. \square

The proof of Lemma 6.1 is now straightforward.

Proof of Lemma 6.1. Let $E = (E_a, \dots, E_b)$ be an undetectable spacetime error satisfying the conditions of Lemma 6.2; otherwise, since the instantaneous code distance (of the Bacon–Shor code) is $2c + 1$, there is nothing to prove. In particular, E_a anticommutes with M_{a+1} . By Lemma 4.10, there must exist an element $S \in \text{ISG}(a)$ such that SE_a commutes with M_{a+1} . By Lemma 6.3, the weight of the first nonidentity time slice of E has weight $\geq 2c$. Combining this with the upper bound $2c$ from [AR24] completes the proof. \square

7 Discussion

We have derived a dichotomy of undetectable errors in bounded-inference dynamical codes that benign errors and only benign errors are correctable. The generating set for the benign errors is as simple as one would guess: the measurement operator inserted immediately after a measurement, called a vacuous error, and a pair of the same Pauli operators inserted on consecutive time steps which would cancel each other but which conjugate measurements that are undisturbed by the insertion, called a sandwich error. Just like Pauli stabilizers leave a static code state invariant, these benign errors leave all but finitely many density matrices invariant in the history of the physical state of the code. The dichotomy is proved by showing that the action of any undetectable error on the history of the code can always be realized by an essentially unique logical operator (trivial or not) of an instantaneous stabilizer code sufficiently far in the future. This is the best one can hope for; some undetectable error may not commute with all instantaneous stabilizers. To bound how far into the future one must go, we have used the bounded-inference property, which is enjoyed by all Floquet (time periodic) codes. Based on the dichotomy, we have defined the code distance of Floquet codes.

Importantly, our results do not assume any time boundary conditions, and are derived only based on the defining data of a dynamical or Floquet code. Even if a dynamical code is defined on a finite time window, for example by destructive single-qubit measurements in the end, the code distance in our sense reproduces what is typically called the “effective distance” of a code that is measured in numerical performance tests in the presence of circuit level noise. Our code distance thus enables intrinsic comparison between Floquet codes, generalizing the role of the code distance in static Pauli stabilizer codes.

Below are further comments and open problems.

Remark 7.1. We have assumed that there is a fixed set of qubits from the beginning of time, on which a dynamical code is defined. This is reasonable if a system of qubits is fabricated once and for all, but does not obviously cover situations in which physical qubits are discarded and replenished frequently. However, our setting can handle the latter case provided that any single qubit is measured in a Pauli basis before it is discarded and any fresh qubit is initialized in Pauli basis before it interacts with any other qubit, which we believe is a reasonable assumption. The discard can be modeled by the completely depolarizing channel $\mathcal{D} : \rho \mapsto \frac{1}{2}\mathbf{1}$ on a qubit q . It is no harm for the purpose of analysis to declare that a fresh qubit that compensates a discarded qubit is the same qubit q that just went through \mathcal{D} . Overall, the qubit q is acted on by a single-qubit Pauli measurement M , and then \mathcal{D} , and then another single-qubit Pauli measurement M' . If M and M' do not commute, then by benign errors we can insert any single-qubit Pauli error at the location of \mathcal{D} . This effectively removes the role of \mathcal{D} , and it suffices to consider a circuit without \mathcal{D} but with M and M' . If M and M' commute, then we may insert a noncommuting measurement for the purpose of analysis in between M and M' , and we may remove \mathcal{D} from further consideration.

Remark 7.2. A logical operator in the steady stage is an undetectable error supported on one time step. Every time step we have an equivalent logical operator for each logical operator we start with. These evolved logical operators set the frame of reference, based on which we can read logical qubit. So, it is *not* meaningful to say that some Floquet code implements some logical Clifford transformation every cycle. For example, the honeycomb code without boundary [HH21] may appear to implement the logical Clifford corresponding to $e \leftrightarrow m$, but this transformation is never used for any computational tasks. A meaningful statement can be made when a Floquet code or its temporary modification such as insertion of a transversal logical unitary gate is *compared* to another code block and the two code blocks may interact.

Problem 7.3. Periodicity in time resembles translation invariance in space. Spatially translation-invariant codes can be compactly represented by polynomial matrices [Haa13], and a similar representation is possible for a space-translation-invariant Floquet codes. In the static translation-invariant case, the excitation map (the map from errors to syndromes) is the adjoint of the stabilizer map. It is unknown however what an analogous excitation map is for spacetime translation-invariant Floquet codes. The ancestry would be a starting point for this question, and it remains to find an explicit generating set of the kernel of \mathbf{ance}_0 on the domain of all deterministic measurements.

Problem 7.4. Many statements in this paper cease to be true in the initial stage of a dynamical code. The initial stage is perhaps the most important for the question of preparation of logical eigenstates. What logical states can be fault-tolerantly initialized without too heavy modification of the initialization step? More concretely, we can ask what logical states can be prepared fault-tolerantly by running the measurement schedule of a Floquet code starting with a product state. In

a similar vein, what logical operators can be measured fault-tolerantly by single-qubit measurements especially if a Floquet code is not CSS?

References

- [AR24] M. Sohaib Alam and Eleanor Rieffel. Dynamical logical qubits in the bacon-shor code. *Physical Review A*, 2024. [arXiv:2403.03291](https://arxiv.org/abs/2403.03291), doi:10.1103/nfxv-3dp7.
- [Bac06] Dave Bacon. Operator quantum error-correcting subsystems for self-correcting quantum memories. *Physical Review A*, 73(1):012340, 2006. [arXiv:quant-ph/0506023](https://arxiv.org/abs/quant-ph/0506023), doi:10.1103/PhysRevA.73.012340.
- [BLN⁺24] Hector Bombin, Daniel Litinski, Naomi Nickerson, Fernando Pastawski, and Sam Roberts. Unifying flavors of fault tolerance with the ZX calculus. *Quantum*, 8:1379, 2024. [arXiv:2303.08829](https://arxiv.org/abs/2303.08829), doi:10.22331/q-2024-06-18-1379.
- [DKLP02] Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill. Topological quantum memory. *J. Math. Phys.*, 43(9):4452–4505, 2002. [arXiv:quant-ph/0110143](https://arxiv.org/abs/quant-ph/0110143), doi:10.1063/1.1499754.
- [dlFOTT⁺25] Julio C. Magdalena de la Fuente, Josias Old, Alex Townsend-Teague, Manuel Rispler, Jens Eisert, and Markus Müller. Xyz ruby code: Making a case for a three-colored graphical calculus for quantum error correction in spacetime. *PRX Quantum*, 6(1):010360, March 2025. [arXiv:2407.08566](https://arxiv.org/abs/2407.08566), doi:10.1103/prxquantum.6.010360.
- [DP23] Nicolas Delfosse and Adam Paetznick. Spacetime codes of clifford circuits. 2023. [arXiv:2304.05943](https://arxiv.org/abs/2304.05943), doi:10.48550/ARXIV.2304.05943.
- [DTB23] Margarita Davydova, Nathanan Tantivasadakarn, and Shankar Balasubramanian. Floquet codes without parent subsystem codes. *PRX Quantum*, 4(2):020341, 2023. [arXiv:2210.02468](https://arxiv.org/abs/2210.02468), doi:10.1103/prxquantum.4.020341.
- [FDB⁺23] Ali Fahimniya, Hossein Dehghani, Kishor Bharti, Sheryl Mathew, Alicia J. Kollár, Alexey V. Gorshkov, and Michael J. Gullans. Fault-tolerant hyperbolic Floquet quantum error correcting codes. 2023. [arXiv:2309.10033](https://arxiv.org/abs/2309.10033), doi:10.48550/ARXIV.2309.10033.
- [FG24] Xiaozhen Fu and Daniel Gottesman. Error correction in dynamical codes. 2024. [arXiv:2403.04163](https://arxiv.org/abs/2403.04163), doi:10.48550/ARXIV.2403.04163.
- [GNFB21] Craig Gidney, Michael Newman, Austin Fowler, and Michael Broughton. A fault-tolerant honeycomb memory. *Quantum*, 5:605, 2021. [arXiv:2108.10457](https://arxiv.org/abs/2108.10457), doi:10.22331/q-2021-12-20-605.
- [GNM22] Craig Gidney, Michael Newman, and Matt McEwen. Benchmarking the planar honeycomb code. *Quantum*, 6:813, 2022. [arXiv:2202.11845](https://arxiv.org/abs/2202.11845), doi:10.22331/q-2022-09-21-813.
- [Got22] Daniel Gottesman. Opportunities and challenges in fault-tolerant quantum computation. 2022. [arXiv:2210.15844](https://arxiv.org/abs/2210.15844), doi:10.48550/ARXIV.2210.15844.
- [Haa13] Jeongwan Haah. Commuting pauli hamiltonians as maps between free modules. *Commun. Math. Phys.*, 324(2):351–399, October 2013. [arXiv:1204.1063](https://arxiv.org/abs/1204.1063), doi:10.1007/s00220-013-1810-2.
- [HH21] Matthew B. Hastings and Jeongwan Haah. Dynamically generated logical qubits. *Quantum*, 5:564, 2021. [arXiv:2107.02194](https://arxiv.org/abs/2107.02194), doi:10.22331/q-2021-10-19-564.

- [HH22] Jeongwan Haah and Matthew B. Hastings. Boundaries for the honeycomb code. *Quantum*, 6:693, 2022. [arXiv:2110.09545](#), [doi:10.22331/q-2022-04-21-693](#).
- [IP15] Pavithran Iyer and David Poulin. Hardness of decoding quantum stabilizer codes. *IEEE Trans. Info. Theory*, 61:5209–5223, 2015. [arXiv:1310.3235](#), [doi:10.1109/TIT.2015.2422294](#).
- [MBG23] Matt McEwen, Dave Bacon, and Craig Gidney. Relaxing hardware requirements for surface code circuits using time-dynamics. *Quantum*, 7:1172, 2023. [arXiv:2302.02192](#), [doi:10.22331/q-2023-11-07-1172](#).
- [PKD⁺23] Adam Paetznick, Christina Knapp, Nicolas Delfosse, Bela Bauer, Jeongwan Haah, Matthew B. Hastings, and Marcus P. da Silva. Performance of planar Floquet codes with Majorana-based qubits. *PRX Quantum*, 4(1):010310, 2023. [arXiv:2202.11829](#), [doi:10.1103/prxquantum.4.010310](#).
- [vdW20] John van de Wetering. Zx-calculus for the working quantum computer scientist. 2020. [arXiv:2012.13966](#), [doi:10.48550/ARXIV.2012.13966](#).
- [Vui21] Christophe Vuillot. Planar Floquet codes. 2021. [arXiv:2110.05348](#), [doi:10.48550/ARXIV.2110.05348](#).
- [XD25] Yichen Xu and Arpit Dua. Fault-tolerant protocols through spacetime concatenation. 2025. [arXiv:2504.08918](#), [doi:10.48550/ARXIV.2504.08918](#).
- [YCC24] Bowen Yan, Penghua Chen, and Shawn X. Cui. Floquet codes from coupled spin chains. 2024. [arXiv:2410.18265](#), [doi:10.48550/ARXIV.2410.18265](#).
- [ZAV23] Zhehao Zhang, David Aasen, and Sagar Vijay. The X-cube Floquet code. *Phys. Rev. B*, 108:205116, 2023. [arXiv:2211.05784](#), [doi:10.1103/PhysRevB.108.205116](#).