DIP Exercise 03, Group-10:
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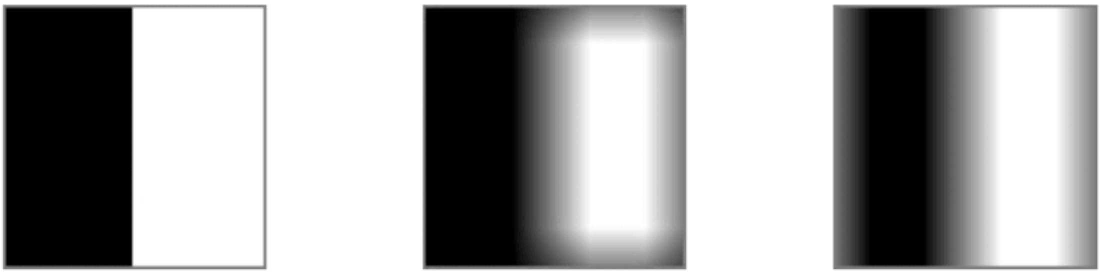
3. Exercise – Theory



a) b) c)

Figure 1

A moving average filter was applied to the image in Figure 1a).

Figure 1b) shows the result, if the convolution is carried out in spatial domain,
Figure 1c) if the convolution is carried out as multiplication in frequency domain.

i) Explain which assumptions lead to the "unexpected" border values in each image and why they are different for both methods.

ii) What steps are necessary for the convolution in spatial domain to produce the result in Fig. 1c)?

iii) What steps are necessary for the convolution by multiplication in frequency domain to produce the result in Fig. 1b)?

i) The result of the convolution in spatial domain(Fig.1b), we see the grey/black boarder in upper/bottom/right side. This happened when we apply moving average filter near the boarder, we assumed that the outside of image was black by padding.
In contrast, we assumed that the image is periodic in frequency domain(Fig.1c). Hence, the area of above the upper side padded by the bottom side and vice versa instead of padding with certain value in Fig.1b). It is applied on right/left side as well.
Therefore we see the different boarder between Fig.1b) and Fig.1c).

ii) To produce Fig.1c) by spatial convolution, we should assume that the image is infinite rather than finite, and a periodic signal. It can be achieved by assuming the image is repeated on each side of the boarder(also in diagonal direction) especially in same size. Practically, we could use circular convolution for that.

iii) To produce Fig.1b), we should assume that the outside of the image(padding) is black, instead of assuming the image is periodic and repeated.

1) Let **a(t) t=0,…,N-1** be a discrete signal and **A(μ) μ=0,…,N-1** the corresponding DFT. What are the values **A(-1),…,A(-N)**? Explain the periodicity. N $\quad A(N-1) \cdots A(0)$

2) How are the following basis-functions related?

$$F_{\mu}(x)=\exp\left(-\frac{2\pi\mu}{N}ix\right)$$

$$F_{-\mu}(x)=\exp\left(-\frac{-2\pi\mu}{N}ix\right) = \exp\left(+\frac{2\pi\mu}{N}ix\right) = \overline{F_{\mu}(x)}$$

$$F_{N-\mu}(x)=\exp\left(-\frac{-2\pi(N-\mu)}{N}ix\right) = \exp\left(+\frac{2\pi(N-\mu)}{N}ix\right) = F_{-\mu}(x)$$

3) Let **A(0)=1000** and **A(μ)=0** for **μ=1,…,N-1.** What is the corresponding signal a(t)?

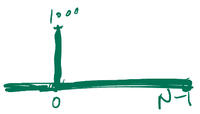
*M is Mu

1) Since DFT spectrum A(M*) is a continuous and periodic function with respect to the index M, the coefficients satisfy A(M+N) = A(M) and A(M-N)=A(N) for all M. Therefore A(-1)=A(N-1), …, A(-N)=A(N-N)=A(0)

2)
1. the basis-function with -M index is a complex conjugate of the basis-function with M index.
2. According to 1), we see F with -M index and F with N-M index are same basis-functions. Also, when sample index x is an integer, exp(-2 Pi ix)= 1. Thus F with N-M index = exp((2 Pi M)/N * ix)) = F with -M. In result, F with N-M is also the complex conjugate of the basis-function with M index.

3) a(t) = 1/N * A(0) * exp(0) = 1000/N for all sample indices t. So, corresponding signal a(t) is non-changing line.

Frequency
$A(0)=1000, \quad A(\mu)=0$ for $(\mu=1 \sim N-1)$



$$A(\mu)=\sum_{x=0}^{N-1} a(x)\exp\left(-\frac{2\pi i}{N}\mu x\right)$$ Forward DFT

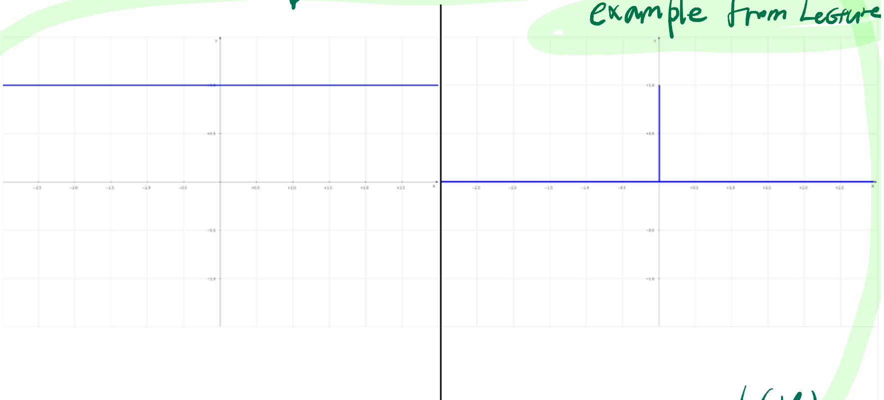
$$\boxed{a(x)}=\frac{1}{N}\sum_{\mu=0}^{N-1}\boxed{A(\mu)}\boxed{\exp\left(\frac{2\pi i}{N}\mu x\right)}$$ Inverse DFT

$$a(t)=a(x)=\frac{1}{N}\sum_{\mu=0}^{N-1}\left(A(\mu)\exp\left(\frac{2\pi i}{N}\mu x\right)\right)$$

$$=\frac{1}{N}\underset{1000}{\cancel{A(0)}}\underbrace{\exp\left(\frac{2\pi i}{N}\cdot\mu x\right)}_{\to 1}$$

$$\approx\frac{1000}{N}$$

example from Lecture



$f(t)=1 \qquad F(\mu)=\delta(\mu) \approx A(\mu)$

4) A Circular shift of the image does not change the Fourier magnitude. It only changes the Fourier phase.

proof) → So the Fourier image looks similar as before.

Let a(x,y) be the image and A(μ,ν) its 2D DFT as defined in the slides,

Forward transform:

$$A(\mu,\nu)=\sum_{x,y=\{0,0\}}^{N-1,M-1} a(x,y)\exp\left(\frac{-2\pi i}{N}\mu x\right)\exp\left(\frac{-2\pi i}{M}\nu y\right)$$
$$=\sum_{x,y=\{0,0\}}^{N-1,M-1} a(x,y)\exp\left(-2\pi i\left(\frac{\mu x}{N}+\frac{\nu y}{M}\right)\right)$$

Inverse transform:

$$a(x,y)=\frac{1}{NM}\sum_{\mu,\nu=\{0,0\}}^{N-1,M-1} A(\mu,\nu)\exp\left(\frac{2\pi i}{N}\mu x\right)\exp\left(\frac{2\pi i}{M}\nu y\right)$$
$$=\frac{1}{NM}\sum_{\mu,\nu=\{0,0\}}^{N-1,M-1} A(\mu,\nu)\exp\left(2\pi i\left(\frac{\mu x}{N}+\frac{\nu y}{M}\right)\right)$$

Now consider a circular shift of the image by (Δx,Δy):

g(*x*,*y*)=*a*((*x*−Δ*x*)mod*N*,(*y*−Δ*y*)mod*M*).

Let $x = x' + \Delta x$, $y = y' + \Delta y$

$$A_{shift} = A(\mu,\nu) = \sum_{x,y} a(x', y') \exp\left(-2\pi i\left(\frac{\mu(x'+\Delta x)}{N} + \frac{\nu(y'+\Delta y)}{M}\right)\right)$$

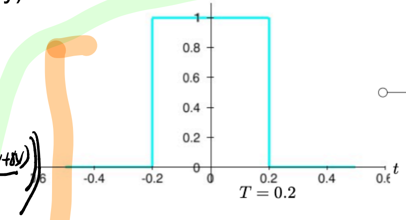
$$= \exp\left(-2\pi i\left(\frac{\mu\Delta x}{N}+\frac{\nu\Delta y}{M}\right)\right)\cdot\sum_{x,y} a(x',y')\exp\left(-2\pi i\left(\frac{\mu x'}{N}+\frac{\nu y'}{M}\right)\right)$$

$$= \underbrace{\exp\left(-2\pi i\left(\frac{\mu\Delta x}{N}+\frac{\nu\Delta y}{M}\right)\right)}\cdot A(\mu,\nu)$$

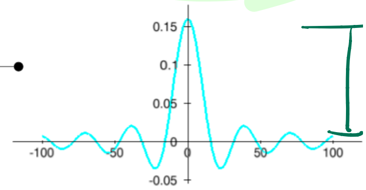
$$\hookrightarrow \exp\Big(i\cdot\underbrace{-2\pi\left(\frac{\mu\Delta x}{N}+\frac{\nu\Delta y}{M}\right)}_{\theta}\Big) = \exp(i\theta)$$

Therefore, we see only change is multiply $\exp(i\theta)$ and it means magnitude will be same and the phase will be changed.

example.



$$f(t)=\begin{cases}1 & |t|\le T\\ 0 & |t|>T\end{cases}$$

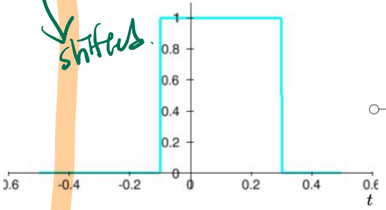


$$F(\mu)=\sqrt{\frac{2}{\pi}}\frac{\sin(\mu T)}{\mu}$$

(Sinc Function)

shifted.

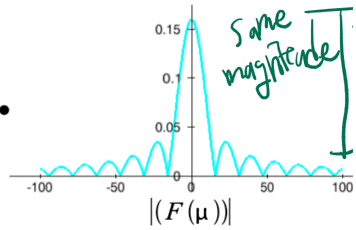
(shifted impulse)



$$f(t)=\begin{cases}1 & |t-0.1|\le T\\ 0 & |t-0.1|>T\end{cases}$$

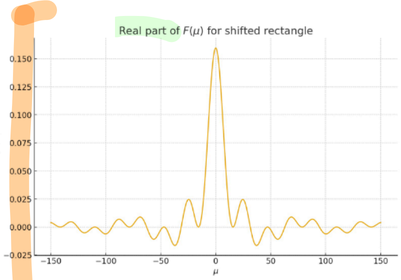
(same amplitude spectrum!

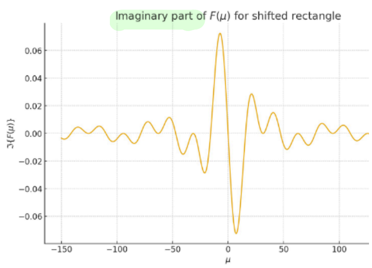
same magnitude



$|(F(\mu))|$

$$F(\mu)=\sqrt{\frac{2}{\pi}}\frac{\sin(\mu T)}{\mu}$$





$$f(t)=\begin{cases}1 & |t-0.1|\le T\\ 0 & |t-0.1|>T\end{cases}$$

$$F(\mu)=\sqrt{\frac{2}{\pi}}\frac{\sin(\mu T)}{\mu}e^{-i\mu}$$

with $T = 0.2$ and $\tau_0 = 0.1$

Filter Technique Comparison Summary - Group 10

December 2, 2025

In this exercise, we compared the processing time of three convolution methods: **Spatial**, **Separable**, and **Frequency-Domain (FFT)**, using an $N \times N$ image and a $K \times K$ filter. The experimental results confirmed the theoretical computational complexities.

Method	**Theoretical Complexity**	**Observed Behavior**
Spatial Convolution	$O(N^2K^2)$	Slowest. Time increases quadratically with both N and K
Separable Filter	$O(N^2 2K)$	Much faster. Time increases linearly with K
Frequency-Domain (FFT)	$O(N^2 \log N)$	Fastest for large images. Almost no dependence on K

Note on Spikes: The measured **spikes** are due to **system noise** (e.g., background tasks), not the algorithms.

Below are the graphs of the different convolution techniques, showing the effects of kernel size and pixel amount on calculation time. We can see that for each method, the calculation time increases with increasing image size (pixel amount).

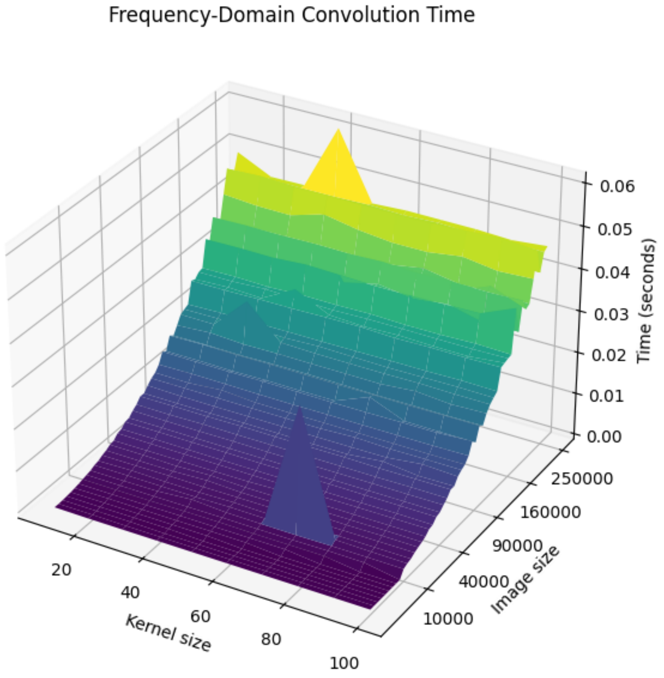


Figure 1: Kernel Size/Pixel Amount to Time of the Frequency-Domain Convolution

We see that frequency convolution does not get affected by the kernel size since it only uses the kernel once for the circular shift.

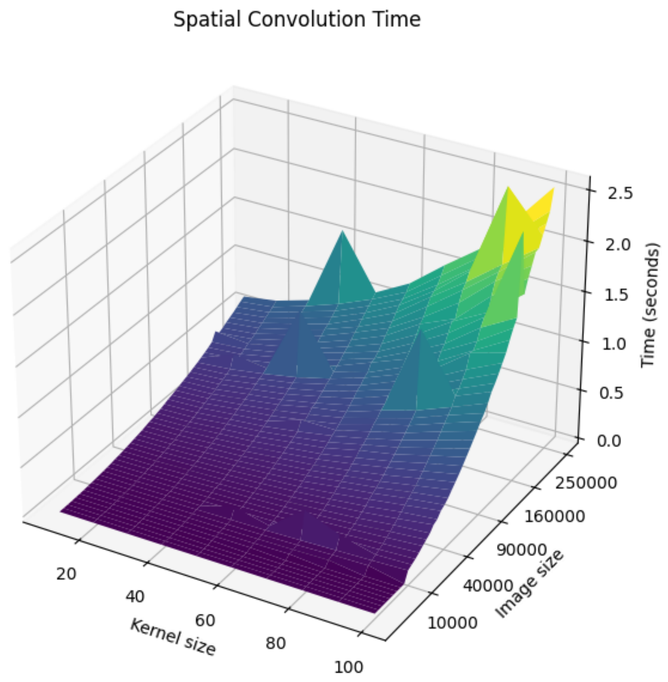


Figure 2: Kernel Size/Pixel Amount to Time of the Spatial Convolution

The spatial convolution is affected by both the kernel size and pixel amount since the kernel and the whole image are both used at every step, and the kernel requires being flipped and moved through the image.

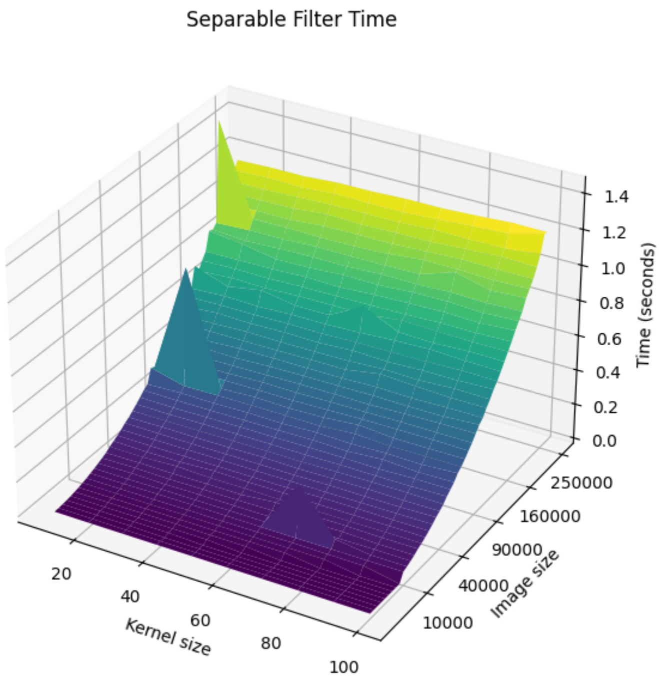


Figure 3: Kernel Size/Pixel Amount to Time of the Separable Convolution

The separable convolution is affected by both the kernel size and the image size, but the image size has a much stronger effect. This is due to the separable filter applying the spatial convolution twice, where the image size plays a stronger role on the calculation time.