컴퓨터그래픽스

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Transformations

- Affine Space & Homogeneous Coordinates
- Linear transformations
- Model transformations
- View transformations

Affine Space & Homogeneous Coordinates

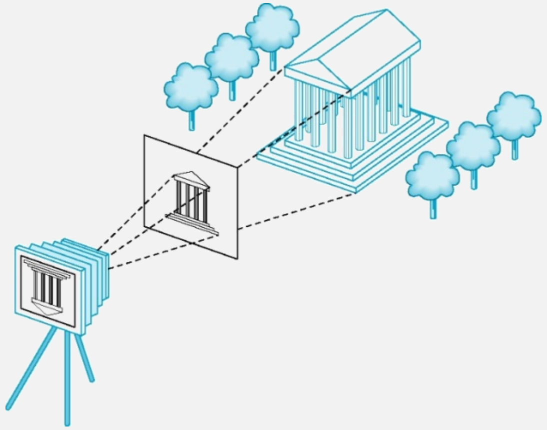
OpenGL ES uses Homogeneous coordinate System

- Graphics cards support homogeneous coordinates
  - 4x1 vectors for 3D points & 3D vectors

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

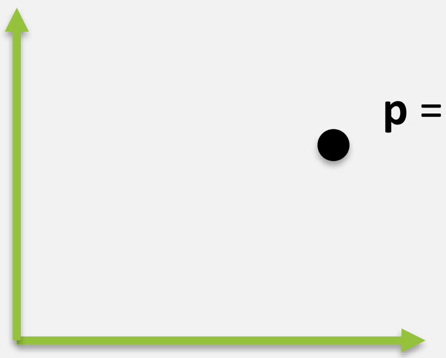
  - In general, the following holds (if a ≠ 0, w ≠ 0)
    - It is related to projective geometry

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \equiv \begin{bmatrix} ax \\ ay \\ az \\ aw \end{bmatrix} \equiv \begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$$



Affine Space

- The affine space contains three types of object
  - Scalars
  - Vectors
  - Points
- Why affine space?
  - We need to clearly distinguish the concepts of vectors and points
    - There was no strong differences in vectors and points, especially in high school



Affine Space

- Scalar ⊕ Vector ⊕ Point
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar opertions
- For any point define
  - $1 \bullet \mathbf{P} = \mathbf{P}$
  - $0 \bullet \mathbf{P} = \mathbf{0}$ (zero vector)

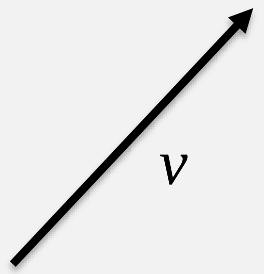
Scalars

- Scalars represent the concept of quantity
  - Ex) 7, 3.14, -1, …
  - Combined with two basic operations
    - Addition
    - Multiplication

- Scalars alone have no geometric propoerties

Vectors

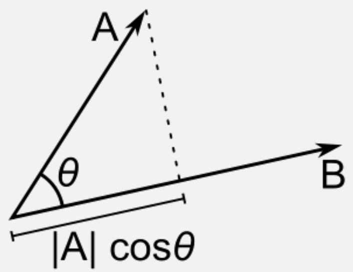
- A vector is a quantity of two attributes
  - Direction
  - Magnitute
- A vector usually represented with a directed line segment

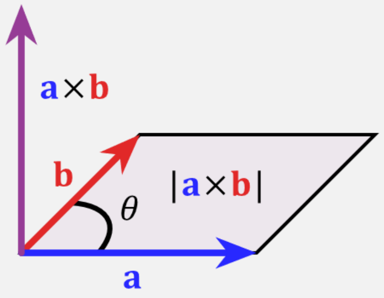


- Examples include
  - Force
  - Velocity

Vector Operations

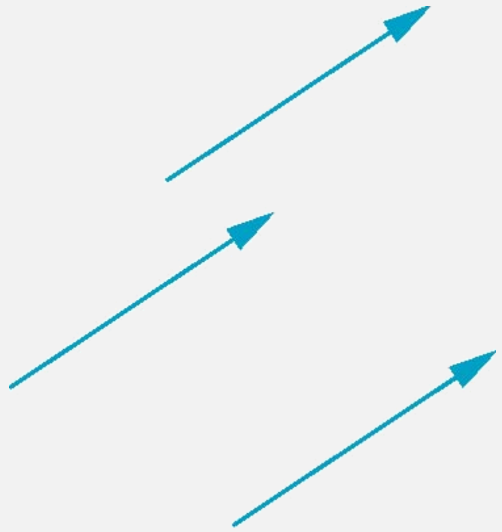
- scalar * vector
  - Ex) 2**v**, -**v**, …
  - Result: a vector
- vector + vector
  - Ex) **u** + **w**
  - Result: a vector
- Products
  - Dot product
    - Ex) **u** • **w**
    - Result: a scalar
      - Physical meaning: amount of projection
  - Cross product
    - Ex) **u** x **w**
    - Result: a vector
      - Physical meaning: amount of rotation





Vectors Lack Point

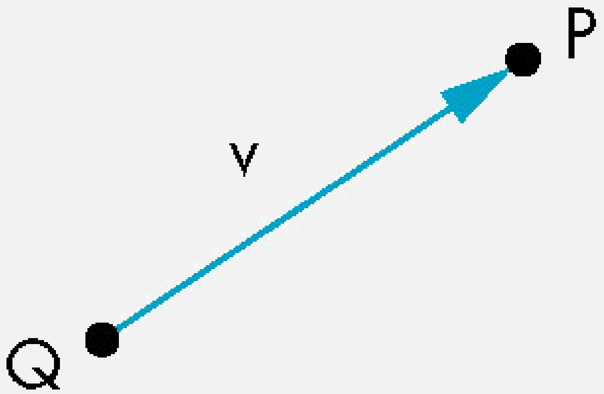
- The following vectors are identical
  - Same length and magnitute



- (Scalars + Vectors) are insufficient for representing geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
    - **P** - **Q** = **v**
    - ~~**P + Q**~~ → no physical meaning
  - Equivalent to point-vector addtion
    - **Q** + **v** = **P**

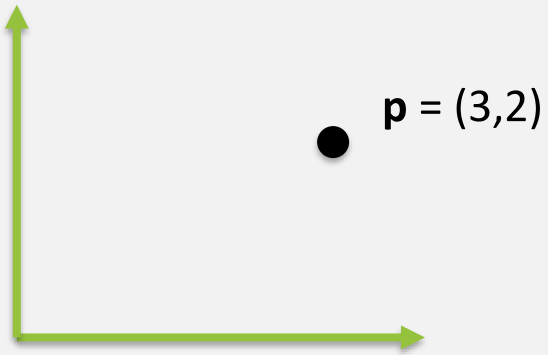


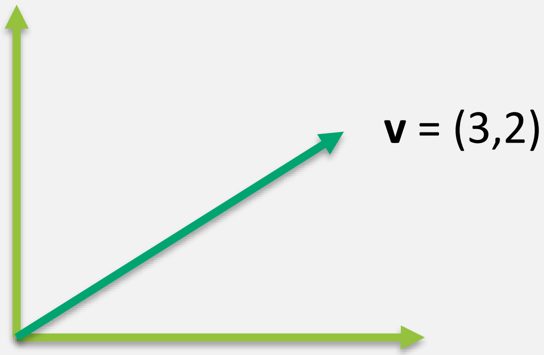
$$v = P - Q$$

$$P = v + Q$$

Homogeneous Coordinates

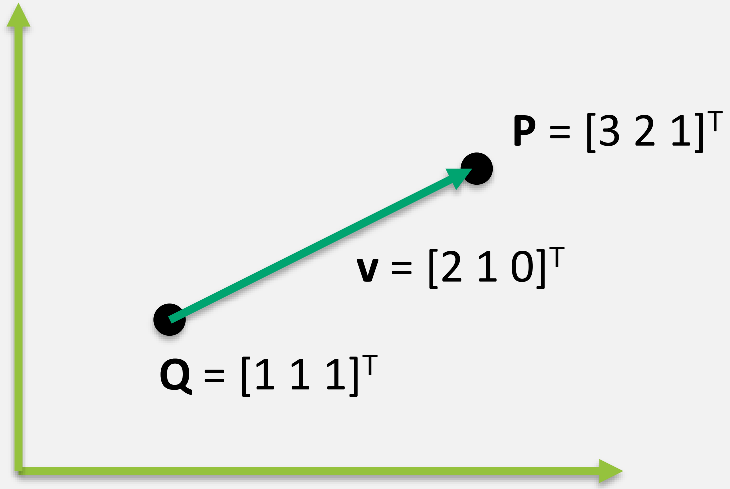
- Homogeneous coordinate systems clearly distnguish the concepts of vectors and points
  - *n*D point is represented with a (*n+1*)D vector, whose last component is 1
  - *n*D vector is represented with a (*n+1*)D vector, whose last component is 0
- Example) 2D points, 2D vectors → 3D vectors
  - $\mathbf{p} = [3\ 2\ 1]^T$
  - $\mathbf{v} = [3\ 2\ 0]^T$





Homogeneous Coordinates

- Homogeneous coordinates hold the operations of affine space
  - E.g.) **P** = $[3\ 2\ \mathbf{1}]^T$, **Q** = $[1\ 1\ \mathbf{1}]^T$, **v** = $[2\ 1\ \mathbf{0}]^T$
    - **v = P - Q** = $[3\ 2\ \mathbf{1}]^T - [1\ 1\ \mathbf{1}]^T = [2\ 1\ \mathbf{0}]^T$
    - **P = Q** + **v** = $[1\ 1\ \mathbf{1}]^T + [2\ 1\ \mathbf{0}]^T = [3\ 2\ \mathbf{1}]^T$
    - 2**v**
    - ~~3**Q**~~



Why Homogeneous Coordinate?

- Graphics cards support homogeneous coordinates
  - 4x1 vectors for 3D points & 3D vectors

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

  - 4x4 matrices for
    - GL_MODELVIEW_MATRIX, GL_PROJECTION_MATRIX, GL_TEXTURE_MATRIX
    - Note: OpenGL ES represents a matrix in the column-major way

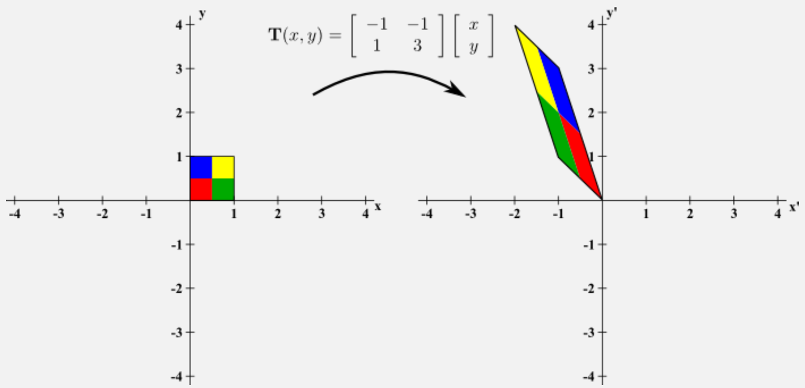
$$\boldsymbol{M} = \begin{bmatrix} m_1 & m_5 & m_9 & m_{13} \\ m_2 & m_6 & m_{10} & m_{14} \\ m_3 & m_7 & m_{11} & m_{15} \\ m_4 & m_8 & m_{12} & m_{16} \end{bmatrix}$$

Linear Transformations

Transformation (변환)

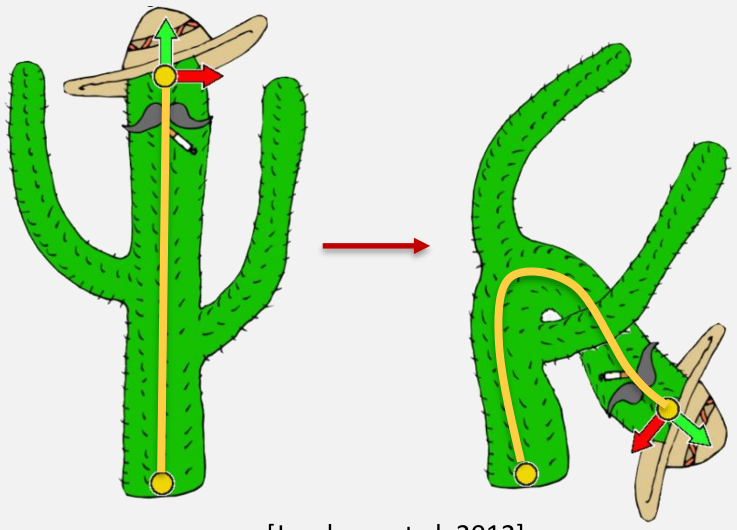
- In computer graphics, transformation refers to
  - Change of shape
    - Linear transformation: line to line
    - Non-linear transformation: line to curve

Linear transformation



http://mathinsight.org/determinant_linear_transformation

Non-linear transformation

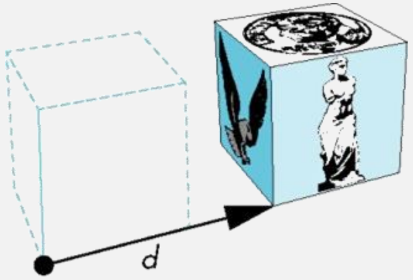


[Jacobson et al. 2012]

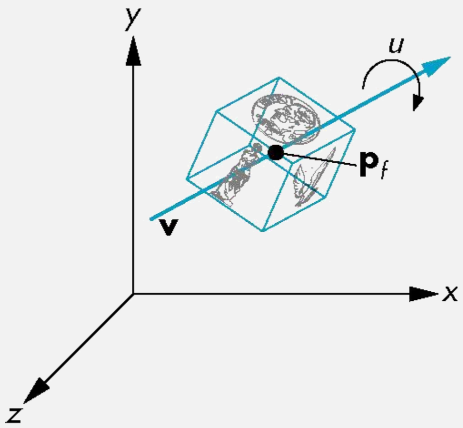
Linear Transformation (선형 변환)

- It can be represented as linear a matrix
- Standard transformation
  - Translation / Rotation / Scaling
- Composition of linear transformation is linear

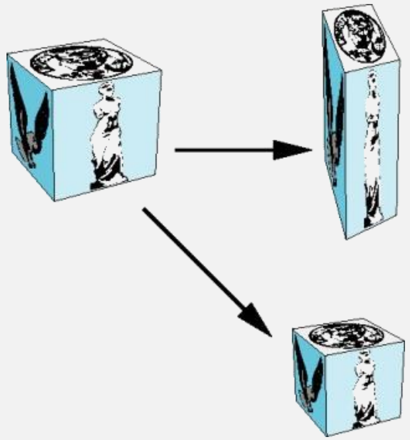
translation



rotation

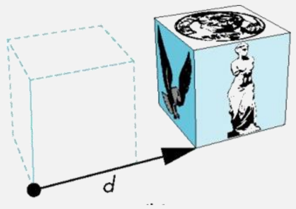


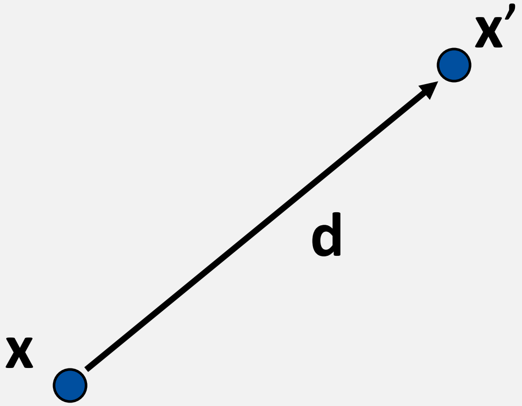
scaling



Translation (이동)

- Move (translate, displace) a point to a new location
  - Translation of an object: every point displaced by same vector



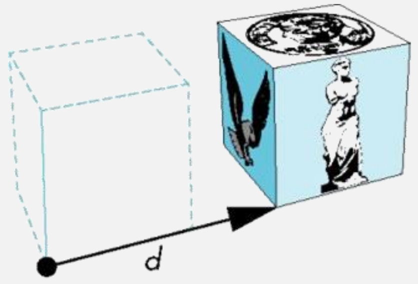


- Displacement determined by a vector **d**
  - Three degrees of freedom for **d**, in 3D case: **d** = (d_x, d_y, d_z)
  - **x'** = **x** + **d**

Translation (이동)

- Translation matrix **T**
  - Translation can also be expressed by using a 4x4 matrix **T** in homogeneous coordinates
  - **x'=Tx**
    - **T** = **T**(d_x, d_y, d_z) =

$$\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



    - glTranslate(d_x, d_y, d_z)
    - This form is better for H/W implementation

Translation (이동) 행렬의 해석: Homogeneous Coordinate 활용

$$\mathbf{T}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{T}(\mathbf{d}) = \mathbf{T}_{\mathbf{d}}$$

- 단, $\mathbf{d} = (d_x, d_y, d_z)$.

Translation about Points or Vectors

Translation about Points

- 위치 $\mathbf{p} = (p_x, p_y, p_z)$를 $\mathbf{d} = (d_x, d_y, d_z)$ 만큼 이동
  - Homogeneous coordinate 이용

$$\mathbf{T_d}\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{d}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{p}+\mathbf{d}\\1\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{p}+\mathbf{d}$

Translation about Vectors

- 벡터 $\mathbf{v} = (v_x, v_y, v_z)$를 $\mathbf{d} = (d_x, d_y, d_z)$ 만큼 이동
  - Homogeneous coordinate 이용한 경우

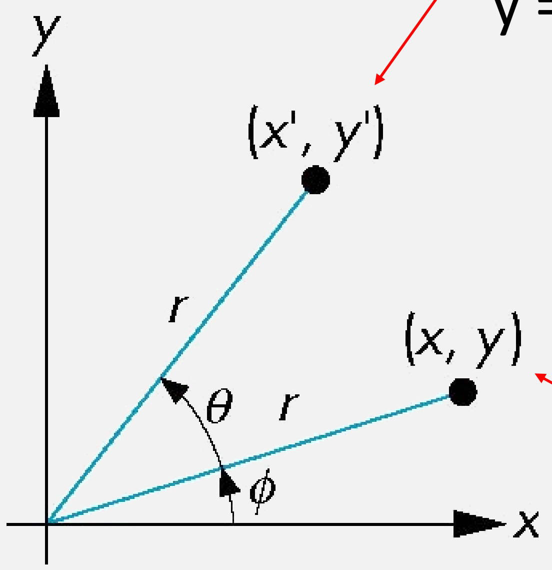
$$\mathbf{T_d}\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{3\times3} & \mathbf{d}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{v}\\0\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{v}$
      - 즉, 벡터 $\mathbf{v}$는 이동 $\mathbf{d}$에 영향이 없음

Rotation (회전)

- Consider rotation about the origin by θ degrees
  - Radius stays the same, angle increases by θ

$$x = r \cos(\phi + \theta)$$
$$y = r \sin(\phi + \theta)$$



$$x = r \cos \phi$$
$$y = r \sin \phi$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation (회전) 행렬의 해석

- 2차원 회전 행렬 해석
  - 고교과정에서 배운 내용에 대한 재해석

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

θ만큼 회전 후, x축 방향

θ만큼 회전 후, y축 방향

Rotation (회전) 행렬의 해석: Homogeneous Coordinate 활용

- 2차원 회전 행렬 해석
  - Homogeneous coordinate를 이용한 해석

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\theta)_{2\times2} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

θ만큼 회전 후, x축 방향

θ만큼 회전 후, y축 방향

원점

Rotation (회전) 행렬의 해석: Homogeneous Coordinate 활용

2D Rotation of Points

- 위치 $\mathbf{p} = (p_x, p_y)$를 θ만큼 회전
  - Homogeneous coordinate를 이용한 해석

$$\mathbf{R}(\theta)\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{R}(\theta)_{2\times2} & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{R}(\theta)_{2\times2}\mathbf{p}\\1\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{R}(\theta)_{2\times2}\mathbf{p}$

2D Rotation of Vectors

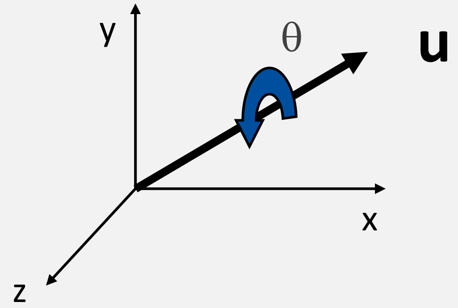
- 벡터 $\mathbf{v} = (v_x, v_y)$를 θ만큼 회전
  - Homogeneous coordinate를 이용한 해석

$$\mathbf{R}(\theta)\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{R}(\theta)_{2\times2} & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{R}(\theta)_{2\times2}\mathbf{v}\\0\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{R}(\theta)_{2\times2}\mathbf{v}$

Rotation (회전)

- General rotation about the origin
  - A rotation by θ rotation an arbitrary axis **u**
    - Quaternion can express general rotation
  - **x'=R$_u$(θ)x**
    - **where,**



$$\mathbf{R_u}(\theta) = \begin{bmatrix} \cos\theta + u_x^2(1-cos\theta) & u_xu_y(1-cos\theta) - u_zsin\theta & u_xu_z(1-cos\theta)+u_ysin\theta & 0 \\ u_yu_x(1-cos\theta)+u_zsin\theta & \cos\theta + u_y^2(1-cos\theta) & u_yu_z(1-cos\theta)-u_xsin\theta & 0 \\ u_zu_x(1-cos\theta)-u_ysin\theta & u_zu_y(1-cos\theta)+u_xsin\theta & \cos\theta+u_z^2(1-cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R_u}(\theta)_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

u축을 중심으로 θ만큼 회전 후, x축 방향

u축을 중심으로 θ만큼 회전 후, y축 방향

u축을 중심으로 θ만큼 회전 후, z축 방향

원점

- glRotate(θ, u_x, u_y, u_z), where **u**=(u_x, u_y, u_z)

Rotation (회전) 행렬의 해석: Homogeneous Coordinate 활용

3D Rotation of Points

- 위치 $\mathbf{p} = (p_x, p_y, p_z)$를 원점 기준, $\mathbf{u}$축을 중심으로 θ만큼 회전
  - Homogeneous coordinate를 이용한 해석

$$\mathbf{R_u}(\theta)\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{R_u}(\theta)_{3\times3} & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{R_u}(\theta)_{3\times3}\mathbf{p}\\1\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{R_u}(\theta)_{3\times3}\mathbf{p}$

3D Rotation of Vectors

- 벡터 $\mathbf{v} = (v_x, v_y, v_z)$를 원점 기준, $\mathbf{u}$축을 중심으로 θ만큼 회전
  - Homogeneous coordinate를 이용한 해석

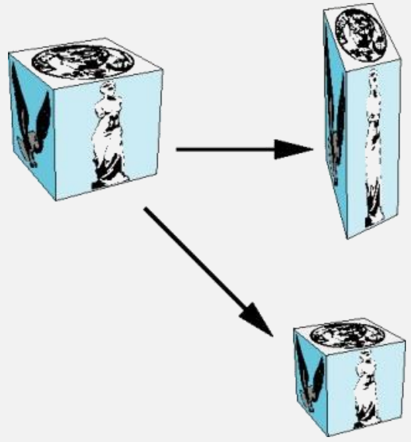
$$\mathbf{R_u}(\theta)\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{R_u}(\theta)_{3\times3} & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{R_u}(\theta)_{3\times3}\mathbf{v}\\0\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{R_u}(\theta)_{3\times3}\mathbf{v}$

Scaling (확대축소)

- Scaling matrix **S**
  - Expand or contract along each axis (fixed point of origin)
  - **x'=Sx**
    - **S** = **S**(s_x, s_y, s_z) =

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



    - glScale(s_x, s_y, s_z)

Scale (확대축소) 행렬의 해석: Homogeneous Coordinate 활용

$$\mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{3\times3}(\mathbf{s}) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{S}(\mathbf{s}) = \mathbf{S}_{\mathbf{s}}$$

- 단, $\mathbf{s} = (s_x, s_y, s_z)$.

Scale (확대축소) 행렬의 해석: Homogeneous Coordinate 활용

3D Scale of Points

- 위치 $\mathbf{p} = (p_x, p_y, p_z)$를 원점 기준으로 $\mathbf{x}$축으로 s_x배, $\mathbf{y}$축으로 s_y배, $\mathbf{z}$축으로 s_z배 확대 축소
  - Homogeneous coordinate를 이용한 해석

$$\mathbf{S}(s_x, s_y, s_z)\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{3\times3}(\mathbf{s}) & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{3\times3}(\mathbf{s})\mathbf{p}\\1\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{S}_{3\times3}(\mathbf{s})\mathbf{p}$

3D Scale of Vectors

- 벡터 $\mathbf{v} = (v_x, v_y, v_z)$를 원점 기준으로 $\mathbf{x}$축으로 s_x배, $\mathbf{y}$축으로 s_y배, $\mathbf{z}$축으로 s_z배 확대 축소
  - Homogeneous coordinate를 이용한 해석

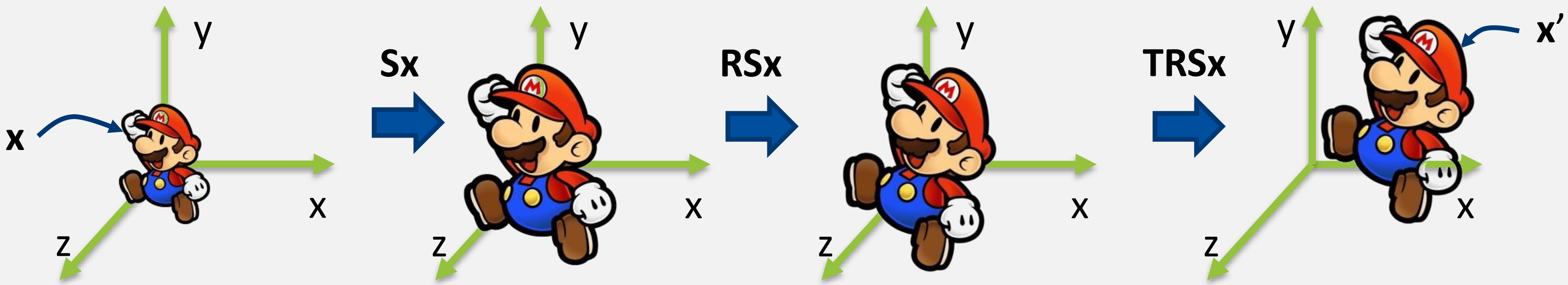
$$\mathbf{S}(s_x, s_y, s_z)\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{3\times3}(\mathbf{s}) & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}\begin{bmatrix}\mathbf{v}\\0\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{3\times3}(\mathbf{s})\mathbf{v}\\0\end{bmatrix}$$

  - Homogeneous coordinate 이용치 않은 경우
    - $\mathbf{S}_{3\times3}(\mathbf{s})\mathbf{v}$

Model Transformations

Composite Transformation (합성 변환)

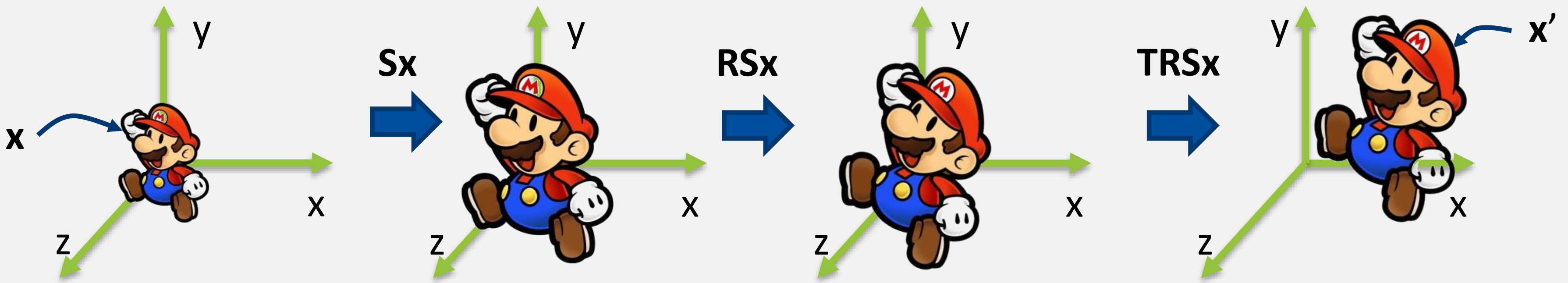
- We can composite transformation by multiplying matrices of rotation, translation, and scaling.
  - Example:
    1) Uniformly scale 2x: **S = S(2,2,2)**
    2) Rotate -30 degrees by +z axis: **R = R_z(-30)**
    3) Translate by (2, 1, 0): **T = T(2,1,0)**
    - **x**' = **T**(**R**(**Sp**)) = **TRSx**
      - **x**: original object
      - **x**': transformed object



Composite Transformation (합성 변환)

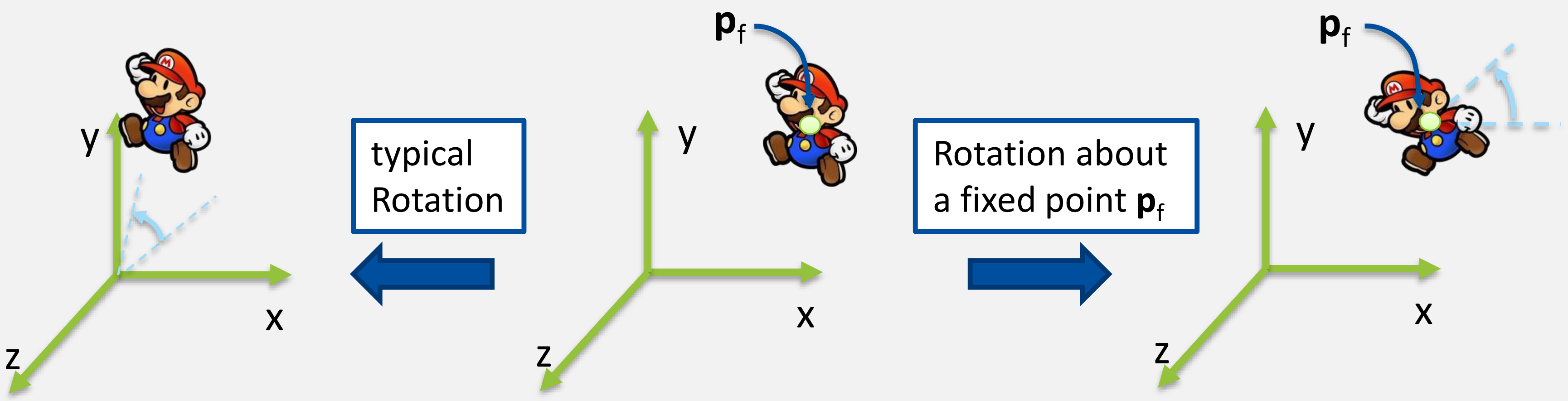
- We can composite transformation by multiplying matrices of rotation, translation, and scaling.
  - Example:
    1) Uniformly scale 2x:
    2) Rotate -30 degrees by +z axis:
    3) Translate by (2, 1, 0):
    - **x**’ = **T**(**R**(**Sp**)) = **TRSx**
      - **x**: original object
      - **x**’: transformed object

```
mat4x4    mat_model, T, R, S;

T = translate(2, 1, 0);      // T
R = rotate(-30, 0, 0, 1);    // R
S = scale(2, 2, 2);          // S

mat_model = T*R*S;

glDrawArrays(…);             // x
```



Rotation about a Fixed Point other than the Origin

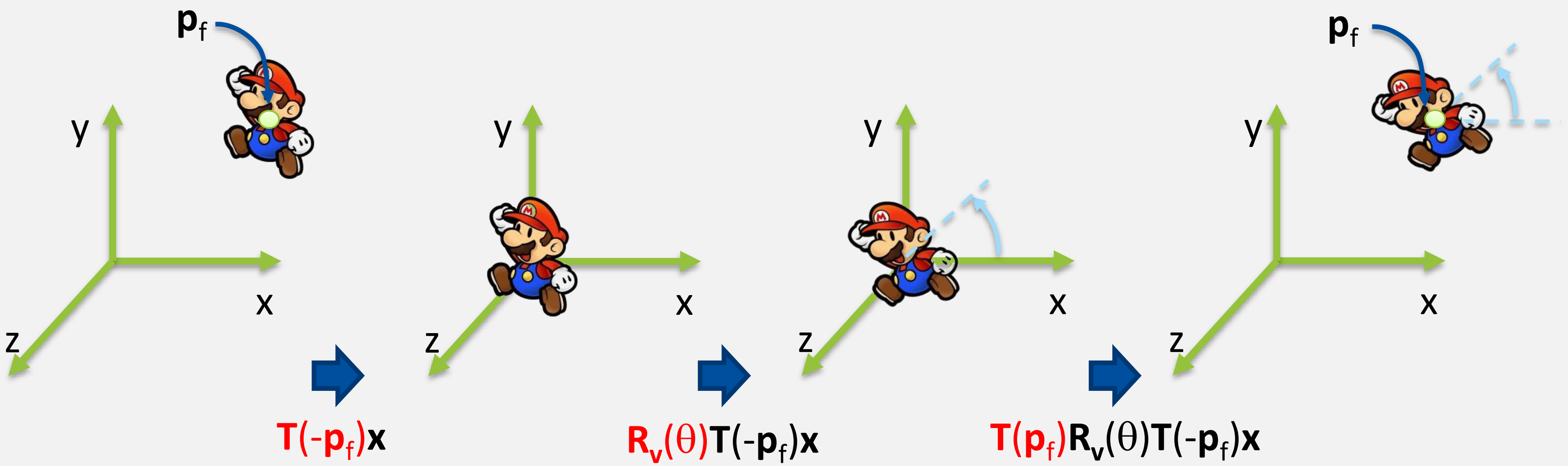
- Rotation: originally, a fixed point is the origin
  - Example
    - Rotate 30 degrees by +z axis: $\mathbf{R} = \mathbf{R}_z\mathbf{(30)}$
- But, ration about a general fixed point $\mathbf{p}_f$ is necessary, in general



Rotation about a Fixed Point other than the Origin

- Composite transformation
  - Move fixed point to origin
  - Rotate
  - Move fixed point back
- $\mathbf{x}' = \mathbf{T}(\mathbf{p}_f)\mathbf{R}_v(\theta)\mathbf{T}(-\mathbf{p}_f)\mathbf{x}$

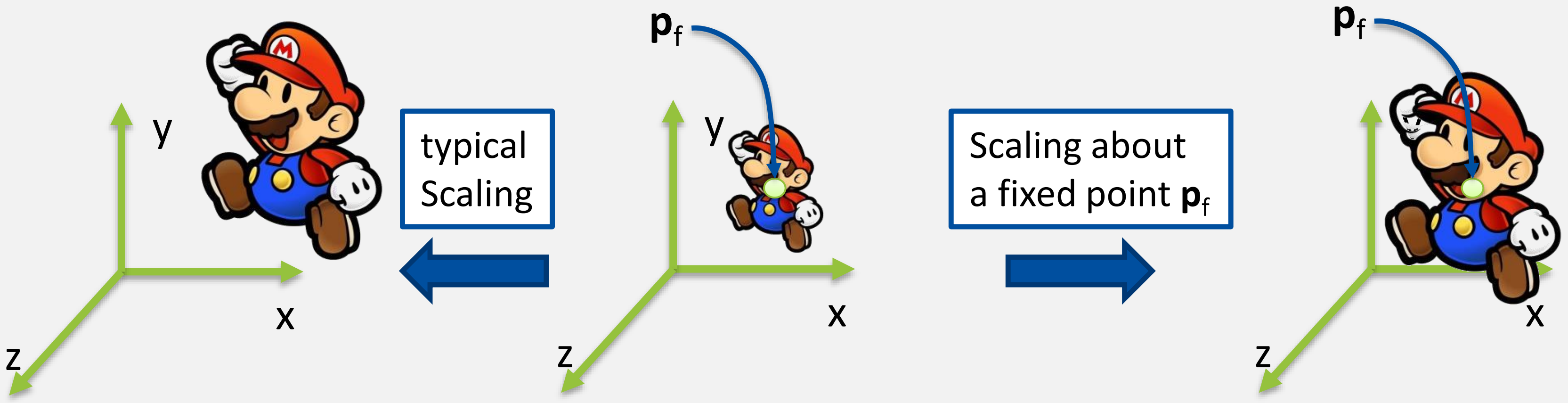
```
mat4x4    mat_model, T1, T2, R;

T2 =  translate(pf_x, pf_y, pf_z);        // T(pf)
R = rotate(theta, vx, vy, vz);            // Rv(θ)
T1 = translate(-pf_x, -pf_y, -pf_z);      // T(-pf)

mat_model = T2*R*T1;

glDrawArrays(…);                          // x
```



Scale about a Fixed Point other than the Origin

- Scaling: originally, a fixed point is the origin
  - Example
    - Uniformly scale 2x: **S**(2,2,2)

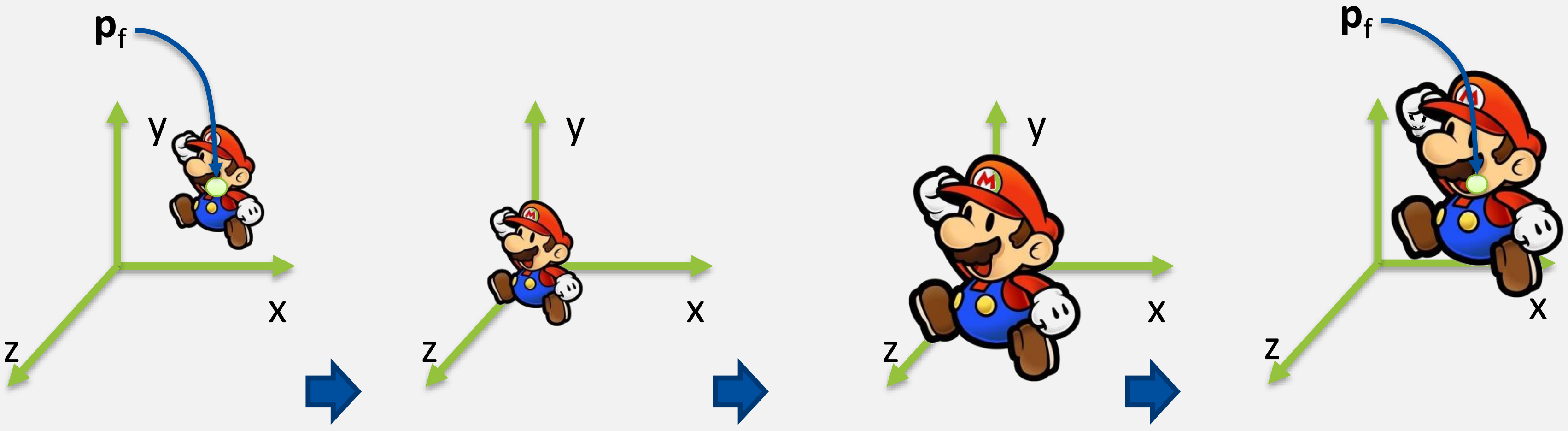
- But, scaling about a general fixed point $\mathbf{p}_f$ is necessary, in general



Scale about a Fixed Point other than the Origin

- Composite transformation
  - Move fixed point to origin
  - Scale
  - Move fixed point back
- $\mathbf{x}' = \mathbf{T}(\mathbf{p}_f)\mathbf{S}(s_x, s_y, s_z)\mathbf{T}(-\mathbf{p}_f)\mathbf{x}$

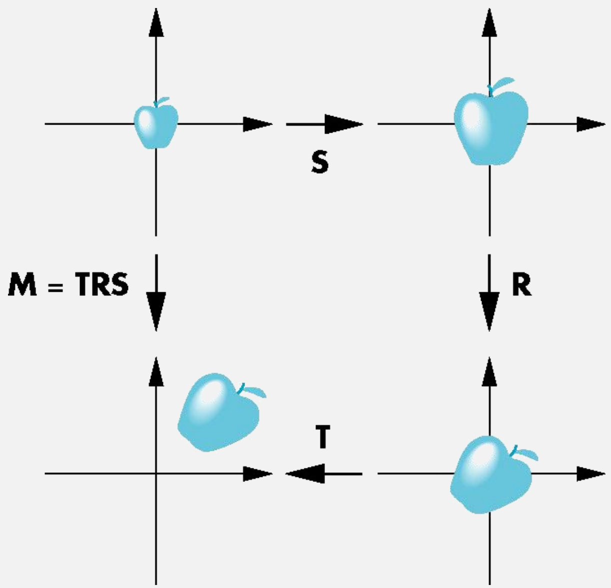
```
mat4x4    mat_model, T1, T2, S;

T2 =  translate(pf_x, pf_y, pf_z);          // T(pf)
S = scale(sx, sy, sz);                      // S(sx, sy, sz)
T1 = translate(-pf_x, -pf_y, -pf_z);        // T(-pf)

mat_model = T2*S*T1;

glDrawArrays(…);                            // x
```



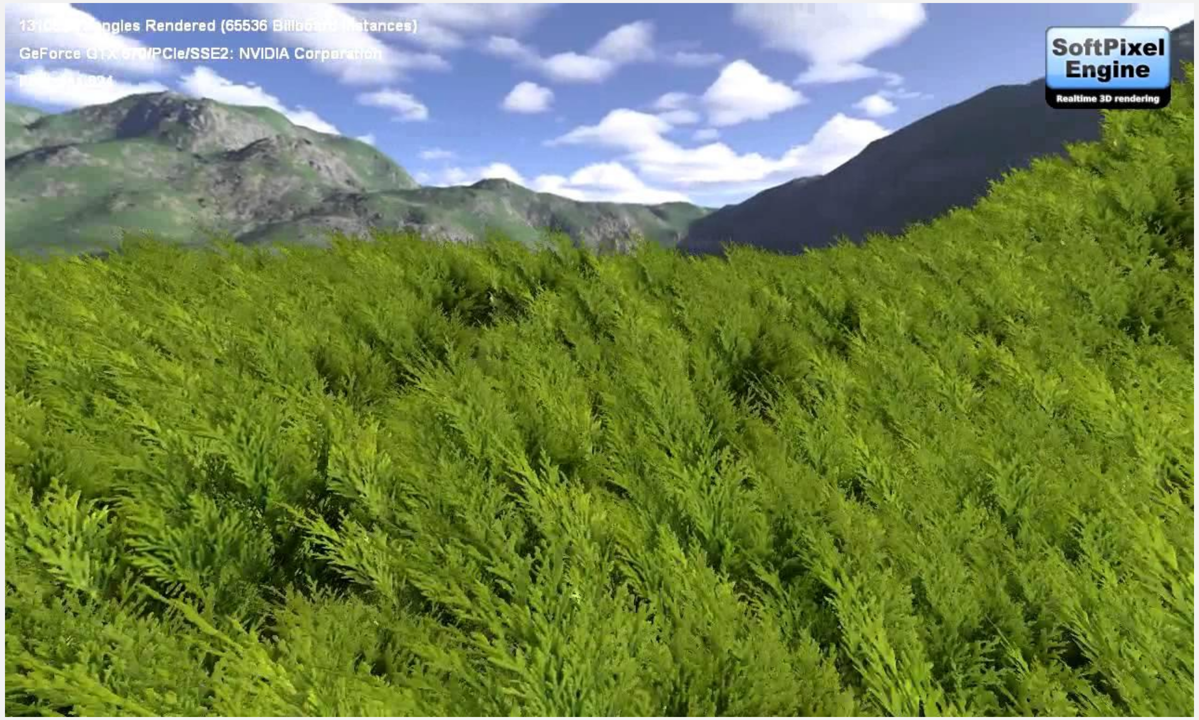
$\mathbf{T}(-\mathbf{p}_f)\mathbf{x}$ $\mathbf{S}(s_x, s_y, s_z)\mathbf{T}(-\mathbf{p}_f)\mathbf{x}$ $\mathbf{T}(\mathbf{p}_f)\mathbf{S}(s_x, s_y, s_z)\mathbf{T}(-\mathbf{p}_f)\mathbf{x}$

Instancing

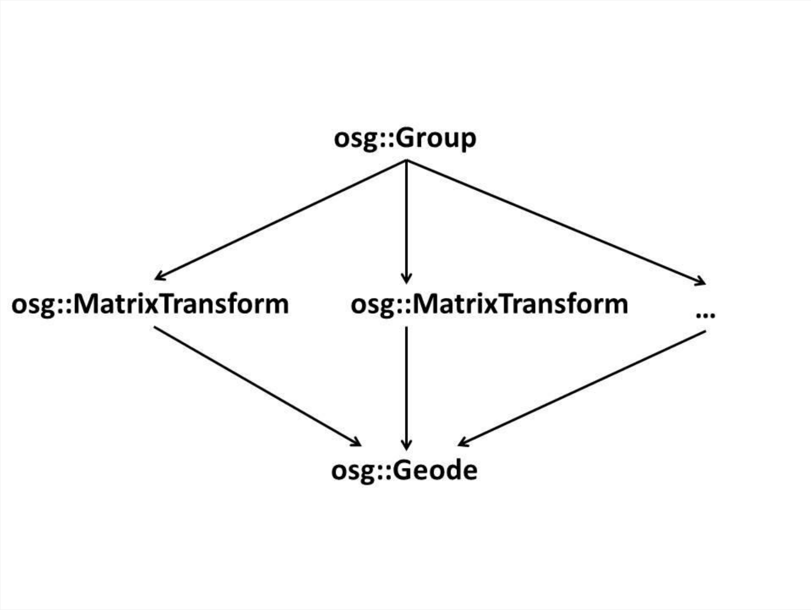
- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size
- We apply an instance transformation to its vertices to
  - Scale
  - Orient
  - Locate



Instancing



- Instanced rendering (youtube)
- Scene graph for instancing



Advanced Transformation

Transformations for Hierarchical Objects

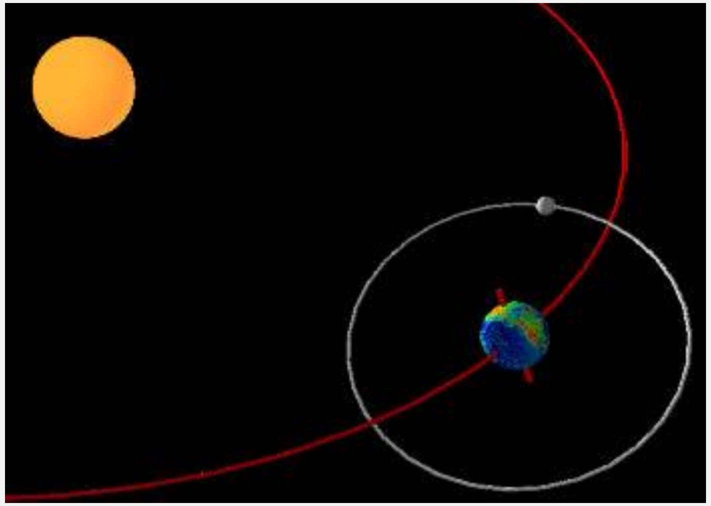
Solar System

- An example of hierarchical objects
  - How can we design transformations for hierarhical objects

(video, youtube)

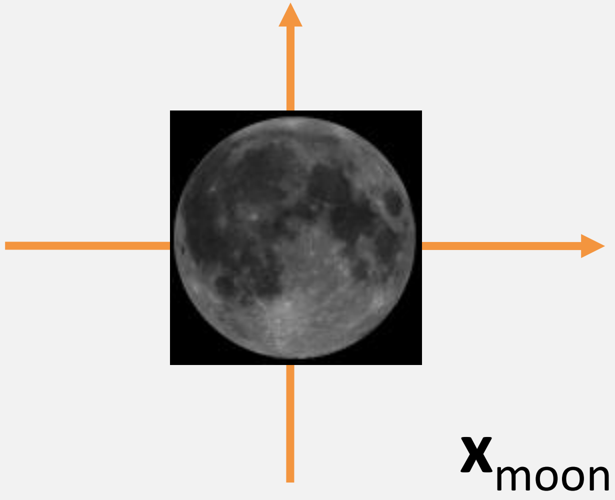
Sun – Earth – Moon

- Sun
- Earth
  - rotating itself
  - orbiting around the sun
- Moon
  - rotating itself
  - orbiting around the earth



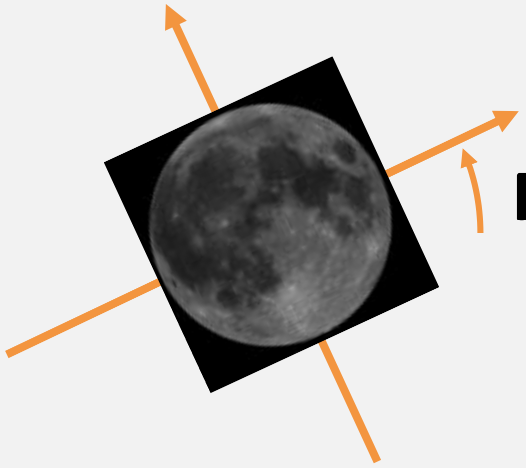
Hierarchical Transformation

- Moon
  - Modeling the moon



Hierarchical Transformation

- Moon
  - Modeling the moon
  - Rotating itself

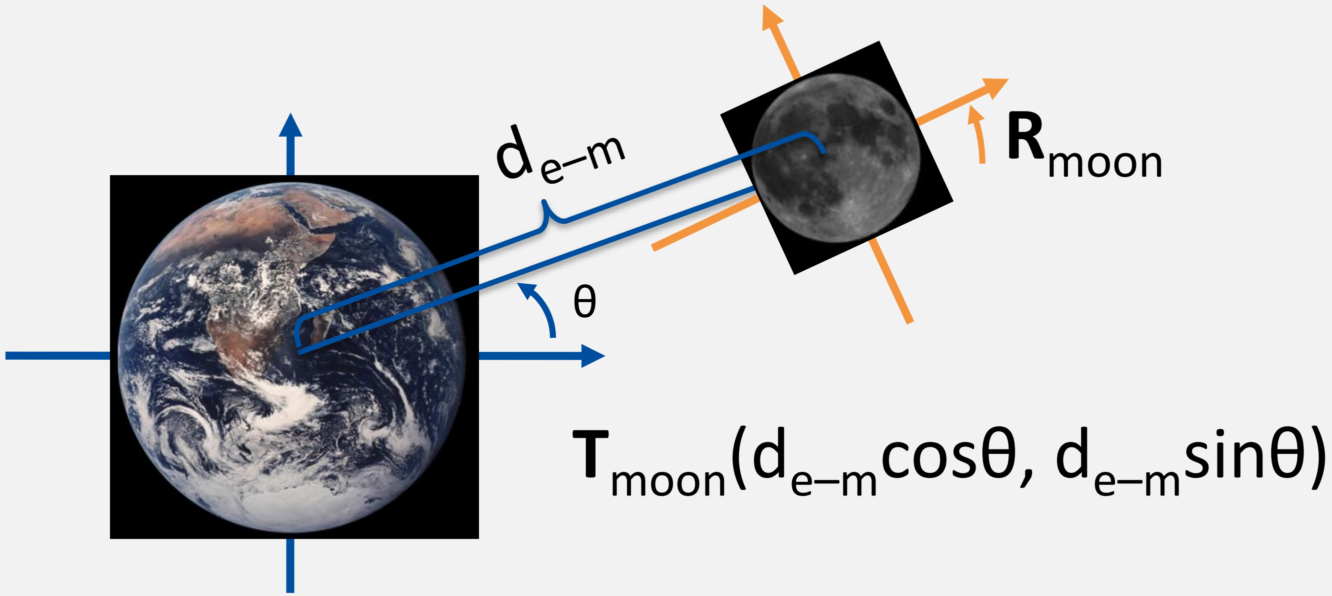


R_{moon} x_{moon}

Hierarchical Transformation

- Earth – Moon
  - The moon is orbiting around the earth

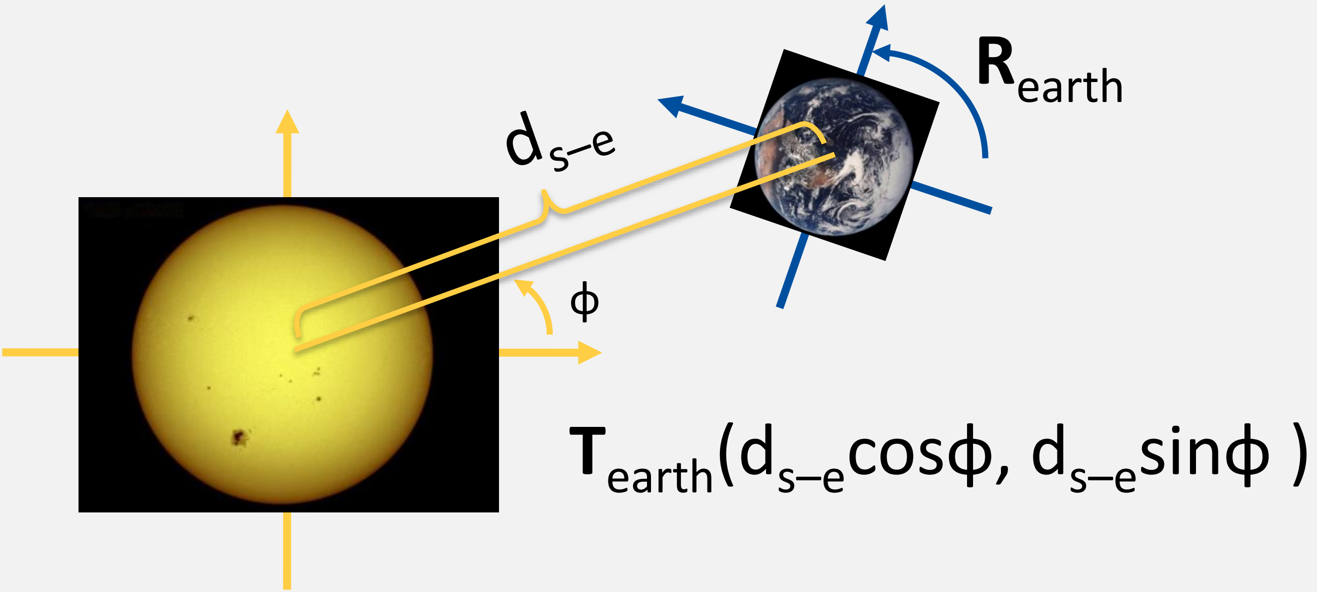
$$\mathbf{x}_{earth} = \mathbf{T}_{moon}(d_{e-m}\cos\theta,\ d_{e-m}\sin\theta)\mathbf{R}_{moon}\ \mathbf{x}_{moon}$$



Hierarchical Transformation

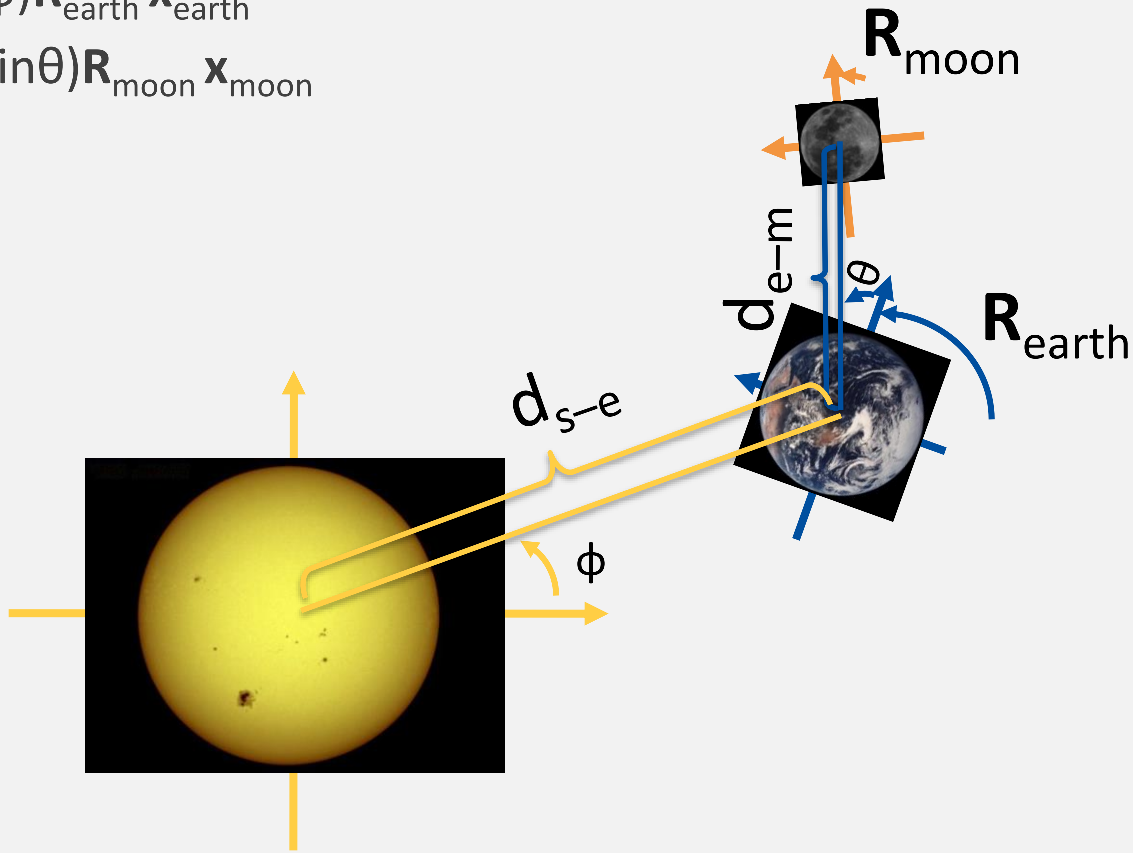
- Sun – Earth
  - The earth is orbiting around the sun

$$\mathbf{x}_{sun}=\mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi)\mathbf{R}_{earth}\ \mathbf{x}_{earth}$$



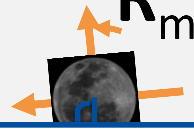
Hierarchical Transformation

- Sun – Earth – Moon
  - $\mathbf{x}_{sun}=\mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi)\mathbf{R}_{earth}\ \mathbf{x}_{earth}$
  - $\mathbf{x}_{earth}=\mathbf{T}_{moon}(d_{e-m}\cos\theta,\ d_{e-m}\sin\theta)\mathbf{R}_{moon}\ \mathbf{x}_{moon}$



Hierarchical Transformation

- Sun – Earth – Moon
  - $\mathbf{x}_{sun}=\mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi)\mathbf{R}_{earth}\ \mathbf{x}_{earth}$
  - $\mathbf{x}_{earth}=\mathbf{T}_{moon}(d_{e-m}\cos\theta,\ d_{e-m}\sin\theta)\mathbf{R}_{moon}\ \mathbf{x}_{moon}$

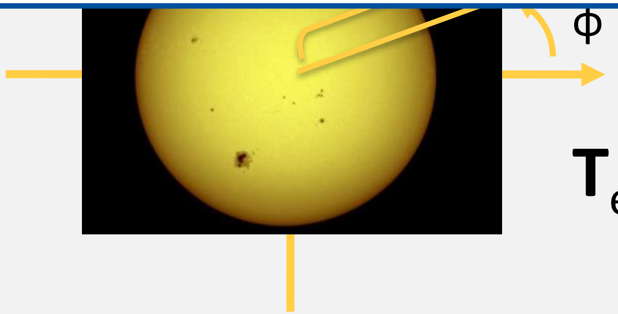


Transformation of the Earth w.r.t. the frame of the Sun

$$\mathbf{x}_{sun}=\mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi)\mathbf{R}_{earth}\ \mathbf{x}_{earth}$$

Transformation of the Moon w.r.t. the frame of the Sun

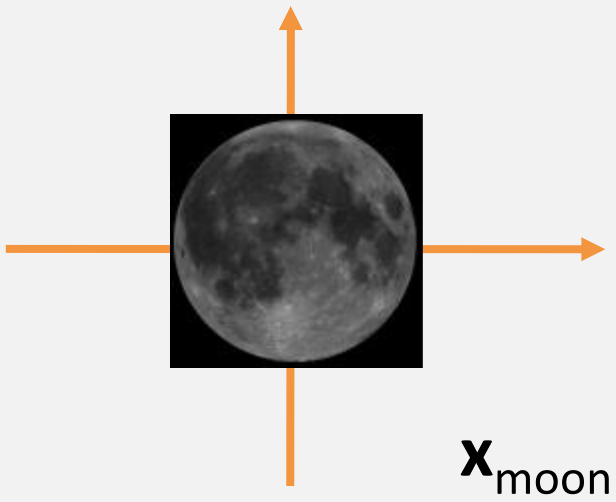
$$\mathbf{x}_{sun}=\mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi)\mathbf{R}_{earth}\mathbf{T}_{moon}(d_{e-m}\cos\theta,\ d_{e-m}\sin\theta)\mathbf{R}_{moon}\ \mathbf{x}_{moon}$$



OpenGL ES 2.x Codes – Hierarchical Transformation

- Moon
  - Modeling the moon

```
void draw_moon()
{
  glBindBuffer(GL_ARRAY_BUFFER, …);
  glEnableVertexAttribArray(…);

  glDrawArrays(…);

  glDisableVertexAttribArray(…);
}
```



OpenGL ES 2.x Codes – Hierarchical Transformation

- Moon
  - Modeling the moon
  - Rotating itself

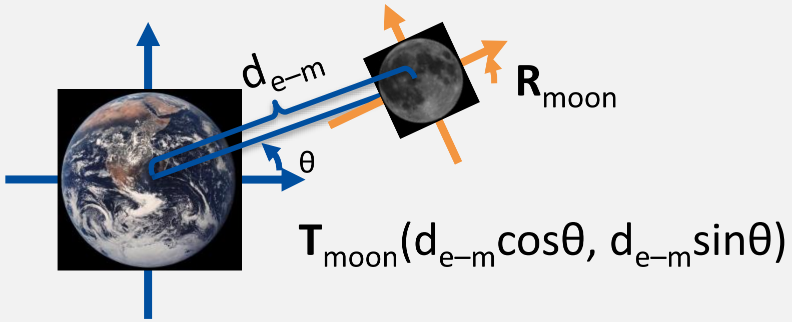
$\mathbf{R}_{\text{moon}}\,\mathbf{x}_{\text{moon}}$

```
mat4x4     g_mat_model;

void draw_earth_system()
{
  mat4x4     R;

  R = rotate(…);    // rotating the Moon

  g_mat_model *= R;
  draw_moon();
}

void draw_moon()   { // … glDrawElements(); … }
```

OpenGL ES 2.x Codes – Hierarchical Transformation

- Earth – Moon
  - The moon is orbiting around the earth

$$\boxed{\mathbf{x}_{earth}=\mathbf{T}_{moon}(d_{e-m}\cos\theta,\ d_{e-m}\sin\theta)\mathbf{R}_{moon}\ \mathbf{x}_{moon}}$$

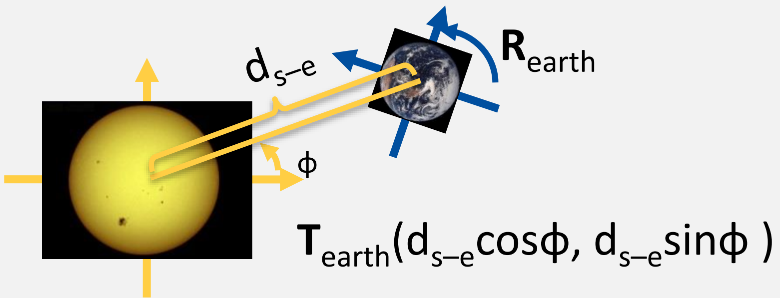



```
mat4x4     g_mat_model;

void draw_earth_system()
{
  mat4x4     T, R;

  draw_earth();

  T = translate(…); // orbiting around the Earth
  R = rotate(…);    // rotating the Moon

  g_mat_model *= T;
  g_mat_model *= R;
  draw_moon();
}

void draw_moon()   { // … glDrawArrays(); … }
void draw_earth()  { // … glDrawArrays(); … }
```

OpenGL ES 2.x Codes – Hierarchical Transformation

- Sun – Earth
  - The earth is orbiting around the sun

$$\mathbf{x}_{sun} = \mathbf{T}_{earth}(d_{s\text{–}e}\cos\phi,\ d_{s\text{–}e}\sin\phi)\mathbf{R}_{earth}\ \mathbf{x}_{earth}$$



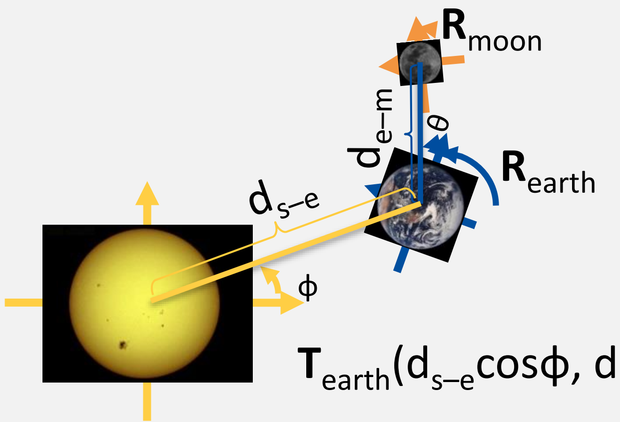

```
mat4x4     g_mat_model;

void draw_sun_system()
{
  mat4x4     T, R;

  g_mat_model.set_identity();

  draw_sun();

  T = translate (…); // orbiting around the Sun
  R = rotate(…);     // rotating the Earth system

  g_mat_model *= T;
  g_mat_model *= R;

  draw_earth_system();
}

void draw_moon()   { // … glDrawArrays(); … }
void draw_earth()  { // … glDrawArrays(); … }
void draw_sun()    { // … glDrawArrays(); … }
```

OpenGL ES 2.x Codes – Hierarchical Transformation

- Sun – Earth – Moon

$$\mathbf{x}_{earth} = \mathbf{T}_{moon}(d_{e-m}\cos\theta,\ d_{e-m}\sin\theta)\mathbf{R}_{moon}\ \mathbf{x}_{moon}$$

$$\mathbf{x}_{sun} = \mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi)\mathbf{R}_{earth}\ \mathbf{x}_{earth}$$



$\mathbf{T}_{earth}(d_{s-e}\cos\phi,\ d_{s-e}\sin\phi\ )$

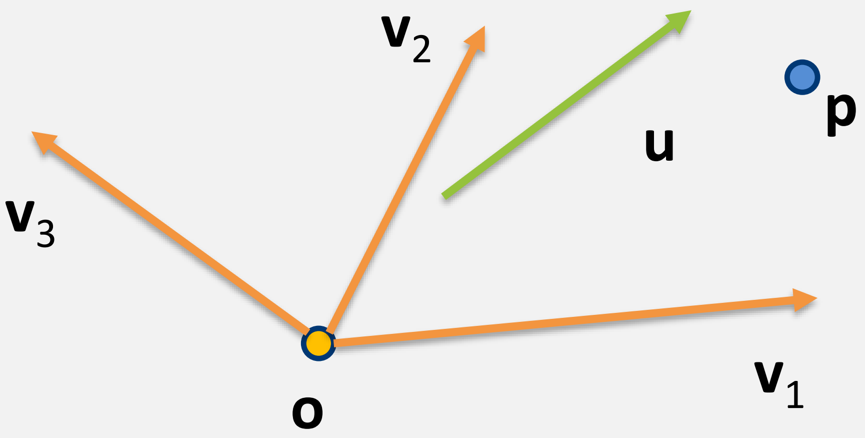
```
mat4x4     g_mat_model;

void draw_sun_system()
{
  mat4x4     T, R;

  g_mat_model.set_identity();

  draw_sun();

  T = translate (…); // orbiting around the Sun
  R = rotate(…);    // rotating the Earth system

  g_mat_model *= T;
  g_mat_model *= R;

  draw_earth_system();
}

void draw_earth_system()
{
  mat4x4     T, R;

  draw_earth();

  T = translate(…); // orbiting around the Earth
  R = rotate(…);    // rotating the Moon

  g_mat_model *= T;
  g_mat_model *= R;
  draw_moon();
}

void draw_moon()   { // … glDrawArrays(); … }
void draw_earth()  { // … glDrawArrays(); … }
void draw_sun()    { // … glDrawArrays(); … }
```

View Transformations

Coordinate System (Frame)

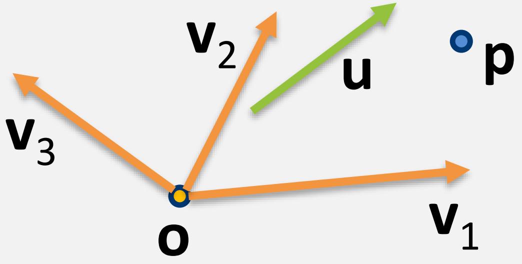
- A Coordinate system (or frame) consists of a set of basis vectors and an origin
  - A set of basis vectors: $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$
  - An origin: $\mathbf{o}$

- How to representing a vector **u** and a point **p** w.r.t. a given coordinate system?



Coordinate System (Frame)

- Consider a set of basis vectors and an origin
  - $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$
  - $\mathbf{o}$
- A vector **u** is written
  - $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n + 0 \cdot \mathbf{o}$
  - The list of scalars, $\{\alpha_1, \alpha_2, \ldots \alpha_n, 0\}$ is the representation of **u** w.r.t. the given coordinate system
- A point **p** is written
  - $\mathbf{p} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_n\mathbf{v}_n + 1 \cdot \mathbf{o}$
  - The list of scalars, $\{\beta_1, \beta_2, \ldots \beta_n, 1\}$ is the representation of **p** w.r.t . the given coordinate system

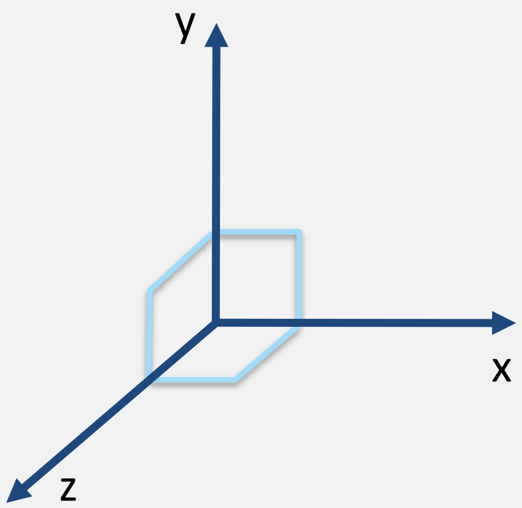


$$\boldsymbol{u} = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\boldsymbol{p} = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]^T = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

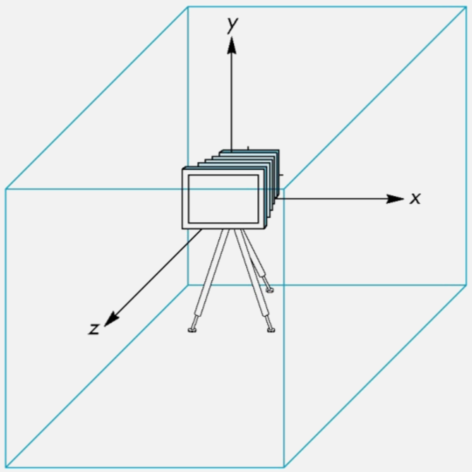
Coordinate System (Frame)

- In OpenGL ES, we just care about ***orthonormal*** frames
  - ***Ortho*** means that the basis vectors are orthogonal to each other
    - x-axis ⊥ y-axis ⊥ z-axis
  - ***normal*** means that the length of each basis vector is 1
    - The unit length of each axis is equal to 1



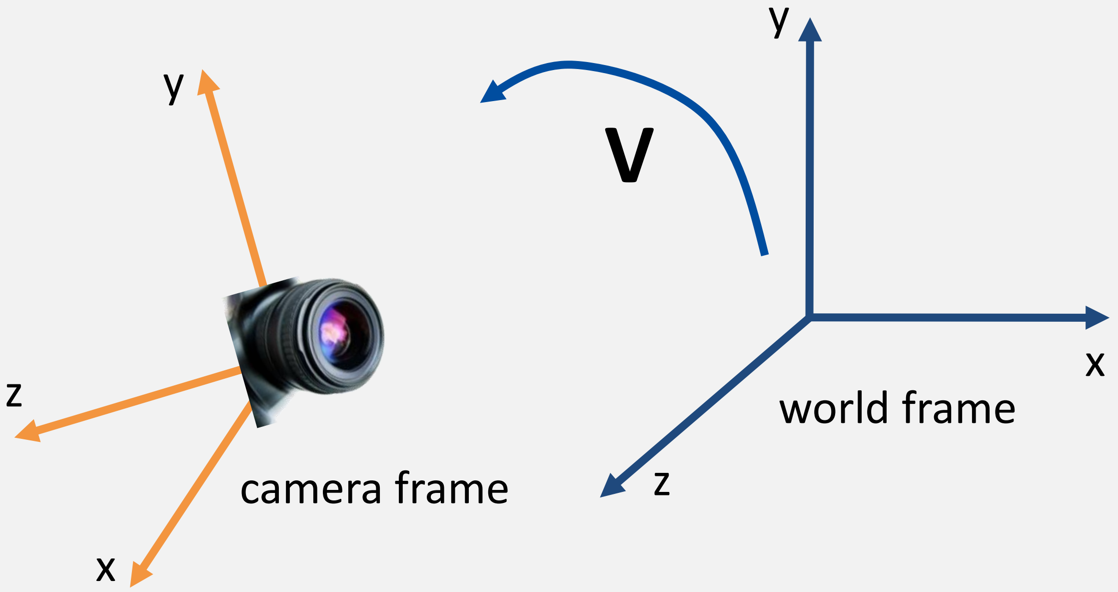
Inside of gluLookAt()

- Initially, OpenGL ES camera coordinate is as follows
  - Center of projection (COP) is placed in the origin
  - Right direction is the positive direction of x-axis
  - Up direction is the positive direction of y-axis
  - Viewing direction is the negative direction of z-axis



Inside of gluLookAt()

- When we apply gluLookAt()
  - Rotate the camera frame on the world frame: $\mathbf{R}_c$
  - Translate the camera frame on the world frame: $\mathbf{T}_c$
  - Therefore, $\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$ and gluLookAt() generates $\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$



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- When we apply gluLookAt()
  - Rotate the camera frame on the world frame: $\mathbf{R}_c$
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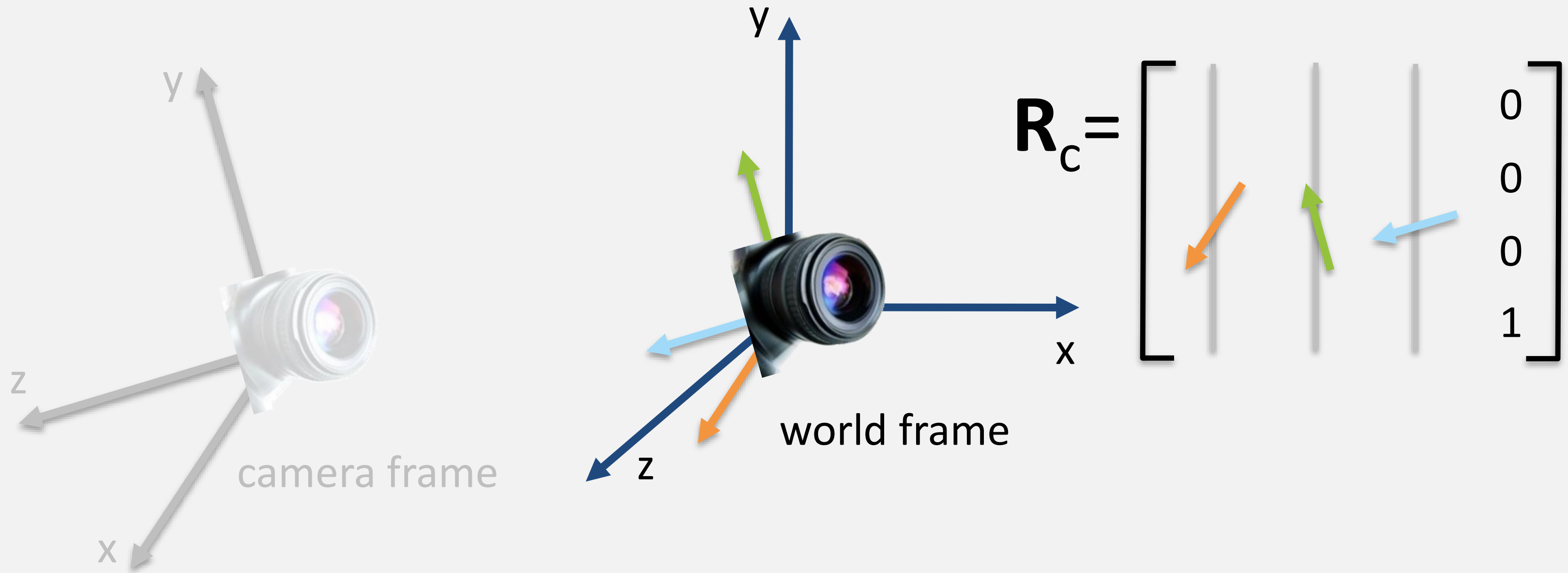
  - Therefore, $\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$ and gluLookAt() generates $\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$



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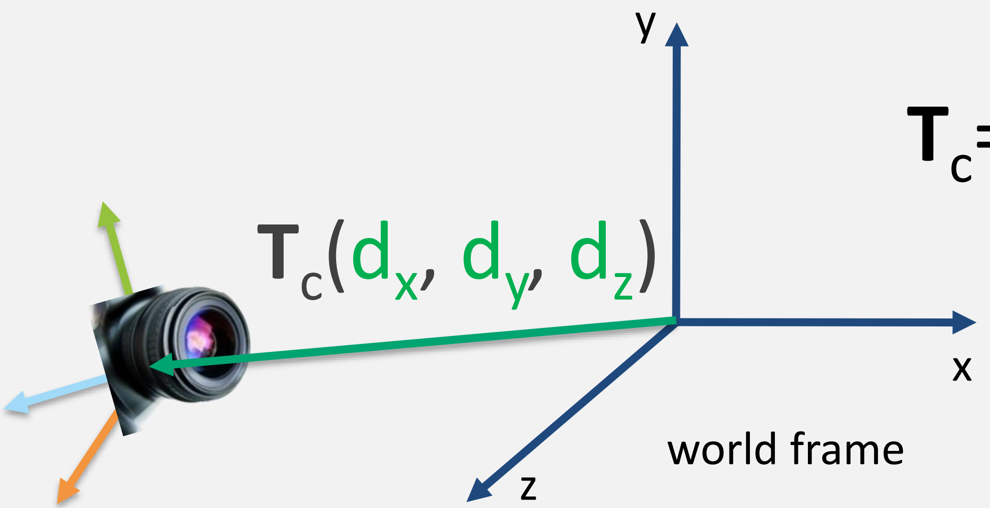
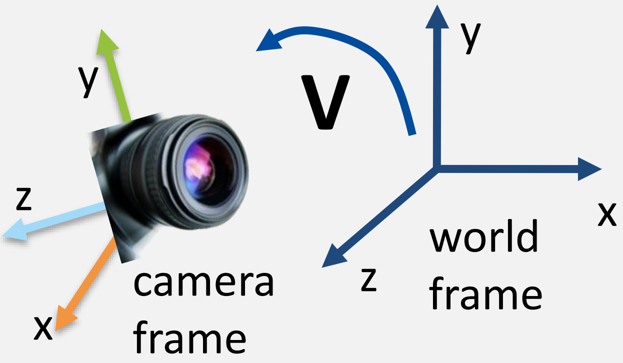


$$\mathbf{T}_c = \begin{bmatrix} 1 & & & dx \\ & 1 & & dy \\ & & 1 & dz \\ & & & 1 \end{bmatrix}$$

Inside of gluLookAt()

- When we apply gluLookAt()
  - Rotate the camera frame on the world frame: $\mathbf{R}_c$
  - Translate the camera frame on the world frame: $\mathbf{T}_c$
  - Therefore, $\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$ and gluLookAt() generates $\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$

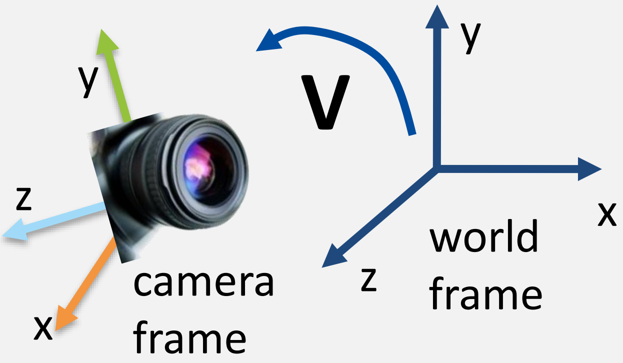


$$\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$$

$$= \begin{bmatrix} 1 & & & dx \\ & 1 & & dy \\ & & 1 & dz \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & & & 0 \\ & & & 0 \\ & & & 1 \end{bmatrix}$$

Inside of gluLookAt()

- When we apply gluLookAt()
  - Rotate the camera frame on the world frame: $\mathbf{R}_c$
  - Translate the camera frame on the world frame: $\mathbf{T}_c$
  - Therefore, $\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$ and gluLookAt() generates $\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$



$$\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$$

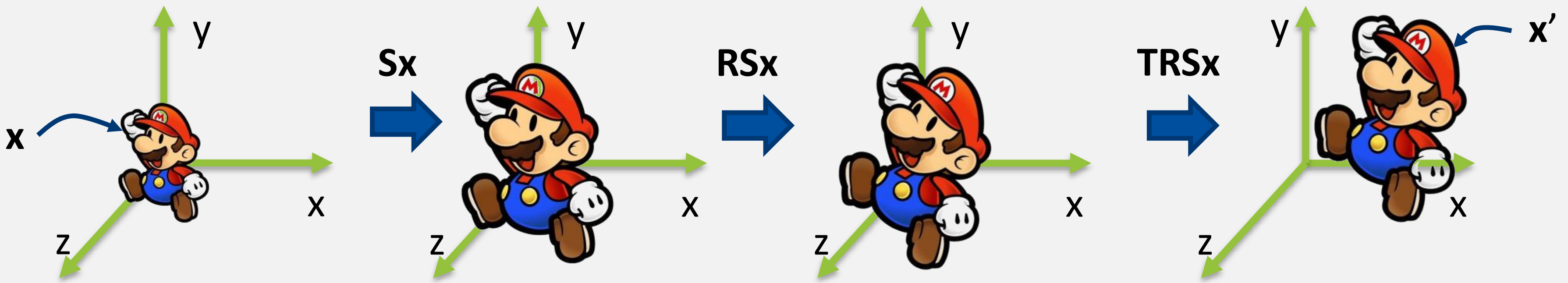
$$= \begin{bmatrix} & & & \\ & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -dx \\ & 1 & & -dy \\ & & 1 & -dz \\ & & & 1 \end{bmatrix}$$

Revisiting Composite Transformation

Composite Transformation (합성 변환)

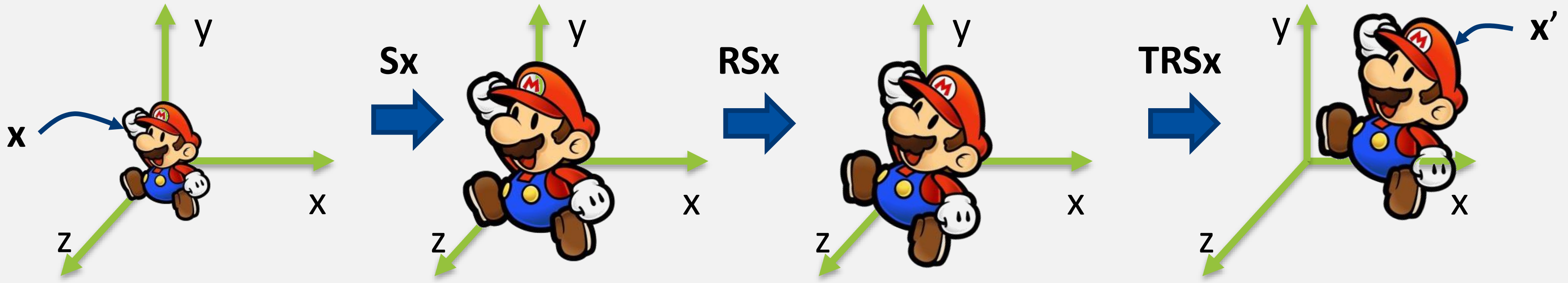
- We can composite transformation by multiplying matrices of rotation, translation, and scaling.
  - Example:
    1) Uniformly scale 2x:
    2) Rotate -30 degrees by +z axis:
    3) Translate by (2, 1, 0):
    - **x**' = **T**(**R**(**Sp**)) = **TRSx**
      - **x**: original object
      - **x**': transformed object

```
mat4x4     mat_model, T, R, S;

T = translate(2, 1, 0);       // T
R = rotate(-30, 0, 0, 1);     // R
S = scale(2, 2, 2);           // S

mat_model = T*R*S;

glDrawArrays(…);              // x
```



Composite Transformation (합성 변환)

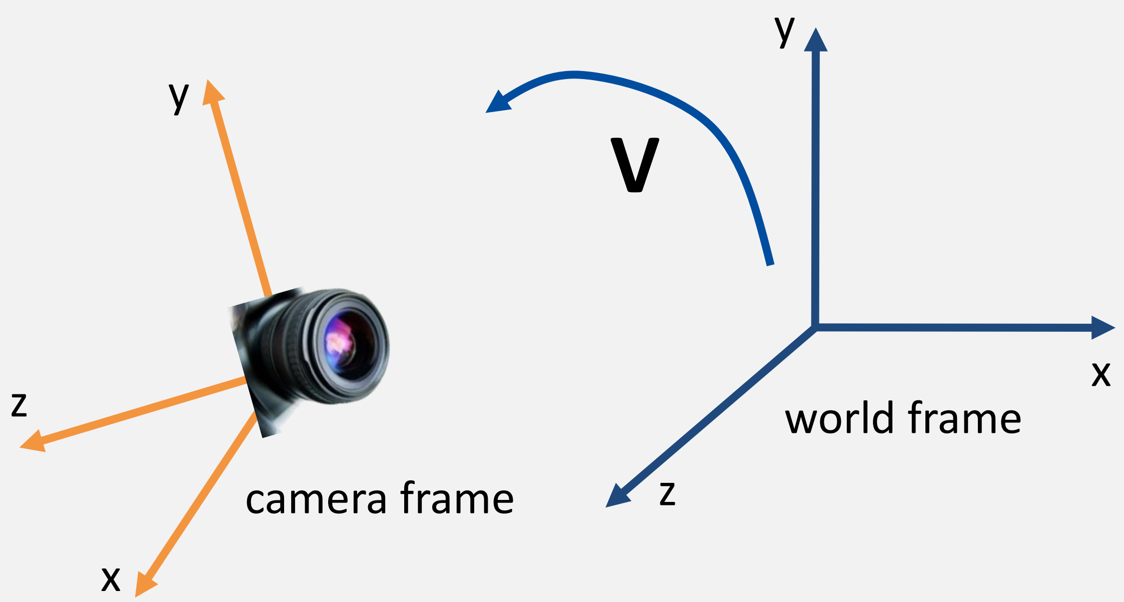
- Composite transformation w/ “scale, rotation, and then translation”



$$\mathbf{T_d R_u}(\theta)\mathbf{S_s} = \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R_u}(\theta)_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S}_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{SR_u}(\theta)_{3\times3} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

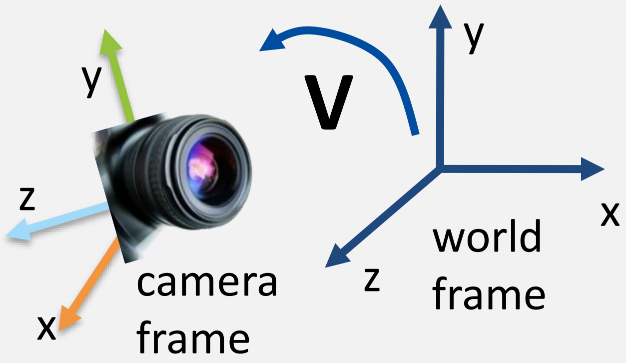
Inside of gluLookAt()

- When we apply gluLookAt()
  - Rotate the camera frame on the world frame: $\mathbf{R}_c$
  - Translate the camera frame on the world frame: $\mathbf{T}_c$
  - Therefore, $\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$ and gluLookAt() generates $\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$



Inside of gluLookAt()

- When we apply gluLookAt()
  - Rotate the camera frame on the world frame: $\mathbf{R}_c$
  - Translate the camera frame on the world frame: $\mathbf{T}_c$
  - Therefore, $\mathbf{V} = \mathbf{T}_c\mathbf{R}_c$ and gluLookAt() generates $\mathbf{V}^{-1} = \mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$



$$\mathbf{V}^{-1}=\mathbf{R}_c^{-1}\mathbf{T}_c^{-1}$$

$$= \begin{bmatrix} & & & \\ & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -dx \\ & 1 & & -dy \\ & & 1 & -dz \\ & & & 1 \end{bmatrix}$$

Inside of gluLookAt()

- ViewMatrix V^{-1}

$$\mathbf{V} = \mathbf{T_d}\mathbf{R_u}(\theta) = \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R_u}(\theta)_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R_u}(\theta)_{3\times3} & \mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\mathbf{V}^{-1} = \mathbf{R_u}^{-1}(\theta)\mathbf{T_d}^{-1} = \mathbf{R_u}^T(\theta)\mathbf{T_d}^{-1} = \begin{bmatrix} \mathbf{R_u}^T(\theta)_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{3\times3} & -\mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{R_u}^T(\theta)_{3\times3} & -\mathbf{R_u}^T(\theta)_{3\times3}\mathbf{d} \\ \mathbf{0}^T & 1 \end{bmatrix}$$