St. Joseph's College of Engineering
St. Joseph's Institute of Technology
Mathematics II (MA6251)
Assignment – I

UNIT II - ORDINARY DIFFERENTIAL EQUATIONS

PART A

1. **Solve** $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$
2. **Solve** $\left(D^3+D^2-D-1\right)y=0$
3. **Solve** $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$
4. **Find the P.I. of** $\left(D^2+3\right)y=\sin 3x$
5. **Find the Particular integral of** $\left(D^2+6\right)y=\sin x\cos x$
6. **Obtain the differential equation in terms of y,** $\frac{dx}{dt}+2x-3y=5t,\ \frac{dy}{dt}-3x+2y=0$
7. **Solve** $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0$
8. **Transform the equation** $x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+2y=x\log x$ **into linear differential equation with constant coefficients.**
9. **Transform the equation** $\left[(2x+3)^2D^2-(2x+3)D-12\right]y=6x$ **into linear differential equation with constant coefficients.**
10. **Transform the equation** $\left[(x-2)D^2-6D+\frac{8}{(x-2)}\right]y=0$ **into linear differential equation with constant coefficients.**

PART B

1 (a) Solve $\left(D^2+4D+3\right)y=e^{-x}\sin x$

SOLUTION:

Given $\left(D^2+4D+3\right)y=e^{-x}\sin x$

The A.E. is $m^2+4m+3=0$

$$(m+1)(m+3)=0$$

$$m=-1,-3$$

$$C.F.=Ae^{-x}+Be^{-3x}$$

$$P.I.=\frac{1}{D^2+4D+3}\left(e^{-x}\sin x\right)=e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$$

$$=e^{-x}\frac{1}{D^2-2D+1+4D-4+3}\sin x=e^{-x}\frac{1}{D^2+2D}\sin x=e^{-x}\frac{1}{-1+2D}\sin x=e^{-x}\frac{2D+1}{(2D)^2-1}\sin x$$

$$=e^{-x}\frac{2D+1}{4D^2-1}\sin x=e^{-x}\frac{2D+1}{-4-1}\sin x=e^{-x}\frac{2D+1}{-5}\sin x$$

$$P.I.=-\frac{e^{-x}}{5}\left[2\cos x+\sin x\right]$$

$$y=C.F.+P.I.=Ae^{-x}+Be^{-3x}-\frac{e^{-x}}{5}\left[2\cos x+\sin x\right]$$

1(b) Solve $\left(D^2-4D+4\right)y=e^{2x}+x^2$

Solution:

The A.E. is $m^2-4m+4=0$

$$(m-2)^2=0$$

$$m=2,2$$

$$C.F.=(Ax+B)e^{2x}$$

$$P.I_1=\frac{1}{D^2-4D+4}e^{2x}$$

$$=\frac{1}{4-8+4}e^{2x}=x\frac{1}{2D-4}e^{2x}=x\frac{1}{4-4}e^{2x}=\frac{x^2}{2}e^{2x}$$

$$P.I_2=\frac{1}{D^2-4D+4}x^2=\frac{1}{4\left(1+\frac{D^2-4D}{4}\right)}x^2=\frac{1}{4}\left(1+\frac{D^2-4D}{4}\right)^{-1}x^2$$

$$=\frac{1}{4}\left(1-\left(\frac{D^2-4D}{4}\right)+\left(\frac{D^2-4D}{4}\right)^2-\cdots\right)x^2=\frac{1}{4}\left(1-\frac{D^2}{4}+D+\frac{D^4+16D^2-8D^3}{16}-\cdots\right)x^2$$

$$=\frac{1}{4}\left(1+D-\frac{D^2}{4}+D^2\right)x^2=\frac{1}{4}\left(1+D+\frac{3D^2}{4}\right)x^2=\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

$$P.I=P.I_1+P.I_2$$

$$\therefore\ y=(Ax+B)e^{2x}+\frac{x^2}{2}e^{2x}+\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$$

2(a) Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=\sin 3x\cos 2x$

SOLUTION:

Given $\left(D^2-4D+3\right)y=\sin 3x\cos 2x$

The A.E. is $m^2-4m+3=0$

$(m-3)(m-1)=0$

$m=1,3$

$C.F.=Ae^x+Be^{3x}$

$$P.I=\frac{1}{D^2-4D+3}(\sin 3x\cos 2x)=\frac{1}{D^2-4D+3}\frac{1}{2}(\sin 5x+\sin x)$$

$$=\frac{1}{D^2-4D+3}\frac{1}{2}\sin 5x+\frac{1}{D^2-4D+3}\frac{1}{2}\sin x$$

$$=P.I_1+P.I_2$$

$$P.I_1=\frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x=\frac{1}{2}\frac{1}{-25-4D+3}\sin 5x=\frac{1}{2}\frac{1}{-4D-22}\sin 5x=-\frac{1}{4}\frac{1}{2D+11}\sin 5x$$

$$=-\frac{1}{4}\frac{1}{(2D+11)}\frac{(2D-11)}{(2D-11)}\sin 5x=-\frac{1}{4}\frac{2D-11}{4D^2-121}\sin 5x=-\frac{1}{4}\frac{2D-11}{4(-25)-121}\sin 5x$$

$$=-\frac{1}{4}\frac{2D-11}{-221}\sin 5x=\frac{1}{884}[10\cos 5x-11\sin 5x]$$

$$P.I_2=\frac{1}{2}\frac{1}{D^2-4D+3}\sin x=\frac{1}{2}\frac{1}{-1-4D+3}\sin x=\frac{1}{2}\frac{1}{2-4D}\sin x$$

$$=\frac{1}{2}\frac{2+4D}{4-16D^2}\sin x=\frac{1}{2}\frac{2\sin x+4\cos x}{20}=\frac{\sin x+2\cos x}{20}$$

$$y=C.F+P.I_1+P.I_2=Ae^x+Be^{3x}+\frac{1}{884}[10\cos 5x-11\sin 5x]+\frac{\sin x+2\cos x}{20}$$

2(b) Solve $\frac{dx}{dt}+2y=-\sin t,\frac{dy}{dt}-2x=\cos t$

SOLUTIO N:

$\frac{dx}{dt}+2y=-\sin t,\frac{dy}{dt}-2x=\cos t$

$(i.e.)\,Dx+2y=-\sin t$........(1); $Dy-2x=\cos t$.......(2)

$(1)\times 2\Rightarrow 2Dx+4y=-2\sin t$...........(3)

$(2)\times D\Rightarrow -2Dx+D^2y=-\sin t$............(4)

$(3)+(4)\Rightarrow \left(D^2+4\right)y=-3\sin t$

The A.E. is $m^2+4=0$

$m=\pm 2i$

$C.F.=A\cos 2t+B\sin 2t$

$$P.I.=\frac{1}{D^2+4}(-3\sin t)=-3\frac{\sin t}{3}=-\sin t$$

$y=A\cos 2t+B\sin 2t-\sin t$

$Dy=-2A\sin 2t+2B\cos 2t-\cos t$

$\therefore (2) \Rightarrow \quad 2x = Dy - \cos t$

$$= -2A\sin 2t + 2B\cos 2t - \cos t - \cos t$$

$$x = -A\sin 2t + B\cos 2t - \cos t$$

3(a) Solve $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$

SOLUTION:

$(D-7)x + y = 0$.............(1)

$-2x + (D-5)y = 0$..........(2)

$(1)\times 2 \Rightarrow \quad 2(D-7)x + 2y = 0$...........(3)

$(2)\times(D-7) \Rightarrow \ -2(D-7)x + (D-5)(D-7)y = 0$.............(4)

$(3)+(4) \Rightarrow \left(D^2 - 12D + 37\right)y = 0$

A.E. is $\quad m^2 - 12m + 37 = 0$

$$m^2 - 12m + 36 + 1 = 0$$

$$(m-6)^2 + 1 = 0$$

$$(m-6)^2 = -1$$

$$m = 6 \pm i$$

$\therefore \quad y = e^{6t}\left(A\cos t + B\sin t\right)$

$(2) \Rightarrow -2x = -(D-5)y$

$$x = \frac{1}{2}Dy - \frac{5}{2}y$$

$$= \frac{1}{2}\left[e^{6t}\left(-A\sin t + B\cos t\right) + 6e^{6t}\left(A\cos t + B\sin t\right)\right] - \frac{5}{2}e^{6t}\left(A\cos t + B\sin t\right)$$

$$x = \frac{1}{2}\left[(A+B)e^{6t}\cos t + (B-A)\mathrm{e}^{6t}\sin t\right]$$

3 (b) Solve $\frac{dx}{dt} + \frac{dy}{dt} + x + y = 10e^t,\ \frac{dx}{dt} - \frac{dy}{dt} + x - y = 0$ given $x(0) = 2,\ y(0) = 3$

SOLUTION:

Given $\frac{dx}{dt} + \frac{dy}{dt} + x + y = 10e^t,\ \frac{dx}{dt} - \frac{dy}{dt} + x - y = 0$

$(i.e.)\ Dx + Dy + x + y = 10e^t$.........(1); $\quad Dx - Dy + x - y = 0$.........(2)

$(1)+(2) \Rightarrow 2Dx + 2x = 10e^t$

$$Dx + x = 5e^t$$

$(D+1)x = 5e^t$......(3)

A.E. is $m + 1 = 0$; $\quad m = -1$

$\therefore \quad C.F. = Ae^{-t}$

$$P.I. = \frac{1}{D+1}5e^t = 5\frac{e^t}{2}$$

$\therefore\ x = C.F + P.I = Ae^{-t} + \frac{5}{2}e^{t}$.......(4)

$(1)-(2) \Rightarrow 2Dy + 2y = 10e^{t}$

$$Dy + y = 5e^{t}$$

$$(D+1)\,y = 5e^{t}$$

A.E. is $m+1=0;\ m=-1$

$$C.F. = Be^{-t}$$

$$P.I. = \frac{1}{D+1}5e^{t} = 5\frac{e^{t}}{2}$$

$\therefore\quad y = Be^{-t} + \frac{5}{2}e^{t}$...........(5)

Given $x(0)=2,\ y(0)=3$

$(4) \Rightarrow 2 = A + \frac{5}{2}$

$$A = 2 - \frac{5}{2} = \frac{-1}{2}$$

$(5) \Rightarrow 3 = B + \frac{5}{2}$

$$B = 3 - \frac{5}{2} = \frac{1}{2}$$

$\therefore\quad (4) \Rightarrow x = \frac{-1}{2}e^{-t} + \frac{5}{2}e^{t}$

$\quad (5) \Rightarrow y = \frac{1}{2}e^{-t} + \frac{5}{2}e^{t}$

4 (a) **Solve $\left(x^2D^2 - 3xD + 4\right)y = x^2\cos(\log x)$**

SOLUTION:

Put $x = e^{z} \Rightarrow \log x = z$

$$xD = D'$$

$$x^2D^2 = D'(D'-1)$$

$(1) \Rightarrow \left[D'(D'-1) - 3D' + 4\right]y = e^{2z}\cos z$

$\left[D'^2 - 4D' + 4\right]y = e^{2z}\cos z$

$\left[D'-2\right]^2 y = e^{2z}\cos z$

The A.E. is

$(m-2)^2 = 0 \qquad m = 2, 2$

$C.F. = (Az+B)e^{2z}$

$$P.I. = \frac{1}{(D'-2)^2}e^{2z}\cos z = e^{2z}\frac{1}{D'^2}\cos z = e^{2z}\frac{1}{D'}(-\sin z) = -e^{2z}\cos z$$

$$\therefore\ y = C.F + P.I = (Az+B)e^{2z} - e^{2z}\cos z = (A\log x + B)x^2 - x^2\cos(\log x)$$

4(b) Solve $\left(x^2D^2 - 2xD - 4\right)y = x^2 + 2\log x$

SOLUTION:

Given $\left(x^2D^2 - 2xD - 4\right)y = x^2 + 2\log x$.............(1)

Put $x = e^z \Rightarrow \log x = z$

$$xD = D'$$

$$x^2D^2 = D'(D'-1)$$

$$(1) \Rightarrow \left[D'(D'-1) - 2D' - 4\right]y = e^{2z} + 2z$$

$$\left[D'^2 - 3D' - 4\right]y = e^{2z} + 2z$$

The A.E. is $m^2 - 3m - 4 = 0$

$$m^2 - 4m + m - 4 = 0$$

$$m(m-4) + (m-4) = 0$$

$$(m-4)(m+1) = 0$$

$$m = 4,\ -1$$

$$C.F. = Ae^{-z} + Be^{4z}$$

$$P.I_1 = \frac{1}{D'^2 - 3D' - 4}e^{2z} = \frac{1}{4-6-4}e^{2z} = \frac{-1}{6}e^{2z}$$

$$P.I_2 = \frac{1}{D'^2 - 3D' - 4}(2z) = \frac{-2}{4}\left[\frac{1}{1 - \frac{D'^2 - 3D'}{4}}\right]z = -\frac{1}{2}\left[1 - \frac{D'^2 - 3D'}{4} + \ldots\right]z$$

$$= -\frac{1}{2}\left[z - \frac{3}{4}(1)\right] = -\frac{1}{2}z + \frac{3}{8}$$

$$\therefore\ y = Ae^{-z} + Be^{4z} - \frac{1}{6}e^{2z} - \frac{1}{2}z + \frac{3}{8}$$

$$= Ae^{-\log x} + Be^{4\log x} - \frac{1}{6}e^{2\log x} - \frac{1}{2}\log x + \frac{3}{8}$$

$$y = \frac{A}{x} + Bx^4 - \frac{1}{6}x^2 - \frac{1}{2}\log x + \frac{3}{8}$$

5 (a) Solve $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

SOLUTION:

Given $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$ (i.e.) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 12\log x$

$\left[x^2D^2 + xD\right]y = 12\log x$..........(1)

Put $x = e^z \Rightarrow \log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

(1) $\Rightarrow \left[D'(D'-1)+D'\right]y = 12z$

$\left[D'^2\right]y = 12z$

The A.E. is $m^2 = 0$

$C.F. = (Az+B)e^{0z} = Az+B$

$P.I. = \frac{1}{D'^2}(12z) = 12\frac{1}{D'}\left(\frac{z^2}{2}\right) = 12\frac{z^3}{6} = 2z^3$

$\therefore\ y = (Az+B)+2z^3 = A\log x + B + 2(\log x)^3$

5(b) Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

SOLUTION:

Given $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$ (*i.e.*) $\left(x^2D^2 - 3xD + 4\right)y = (1+x)^2$ ....(1)

Put $x = e^z \Rightarrow \log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

(1) $\Rightarrow \left[D'(D'-1) - 3D' + 4\right]y = \left(1+e^z\right)^2$

$\left[D'^2 - D' - 3D' + 4\right]y = 1 + e^{2z} + 2e^z$

$\left[D'-2\right]^2 y = e^{0z} + e^{2z} + 2e^z$

The A.E. is $(m-2)^2 = 0$

$m = 2, 2$

$C.F. = (Az+B)e^{2z} = (A\log x + B)x^2$

$P.I_1 = \frac{1}{(D'-2)^2}e^{0z} = \frac{1}{4}e^{0z} = \frac{1}{4}$

$P.I_2 = \frac{1}{(D'-2)^2}e^{2z}$

$= \frac{1}{(2-2)^2}e^{2z}$

$= z\frac{1}{2(D'-2)}e^{2z}$

$$=\frac{z}{2}\frac{1}{2-2}e^{2z}$$

$$=\frac{z^2}{2}e^{2z}=\frac{(\log x)^2 x^2}{2}$$

$$P.I_3=\frac{1}{(D'-2)^2}2e^z=2e^z\frac{1}{(1-2)^2}=2e^z=2x$$

$$\therefore\quad y=C.F+P.I_1+P.I_2+P.I_3=(A\log x+B)x^2+\frac{1}{4}+\frac{(\log x)^2 x^2}{2}+2x$$

6 (a) Solve $(3x+2)^2\, y''+3(3x+2)\,y'-36y=3x^2+4x+1$

SOLUTION:

Given $(3x+2)^2\, y''+3(3x+2)\,y'-36y=3x^2+4x+1$

Put $3x+2=e^z \Rightarrow \log(3x+2)=z$

$$x=\frac{e^z}{3}-\frac{2}{3}$$

Let $(3x+2)D=3D'$

$(3x+2)^2 D^2=9D'(D'-1)$

$$\left[9D'(D'-1)+3(3D')-36\right]y=3\left[\frac{e^z}{3}-\frac{2}{3}\right]^2+4\left[\frac{e^z}{3}-\frac{2}{3}\right]+1$$

$$\left[9D'^2-9D'+9D'-36\right]y=3\left[\frac{e^{2z}}{9}+\frac{4}{9}-\frac{4}{9}e^z\right]+\frac{4}{3}e^z-\frac{8}{3}+1$$

$$\left[9D'^2-36\right]y=\frac{e^{2z}}{3}-\frac{1}{3}$$

$$\left[D'^2-4\right]y=\frac{1}{27}e^{2z}-\frac{1}{27}$$

The A.E. is $m^2-4=0$

$m=\pm 2$

$$C.F.=Ae^{2z}+Be^{-2z}=A(3x+2)^2+B(3x+2)^{-2}$$

$$P.I_1=\frac{1}{D'^2-4}\left(\frac{e^{2z}}{27}\right)=\frac{1}{27}\frac{1}{4-4}e^{2z}=\frac{1}{27}z\frac{1}{2D'}e^{2z}=\frac{z}{54}\frac{e^{2z}}{2}=\frac{ze^{2z}}{108}=\frac{\log(3x+2)}{108}(3x+2)^2$$

$$P.I_2=\frac{1}{D'^2-4}\left(\frac{e^{0z}}{27}\right)=-\frac{1}{108}$$

$$y=C.F+P.I_1-P.I_2=A(3x+2)^2+B(3x+2)^{-2}+\frac{\log(3x+2)}{108}(3x+2)^2+\frac{1}{108}$$

6(b) **Solve** $\left(1+x\right)^2 \frac{d^2y}{dx^2}+\left(1+x\right)\frac{dy}{dx}+y=2\sin\left[\log\left(1+x\right)\right]$

SOLUTION:

$$\left(1+x\right)^2 \frac{d^2y}{dx^2}+\left(1+x\right)\frac{dy}{dx}+y=2\sin\left[\log\left(1+x\right)\right]$$

Put $1+x=e^z$

$$z=\log\left(1+x\right)$$

Then $\left(x+1\right)D=D'$

$$\left(x+1\right)^2 D^2=D'\left(D'-1\right)$$

$$\therefore \quad \left[D'\left(D'-1\right)+D'+1\right]y=2\sin z$$

$$\left[D'^2+1\right]y=2\sin z$$

The A.E. is $m^2+1=0$

$$m=\pm i$$

$$C.F.=A\cos z+B\sin z=A\cos\left[\log\left(1+x\right)\right]+B\sin\left[\log\left(1+x\right)\right]$$

$$P.I.=\frac{1}{D'^2+1}2\sin z=2\frac{1}{D'^2+1}\sin z=2\frac{1}{-1+1}\sin z$$

$$=2z\frac{1}{2D'}\sin z=z\frac{1}{D'}\sin z=z\left(-\cos z\right)=-\log\left(1+x\right)\cos\left[\log\left(1+x\right)\right]$$

$$\therefore \quad y=C.F.+P.I$$

$$=A\cos\left[\log\left(1+x\right)\right]+B\sin\left[\log\left(1+x\right)\right]-\log\left(1+x\right)\cos\left[\log\left(1+x\right)\right]$$

7 (a) **Solve** $\left(2+x\right)^2 \frac{d^2y}{dx^2}-\left(2+x\right)\frac{dy}{dx}+y=2+x$

SOLUTION:

Put $2+x=e^z \Rightarrow \log\left(2+x\right)=z$

$$x=e^z-2$$

Let $\left(2+x\right)D=D'$

$$\left(2+x\right)^2 D^2=D'\left(D'-1\right)$$

Then

$$\left[D'\left(D'-1\right)-D'+1\right]y=e^z$$

$$\left[D'^2-2D'+1\right]y=e^z$$

$$\left[D'-1\right]^2 y=e^z$$

The A.E. is

$$(m-1)^2=0$$

$$m=1,1$$

$$\text{C.F.}=(\text{Az+B})e^z=(A\log(2+x)+B)(2+x)$$

$$P.I.=\frac{1}{[D'-1]^2}e^z=z\frac{1}{2D'}e^z=\frac{ze^z}{2}=\frac{1}{2}[\log(2+x)](2+x)$$

$$y=(A\log(2+x)+B)(2+x)+\frac{1}{2}[\log(2+x)](2+x)$$

7 (b) Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+4y=\sec 2x$

SOLUTION:

Given $(D^2+4)y=\sec 2x$

The A.E. is

$m^2+4=0 \qquad m=\pm 2i$

$C.F.=c_1\cos 2x+c_2\sin 2x$

$f_1=\cos 2x,\quad f_2=\sin 2x$

$f_1'=-2\sin 2x,\quad f_2'=2\cos 2x$

$f_1f_2'-f_1'f_2=2(\cos^2 2x+\sin^2 2x)=2$

$$P=-\int\frac{f_2X}{f_1f_2'-f_1'f_2}dx=-\int\frac{\sin 2x\sec 2x}{2}dx=-\frac{1}{2}\int\tan 2x\,dx=-\frac{1}{4}\log[\sec 2x]$$

$$Q=\int\frac{f_1X}{f_1f_2'-f_1'f_2}dx=\int\frac{\cos 2x\sec 2x}{2}dx=\frac{1}{2}\int dx=\frac{x}{2}$$

$$P.I.=f_1P+f_2Q=-\frac{1}{4}\cos 2x\log[\sec 2x]+\sin 2x\left(\frac{x}{2}\right)$$

$$y=c_1\cos 2x+c_2\sin 2x-\frac{1}{4}\cos 2x\log[\sec 2x]+\sin 2x\left(\frac{x}{2}\right)$$

8(a) Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+y=\tan x$

SOLUTION:

Given $(D^2+1)y=\tan x$

The A.E. is $m^2+1=0$

$m=\pm i$

$C.F.=c_1\cos x+c_2\sin x$

$f_1=\cos x,\quad f_2=\sin x$

$f_1'=-\sin x,\quad f_2'=\cos x$

$f_1f_2'-f_1'f_2=\cos^2 x+\sin^2 x=1$

$$P=-\int \frac{f_2 X}{f_1 f_2'-f_1' f_2}dx=-\int \frac{\sin x\ \tan x}{1}dx=-\int \frac{\sin^2 x}{\cos x}dx=-\int \frac{1-\cos^2 x}{\cos x}dx$$

$$=-\int(\sec x-\cos x)\,dx=-\log(\sec x+\tan x)+\sin x$$

$$Q=\int \frac{f_1 X}{f_1 f_2'-f_1' f_2}dx=\int \frac{\cos x\ \tan x}{1}dx=\int \sin x\,dx=-\cos x$$

$$P.I.=f_1P+f_2Q=\cos x\left[-\log(\sec x+\tan x)+\sin x\right]-\sin x.\cos x$$

$$y=c_1\cos x+c_2\sin x+\cos x\left[-\log(\sec x+\tan x)+\sin x\right]-\sin x.\cos x$$

8 (b) Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+y=x\sin x$

SOLUTION:

Given $\left(D^2+1\right)y=x\sin x$

The A.E. is

$m^2+1=0$

$m=\pm i$

$C.F.=c_1\cos x+c_2\sin x$

$f_1=\cos x,\quad f_2=\sin x$

$f_1'=-\sin x,\quad f_2'=\cos x$

$f_1 f_2'-f_1' f_2=\cos^2 x+\sin^2 x=1$

$$P=-\int \frac{f_2 X}{f_1 f_2'-f_1' f_2}dx=-\int \frac{\sin x\ x\sin x}{1}dx=-\int x\sin^2 x\,dx=-\int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=-\frac{1}{2}\int(x-x\cos 2x)\,dx=-\frac{1}{2}\left[\frac{x^2}{2}\right]+\frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right)-(1)\left(\frac{-\cos 2x}{4}\right)\right]$$

$$=-\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x$$

$$Q=\int \frac{f_1 X}{f_1 f_2'-f_1' f_2}dx=\int \frac{\cos x\ x\sin x}{1}dx=\frac{1}{2}\int x\sin 2x\,dx$$

$$=\frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]=-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x$$

$$P.I.=f_1P+f_2Q=\cos x\left[-\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x\right]+\sin x\left[-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x\right]$$

$$y=c_1\cos x+c_2\sin x+\cos x\left[-\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x\right]+\sin x\left[-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x\right]$$