Unit I - Fourier series

Fourier series in an interval of length 2ℓ

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)}$$

Fourier series of f(x) in $(0,2\ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{1}{\ell}\int_0^{2\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Fourier series of f(x) in $(-\ell,\ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Even Function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = 0$$

Odd Function

$$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi x}{\ell}\right)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Convergence of Fourier Series:

- At a continuous point x = a, Fourier series converges to f(a)
- At end point c or c+2*l* in (c, c+2*l*), Fourier series converges to $\dfrac{f(c)+f(c+2\ell)}{2}$
- At a discontinuous point x = a, Fourier series converges to $\dfrac{f(a-)+f(a+)}{2}$

Fourier series in the Interval of length 2π

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

Fourier Series of f(x) in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Fourier Series of f(x) in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Even Function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Odd Function

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin nx\right)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Half Range Fourier series

Fourier Cosine Series | Fourier Sine Series

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{\ell n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{2}{\ell}\int_0 f(x)\,dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi x}{\ell}\right)$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Convergence of Fourier Cosine series:

- At a continuous point x = a, Fourier cosine series converges to f(a).
- At end point 0 in(0,*l*), Fourier cosine series converges to f(0+)
- At end point *l* in(0,*l*), Fourier cosine series converges to f(*l*-)

Convergence of Fourier Sine series:

- At a continuous point x = a, Fourier Sine series converges to f(a).
- At both end points Fourier Sine series converges to 0.

Harmonic Analysis:

$$a_0 = 2\left[\frac{\sum y}{N}\right], \quad a_n = 2\left[\frac{\sum y\cos\left(\frac{n\pi x}{\ell}\right)}{N}\right], \quad b_n = 2\left[\frac{\sum y\sin\left(\frac{n\pi x}{\ell}\right)}{N}\right]$$

Parseval's Theorem:

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{\ell} + b_n\sin\frac{n\pi x}{\ell}\right)$ is the Fourier series of f(x) in (c, c+2*l*),

Then $\overline{y}^2 = \frac{a_0^{\,2}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})$ (or) $\frac{1}{2\ell}\int_c^{c+2\ell}[f(x)]^2\,dx = \frac{a_0^{\,2}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})$

Root Mean Square Value:

$\overline{y}^2$ is the effective value (or) Root Mean square (RMS) value of the function y = f(x), which is given by

$$\overline{y} = \sqrt{\frac{\int_c^{c+2\ell}[f(x)]^2\,dx}{2\ell}}$$

Some Important Results:

1. Sin nπ =0 for all integer values of n
2. Cos nπ= $(-1)^n$ for all integer values of n
3. Cos2nπ=1 for all integer values of n
4. Sin2nπ = 0 for all integer values of n
5. If f(–x) = f(x) then f(x) is even and If f(–x) = – f(x) then f(x) is odd.

6. $f(x) = \begin{cases} \varphi_1(x) & (-\ell,0) \\ \varphi_2(x) & (0,\ell) \end{cases}$ is even if either $\varphi_1(-x) = \varphi_2(x)$ or $\varphi_2(-x) = \varphi_1(x)$

7. $f(x) = \begin{cases} \varphi_1(x) & (-\ell,0) \\ \varphi_2(x) & (0,\ell) \end{cases}$ is odd if either $\varphi_1(-x) = -\varphi_2(x)$ or $\varphi_2(-x) = -\varphi_1(x)$

8. $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

9. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

10. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

11. $\int u\,dv = uv_1 - u'v_2 + u''v_3 - \ldots\ldots\ldots$ Where

$u' = \frac{du}{dx},\ u'' = \frac{d^2u}{dx^2},\ \ldots\ldots\ v_1 = \int dv,\ v_2 = \int v_1\,dx,\ v_3 = \int v_2\,dx \ldots\ldots\ldots\ldots$