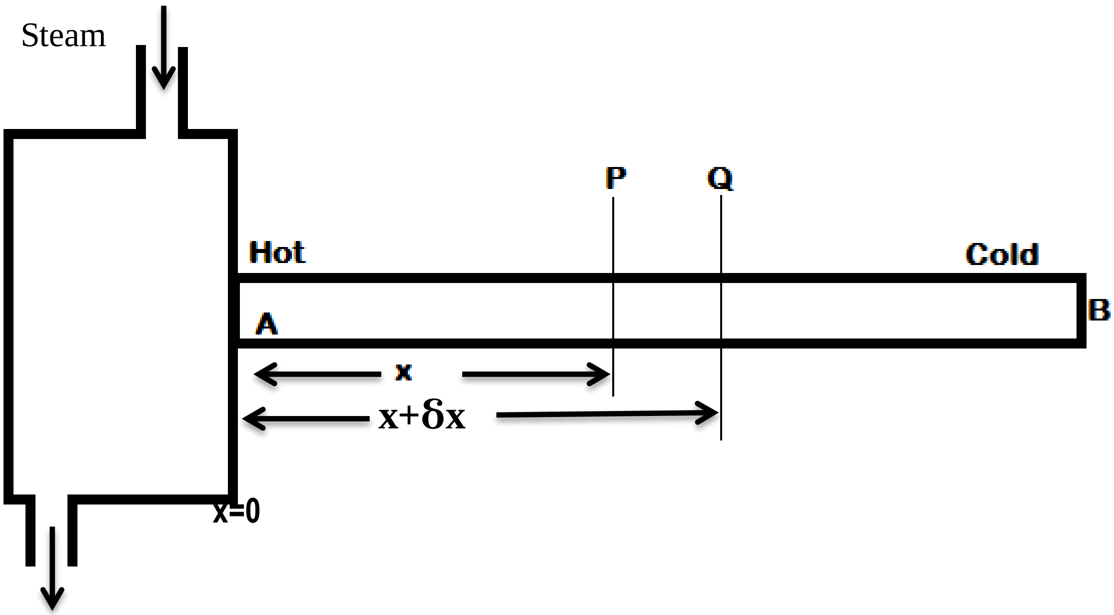
RECTILINEAR FLOW OF HEAT THROUGH A ROD

Consider a long rod AB of uniform cross section heated at one end A as shown in figure.

Then there is flow of heat along the length of the bar and heat is also radiated from its surface. B is the cold end.



consider the flow of heat between the sections P and Q at distance x and $x+\delta x$ from the hot end.

Excess temperature at section P above the surroundings = θ

Temperature gradient at section P = $\frac{d\theta}{dx}$

Excess temperature at section Q = $\theta + \frac{d\theta}{dx}\boldsymbol{\delta x}$

∴ Temperature gradient at Q = $\frac{d}{dx}\left(\theta + \frac{d\theta}{dx}\boldsymbol{\delta x}\right)$

$$= \frac{d\theta}{dx} + \frac{d^2\theta}{dx^2}\boldsymbol{\delta x}$$

Heat entering throught P in one second

$$Q_1 = -KA\,\frac{d\theta}{dx}$$

Heat leaving through Q in one second

$$Q_2 = -KA\left(\frac{d\theta}{dx} + \frac{d^2\theta}{dx^2}\boldsymbol{\delta x}\right)$$

$$Q_2 = -\,KA\,\frac{d\theta}{dx} - KA\,\frac{d^2\theta}{dx^2}\boldsymbol{\delta x}$$

$\therefore$ Net gain of heat by element $\boldsymbol{\delta x}$ in one second

$$Q = Q_1 - Q_2$$

$$Q = -KA\,\frac{d\theta}{dx} - \left(-\,KA\,\frac{d\theta}{dx} - KA\,\frac{d^2\theta}{dx^2}\boldsymbol{\delta x}\right)$$

$$\therefore \quad Q = KA\,\frac{d^2\theta}{dx^2}\boldsymbol{\delta x} \qquad \textbf{(1)}$$

<u>Before the steady state is reached</u>

Before the steady state is reached, the amount of heat Q is used in two ways. A part of the heat is used in raising the temperature of the rod and the remaining heat is lost by radiation from the surface.

Heat absorbed / second to raise the temperature of the rod

$$= \text{mass x specific heat capacity x } \frac{d\theta}{dx}$$

$$= (A \text{ x } \boldsymbol{\delta x})\,\rho \text{ x } S \text{ x } \frac{d\theta}{dt} \qquad (2)$$

Heat lost / second due to radiation

$$= E\,p\,\boldsymbol{\delta x}\,\boldsymbol{\theta} \qquad (3)$$

Amount of heat (Q) = Amount of heat absorbed + Amount of heat lost

$$Q = (A \text{ x } \boldsymbol{\delta x})\,\rho \text{ x } S \text{ x } \frac{d\theta}{dt} + E\,p\,\boldsymbol{\delta x}\,\boldsymbol{\theta} \qquad (4)$$

Compare the eqns (1) and (2)

$$KA\ \frac{d^2\theta}{dx^2}\boldsymbol{\delta x} = (\mathrm{A\ x}\ \boldsymbol{\delta x})\ \rho\ \mathrm{x\ S\ x}\ \frac{d\theta}{dt} + \mathrm{E\ p}\ \boldsymbol{\delta x}\ \boldsymbol{\theta}$$

Dividing both LHS and RHS of the above equation by KA $\boldsymbol{\delta x}$**,** we have,

$$\frac{d^2\theta}{dx^2} = \frac{\rho\ S}{K}\frac{d\theta}{dt} + \frac{E\ p}{KA}\ \boldsymbol{\theta} \qquad \textbf{(5)}$$

This equation is the standard differential equation for the flow of heat through the rod.

Special cases:-

Case – 1: when heat lost by radiation is negligible.

If the rod is completely covered by insulating materials, then there is no loss of heat due to radiation.

Hence E p $\boldsymbol{\delta x}\ \boldsymbol{\theta} =$ 0

The total heat gained by the rod is completely used to raise the temperature of the rod.

From eqn (5)

$$\frac{d^2\theta}{dx^2} = \frac{\rho\ S}{K}\frac{d\theta}{dt} = \frac{1}{h}\frac{d\theta}{dt}$$

Here h = $\frac{K}{\rho\ S}$ is thermal diffusivity of the rod.

Case – 2: After the steady state is reached.

After the steady state is reached, there is no raise of temperature

Hence $\frac{d\theta}{dt} = 0$,

From eqn (5)

$$\frac{d^2\theta}{dx^2} = \frac{E\ p}{KA}\ \boldsymbol{\theta} \qquad \text{taking } \boldsymbol{\mu^2} = \frac{E\ p}{KA}\ \boldsymbol{\theta}$$

$$\frac{d^2\theta}{dx^2} = \boldsymbol{\mu^2}\ \boldsymbol{\theta} \tag{6}$$

It is a second order differential equation

The general solution of this equation is

$$\boldsymbol{\theta = A\ e^{+\mu x} + Be^{-\mu x}} \tag{7}$$

A & b are arbitrary constants.

Suppose the bar is of infinite length,

Excess temperature above the surrounding of the rod of the hot end = θ_0

Temperature at the cold end = 0

First boundary condition is at $x = 0, \theta = \theta_0$

Eqn (7) becomes

θ_0 = A + B

Second boundary condition is at $x = \infty$, $\theta = 0$,

Eqn (7) becomes

$0 = \boldsymbol{A\ e^{\infty}}$

A = 0, because $\boldsymbol{e^{\infty}} \neq \boldsymbol{0}$

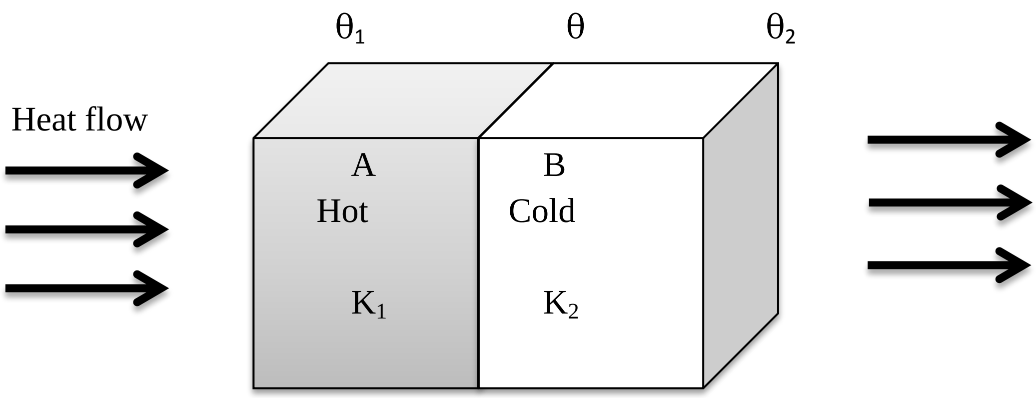
Then θ_0 = B

$\therefore \theta = \theta_0 \boldsymbol{e^{-\mu x}}$

This above equation represents the excess temperature of a point at a distance x from the hot end after the steady state is reached and it exponentially falls from hot end.

HEAT CONDUCTION THROUGH A COMPOUND MEDIA (SERIES AND PARALLEL)

Consider a composite slab of two different materials, A & B of thermal conductivity K_1 & K_2 respectively. Let the thickness of these two layers A & B be d_1 and d_2 respectively



Let the temperature of the end faces be θ_1 & θ_2 and temperature at the contact surface be θ, which is unknown. Heat will flow from A to B through the surface of contact only if $\theta_1 > \theta_2$. After steady state is reached heat flowing per second (Q) through every layer is same. A is the area of cross section of both layers

Amount of heat flowing per sec through A Q = $(K_1\, A\, (\theta_1 - \theta\,))/d_1$ (1)

Amount of heat flowing per sec through B Q = $(K_2\, A\, (\theta - \theta_2\,))/d_2$ (2)

Since the amount of heat flowing through A and B are equal,

Equation (1) = Equation (2)

$$(K_1\, A\, (\theta_1 - \theta\,))/d_1 = (K_2\, A\, (\theta - \theta_2\,))/d_2$$

$$\frac{K_1\theta_1}{d_1} - \frac{K_1\theta}{d_1} = \frac{K_2\theta}{d_2} - \frac{K_2\theta_2}{d_2}$$

Rearranging and we get,

$$\frac{K_1\theta}{d_1} + \frac{K_2\theta}{d_2} = \frac{K_2\theta_2}{d_2} + \frac{K_1\theta_1}{d_1}$$

$$\therefore \theta = \frac{\frac{K_1\,\theta_1}{d_1} + \frac{K_2\,\theta_2}{d_2}}{\frac{K_1}{d_1} + \frac{K_2}{d_2}} \qquad (3)$$

By substituting the values of θ in eqn (1), we get,

$$Q = \frac{K_1\,A\left(\theta_1 - \frac{\frac{K_1\,\theta_1}{d_1} + \frac{K_2\,\theta_2}{d_2}}{\frac{K_1}{d_1} + \frac{K_2}{d_2}}\right)}{d_1}$$

or $$Q = \frac{K1K2A(\theta1 - \theta2)}{K1d2+K2d1}$$

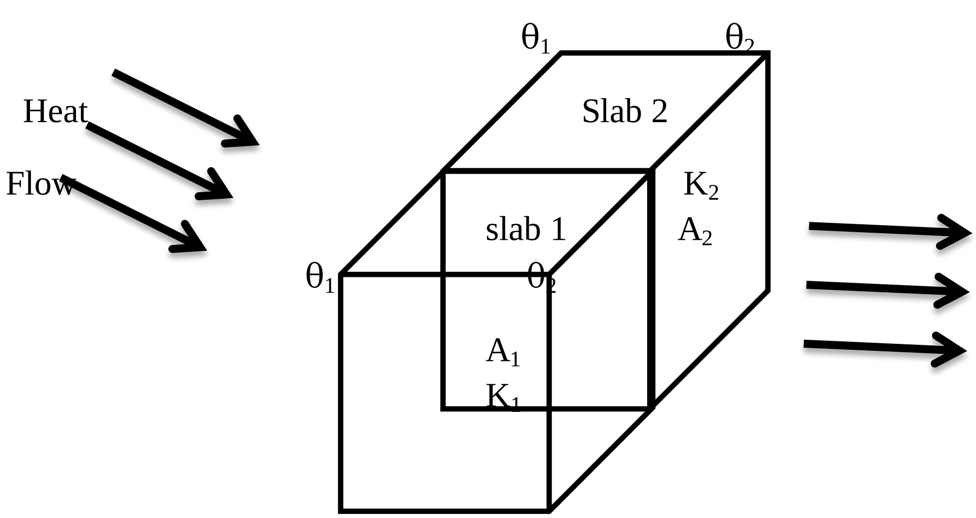
finally we get,

$$Q = \frac{A\,(\theta_1 - \theta_2)}{\frac{d_1}{K_1} + \frac{d_2}{K_2}} \qquad (4)$$

The above equation gives the amount of heat conducted by two layers in series.

(ii) <u>BODIES IN PARALLEL</u>

Consider a compound wall of two different materials A and B of thermal conductivities K_1 and K_2 and of thickness d_1 and d_2 respectively. These two material layers are arranged in parallel.



The opposite faces of the material are kept at temperatures θ_1 & θ_2 and A_1 & A_2 be the areas of cross-section of the materials.

Then

The amount of heat flowing through the first slab $Q_1 = \frac{K_1 A_1 (\theta_1 - \theta_2)}{d_1}$

The amount of heat flowing through the second slab $Q_2 = \frac{K_2 A_2 (\theta_1 - \theta_2)}{d_2}$

The total heat flowing through these two slabs per second $Q = Q_1 + Q_2$

$$Q = \frac{K_1 A_1 (\theta_1 - \theta_2)}{d_1} + \frac{K_2 A_2 (\theta_1 - \theta_2)}{d_2}$$

$$Q = \left(\frac{K_1 A_1}{d_1} + \frac{K_2 A_2}{d_2}\right)(\theta_1 - \theta_2)$$

the above equation gives us the amount of heat flowing through compound wall of two layers in parallel.

RADIAL FLOW OF HEAT

In this method heat flows from the inner side towards the other side along the radius of the cylindrical shell

CYLINDRICAL SHELL METHOD

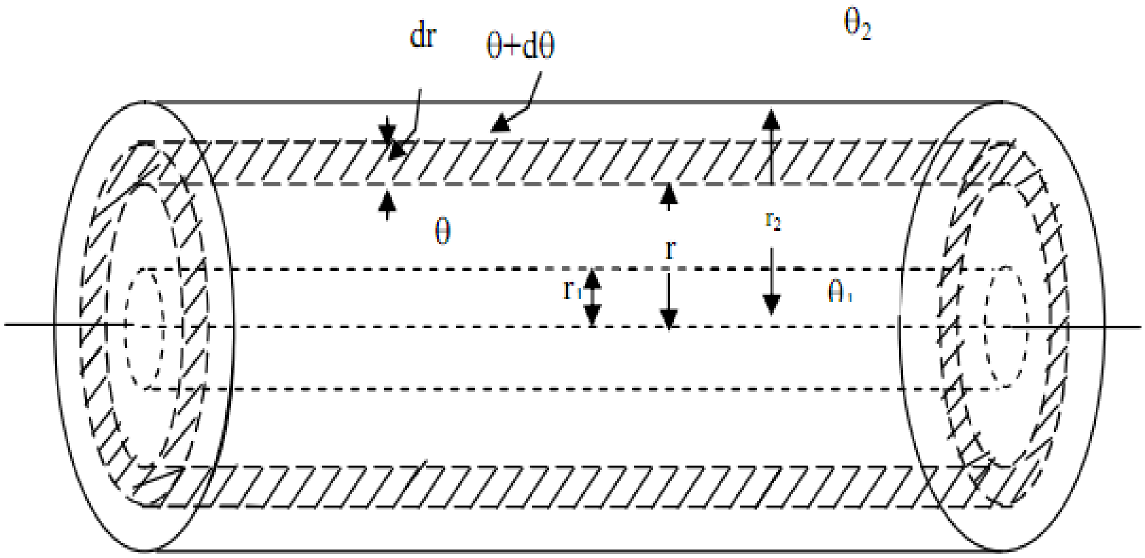
Consider a cylindrical tube of length l, inner radius r_1 and outer radius r_2. The tube carries steam or some hot liquid. After the steady state is reached, the temperature on the inner surface is θ_1 and on the outer surface is θ_2 in such a way $\theta_1 > \theta_2$. Heat is conducted radially across the wall of the tube. Consider an element of thickness dr and length l at a distance r from the axis.

Amount of heat flowing per second through this element

$$Q = -KA\frac{d\theta}{dr} \qquad (1)$$

Here $A = 2\pi rl$

$$Q = -K\,2\pi r l\,\frac{d\theta}{dr} \qquad (2)$$



After steady state is reached, the amount of heat flowing (Q) through all the imaginary cylinders is same.

Rearranging the equation (2), we get

$$\frac{dr}{r} = -K\frac{2\pi\, l}{Q} \qquad (3)$$

$$Q\int_{r_1}^{r_2}\frac{dr}{r} = 2\pi Kl\int_{\theta_1}^{\theta_2} d\theta$$

$$Q\left[\log_e \frac{r_2}{{}_1 1}\right] = -2\pi Kl\left[\theta_1 - \theta_{{}_1}\right]$$

$$Q\left(\log_e \frac{r_2}{r_1}\right) = -2\pi Kl\left(\theta_2 - \theta_1\right)$$

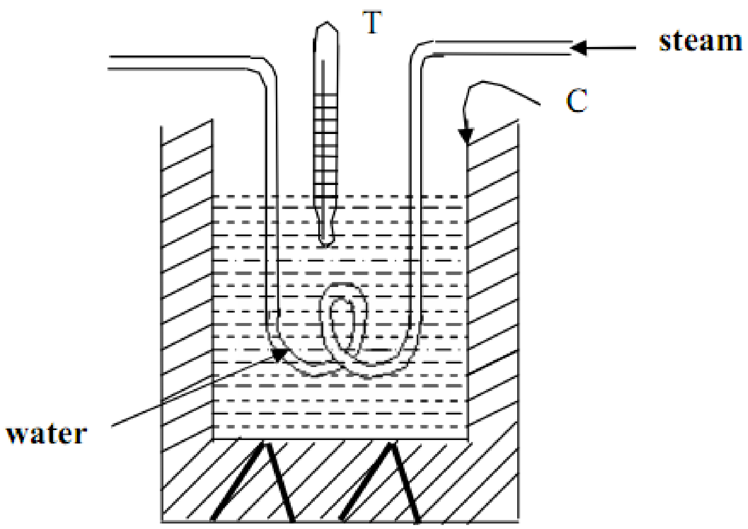
$$K = \frac{Q \log_e \frac{r_2}{r_1}}{2\pi(\theta_1 - \theta_2)}$$

$$K = \frac{Q \times 2.3026 \times \log_{10} \frac{r_2}{r_1}}{2\pi l(\theta_1 - \theta_2)}$$

DETERMINATION OF THERMAL CONDUCTIVITY OF RUBBER

It is based on the principle of radial flow of heat through a cylindrical shell.

A known quantity of water is taken in calorimeter C. A rubber tubing whose inner and outer radii arc r_1 and r_2 is taken and a known length (50 cm) of it is immersed in water as shown in the above figure. The initial temperature of water is noted. Let it be θ_1. Steam is passed through the rubber tubing for a known time t seconds. Let the final temperature of water be θ_2 after applying radiation correction.



K of rubber:

Let the temperature of steam is θ_3. The average temperature on the outer surface of rubber tubing is, $\theta_4 = \left(\frac{\theta_1 + \theta_2}{2}\right)$

Calculation:

Let the mass of water = m.

Water equivalent of the calorimeter (ws)= W

Rise in temperature $(\theta_2 - \theta_1)$

Heat gained by water = $(m+w)(\theta_2 - \theta_1)$

Quantity of heat flowing per second,

$$Q = \frac{(m+w)(\theta_2 - \theta_1)}{t}$$

$$K = \frac{Q \times 2.3026 \times \log_{10} \frac{r_2}{r_1}}{2\pi l(\theta_1 - \theta_2)}$$

Substituting the values of Q, we get

$$K = \frac{(m+w)(\theta_2 - \theta_1) \times \frac{r_2}{r_1} \times 2.3026 \times \log_{10} \frac{r_2}{r1}}{2\pi l\left(\theta_3 - \frac{\theta_1 + \theta_2}{2}\right)t}$$

Thus K for rubber calculated.