# St. JOSEPH'S COLLEGE OF ENGINEERING, CHENNAI-119.
# St. JOSEPH'S INSTITUTE OF TECHNOLOGY, CHENNAI-119.

**COURSE: B.E./B.TECH ( COMMON TO ALL BRANCHES) - FIRST SEMESTER**

**MA6151/MATHEMATICS – I**
**UNIT –V MULTIPLE INTEGRALS**

## INTRODUCTION

Integration can be defined as the reverse process of differentiation or as the limit of a sum. Let the function f(x) be defined on a closed interval [a, b] and is piecewise continuous there. Then the Riemann integral $\int_a^b f(x)dx$ is defined as

$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \text{Max}\,\lambda_i \to 0}} \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$, where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition of [a, b] into closed subintervals, $s_i \in [x_{i-1}, x_i]$ and $\lambda_i = |x_i - x_{i-1}|$.

The extensions of Riemann integral to two and three dimensions are called double and triple integrals or in general multiple integrals. In this chapter, we discuss the various methods of evaluating the double integrals $\iint_R f(x, y)\, dx\, dy$ and the triple integrals $\iiint_V f(x, y, z)\, dx\, dy\, dz$, when f is a function, continuous inside and on the boundary of the region R or V. One can also think of the multiple integral in the n-dimensional space $\mathbb{R}^n$, denoted by $\iint_V \cdots \int f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n$

## DOUBLE INTEGRALS

### Definition

Let f(x, y) be a continuous function in the closed and bounded region R in the two dimensional space $\mathbb{R}^2$. Let the region R be subdivided in any manner into n subregions $\Delta R_1, \Delta R_2, \ldots, \Delta R_n$ of areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. Let $(\varsigma_i, \eta_i)$ be any point in the $i^{th}$ subregion $\Delta R_i$. Then the **double integral** of f(x, y) over the region R is defined as $\lim_{\substack{n\to\infty \\ \text{Max}\,|\Delta A_i| \to 0}} \sum_{i=1}^{n} f(\varsigma_i, \eta_i)\,\Delta A_i$ and is denoted by $\iint_R f(x, y)\, dA$ or $\iint_R f(x, y)\, dx\, dy$

## APPLICATIONS OF DOUBLE INTEGRALS

1. **Area of a region.**

   If f(x, y) = 1 then $\iint_R dx\, dy = \lim_{\substack{n\to\infty \\ \text{Max}\,|\Delta A_i| \to 0}} \sum_{i=1}^{n} \Delta A_i = A,$ the area of the region R.

2. **Volume under a surface.**

   If z = f(x, y) is a surface in $\mathbb{R}^3$ and z is non-negative over the region R then

$\iint_R f(x, y)\, dx\, dy = \iint_R z\, dx\, dy$ gives the volume of the region below the surface and above the region R in the xy-plane, and bounded by the cylinder $0 \le z \le f(x, y), (x, y) \in \partial R$ ,the boundary of the region R.

**3. Average.**

$\frac{1}{A}\iint_R f(x, y)\, dx\, dy$ gives the average value of f(x, y) over R, where A is the area of the region R.

**4. Centre of gravity.**

Let f(x, y) = ρ(x, y) be the density function (mass per unit area) of a distribution of mass in the region R in the xy-plane. Then

(i) $M = \iint_R f(x, y)\, dx\, dy$ gives the **total mass** of R.

(ii) $\bar{x} = \frac{1}{M}\iint_R x.f(x, y)\, dx\, dy,\ \bar{y} = \frac{1}{M}\iint_R y.f(x, y)\, dx\, dy$ gives the coordinates of the **centre of gravity** $(\bar{x}, \bar{y})$ of the mass M in R.

**5.Moment of Inertia.**

$I_x = \iint_R y^2.f(x, y)\, dx\, dy$ and $I_y = \iint_R x^2.f(x, y)\, dx\, dy$ gives the **moment of inertia** of the mass M in R about the x and y axes respectively. $I_0 = I_x + I_y$ is the moment of inertia of the mass about the origin.

## EVALUATION OF DOUBLE INTEGRALS

Let f(x, y) be a function defined and continuous in a region R in the xy-plane.

To evaluate the double integral $I = \iint_R f(x, y)\, dx\, dy,$ (1)

one has to perform two successive integrations. We consider the following three cases.

**Case 1. All the limits are constants.**

$$R = \{ (x, y) / a \le x \le b, c \le y \le d \}$$

R is a rectangular region with sides parallel to the coordinate axes. For any fixed $x \in [a, b]$, consider the integral $\int_c^d f(x, y)\, dy$ . The value of this integral is a function of x and hence can be integrated w.r.t x and we get $\int_a^b \left[ \int_c^d f(x, y)\, dy \right] dx$ (2).

Here, we first integrate along the vertical strip PQ (Fig. 1.1) and then slide it from AD to BC parallel to the x-axis. Similarly we can define $\int_c^d \left[ \int_a^b f(x, y)\, dx \right] dy$ (3)

i.e., we first integrate along the horizontal strip P′Q′ and then slide it from AB to DC, parallel to the y-axis. The integrals (2) and (3) are called **iterated integrals.** Both the integrals give the same value. Thus for double integrals over a rectangular region with sides parallel to the coordinate axes or equivalently, for double integrals in which both pairs of limits are constants, it does not matter whether we first integrate w.r.t x and then w.r.t y or vice versa.

**Case 2. Limits for x are constants.**

R = {(x, y) / $\varphi(x) \le y \le \chi(x)$, $a \le x \le b$} where $\varphi(x)$ and $\chi(x)$ are continuous functions such that $\varphi(x) \le \chi(x)$ for all $x \in [a, b]$. (Fig. 5.2) Then

$$I = \int_{x=a}^{b}\left[\int_{y=\varphi(x)}^{\chi(x)} f(x,y)\,dy\right]dx \qquad (4)$$

While evaluating the inner integral in (4), x is treated as a constant. The iterated integral in the right hand side of (4) is also written as $\int_a^b\int_{\varphi(x)}^{\chi(x)} f(x,y)\,dy\,dx$ or $\int_a^b dx \int_{\varphi(x)}^{\chi(x)} f(x,y)\,dy$.

**Case 3. Limits for y are constants.**

R = {(x, y) / $\varphi(y) \le x \le \chi(y)$, $c \le y \le d$} where $\varphi(y)$ and $\chi(y)$ are continuous functions such that $\varphi(y) \le \chi(y)$ for all $y \in [c, d]$. (Fig. 5.3) Then

$$I = \int_{y=c}^{d}\left[\int_{x=\varphi(y)}^{\chi(y)} f(x,y)\,dx\right]dy \qquad (5)$$

While evaluating the inner integral in (5), y is treated as a constant. The iterated integral in the right hand side of (5) can also be written as $\int_c^d\int_{\varphi(y)}^{\chi(y)} f(x,y)\,dx\,dy$ or $\int_c^d dy \int_{\varphi(y)}^{\chi(y)} f(x,y)\,dx$.

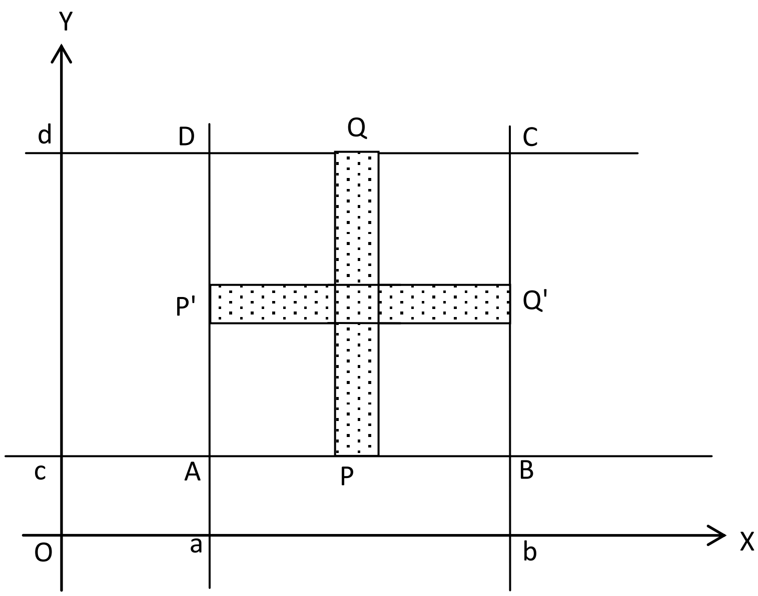



Fig. 5.1

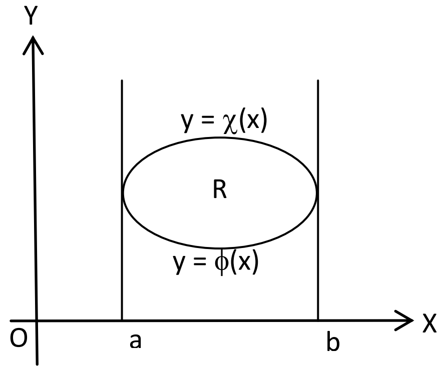



Fig. 5.2

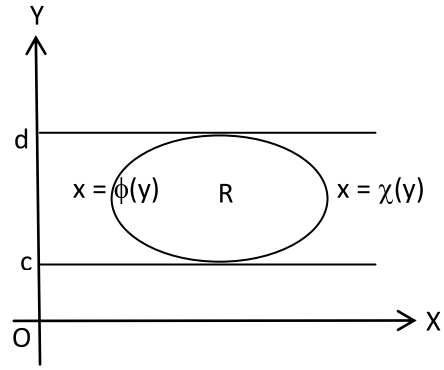



Fig. 5.3

If the region of integration R is such that it cannot be written in one of the forms discussed above, then we divide R into a finite number of subregions $R_1, R_2, \ldots, R_m$ such that each of these subregions can be represented in one of the above forms and we get the double integral over R by adding the integrals over these subregions.

i.e., $$\iint_R f(x, y)\, dx\, dy = \sum_{i=1}^{m} \left[ \iint_{R_i} f(x, y)\, dx\, dy \right] \qquad (6)$$

For example, in Fig 5.4, $R = R_1 \cup R_2$ is a partition of R and we have

$$\iint_R f(x, y)\, dx\, dy = \iint_{R_1} f(x, y)\, dx\, dy + \iint_{R_2} f(x, y)\, dx\, dy$$

$$= \int_a^c \left[ \int_{\varphi_1(x)}^{\chi(x)} f(x, y)\, dy \right] dx + \int_c^b \left[ \int_{\varphi_2(x)}^{\chi(x)} f(x, y)\, dy \right] dx$$

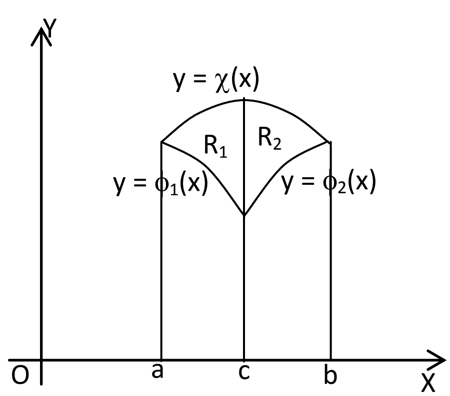



Figure 5.4

**DOUBLE INTEGRATION IN POLAR COORDINATES**

(i) To evaluate the double integral $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ where $r_1$ and $r_2$ are functions of $\theta$ and $\theta_1$, $\theta_2$ are constants, we first integrate w.r.t r between the limits $r = r_1$ and $r = r_2$ keeping $\theta$ fixed and the resulting expression is integrated w.r.t $\theta$ from $\theta_1$ to $\theta_2$. The order of integration is changed when $\theta_1$ and $\theta_2$ are functions of r and $r_1$, $r_2$ are constants.

(ii) The double integral in Cartesian coordinates $I = \iint_R f(x,y)\,dx\,dy$, can be transformed into a double integral in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$. Thus $I = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

**Note:** If A is the area of the region R enclosed by curves whose equations are in polar coordinates, then $A = \iint_R r\,dr\,d\theta$.

**1.** Evaluate the double integral $\int_0^3\int_1^2 xy(x+y)\,dx\,dy$

**Solution:**

Since all the limits are constants, order of integration is immaterial.

$$\int_0^3\int_1^2 xy(x+y)\,dx\,dy = \int_0^3\left[\int_1^2 (x^2y + xy^2)\,dx\right]dy$$

$$= \int_0^3\left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_1^2 dy$$

$$= \int_0^3\left[\frac{8}{3}y + 2y^2 - \frac{y}{3} - \frac{y^2}{2}\right]dy$$

$$= \int_0^3\left[\frac{7}{3}y + \frac{3y^2}{2}\right]dy$$

$$= \left[\frac{7}{6}y^2 + \frac{3}{6}y^3\right]_0^3$$

$$= \frac{7}{6}.9 + \frac{3}{6}.27 = 24$$

**2.** Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

**Solution:**

Since all the limits are constants, the order of integration is immaterial.

$$I = \int_0^{\pi/2}\left[\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\,dr\right]d\theta$$

$$= \int_0^{\pi/2} \left[ \frac{-1}{2} \frac{1}{r^2 + a^2} \right]_0^{\infty} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2a^2} d\theta$$

$$= \left[ \frac{1}{2a^2} \theta. \right]_0^{\pi/2} = \frac{\pi}{4a^2}$$

**3.** Evaluate $\int_0^a \int_0^{\sqrt{a^2 - x^2}} y^3 dy\, dx$ .

**Solution:**

Since the inner limits are functions of x, first integrate w.r.t. y.

$$I = \int_0^a \left[ \int_0^{\sqrt{a^2-x^2}} y^3 dy \right] dx$$

$$= \int_0^a \left[ \frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{1}{4} \left( a^2 - x^2 \right)^2 dx$$

$$= \frac{1}{4} \int_0^a \left( a^4 - 2a^2 x^2 + x^4 \right) dx$$

$$= \frac{1}{4} \left[ a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right]_0^a$$

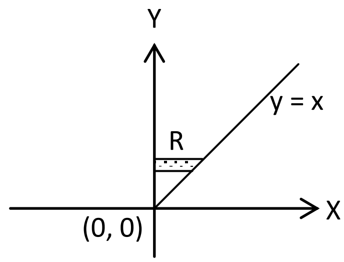
$$= \frac{1}{4} \left[ a^5 - \frac{2}{3} a^5 + \frac{a^5}{5} \right]$$

$$= \frac{2a^5}{15}$$

**4.** Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

**Solution:**

The region of integration is bounded by the lines y = x and x = 0. The evaluation of the double integral will be easier if we first integrate w.r.t x. Hence change the order of integration.




$$I = \int_{y=0}^{\infty}\left[\int_{x=0}^{y} \frac{e^{-y}}{y} dx\right] dy$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y}.x\right]_0^y dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$= 1$

**5.** Evaluate $\int_0^{2\pi} \int_{a \sin\theta}^{a} r\, dr\, d\theta$.

**Solution:**

Since the inner limits are functions of θ, first integrate w.r.t r.

$$I = \int_0^{2\pi}\left[\int_{a\sin\theta}^{a} r\, dr\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{a\sin\theta}^{a} d\theta = \int_0^{2\pi}\left(\frac{a^2}{2} - \frac{a^2}{2}\sin^2\theta\right) d\theta = \frac{a^2}{2}\int_0^{2\pi}\left(1 - \sin^2\theta\right) d\theta$$

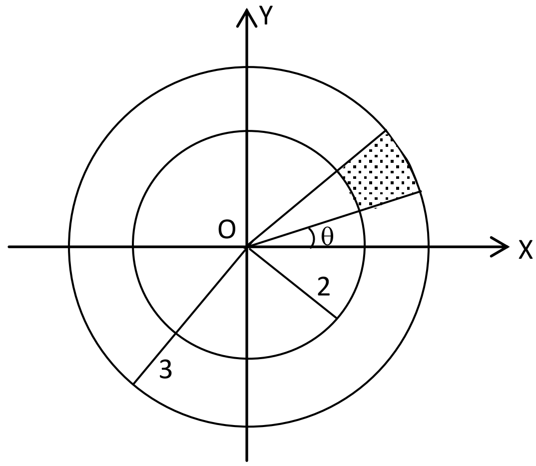
$$= \frac{a^2}{2}\int_0^{2\pi}\frac{1+\cos 2\theta}{2} d\theta \qquad (\text{Since } 1 - \sin^2\theta = \cos^2\theta = \frac{1+\cos 2\theta}{2})$$

$$= \frac{a^2}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} \qquad = \frac{a^2}{2}[\pi] = \frac{\pi a^2}{2}$$

**6.** Evaluate $\iint_R \sqrt{x^2 + y^2}\, dx\, dy$ where R is the region in the xy-plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

**Solution:**

Let $I = \iint_R \sqrt{x^2 + y^2}\, dx\, dy$




We shall evaluate this by transforming it to polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$. Then $\sqrt{x^2+y^2} = r$

$$I = \iint_R r.r.dr.d\theta = \int_0^{2\pi}\left[\int_{r=2}^{3} r^2 dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta$$

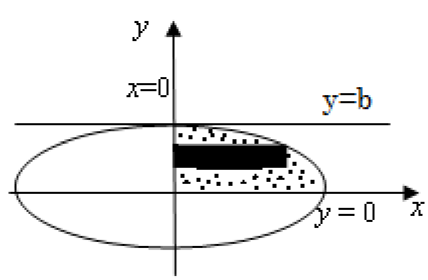
$$= \frac{19}{3}\int_0^{2\pi} d\theta = \frac{19}{3}2\pi = \frac{38\pi}{3}$$

**7.** Find the area of the ellipse $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$

**Solution:**

Area of ellipse $= 4\times$ area of quadrant

$x$ varies from $x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$ and $y$ varies from $y=0$ to $y=b$




$$\therefore \text{ The required area } = 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$$

$$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= 4\int_0^b \left[\frac{a}{b}\sqrt{b^2-y^2} - 0\right] dy$$

$$= \frac{4a}{b}\int_0^b \sqrt{b^2 - y^2}\ dy$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right) + \frac{y}{2}\sqrt{b^2 - y^2}\right]_0^b$$

$$= \frac{4a}{b}\left[\left(\frac{b^2}{2}\sin^{-1}(1) + 0\right) - (0 + 0)\right]$$

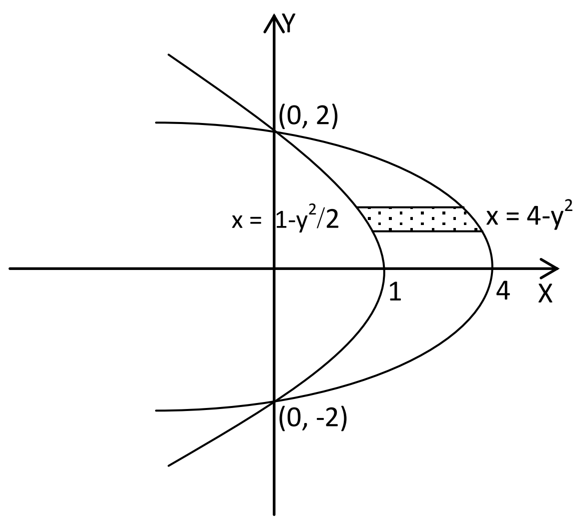
$$= \frac{4a}{b}\left(\frac{b^2}{2}\frac{\pi}{2}\right) = \pi ab$$

**8.** Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$.

**Solution:**

If R is the region bounded by the parabolas, then $A = \iint_R dx\,dy$

The parabolas meet the x-axis at x = 1 and x = 4.
Both the parabolas meet the y-axis at the points (0, 2) and (0, -2).
The region R is symmetric about the x-axis.




$$A = 2\int_0^2 \left[\int_{1-y^2/4}^{4-y^2} dx\right] dy$$

$$= 2\int_0^2 \left[\left(4 - y^2\right) - \left(1 - \tfrac{y^2}{4}\right)\right] dy$$

$$= 2\int_0^2 \left(3 - 3\tfrac{y^2}{4}\right) dy$$

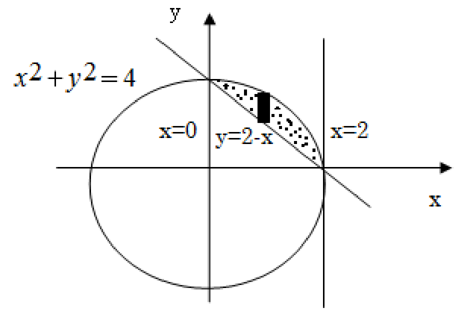
$$= 6\left[ y - \frac{y^3}{12} \right]_0^2 = 6\left[2 - \frac{8}{12}\right]$$

= 8 square units.

**9.** Find the smaller of the area bounded by $\boldsymbol{y = 2 - x}$ and $\boldsymbol{x^2 + y^2 = 4}$

**Solution:**

Given $y = 2 - x$ ---------(1) & $x^2 + y^2 = 4$ ---------(2)




Solving (1) and (2) we get $x = 0,\ x = 2$

$x = 0 \Rightarrow y = 2$ and $x = 2 \Rightarrow y = 0$.

Therefore the point of intersection of (1) and (2) is $(0,2)$ and $(2,0)$.

$x$ varies from $x = 0$ to $x = 2$ and $y$ varies from $y = 2 - x$ to $y = \sqrt{4 - x^2}$

$$\therefore \quad \text{The required area } = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\, dx$$

$$= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}}\, dx \quad = \int_0^2 \left[ \sqrt{4 - x^2} - (2 - x) \right] dx$$

$$= \int_0^2 \sqrt{4 - x^2}\, dx - \int_0^2 (2 - x)\, dx = \left[ \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) \right]_0^2 - \left[ 2x - \frac{x^2}{2} \right]_0^2$$

$$= \left[ \left(0 + 2\frac{\pi}{2}\right) - (0 + 0) \right] - \left[ (4 - 2) - (0 - 0) \right] \quad = \pi - 2 \text{ square units.}$$

**10.** Find the area of the cardioid $\boldsymbol{r = a(1 + \cos\theta)}$.

**Solution:**

The curve is symmetrical about the initial line.

$$\text{The required area } = 2 \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1+\cos\theta)} r\ dr\, d\theta$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi}\left[a^2(1+\cos\theta)^2 - 0\right]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\cos^2\theta+2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

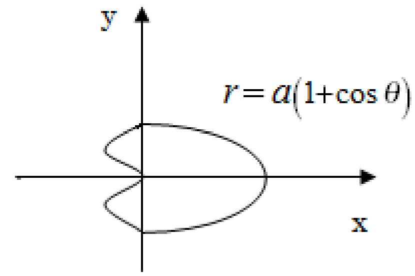
$$= a^2\int_0^{\pi}\left[1+\frac{1}{2}+\frac{\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left[\frac{3}{2}+\frac{\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta+\frac{\sin 2\theta}{4}+2\sin\theta\right]_0^{\pi}$$

$$= a^2\left[\left(\frac{3}{2}\pi+0+0\right)-(0+0+0)\right]$$

$$= \frac{3}{2}\pi a^2.$$




**11.** Find the area lying inside the circle $\boldsymbol{r = a\sin\theta}$ and outside the cardioid $\boldsymbol{r = a(1-\cos\theta)}$.

**Solution:**

Eliminating $r$ between the equations of two curves, $\sin\theta = 1-\cos\theta$ or $\sin\theta+\cos\theta = 1$

Squaring $\quad 1+\sin 2\theta = 1$ or $\sin 2\theta = 0$

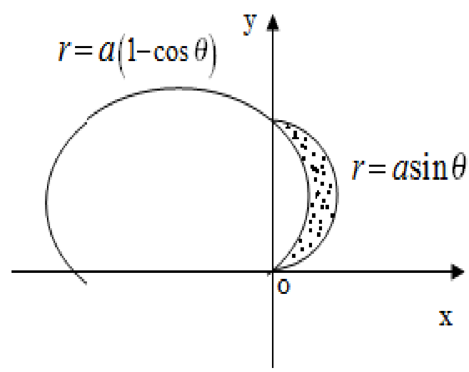
$\therefore \quad 2\theta = 0$ or $\pi$

(i.e.) $\quad \theta = 0$ or $\frac{\pi}{2}$

For the required area, $r$ varies from $a(1-\cos\theta)$ to $a\sin\theta$ and $\theta$ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \text{The required area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$




$$= \frac{1}{2}\int_0^{\pi/2} a^2\left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$$

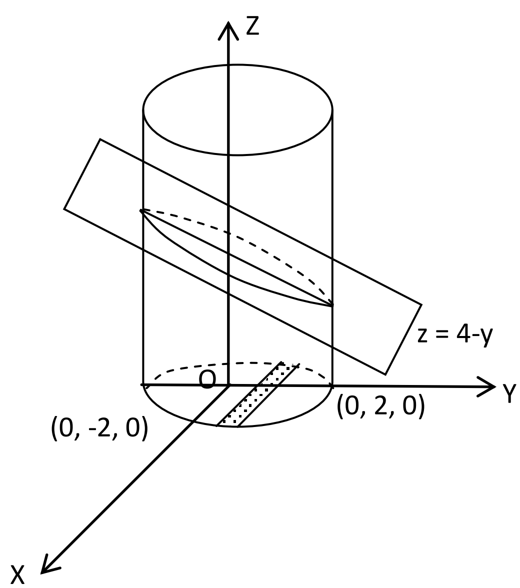
$$= \frac{a^2}{2}\int_0^{\pi/2} \left[\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left[-2\cos^2\theta + 2\cos\theta\right] d\theta$$

$$= a^2\int_0^{\pi/2} \left[-\cos^2\theta + \cos\theta\right] d\theta$$

$$= a^2\left[-\frac{1}{2}\frac{\pi}{2} + 1\right] \quad = a^2\left(1 - \frac{\pi}{4}\right)$$

**12.** Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.
**Solution:**



The required volume lies below the surface z = 4-y and is bounded by the cylinder $x^2 + y^2 = 4$ and plane z = 0.

Let R be the region in the xy-plane bounded by the circle $x^2 + y^2 = 4$, z = 0.

Then $V = \iint_R z\,dx\,dy$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y)\,dx\right]dy$$

$$= \int_{-2}^{2}\Big[(4-y)\,x\Big]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= \int_{-2}^{2} 2(4-y)\sqrt{4-y^2}\,dy$$

$$= 16\int_{0}^{2}\sqrt{4-y^2}\,dy \qquad \text{(Second term vanishes being an odd function of y)}$$

$$= 16\int_{0}^{\pi/2} 2\cos\theta\, 2\cos\theta\, d\theta \qquad \text{(By putting } y = 2\sin\theta\text{)}$$

$$= 64\int_{0}^{\pi/2}\cos^2\theta\, d\theta = 32\int_{0}^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= 32\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 16\,\pi \text{ cubic units.}$$

**CHANGE OF ORDER OF INTEGRATION**

In a double integral, if the limits of integration are constants, then the order of integration is not important. One can integrate w.r.t x and then w.r.t y or vice versa. But in a double integral with variable limits, the change of order of integration changes the limits of integration. Sometimes it is convenient to evaluate a double integral by changing the order of integration. In such cases the limits of integration are suitably modified. It may be necessary to split the given region of integration into subregions.

**13.** Change the order of integration in $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ and hence evaluate it.

**Solution:**

Given $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

The limits for $y$ varies from $y = 0$ to $y = a$ and $x$ varies from $x = y$ to $x = a$.

The region of integration is bounded by $y = 0$, $y = a$, $x = y$ and $x = a$.

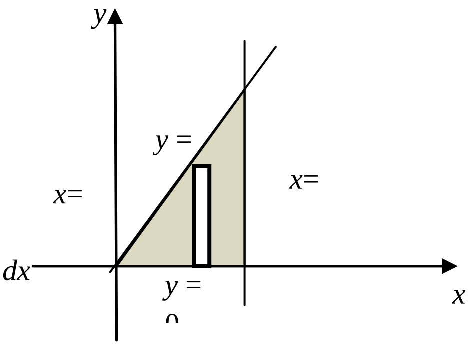
After changing the order of integration, the limits for $x: \quad 0 < x < a$, $y: \quad 0 < y < x$

$$\therefore \quad \int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy = \int_0^a \int_0^x \frac{x}{x^2 + y^2}\, dy\; dx$$

$$= \int_0^a \left[ \tan^{-1}\left(\frac{y}{x}\right) \right]_{y=0}^{y=x} dx$$

$$= \int_0^a \left[ \tan^{-1}(1) - \tan^{-1}(0) \right] dx = \int_0^a \left[ \frac{\pi}{4} - 0 \right] dx$$

$$= \frac{\pi}{4} \int_0^a dx = \frac{\pi}{4} [x]_0^a = \frac{\pi}{4} [a - 0]$$




$$\boxed{\int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy = \frac{\pi}{4} a}$$

**14.** Change the order of integration in $\int_0^\infty \int_0^y \boldsymbol{y e^{\frac{-y^2}{x}}\, dx\, dy}$ and hence evaluate it.

**Solution:**

The region of integration is bounded by $y = 0, y = \infty, x = 0$ and $x = y$.

Here $y$ varies from $y = 0$ to $y = \infty$ and $x$ varies from $x = 0$ to $x = y$. After changing the order of integration, the limits for

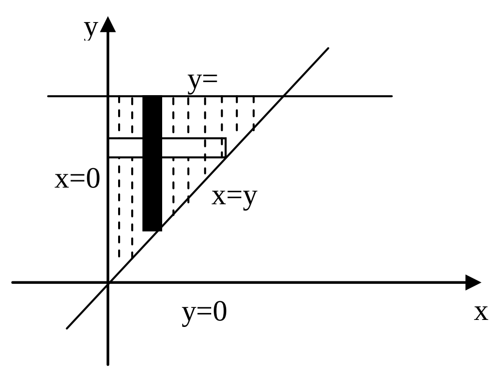
$x: \quad 0 < x < \infty$

$y: \quad x < y < \infty$

$$\therefore \quad \int_0^\infty \int_0^y y e^{\frac{-y^2}{x}}\, dx\, dy = \int_0^\infty \int_x^\infty y e^{\frac{-y^2}{x}}\, dy\, dx$$

$$= \frac{1}{2} \int_0^\infty \int_x^\infty 2y e^{\frac{-y^2}{x}}\, dy\, dx$$

$$= \frac{1}{2} \int_0^\infty \left[ \int_x^\infty e^{\frac{-y^2}{x}}\, d\left(y^2\right) \right] dx$$



$$= \frac{1}{2}\int_0^\infty \left[\frac{e^{\frac{-y^2}{x}}}{-\frac{1}{x}}\right]_x^\infty dx = \frac{1}{2}\int_0^\infty \left[-xe^{\frac{-y^2}{x}}\right]_x^\infty dx = \frac{1}{2}\int_0^\infty \left[0-\left(-xe^{-x}\right)\right] dx$$

$$= \frac{1}{2}\int_0^\infty xe^{-x}\,dx = \frac{1}{2}\left[x\frac{e^{-x}}{-1}-(1)\frac{e^{-x}}{(-1)^2}\right]_0^\infty = -\frac{1}{2}\left[xe^{-x}+e^{-x}\right]_0^\infty$$

$$= \frac{-1}{2}\left[(0+0)-(0+1)\right]$$

$$\boxed{\int_0^\infty\int_0^y y\,e^{\frac{-y^2}{x}}\,dx\,dy = \frac{1}{2}}$$

**15.** Change the order of integration and hence evaluate $\displaystyle\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} \boldsymbol{xy\,dx\,dy}$ .

**Solution :**

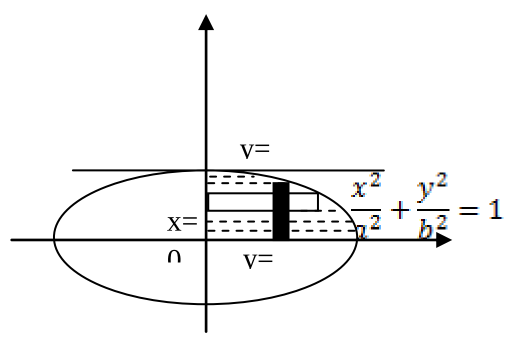
The region of integration is bounded by $y=0$, $y=b$, $x=0$ and $x=\frac{a}{b}\sqrt{b^2-y^2}$

Here y varies from $y=0$ to $y=b$ and x varies from $x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$

After changing the order of integration, the limits for

$x: \quad 0<x<a$

$y: \quad 0<y<\frac{b}{a}\sqrt{a^2-x^2}$




$$\therefore \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\;dx\,dy = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\;dy\,dx$$

$$= \int_0^a \left[\frac{xy^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{x\frac{b^2}{a^2}\left(a^2-x^2\right)}{2}-0\right] dx$$

$$= \frac{b^2}{2a^2}\int_0^a \left(a^2x - x^3\right)dx = \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

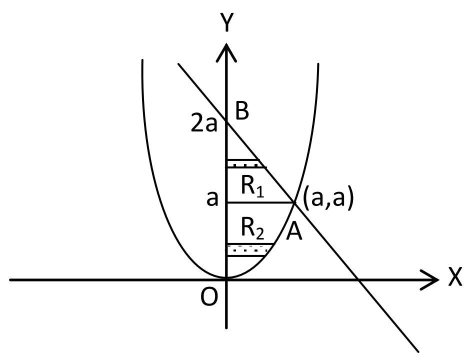
$$= \frac{b^2}{2a^2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0-0)\right] = \frac{b^2}{2a^2}\left(\frac{a^4}{4}\right)$$

$$\boxed{\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\ dx\,dy = \frac{a^2b^2}{8}}$$

**16.** Change the order of integration of the double integral $\int_0^a \int_{x^2/a}^{2a-x} xy\ dy\ dx$ and hence evaluate it.

**Solution:**

The region of integration is bounded by $y = \frac{x^2}{a}$, y = 2a-x, x = 0 and x = a. This region R is in the first quadrant, bounded by the parabola $x^2$ = ay, the line x + y = 2a and the y-axis .




To change the order of integration, divide the region R into two subregions $R_1$ and $R_2$. $R_1$ from y = 0 to y = a, and $R_2$ from y = a to y = 2a.

$$\therefore \text{I} = \iint_{R_1} xy\ dx\ dy + \iint_{R_2} xy\ dx\ dy$$

$$= \int_0^a \left[\int_0^{\sqrt{ay}} xy\ dx\right]dy + \int_a^{2a}\left[\int_0^{2a-y} xy\ dx\right]dy$$

$$= \int_0^a \left[y.\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a}\left[y.\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a \frac{ay^2}{2}dy + \int_a^{2a}\frac{y}{2}(2a-y)^2\,dy$$

$$= \int_0^a \frac{ay^2}{2}\,dy + \int_a^{2a} \left(2a^2y - 2ay^2 + \tfrac{1}{2}y^3\right)dy$$

$$= \frac{a^4}{6} + \left[a^2y^2 - \frac{2ay^3}{3} + \frac{y^4}{8}\right]_a^{2a}$$

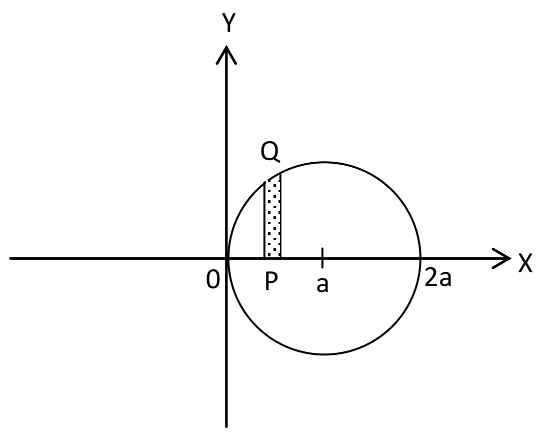
$$= \frac{a^4}{6} + 4a^4 - \frac{16}{3}a^4 + 2a^4 - a^4 + \frac{2}{3}a^4 - \frac{a^4}{8}$$

$$= \frac{3a^4}{8}$$

**17.** Change the order of integration and then evaluate $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\,dx$.

**Solution:**

The region of integration is bounded by $x = a \pm \sqrt{a^2 - y^2}$ and the lines y = 0 and y = a. i.e., the circle $x^2 + y^2 - 2ax = 0$ and the x-axis (Fig. 5.20).




To change the order of integration, first integrate w.r.t y along the strip PQ between y = 0 and $y = \sqrt{2ax - x^2}$ and then slide this vertical strip from x = 0 to x = 2a.

$$\therefore I = \int_0^{2a} \left[\int_0^{\sqrt{2ax-x^2}} dy\right] dx$$

$$= \int_0^{2a} \sqrt{2ax - x^2}\,dx$$

$$= \int_0^{2a} \sqrt{a^2 - (x-a)^2}\,dx = \left[\frac{1}{2}.x.\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}\right]_0^{2a} = \frac{a^2}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= \frac{\pi a^2}{2}$$

**CHANGE OF VARIABLES**

By changing the variables using a suitable transformation, a given integral can be transformed into a simpler integral involving the new variables. Thus in certain cases, the evaluation of a double or triple integral becomes easier, when we change the given variables into a new set of variables. Considering the region of integration and the integrand a suitable transformation is chosen for the change of variables.

**CHANGE OF VARIABLES IN DOUBLE INTEGRALS**

Let f(x, y) be continuous over a region R in the xy-plane. Let the variables x, y be changed to new variables u, v by the transformation x = φ(u, v), y = χ(u, v) where the functions φ(u, v) and χ(u, v) are continuous and have continuous first order partial derivatives in the region R′ in the uv-plane, which corresponds to the region R in the xy-plane. Then $\iint_R f(x,y)\,dx\,dy = \iint_{R'} F(u,v)\,|J|\,du\,dv$ where F(u, v) = f(φ(u, v), χ(u, v)) and $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$ is the **functional determinant** or **Jacobian of transformation** from (x, y) to (u, v) coordinates.

For example, to change Cartesian coordinates (x, y) to polar coordinates (r, θ), we have x = r cosθ, y = r sin θ. Then,

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

where R′ is the region in the rθ-plane corresponding to R in the xy-plane

**18.** Evaluate $\iint_R (x+y)^2\,dxdy$, where R is the parallelogram in the xy-plane with vertices $(1,0),(3,1),(2,2),(0,1)$ using the transformation $u = x+y$ and $v = x-2y$.

**Solution:**

The vertices $A(1,0), B(3,1), C(2,2), D(0,1)$ of the parallelogram ABCD in the xy-plane become $A'(1,1), B'(4,1), C'(4,-2), D'(1,-2)$ in the uv-plane under the given transformation.

The region R in the xy-plane becomes the region $R'$ in the uv-plane which is the rectangle bounded by the lines $u=1, u=4,$ and $v=-2, v=1$

Solving the given equations , weget $x = \frac{1}{3}(2u+v),\; y = \frac{1}{3}(u-v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

$$\therefore \quad \iint_R (x+y)^2\, dxdy = \iint_{R'} u^2 |J|\, dudv = \int_{-2}^{1}\int_{1}^{4} u^2 \frac{1}{3} dudv = \int_{-2}^{1} \frac{1}{3}\left(\frac{u^3}{3}\right)_1^4 dv = \int_{-2}^{1} 7dv = 21$$

**19.** By using the transformation $\boldsymbol{x+y=u,\; y=uv,}$ show that $\displaystyle\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\, dy\, dx = \frac{e-1}{2}$

**Solution:**

Given $x+y=u,\; y=uv,\; x=u(1-v)$

$Now\ y=0 \Rightarrow u=0(or)v=0$

$y=1-x \Rightarrow u=1, x=0 \Rightarrow u=0(or)v=1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

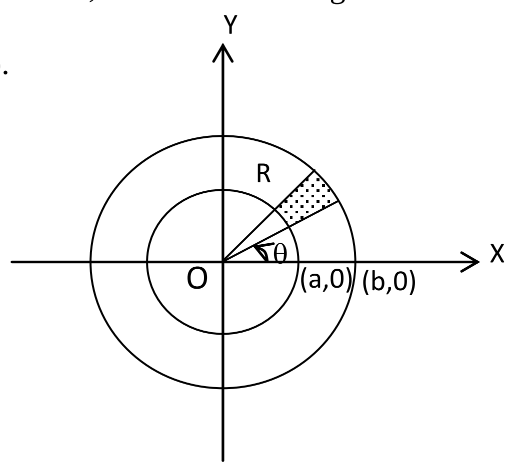
$$\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\, dy\, dx = \iint_{R'} e^{\frac{uv}{u}} |J|\, du\, dv$$

$$= \int_0^1\int_0^1 e^v u\, du\, dv$$

$$= \frac{1}{2}\int_0^1 e^v dv$$

$$= \frac{e-1}{2}$$

**20.** Evaluate $\displaystyle\iint_R \frac{xy}{\sqrt{x^2+y^2}}\, dx\, dy$, after transforming to polar coordinates, where R is the region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ in the first quadrant (a < b).




**Solution:**

Put $x = r\cos\theta$, $y = r\sin\theta$

Then $dxdy = r\, dr\, d\theta$, and

$$\frac{xy}{\sqrt{x^2+y^2}} = \frac{r\cos\theta\, r\sin\theta}{r}$$

$= r\cos\theta\sin\theta$

The limits of integration are $a \le r \le b$ and $0 \le \theta \le \pi/2$.

$$\therefore I = \int_0^{\pi/2}\int_a^b r\cos\theta\sin\theta\, r dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_a^b \cos\theta\sin\theta\ d\theta = \frac{1}{3}(b^3-a^3)\int_0^{\pi/2}\cos\theta\sin\theta\ d\theta$$

$$= \frac{1}{3}(b^3-a^3)\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{6}(b^3-a^3)$$

**21.** Transform the integral into polar co-ordinates and hence evaluate $\displaystyle\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$

**Solution:**

The region of integration is bounded by $y=0$, $y=a$, $x=y$ and $x=a$.

Let us transform this integral in polar co-ordinates by taking $x=r\cos\theta$, $y=r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

$$\therefore \int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^3\cos^2\theta}{\sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)}}\,dr\,d\theta \quad = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^3\cos^2\theta}{r}\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} r^2\cos^2\theta\,dr\,d\theta \qquad = \int_0^{\pi/4}\cos^2\theta\left[\frac{r^3}{3}\right]_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4}\cos^2\theta\left[\frac{a^3\sec^3\theta}{3}-0\right]d\theta \qquad = \frac{a^3}{3}\int_0^{\pi/4}\cos^2\theta\,\frac{1}{\cos^3\theta}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta \qquad = \frac{a^3}{3}\left[\log\left(\sec\theta+\tan\theta\right)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)-\log(1+0)\right] \qquad = \frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)-0\right]$$

$$\boxed{\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy = \frac{a^3}{3}\log\left(\sqrt{2}+1\right)}$$

**22.** Transform the integral into polar co-ordinates and hence evaluate $\displaystyle\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy$

**Solution:**

Let us transform this integral in polar co-ordinates by taking $x = r\cos\theta,\ y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

$$\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta \quad = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,\frac{1}{2}d\left(r^2\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[-e^{-r^2}\right]_0^\infty d\theta \quad = -\frac{1}{2}\int_0^{\pi/2}\left[0-1\right]d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta$$

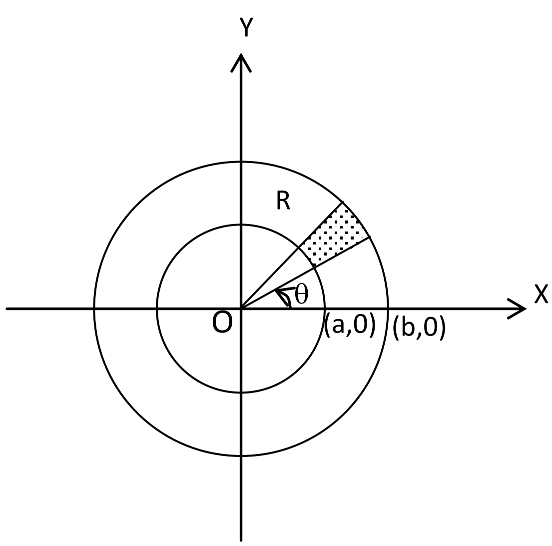
$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$\boxed{\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy = \frac{\pi}{4}}$$

**23.** By transforming into polar coordinates, Evaluate $\iint_R \frac{\mathbf{x^2y^2}}{\sqrt{\mathbf{x^2+y^2}}}\,\mathbf{dxdy}$ over the annular region R between the circles $\mathbf{x^2+y^2=a^2}$ and $\mathbf{x^2+y^2=b^2}$, $(\boldsymbol{b>a})$

**Solution:**

Put $x = r\cos\theta, y = r\sin\theta$, then $x^2+y^2 = a^2 \Rightarrow r = a,$ $x^2+y^2 = b^2 \Rightarrow r = a$ and $\theta$ varies from 0 to $2\pi$



$$\iint \frac{x^2y^2}{x^2+y^2}dxdy = \int_0^{2\pi}\int_a^b \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} rdr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\sin^2\theta\left(\frac{r^4}{4}\right)_a^b d\theta$$

$$= \left(\frac{b^4-a^4}{4}\right)\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta$$

$$= \left(\frac{b^4-a^4}{4}\right)4\int_0^{\pi/2}\cos^2\theta\sin^2\theta d\theta$$

$$\iint \frac{x^2y^2}{x^2+y^2}dxdy = \frac{\pi}{16}\left(b^4-a^4\right)$$

**AREA OF CURVED SURFACE**

Surface Area $S = \iint_{S'}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$

**24.** Find the surface area of the sphere $\boldsymbol{x^2+y^2+z^2=a^2}$

**Solution:**

Given $x^2+y^2+z^2=a^2 \Rightarrow z^2 = a^2-x^2-y^2$

Surface Area $S = \iint_{S'}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy = 2\iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$

Let us transform this integral in polar co-ordinates by taking $x = r\cos\theta,\ y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

$$S = 2\int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2}}rdrd\theta$$

$$= -2a\int_0^{2\pi}\left(\sqrt{a^2-r^2}\right)_0^a d\theta$$

$$= 4\pi a^2\ sq.units$$

**25.** Find the area of the portion of the cylinder $\boldsymbol{x^2+z^2=4}$ lying inside the cylinder $\boldsymbol{x^2+y^2=4}$

**Solution:**

Given $x^2+y^2=4 \Rightarrow y=\sqrt{4-x^2}$

Surface Area $S = \iint_{S'}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dzdx$

$$= 8\int_0^2 \int_0^{\sqrt{2^2-x^2}} \frac{2}{\sqrt{2^2 - x^2}} dzdx$$

$$= 8\int_0^2 \frac{2}{\sqrt{2^2 - x^2}} \sqrt{2^2 - x^2}\, dx$$

$$= 16\int_0^2 dx$$

$$= 32 \text{ sq.units.}$$

**TRIPLE INTEGRALS**

Triple integrals are defined in a manner similar to a double integral.

**Definition 1.3.1** Let f(x, y, z) be a function defined and continuous in a closed and bounded region R in $\mathbb{R}^3$. Divide R into n sub regions $\Delta R_1, \Delta R_2, \ldots, \Delta R_n$ of volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$. Let $(\varsigma_i, \eta_i, \delta_i)$ be any point in the $i^{th}$ subregion $\Delta R_i$. Then the **triple integral** of f(x, y, z) over the region R is defined as $\underset{\substack{n\to\infty \\ \text{Max}|\Delta V_i|\to 0}}{\text{Lim}} \sum_{i=1}^{n} f(\varsigma_i, \eta_i, \delta_i).\Delta V_i$ and is denoted by $\iiint_R f(x,y,z)\, dV$ or $\iiint_R f(x,y,z)\, dx\, dy\, dz$.

The triple integral can be computed by expressing it as the iterated integral

$$\int_{x=a}^{b}\left[\int_{y=\varphi(x)}^{\chi(x)}\left[\int_{z=g(x,y)}^{h(x,y)} f(x,y,z)dz\right]dy\right]dx$$

The order of integration can be changed by suitably modifying the limits of integration and if necessary by subdividing the region R into a finite number of subregions, in which case the triple integral over R is obtained by summing the triple integrals over these subregions. The properties of triple integrals are similar to those of double integrals.

**APPLICATIONS OF TRIPLE INTEGRALS**

**1. Volume of a region**

If f(x, y, z) = 1, then the triple integral over R gives the volume V of the region R.

i.e., $V = \iiint_R dx\, dy\, dz.$

**2.Centre of gravity**

Let f(x, y, z) = ρ(x, y, z) be the density of a mass in the region R.

Then $M = \iiint_R f(x, y, z)\, dx\, dy\, dz$ gives the total mass of the solid in R

If $\bar{x} = \frac{1}{M}\iiint_R x.f(x, y, z)\, dx\, dy\, dz$

$$\bar{y} = \frac{1}{M}\iiint_R y.f(x, y, z)\, dx\, dy\, dz$$

$\bar{z} = \frac{1}{M}\iiint_R z.f(x, y, z)\,dx\,dy\,dz$ , where f(x, y, z) = ρ(x, y, z), then $(\bar{x}, \bar{y}, \bar{z})$ is the **centre of mass** or**centre of gravity** of the solid of mass M in the region R.

**3. Moment of inertia**

If f(x, y, z) = ρ(x, y, z) is the density function of a mass in the region R, then

$I_x = \iiint_R (y^2 + z^2).f(x, y, z)\,dx\,dy\,dz$

$I_y = \iiint_R (x^2 + z^2).f(x, y, z)\,dx\,dy\,dz$

$I_z = \iiint_R (x^2 + y^2).f(x, y, z)\,dx\,dy\,dz$ are the **moments of inertia** of the mass about the x-axis, y-axis and z-axis respectively.

**26.** Evaluate: $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,dx\,dy\,dz$

**Solution:**

$$\text{Let I} = \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[e^z\right]_0^{x+y} dy\,dx = \int_0^1 \int_0^{1-x} \left[e^{x+y} - e^0\right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[e^{x+y} - 1\right] dy\,dx = \int_0^1 \left[e^x e^y - y\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[\left(e^x\,e^{1-x} - (1-x)\right) - \left(e^x\right)\right] dx = \int_0^1 \left[e^1 - (1-x) - e^x\right] dx$$

$$= \int_0^1 \left[e - 1 + x - e^x\right] dx \left[ex - x + \frac{x^2}{2} - e^x\right]_0^1$$

$$= \left[\left(e - 1 + \frac{1}{2} - e\right) - (0 - 0 + 0 - 1)\right]$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

**27. Evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2 - x^2 - y^2 - z^2}}$.

**Solution:**

Let $$I=\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}}dy\,dx$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\sin^{-1}(1)-\sin^{-1}(0)\right]dy\,dx$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\frac{\pi}{2}-0\right]dy\,dx \quad =\frac{\pi}{2}\int_0^a\left[y\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=\frac{\pi}{2}\int_0^a\sqrt{a^2-x^2}\,dx \quad =\frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{\pi}{2}\left[\left(0+\frac{a^2}{2}\frac{\pi}{2}\right)-(0+0)\right]$$

$$\boxed{\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}=\frac{\pi^2a^2}{8}}$$

**28.** **Evaluate $\iiint_V\frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes $x=0, y=0$, $z=0, x+y+z=1$ .**

**Solution:**

Here $z$ varies from $z=0$ to $z=1-x-y$

$y$ varies from $y=0$ to $y=1-x$

$x$ varies from $x=0$ to $x=1$

$$\therefore\iiint_V\frac{dz\,dy\,dx}{(x+y+z+1)^3}=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z+1)^{-3}dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[2^{-2}-(x+y+1)^{-2}\right]dy\,dx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-(x+y+1)^{-2}\right]dy\,dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}y-\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}y+(x+y+1)^{-1}\right]_0^{1-x} dx \quad =-\frac{1}{2}\int_0^1\left[\left(\frac{1}{4}(1-x)+2^{-1}\right)-\left(0+(x+1)^{-1}\right)\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}-\frac{x}{4}+\frac{1}{2}-\frac{1}{1+x}\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{3}{4}-\frac{x}{4}-\frac{1}{1+x}\right]dx \quad =-\frac{1}{2}\left[\frac{3}{4}x-\frac{x^2}{8}-\log(1+x)\right]_0^1$$

$$=-\frac{1}{2}\left[\left(\frac{3}{4}-\frac{1}{8}-\log 2\right)-(0-0-0)\right]$$

$$\boxed{\iiint_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}=\frac{1}{2}\log 2-\frac{5}{16}}$$

**29.** Find the volume of the sphere $\boldsymbol{x^2+y^2+z^2=a^2}$.

**Solution:**

Volume $V=8\times$ Volume in an octant

$z$ varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

$y$ varies from $y=0$ to $y=\sqrt{a^2-x^2}$

$x$ varies from $x=0$ to $x=a$

$$\therefore \qquad V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[z\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{a^2-x^2-y^2} - 0\right] dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2 - y^2}\, dy\, dx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[\left(0 + \frac{a^2-x^2}{2}\frac{\pi}{2}\right) - (0+0)\right] dx$$

$$= 2\pi\int_0^a \left(a^2-x^2\right) dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[\left(a^3 - \frac{a^3}{3}\right) - (0-0)\right]$$

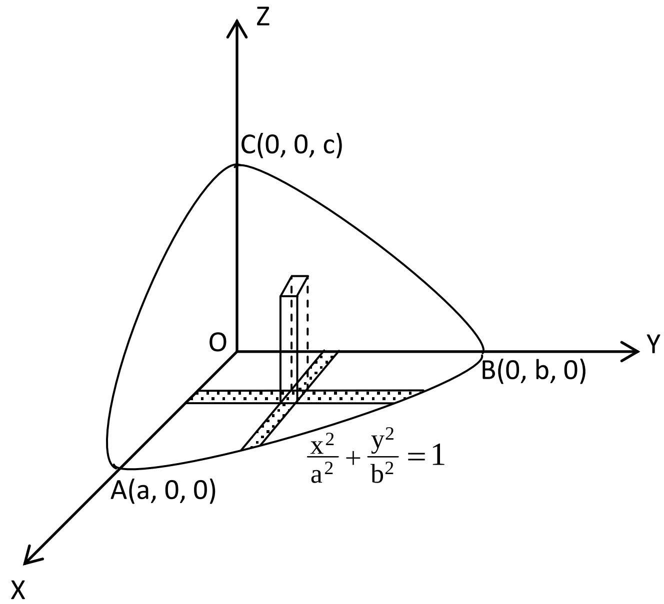
$$\boxed{V = \frac{4}{3}\pi a^3}.$$

**30.** Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

**Solution:**

The cross section of the ellipsoid by the xy-plane is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

To obtain the required volume, first integrate w.r.t z from z = 0 to $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$,




then w.r.t y from y = 0 to $y = b\sqrt{1-x^2/a^2}$ and finally w.r.t x from x = 0 to x = a

$$\therefore V = 8\int_0^a \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= 8\int_0^a \int_0^{\rho} \frac{c}{b}\sqrt{\rho^2-y^2}\,dy\,dx, \text{ where } \rho = b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{\rho^2-y^2}+\frac{\rho^2}{2}\sin^{-1}\left(y/\rho\right)\right]_0^{\rho} dx$$

$$= \frac{8c}{b}\int_0^a \frac{\rho^2}{2}\frac{\pi}{2}dx$$

$$= 2\pi bc\int_0^a \left(1 - x^2/a^2\right)dx$$

$$= 2\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a$$

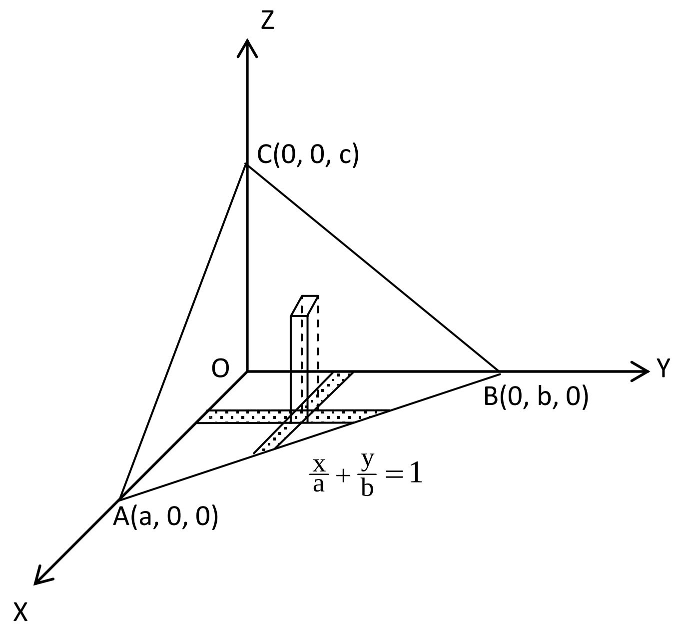
$$= 2\pi bc\left(a - \frac{a}{3}\right)$$

$$= \frac{4}{3}\pi abc \text{ cubic units.}$$

**31.** Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

**Solution:**

To obtain the required volume, first integrate w.r.t z from z = 0 to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$, then w.r.t y from y = 0 to $y=b\left(1-\frac{x}{a}\right)$ and finally w.r.t x from x = 0 to x = a




$$\therefore V = \int_0^a \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx.$$

$$= \int_0^a \int_0^{\rho} \frac{c}{b}(\rho-y)\,dy\,dx.$$

where $\rho=b\left(1-\frac{x}{a}\right)$

$$= \frac{c}{b}\int_0^a \left(\rho y-\frac{y^2}{2}\right)_0^{\rho} dx$$

$$= \frac{c}{b}\int_0^a \frac{\rho^2}{2}dx$$

$$= \frac{c}{2b}\int_0^a \frac{b^2}{a^2}(a-x)^2 dx$$

$$= \frac{bc}{2a^2}\left[\frac{(a-x)^3}{-3}\right]_0^a$$

$$= \frac{bc}{2a^2}.\frac{a^3}{3}$$

$$V=\frac{abc}{6}$$