UNIT I - VECTOR CALCULUS

PART A

1. If $\phi = 3x^2y - y^3z^2$, find gradϕ at (1, –1, 2)

Solution: $\frac{\partial\phi}{\partial x} = 6xy,\quad \frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2,\quad \frac{\partial\phi}{\partial z} = -2y^3z$

$grad\phi = 6xy\vec{i} + \left(3x^2 - 3y^2z^2\right)\vec{j} - 2y^3z\vec{k}$

$\left(grad\,\phi\right)_{(1,\ -1,\ 2)} = -6\vec{i} - 9\vec{j} + 4\vec{k}$

2. Find $|\nabla\phi|$ *if* $\phi = 2xz^4 - x^2y$ at (2, -2, -1)

Solution: $\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}$

$= \vec{i}\left(2z^4 - 2xy\right) + \vec{j}\left(-x^2\right) + \vec{k}\left(8xz^3\right)$

$(\nabla\phi)_{(2,\ -2,\ -1)} = 10\vec{i} - 4\vec{j} - 16\vec{k} \Rightarrow |\nabla\phi| = \sqrt{100+16+256} = \sqrt{372}$

3. Find the directional derivative of ϕ = 3x²+2y–3z at (1, 1, 1) in the direction $2\vec{i} + 2\vec{j} - \vec{k}$

Solution: $\nabla\phi\cdot n = \left[(6x\vec{i} + 2\vec{j} - 3\vec{k})\cdot\left(\frac{2\vec{i}+2\vec{j}-\vec{k}}{3}\right)\right]_{(1,1,1)} = \frac{19}{3}$

4. Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point (1,–1,2)

Solution: $\phi = x^2 + xy + z^2$, $[\nabla\phi]_{(1,-1,2)} = \left[(2x+y)\vec{i} + x\vec{j} + 2z\vec{k}\right]_{(1,-1,2)} = \vec{i} + \vec{j} + 4\vec{k}$

$\therefore\quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}$

5. Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1)

Solution: Let $\phi_1 = y^2 - x\log z - 1$

$\nabla\phi_1 = -\log z\ \vec{i} + 2y\vec{j} - \frac{x}{z}\vec{k}$, $(\nabla\phi_1)_{(1,1,1)} = 2\vec{j} - \vec{k}$ and $|\nabla\phi_1| = \sqrt{5}$

Let $\phi_2 = x^2y - 2 + z$

$\nabla\phi_2 = \vec{i}(2xy) + \vec{j}x^2 + \vec{k}(1)$, $(\nabla\phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$ and $|\nabla\phi_2| = \sqrt{6}$

$cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{0+2-1}{\sqrt{30}} \Rightarrow \theta = cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$

6. In what direction from (–1,1,2) is the directional derivative of $\phi = xy^2z^3$ a maximum. Find also the magnitude of this maximum.

Solution: Given $\phi = xy^2z^3$

$\nabla\phi = (y^2z^3)\vec{i} + (2xyz^3)\vec{j} + (3xy^2z^2)\vec{k}$ and $\nabla\phi$ at (1,1,2) = $\mathbf{8}\vec{i} - \mathbf{16}\vec{j} - \mathbf{12}\vec{k}$

$\therefore$ The maximum directional derivative occurs in the direction of $\nabla\phi = \mathbf{8}\vec{i} - \mathbf{16}\vec{j} - \mathbf{12}\vec{k}$

The magnitude of this max. directional derivative = $|\nabla\phi| = \sqrt{\mathbf{464}}$

7. Prove that $\nabla\left(r^n\right) = nr^{n-2}\vec{r}$

Solution:

$$\nabla\left(r^n\right)=\sum \vec{i}\,\frac{\partial r^n}{\partial x} = \sum \vec{i}\,nr^{n-1}\frac{x}{r} = \sum \vec{i}\,nr^{n-2}x = nr^{n-2}(x\vec{i}+y\vec{j}+z\vec{k}) = nr^{n-2}\vec{r}$$

8. **If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, find div curl $\vec{F}$**

Solution:

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

div (curl $\vec{F}$) = $\nabla\cdot\left(\nabla\times\vec{F}\right) = 0$

9. **Find '*a*', such that (3x-2y +z) $\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.**

Solution: div $\vec{F}$ = $\nabla\cdot$[(3x-2y +z) $\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$] = 3 + a + 2 = 5 + a

div $\vec{F}$ = 0 $\Rightarrow$ a = –5

10. **If $\vec{A}$ and $\vec{B}$ are irrotational vectors prove that $\vec{A}\times\vec{B}$ is solenoidal.**

Solution: $\vec{A}$ is irrotational $\Rightarrow$ curl $\vec{A}$ = 0 and $\vec{B}$ is irrotational $\Rightarrow$ curl $\vec{B}$ = 0

$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\text{curl}\,\vec{A}) - \vec{A}\cdot(\text{curl}\,\vec{B}) = \vec{B}\cdot 0 - \vec{A}\cdot 0 = 0$

$\therefore$ $\vec{A}\times\vec{B}$ is solenoidal.

11. **Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational.**

Solution:

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2-3z^2) + \vec{k}(6x-6x) = 0$$

$\therefore$ $\vec{F}$ is irrotational

12. **If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F}.d\vec{r}$ where C is the part of the curve $y = x^3$ between x =1 and x =2**

Solution: $y = x^3 \Rightarrow dy = 3x^2\,dx$

$$\int_c \vec{F}\cdot d\vec{r} = \int_c (5xydx + 2ydy) = \int_1^2 (5x^4 + 6x^5)dx = \left[x^5 + x^6\right]_1^2 = 31 + 63 = 94$$

13. **If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$ from (0,0) to (1,1) along the path $y = x$**

Solution: $\int_c \vec{F}\cdot d\vec{r} = \int_c x^2dx + xy^2dy \qquad (\because y = x)$

$$= \int_0^1 (x^2 + x^3)dx = \frac{7}{12}$$

14. **If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\ \vec{k}$ .Check whether the integral $\int_C \vec{F}\cdot dr$ is independent of the path C**

Solution: This integral is independent of the path of integration if $\nabla\times\vec{F}=0$

$$\nabla\times\vec{F}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy-3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}=0$$

Hence the line integral is independent of path.

15. State Stoke's theorem.

Statement: If S is an open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in S and on C, then $\iint_s curl\vec{F}\cdot\hat{n}ds=\int_c \vec{F}\cdot d\vec{r}$

Where $\hat{n}$ is the unit vector normal to the surface.

16. State Green's Theorem

Statement: If M(x ,y) and N(x ,y) are continuous function with continuous partial derivatives in a region R of the xy plane bounded by a simple closed curve C, then $\oint_C \mathrm{M}dx+\mathrm{N}dy=\iint_R\left(\frac{\partial \mathrm{N}}{\partial x}-\frac{\partial \mathrm{M}}{\partial y}\right)dxdy$

Where C is the curve described in positive direction.

17. State Gauss Divergence Theorem.

Statement: If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous partial derivative in V and on S , then $\iint_S \vec{F}\cdot\hat{n}\,ds=\iiint_V div\vec{F}\ dv$

Where $\hat{n}$ is the unit vector normal to the surface.

18. Using Divergence theorem, evaluate $\iint_s xdydz+ydzdx+zdxdy$ over the surface of the sphere $x^2+y^2+z^2=a^2$

Solution: By Divergence theorem, $\iint_S \vec{F}\cdot\hat{n}\,ds=\iiint_V div\vec{F}\ dv$

$$\iint_s xdydz+ydzdx+zdxdy=\iiint_v\nabla\cdot\vec{F}dv=\iiint_v 3dv=3\left(\frac{4}{3}\pi a^3\right)=4\pi a^3$$

19. Find the area of a circle of radius '*a*' using Green's theorem.

Solution: We know that the Green's theorem is $\oint_C \mathrm{M}dx+\mathrm{N}dy=\iint_R\left(\frac{\partial \mathrm{N}}{\partial x}-\frac{\partial \mathrm{M}}{\partial y}\right)dxdy$

Area $=\frac{1}{2}\int_c(xdy-ydx)$ on $x^2+y^2=a^2$, We have x = a cos θ y= a sinθ , θ: 0→2π

Therefore Area $=\frac{1}{2}\int_0^{2\pi}(a^2\cos^2\theta+a^2\sin^2\theta)d\theta=\frac{a^2}{2}\int_0^{2\pi}d\theta=\pi a^2$

20. If S is any closed surface enclosing a volume V and $\vec{F}=ax\vec{i}+by\vec{j}+cz\vec{k}$, prove that $\iint_S \vec{F}\cdot n\,ds=(a+b+c)V$

Solution: $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\;dv$

div $\vec{F}$ =a+b+c; $\iint_S \vec{F}\cdot\vec{n}ds = \iiint_V div\vec{F}\cdot dv$ =(a+b+c) $\iiint_V dv$ =(a+b+c)V

PART B

1(a) **Find the directional derivative of ϕ = 2xy + z^2 at the point (1,-1,3) in the direction of $\vec{i}+2\vec{j}+2\vec{k}$**

(b) **Show that $\vec{F}=(4xy-z^3)\vec{i}+2x^2\vec{j}-3xz^2\vec{k}$ is irrotational and find its scalar potential.**

2(a) **Find the angle between the normals to the surface xy^3z^2 =4 at the points (-1,-1,2) and (4,1,-1)**

(b) **Find the constants *a, b, c* so that $\vec{F}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ is irrotational.**

3(a) **If $\vec{r}$ is the position vector of the point (*x,y,z*), Prove that $\nabla^2 r^n = n(n+1)r^{n-2}$**

(b) **Prove that $\nabla^2\left(r^n\vec{r}\right)=n(n+3)r^{n-2}\vec{r}$**

4(a) **Find the work done when a force $\vec{F}=(x^2-y^2+x)\vec{i}-(2xy+y)\vec{j}$ moves a particle in the XY -Plane from (0, 0) to (1,1) along the parabola $y^2 = x$**

(b) **Evaluate $\int_C\left((x^2+xy)dx+(x^2+y^2)dy\right)$ where C is the square bounded by the lines x=0, x=1, y =0 and y=1**

5 **Verify Green's theorem for $\int_C\left(x^2dx-xydy\right)$ where C is the boundary of the square formed by the lines x = 0, y = 0, x = a, y = a**

6 **Verify Green's theorem in the XY plane for $\int_C\left[\left(3x-8y^2\right)dx+\left(4y-6xy\right)dy\right]$ where C is the boundary of the region given by x = 0, y = 0, x+y = 1**

7 **Verify Gauss- Divergence theorem for $\vec{A}=4xz\vec{i}-y^2\vec{j}+yz\vec{k}$ over the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z = 1**

8 **Evaluate $\iint_s \vec{F}.n\,ds$ where $\vec{F}=4x\vec{i}-2y^2\vec{j}+z^2\vec{k}$ and S is the surface bounding the region $x^2+y^2=4,\quad z=0$ and $z=3$**

9 **Verify Stoke's theorem for $\vec{F}=x^2\vec{i}-xy\vec{j}$ in the square region in the XY- plane bounded by the lines x = 0, y = 0, *x* = *a* and *y* = *a***

10(a) **Evaluate the integral, $\int_C\left[\left(x+y\right)dx+\left(2x-z\right)dy+\left(y+z\right)dz\right]$ where C is the boundary of the triangle with vertices (2,0,0), (0,3,0)and (0,0,6) using Stoke's theorem.**

(b) **If $\vec{F}=(2x^2-3z)\vec{i}-2xy\vec{j}-4x\vec{k}$, Evaluate $\iiint_V \nabla\times\vec{F}\,dV$ where V is the region bounded by x = 0, x = 1, y = 0, y =2, z = 0, z = 3**

UNIT II - ORDINARY DIFFERENTIAL EQUATIONS

PART A

1. Solve $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

Solution: Given $\left(D^2-5D+6\right)y=0$

The Auxiliary equation (A.E) is $m^2-5m+6=0$

$$(m-2)(m-3)=0$$

$m_1=2,\ m_2=3$ The roots are real and distinct.

Complementary function is (C.F) $=Ae^{m_1x}+Be^{m_2x}=Ae^{2x}+Be^{3x}$, Since $R.H.S=0$ $\therefore P.I.=0$

$\therefore$ The general solution is $y=Ae^{2x}+Be^{3x}$

2. Solve $\left(D^3+D^2-D-1\right)y=0$

Solution: The A.E. is $m^3+m^2-m-1=0$

$m^2(m+1)-1(m+1)=0$

$\left(m^2-1\right)(m+1)=0$

$m^2=1,\ m=-1$ $\quad m=\pm1,\ m=-1$ $\quad m_1=1,\ m_2=m_3=-1$

Roots are real, distinct and equal

$\therefore C.F.=Ae^{m_1x}+(Bx+C)e^{m_2x}=Ae^{x}+(Bx+C)e^{-x}$

$\because R.H.S.=0, \therefore P.I.=0$ $\quad\therefore\quad y=Ae^{x}+(Bx+C)e^{-x}$

3. Solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$

Solution: Given $\left(D^2-6D+13\right)y=0$

The Auxiliary equation (A.E) is $m^2-6m+13=0$

$m=3\pm 2i$ $\quad(\alpha\pm i\beta)$ $\therefore$ The roots are complex $(\alpha=3,\beta=2)$

C.F. $=e^{\alpha x}\left(A\cos\beta x+B\sin\beta x\right)=e^{3x}\left(A\cos 2x+B\sin 2x\right)$, $\quad\because\ R.H.S=0$ $\quad\therefore P.I.=0$

$\therefore\ y=e^{3x}\left(A\cos 2x+B\sin 2x\right)$

4. Find the P.I. of $\left(D^2+2D+2\right)y=\cosh x$

Solution: P.I. $=\frac{1}{D^2+2D+2}\cosh x=\frac{1}{D^2+2D+2}\left(\frac{e^x+e^{-x}}{2}\right)$ $\quad\left(\because \cosh x=\frac{e^x+e^{-x}}{2}\right)$

$$=\frac{1}{2}\left[\frac{1}{D^2+2D+2}e^{x}+\frac{1}{D^2+2D+2}e^{-x}\right]=\frac{1}{2}\left[\frac{e^{x}}{1^2+2(1)+2}+\frac{e^{-x}}{(-1)^2+2(-1)+2}\right]$$

$$\therefore \qquad \text{P.I.} = \frac{1}{2}\left(\frac{e^{x}}{5} + e^{-x}\right)$$

5. Find the P.I. of $\left(D^2+3\right)y = \sin 3x$

Solution: $\text{P.I.} = \dfrac{1}{D^2+3}\sin 3x \qquad \left(D^2 = -a^2 = -9\right)$

$$= \frac{\sin 3x}{-9+3}$$

$$\therefore \qquad P.I. = -\frac{\sin 3x}{6}$$

6. Find the P.I. of $\left(D^2+2\right)y = x^2$

Solution: $P.I. = \dfrac{1}{2+D^2}x^2 = \dfrac{x^2}{2\left[1+\dfrac{D^2}{2}\right]} = \dfrac{1}{2}\left[1+\dfrac{D^2}{2}\right]^{-1}x^2$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{D^4}{4}-\cdots\right]x^2 \qquad \left[\because \left(1+x\right)^{-1} = 1-x+x^2-\cdots\right]$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}\right]x^2 = \frac{1}{2}\left[x^2-\frac{D^2\left(x^2\right)}{2}\right] \qquad \text{(Omitting Higher terms of } D^2\text{)}$$

$$= \frac{1}{2}\left[x^2-\frac{D\left(2x\right)}{2}\right] = \frac{1}{2}\left[x^2-1\right]$$

7. Find the Particular integral of $\left(D^2+4D+4\right)y = e^{-2x}x$

Solution: $\text{P.I} = \dfrac{e^{-2x}x}{(D+2)^2} = e^{-2x}\dfrac{x}{((D-2)+2)^2} = e^{-2x}\dfrac{1}{D^2}(x) = e^{-2x}\dfrac{1}{D}\displaystyle\int x\,dx = e^{-2x}\int\frac{x^2}{2}dx = \frac{e^{-2x}x^3}{6}$

8. Find the Particular integral of $\left(D^2+6\right)y = \sin x\cos x$

Solution: $P.I. = \dfrac{1}{D^2+6}\sin x\,\cos x \qquad \left(\because \sin 2x = 2\sin x\cos x \Rightarrow \sin x\cos x = \dfrac{\sin 2x}{2}\right)$

$$= \frac{1}{2}\left(\frac{\sin 2x}{-4+6}\right) \qquad \left(D^2 = -\left(2^2\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sin 2x}{2}\right) = \frac{\sin 2x}{4}$$

9. Find the solution of x from $\dfrac{dy}{dt} = x,\ \dfrac{dx}{dt} = y$

Solution: Given $Dy = x,\ Dx = y$

$Dy - x = 0$ -------- (1) $-y + Dx = 0$ ------------ (2)

Eliminate y from (1) and (2), we get

$\left(D^2 - 1\right)x = 0$

A.E. is $m^2 - 1 = 0$, $m = \pm 1$

$C.F. = Ae^t + Be^{-t}$ Since $R.H.S. = 0 \Rightarrow P..I. = 0 \ \therefore\ x(t) = Ae^t + Be^{-t}$

10. Obtain the differential equation in terms of y, $\frac{dx}{dt} + 2x - 3y = 5t,\ \frac{dy}{dt} - 3x + 2y = 0$

Solution: $(D+2)x - 3y = 5t$ -------- (1) $-3x + (D+2)y = 0$ ------------- (2)

Eliminate x from (1) and (2), we get

$(1) \times 3 \Rightarrow \quad \cancel{3(D+2)x} - 9y = 15t$

$(2) \times (D+2) \Rightarrow \quad \cancel{-3(D+2)x} + (D+2)^2 y = 0$

$$\left((D+2)^2 - 9\right)y = 15t$$

$$\left(D^2 + 4D + 4 - 9\right)y = 15t$$

$$\left(D^2 + 4D - 5\right)y = 15t$$

11. Write Cauchy's homogeneous linear equation.

Answer: $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X$

12. Solve $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0$

Solution: $\left(x^2 D^2 + 4xD + 2\right)y = 0$

$x = e^z,\ z = \log x,\ xD = \theta,\ x^2 D^2 = \theta(\theta - 1)$

$\left[\theta(\theta-1) + 4\theta + 2\right]z = 0 \Rightarrow \left(\theta^2 + 3\theta + 2\right)z = 0$

$m^2 + 3m + 2 = 0 \Rightarrow m = -2, -1$

$z = Ae^{-2z} + Be^{-z}$

$y = Ae^{-2\log x} + Be^{-\log x}$ $\therefore\ y = \frac{A}{x^2} + \frac{B}{x}$

13. Transform the equation $x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+2y=x\log x$ into linear differential equation with constant coefficients.

Solution: $\left(x^2D^2+6xD+2\right)y=x\log x$

$x=e^z,\ z=\log x,\ xD=\theta,\ x^2D^2=\theta(\theta-1)$

$\left[\theta(\theta-1)+6\theta+2\right]y=e^z z\Rightarrow\left(\theta^2+5\theta+2\right)y=e^z z$

14. Transform the equation $(x^2D^2-xD+1)y=\left(\frac{\log x}{x}\right)^2$ into linear differential equation with constant coefficients.

Solution: $\left(x^2D^2+6xD+2\right)y=x\log x$

$x=e^z,\ z=\log x,\ xD=\theta,\ x^2D^2=\theta(\theta-1)$

$\therefore\left(\theta^2-2\theta+1\right)y=\left(ze^{-z}\right)^2$

15. Write Legendre's linear equation.

Answer: $(a+bx)^n\frac{d^ny}{dx^n}+A_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+A_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+...+A_ny=f(x)$

A_1, A_2, … are constants.

16. Transform the equation $\left[(2x+3)^2D^2-(2x+3)D-12\right]y=6x$ into linear differential equation with constant coefficients.

Solution: Put $2x+3=e^z,\ z=\log(2x+3),\quad x=\frac{e^z-3}{2}$

$$(2x+3)^2D^2=4\left(\theta^2-\theta\right)$$

$$(2x+3)D=2\theta$$

Hence the D.E is $\left(4\theta^2-6\theta-12\right)y=3e^z-9$

17. Transform the equation $\left[(3x+5)^2D^2-6(3x+5)D+8\right]y=0$ into linear differential equation with constant coefficients.

Solution: Put $3x+5=e^z,\ z=\log(3x+5)$

$(3x+5)^2D^2=9\left(\theta^2-\theta\right)$

$(3x+5)D=3\theta$

Hence the given equation becomes $\left[9\left(\theta^2-\theta\right)-18\theta+8\right]y=0\Rightarrow\left(9\theta^2-27\theta+8\right)y=0$

18. Transform the equation $\left[(x-2)D^2-6D+\frac{8}{(x-2)}\right]y=0$ into linear differential equation with constant coefficients.

Solution: Pre multiply (x–2) on both sides $\left[(x-2)^2 D^2 - 6(x-2)D + 8\right] y = 0$

$(x-2) = e^z, z = \log(x-2)$

$(x-2)^2 D^2 = 1^2(\theta^2 - \theta) = \theta^2 - \theta$

$(x-2)\ D = 1\ \theta = \theta$

$\therefore (1)$ implies

$\left[\left(\theta^2 - \theta\right) - 6\theta + 8\right] y = 0$

$\left[\theta^2 - 7\theta + 8\right] y = 0$

19. Write down the P.I. formula of solving ODE using Method of variation of parameters.

Solution: $P.I. = P\ f_1 + Q\ f_2$

Where $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$; $Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$

20. Find Q from the given C.F and $\left(D^2 + 4\right) y = 4 \tan 2x$

Solution:

$C.F = C_1 \cos 2x + C_2 \sin 2x$

$f_1 = \cos 2x,\ \ f_2 = \sin 2x$

$f_1 f_2' - f_2 f_1' = \cos 2x\left(2\cos 2x\right) - \sin 2x\left(-2\sin 2x\right) = 2\left(\cos^2 2x + \sin^2 2x\right) = 2$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} dx = \int \frac{\cos 2x\left(4\tan 2x\right)}{2} dx = 2\int \sin 2x\, dx = -\cos 2x + c$$

PART B

1 (a) Solve $\left(D^2 + 16\right) y = \cos^2 x$

(b) Solve $\left(D^2 + 4D + 3\right) y = e^{-x} \sin x$

2 (a) Solve $\left(D^2 - 4D + 4\right) y = e^{2x} + x^2$

(b) Solve $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin 3x\ \cos 2x$

3 (a) Solve $\frac{dx}{dt} + 2y = -\sin t, \frac{dy}{dt} - 2x = \cos t$

(b) Solve $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$

4 (a) Solve $\frac{dx}{dt} + \frac{dy}{dt} + x + y = 10e^t, \frac{dx}{dt} - \frac{dy}{dt} + x - y = 0$ given $x\left(0\right) = 2,\ y\left(0\right) = 3$

(b) Solve $Dx + y = e^t,\ x - Dy = t$

5 (a) Solve $\left(x^2 D^2 - 3xD + 4\right) y = x^2 \cos\left(\log x\right)$

(b) **Solve** $\left(x^2D^2-2xD-4\right)y=x^2+2\log x$

6 (a) **Solve** $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$

(b) **Solve** $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=(1+x)^2$

7 (a) **Solve** $(3x+2)^2y''+3(3x+2)y'-36y=3x^2+4x+1$

(b) **Solve** $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin\left[\log(1+x)\right]$

8 (a) **Solve** $(2+x)^2\frac{d^2y}{dx^2}-(2+x)\frac{dy}{dx}+y=2+x$

(b) **Solve** $(2x-3)^2y''-2(2x-3)y'-12y=6x-9$

9 (a) **Solve by the method of variation of parameters** $\frac{d^2y}{dx^2}+4y=\sec 2x$

(b) **Solve by the method of variation of parameters** $\frac{d^2y}{dx^2}+y=\tan x$

10(a) **Solve** $\left(D^2-4D+4\right)y=e^{2x}$ **by method of variation of parameters.**

(b) **Solve by the method of variation of parameters** $\frac{d^2y}{dx^2}+y=x\sin x$

UNIT- III LAPLACE TRANFORM

PART A

1. **State under which conditions Laplace transform of *f*(*t*) exists.**

Answer: The Laplace transform of *f*(*t*) exists if

(i) *f*(*t*) is piecewise continuous in [*a*,*b*] where *a* >0 .

(ii) *f*(*t*) is of exponential order.

2. **Find the Laplace transform of** $e^{-2t}t^{1/2}$**.**

Solution: $L\left[e^{-2t}t^{1/2}\right]=L\left[t^{1/2}\right]_{s\to s+2}$ $\quad\because$ *If* $L[f(t)]=F(s),\, then\, L\left[e^{-at}f(t)\right]=F(s)\big|_{s\to s+2}$

$$=\left[\Gamma\left(\frac{1}{2}+1\right)\right]_{s\to s+2}=\frac{\sqrt{\pi}}{2(s+2)^{3/2}}$$

3. **If L[f(t)] = F(s), prove that L{f(t/5)}= 5 F(5s).**

Solution: $L\left[f\left(\frac{t}{5}\right)\right]=\int_0^\infty e^{-st}f\left(\frac{t}{5}\right)dt$

put $\frac{t}{5}=u \Rightarrow$ 5du = dt

$$\therefore L\left[f\left(\frac{t}{5}\right)\right] = \int_0^\infty e^{-5su} f(u)5du = 5\int_0^\infty e^{-(5s)u} f(u)du = 5\,F(5s)$$

4. Find the Laplace transform of unit step function.

Solution: The unit step function is defined as $u_a(t) = \begin{cases} 0, t < a \\ 1, t > a,\ a \geq 0 \end{cases}$

We know that, $L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt$

Therefore $L[u_a(t)] = \dfrac{e^{-as}}{s}$

5. Find the Laplace transform of $f(t) = \cos^2 3t$.

Solution: $L[\cos^2 3t] = \dfrac{L(1) + L(\cos 6t)}{2}$ $\qquad \because \cos^2 t = \dfrac{1+\cos 2t}{2}$

$= \dfrac{1}{2s} + \dfrac{s}{2(s^2+36)}$ $\qquad \because L(1) = \dfrac{1}{s}, L(\cos at) = \dfrac{s}{s^2+a^2}$

$= \dfrac{s^2+18}{s(s^2+36)}$

6. Does $L\left[\dfrac{\cos at}{t}\right]$ exist?

Solution: $\underset{t\to 0}{lt} \dfrac{f(t)}{t} = \underset{t\to 0}{lt} \dfrac{\cos at}{t} = \dfrac{1}{0} = \infty$

$\therefore\ L\left[\dfrac{\cos at}{t}\right]$ does not exist.

7. Obtain the Laplace transform of sin2t – 2tcos2t in the simplified form.

Solution: $L[\sin 2t - 2t\cos 2t] = L[\sin 2t] - 2L[t\cos 2t]$

$$= \frac{2}{s^2+4} - (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{2}{s^2+4} + \left(\frac{4-s^2}{\left(s^2+4\right)^2}\right)$$

$$= \frac{s^2+12}{\left(s^2+4\right)^2}$$

8. Find $L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right]$

Solution: $L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right] = L^{-1}\left[\dfrac{(s+1)+1}{(s+1)^2+1}\right]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$= e^{-t}$ (cos t + sin t)

9. What is the Laplace transform of f(t) = f(t +10), 0< t < 10?

Solution: Given that f(t) is a periodic function with period 10

$$L\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

Put p=10, $L\{f(t)\} = \frac{1}{1-e^{-10s}}\int_0^{10} e^{-st} f(t)dt$

10. If L {f(t)} = $F(S)$, find the value of $\int_0^\infty f(t)dt$

Solution: $\int_0^\infty f(t)dt = \left[\int_0^\infty e^{-st} f(t)dt\right]_{s=0} = [L[f(t)]]_{s=0} = \left[\frac{s+2}{s^2+4}\right]_{s=0} = \frac{1}{2}$

11. Find $L^{-1}\left(\frac{s}{(s+2)^3}\right)$

Solution: $L^{-1}\left(\frac{s}{(s+2)^3}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^3}\right)$

$$= L^{-1}\left(\frac{1}{(s+2)^2}\right) - 2\, L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t}L^{-1}\left(\frac{2}{s^3}\right) = e^{-2t}t(1-t)$$

12. Find the Laplace transform $\sin^3 2t$

Solution: $L[\sin^3 2t] = \frac{1}{4}L[3\sin 2t - \sin 6t] = \frac{3}{4}L[\sin 2t] - \frac{1}{4}L[\sin 6t]$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right) = \frac{3}{2}\left(\frac{1}{s^2+4}\right) - \frac{6}{4}\left(\frac{1}{s^2+36}\right)$$

13. Find $L^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right)$

Solution: Let F(s) = $L^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right)$

$$F'(s) = \frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right) = \frac{-1}{s^2+1}$$

$$\therefore L^{-1}\left(F'(s)\right) = -\sin t; \qquad L^{-1}\left(F(s)\right) = \frac{-1}{t}L^{-1}\left[F'(s)\right]$$

$$\therefore L^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right) = \frac{\sin t}{t}$$

14. Solve using Laplace transform $\frac{dy}{dt} + y = e^{-t}$ given that y(0)=0.

Solution: Taking L.T. on both sides, we get L[y′] + L[y] = L[e^{-t}]

$$(s+1)\, L[y] = \frac{1}{s+1}$$

$$L[y] = \frac{1}{(s+1)^2}; \qquad \therefore\ y = L^{-1}\left(\frac{1}{(s+1)^2}\right) = t\,e^{-t}$$

15. Give an example for a function that do not have Laplace transform.

Solution: Consider f(t)=e^{t^2}, since $\underset{t\to\infty}{lt}\ e^{-st}e^{t^2} = \infty$, hence e^{t^2} is not of exponential order function.

Hence f(t)=e^{t^2}, does not have Laplace transform.

16. Can F(s) = $\frac{s^3}{(s+1)^2}$ be the transform of some f(t)?

Solution: $\underset{s\to\infty}{lt}\ F(s) = \underset{s\to\infty}{lt}\ \frac{s^3}{(s+1)^2} \neq 0$

Hence F(s) cannot be Laplace transform of f(t).

17. Evaluate $\int_0^t \sin u\cos(t-u)\,du$

Solution: Let $L\left[\int_0^t \sin u\cos(t-u)\,du\right] = L[\sin t * \cos t]$

= L[sint] L[cost] (by Convolution theorem)

$$= \frac{s}{\left(s^2+1\right)}\frac{1}{\left(s^2+1\right)}$$

$$= \frac{s}{\left(s^2+1\right)^2}$$

$$\int_0^t \sin u\cos(t-u)\,du = L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right] = \frac{t}{2}\sin t$$

18. Give an example for a function having Laplace transform but not satisfying the continuity condition.

Answer: f (t) = $t^{-1/2}$ has Laplace transform even though it does not satisfy the continuity condition. i.e. It is not piecewise continuous in (0,∞) as $\underset{t \to 0}{lt} f(t) = \infty$

19. **Define a Periodic function and give examples.**

Definition: A function f (t) is said to be periodic function if f(t + p) =f(t) for all t. The least value of p> 0 is called the period of f(t). For example, sin*t* and cos*t* are periodic functions with period 2π

20. **State the Convolution theorem.**

Answer: The convolution of two functions f(t) and g(t) is defined as $f(t) * g(t) = \int_0^t f(u) g(t-u) du$

PART B

1 (a) **Find the Laplace transform of (i) $t^2 e^{-t} \cos t$ (ii) cosh*at* cos*at***

(b) **Find $L\left[t^2 e^t \sin t\right]$**

2 (a) **Find $L\left[\frac{\sin^2 t}{t}\right]$**

(b) **Find the Laplace transform of $e^{-4t} \int_0^t t \sin 3t dt$**

3 (a) **Evaluate $\int_0^\infty te^{-2t} \cos t dt$ using Laplace transform.**

(b) **Verify initial and final value theorems for the function f(t) = 1 + e^{-t}(sin t + cos t)**

4 (a) **Find the Laplace transform of the Periodic function f(t) = $\begin{cases} k, & 0 \le t \le a \\ -k, & a \le t \le 2a \end{cases}$ and f(t+2a) = f(t) for all t**

(b) **Find the Laplace transform of f(t) = $\begin{cases} t, & 0 \le t \le a \\ 2a-t, & a \le t \le 2a \end{cases}$ and f(t+2a) = f(t) for all t**

5 (a) **Find the Laplace transform of the rectangular wave given by $f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$**

(b) **Find the inverse Laplace transform of $\frac{1}{(s+1)(s^2+4)}$**

6 (a) **Find $L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right]$**

(b) **Find the inverse Laplace transform of $\log\left(\frac{1+s}{s^2}\right)$**

7 (a) **Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.**

(b) **Using Convolution theorem find the inverse Laplace transform of $\frac{2}{(s+1)(s^2+4)}$**

8 (a) **Using Convolution theorem find** $L^{-1}\left[\dfrac{s}{\left(s^2+a^2\right)^2}\right]$

(b) **Solve Using Convolution theorem find** $L^{-1}\left[\dfrac{s^2}{\left(s^2+9\right)\left(s^2+25\right)}\right]$

9 (a) **Solve the equation $y''+9y=\cos 2t$ with $y(0)=1$ $y\left(\frac{\pi}{2}\right)=-1$**

(b) **Solve** $y''+2y'-3y=\sin t$, **given** $y(0)=0,\ y'(0)=0$

10(a) **Using Laplace transform solve the differential equation $y''-3y'-4y=2e^{-t}$ with $y(0)=y'(0)=1$.**

(b) **Determine y which satisfies the equation** $\dfrac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t,\ y(0)=1$

UNIT-IV ANALYTIC FUNCTIONS

PART A

1. **Define an analytic function (or) harmonic function (or) Regular function.**
Answer: A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

2. **Define an entire function.**
Answer: A function which is analytic everywhere in the finite plane is called an entire function. An entire function is analytic everywhere except at $z=\infty$
Ex. e^z, sinz, cosz, sinhz, coshz

3. **State the necessary condition for *f*(*z*) to be analytic [Cauchy – Riemann Equations].**
Answer: The necessary conditions for a complex function $f(z)=u(x,y)+iv(x,y)$ to be analytic in a region R are $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ **and** $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}$ (i.e) $u_x=v_y$ and $v_x=-u_y$

4. **State the sufficient conditions for *f*(*z*) to be analytic.**
Answer: If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x=v_y$ and $u_y=-v_x$. Then the function f(z) is analytic in a domain D.

5. **State the polar form of the C – R equations.**
Answer: In Cartesian coordinates any point z is $z=x+iy$
In polar coordinates it is $z=re^{i\theta}$ where r is the modulus and θ is the argument. Then the C- R equation in polar coordinates is given by
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}\quad,\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

6. **Define harmonic function.**
Answer: A real function of two variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function.

7. **Define conjugate harmonic function.**
Answer: If u and v are harmonic functions such that u + iv is analytic, then each is called the conjugate harmonic function of the other.

8. **Define conformal mapping.**
Answer: Consider the transformation $w=f(z)$, where f(z) is a single valued function of z, a point z_0 and any two curves C_1 and C_2 passing through z_0 in the Z plane,will be mapped onto a point w_0 and two

curves C'_1 and C'_2 in the W plane. If the angle between C_1 and C_2 at z_0 is the same as the angle between C_1' and C_2' at w_0 both in magnitude and direction, then the transformation w = f(z) is said to be conformed at the point z_0.

9. Define Isogonal transformation.

Answer: A transformation under which angles between every pair of curves through a point are preserved in magnitude but opposite in direction is said to be isogonal at that point.

10. Define Bilinear transformation.

Answer: The transformation $w = \frac{az+b}{cz+d}$, ad – bc $\neq 0$ where a, b,c,d are complex numbers is called a bilinear transformation. This is also called as Mobius or linear fractional transformation.

11. Define Cross Ratio.

Answer: Given four points z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is called the cross ratio of the four points.

12. Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic at $z = 0$.

Solution: Let z = x + iy and $\bar{z} = x - iy$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

$$u = x^2 + y^2 \quad , \quad v = 0$$

$$u_x = 2x \quad , \quad v_x = 0$$

$$u_y = 2y \quad , \quad v_y = 0$$

So the CR equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at z = 0. So f(z) may be differentiable only at z = 0. Now $u_x = 2y$, $v_y = 0$ and $u_y = 2y$, $v_x = 0$ are continuous everywhere and in particular at (0, 0). So f(z) is differentiable at z = 0 only and not analytic there.

13. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not?

Solution: Let $f(z) = 2xy + i(x^2 - y^2)$

$u = 2xy \quad ; v = x^2 - y^2$

$u_x = 2y$, $v_y = -2y$ and $u_y = 2x$, $v_x = 2x$

$u_x \neq v_y$ and $u_y \neq -v_x$

CR equations are not satisfied.

Hence f(z) is not an analytic function

14. Determine whether the function $2xy + i(y^2 - x^2)$ is analytic or not?

Solution: Let $f(z) = 2xy + i(y^2 - x^2)$

$u = 2xy \quad ; v = y^2 - x^2$

$u_x = 2y$, $v_y = 2y$ and $u_y = 2x$, $v_x = -2x$

$u_x = v_y$ and $u_y = -v_x$

CR equations are satisfied.

Hence f(z) is an analytic function

15. Prove that an analytic function whose real part is constant must itself be a constant.

Solution: Let f(z) = u + iv

Given u = constant. $\Rightarrow u_x = 0$ and $u_y = 0$

By CR equations

$u_x = 0 \Rightarrow, v_y = 0$; $u_y = 0 \Rightarrow v_x = 0$

$f'(z)$ = $u_x + iv_x = 0 + i0$

$f'(z) = 0 \Rightarrow f(z) = c$

f(z) is a constant.

16. Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic.

Solution: Given $u = 2x - x^3 + 3xy^2$

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = -6x \; ; \qquad \frac{\partial u}{\partial y} = 6xy \text{ and } \frac{\partial^2 u}{\partial y^2} = 6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$

Hence u is harmonic

17. Find a function w such that $w = u + iv$ is analytic, if $u = e^x \sin y$

Solution: Given $u = e^x \sin y$

$$\phi_1(x,y) = u_x = e^x \sin y \quad ; \quad \phi_2(x,y) = u_y = e^x \cos y$$

$$\phi_1(z,0) = e^z(0) = 0 \quad ; \quad \phi_2(z,0) = e^z$$

By Milne Thomson's method

$$f(z) = \int \phi_1(z,0)dz - i\int \phi_2(z,0)dz = 0 - i\int e^z dz = -ie^z + C$$

18. Obtain the invariant points (fixed points) of the transformation

Solution: $w = 2 - \frac{2}{z}$ **,** The invariant points are given by $z = 2 - \frac{2}{z}$

$$z = \frac{2z-2}{z} \Rightarrow z^2 = 2z - 2 \Rightarrow z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

19. Define a critical point of the bilinear transformation.

Answer: The point at which the mapping w = f(z) is not conformal, (i.e) $f'(z) = 0$ is called a critical point of the mapping.

20. Find the critical point of the transformation $w^2 = (z-\alpha)(z-\beta)$

Answer: $w^2 = (z-\alpha)(z-\beta)$

$$2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$= 2z - (\alpha+\beta)$$

$$\frac{dw}{dz} = 0 \Rightarrow z = \frac{(\alpha+\beta)}{2}$$

$$\frac{dz}{dw} = 0 \Rightarrow z = \alpha, \beta$$

Therefore the critical points are $z = \frac{(\alpha+\beta)}{2}, \alpha, \beta$

PART B

1 (a) Show that the function $f(z) = |z^2|$ is differentiable at $z = 0$ but not analytic at $z = 0$

(b) Test the analyticity of the function $w = \tan z$

2 (a) The function f(z) = u + iv is analytic, show that u = constant and v = constant are orthogonal.

(b) Prove that an analytic function with constant modulus is constant.

3 (a) Prove that every analytic function w = u + iv can be expressed as a function of z alone, not as a function of $\bar{z}$

(b) If f(z) is an analytic function, prove that $\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}|f(z)|^2=4|f'(z)|^2$

4 (a) Show that the function $u=\frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find f(z).

(b) If $\varphi=3x^2y-y^3$, find ψ where $w=\varphi+i\psi$ is an analytic function.

5 (a) Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y-\cos 2x}$

(b) Find the regular function whose imaginary part is $e^{-x}(x\cos y+y\sin y)$

6 (a) If *f*(*z*) = *u* + *iv* is an analytic function and $u-v=e^x(\cos y-\sin y)$ find *f*(*z*) in terms of *z*

(b) Find the analytic function *f*(*z*) = *u* + *iv* given that $2u+v=e^x(\cos y-\sin y)$

7 (a) Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$

(b) Find the image of the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ under the transformation $w=\frac{1}{z}$

8 (a) Discuss the transformation of $w=e^z$

(b) Find the image of the circle $|z-1|=1$ under the transformation $w=z^2$

9 (a) Find the bilinear transformation of the points –1, 0, 1 in z- plane onto the points 0, *i*, 3*i* in w- plane.

(b) Find the bilinear transformation that maps *z* = 1, i, –1 into *w* = 2, *i*, –2

10(a) Find the bilinear transformation which maps the points $0,1,\infty$ in z-plane into itself in w-plane.

(b) Find the bilinear transformation which maps the points $z=\infty,i,0$ into $w=0,i,\infty$ respectively.

UNIT-V COMPLEX INTEGRATION

PART A

1. State Cauchy's Integral formula for Complex Integration.

Statement: If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if 'a' is any point with in C, then $f(a)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-a}dz$

2. State Cauchy's integral formula for derivative of an analytic function.

Statement: If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if 'a' is any point with in C, then $f(a)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-a}dz$, $f'(a)=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-a)^2}dz$

$f''(a)=\frac{2!}{2\pi i}\int_C\frac{f(z)}{(z-a)^3}dz$ and in general $f^{(n)}(a)=\frac{n!}{2\pi i}\int_C\frac{f(z)}{(z-a)^{n+1}}dz$

3. What is the value of $\int_C e^z dz$, where C is $|z|=1$?

Answer: Since $f(z)=e^z$ is analytic and its derivative is continuous at all points inside the unit circle

$|z|=1$. Therefore by Cauchy's integral theorem $\int_C f(z)dz = 0$

4. **Evaluate $\int_C \frac{\cos \pi z}{z-1}dz$ where C is $|z|=2$**

Solution: We know that by Cauchy's integral formula $\int_C \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i\, f(a) & \text{if } a \text{ lies inside C} \\ 0 & \text{if } a \text{ lies outside C} \end{cases}$

Given $\int_C \frac{\cos \pi z}{z-1}dz$, Here $f(z)=\cos \pi z$ and $a = 1$ lies inside $|z|=2$

Therefore by Cauchy's integral Formula $\int_C \frac{\cos \pi z}{z-1}dz = 2\pi i(\cos \pi z)_{z=1} = -2\pi i$

5. **Evaluate $\int_C \frac{3z^2+7z+1}{z+1}dz$, where C is $|z|=\frac{1}{2}$**

Solution: By Cauchy's integral formula $\int_C \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i\, f(a) & \text{if } a \text{ lies inside C} \\ 0 & \text{if } a \text{ lies outside C} \end{cases}$

Given $\int_C \frac{3z^2+7z+1}{z+1}dz$, f (z) = $3z^2+7z+1$ and a = –1 lies outside $|z|=\frac{1}{2}$.

Therefore $\int_C \frac{3z^2+7z+1}{z+1}dz = 0$

6. **Evaluate $\int_C \frac{e^{2z}}{\left(z^2+1\right)}dz$ where C is $|z|=\frac{1}{2}$**

Solution: By Cauchy's integral formula $\int_C \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i\, f(a) & \text{if } a \text{ lies inside C} \\ 0 & \text{if } a \text{ lies outside C} \end{cases}$

Given $\int_c \frac{e^{2z}}{z^2+1}dz = \int_c \frac{e^{2z}}{(z+i)(z-i)}dz$, $f(z)=e^{2z}$ and $a = \pm i$ lies outside c.

Therefore $\int_c \frac{e^{2z}}{z^2+1}dz = 0$

7. **Evaluate $\int_C \frac{e^{2z}}{(z+2)^4}dz$ where C is $|z|=1$ using Cauchy's integral formula.**

Solution: By Cauchy's integral formula $\int_C \frac{f(z)}{z-a}dz = \begin{cases} 2\pi i\, f(a) & \text{if } a \text{ lies inside C} \\ 0 & \text{if } a \text{ lies outside C} \end{cases}$

Given $\int_C \frac{e^{2z}}{(z+2)^4}dz$, $f(z)=e^{2z}$ and a = -2 lies outside C.

Hence $\int_C \frac{e^{2z}}{(z+2)^4}dz = 0$

8. Obtain the Taylor's series expansion of log(1+z) when $|z| = 0$

Solution: Let $f(z)$ =log(1+z) $\qquad f(0) = \log 1 = 0$

$$f'(z) = \frac{1}{1+z} \qquad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z) = \frac{2}{(1+z)^3} \qquad f'''(0) = 2$$

$$f^{iv}(z) = \frac{-6}{(1+z)^4} \qquad f^{iv}(0) = -6$$

$$\log(1+z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \ldots\ldots$$

$$\log(1+z) = 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots$$

9. Find the region of convergence to expand cos *z* in Taylor's series.

Solution: Let f(z) = cos z

$$f^n(z) = \cos\left(z + \frac{n\pi}{2}\right)$$

$$f^n\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{4}\right)}{n!}\left(z - \frac{\pi}{4}\right)^n = \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}{n!}\left(z - \frac{\pi}{4}\right)^n$$

The region of convergence is $\left|z - \frac{\pi}{4}\right| < \infty$

10. Expand $f(z) = \frac{1}{(1+z)}$ in the region $|Z| < 1$ Using this result expand $\tan^{-1}z$ in powers of *z*.

Solution: $f(z) = (1+z)^{-1}$ By using binomial series expansions, $(1+z)^{-1} = 1 - z + z^2 - z^3 + \ldots$

$$\frac{1}{(1+z)} = 1 - z + z^2 - z^3 + \ldots \quad \_\_\_\_\_\_\_\_\_\_ (1)$$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \ldots \quad \_\_\_\_\_ (2)$$

If f(z)=tan^{-1}z, then $f'(z) = \frac{1}{1+z^2}$

Hence by integrating (2) with respect to z, $\tan^{-1}z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots$

11. State Laurent's series.

Solution: If C_1, C_2 are two concentric circles with centre z=a and radii r_1 and r_2 $\left(r_1 < r_2\right)$ and if f(z) is analytic inside and on the annular region between C_1 and C_2, then, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} b_n (z-a)^{-n}$$,where $a_n = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{(z-a)^{n+1}}dz$ and

$b_n = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{(z-a)^{-n+1}}dz$ where C is any circle lying between C_1 and C_2 with centre at $z = a$ for all n.

12. Obtain the Laurent expansion of the function $\dfrac{e^z}{z^2}$ in the neighbourhood of its singular point. Hence find the residue at that point.

Solution: z=0 is a pole of order 2 $f(z) = \dfrac{e^z}{z^2}$ becomes $f(z) = \dfrac{1}{z^2}\left[1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dots\right]$

$f(z) = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{z^2}\left[\dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dots\right]$ Residue of f(z) = Coefficient of $\dfrac{1}{z}$ =1

13. Obtain the Laurent expansion of the function $\dfrac{e^z}{(z-1)^2}$ in the neighbourhood of its Singular point. Hence find the residue at that point.

Solution: z = 1 is a pole of order 2

Put z–1=u .Then $f(z) = \dfrac{e^z}{(z-1)^2}$ becomes $f(z) = \dfrac{e.e^u}{u^2} = \dfrac{e}{u^2}\left[1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \dots\right]$

$$f(z) = \frac{e}{u^2} + \frac{e}{u} + \frac{e}{u^2}\left[\frac{u^2}{2!} + \frac{u^3}{3!} + \dots\right]$$

$$f(z) = \frac{e}{(z-1)^2} + \frac{e}{(z-1)} + \frac{e}{(z-1)^2}\left[\frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \dots\right]$$

Residue of $f(z)\big|_{z=1}$ = Coefficient of $\dfrac{1}{z-1}$ = e

14. Find the Singular points of $f(z) = \dfrac{\sin z}{(z+1)(z-2)}$

Solution: Since $f(z)$ is not analytic at $z = -1$ and $z = 2$, Hence the singular points are $z = -1$ and $z = 2$

15. What is the Nature of the singularity at z=0 of the function $\dfrac{\sin z - z}{z^3}$.

Solution: f(z) = $\dfrac{\sin z - z}{z^3}$ The function f(z) is not defined at z = 0

But by L'Hospital's rule,

$$\lim_{z\to 0}\frac{\sin z - z}{z^3} = \lim_{z\to 0}\frac{\cos z - 1}{3z^2} = \lim_{z\to 0}\frac{-\sin z}{6z} = \lim_{z\to 0}\frac{-\cos z}{6} = \frac{-1}{6}$$

Therefore the limit exists and is finite. Hence z = 0 is a removable singularity.

16. Define essential singularity with an example.

Solution: If the principal part contains an infinite number of non zero terms, then $z = z_o$ is known as a essential singularity.

$$f(z) = e^{\frac{1}{z}} = 1 + \frac{\frac{1}{z}}{1!} + \frac{\frac{1}{z^2}}{2!} + \ldots$$ has z = 0 as an essential singularity.

Since f(z) is an infinite series of negative powers of z.

17. Define removable singularity with an example.

Solution: If the principal part of f(z) contains no terms, That is $b_n = 0$ for all n, then the singularity $z = z_o$ is known as the removable singularity of f(z).

Example, $f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right]$ $= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - - - -$

There is no negative power of z. Hence z = 0 is a removable singularity.

18. State Cauchy's residue theorem.

Solution: If $f(z)$ is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \ldots a_n$ inside C, then $\int_C f(z)dz = 2\pi i$ (Sum of the residues of f(z) at these singular points).

19. Find the residue of the function $f(z) = \frac{z^2}{(z-1)(z-2)^2}$ at a simple pole.

Solution: Given $f(z) = \frac{z^2}{(z-1)(z-2)^2}$. Here z = 1 is a simple pole

$$\text{Res}\left[f(z)\right]_{z=1} = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z-2)^2} = 1$$

20. Find the poles and residues of $f(z) = \frac{z}{z^2 - 3z + 2}$

Solution: Poles of $f(z)$ are z = 1, 2

$$\text{Res}\left[f(z)\right]_{z=1} = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z-2)} = -1$$

$$\text{Res}\left[f(z)\right]_{z=2} = \lim_{z\to 2}(z-2)\frac{z}{(z-1)(z-2)} = 2$$

PART B

1(a) Using Cauchy's integral formula, find $\int_C \frac{z+4}{z^2+2z+5}dz$, where C is $|z+1-i| = 2$

(b) Using Cauchy's integral formula, evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-1)}dz$, where C is $|z| = 3$

2(a) Using Cauchy's integral formula, evaluate $\int_C \frac{z}{(z-1)(z-2)^2}dz$, where C is $|z-2| = \frac{1}{2}$

(b) **Using Cauchy's integral formula, evaluate** $\int_C \frac{1}{(z-2)\ \ (z+1)^2}dz$ **, where C is** $|z| = \frac{3}{2}$

3(a) **Expand** $f(z) = \frac{1}{z}$ **as a Taylor's series about** $z = 1$ **and** $z = 2$

(b) **Find the Taylor's series expansion of f(z)=** $\frac{z}{(z+1)(z-3)}$ **, about z = 0**

4(a) **Expand f(z) =** $\frac{z^2-1}{z^2+5z+6}$ **in a Laurent's series expansion for** $|\ z\ | > 3$ **and** $2 < |\ z\ | < 3$

(b) **Obtain the Laurent's series expansion for the function f(z) =** $\frac{4z}{\left(z^2-1\right)\left(z-4\right)}$ **in**

$|\ z-1\ | > 4$ **and** $2 < |\ z-1\ | < 3$

5(a) **Find the residues of f(z)=** $\frac{z^2}{(z-1)(z+2)^2}$ **at its isolated singularities using Laurent's series expansion.**

(b) **Evaluate f(z)=** $\frac{1}{(z+1)(z+3)}$ **in Laurent's series valid for the regions** $|\ z| > 3$ **and** $1 < |z| < 3$

6(a) **Using Cauchy's residue theorem evaluate** $\int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz$ **, where C is** $|z| = 2$

(b) **Evaluate** $\int_C \frac{z-1}{\left(z+1\right)^2(z-2)}dz$ **, where C is** $|z-i| = 2$ **using Cauchy's residue theorem**

7 **Evaluate** $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta}$ **, using contour integration.**

8 **Evaluate** $\int_0^{2\pi} \frac{d\theta}{13+4\sin\theta}$ **, using contour integration.**

9(a) **Evaluate** $\int_{-\infty}^{\infty} \frac{dx}{x^4+8x^2+16}$, **, using contour integration.**

(b) **Evaluate** $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a > 0, b > 0$**, using contour integration.**

10(a) **Evaluate** $\int_0^{\infty} \frac{\cos ax\, dx}{x^2+1}$, $a > 0$, **using contour integration.**

(b) **Evaluate** $\int_0^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx$, **using contour integration.**