ST. JOSEPH'S COLLEGE OF ENGINEERING

ST. JOSEPH'S INSTITUTE OF TECHNOLOGY

OMR, CHENNAI - 600 119.

I YEAR B.E / B.TECH

UNIT - I

MATRICES Sub. code : MA6151

1. a). Symmetric matrix :-

A Given matrix A is said to be Symmetric if $A = A^T$ (Transpose of A).

b). Orthogonal matrix :-

A Sq. matrix A with real elts is said to be orthogonal if $AA' = A'A = I$

2. Let A be a Sq. matrix. If $|A| \neq 0$ then the gn vectors are linearly Independent. If $|A| = 0$ it is linearly dependent.

3. characteristic Eqns, Eigenvalues, Eigenvectors & Cayley-hamilton thm.

(i). Let A be the gn Square matrix

(ii) compute $|A - \lambda I| = 0$ where 'I' is the unit matrix

(iii) $|A - \lambda I| = 0$ is called the characteristic Eq. (in terms of λ)

(iv) Solve the characteristic Eq. Values of A are called as Eigenvalues

(V). For Each Eigenvalue find the Corresponding Eigenvectors X which Satisfies the Eqns. $(A-\lambda I)X=0$.

(Vi). In the characteristic Eq. If we put $\lambda = A$ & Let I be the unit matrix (If A satisfies its own characteristic Eq. then C-H Thm is proved) then the answer is a null matrix which Implies C-H Thm is verified.

4. The matrix of the Q.F

$$X'AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x^3 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{31}x_3x_1.$$

is gn by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The Elts of A is obtained as follows

In the I Row : a_{11} = Coeffi. of x_1^2

$a_{12} = 1/2$ " " x_1x_2

$a_{13} = 1/2$ " " x_1x_3

II Row : $a_{21} = 1/2$ " " x_1x_2

a_{22} = Coeffi. of x_2^2

$a_{23} = 1/2$ " " x_2x_3

III Row: $a_{31} = \frac{1}{2}$ coeffi. of $x_1 x_3$

$a_{32} = \frac{1}{2}$ " " $x_2 x_3$

$a_{33} =$ Coeffi. of x_3^2.

5. Reduction of Q.F to Canonical form.

(i) Find the Chara. Eq., Eigenvalues, Eigenvectors of A.

(ii) corresponding to a Eigenvector find the normalised Eigenvectors.

(iii) Find the normalised modal matrix 'P'

(iv) $P' = (P)^T$

(v) Orthogonal transformation $X = PY$

(vi) $Y'(P'AP)Y$ gives the sq. Q.F where

$$Y' = (y_1 \ y_2 \ y_3) \quad \& \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

6. Diagonalisation of a matrix:

(i) For the matrix A find $|A - \lambda I| = 0$

(ii) Find the Eigenvalues, Eigenvectors of A

(iii) The Eigenvectors Corresponding to Eigenvalues are written column wise & it is denoted by B.

(iv) Find $B^{-1} = \frac{Adj. B}{|B|}$

(v) $D = B^{-1}AB$; D is called the diagonal matrix

(vi) $B^{-1}A^{n}B = D^{n}$.

7. Nature of a Q.F.

Let $X'AX$ be the gn Q.F in the

Variables $x_1, x_2 \ldots x_n$.

(w) $X'AX = d_1x_1^2 + d_2x_2^2 + \ldots + d_nx_n^2 \rightarrow (1)$.

Let the rank of A be r.

Then $X'AX$ contains only 'r' terms.

The no. of +ve in (1) is called the Index of the Q.F. & it is denoted by 's'

The diff. b/w the no. of +ve terms and the -ve terms is called the Signature of the Q.F.

(ie) Signature = No. of +ve terms − No. of −ve terms.

$= s - (\text{total no. of terms} - \text{+ve terms})$

$= s - (\text{rank of } A - s)$

$= s - (r - s)$

$\therefore$ Signature $= 2s - r$.

where s - no. of +ve terms ; r - rank of A.

Let $X'AX$ be the gn real Q.F, where 'A' is the matrix of Q.T.

Let the Eigenvalues of A be $\lambda_1, \lambda_2, \lambda_3$. Now the Q.F $X'AX$ is said to be

(a) +ve definite – If all the Eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are +ve.

(b) –ve definite – " " " " " " –ve

(c) positive semidefinite – If atleast one Eigenvalue is zero & the remaining are +ve.

(d) negative semidefinite – If atleast one Eigenvalue is zero & the remaing are –ve.

(e) Indefinite – If some Eigenvalues are +ve & some Eigenvalues are –ve.

CHAPTER - I

MATRICES

CHARACTERISTIC EQUATION.

Let A be a given matrix. Let λ be a scalar. The equation $\det[A-\lambda I] = 0$ (or) $|A-\lambda I| = 0$ is called the characteristic equation of the matrix A.

Example 1: Find the characteristic equation of $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Solution: The characteristic equation is $|A-\lambda I| = 0$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 0 = 0$$

$$(1-\lambda)^2 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

Example 2: Find the characteristic equation of $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

The characteristic eqn is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 6 = 0$$

$$4-\lambda-4\lambda+\lambda^2+6 = 0$$

$$\lambda^2-5\lambda+10 = 0$$

Example 3: Find the characteristic equation of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution : Let the characteristic equation be

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \quad \cdots\to (1)$$

where, $a_1 = \begin{cases} \text{Sum of leading} \\ \text{diagonal element} \end{cases}$

$$= -2+1+0$$

$$= -1 \quad \cdots\to (2)$$

$a_2 = \begin{cases} \text{Sum of the minors of} \\ \text{the leading diagonal} \\ \text{elements} \end{cases}$

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 0-12+0-3-2-4$$

$$= -21 \quad \cdots\to (3)$$

$a_3 = |A|$

$$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$= -2(0-12) - 2(0-6) - 3(-4+1)$

$= 24 + 12 + 9$

$= 45$ ....→ (4)

substituting (2), (3) and (4) in (1), we get

The characteristic equation is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

EIGEN VALUES AND EIGEN VECTORS.

Definition: Eigen Values.

The values of λ obtained from the characteristic equation $|A - \lambda I| = 0$ are called Eigenvalues of 'A'. [Or Latent Values of A or characteristic values of A].

Definition: Eigen vectors.

Let A be a square matrix of order 3 and λ be a scalar (eigen value). The column matrix $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ which satisfies $(A - \lambda I)X = 0$ is called Eigen vectors (or) Latent vectors (or) characteristic vector.

TO FIND EIGENVALUES AND EIGENVECTORS OF A GIVEN MATRIX

Example 1 : Find the eigenvalues of matrix $\begin{bmatrix} 6 & 10 \\ 14 & 25 \end{bmatrix}$

Solution :

The characteristic equation is

$$\begin{vmatrix} 6-\lambda & 10 \\ 14 & 25-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(25-\lambda) - 140 = 0$$

$$150 - 6\lambda - 25\lambda + \lambda^2 - 140 = 0$$

$$\lambda^2 - 31\lambda + 10 = 0$$

$$\lambda = \frac{31 \pm \sqrt{961 - 40}}{2}$$

$$= \frac{31 \pm \sqrt{921}}{2}$$

Example 2 : Find the eigenvalues of the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Solution :

The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 3 = 0$$

$$8 - 6\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda-1)(\lambda-5) = 0$$

$$\lambda = 5 \text{ (or) } 1$$

$\therefore$ The eigenvalues are 1 and 5.

PROBLEMS ON NON SYMMETRIC MATRICES WITH NON REPEATED EIGEN VALUES

Example 1: Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

Solution: step 1: To find characteristic equation and eigenvalues:

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 3 = 0$$

$$-1 - \lambda + \lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda = \pm 2$$

$\therefore$ The eigenvalues are $\boxed{\lambda = 2, -2}$

step 2: To find eigenvectors:

The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is given by the equation

$$(A - \lambda I) X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (1-\lambda) x_1 + x_2 &= 0 \\ 3x_1 + (-1-\lambda) x_2 &= 0 \end{aligned}\right\} \rightarrow (A)$$

Case (i): when $\lambda = 2$, we get from Ⓐ

$$-x_1 + x_2 = 0 \quad \rightarrow (1)$$

$$3x_1 - 3x_2 = 0 \quad \rightarrow (2)$$

From equations (1) and (2), we get

$$x_1 = x_2$$

put $x_2 = k \Rightarrow x_1 = k$

The eigenvector is $X_1 = \begin{pmatrix} k \\ k \end{pmatrix}$

(or) $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (put $k=1$)

Case (ii): when $\lambda = -2$, we get from Ⓐ

$$3x_1 + x_2 = 0$$

$$3x_1 + x_2 = 0$$

$$\Rightarrow \quad x_2 = -3x_1$$

put $x_1 = k \Rightarrow x_2 = -3k$

$$\therefore \quad X_2 = \begin{pmatrix} k \\ -3k \end{pmatrix}$$

The simplest eigenvector

$X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ [put $k=1$]

Conclusion:

characteristic equation	Eigenvalues	Eigenvectors
$\lambda^2 - 4 = 0$	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	$\lambda_2 = -2$	$X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$a_3 = |A| = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix}$$

$$= 2(-3-2) - 2(-6+7) + 0(4+7)$$

$$= -10-2$$

$$= -12 \quad \rightarrow (4)$$

Substituting (2), (3) and (4) in (1), we get the characteristic equation is

$$\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0$$

$$\lambda^3 - 13\lambda + 12 = 0$$

step 2: To find eigenvalues.

$$\lambda^3 - 13\lambda + 12 = 0$$

when $\lambda = 1$

$$1 - 13 + 12 = 0$$

$\therefore$ 1 is a root.

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & \boxed{1 \quad 1 \quad -12} & & & 0 \end{array}$$

$$\lambda^2 + \lambda - 12 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1+48}}{2}$$

$$= 3 \text{ or } -4$$

$\therefore$ Eigenvalues are $\lambda = 1, 3, -4$.

Example 2: Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$$

Solution:

Step 1: To find characteristic equation

Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$

The characteristic equation is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \rightarrow (1)$$

where $a_1 = \left\{ \begin{array}{l} \text{Sum of leading} \\ \text{diagonal elements} \end{array} \right.$

$= 2+1-3$

$= 0 \qquad \rightarrow (2)$

$a_2 = \left\{ \begin{array}{l} \text{sum of the minors} \\ \text{of the leading diagonal} \\ \text{elements} \end{array} \right.$

$= \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$

$= (-3-2) + (-6-0) + (2-4)$

$= -5-6-2$

$= -13 \qquad \rightarrow (3)$

(a)

Step 3: To find Eigen vectors:

The Eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is gn by the Eq.

$$(A - \lambda I)X = 0.$$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + 0x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 + x_3 &= 0 \\ -7x_1 + 2x_2 + (-3-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (A)$$

Case (i). when $\lambda = 1$ we get from (A)

$$x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 + 0x_2 + x_3 = 0$$

$$-7x_1 + 2x_2 - 4x_3 = 0$$

Considering 1st two Eqns. & using cross rule method we have

$$x_1 = 2k \qquad x_2 = -k \qquad x_3 = -4k$$

The Eigen vector is $X_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ (put $k=1$)

Case (ii) when $\lambda = 3$ we get from (A)

$$-x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 - 2x_2 + 0x_3 = 0$$

$$-7x_1 + 2x_2 - 6x_3 = 0$$

Considering first two Eqns. & applying rule of Cross multiplication we have

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k$$

$x_1 = 2k \qquad x_2 = k \qquad x_3 = -2k$

The Eigenvector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ (put $k=1$)

Case (iii) when $\lambda = -4$ we get from (A)

$6x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + 5x_2 + x_3 = 0$

$-7x_1 + 2x_2 + x_3 = 0$

Applying cross rule in 1^{st} two Eqns.

$x_1 = 2k \qquad x_2 = -6k \qquad x_3 = 26k$.

The Eigenvector is $X_3 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$ (put $k = 1/2$)

Characteristic Eq.	Eigenvalues	Eigenvectors
	$\lambda = 1$	$X_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$
$\lambda^3 - 13\lambda + 12 = 0$	$\lambda_2 = 3$	$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$
	$\lambda_3 = 4$	$X_3 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$

PROBLEMS ON NON-SYMMETRIC MATRICES WITH REPEATED EIGENVALUES

1). Find the Eigenvalues & Eigenvectors of $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

The characteristic Eq. is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$\lambda = 2, 2, 2$ .. (Eigenvalues)

The Eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is gn by the Eq.

$$(A - \lambda I) X = 0.$$

$$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case(i): when $\lambda = 2$.

Eq. (1) becomes

$$0x_1 + x_2 + 0x_3 = 0 \longrightarrow (2)$$

$$0x_1 + 0x_2 + x_3 = 0 \longrightarrow (3)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \longrightarrow (4)$$

Taking Eq. (2) & (3) & applying Cross rule method

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = k$$

$$x_1 = k, \quad x_2 = 0, \quad x_3 = 0.$$

$\therefore$ The Eigenvector is $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (Taking $k=1$)

$\therefore$ 2^{nd} & 3^{rd} Eigenvector is also same as X_1.

Characteristic Eq.	Eigenvalues	Eigenvectors
	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$(2-\lambda)^3 = 0$	$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\lambda_3 = 2$	$X_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ These 3 Eigen vectors are linearly dependent.

PROBLEMS ON SYMMETRIC MATRICES WITH DIFFERENT EIGENVALUES.

1). Find the Eigenvalues & Eigenvectors of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

Sol:- To find characteristic Eq. & Eigenvalues.

The characteristic Eq. is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(3-\lambda)^2-1\right]=0.$

$1-\lambda=0$ or $9+\lambda^2-6\lambda-1=0$

$\lambda=1$ or $\lambda^2-6\lambda+8=0.$

$\lambda=1$ or $\lambda=\frac{6\pm 2}{2}.$

$\lambda=1$ or $\lambda=4$ or $2.$

$\therefore$ The Eigenvalues are $\lambda=1,2\ \&\ 4$

The Eigen vector $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is gn by

$(A-\lambda I)X=0.$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1+0x_2+0x_3&=0 \\ 0x_1+(3-\lambda)x_2-x_3&=0 \\ 0x_1-x_2+(3-\lambda)x_3&=0 \end{aligned}\right\}\longrightarrow (A)$$

when $\lambda=1$ we get from (A)

$0x_1+0x_2+0x_3=0$

$0x_1+2x_2-x_3=0$

$0x_1-x_2+2x_3=0.$

Taking 2^{nd} & 3^{rd} Eqns. & applying Cross rule method we get

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0} = k \quad (\text{say})$$

$$x_1 = 3k \qquad x_2 = 0 \qquad x_3 = 0.$$

$\therefore$ The Eigen vector is $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Case (ii). when $\lambda = 2$ we get from (A)

$$-x_1 + 0x_2 + 0x_3 = 0.$$

$$0x_1 + x_2 - x_3 = 0$$

$$0x_1 - x_2 + x_3 = 0$$

Taking 1^{st} two Eqns. & applying Cross rule method

we get $\dfrac{x_1}{0} = \dfrac{x_2}{-1} = \dfrac{x_3}{-1} = k \quad (\text{say})$

$$x_1 = 0 \qquad x_2 = -k \qquad x_3 = -k$$

$\therefore$ The Eigenvector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ (Taking $k = -1$)

Case (iii) when $\lambda = 4$ we get from (A)

$$-3x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0.$$

Taking 1^{st} two Eqns. & applying cross rule method, we get

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3} = k$$

$$x_1 = 0k \qquad x_2 = -3k \qquad x_3 = 3k$$

$\therefore$ The simplest Eigenvector is $X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ (Taking $k = 1/3$)

Characteristic Eq.	Eigenvalues	Eigenvectors
$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$	$\lambda_1 = 1$	$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
	$\lambda_3 = 4$	$X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

PROBLEMS ON SYMMETRIC MATRICES WITH REPEATED EIGENVALUES

1). Find the Eigenvalues & Eigenvectors of $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

Sol:-

The characteristic Eq.

$$\lambda^3 - a_1 \lambda^2 + a_2 \lambda - a_3 = 0.$$

where a_1 = Sum of the leading diagonal elts

$$= 2 + 2 + 2$$

$$= 6.$$

$a_2 = \left\{ \begin{array}{l} \text{Sum of the minors of the} \\ \text{leading diagonal elts.} \end{array} \right.$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 4-1+4-1+4-1$$

$$= 9.$$

$$a_3 = |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 6-1-1$$

$$= 4$$

we get the characteristic Eqn. as

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

To find Eigen values.

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

when $\lambda = 1$, $(1)^3 - 6(1)^2 + 9(1) - 4$

$$= 0.$$

$\therefore \lambda = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 9 & -4 \\ & 0 & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 4 = 0.$$

$$\lambda = \frac{5 \pm \sqrt{25-16}}{2}$$

$$= \frac{5 \pm 3}{2}$$

$$= 4 \text{ or } 1.$$

$\therefore$ Eigen values are $\lambda = 1, 1, 4$.

To find Eigen vectors :-

The Eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is gn by the Eq.

$$(A-\lambda I)\, X = 0.$$

$$\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 - x_2 + x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 - x_3 &= 0. \\ x_1 - x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (A).$$

Case (i). when $\lambda = 1$ we get from (A)

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

The above Eqns. are the same

$$x_1 - x_2 + x_3 = 0.$$

$$x_1 = x_2 - x_3.$$

By putting $x_2 = k$ $x_3 = k$ we get

$$x_1 = k_1 - k_2.$$

The Eigenvector is $X_1 = \begin{pmatrix} k_1 - k_2 \\ k_1 \\ k_2 \end{pmatrix}$

The Simplest Eigenvector is obtained by putting

$k_1 = 1$ $k_2 = 0$ we get

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

One more Eigenvector is obtained by putting

$k_1 = 0$ $k_2 = -1$ we get

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Case (ii) when $\lambda = 4$ we get from A

$$-2x_1 - x_2 + x_3 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

$$x_1 - x_2 - 2x_3 = 0.$$

Taking 1st & 2nd Eqns. & applying Cross rule method, we get

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3} = k.$$

$x_1 = 3k$ $x_2 = -3k$ $x_3 = 3k$.

$\therefore$ The Simplest Eigenvector is $X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(Putting $k = 1/3$)

Characteristic Eq.	Eigenvalues	Eigenvectors
$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$	$\lambda_1 = 1$	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
	$\lambda_2 = 1$	$X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
	$\lambda_3 = 4$	$X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

PROPERTIES OF EIGENVALUES:

PROPERTY 1:

Sum of Eigenvalues is Equal to the Sum of the diagonal elements of the matrix

1). Find the Sum of the Eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

Sum of the Eigenvalues = $\left\{ \begin{array}{l} \text{Sum of leading} \\ \text{diagonal elts.} \end{array} \right.$

$= 1+1$

$= 2.$

2). Find the Sum of the Eigenvalues of $A = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 1 & -3 \\ -4 & -2 & 13 \end{pmatrix}$

Sum of the Eigenvalues = $\left\{ \begin{array}{l} \text{Sum of leading} \\ \text{diagonal elts} \end{array} \right.$

$= 2 + 1 + 13$

$= 16.$

PROPERTY 2:

Product of Eigenvalues is Equal to its determinant Value.

1). Find the product of the Eigenvalues of

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$$

Product of the Eigenvalues $= |A|$

$$= \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 15 - 1$$

$$= 14.$$

2). Find the product of the Eigenvalues of A

$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

Product of the Eigenvalues $= |A|$

$$= \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 42 - 2 - 4$$

$$= 36.$$

PROPERTY 3:

Every Square matrix & its transpose have the Same Eigenvalues.

1). If 2, −2 are the Eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ then find Eigen values of A^T

Eigenvalues of A = Eigen values of A^T

$\therefore$ Eigenvalues of $A^T = 2, -2$.

2). If 2, 2, 3 are the Eigenvalues of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ then find the Eigenvalues of $A^T = B$.

Let $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$

$$A^T = \begin{pmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{pmatrix} = B.$$

$\because$ the matrix B is a transpose of A.

the Eigen Values of B are 2, 2, 3.

3). Find the Sum & product of the Eigenvalues of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Sum of the Eigenvalues = $\left\{ \begin{array}{l} \text{Sum of leading} \\ \text{diagonal elts.} \end{array} \right.$

$= -2+1+0$

$= -1.$

Product of the Eigenvalues = $\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= 45.$

4). The product of two Eigenvalues of the matrix A

$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the 3rd Eigenvalue.

Let the Eigenvalues be $\lambda_1, \lambda_2, \lambda_3$.

Gn. $\lambda_1 \lambda_2 = 16.$ $\longrightarrow (1)$

But $\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 6(9-1) + 2(-6+2) + 2(2-6)$

$= 32.$ $\longrightarrow (2)$

Sub (2) in (1) $\boxed{\lambda_3 = 2}$

PROPERTY 4 :-

If $\lambda_1, \lambda_2 \ldots \lambda_n$ are Eigenvalues of matrix A, then the Inverse A^{-1} has the Eigenvalues $1/\lambda_1, 1/\lambda_2 \ldots 1/\lambda_n$

1). If the Eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are $2, -2$ then find the Eigenvalues of A^{-1}.

Eigenvalues of A are λ_1, λ_2.

Eigenvalues of $A^{-1} = 1/\lambda_1, 1/\lambda_2$.

$\Rightarrow$ The Eigenvalues of A^{-1} are $1/2, -1/2$.

2). The matrix A is $\begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Find the Eigenvalues of A^{-1}.

The Eigenvalues of A are $-1, -3, 2$.

$\therefore$ the Eigen Values of A^{-1} are $\frac{1}{-1}, \frac{1}{-3}, 1/2$.

PROPERTY 5 :-

If $\lambda_1, \lambda_2 \ldots \lambda_n$ are Eigenvalues of matrix A, then the matrix $A + kI$ has the EigenValues $k+\lambda_1, k+\lambda_2 \ldots k+\lambda_n$.

PROPERTY 6 :-

If $\lambda_1, \lambda_2 \ldots \lambda_n$ are Eigenvalues of matrix A, then the matrix $A - kI$ has the Eigenvalues $\lambda_1 - k, \ldots \lambda_n - k$.

1). Form the matrix whose Eigenvalues are $\alpha-5,\ \beta-5,\ \gamma-5$ where α, β, γ are the Eigenvalues of $A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$.

If the matrix A has the Eigenvalues λ_1, λ_2 & λ_3 then the matrix $A-kI$ has the Eigenvalues $\lambda_1 - k,\ \lambda_2 - k,\ \lambda_3 - k$

$$A - 5I = \begin{pmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{pmatrix}$$

2). If the Eigenvalues of the matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$ are $-1, -3, 2$ then find the Eigenvalues of $A+2I$ & $A-3I$.

If $\lambda_1, \lambda_2, \lambda_3$ are the Eigenvalues of A, then the Eigenvalues of $A+kI$ are $k+\lambda_1,\ k+\lambda_2,\ k+\lambda_3$.

Here $\lambda_1 = -1,\quad \lambda_2 = -3,\quad \lambda_3 = 2$.

$\therefore$ The Eigenvalues of $A+2I$ are $1, -1, 4$

$\therefore$ The Eigenvalues of $A-3I$ are $-4, -6, -1$.

PROPERTY 7:-

If $\lambda_1, \lambda_2 \dots \lambda_n$ are Eigenvalues of matrix A, then the matrix A^2 has the Eigenvalues

$\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$.

PROPERTY 8:-

If $\lambda_1, \lambda_2 \dots \lambda_n$ are Eigenvalues of matrix A, then the matrix kA has the Eigenvalues

$k\lambda_1, k\lambda_2 \dots k\lambda_n$.

1). If $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ then find the Eigenvalues of $3A^3 + 5A^2 - 6A + 2I$.

The Eigenvalues of A are 1, 3, -2.

Eigenvalues of A^3 are 1, 27, -8

Eigenvalues of A^2 are 1, 9, 4

Eigenvalues of A are 1, 3, -2

Eigenvalues of I are 1, 1, 1.

∴ The Eigenvalues of $3A^3 + 5A^2 - 6A + 2I$

1st Eigenvalue $= 3(1)^3 + 5(1)^2 - 6(1) + 2 = 4$

2nd " $= 3(27) + 5(9) - 6(3) + 2(1) = 110$

3rd " $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

∴ The req. Eigenvalues are 4, 110, 10.

CAYLEY - HAMILTON THEOREM

Every Square matrix Satisfies its own characteristic Eqn.

1). Verify Cayley - hamilton theorem & hence find A^{-1} & A^4

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

Step 1: To find characteristic Eq.

The characteristic Eq. is

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \longrightarrow (1).$$

where $a_1 = 3+5+3 = 11 \longrightarrow (2)$

$$a_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix}$$

$$= 38 \longrightarrow (3)$$

$$a_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 40 \longrightarrow (4).$$

Sub (2) (3) & (4) in (1).

we get $\lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0.$

Step 2: Verification:

To verify C-H Thm

T.P $A^3 - 11A^2 + 38A - 40I = 0.$

$$A^2 = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{pmatrix}$$

$$\therefore\ A^3 - 11A^2 + 38A - 40\,I$$

$$= \begin{pmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{pmatrix} - 11 \begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix} + 38 \begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} - 40 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25-99+114-40 & 39-77+38+0 & 17-55+38+0 \\ -61+99-38+0 & 125-275+190-40 & -61+99-38+0 \\ 39-77+38+0 & -39+77-38+0 & 47-121+114-40 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence C-H thm is verified

Step 3: To find A^{-1}

W.K.T $A^3 - 11A^2 + 38A - 40I = 0.$ $\longrightarrow (5)$

Pre $\times^{ly}$ by A^{-1}

$$\Rightarrow A^{-1}(A^3 - 11A^2 + 38A - 40I) = A^{-1}.0.$$

$$\Rightarrow A^2 - 11A + 38 - 40A^{-1} = 0.$$

$$A^{-1} = \frac{1}{40}(A^2 - 11A + 38I)$$

$$= \frac{1}{40}\left[\begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix} - \begin{pmatrix} 33 & 11 & 11 \\ -11 & 55 & -11 \\ 11 & -11 & 33 \end{pmatrix} + \begin{pmatrix} 38 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 38 \end{pmatrix}\right]$$

$$= \frac{1}{40}\begin{bmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

$$= 1/20 \begin{pmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{pmatrix}$$

To find A^4.

$\times^{ly}$. Eq. (5) by A, we get

$$A^4 - 11A^3 + 38A^2 - 40A = 0.$$

$$A^4 = 11A^3 - 38A^2 + 40A.$$

$$= 11\begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix} - 38\begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} + 40\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 275 & 429 & 187 \\ -671 & 1375 & -671 \\ 429 & -429 & 517 \end{bmatrix} - \begin{bmatrix} 342 & 266 & 190 \\ -342 & 950 & -342 \\ 266 & -266 & 418 \end{bmatrix} + \begin{bmatrix} 120 & 40 & 40 \\ -40 & 200 & -40 \\ 40 & -40 & 120 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 53 & 203 & 37 \\ -369 & 625 & -369 \\ 203 & -203 & 219 \end{bmatrix}$$

2). Using Cayley - hamilton thm Evaluate

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$

& the matrix is gn by $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Sol:-

To find the characteristic Eq.

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0. \longrightarrow (1).$$

$$a_1 = 2+1+2 = 5 \longrightarrow (2)$$

$$a_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 7 \longrightarrow (3)$$

$$a_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 3 \longrightarrow (4)$$

Sub (2) (3) & (4) in (1).

we get $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

To Evaluate $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) - 15A^2 + 5A - I.$$

$$= A^5(0) + A(0) - 15A^2 + 5A - I.$$

$\left[\because A^3 - 5A^2 + 7A - 3I = 0 \text{ By C-H Thm}\right]$.

$$= -15A^2 + 5A - I.$$

$$= -15\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -66 & -55 & -55 \\ 0 & -11 & 0 \\ -55 & -55 & -66 \end{bmatrix}$$

$$= -11\begin{bmatrix} 6 & 5 & 5 \\ 0 & 1 & 0 \\ 5 & 5 & 6 \end{bmatrix}.//$$

DIAGONALISATION OF A MATRIX.

Diagonalisation of a matrix A is the process of reducing A to a diagonal form. A square matrix A of order n with n linearly indep. eigenvectors can be diagonalised by a similarity trans. $D = B^{-1}AB$, where B is the matrix whose columns are the eigenvectors of A.

1). Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ Find the matrix $P \ni P^{-1}AP$ is a diagonal matrix

Step 1 :- To find characteristic Eq.

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \longrightarrow (1).$$

where $a_1 = 12 \longrightarrow (2).$

$$a_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 36 \longrightarrow (3).$$

$$a_3 = |A|$$

$$= 32 \longrightarrow (4)$$

Sub (2), (3) (4) in (1).

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$$

To find Eigen values:-

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$$

when $\lambda = 2$, $(2)^3 - 12(2)^2 + 36(2) - 32$

$$= 8 - 48 + 72 - 32$$

$$= 0.$$

$\therefore \lambda = 2$ is a root.

$$\begin{array}{c|cccc} & 1 & -12 & 36 & -32 \\ 2 & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0.$$

$$\lambda = \frac{10 \pm 6}{2} = 8 \text{ or } 2.$$

$\therefore$ Eigenvalues are $\lambda = 2, 2, 8$.

To find Eigenvectors-

The Eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is gn by the Eq.

$$(A - \lambda I)X = 0.$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (A)$$

Case (i). when $\lambda = 2$ we get from (A).

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0.$$

The above Eq. are the same.

$$2x_1 - x_2 + x_3 = 0.$$

$$x_2 = 2x_1 + x_3.$$

By putting $x_1 = k_1$ $x_2 = k_2$

we get $x_2 = 2k_1 + k_2$

The Eigenvector is $X_1 = \begin{pmatrix} k_1 \\ 2k_1 + k_2 \\ k_2 \end{pmatrix}$.

$\therefore$ The simplest Eigenvector is obtained by putting $k_1 = 1$ $k_2 = 0$ we get

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

One more Eigenvector is obtained by putting $k_2 = 1$ $k_1 = 0$ we get

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Case (ii). when $\lambda = 8$, we get from (A)

$$-2x_1 - 2x_2 + 2x_3 = 0 \Longrightarrow -x_1 - x_2 + x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Taking 1^{st} & 2^{nd} Eq. & applying Cross rule method, we get

$$\frac{x_1}{1+5} = \frac{x_2}{-2-1} = \frac{x_3}{5-2}.$$

$$\frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3} = k.$$

$x_1 = 6k$, $x_2 = -3k$, $x_3 = 3k$.

$\therefore$ The Eigenvector is $X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. (By taking $k = 1/3$)

Characteristic Eq.	Eigenvalues	Eigenvectors
$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$	$\lambda_1 = 2$	$X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$
	$\lambda_2 = 2$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
	$\lambda_3 = 8$	$X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

To find $P^{-1}AP$:

$$\therefore P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

To find B^{-1}

The co-factor of the elt $0 = 4$

" $1 = -2$

" $2 = -2$

" $1 = 1$

" $3 = -2$

" $-1 = 1$

" $1 = -7$

" $1 = 2$

" $1 = -1$

40

Adj. of $P = \begin{pmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{pmatrix}$

$$|P| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2 - 4$$

$$|P| = -6.$$

Hence, the Inverse of the matrix B is

$$P^{-1} = \frac{\text{Adj. } P}{|P|}$$

$$\therefore P^{-1} = -\frac{1}{6} \begin{pmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{pmatrix}$$

$$\therefore P^{-1}AP = -\frac{1}{6} \begin{pmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= -\frac{1}{6} \begin{pmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -48 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

QUADRATIC FORM :-

1). Reduce the Q.F $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ Into a canonical form & find the nature of the Q.F

Sol:-

The gn Q.F is $X'AX$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $X' = (x_1 \; x_2 \; x_3)$

Step, 1:- To Find the matrix of the Q.F :-

The matrix of the quadratic form is

$$A = \begin{pmatrix} \text{Coeffi. of } x^2 & \tfrac{1}{2}\text{ Coeffi. of } xy & \tfrac{1}{2}\text{ coeffi. of } xz \\ \tfrac{1}{2}\text{ coeff. of } yz & \text{coeff. of } y^2 & \tfrac{1}{2}\text{ coeff. of } yz \\ \tfrac{1}{2}\text{ Coeff. of } zx & \tfrac{1}{2}\text{ coeff. of } zy & \text{coeff. of } z^2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

$= 48 - 8 - 8 = 32 \longrightarrow 4)$.

Sub (2), (3) & (4) in (I) we get the chara. Eq.

is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

To find Eigenvalues

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \longrightarrow (I)$$

when $\lambda = 2$ $(2)^3 - 12(2)^2 + 36(2) - 32$

$= 0$.

$\therefore \lambda = 2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$\lambda^2 - 10\lambda + 16 = 0.$

$$\lambda = \frac{10 \pm \sqrt{100 - 64}}{2}$$

$$= \frac{10 \pm 6}{2} = 8 \text{ or } 2.$$

$\therefore$ Eigenvalues are $\lambda = 2, 2, 8$

To find Eigenvectors :-

The Eigenvectors $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is gn by the Eq.

$$(A - \lambda I) X = 0.$$

$$\begin{pmatrix} (6-\lambda) & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} (6-\lambda)x_1 - 2x_2 + 2x_3 = 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 = 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 = 0 \end{array}\right\} \longrightarrow (A).$$

Case (i). when $\lambda = 8$,

$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - x_3 = 0$$
$$2x_1 - x_2 - 5x_3 = 0.$$

Taking 1^{st} two Eqns. & using cross rule method we get

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k \text{ (say)}$$

$$x_1 = 12k, \quad x_2 = -6k, \quad x_3 = 6k$$

The Eigenvector is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ (put $k = 1/6$)

Case (ii) when $\lambda = 2$

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0.$$

All the above Eqns. are the Same.

$$2x_1 - x_2 + x_3 = 0$$

$$\Rightarrow x_2 = 2x_1 + x_3.$$

Putting $x_1 = 1$, $x_3 = 0$ we get $x_2 = 2$.

$\therefore$ The Eigenvector $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Let the 3rd Eigenvector be $X_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Since we are going to diagonalize the matrix thro' Orthogonal trans. we have to find X_3 which is orthogonal to X_2

Let X_3 is orthogonal to X_2

$\therefore x+2y+0z=0 \quad \rightarrow (1A)$

$\because X_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Satisfies $2x-y+z=0$

we get $2x-y+z=0 \Rightarrow z=y-2x \rightarrow (B)$

$(1A) \Rightarrow x=-2y$

$\therefore z=5y$

Putting $y=1$, we get $x=-2, z=5$

$\Rightarrow X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$

Step 5: Normalized Eigenvectors.

	X_1	X_2	X_3
Eigenvectors	$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$
Normalised form	$\begin{pmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$	$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{pmatrix}$

$\Rightarrow$ to find modal matrix:

The normalised modal matrix is

$$P = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix} \ \& \ P^1 = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{pmatrix}$$

$\Rightarrow$ To find $X'AX$:

let $X = PY \longrightarrow 5)$ be the orthogonal trans.

Sub (5) in (1).

$$X'AX = (PY)'A(PY)$$
$$= Y'(P'AP)Y.$$

To find $P'AP$:

$$AP = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix}$$

$$= \begin{pmatrix} 16/\sqrt{6} & 2/\sqrt{5} & -4/\sqrt{30} \\ -8/\sqrt{6} & 4/\sqrt{5} & 2/\sqrt{30} \\ 8/\sqrt{6} & 0 & 60/\sqrt{30} \end{pmatrix}$$

$$P'AP = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 16/\sqrt{6} & 2/\sqrt{5} & -4/\sqrt{30} \\ -8/\sqrt{6} & 4/\sqrt{5} & 2/\sqrt{30} \\ 8/\sqrt{6} & 0 & 60/\sqrt{30} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$$ is a diagonal matrix.

To find $y'(P'AP)y$ (Canonical form)

$$y'(P'AP)y = (y_1\ y_2\ y_3)\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

The Q.F is a +ve definite.

INDEX & SIGNATURE OF THE REAL Q.F.

1). Find the Index & Signature of the Q.F.

$3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$

Sol: The matrix of the Q.F is

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

The rank of A is 3.

The canonical form of the above Q.F is

$2y_1^2 + 3y_2^2 + 6y_3^2$.

Now Index (s) = No. of +ve terms

$= 3$.

Rank $(r) = 3$

$\therefore$ Signature $= 2s - r$

$= 6 - 3 = 3$ //.