ASSIGNMENT 3

UNIT – IV: QUEUEING THEORY

PART – A

1 What are the basic characteristics of a queueing system?

The basic characteristics of a queueing system are

1. Arrival pattern of customer
2. Service pattern of customer
3. Number of service channels
4. System capacity
5. Queue discipline

2 What do you mean by transient state and steady state queueing system?

A queueing system is in transient state when its operating characteristics are depend on time. It is in steady state when the characteristics are independent on time.

3 State the various queue disciplines in queueing model.

This is the manner by which customer are selected for service when a queue has formed. The most common queues discipline are

1. FIFO –First In First Out (OR) FCFS – First Come First served
2. LIFO – Last In First Out (OR) LCFS – Last Come First Served
3. SIRO – Selection In Random Order
4. PIR – Priority in Section

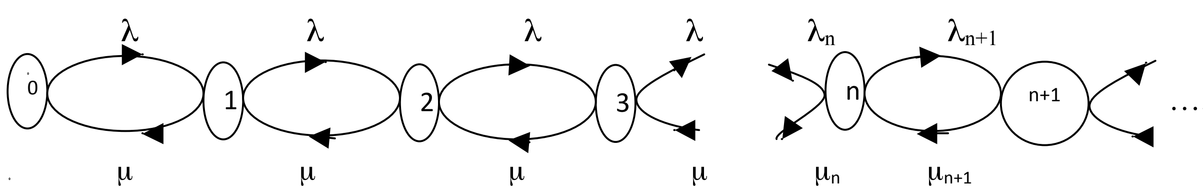
4 In an $(M/M/1):(\infty/FIFO)$ queueing model if $\lambda = 8/hr,\ \mu = 10/hr$, what is the average waiting time in the queue?

Solution:

$\lambda = 8/hr,\ \mu = 10/hr$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{10(10-8)} = 0.4\text{hr}$$

5 Draw the state transition diagram for M/M/1 queueing model.



This model is based on the birth and death process. Arrivals are births and departures are deaths. When the system is in state n if an arrival comes, it goes to state n+1 and if a departure is there then the system goes to state n-1. The number of customers in the system is the state of the system.

6 In an $(M/M/2):(5/FIFO)$ queueing model if $\lambda = 3$, $\mu = 4$ and P_0=0.4564 find the effective arrival rate λ'.

Solution:

Given : $\lambda = 3$, $\mu = 4$ and P_0=0.4564

Number of servers $c = 2$
Capacity = k = 5

Effective arrival rate $\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)P_n\right] = 3\left[2 - \sum_{n=0}^{1}(2-n)P_n\right]$

$\lambda' = 3\left[2 - \left(2P_0 + P_1\right)\right]$ where $P_1 = \left(\frac{\lambda}{\mu}\right)P_0$ here $\frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

$\lambda' = 3\left[2 - (2\times 0.4564) + (0.75\times 0.4564)\right] = 3(1.4295) = 4.2885$

7 What is the probability that an arrival to an infinite capacity 3 server Poisson queueing system with $\frac{\lambda}{\mu} = 2$ *and* $P_0 = \frac{1}{6}$ enters the service without waiting.

Solution:

Given $\frac{\lambda}{\mu} = 2$ *and* $P_0 = \frac{1}{6}$ and The number of server c = 3

P(a customer enters service without waiting) = $P(N_s < 3) = P_0 + P_1 + P_2$

w.k.t $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad , \; n \le c$

$P_1 = \frac{1}{1!}2^1\left(\frac{1}{6}\right) = 0.3333$

$P_2 = \frac{1}{2!}2^2\left(\frac{1}{6}\right) = 0.3333$

$\therefore$ P(customer enters service without waiting) = $P(N_s < 3) = 0.1667 + 0.3333 + 0.3333 = 0.8333$

8 Which queue is called to be the queue with discouragement?
If a customer is discouraged to join the queue expecting a long waiting time or having the impatience in getting the service, the queueing model is said to be the queue with discouragement.

PART B

1 a A TV repairman finds that the time spend on his job has an exponential distribution with mean 30 minutes. If he repair sets in the order in which they came in and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 hour day, what is the repairman’s expected idle time each day? How many jobs are ahead of average set just

brought?

Solution:

Model identification:

since there is only one repair man and the capacity of the system is infinity. Hence this problem comes under the model (M/M/1);(∞/FCFS).

Given Data:

Arrival rate $\lambda = 10$ per 8hr day

Service rate $\frac{1}{\mu} = 30 \Rightarrow \mu = \frac{1}{30}$ per min

$(i.e)\ \mu = \frac{60}{30} = 2/hr \Rightarrow (i.e)\ \mu = 8 \times 2 = 16$ per 8hr day

i) The repairman's idle time $= P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{16} = \frac{6}{16} = \frac{3}{8}$ / day

The Expected idle time $= 8 \times \frac{3}{8} = 3\ hrs$

ii) The average number of jobs in the system $= L_s = \frac{\lambda}{\mu - \lambda} = \frac{10}{16-10} = 1.667$; 2jobs.

Another method:

Arrival rate $\lambda = \frac{10}{8} = \frac{5}{4}$per hr

Service rate $\mu = \frac{1}{30}$ per min

$(i.e)\ \mu = \frac{60}{30} = 2/hr$

i) The repairman's idle time $= 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{16} = \frac{6}{16} = \frac{3}{8}$

The Expected idle time each day $= 8 \times \frac{3}{8} = 3\ hrs$

ii) Number of the jobs ahead of the
average set brought in = The average number of jobs in the system

$$= L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{5}{4}}{2 - \frac{5}{4}} = 1.667\ ;\ 2\text{jobs.}$$

b A Telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.

(1) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

(2) If the subscribers will wait and are serviced in turn, what is the expected waiting time?

Solution:

Model identification: Since there are two operators, infinite capacity.

Hence this problem comes under the model (M/M/c);(∞/FCFS).

Given data:

Arrival rate $\lambda = 15/\text{hr}$

Service rate $\mu = \dfrac{1}{5}$ per min

(i.e) $\mu = \dfrac{60}{5} = 12$ per hr

Number of servers $c = 2$

w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n + \dfrac{\mu c}{c!(\mu c - \lambda)}\left(\dfrac{\lambda}{\mu}\right)^c\right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{15}{12}\right)^n + \frac{2\times 12}{2!((2\times 12)-15)}\left(\frac{15}{12}\right)^2\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(1.25)^n + \frac{24}{18}(1.25)^2\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(1.25)^n + 2.078\right]^{-1}$$

$$P_0 = \left[(1+1.25)+2.078\right]^{-1} = 4.328^{-1} = 0.2311$$

i) the probability that a subscriber will have to wait for his service is

$$P(N_s \geq 2) = \frac{\left(\frac{15}{12}\right)^c}{2!\left(1-\frac{15}{24}\right)}(0.2311) \qquad \because P(N_s \geq c) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)}P_0$$

$$P(N_s \geq 2) = \frac{(1.25)^2}{0.75}(0.2311) = 0.4814$$

ii) If the subscriber will wait and are serviced in turn then the expected waiting time

$$W_s = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

Where $L_q = \dfrac{1}{c.c!}\left(\dfrac{\lambda}{\mu}\right)^{c+1}\left(1-\dfrac{\lambda}{\mu c}\right)^{-2} P_0$

$$= \frac{1}{(2)2!}(1.25)^3\left(1-\frac{15}{24}\right)^{-2}(0.2311)$$

$$= \frac{1}{4}(1.25)^3\left(1-\frac{15}{24}\right)^{-2}(0.2311)$$

$$= 1.953\times 7.111\times 0.2311 = 0.8023$$

$$\therefore W_s = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{3.209}{15} + \frac{1}{12} = 0.1368 \text{ hr}$$

c Consider a single server queueing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible number calling units in the system is two. Find the steady state probability distribution of the number of calling units in the system and the expected number of calling units in the system.

Solution:

<u>Model identification</u>:

since there is only one server and the maximum number of calling source is 2, capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k/FCFS).

<u>Given Data:</u>

Arrival rate $\lambda = 3$ per hr

Service rate $\mu = \frac{1}{0.25} = \frac{1}{1/4}$ per hr (i.e) $\mu = 4$ per hr

Capacity of the system $k = 2$

$$\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$$

i) Steady state Probability distribution of the number of calling units is

$P_n, n \geq 0$ $(i.e)$ To find P_0, P_1, P_2

$$P_n = \frac{(1-\rho)\rho^n}{1-\rho^{k+1}} = \frac{(1-0.75)(0.75)^n}{1-0.75^3} = (0.4324)(0.75)^n$$

$$P_0 = (0.4324)(0.75)^0 = 0.4324$$

$$P_1 = (0.4324)(0.75)^1 = 0.3243$$

$$P_2 = (0.4324)(0.75)^2 = 0.2432$$

ii) The average number of calling units in the system is

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} \quad \text{since } \lambda \neq \mu \qquad \text{(or) } L_s = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$L_s = \frac{0.75}{1-0.75} - \frac{(2+1)(0.75)^3}{1-(0.75)^3} = 0.81$$

Another method to find L_s :

$$L_s = \sum_{n=0}^{k} nP_n = \sum_{n=0}^{2} n(0.43)(0.75)^n = 0.43\left[0 + 0.75 + 2(0.75)^2\right] = 0.81$$

2 a A supermarket has 2 girls running up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 per hour, find the following:

(i) What is the probability of having to wait for service?

(ii) What is the expected percentage of idle time for each girl?

(iii) What is the expected length of customer's waiting time?

(iv) What is the expected number of idle girls at any time?

Solution:

Model identification: Since there are two girls and infinite capacity.

Hence this problem comes under the model (M/M/c);(∞/FCFS).

Given data:

Arrival rate $\lambda = 10$ per hr

Service rate $\mu = \frac{1}{4}$ per min (i.e) $\mu = \frac{60}{4} = 15$ per hr.

Number of servers $c = 2$

w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\mu c}{c!(\mu c - \lambda)}\left(\frac{\lambda}{\mu}\right)^c \right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{10}{15}\right)^n + \frac{2\times 15}{2!((2\times 15)-10)}\left(\frac{10}{15}\right)^2 \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(0.6667)^n + \frac{30}{40}(0.6667)^2 \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(0.6667)^n + 0.3333 \right]^{-1}$$

$$P_0 = \left[(1+0.6667)+0.3333\right]^{-1} = 2^{-1} = 0.5$$

i) The probability that a customer has to wait for the service is

$$P(N_s \geq 2) = \frac{\left(\frac{10}{15}\right)^2}{2!\left(1-\frac{10}{15\times 2}\right)}(0.5) \qquad \text{Q } P(N_s \geq c) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)}P_0$$

$$P(N_s \geq 2) = \frac{(0.6667)^2}{1.3333}(0.5) = \frac{1}{6} = 0.1667$$

ii) Probability of time that a girl is busy $= \rho = \frac{\lambda}{c\mu} = \frac{10}{2\times 15} = \frac{1}{3}$

$\therefore$ Probability of time when a girl is idle = 1- Probability of time that a girl is busy

$$= 1-\frac{1}{3} = \frac{2}{3}$$

$\therefore$ Percentage of idle time of for each girl $= \frac{2}{3}\times 100 = 67\%$

iii) Expected waiting time of customer = $W_s = \frac{L_q}{\lambda} + \frac{1}{\mu}$

Where $L_q = \frac{1}{c.c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left(1-\frac{\lambda}{\mu c}\right)^{-2} P_0$

$$= \frac{1}{(2)2!}(0.6667)^3\left(1-\frac{10}{30}\right)^{-2}(0.5) = 0.083$$

$$\therefore W_s = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{0.083}{10} + \frac{1}{15} = 0.0083 \text{ hrs}$$

iv) The expected idle no of girl:
E(idle no of girl)=?

No.of idle girls:	2	1	0
Probability	P_0	P_1	P_2

E(idle time for each girl)=2 P_0 +1 P_1 +0 P_2

Now, P_0=0.5

w.k.t $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad , \quad 0 \le n < c$

$$P_1 = \frac{1}{1!}\left(\frac{10}{15}\right)^1 (0.5) = 0.333 = \frac{1}{3}$$

$$P_2 = \frac{1}{2!}\left(\frac{10}{15}\right)^2 (0.5) = 0.1111 = \frac{1}{9}$$

E(idle no of girl)=2 P_0 +1 P_1 +0 P_2 $= 2\times\frac{1}{9} + 1\times\frac{1}{3} + 0 = \frac{5}{9}$

$\therefore$ The expected percentage idle time for each girl $= \frac{5}{9} \times 100 = 55.56\%$

b Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 5 trains, find the probability that the yard is empty and the average number of trains in the system, given that the inter arrival time and service time are following exponential distribution.

Solution:

Model identification:

since there is only one server and the maximum number of customers allowed is 2, capacity of the system which is finite. Hence this problem comes under the model (M/M/1);(k/FCFS).

Given Data:

Arrival rate $\lambda = \frac{1}{15}$ per min

Service rate $\mu = \frac{1}{33}$ per min

Capacity of the system $k = 5$

$$\rho = \frac{\lambda}{\mu} = \frac{33}{15} = 2.2$$

i) Probability that the yard is empty $= P_0 = \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \dfrac{1-\rho}{1-\rho^{k+1}} = \dfrac{1-2.2}{1-2.2^6} = 0.01068$

ii) Average number of trains in the system:

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} \quad \text{since } \lambda \neq \mu \qquad \text{(or) } L_s = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$L_s = \frac{2.2}{1-2.2} - \frac{(5+1)(2.2)^6}{1-(2.2)^6} = (-1.8333) - \left(\frac{680.28}{-112.37}\right) = 4.221$$

Another method to find L_s :

$$L_s = \sum_{n=0}^{5} nP_n = P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5$$

$$L_s = \rho P_0 + 2\rho^2 P_0 + 3\rho^3 P_0 + 4\rho^4 P_0 + 5\rho^5 P_0$$

$$L_s = P_0\left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5\right]$$

$$L_s = P_0\left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5\right]$$

$$= 0.01168\left[2.2 + 2\times 2.2^2 + 3\times 2.2^3 + 4\times 2.2^4 + 5\times 2.2^5\right]$$

$$L_s = 4.221$$

c Find the system size probabilities for an $M/M/S$: $FCFS/\infty/\infty$ queueing system under steady state conditions. Also obtain the expression for average number of customers in the system.

Solution:

Assume $S = c \Rightarrow (M/M/c);(\infty/FCFS)$

This model represents a queueing system with poisson arrivels, exponential service time, multiple servers, infinite capacity and FCFS queue service from a single queue.

If $n < c$, then only n of the c servers will be busy and others are idle and hence mean service rate will be $n\mu$.

If $n \geq c$, all c servers will be busy and hence the mean service rate is $c\mu$

$$\therefore \mu_n = \begin{cases} n\mu & ,0 \leq n < c \\ c\mu & ,n \geq c \end{cases} \quad \text{and } \lambda_n = \lambda \ \forall n$$

By birth and death process

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\mu_4\ldots\mu_n}P_0$$

Case i: When $0 \leq n < c$

$$P_n = \frac{\lambda\lambda\lambda\lambda\ldots\lambda\,(n\text{ times})}{1\mu\,2\mu\,3\mu\,4\mu\ldots\text{n}\,\mu}P_0$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)P_0$$

Case ii: when $n \geq c$

$$P_n = \frac{\lambda\lambda\lambda\lambda\ldots\ldots(n\text{ times})}{\mu_1\mu_2\ldots\mu_c\ldots\mu_{c+1}\ldots\mu_n}P_0$$

$$= \frac{\lambda^n}{\{1\mu\, 2\mu\, 3\mu ... (c-1)\mu\}\{c\mu\, c\mu ... (c-(c-1))\, t\text{imes}\}} P_0$$

$$= \frac{\lambda^n}{(c-1)!\, \mu^{c-1} c^{n-c+1} \mu^{n-c+1}} P_0$$

$$= \frac{\lambda^n}{(c-1)!\, c\, \mu^{c-1} c^{n-c} \mu^{c-1} \mu^{-(c-1)}} P_0$$

$$= \frac{\lambda^n}{(c-1)!\, \mu^{c-1} c^{n-c+1} \mu^{n-c+1}} P_0$$

$$P_n = \frac{1}{\text{c!c}^{n-c}} \left(\frac{\lambda}{\mu}\right)^n P_0, n \geq c$$

$$P_n = \begin{cases} \frac{1}{\text{c!c}^{n-c}} \left(\frac{\lambda}{\mu}\right)^n P_0 & , n \geq c \\ \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 & , 0 \leq n < c \end{cases}$$

To find P_0

Since $\sum_{n=0}^{\infty} P_n = 1$

$$\Rightarrow \sum_{n=0}^{c-1} P_n + \sum_{n=c}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=c}^{\infty} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow P_0 \left[\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n\right] = 1 \quad \text{---------(1)}$$

Consider

$$\Rightarrow \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{c!c^{-c}} \sum_{n=c}^{\infty} \left(\frac{\lambda}{\mu c}\right)^n$$

$$= \frac{1}{c!c^{-c}} \left[\left(\frac{\lambda}{\mu c}\right)^c + \left(\frac{\lambda}{\mu c}\right)^{c+1} + \left(\frac{\lambda}{\mu c}\right)^{c+2} + ...\right]$$

$$= \frac{1}{c!c^{-c}} \left(\frac{\lambda}{\mu c}\right)^c \left[1 + \left(\frac{\lambda}{\mu c}\right)^1 + \left(\frac{\lambda}{\mu c}\right)^2 + \left(\frac{\lambda}{\mu c}\right)^3 + ...\right]$$

$$= \frac{1}{c!c^{-c}} \left(\frac{\lambda}{\mu c}\right)^c \left[1 - \left(\frac{\lambda}{\mu c}\right)\right]^{-1}$$

$$= \frac{1}{c!c^{-c}} \left(\frac{\lambda}{\mu c}\right)^c \left[\frac{\mu c - \lambda}{\mu c}\right]^{-1}$$

$$\Rightarrow \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^{n} = \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^{c}\left[\frac{\mu c}{\mu c-\lambda}\right]$$

$$(1) \ \Rightarrow P_0 = \left[\sum_{n=0}^{c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n} + \sum_{n=c}^{\infty}\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^{n}\right]^{-1}$$

$$\Rightarrow P_0 = \left[\sum_{n=0}^{c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n} + \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^{c}\left(\frac{\mu c}{\mu c-\lambda}\right)\right]^{-1}$$

To find the average number of customer in the system: (L_s)

$$L_s = L_q + \frac{\lambda}{\mu}$$

Where

$L_q =$ Average number of customer in the queue

$$L_q = \sum_{n=c}^{\infty}(n-c)P_n$$

$$L_q = \sum_{n=c}^{\infty}(n-c)\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^{n}P_0$$

$$= \frac{1}{c!c^{-c}}P_0\sum_{n=c}^{\infty}(n-c)\left(\frac{\lambda}{\mu c}\right)^{n}$$

$$= \frac{1}{c!c^{-c}}P_0\left[0+(c+1-c)\left(\frac{\lambda}{\mu c}\right)^{c+1} + (c+2-c)\left(\frac{\lambda}{\mu c}\right)^{c+2} + (c+3-c)\left(\frac{\lambda}{\mu c}\right)^{c+3} + \ldots\right]$$

$$= \frac{1}{c!c^{-c}}P_0\left(\frac{\lambda}{\mu c}\right)^{c+1}\left[1+(2)\left(\frac{\lambda}{\mu c}\right)^{1} + (3)\left(\frac{\lambda}{\mu c}\right)^{2} + \ldots\right]$$

$$= \frac{1}{c!c^{-c}}P_0\left(\frac{\lambda}{\mu c}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}$$

$$L_q = \frac{1}{c\ c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}P_0$$

$$\therefore L_s = L_q + \frac{\lambda}{\mu} = \frac{1}{c\ c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}P_0 + \frac{\lambda}{\mu}$$

3 a In a cinema theatre people arrive to purchase tickets at the average rate of 6 per minute and it takes 7.5 seconds on the average to purchase a ticket. If a person arrives just 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket. Can he expect to be seated for the start of the picture and what is the probability that he will be seated when the film starts?

Solution:

Model identification:

since there is only one counter and the arrival of persons are infinite, capacity of the system is infinie. Hence this problem comes under the model (M/M/1);(∞/FCFS).

Given Data:

Arrival rate $\lambda = 6$ per min

Service rate $\mu = \frac{1}{7.5}$ per sec $\Rightarrow \mu = \frac{60}{7.5} = 8$ per min

i) Expected total time required to purchase the ticket and to reach the seat

=waiting time in the system + time to reach the seat = $W_s + 1.5$.

Where $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$

Expected total time required to purchase the ticket and to reach the seat =0.5+1.5=2 min

ii) P(he will be the seated for the start of the picture)

=P(Total time < 2 min)

$$= P\left(W_s < \frac{1}{2}\right) = 1 - P\left(W_s > \frac{1}{2}\right) = 1 - e^{-(\mu-\lambda)t} = 1 - e^{-(8-6)\frac{1}{2}} = 0.63$$

b A group of engineers has two terminals available to aid their calculations. The average computing job requires 20 minutes of terminal time and each engineer requires some computation one in half an hour. Assume that these are distributed according to an exponential distribution. If there are 6 engineers in the group find the expected number of engineers waiting to use the terminals in the computing center.

Solution:

Since there are 2 terminals, also since there are 6 engineers in the group, the capacity of the system is finite.

Hence this problem comes under the model (M/M/c);(k/FCFS)

Given data:

Arrival rate $\lambda = \frac{1}{1/2}$=2 per hr

Service rate $\mu = \frac{1}{20}$ per min (i.e) $\mu = \frac{60}{20} = 3$ per hr.

Number of servers $c = 2$

Capacity = $k = 6$

Expected number of engineers waiting to use in the computing center = L_s

$$L_s = L_q + \frac{\lambda'}{\mu}$$

Where $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c + \sum_{n=c}^{k}\left(\frac{\lambda}{\mu c}\right)^{n-c} \right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{2}{3}\right)^n + \frac{1}{2!}\left(\frac{2}{3}\right)^2 + \sum_{n=2}^{6}\left(\frac{2}{3\times 2}\right)^{n-2} \right]^{-1}$$

$$= \left[1 + \frac{2}{3} + \frac{1}{2} \times \left(\frac{2}{3}\right)^2 \left\{ 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 \right\} \right]^{-1} = 0.5003$$

And $\rho = \frac{\lambda}{\mu c} = \frac{1}{3}$

$$L_q = \left(\frac{\lambda}{\mu}\right)^c \frac{\rho}{c!(1-\rho)^2} \left\{ 1 - \rho^{k-c} - (k-c)(1-\rho)\rho^{k-c} \right\} P_0$$

$$L_q = \left(\frac{2}{3}\right)^2 \frac{\frac{1}{3}}{2!\left(1 - \frac{1}{3}\right)^2} \left\{ 1 - \left(\frac{1}{3}\right)^{6-2} - (6-2)\left(1 - \frac{1}{3}\right)\left(\frac{1}{3}\right)^{6-2} \right\} (0.5003) = 0.0796$$

Effective arrival rate $\lambda' = \mu \left[c - \sum_{n=0}^{c-1} (c-n) P_n \right] = 3 \left[2 - \sum_{n=0}^{1} (2-\text{n}) P_n \right]$

$\lambda' = 3\left[2 - \left(2P_0 + P_1\right)\right]$ where $P_1 = \left(\frac{\lambda}{\mu}\right) P_0$

$$\lambda' = 3\left[2 - \left(2 \times 0.5003 + \frac{2}{3} \times 0.5003\right)\right] = 1.9976$$

Expected number of engineers waiting to use in the computing center = L_s

$$L_s = L_q + \frac{\lambda'}{\mu} = 0.0796 + \frac{1.9976}{3} = 0.7455$$

c Self service system is followed in a super market at a metropolis. The customer arrivals occur according to a Poisson distribution with mean 40 per hour. Service time per customer is exponentially distributed with mean 6 minutes. Find the expected number of customers in the system and what is the percentage of time that the facility is idle?

Solution:

<u>Model identification</u>:

It is self service model and the customer himself is treated as server. The number of server is unlimited . Hence this problem comes under the model M/M/∞ queues.

<u>Given Data:</u>

Arrival rate λ = 40 per hr

Service rate $\mu = \frac{1}{6}$ per min

$(i.e)\ \mu = \frac{60}{6} = 10 / hr$

i) The average (Expected) number of customer in the system

$L_s = \frac{\lambda}{\mu} = \frac{40}{10} = 4$ customer

ii) P(the facility is idle) = $P_0 = e^{-\frac{\lambda}{\mu}} = e^{-4} = 0.0183$

Percentage of idle facility = 0.0183x100=1.83%

4 a A telephone exchange receives one call every 4 minutes and connects one call every 3

minutes. If the rate of arrival follows Poisson distribution and the service rate follows exponential distribution find the average waiting time for a call in the queue and in the system.

Model identification:

since there are one server and the capacity of the system is infinity. Hence this problem comes under the model (M/M/1);(∞/FCFS).

Given Data:

Arrival rate $\lambda = \frac{1}{4}$ per min

Service rate $\mu = \frac{1}{3}$ per min

i) The average waiting time for a call in the System

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{3} - \frac{1}{4}} = 12 \text{ min}$$

ii) The average waiting time for a call in the queue

$$W_q = W_s - \frac{1}{\mu} = 12 - 3 = 9 \text{ min}$$

b A petrol pump has two pumps. The service times follow the exponential distribution with mean 4 minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service and what is the probability that the pumps remain idle?

Solution:

Model identification: Since the petrol bunk has 2 pumps, infinite capacity.

Hence this problem comes under the model (M/M/c);(∞/FCFS).

Given data:

Arrival rate $\lambda = 10$ per hr

Service rate $\mu = \frac{1}{4}$ per min (i.e) $\mu = \frac{60}{4} = 15$ per hr.

Number of servers $c = 2$

w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \frac{\mu c}{c!(\mu c - \lambda)} \left(\frac{\lambda}{\mu} \right)^c \right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!} \left(\frac{10}{15} \right)^n + \frac{2 \times 15}{2!((2 \times 15) - 10)} \left(\frac{10}{15} \right)^2 \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!} (0.6667)^n + \frac{30}{40} (0.6667)^2 \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!} (0.6667)^n + 0.3333 \right]^{-1}$$

$$P_0 = \left[(1 + 0.6667) + 0.3333 \right]^{-1} = 2^{-1} = 0.5$$

i) The probability that a customer has to wait for the service is

$$P(N_s \geq 2) = \frac{\left(\frac{10}{15}\right)^c}{2!\left(1-\frac{10}{15\times 2}\right)}(0.5) \qquad \because P(N_s \geq c) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)}P_0$$

$$P(N_s \geq 2) = \frac{(0.6667)^2}{1.3333}(0.5) = \frac{1}{6} = 0.1667$$

ii) Probability of time that a pump is busy $= \rho = \frac{\lambda}{c\mu} = \frac{10}{2\times 15} = \frac{1}{3}$

$\therefore$ Probability of time when a pump is idle = 1- Probability of time that a pump is busy

$$= 1-\frac{1}{3} = \frac{2}{3}$$

c A shipping company has a single unloading dock with ships arriving in a Poisson fashion at an average rate of 3 per day. The unloading time distribution for a ship with n unloading crews is found to be exponent with average unloading time $\frac{1}{2n}$ days. The company has a large labor supply without regular working hours, and to avoid long waiting times, the company has a policy of using as many unloading crews as there are ships waiting in line or being unloaded. Find the average number of unloading crews working at any time and the probability that more than 4 crews will be needed.

Solution:

Arrival rate $\lambda = 3$ ships/day

Service rate $\mu_n = n\mu = \frac{1}{1/2n} = 2n \Rightarrow \mu = 2$ ships/day for one unloading crew.

i) Expected number of unloading crews is equal to the number of ship in the system

$$L_s = \frac{\lambda}{\mu} = \frac{3}{2} = 1.5$$

ii) $P_n = e^{-\frac{\lambda}{\mu}}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \quad n = 0,1,2,3,..$

$$P_n = e^{-1.5}\frac{(1.5)^n}{n!}, \; n = 0,1,2,3,...$$

P(more than 4 crews will be needed)

= P(at least 5 ships in the system)

$$= P(N \geq 5) = 1-\sum_{n=0}^{4} P_n = 1-\sum_{n=0}^{4} e^{-1.5}\frac{(1.5)^n}{n!}$$

$$= 1-e^{-1.5}\left[1+1.5+\frac{(1.5)^2}{2!}+\frac{(1.5)^3}{3!}+\frac{(1.5)^4}{4!}\right]$$

$$= 1-0.2231[1+1.5+1.125+0.5625+0.2109]$$

$$= 1-0.98129 = 0.0187$$