## St.JOSEPH's COLLEGE OF ENGINEERING, CHENNAI-119

## DEPARTMENT OF MATHEMATICS
## FORMULA LIST
## UNIT-I ORDINARY DIFFERENTIAL EQUATIONS

1. General form of linear Differential equation of the $n^{th}$ order with constant co-efficients is

$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X$ ..............(A) Where $a_0, a_1, a_2, \ldots a_n$ are constants, X is the functions of x alone.

i.e $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = X$

here $D = \frac{d}{dx} \quad D^2 = \frac{d^2}{dx^2} \quad D^n = \frac{d^n}{dx^n}$

2. <u>General Solution</u> of equation of (A) is y = Complimentary Function + Particular Integral.

3. The General form of the linear differential equation of second order is

$$\frac{d^2 y}{dx^2} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 y = X$$

$(D^2 + a_1 D + a_2) y = X$ .................> (B)

<u>Finding Complimentary function:</u>

Step 1: Finding the Auxiliary equation by putting $D^2 y = m^2 \quad Dy = m$

$$m^2 + a_1 m + a_2 = 0$$

Step2: Compare the roots of m & Write the complimentary function

| | Roots of A.E | C.F |
|---|---|---|
| 1. | Roots are real and different $m_1, m_2 \quad (m_1 \neq m_2)$ | $Ae^{m_1 x} + Be^{m_2 x}$ |
| 2. | Roots are real and equal $m_1, m_2 \quad (m_1 = m_2) = m$ | $(Ax + B)e^{mx}$ |
| 3 | Roots are imaginary $\alpha \pm i\beta$ | $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ |

4. Consider the fourth order differential equation $(D^4 + a_1 D^3 + a_2 D^2 + a_3 D + a_4) y = X$

<u>Finding Complimentary function:</u>

Step 1: Finding the Auxiliary equation by putting $D^4 y = m^4, \; D^3 y = m^3 \; D^2 y = m^2 \quad Dy = m$

$$m^4 + a_1 m^3 + a_2 m^2 + a_3 m + a_4 = 0$$

Step2: Solving A.E we get the roots $m_1, m_2, m_3, m_4$

Step3: Compare the roots of m & Write the complimentary function

| | Roots of A.E | C.F |
|---|---|---|
| 1. | Roots are real and different $m_1, m_2, m_3, m_4 \; (m_1 \neq m_2 \neq m_3 \neq m_4)$ | $Ae^{m_1 x} + Be^{m_2 x} + Ae^{m_3 x} + Ae^{m_4 x}$ |
| 2. | Roots are real and equal $m_1, m_2, m_3, m_4 \; (m_1 = m_2 = m_3 = m_4)$ | $(Ax^3 + Bx^2 + Cx + D)e^{mx}$ |
| 3 | Two Roots are real and equal & Two Roots are real and distinct. $m_1, m_2, m_3, m_4$ | $(Ax + B)e^{mx} + Ce^{m_3 x} + De^{m_4 x}$ |

| | | |
|---|---|---|
| | $(m_1 = m_2 = m,\ \ m_3 = m_4)$ | |
| 4 | Two pairs of equal roots.<br>$m_1, m_2, m_3, m_4$<br>$(m_1 = m_2 = m_1,\ \ m_3 = m_4 = m_2)$ | $(Ax+B)e^{m_1 x} + (Cx+D)e^{m_2 x}$ |
| 5 | Two Roots are real and equal & other Two Roots are imaginary.<br>$m_1, m_2, m_3, m_4$<br>$(m_1 = m_2 = m_1,\ \ \alpha \pm i\beta)$ | $e^{\alpha x}(A\cos\beta x + B\sin\beta x) + (Cx+D)e^{mx}$ |
| 6 | Two Roots are real and different & other Two Roots are imaginary.<br>$m_1, m_2, m_3, m_4$<br>$(m_1 \neq m_2\ \ \alpha \pm i\beta)$ | $e^{\alpha x}(A\cos\beta x + B\sin\beta x) + Ce^{m_1 x} + De^{m_2 x}$ |
| 7 | Two pairs of imaginary roots are equal<br>$(m_1 = m_2 = \alpha + i\beta\ \&\ m_1 = m_2 = \alpha - i\beta)$ | $e^{\alpha x}[(Ax+B)\cos\beta x + (Cx+D)\sin\beta x]$ |

Rules of finding Particular Integral.

| Type | X | P.I |
|---|---|---|
| I | $e^{ax}$ | P.I= $\frac{1}{f(D)} e^{ax}$<br>Replace D by ‘a’<br>P.I= $e^{ax}\frac{1}{f(a)},\ \ f(a) \neq 0$ If f(a)=0 then<br>P.I= $xe^{ax}\frac{1}{f'(a)},\ \ f'(a) \neq 0$ if $f'(a) = 0$ then<br>P.I= $x^2 e^{ax}\frac{1}{f''(a)},\ \ f''(a) \neq 0$ if $f''(a) = 0$<br>Continuing this process until $Dr \neq 0$ |
| II | $\sin ax\ (or)\ \cos ax$ | P.I= $\frac{1}{f(D)} \sin ax\ (or)\ \cos ax$<br>Replace $D^2$ by $-(a^2)$ if $Dr = 0$ continuing the above process. |
| III | $x^n$ ( n being a +ve integer) | P.I= $\frac{1}{f(D)} x^n$<br>P.I= $[f(D)]^{-1} x^n$<br>Expand $[f(D)]^{-1}$ by using the following formula<br>1) $(1+D)^{-1} = 1 - D + D^2 - D^3 + ....$<br>2) $(1-D)^{-1} = 1 + D + D^2 + D^3 + ....$<br>3) $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + ....$<br>4) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + ....$ |
| IV | $e^{ax}\ V$<br>Where<br>$V = \sin ax\ (or)\ \cos ax\ (or)\ x^n$ | P.I= $\frac{1}{f(D)} e^{ax}[\sin ax\,(or)\cos ax\ (or)\ x^n]$ |

| | | |
|---|---|---|
| | | Replace $D$ by $D+a$<br>P.I= $e^{ax}\frac{1}{f(D+a)}[\sin ax(\text{or})\cos ax\ (\text{or})\ x^n]\ e^{ax}$<br>Then apply Type II (or) Type III depending on X. |
| V | $x^n[\sin ax\ (\text{or})\ \cos ax]$ | P.I= $\frac{1}{f(D)}x^n\ \sin ax\ (\text{or})x^n\ \cos ax$<br>Apply $\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$<br>(or)<br>1) P.I= $\frac{1}{f(D)}x^n\ \cos ax$<br>P.I=R.P. of $\frac{1}{f(D)}x^n e^{iax}$<br>Then apply Type ( IV)<br>2) P.I= $\frac{1}{f(D)}x^n\ \sin ax$<br>P.I = I.P. of $\frac{1}{f(D)}x^n e^{iax}$<br>Then apply Type ( IV) |

5 Method of Variation of parameter:

$$\frac{d^2y}{dx^2}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2y = X..............>(1)$$

Complimentary function of (1) is

C.F= $Ay_1+By_2$

Where A & B are constants and $y_1\ \&\ y_2$ are functions of x.

Then

P.I = $Py_1+Qy_2$

Where $P=-\int\frac{y_2X}{y_1y_1'-y_1'y_2}dx$, $P=-\int\frac{y_1X}{y_1y_1'-y_1'y_2}dy$

General Solution $y = Ay_1+By_2+P.I$

6 Linear Differential Equations With variable coefficients:

1. Homogeneous equations of Euler type: (Cauchy's type)

An equation of the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+...+a_ny = X$$

$$(x^nD^n+a_1x^{n-1}D^{n-1}+a_2x^{n-2}D^{n-2}+...+a_n)y = X.........(A)$$

Where X is the function of x alone.

Equation (A) can be reduced to linear differential equation with constant coefficients by putting $x = e^z$, $z=\log x$

$xD = D'$

and $x^2D^2 = D'(D'-1)$

$x^3D^3 = D'(D'-1)(D'-2)$ and so on

7 Legendre's Linear Differential Equations.

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = X$$

Where X is the function of x alone.

Equation (A) can be reduced to linear differential equation with constant coefficients by putting $ax+b = e^z$, $z = \log(ax+b)$

$(ax+b)D = aD'$

and $(ax+b)^2 D^2 = a^2 D'(D'-1)$

$(ax+b)^3 D^3 = a^3 D'(D'-1)(D'-2)$ and so on