UNIT - I

VECTOR CALCULUS

PART - A

1. If $\phi = 3x^2y - y^3z^2$, find gradϕ at (1, –1, 2)
2. Find the directional derivative of ϕ = $3x^2+2y-3z$ at (1, 1, 1) in the direction $2\vec{i} + 2\vec{j} - \vec{k}$
3. Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point (1,–1,2)
4. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1)
5. Prove that $\nabla\left(r^n\right) = nr^{n-2}\vec{r}$
6. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, find div curl $\vec{F}$
7. Find 'a', such that (3x-2y +z) $\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.
8. Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational.
9. If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\,\vec{k}$. Check whether the integral $\int_C \vec{F}\cdot dr$ is independent of the path C
10. State Gauss Divergence Theorem.

PART - B

1.a). Find the directional derivative of $\phi = 2xy + z^2$ at the point (1, -1, 3) in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution:

Formula: directional derivative $= \nabla\phi \cdot \hat{n}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

given $\phi = 2xy + z^2$ and $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$\frac{\partial\phi}{\partial x} = 2y, \quad \frac{\partial\phi}{\partial y} = 2x, \quad \frac{\partial\phi}{\partial z} = 2z$$

$$\therefore \nabla\phi = 2y\vec{i} + 2x\vec{j} + 2z\vec{k}$$

$$\nabla\phi\big|_{(1,-1,3)} = -2\vec{i} + 2\vec{j} + 6\vec{k}$$

$$\hat{n} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{1^2+2^2+2^2}} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3} \qquad \left(\because \hat{n} = \frac{\vec{a}}{|\vec{a}|}\right)$$

$$\nabla\phi\cdot\hat{n} = -2\vec{i}+2\vec{j}+6\vec{k} \cdot \left(\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right)$$

$$= \frac{-2+4+12}{3}$$

$$= \frac{14}{3}$$

$\therefore$ directional derivative $= \frac{14}{3}$

1. b). Show that $\vec{F} = (4xy - z^3)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k}$ is irrotational and find its scalar potential.

Solution :

$\vec{F}$ is irrotational if $\nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy-z^3 & 2x^2 & -3xz^2 \end{vmatrix}$$

$$\nabla \times \vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial z}(4xy - z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right]$$

$$= \vec{i}(0-0) - \vec{j}(-3z^2 + 3z^2) + \vec{k}(4x - 4x)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\nabla \times \vec{F} = 0$$

$\Rightarrow \vec{F}$ is irrotational.

Scalar potential

$$\vec{F} = \nabla \phi$$

$$(4xy - z^3)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$\frac{\partial \phi}{\partial x} = 4xy - z^3$$

$$\int \partial \phi = \int (4xy - z^3)\,\partial x$$

$$\phi_1 = \cancel{4}^{2} y \frac{x^2}{\cancel{2}} - z^3 x + c_1$$

$$\frac{\partial \phi}{\partial y} = 2x^2$$

$$\int \partial \phi = \int (2x^2)\,\partial y$$

$$\phi_2 = 2x^2 y + c_2$$

$$\frac{\partial \phi}{\partial z} = -3xz^2$$

$$\int \partial \phi = \int (-3xz^2)\,\partial z$$

$$\phi_3 = -\cancel{3}x \frac{z^3}{\cancel{3}} + c_3$$

$$\therefore \phi = 2x^2 y - xz^3 + c$$

2. a). Find the angle between the normals to the surface $xy^3z^2=4$ at the points $(-1,-1,2)$ and $(4,1,-1)$

Solution:

Formula: $$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

given $\phi = xy^3z^2-4$

$$\nabla\phi = y^3z^2\vec{i} + 3xy^2z^2\vec{j} + 2xy^3z\,\vec{k}$$

Let $\nabla\phi_1 = \nabla\phi\big|_{(-1,-1,2)}$ and $\nabla\phi_2 = \nabla\phi\big|_{(4,1,-1)}$

$$\therefore \nabla\phi_1 = -4\vec{i}-12\vec{j}+4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{(-4)^2+(-12)^2+(4)^2}$$

$$= \sqrt{16+144+16}$$

$$= \sqrt{176} = 4\sqrt{11}$$

$$\nabla\phi_2 = \vec{i}+12\vec{j}-8\vec{k}$$

$$|\nabla\phi_2| = \sqrt{(1)^2+(12)^2+(-8)^2}$$

$$= \sqrt{1+144+64}$$

$$= \sqrt{209}$$

$$\cos\theta = \frac{(-4\vec{i}-12\vec{j}+4\vec{k})\cdot(\vec{i}+12\vec{j}-8\vec{k})}{\sqrt{176}\times\sqrt{209}}$$

$$\cos\theta = \frac{-4-144-32}{4\sqrt{11}\times\sqrt{209}}$$

$$= \frac{-180}{4\sqrt{2299}}$$

$$= \frac{-45}{\sqrt{2299}}$$

$$\theta = \cos^{-1}\left(\frac{-45}{\sqrt{2299}}\right)$$

2.b). Find the constants a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational

Solution:

$\vec{F}$ is irrotational if $\nabla\times\vec{F}=0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] +$$
$$\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] +$$
$$\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right] = 0\vec{i}+0\vec{j}+0\vec{k}$$

$$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0\vec{i}+0\vec{j}+0\vec{k}$$

$$\Rightarrow\ c+1=0,\quad 4-a=0,\quad b-2=0$$

$\Rightarrow \quad c = -1, \quad a = 4, \quad b = 2$

$\therefore \quad a = 4, \quad b = 2, \quad c = -1.$

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3. a). Prove that $\nabla^2 (r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$

Proof:

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow \frac{\partial \vec{r}}{\partial x} = \vec{i}, \quad \frac{\partial \vec{r}}{\partial y} = \vec{j}, \quad \frac{\partial \vec{r}}{\partial z} = \vec{k}$$

also $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now

$$\nabla^2 (r^n \vec{r}) = \Sigma \frac{\partial^2}{\partial x^2} (r^n \vec{r})$$

$$= \Sigma \frac{\partial}{\partial x} \left(r^n \frac{\partial \vec{r}}{\partial x} + n r^{n-1} \frac{\partial r}{\partial x} \vec{r} \right)$$

$$= \Sigma \frac{\partial}{\partial x} \left(r^n \vec{i} + n r^{n-1} \frac{x}{r} \vec{r} \right)$$

$$= \Sigma \frac{\partial}{\partial x} \left(r^n \vec{i} + n r^{n-2} x \vec{r} \right)$$

$$= \Sigma \left(n r^{n-1} \frac{\partial r}{\partial x} \vec{i} + n r^{n-2} x \frac{\partial \vec{r}}{\partial x} + n r^{n-2} \vec{r} (1) + n(n-2) r^{n-3} \frac{\partial r}{\partial x} x \vec{r} \right)$$

$$= \Sigma \left(n r^{n-1} \frac{x}{r} \vec{i} + n r^{n-2} x \vec{i} + n r^{n-2} \vec{r} + n(n-2) r^{n-3} \frac{x}{r} x \vec{r} \right)$$

$$\nabla^2(r^n \vec{r}) = \Sigma\left(n r^{n-2} x \vec{i} + n r^{n-2} x \vec{i} + n r^{n-2} \vec{r} + n(n-2) r^{n-4} x^2 \vec{r} \right)$$

$$= n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k}) + n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k}) + 3n r^{n-2} \vec{r}$$

$$+ n(n-2) r^{n-4} \vec{r} \, (x^2+y^2+z^2)$$

$$= n r^{n-2} \vec{r} + n r^{n-2} \vec{r} + 3n r^{n-2} \vec{r} + n(n-2) r^{n-4} \vec{r} \, r^2$$

$$= 5n r^{n-2} \vec{r} + n(n-2) r^{n-2} \vec{r}$$

$$= n r^{n-2} \vec{r} \, (5 + n - 2)$$

$$= n r^{n-2} \vec{r} \, (n+3)$$

$$\nabla^2(r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$$

Hence Proved

3. b). Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle in the xy-plane from $(0,0)$ to $(1,1)$ along the parabola $y^2 = x$

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$

we know that

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

Now

$$\vec{F}\cdot d\vec{r} = (x^2 - y^2 + x)\,dx - (2xy + y)\,dy$$

given

$$y^2 = x$$

$$\Rightarrow 2y\,dy = dx$$

$$\therefore \vec{F}\cdot d\vec{r} = (x^2 - x + x)\,dx - (2y^3 + y)\,dy$$

$$= x^2\,dx - (2y^3 + y)\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 x^2\,dx - \int_0^1 (2y^3 + y)\,dy$$

$$= \frac{x^3}{3}\Big|_0^1 - \left(\frac{2y^4}{4} + \frac{y^2}{2}\right)\Big|_0^1$$

$$= \frac{x^3}{3}\Big|_0^1 - \left(\frac{y^4}{2} + \frac{y^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{3}(x^3)\Big|_0^1 - \frac{1}{2}(y^4 + y^2)\Big|_0^1$$

$$= \frac{1}{3}(1-0) - \frac{1}{2}(2-0)$$

$$= \frac{1}{3} - 1$$

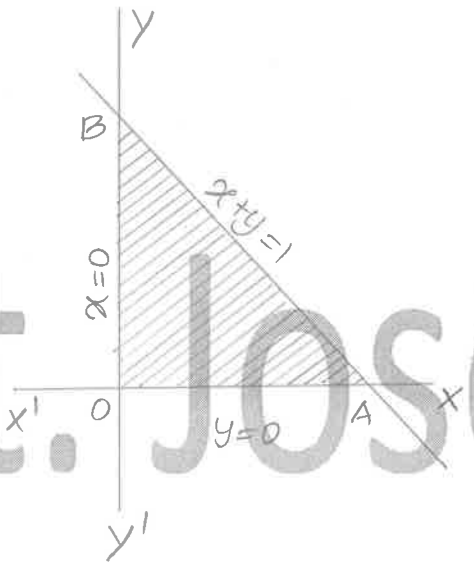
$$\int_C \vec{F}\cdot d\vec{r} = -\frac{2}{3}$$

$$\therefore \text{Work done} = \int_C \vec{F}\cdot d\vec{r} = -\frac{2}{3}$$

4. Verify Green's theorem in the xy plane for $\int_C [(3x-8y^2)\,dx + (4y-6xy)\,dy]$ where C is the boundary of the region given by $x=0,\ y=0,\ x+y=1$

Solution :

Formula : $$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



given $\int_C [(3x-8y^2)\,dx + (4y-6xy)\,dy]$

$$\left.\begin{aligned} M &= 3x-8y^2 \Rightarrow \frac{\partial M}{\partial y} = -16y \\ N &= 4y-6xy \Rightarrow \frac{\partial N}{\partial x} = -6y \end{aligned}\right\} \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y+16y = 10y$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^{1-y} 10y\,dx\,dy$$

$$= \int_0^1 10y \left. x \right|_0^{1-y} dy$$

$$= \int_0^1 10y\,[(1-y)-0]\,dy$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 10\int_0^1 y - y^2\,dy$$

$$= 10 \left. \frac{y^2}{2} - \frac{y^3}{3} \right|_0^1$$

$$= 10\left[\left(\tfrac{1}{2} - \tfrac{1}{3}\right) - 0\right]$$

$$= 10 \times \tfrac{1}{6}$$

$$\text{R.H.S} = \tfrac{5}{3} \quad \text{——} (1)$$

L.H.S

$$\int_C \left[(3x - 8y^2)\,dx + (4y - 6xy)\,dy\right] = \int_{OA} + \int_{AB} + \int_{BO}$$

along OA:

$$y = 0 \Rightarrow dy = 0$$

$$\int_{OA} (3x - 8y^2)\,dx + (4y - 6xy)\,dy = \int_0^1 3x\,dx$$

$$= 3 \left. \frac{x^2}{2} \right|_0^1$$

$$= 3\left(\tfrac{1}{2} - 0\right)$$

$$= \tfrac{3}{2}$$

along AB:

$$x + y = 1 \Rightarrow y = 1 - x, \quad dy = -dx$$

$$\int_{AB} (3x - 8y^2)\,dx + (4y - 6xy)\,dy = \int_1^0 (3x - 8(1-x)^2)\,dx$$
$$+ (4(1-x) - 6x(1-x))(-dx)$$

$$= \int_1^0 [3x - 8(1-2x+x^2) - 4(1-x) + 6x(1-x)]\, dx$$

$$= \int_1^0 (-14x^2 + 29x - 12)\, dx$$

$$= -14\frac{x^3}{3} + 29\frac{x^2}{2} - 12x \Big|_1^0$$

$$= \left[0 - \left(-\frac{14}{3} + \frac{29}{2} - 12\right)\right]$$

$$= \frac{13}{6}$$

along BO:

$x=0 \Rightarrow dx=0$

$$\int_{BO} (3x-8y^2)\,dx + (4y-6xy)\,dy = \int_1^0 4y\,dy$$

$$= 4\frac{y^2}{2}\Big|_1^0$$

$$= 0-2$$

$$= -2.$$

$$\therefore \int_C (3x-8y^2)\,dx + (4y-6xy)\,dy = \frac{3}{2} + \frac{13}{6} - 2$$

$$= \frac{5}{3} \quad -(2)$$

from (1) & (2)

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem verified

5. Verify Gauss-divergence theorem for $\vec{A} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$

Solution:

Formula : $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv$

Given $\vec{A} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\operatorname{div}\vec{A} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$\operatorname{div}\vec{A} = 4z - y$$

R.H.S

$$\iiint_V \operatorname{div}\vec{A}\, dv = \int_0^1\int_0^1\int_0^1 (4z-y)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 (4z-y)\, x\Big|_0^1\, dy\, dz$$

$$= \int_0^1\int_0^1 (4z-y)\,[1-0]\, dy\, dz$$

$$= \int_0^1\int_0^1 (4z-y)\, dy\, dz$$

$$\iiint_V \operatorname{div} \vec{A}\, dv = \int_0^1 \left(4zy - \frac{y^2}{2}\right)\Big|_0^1 dz$$

$$= \int_0^1 \left[\left(4z - \tfrac{1}{2}\right) - 0\right] dz$$

$$= \int_0^1 \left(4z - \tfrac{1}{2}\right) dz$$

$$= 4\frac{z^2}{2} - \frac{1}{2}z \Big|_0^1$$

$$= \left(\frac{4}{2} - \frac{1}{2}\right) - 0$$

$$R.H.S = \frac{3}{2} \quad \text{——(1)}$$

L.H.S

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

along $x=0$

$x=0, \quad \hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{A} \cdot \hat{n}\, ds = \int_0^1\int_0^1 (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{i})\, dy\, dz$$

$$= \int_0^1\int_0^1 -4xz\, dy\, dz \quad (\text{but } x=0)$$

$$= \int_0^1\int_0^1 0\, dy\, dz = 0$$

along $x=1$

$x=1, \qquad \hat{n} = \vec{i}$

$$\iint_{S_2} \vec{A}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot\vec{i}\; dy\, dz$$

$$= \int_0^1\int_0^1 4xz\; dy\, dz \qquad (\because x=1)$$

$$= \int_0^1\int_0^1 4z\, dy\, dz$$

$$= \int_0^1 4zy\Big|_0^1 dz$$

$$= \int_0^1 4z\,(1-0)\; dz$$

$$= \int_0^1 4z\, dz$$

$$= 4\cdot \frac{z^2}{2}\Big|_0^1$$

$$= 2-0$$

$$= 2.$$

along $y=0$

$y=0, \qquad \hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})(-\vec{j})\; dx\, dz$$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\, ds = \int_0^1\int_0^1 y^2\, dx\, dz$$

$$= \int_0^1\int_0^1 0\, dx\, dz \qquad (\because\ y=0)$$

$$= 0$$

along $y=1$

$y=1, \qquad \hat{n} = \vec{j}$

$$\iint_{S_4} \vec{A}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\,\vec{j}\ dx\, dz$$

$$= \int_0^1\int_0^1 -y^2\, dx\, dz$$

$$= \int_0^1\int_0^1 -1\, dx\, dz \qquad (\because\ y=1)$$

$$= -\int_0^1 x\Big|_0^1\, dz$$

$$= -\int_0^1 (1-0)\, dz$$

$$= -\int_0^1 dz$$

$$= -\left(z\Big|_0^1\right)$$

$$= -(1-0)$$

$$= -1$$

along $z=0$

$z=0$, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{A}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot(-\vec{k})\, dx\, dy$$

$$= \int_0^1\int_0^1 -yz\, dx\, dy$$

$$= \int_0^1\int_0^1 0\, dx\, dy \quad (\because z=0)$$

$$= 0$$

along $z=1$

$z=1$, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{A}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot\vec{k}\, dx\, dy$$

$$= \int_0^1\int_0^1 yz\, dx\, dy$$

$$= \int_0^1\int_0^1 y\, dx\, dy \quad (\because z=1)$$

$$= \int_0^1 y\, x\Big|_0^1\, dy$$

$$= \int_0^1 y\,(1-0)\, dy$$

$$\iint_{S_6} \vec{A} \cdot \hat{n}\, ds = \int_0^1 y\, dy$$

$$= \frac{y^2}{2}\Big|_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

$$\therefore \iint_S \vec{A} \cdot \hat{n}\, ds = 0 + 2 + 0 - 1 + 0 + \frac{1}{2}$$

$$L.H.S = \frac{3}{2} \quad \text{—— (2)}$$

From (1) & (2)

$$L.H.S = R.H.S$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \operatorname{div} \vec{F}\, dV$$

Hence Gauss - divergence theorem verified.

6. Verify Stoke's theorem for $\vec{F} = x^2\vec{i} - xy\vec{j}$ in the square region in the xy-plane bounded by the lines $x=0$, $y=0$, $x=a$ and $y=a$

Solution:

Formula : $\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds = \int_C \vec{F} \cdot d\vec{r}$

Given

$$\vec{F} = x^2\vec{i} - xy\,\vec{j}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-xy)\right] - \vec{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-y-0)$$

$$\text{curl } \vec{F} = -y\,\vec{k}$$

R.H.S

$$\iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds = \int_0^a \int_0^a -y\vec{k}\cdot\vec{k}\, dx\, dy \quad (\because \hat{n} = \vec{k})$$

$$= \int_0^a \int_0^a -y\, dx\, dy$$

$$= \int_0^a -y \cdot x\Big|_0^a dy$$

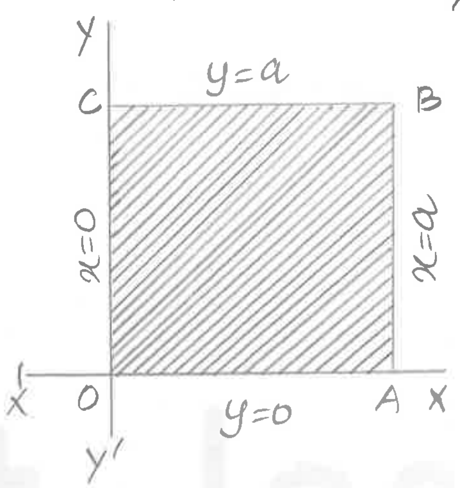
$$= \int_0^a -y\,(a-0)\, dy$$

$$= -a \int_0^a y\, dy$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = -a\left.\frac{y^2}{2}\right|_0^a$$

$$= -a\cdot\left(\frac{a^2}{2}-0\right)$$

$$R.H.S = -\frac{a^3}{2} \quad \text{—(1)}$$



L. H. S

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Now

$$\vec{F} = x^2\vec{i} - xy\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\therefore \vec{F}\cdot d\vec{r} = x^2dx - xy\,dy$$

along OA $\quad y=0, \quad dy=0 \Rightarrow$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2dx$$

$$= \left.\frac{x^3}{3}\right|_0^a = \frac{a^3}{3} - 0 = \frac{a^3}{3}$$

along AB

$x = a, \quad dx = 0 \Rightarrow$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^a -ay \, dy$$

$$= -a \int_0^a y \, dy$$

$$= -a \left. \frac{y^2}{2} \right|_0^a$$

$$= -a \left(\frac{a^2}{2} - 0 \right)$$

$$= -\frac{a^3}{2}$$

along BC

$y = a, \quad dy = 0 \Rightarrow$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2 \, dx$$

$$= \left. \frac{x^3}{3} \right|_a^0$$

$$= 0 - \frac{a^3}{3}$$

$$= -\frac{a^3}{3}$$

along CO

$x = 0, \quad dx = 0 \Rightarrow$

$$\int_{C_0} \vec{F}\cdot d\vec{r} = \int_a^0 0\,dy$$

$$= 0$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \frac{\cancel{a^3}}{\cancel{3}} - \frac{a^3}{2} - \frac{\cancel{a^3}}{\cancel{3}} + 0$$

$$L.H.S = -\frac{a^3}{2} \quad \text{—— (2)}$$

From (1) & (2)

$$L.H.S = R.H.S$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

Hence Stoke's theorem verified.

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7. a). Evaluate $\int_C [(x^2+xy)\,dx + (x^2+y^2)\,dy]$ where C is the square bounded by the lines $x=0, x=1, y=0, y=1$.

Solution:

By Stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

given

$$\vec{F}\cdot d\vec{r} = (x^2+xy)\,dx + (x^2+y^2)\,dy$$

$$\Rightarrow \vec{F} = (x^2+xy)\,\vec{i} + (x^2+y^2)\,\vec{j}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy & x^2+y^2 & 0 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x^2+y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+xy)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2+xy)\right]$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2x-x)$$

$$\text{curl } \vec{F} = x\,\vec{k}$$

$$\therefore \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 x\,\vec{k}\cdot\vec{k}\; dx\,dy \quad (\because \hat{n} = \vec{k})$$

$$= \int_0^1\int_0^1 x\, dx\, dy$$

$$= \int_0^1 \left.\frac{x^2}{2}\right|_0^1 dy$$

$$= \int_0^1 \left(\tfrac{1}{2} - 0\right) dy$$

$$= \tfrac{1}{2}\int_0^1 dy$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \tfrac{1}{2}\,y\Big|_0^1$$

$$= \tfrac{1}{2}(1-0)$$

$$= \tfrac{1}{2}.$$

$$\therefore \int_C \left[(x^2+xy)\,dx + (x^2+y^2)\,dy\right] = \tfrac{1}{2}.$$

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7. b). If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, Evaluate $\iiint_V \nabla\times\vec{F}\,dv$ where V is the region bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$.

Solution:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2-3z)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2-3z)\right]$$

$$= \vec{i}(0-0) - \vec{j}(-4+3) + \vec{k}(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

$$\iiint_V \nabla \times \vec{F}\, dv = \int_0^1 \int_0^2 \int_0^3 (\vec{J} - 2y\vec{K})\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^2 z\vec{J} - 2yz\vec{K} \Big|_0^3 dy\, dx$$

$$= \int_0^1 \int_0^2 (3-0)\vec{J} - 2y(3-0)\vec{K}\, dy\, dx$$

$$= \int_0^1 \int_0^2 (3\vec{J} - 6y\vec{K})\, dy\, dx$$

$$= \int_0^1 3y\vec{J} - \cancel{6}^{3} \frac{y^2}{\cancel{2}} \vec{K} \Big|_0^2 dx$$

$$= \int_0^1 3(2-0)\vec{J} - 3(4-0)\vec{K}\, dx$$

$$= \int_0^1 6\vec{J} - 12\vec{K}\, dx$$

$$= 6x\vec{J} - 12x\vec{K} \Big|_0^1$$

$$= 6(1-0)\vec{J} - 12(1-0)\vec{K}$$

$$\iiint_V \nabla \times \vec{F}\, dv = 6\vec{J} - 12\vec{K}$$