Problem:

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \qquad \text{ROC: } |z| > 1$$

using Long-division method.

Soln: ROC: $|z|>1$, ROC is outside of a circle, $\therefore x(n)$ is causal.

$x(n),\ n \geq 0$

$$
\begin{array}{r|l}
 & 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + 1.9375z^{-4} \\
\hline
1 - 1.5z^{-1} + 0.5z^{-2} & 1 \\
 & \underset{(-)}{1} \underset{(+)}{-} 1.5z^{-1} \underset{(-)}{+} 0.5z^{-2} \\
\hline
 & 1.5z^{-1} - 0.5z^{-2} \\
 & \underset{(-)}{1.5z^{-1}} \underset{(+)}{-} 2.25z^{-2} \underset{(-)}{+} 0.75z^{-3} \\
\hline
 & 1.75z^{-2} - 0.75z^{-3} \\
 & \underset{(-)}{1.75z^{-2}} \underset{(+)}{-} 2.625z^{-3} \underset{(-)}{+} 0.875z^{-4} \\
\hline
 & 1.875z^{-3} - 0.875z^{-4} \\
 & 1.875z^{-3} - 0.875z^{-4} \\
 & \underset{(-)}{1.875z^{-3}} \underset{(+)}{-} 2.8125z^{-4} \underset{(-)}{+} 0.9375z^{-5} \\
\hline
 & 1.9375z^{-4} - 0.9375z^{-5}
\end{array}
$$

$$x(n) = \{\underset{\uparrow}{1}, 1.5, 1.75, 1.875, 1.9375 \ldots\}.$$

Determine $x(n)$ of

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}} \quad \text{for ROC } |z| > 1$$

using long division method.

Soln:

ROC $|z| > 1$, $\therefore x(n)$ is causal.

$x(n),\ \forall\ n \geq 0$.

$$1-2z^{-1}+z^{-2} \,\big)\; 1+2z^{-1} \qquad \text{quotient: } \ldots 7z^{-2}+10z^{-3}+13z^{-4}+16z^{-5}$$

$$\underset{(-)}{1}\ \underset{(+)}{-}\ 2z^{-1}\ \underset{(-)}{+}\ z^{-2}$$

$$4z^{-1}-z^{-2}$$

$$\underset{(-)}{4z^{-1}}\ \underset{(+)}{-}\ 8z^{-2}\ \underset{(-)}{+}\ 4z^{-3}$$

$$7z^{-2}-4z^{-3}$$

$$\underset{(-)}{7z^{-2}}\ \underset{(+)}{-}\ 14z^{-3}\ \underset{(-)}{+}\ 7z^{-4}$$

$$10z^{-3}-7z^{-4}$$

$$\underset{(-)}{10z^{-3}}\ \underset{(+)}{-}\ 20z^{-4}\ \underset{(-)}{+}\ 10z^{-5}$$

$$13z^{-4}-10z^{-5}$$

$$\underset{(-)}{13z^{-4}}\ \underset{(+)}{-}\ 26z^{-5}\ \underset{(-)}{+}\ 13z^{-6}$$

$$16z^{-5}-13z^{-6}$$

$$\therefore x(n)=\{\underset{\uparrow}{1},4,7,10,13,16,\ldots\}.$$

$\gamma \longrightarrow \gamma.$

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \quad \text{for ROC: } |z| < 0.5$$

using long-division method.

Soln:

ROC: $|z| < 0.5$, $x(n)$ is a ~~non~~ anti causal signal.

To find:

$x(n), \ \forall\, n \le 0$.

Divisor: $\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1$

Quotient: $2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6$.

$$1$$

$$(-)\ 1 - 3z + 2z^2$$

$$0 + 3z - 2z^2$$

$$3z - 9z^2 + 6z^3$$

$$(-)\quad (+)\quad (-)$$

$$7z^2 - 6z^3$$

$$7z^2 - 21z^3 + 14z^4$$

$$(-)\quad (+)\quad (-)$$

$$15z^3 - 14z^4$$

$$15z^3 - 45z^4 + 30z^5$$

$$(-)\quad (+)\quad (-)$$

$$31z^4 - 30z^5$$

$$X(z) = \dots + 62z^6 + 30z^5 + 14z^4 + 6z^3 + 2z^2$$

$$\therefore x(n) = \{\dots, 62, 30, 14, 6, 2, 0, 0\}.$$

Problem:

Determine $x(n)$ of

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}} \quad \text{for ROC: } |z| < 0.5$$

using long division method.

Soln: ROC: $|z| < 0.5$, $x(n)$ is anticausal signal.

To find:

$x(n)$, $\forall\, n \leq 0$.

$$
\begin{array}{r|l}
 & 2z + 5z^2 + 8z^3 + 11z^4 + 14z^5 + 17z^6 \\
\hline
z^{-2} - 2z^{-1} + 1 & 2z^{-1} + 1 \\
 & \underline{\overset{(-)}{2z^{-1}} \overset{(+)}{-4} \overset{(-)}{+2z}} \\
 & 5 - 2z \\
 & \underline{\overset{(-)}{5} \overset{(+)}{-10z} \overset{(-)}{+5z^2}} \\
 & 8z - 5z^2 \\
 & \underline{8z \overset{(+)}{-16z^2} \overset{(-)}{+8z^3}} \\
 & 11z^2 - 8z^3 \\
 & \underline{\overset{(-)}{11z^2} \overset{(+)}{-22z^3} \overset{(-)}{+11z^4}} \\
 & 14z^3 - 11z^4 \\
 & \underline{\overset{(-)}{14z^3} \overset{(+)}{-28z^4} \overset{(-)}{+14z^5}} \\
 & 17z^4 - 14z^5
\end{array}
$$

$\therefore X(z) = \cdots + 17z^6 + 14z^5 + 11z^4 + 8z^3 + 5z^2 + 2z$

$\therefore x(n) = \{\ldots, 17, 14, 11, 8, 5, 2, \underset{\uparrow}{0}\}$.

Partial Fraction Expansion Method:

If $X(z)$ is a rational function, $X(z)$ can be given as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

* $X(z)$ is called proper rational function if $M < N$ & $a_N \neq 0$.
* $X(z)$ is called improper rational function if $M \geq N$.

Note:

An improper rational function can be expressed as sum of proper rational function and a polynomial.

$$X(z) = \frac{N(z)}{D(z)} = C_0 + C_1 z^{-1} + \cdots + C_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D(z)}.$$

Expanding Proper rational function:

To eliminate negative powers of z, we multiply both denominator & numerator by z^N.

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}.$$

There are two cases:

(i) If $X(z)$ has distinct poles:

Let the poles be $p_1, p_2, \dots, p_N$

Then $\frac{X(z)}{z}$ can be expanded as

$$\frac{X(z)}{z} = \frac{C_1}{(z-p_1)} + \frac{C_2}{(z-p_2)} + \dots + \frac{C_N}{(z-p_N)}$$

The coefficients $C_1, C_2, \dots, C_N$ can be obtained by the formula.

$$\boxed{C_k = \frac{(z-p_k)\,X(z)}{z}\bigg|_{z=p_k} \quad k=1,2,\dots N.}$$

(ii) If $X(z)$ has Multiple order poles:

Simple case. Let

$$\frac{X(z)}{z} = \frac{1}{(z-p_2)^2(z-p_1)},$$ This can be expanded as

$$\frac{X(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + \frac{C_3}{(z-p_2)^2}$$

$$C_1 = (z-p_1)\frac{X(z)}{z}\bigg|_{z=p_1}$$

$$C_2 = \frac{d}{dz}\left[(z-p_2)^2\frac{X(z)}{z}\right]\bigg|_{z=p_2}$$

$$C_3 = (z-p_2)^2\frac{X(z)}{z}\bigg|_{z=p_2}.$$

Note: If there are l poles

$$\frac{X(z)}{z} = \frac{R(z)}{(z-p_k)^l}$$

The signal is anticausal.

$$x(n) = -2(1)^n u(-n-1) + (0.5)^n u(-n-1) \qquad \left(\because Z\{-a^n u(-n-1)\} = \frac{1}{1-az^{-1}}, |z|<|a|\right)$$

$$\therefore \boxed{x(n) = [-2 + 0.5^n]\, u(-n-1)}$$

Case (iii) ROC: $0.5 < |z| < 1$.

The ROC corresponds to two sided signal.

The ROC is overlapping of $|z| > 0.5$ & $|z| < 1$

$\therefore$ Pole $P_1 = 0.5$, corresponds to causal signal

Pole $P_2 = 1$, corresponds to anticausal signal.

$$Z\{a^n u(n)\} = \frac{1}{1-az^{-1}}, |z| > |a|$$

$$Z\{-a^n u(-n-1)\} = \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$$\therefore \boxed{x(n) = -2u(-n-1) - 1(0.5)^n u(n)}$$

Assignment Problem: Determine $x(n)$ of

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

using partial fraction method for the following conditions.

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$.

Soln:

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

The highest power in the denominator is $N=2$,
$\therefore$ Multiply Num & Den of $X(z)$ by z^2 to eliminate negative power terms.

$$X(z) = \frac{z^2+2z}{z^2-2z+1}$$

$$\frac{X(z)}{z} = \frac{z+2}{z^2-2z+1}.$$

$$z^2-2z+1 = (z-1)(z-1)$$

$$\frac{X(z)}{z} = \frac{(z+2)}{(z-1)^2}.$$

* $\frac{X(z)}{z}$ has second order pole.

$$\frac{X(z)}{z} = \frac{C_1}{(z-1)} + \frac{C_2}{(z-1)^2}.$$

$$C_1 = \frac{d}{dz}\left[(z-1)^2 \frac{X(z)}{z}\right]\Bigg|_{z=1}$$

$$C_2 = (z-1)^2 \frac{X(z)}{z}\Bigg|_{z=1}$$

$$C_1 = \frac{d}{dz}\left[\cancel{(z-1)^2} \frac{(z+2)}{\cancel{(z-1)^2}}\right]\Bigg|_{z=1}$$

$$\boxed{C_1 = 1}, \quad C_2 = \cancel{(z-1)^2} \cdot \frac{(z+2)}{\cancel{(z-1)^2}}\Bigg|_{z=1}$$

$$\boxed{C_2 = 3}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{(z-1)} + \frac{3}{(z-1)^2}.$$

(i) Roc: $|z|>1$, $\therefore$ The Roc corresponds to Causal Signal.

$$X(z) = \frac{1}{1-z^{-1}} + \frac{3z}{(z-1)^2} = \frac{1}{1-z^{-1}} + \frac{3z^{-1}}{(1-z^{-1})^2}$$

$Z\{u(n)\} = \frac{1}{1-z^{-1}}$, $Z\{n\,u(n)\} = \frac{z^{-1}}{(1-z^{-1})^2}.$

Roc: $|z|>|a|$ Roc: $|z|>|a|$

$\therefore$

$$\boxed{x(n) = u(n) + 3n\,u(n).}$$

(ii) Roc: $|z| < 0.5$ Roc corresponds to anticausal signal.

$$X(z) = \frac{1}{1-z^{-1}} + \frac{3z^{-1}}{(1-z^{-1})^2}$$

$Z\{-u(-n-1)\} = \frac{1}{1-z^{-1}}$ $z\{-n\,u(-n-1)\} = \frac{z^{-1}}{(1-z^{-1})^2}$

Roc: $|z|<|a|$ Roc: $|z|<|a|$

$$\therefore \boxed{x(n) = -u(-n-1) - 3n\,u(-n-1)}$$

x ———— x.