## QUESTION BANK FOR MATHEMATICS – I (MA6151)
## I-YEAR B.E./ B.TECH. ( COMMON TO ALL BRANCHES )
## UNIT – I MATRICES
### PART – A

**1. Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & 3 \end{bmatrix}$

**Ans:** Sum of the eigen values = Sum of the main diagonal elements = 1+0+3 = 4
Product of the eigen values = |A|= – 13

**2. If 3 and 15 are the two eigen values of** $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$**, find | A | , without expanding the determinant.**

**Ans:** If $\lambda$ is the third eigen value of A, then $3 + 15 + \lambda = 8 + 7 + 3 \Rightarrow \lambda = 0$
We know that, $|A|$ = product of eigen values = (0)(3)(15) = 0

**3. The product of two eigen values of the matrix** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third eigen value.**

**Ans:** Let $\lambda_1 , \lambda_2 , \lambda_3$ be the eigen values of the given matrix , then $\lambda_1\lambda_2\lambda_3 = |A|$
$\Rightarrow (16)\lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$ [ since product of two eigen values is 16]
$\Rightarrow (16)\lambda_3 = 32 \Rightarrow \lambda_3 = 2$

**4. One of the eigen values of** $\begin{bmatrix} 7 & 4 & 4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ **is -9, Find the other two eigen values.**

**Ans:** If $\lambda_1 , \lambda_2$ be the other two eigen values, then
$\Rightarrow \lambda_1 + \lambda_2 - 9 = 7 - 8 - 8 = -9$ (since sum of the eigen values = sum of the leading diagonal elements)
$\Rightarrow \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 = -\lambda_2$ …(1)
$\Rightarrow -9\lambda_1\lambda_2 = |A| = 441$ ( since product of the eigen values = | A | )
$\Rightarrow \lambda_1\lambda_2 = -49 \Rightarrow \lambda_1 = \frac{-49}{\lambda_2}$ …… (2)
substitute in (1) we get, $\lambda_2 = \pm 7$
(1) => $\lambda_1 = \pm 7$. Hence the other two eigen values are 7 and -7.

**5. Find the eigen values of $A^2$ , if** $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

**Ans:** In a triangular matrix, the main diagonal values are the eigen values of the matrix.
Here 3, 2, 5 are the eigen values of A. Hence the eigen values of $A^2 = 3^2, 2^2, 5^2$=9,4,25.

**6. If -2,3,6 are the eigen values of a 3 × 3 matrix A, then what are the eigen values of** $6A^{-1}$ and $A^T$ ?

**Ans: Eigen values of $A^{-1}$=** $\frac{-1}{2}, \frac{1}{3}, \frac{1}{6}$

$\therefore$ Eigen values of $6A^{-1} = \frac{-6}{2}, \frac{6}{3}, \frac{6}{6} = -3, 2, 1.$

Eigen values of $A^T$ = Eigen values of $A = -2, 3, 6$

**7. State Cayley Hamilton theorem.**

**Ans:** Every square matrix satisfies its own characteristics equation.

**8. Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, Find $A^{-1}$ using Cayley – Hamilton theorem.**

**Ans:** The characteristics equation is $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$. By Cayley – Hamilton theorem $A^2 - 4A - 5I = 0$

Pre multiplying by $A^{-1}$, we get $A - 4I - 5A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{5}[A - 4I] = \begin{bmatrix} \frac{-3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-1}{5} \end{bmatrix}$$

**9. If $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, find the corresponding eigen value.**

**Ans:** $(A - \lambda I)X = 0 \Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$(-2-\lambda)(1) + 2(2) + (-3)(-1) = 0 \Rightarrow \lambda = 5.$

**10. Find the eigen vector of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ corresponding to the eigen value 2.**

**Ans:** The eigen vectors are given by $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 2$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = 0$ & $x_1 = 0$. Therefore the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

**11. Find the constants a and c such that the matrix $\begin{pmatrix} a & 4 \\ 1 & c \end{pmatrix}$ has 3 &–2 as eigen values.**

**Ans:** Sum of the eigen values = Trace of the matrix $\Rightarrow a + c = 3-2 = 1$-----(1)

Product of the eigen values = Determinant of the matrix

$$ac- 4 = (3)(-2) = -6 \Rightarrow ac = -2 \quad \therefore c = -2/a$$

sub c in (1) $a + c = 1 \Rightarrow a + (-2/a) =1 \Rightarrow a = -1, 2 \Rightarrow c = 2,-1$

**12. Determine $\lambda$ so that $\lambda (x^2 + y^2 + z^2) + 2xy - 2xz + 2zy$ is positive definite.**

**Ans:** The matrix of the given quadratic form is $A = \begin{pmatrix} \lambda & 1 & -1 \\ 1 & \lambda & 1 \\ -1 & 1 & \lambda \end{pmatrix}$

$D_1 = \lambda$, $D_2 = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$ & $D_3 = |A| = (\lambda+1)^2(\lambda -2)$

The Quadratic form is positive definite if $D_1$, $D_2$ & $D_3 > 0 \Rightarrow \lambda > 2$

**13. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigen values of an n x n matrix A, then show that $\lambda_1^3, \lambda_2^3, \lambda_3^3, \ldots, \lambda_n^3$ are the eigen values of $A^3$.**

**Ams:** Let $\lambda_r$ be the eigen value of A with the eigen vector $X_r$, then $AX_r = \lambda_r X_r$

Consider,
$$\begin{aligned} A^3X_r &= A^2(AX_r) \\ &= A^2(\lambda_r X_r) \\ &= \lambda_r A(AX_r) \\ &= \lambda_r A(\lambda_r X_r) = \lambda_r^2(AX_r) = \lambda_r^3 X_r \end{aligned}$$

**14. Determine the nature of the following quadratic form: $f(x_1,x_2,x_3) = x_1^2 + 2x_2^2$.**

**Ans:** The matrix of the given quadratic form is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The above matrix is a diagonal matrix, so the diagonal elements are their Eigen values.
(i.e) 1,2,0 . Here two eigen values are positive another one is zero,
$\therefore$ the nature of the quadratic form is positive semi definite.

**15. If 1 & 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of $A^2$, adj A and A+7I.**

**Ans:** The eigen values of $A^2$ are $1^2, 2^2 = 1, 4$

The eigen values of A+7I are 1+7 , 2+7 = 8, 9.

The eigen values of adj A are $\frac{|A|}{1}, \frac{|A|}{2} = \frac{2}{1}, \frac{2}{2} = 2,1$ [since

$A^{-1} = \frac{1}{|A|} adjA \Rightarrow adjA = |A| A^{-1}$ and $|A| = 2$

**16. Find the characteristic equation of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$**

**Ans:** The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1$ = Sum of the main diagonal elements = 3;

$S_2$ = Sum of the minors of the main diagonal elements = –1

$S_3 = |A| = -9$. Therefore the characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$.

**17. Find the Rank, index and signature of the Quadratic form whose Canonical form is $x_1^2 + 2x_2^2 - 3x_3^2$**

**Ans:** Rank (r) = Number of terms in the C.F = 3 ,

Index (p) = Number of Positive terms in the C.F = 2

Signature (s) = 2p – r =1

**18. Show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.**

**Ans:** $$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^TA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal.

**19. Write the matrix of the quadratic form $3x_1^2 + 5x_2^2 - 5x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_3x_1$**

**Ans:** The matrix of the quadratic form is given by

$a_{11}$ = coeff of $x_{11}$ = 3 $\qquad$ $a_{22}$ = coeff of $x_{22}$ = 5

$a_{33}$ = coeff of $x_{33}$ = -5 $\qquad$ $a_{12} = a_{21} = \frac{1}{2}(\text{coeff of } x_1x_2) = \frac{-2}{2} = -1$

$a_{13} = a_{31} = \frac{1}{2}(\text{coeff of } x_1x_3) = \frac{6}{2} = 3$ $\qquad$ $a_{23} = a_{32} = \frac{1}{2}(\text{coeff of } x_1x_3) = \frac{2}{2} = 1$

$$\Rightarrow A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & -5 \end{bmatrix}$$

**20. Write down the quadratic form for the given matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 3 & -1 \\ -2 & -1 & -4 \end{bmatrix}$**

**Ans: Quadratic form is** $X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 3 & -1 \\ -2 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow$ $2x_1^2 + 3x_2^2 - 4x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_3x_1$

## PART – B

**1(a) Find the eigen values and eigen vectors of the matrix A =** $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

**Hints:** The Characteristic equation is $|A-\lambda I|=0$

Characteristic equation: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

Solve the characteristic equation and get the eigen values $\lambda = 1,2,3$

Eigen vectors are given by $|A-\lambda I|X = 0$

Eigen vectors: $\lambda = 1 \Rightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \lambda = 2 \Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \lambda = 3 \Rightarrow X_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix};$

**(b) Using Cayley-Hamilton theorem, find $A^{-1}$ and $A^4$, if A =** $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

**Hints:** The Characteristic equation is $|A-\lambda I|=0$

Characteristic equation : $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$

Cayley-Hamilton theorem states that " Every Square matrix satisfies its own characteristic equation"

$\Rightarrow A^3 - 5A^2 + 9A - I = 0$

Pre multiplying A on both sides and get $A^4 = 5A^3 - 9A^2 + A$.

Pre multiplying $A^{-1}$ on both sides and get $A^{-1} = A^2 - 5A + 9I$

$$A^4 = \begin{pmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

**2(a) Find the Eigen values and Eigen vectors of the matrix A =** $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

**Hints:** The Characteristic equation is $|A-\lambda I|=0$

Characteristic equation: $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

Eigen values $\lambda=1,1,5$

$$\text{Eigen vectors}: \lambda=5,\ X_1=\begin{bmatrix}1\\1\\1\end{bmatrix};\ \lambda=1,\ X_2=\begin{bmatrix}1\\0\\-1\end{bmatrix};\ \lambda=1,\ X_3=\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

**(b) Using Cayley-Hamilton theorem, Evaluate the matrix equation**

$$\boldsymbol{A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I} \textbf{ for A = } \begin{pmatrix}\mathbf{2}&\mathbf{1}&\mathbf{1}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{1}&\mathbf{1}&\mathbf{2}\end{pmatrix}$$

**Hints:** The characteristic equation is $|A-\lambda I|=0$

Characteristic equation : $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Cayley-Hamilton theorem states that " Every Square matrix satisfies its own characteristic equation"

$\Rightarrow A^3 - 5A^2 + 7A - 3 = 0$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$
$$= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + A\left(A^3 - 5A^2 + 7A - 3I\right) - 15A^2 + 5A - I$$
$$= -15A^2 + 5A - I = -11\begin{pmatrix}6&5&5\\0&-11&0\\-55&-55&-66\end{pmatrix}$$

**3(a) Find the eigen values and eigen vectors of the matrix** $\begin{pmatrix}\mathbf{10}&\mathbf{-2}&\mathbf{-5}\\\mathbf{-2}&\mathbf{2}&\mathbf{3}\\\mathbf{-5}&\mathbf{3}&\mathbf{5}\end{pmatrix}$

**Hints:** The Characteristic equation is $|A-\lambda I|=0$

Characteristic equation: $\lambda^3 - 17\lambda^2 + 42\lambda = 0$.

Solve the characteristic equation and get the eigen values $\lambda = 0, 3, 14$

Eigen vectors are given by $|A-\lambda I|X = 0$

$$\text{Eigen vectors: } \lambda=0 \Rightarrow X_1=\begin{pmatrix}1\\-5\\4\end{pmatrix};\lambda=3\Rightarrow X_2=\begin{pmatrix}1\\1\\1\end{pmatrix};\lambda=14\Rightarrow X_3=\begin{pmatrix}-3\\1\\2\end{pmatrix};$$

**(b) Find the matrix A whose eigen values are 2, 3 and 6 and eigen vectors are** $\boldsymbol{\{1,\ 0,\ -1\}^T,\ \{1,\ 1,\ 1\}^T,\ \{1,\ -2,\ 1\}^T}$

**Hints:** Since the given eigen vectors $\{1,\ 0,\ -1\}^T$, $\{1,\ 1,\ 1\}^T$, $\{1,-2,1\}^T$ are pairwise orthogonal, we know by orthogonal reduction $N^T AN = D$

$$\Rightarrow A = NDN^T = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

**4(a) Find the eigen values and eigen vectors of the matrix** $\begin{pmatrix} \mathbf{6} & \mathbf{-6} & \mathbf{5} \\ \mathbf{14} & \mathbf{-13} & \mathbf{10} \\ \mathbf{7} & \mathbf{-6} & \mathbf{4} \end{pmatrix}$

**Hints:** The Characteristic equation is $|A-\lambda I| = 0$

Characteristic equation: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

Solve the characteristic equation and get the eigen values $\lambda = -1, -1, -1$

Eigen vectors are given by $|A-\lambda I|X = 0$

Eigen vectors: $\lambda = -1 \Rightarrow X_1 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}; \lambda = -1 \Rightarrow X_2 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}; \lambda = -1 \Rightarrow X_3 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix};$

**(b) Verify that A =** $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} \end{pmatrix}$ **satisfies its own characteristic equation and hence find** $\mathbf{A^4}$

**Hints:** The Characteristic equation is $|A-\lambda I| = 0$

Characteristic equation : $\lambda^2 - 5 = 0$

Cayley-Hamilton theorem states that " Every Square matrix satisfies its own characteristic equation"

$\Rightarrow A^2 - 5I = 0$

Verification: Find $A^2$ as $A^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ and then prove $A^2 - 5I = 0$

To find $A^4$ : Pre multiply $A^2$ on both sides and get $A^4 - 5A^2 = 0$

$$\Rightarrow A^4 = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

**5 Verify Cayley- Hamilton theorem for the matrix A=** $\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$ **and also find** $\mathbf{A^4}$ **and** $\mathbf{A^{-1}}$

**Hints:** The characteristic equation is $|A-\lambda I| = 0$

⇨ The characteristic Equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$.

Cayley-Hamilton theorem states that " Every Square matrix satisfies its own characteristic equation "

$\Rightarrow A^3 - 6A^2 + 8A - 3I = 0$ ------(1)

To verify Cayley Hamilton theorem, we have to verify $A^3 - 6A^2 + 8A - 3I = 0$

Find $A^2$ and $A^3$ as $A^2 = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$; $A^3 = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$;

$$A^3 - 6A^2 + 8A - 3I = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 6\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 8\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Cayley Hamilton Theorem is verified.

**To Find** $A^4$

Premultiply (1) by A

$(A^3 - 6A^2 + 8A - 3I)A = 0 \quad \Rightarrow A^4 - 6A^3 + 8A^2 - 3A = 0$

$\Rightarrow A^4 = 6A^3 - 8A^2 + 3A$

$$\Rightarrow A^4 = \begin{pmatrix} 124 & 123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

**To Find** $A^{-1}$

Premultiply (1) by $A^{-1}$

$A^{-1}(A^3 - 6A^2 + 8A - 3I) = 0 \quad \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

**6 Diagonalise the matrix A=** $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$ **by means of an orthogonal transformation.**

**Hints:**

- The characteristic equation is $\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$.
- $\lambda = -2, 6, 6$

Case (1) : $\boldsymbol{\lambda = -2} \Rightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case (2) : $\boldsymbol{\lambda = 6} \Rightarrow X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X_3$ should be orthogonal with $X_1$ and $X_2$.

$$X_1^T X_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow (a-c) = 0$$

$$X_2^T X_3 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow (a+c) = 0$$

Solving the above equations, we get a = c =0 and b can take any value.

Choose b = 1 we get $X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ Clearly the eigen vectors are pair wise orthogonal.

The Normalised modal matrix are $N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

$$N^T A N = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D(-2,6,6)$$

**7 Diagonalise the matrix $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by means of an orthogonal transformation.**

**Hints:** The characteristic equation is $|A - \lambda I| = 0$

- ⇨ The characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$.
- ⇨ $\lambda = 1, 4, 4$

Case (1) : $\boldsymbol{\lambda = 1} \Rightarrow X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Case (2) : $\boldsymbol{\lambda = 4} \Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X_3$ should be orthogonal with $X_1$ and $X_2$.

$$X_1^T X_3 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow (-a + b + c) = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow (b - c) = 0$$

Solving the above equations, we get $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Clearly the eigen vectors are pair wise orthogonal.

The Normalised modal matrix are $N = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N^T A N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D(1,4,4)$$

**8 Reduce quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to canonical form through an orthogonal transformation. Also find the index, nature and rank of the quadratic form.**

**Hints:** The symmetric matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

- ⇨ The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.
- ⇨ $(\lambda)(\lambda-3)(\lambda-15) = 0$
- ⇨ $\lambda = 0,3,15$

Case (1) : $\boldsymbol{\lambda = 0}$ $\Rightarrow$ Eigen vector $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Case (2) : $\boldsymbol{\lambda = 3}$ $\Rightarrow$ Eigen vector $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Case (3) : $\boldsymbol{\lambda = 15}$ $\Rightarrow$ Eigen vector $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

It is clear that $X_1^T X_2 = X_1^T X_3 = X_2^T X_3 = 0$

Normalized matrix $N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

$$N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D\,(0,3,15)$$

Canonical Form is $Y^T(N^TAN)Y = Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= 3{y_2}^2 + 15{y_3}^2$$

Rank (r) : 2 ( No of non zero rows )

Index (p) : 2 ( No of Positive terms )

Signature ( s) = 2p – r = 2(2) – 2= 2

Nature : Positive semi definite

**9 Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to canonical Form through an orthogonal transformation and hence show that it is positive semi definite. Also give a non-zero set of values $(x_1, x_2, x_3)$ which makes the quadratic form zero.**

**Hints:** The symmetric matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

- ⇨ The characteristic Equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$.
- ⇨ $(\lambda)(\lambda-1)(\lambda-3) = 0$
- ⇨ $\lambda = 0, 1, 3$

Consider $\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ -----(1)

**Case (1) : λ = 0** ⇒ Eigen vector is $X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

**Case (2)** : **λ = 1** ⇒ Eigen vector is $X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

**Case (3)** : **$\lambda$ = 3** $\Rightarrow$ Eigen vector is $X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

It is clear that $X_1^T X_2 = X_1^T X_3 = X_2^T X_3 = 0$

$$\text{Normalized modal matrix } N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D\,(0,1,3)$$

Canonical Form is $Y^T(N^T AN)Y = Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= y_2^2 + 3y_3^2$$

The orthogonal transformation is

$$X = NY$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3 \; , \; x_1 = \frac{1}{\sqrt{3}} y_1 + 0 y_2 + \frac{2}{\sqrt{6}} y_3 \; , \; x_1 = \frac{-1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

The canonical form contains only two terms both are positive, it is positive semi definite.
Quadratic form becomes zero, when $y_2 = 0, y_3 = 0$ & $y_1$ is arbitrary.
Taking $y_1 = \sqrt{3}, y_2 = 0$ & $y_3 = 0$ we get $x_1 = 1, x_2 = 1$ & $x_3 = -1$
For these values of $x_1, x_2$ & $x_3$ the required quadratic form is zero.

**10 Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by an orthogonal reduction. Hence find its rank and nature.**

**Hints:** The symmetric matrix $A=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The characteristic equation is $|A-\lambda I|=0$

- ⇨ The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.
- ⇨ $\lambda = 8, 2, 2$

Case (1) : $\boldsymbol{\lambda = 8}$ $\Rightarrow$ Eigen vector $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Case (2) : $\boldsymbol{\lambda = 2}$ $\Rightarrow$ Eigen vector $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Case (3) : $\boldsymbol{\lambda = 2}$

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X_3$ should be orthogonal with $X_1$ and $X_2$

$X_1^T X_3 = X_2^T X_3 = 0$

$\Rightarrow$ Eigen vector $X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$

Normalized modal matrix $N=\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$

$$N^T A N = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D\,(8, 2, 2)$$

Canonical Form is $Y^T(N^TAN)Y = Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Rank (r) : 3 ( No of non zero rows )

Nature: Positive definite.

## UNIT – 2 SEQUENCES AND SERIES

## PART – A

**1. Define Bounded below Sequence**

**Ans:** A Sequence $\{a_n\}$ is said to be bounded below if there exists ‘ m ’ in R such that $m \le a_n$ for every n. Here ‘m ’ is called lower bound to the sequence $\{a_n\}$

**2. Define Bounded above sequence**

**Ans:** A Sequence $\{a_n\}$ is said to be bounded above if there exists ‘ M ’ in R such that $a_n \le M$ for every n. Here ‘M’ is called upper bound to the sequence $\{a_n\}$

**3. Define Bounded Sequence**

**Ans:** A sequence $\{a_n\}$ is said to be bounded when it is bounded below and bounded above

**4. Define Monotonically increasing and Monotonically decreasing sequence with examples**

**Ans:** A sequence $\{ a_n \}$ is said to be monotonically increasing if $a_n \le a_{n+1}$, for every n
A sequence $\{ a_n \}$ is said to be monotonically decreasing if $a_{n+1} \le a_n$, for every n
Eg. $a_n = \{ n \}$ is monotonically increasing
$a_n = \{ -n \}$ is monotonically decreasing

**5. Write any two properties of series.**

**Ans:** (1) The convergence of a series is not affected by the suppression of a finite number of its terms.

(2) If the series $u_1 + u_2 + u_3 \ldots$ converges and its sum S, then the series $c\,u_1 + c\,u_2 + c\,u_3 \ldots$

where $c$ is some fixed number, also converges, and its sum is $c\,S$.

**6. Write the Comparison test for convergence**

**Ans:** (a) If there are two series of positive terms $\sum u_n$ *and* $\sum v_n$ such that

(i) $\sum v_n$ Converges (ii) $u_n \le v_n$ for all values of n . Then $\sum u_n$ also converges.

(b) If there are two series of positive terms $\sum u_n$ *and* $\sum v_n$ such that

(i) $\sum v_n$ Diverges (ii) $v_n \le u_n$ for all values of n . Then $\sum u_n$ also diverges.

**7. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n!}$**

**Ans:** Clearly $2^n < n!$ for all n > 3.

$$\Rightarrow \frac{1}{2^n} > \frac{1}{n!} \Rightarrow \sum \frac{1}{n!} < \sum \frac{1}{2^n} \text{ for all } n > 3$$

But the series $\sum \frac{1}{2^n}$ is a geometric series with common ratio r = $\frac{1}{2}$<1 which is convergent.

Hence $\sum_{n=1}^{\infty}\frac{1}{n!}$ is also convergent by comparison test

**8. Write the Integral test for convergence**

**Ans:** A positive term series f(1) + f(2) + f(3) +……+ f(n) +… where f(n) decreases as n increases converges or diverges according as the integral $\int_1^{\infty} f(x)\,dx$ is finite or infinite.

**9. Test the convergence of $\sum_1^{\infty}\frac{1}{n\log n}$**

**Ans:** $\sum_1^{\infty}\frac{1}{n\log n}$ is a positive term series decreases as n increases after n ≥ 2.

So we can apply integral test.

$$\int_2^{\infty}\frac{1}{x\log x}dx = \int_2^{\infty}\frac{(1/x)}{\log x}dx = \left[\log(\log x)\right]_2^{\infty} = \infty$$

By integral test, the series diverges.

**10. Write D' Alembert's Ratio Test for convergence**

**Ans:** The series of $\sum u_n$ of positive terms is convergent if $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} < 1$ and diverges if

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} > 1$$

**11. Test the convergence of $\sum \frac{n!\,2^n}{n^n}$**

**Ans:** Let $u_n = \frac{n!\,2^n}{n^n}$, $u_{n+1} = \frac{(n+1)!\,2^{(n+1)}}{(n+1)^{(n+1)}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left( \frac{(n+1)!\, 2^{(n+1)}}{(n+1)^{(n+1)}} \right) \left( \frac{n^n}{n!\, 2^n} \right)$$

$$= \lim_{n\to\infty} \left( 2 \left( \frac{n}{n+1} \right)^n \right) = \lim_{n\to\infty} \left( 2 \left( \frac{1}{1+\frac{1}{n}} \right)^n \right) = \frac{2}{e} < 1$$

By Ratio Test, $\sum \frac{n!\, 2^n}{n^n}$ converges.

**12. Show that the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ is a convergent series.**

**Ans:** The terms of the given series are alternately positive and negative.

Clearly 1. $1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \ldots$ (numerically)

2. Here $u_n = \frac{1}{\sqrt{n}}$ ; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Hence by Leibnitz's rule the given series is convergent.

**13. Write the Leibnitz's test for convergence**

**Ans:** An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots..$ converges if

(i) Each term is numerically less than its preceding term (i.e.) $u_1 > u_2 > u_3 \ldots.$

(ii) $\lim_{n\to\infty} u_n = 0$

**14. Define Absolute convergence**

**Ans:** If the series of arbitrary terms $u_1 + u_2 + u_3 + \ldots.. + u_n + \ldots.$ be such that the series $|u_1| + |u_2| + |u_3| + \ldots. + |u_n| + \ldots$ is convergent, then the series $\sum u_n$ is said to be absolute convergent

**15. Define Conditional convergence**

**Ans:** If $\sum u_n$ is convergent and $\sum |u_n|$ is divergent, then $\sum u_n$ is said to be conditionally convergent.

**16. Test the convergence of the series 5 – 4 – 1+ 5 – 4 –1 +5 – 4 – 1+…..**

**Ans:** For the given series $S_n$ = 5 – 4 –1 + 5 –4 –1 +5 –4 –1+….. to n terms

= 0, 5, 1 according to the number of terms are 3n,3n+1,3n+2

$\lim_{n\to\infty} S_n =$ does not tend to a unique limit

Clearly the series oscillates finitely

**17. Test the convergence of the series $\frac{5}{2}-\frac{7}{4}+\frac{9}{6}-\frac{11}{8}+......$**

**Ans:** Clearly the series is an alternating series**.** It is of the form $\sum(-1)^n u_n$ where

$$u_n=\frac{2n+3}{2n}$$

⇨ $u_{n+1}=\frac{2n+5}{2n+2}$ and $u_n-u_{n+1}=\frac{2n+3}{2n}-\frac{2n+5}{2n+2}=\frac{6}{2n(2n+2)}>0,\forall n$

⇨ $\lim\limits_{n\to\infty}u_n=1\neq 0$

By Leibnitz Rule , The given series oscillates

**18. Test the convergence of the series $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{7.8}-......$**

**Ans:** Clearly the series is an alternating series**.** It is of the form $\sum(-1)^{n-1}u_n$ where

$$u_n=\frac{1}{(2n-1)(2n)}$$

⇨ $u_{n+1}=\frac{1}{(2n+1)(2n+2)}$ and $u_n>u_{n+1},\forall n$

⇨ $\lim\limits_{n\to\infty}u_n=0$

By Leibnitz Rule , The given series is converges

**19. Prove that $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-......$ converges absolutely.**

**Ans:** The series $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-......$ is an alternating series

It is of the form $\sum(-1)^{n-1}u_n$ where $u_n=\frac{\sin nx}{n^3}$

clearly $|u_n|=\left|\frac{\sin nx}{n^3}\right|\leq\frac{1}{n^3}$ $\forall n.$

Clearly $\sum\frac{1}{n^3}$ is convergent. By comparison test $\sum|u_n|$ also converges.

⇨ Given series is absolutely convergent.

**20. Test the convergence of the series $\log 2+\log\left(\frac{3}{2}\right)+\log\left(\frac{4}{3}\right)+\log\left(\frac{5}{4}\right)+......$**

**Ans:** For the given series $S_n=\log 2+\log\left(\frac{3}{2}\right)+\log\left(\frac{4}{3}\right)+\log\left(\frac{5}{4}\right)+...+\log\left(\frac{n+1}{n}\right)$

$$=\log\left[2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right).......\left(\frac{n+1}{n}\right)\right]=\log(n+1)$$

$\lim_{n\to\infty} S_n = \infty$

⇨ The Series is divergent.

**PART B**

**1(a) Discuss the convergence of the series** $\sum \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$

**Hints:** We have $u_n = \sum \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$

Choose $v_n = \frac{1}{n^2}$

As $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2} \times n^2$

$= \frac{\sqrt{2}}{4}$ (a finite quantity)

So both $\sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n$ converges so $\sum u_n$ also converges.

**(b) Show that the series** $e^{-1} + 2e^{-2} + 3e^{-3} + ... + ne^{-n} + ...$**converges.**

**Hints:** Here $f(x) = xe^{-x}$

Then $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} ne^{-n}$

f(x) > 0 and is decreasing in $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-x}\,dx = 2e^{-1} = \frac{2}{e}\text{(finite)}$$

$\int_1^{\infty} f(x)\,dx$ converges.

$\therefore$ By Integral test, $\sum_1^{\infty} f(n)$ also converges.

**2(a) Show that the harmonic series** $\sum \frac{1}{n^p}$ **converges if p >1 and diverges if p** $\le 1$

**Hints: Case (i) when p > 1**

$$\frac{1}{1^p} = 1,\ \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}} \qquad \because \frac{1}{3^p} < \frac{1}{2^p}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{\left(2^{p-1}\right)^2}$$

$$\because \frac{1}{5^p} < \frac{1}{4^p}, \frac{1}{6^p} < \frac{1}{4^p} \ etc.$$

Similarly, the sum of next eight terms

$$\frac{1}{8^p} + \frac{1}{9^p} + \ldots.. \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \ldots \frac{1}{8^p} = \frac{1}{8^{p-1}} = \frac{1}{\left(2^{p-1}\right)^3}$$

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots.. + \frac{1}{n^p} + \ldots$$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p}\right) + \ldots\ldots\ldots \quad \text{------(1)}$$

Each term of (1) after the first is less than the corresponding term in

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \ldots\ldots\ldots \quad \text{-------(2)}$$

But (2) is a G.P. whose common ratio $= \frac{1}{2^{p-1}}$ =1

$\therefore$ (2) is convergent $\Rightarrow$ (1) is convergent

**Case (ii) when p= 1**

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2}, \quad \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad , \quad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \ldots. \quad \text{--------------} \quad (1)$$

Each term of (1) after the second is greater than the corresponding term is

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ldots. \quad \text{---------------} \quad (2)$$

But after the second term (2) is a G.P whose common ratio = 1

$\therefore$ (2) is divergent $\Rightarrow$ (1) is divergent

**Case (iii) When p < 1**

$$p < 1 \Rightarrow n^p < n \Rightarrow \frac{1}{n^p} > \frac{1}{n} \quad \forall n$$

But the series $\sum \frac{1}{n}$ is divergent ( by case (ii) ). Hence $\sum \frac{1}{n^p}$ is also divergent

**(b) Using D'Alembert's ratio test show that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.**

**Hints:**

$$u_n = \frac{n^2}{2^n}$$

$$u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$$

$\therefore$ By D'Alembert's ratio test, $\sum u_n$ is a convergent series.

**3(a)** **Discuss the convergence of the series** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$ **, ( $p > 0$ )**

**Hints:** Here $u_n = \dfrac{1}{n\,(\log n)^p}$ $\quad \therefore\ f(x) = \dfrac{1}{x\,(\log x)^p}$

For $x \geq 2$ , $p > 0$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ By Cauchy's Integral test $\displaystyle\sum_{n=2}^{\infty} u_n$ as $\displaystyle\int_2^{\infty} f(x)\,dx$ is finite or infinite.

**Case (i) when p $\neq 1$**

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} (\log x)^{-p}\,\frac{dx}{x} = \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^{\infty}$$

**Sub case (i)**

**when p > 1 , p – 1 is +ve, so that**

$$\int_2^{\infty} f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^{\infty} = -\frac{1}{p-1}\left[\frac{1}{(\log 2)^{p-1}}\right] = \textit{finite}$$

$$\Rightarrow \int_2^{\infty} f(x)\,dx \text{ is finite} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ converges}$$

**Sub case (ii)**

**when p < 1 , 1– p is +ve , so that**

$$\int_2^{\infty} f(x)\,dx = \frac{1}{1-p}\left[(\log x)^{1-p}\right]_2^{\infty} = \frac{1}{1-p}\left[\infty - (\log 2)^{1-p}\right] = \infty$$

$$\Rightarrow \int_2^{\infty} f(x)\,dx \text{ is infinite} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges}$$

**Case (ii)**

when $p = 1$ , $f(x) = \dfrac{1}{x\,(\log x)}$

$$\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{dx}{x \log x} = \left[\log\,(\log x)\,\right]_2^{\infty} = \infty$$

$$\Rightarrow \int_2^{\infty} f(x)\, dx \quad \text{is infinite} \Rightarrow \sum_{n=2}^{\infty} u_n \ \text{ diverges}$$

Hence $\sum_{n=2}^{\infty} u_n$ converges if p > 1 and diverges $0 < p \le 1$.

**(b) Test the convergence of the series $\sin\pi + \frac{1}{4}\sin\frac{\pi}{2} + \frac{1}{9}\sin\frac{\pi}{3} + \ldots$ using Intergral test**

**Hints:**

Let $f(x) = \frac{1}{x^2}\sin\frac{\pi}{x}$

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{\pi}{n}$$

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^2}\sin\frac{\pi}{x} = \frac{2}{\pi}(\text{finite})$$

$\therefore \int_1^{\infty} f(x)\, dx$ converges.

$\therefore$By Integral test, $\sum_1^{\infty} f(n)$ also converges.

**4(a) Test the convergence of the series $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{16}{17}x^3 + \ldots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots, x > 0$**

**Hints:**

The general term $u_n = \frac{2^n - 2}{2^n + 1}x^{n-1}$, $u_{n+1} = \frac{2^{n+1} - 2}{2^{n+1} + 1}x^n$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\frac{2^{n+1} - 2}{2^{n+1} + 1} . \frac{x^n}{2^n - 2} . \frac{2^n + 1}{x^{n-1}}\right]$$

$$= \lim_{n\to\infty}\left(\left(\frac{2 - \frac{2}{2^n}}{2 + \frac{1}{2^n}}\right)\left(\frac{1 + \frac{1}{2^n}}{1 - \frac{2}{2^n}}\right)(x)\right) = \left(\left(\frac{2-0}{2+0}\right)\left(\frac{1+0}{1-0}\right)(x)\right) = x$$

Thus by Ratio test $\sum u_n$ converges if x<1 and diverges for x >1

But it fails when x=1

So let x=1 then $u_n = \frac{2^n - 2}{2^n + 1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}} = 1 \neq 0$$

$\therefore \sum u_n$ diverges when $x = 1$

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$

**(b) Test for convergence $\sum_{n=1}^{\infty} \sqrt{n^4+1} - n^2$ by Comparison test**

**Hints:**

Here $\sum_{n=1}^{\infty} \sqrt{n^4+1} - n^2 = \frac{1}{\sqrt{n^4+1}+n^2}$

Choose $v_n = \frac{1}{n^2}$

$\sum v_n = \sum \frac{1}{n^2}$ is convergent

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$(finite)

Since $\sum v_n$ is convergent $\sum u_n$ is also convergent.

**5(a) Examine the convergence of the series $\sum_{1}^{\infty} \frac{n}{(n+1)(n+2)} x^n$**

**Hints:** The general term $u_n = \frac{n}{(n+1)(n+2)} x^n$, $u_{n+1} = \frac{n+1}{(n+2)(n+3)} x^{n+1}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left( \frac{n+1}{(n+2)(n+3)} x^{n+1} \right) \left( \frac{(n+1)(n+2)}{n} \frac{1}{x^n} \right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n(n+3)} x = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{3}{n}\right)} = x$$

$\therefore \sum u_n$ is convergent if $x < 1$, if $x > 1$, $\sum u_n$ diverges

If x=1, the series becomes $u_n = \frac{n}{(n+1)(n+2)}$

Choose $v_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left( \frac{n}{(n+1)(n+2)} \right)\left(\frac{n}{1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1 (\text{a finite quantity})$$

Both $\sum u_n$ & $\sum v_n$ behave alike, but $\sum v_n = \sum \frac{1}{n}$ is divergent

$\therefore \sum u_n$ also divergent by comparison test

**(b)** **Test the convergence of the series** $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} \ldots (0 < x < 1).$

**Hints:** The given series is an alternating series.

$$u_n = \frac{x^n}{1+x^n}$$

$$\therefore u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})}$$

when x is +ve and less than 1, $1-x<1$ and also $u_{n+1} < u_n$

$\therefore \lim_{n\to\infty} u_n = 0$

$\therefore$ By Leibnitz's test, The series is convergent.

**6(a)** **Test the convergence of the series** $\sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{n}+\sqrt{n+1}} \right]$

**Hints:** We have $u_n = \left[ \frac{1}{\sqrt{n}+\sqrt{n+1}} \right]$

$$= \frac{1}{n+1-n}\left[\sqrt{n+1}-\sqrt{n}\right] = \sqrt{n+1}-\sqrt{n}$$

$$= \sqrt{n}\left[\left(1+\frac{1}{n}\right)^{1/2} - 1\right] = \sqrt{n}\left[1+\frac{1}{2n}-\frac{1}{8n^2}+\ldots-1\right]$$

$$= \frac{1}{\sqrt{n}}\left[\frac{1}{2}-\frac{1}{8n}+\ldots\right]$$

Choose $v_n = \frac{1}{\sqrt{n}}$

As $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{8n} + \dots\right) = \frac{1}{2}$ (a finite quantity)

So both $\sum u_n$ *and* $\sum v_n$ converge or diverge together but $\sum v_n$ diverges *so* $\sum u_n$ also diverges.

**(b) Discuss the convergence and divergence of the following series:**

$$\mathbf{\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \dots}$$

**Hints:** The given series is an alternating series.

$$u_n = \frac{1+2+3+\dots+n}{(n+1)^n} = \frac{n}{2(n+1)^2}$$

$$u_n - u_{n+1} = \frac{-n^2+n+1}{2n^2(n+1)^2} < 0 \quad \text{for } n > 1$$

Also $\lim_{n\to\infty} u_n = 0$

$\therefore$ By Leibnitz's test, The series is convergent.

**7(a) Examine the series $1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots$ for absolute convergence.**

**Hints:** We have $\sum u_n = \sum (-1)^n \frac{x^{2n}}{2^2 4^2 6^2 \dots (2n)^2}$

$$|u_n| = \frac{|x|^{2n}}{2^2 4^2 6^2 \dots (2n)^2} \; ; \; |u_{n+1}| = \frac{|x|^{2n+2}}{2^2 4^2 6^2 \dots (2n)^2 (2n+2)^2}$$

$$\frac{|u_{n+1}|}{|u_n|} = \frac{|x|^{2n+2}}{2^2 4^2 6^2 \dots (2n)^2 (2n+2)^2} \frac{2^2 4^2 6^2 \dots (2n)^2}{|x|^{2n}} = \frac{|x|^2}{(2n+2)^2}$$

$$\lim_{n®¥} \frac{|u_{n+1}|}{|u_n|} = \frac{|x|^2}{(2n+2)^2} = 0 < 1$$

Hence by Ratio test $\sum |u_n|$ is convergent for all values of x.

i.e. $\sum u_n$ is convergent absolutely for all values of x.

**(b) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.3.5\dots(2n-1)}{2.4.6\dots2n} . x^{n-1}$, x > 0 by Ratio test.**

**Hints:**

Here $u_n = \frac{1.3.5...(2n-1)}{2.4.6...2n} x^{n-1}$

$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\frac{1}{x}$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$

If $\frac{1}{x} > 1$ i.e., if $x < 1$ then by ratio test, $\sum u_n$ converges.

If $\frac{1}{x} < 1$ i.e., if $x > 1$ then by ratio test, $\sum u_n$ diverges.

If $x = 1$, the ratio test fails.

when $x = 1$, we have $u_n = \frac{1}{2} < 1$

$\therefore \lim_{n\to\infty} u_n \neq 0$

$\therefore$ The series is divergent.

**8(a) Test the convergence of the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ...$ by Comparison test**

**Hints:** $u_n = \frac{1}{n(n+1)(n+2)}$ Take $v_n = \frac{1}{n^3}$

We have $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$, finite & non zero

$\therefore \sum u_n$ & $\sum v_n$ converges or diverges together.

But the series $\sum v_n = \sum \frac{1}{n^3}$ is convergent.

so $\sum u_n$ also convergent.

**(b) Test for the convergence of the series $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + ...$ by D' Alembert's ratio test**

**Hints:** Here $u_n = \frac{x^n}{n(n+1)}$

$\frac{u_n}{u_{n+1}} = \frac{n}{n+2} x$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = x$

If $x > 1, \sum u_n$ diverges.

If $x < 1, \sum u_n$ converges.

If $x = 1$, the ratio test fails.

when $x = 1$, we have $u_n = \dfrac{1}{n(n+1)}$

Take $v_n = \dfrac{1}{n^2}$

$\therefore \lim\limits_{n\to\infty} u_n = 1 \neq 0$

By Comparison test, both $\sum u_n$ and $\sum v_n$ will converge or diverge.

But $\sum v_n = \dfrac{1}{n^2}$ is a convergent series.

$\therefore$ The series $\sum u_n$ is convergent.

**9(a) Test for the convergence, absolute convergence and conditional convergence of the series $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + ...$**

**Hints:**

$u_n = \dfrac{1}{3n-2}$

Clearly (i) $1 > \dfrac{1}{4} > \dfrac{1}{7} > \dfrac{1}{10} > .....$

(ii) $\lim\limits_{n\to\infty} u_n = 0$

$\therefore$ By Leibnitz's test the given series is convergent.

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{3n-2}$$

Let $v_n = \dfrac{1}{n}$. $\therefore \sum v_n$ is divergent.

$\therefore \sum\limits_{n=1}^{\infty} |u_n|$ is also divergent.

$\therefore$ The given series is convergent but $\sum\limits_{n=1}^{\infty} |u_n|$ is divergent.

$\therefore$ The given series converges conditionally.

**(b) Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + ...$ for**

**(i) absolute convergence**

**(ii) conditional convergence**

**Hints:**

The given series is an alternating series with $u_n = \frac{1}{n\sqrt{n}}$

Clearly (i) $1 > \frac{1}{2\sqrt{2}} > \frac{1}{3\sqrt{3}} > \frac{1}{4\sqrt{4}} > ......$ (numerically)

(ii) $\lim_{n\to\infty} u_n = 0$

$\therefore$ By Leibnitz's test the given series is convergent.

Now $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent.

$\therefore$ The given series is absolute convergent.

**10(a) Test for the convergence of** $\sum_{n=1}^{\infty} (-1)^n (n+1) x^n, x < \frac{1}{2}$

**Hints:**

$u_n = (n+1)x^n$

Now $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = |x| < \frac{1}{2}$

By ratio test,

The series $\sum |u_n|$ is convergent, $x < \frac{1}{2}$

$\therefore$ The given series is absolutely convergent and hence convergent.

**(b) Test the convergence of the series** $\sum_{0}^{\infty} ne^{-n^2}$

**Hints:**

Let $\sum_{1}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} f(n)$

$f(x) > 0$ and decreasing in $[1, \infty)$

$\therefore \int_1^{\infty} f(x)dx = \int_1^{\infty} xe^{-x^2} dx = \frac{1}{4e}$

$\int_1^{\infty} f(x)dx$ converges.

Hence $\sum_{n=1}^{\infty} f(n)$ also converges.

## UNIT – III APPLICATIONS OF DIFFERENTIAL CALCULUS

### PART – A

**1. Find the radius of curvature at** $x = \frac{\pi}{2}$ **on the curve** $y = 4\sin x$

**Ans:** $y = 4\sin x$

Differentiating w.r.t $x$,

$y_1 = 4\cos x,\ y_2 = -4\sin x$

At $x = \frac{\pi}{2}$, $y_1 = 0$, and $y_2 = -4$

The radius of curvature is $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+0)^{\frac{3}{2}}}{-4} = -\frac{1}{4} \quad \therefore \quad \rho = \frac{1}{4}$

**2.** **Find the radius of curvature of $y = e^x$ at the point where the curve cuts the $y$ axis.**

**Ans:** The curve $y = e^x$ cuts the $y$ axis at $x = 0$

$y_1 = e^x$, and $y_2 = e^x$

At $x = 0,\ y_1 = 1$ and $y_2 = 1$

The radius of curvature is $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+1)^{\frac{3}{2}}}{1} = 2^{3/2} = 2\sqrt{2}$

**3.** **Find the curvature of the curve $x^2 + y^2 - 2x - 4y - 4 = 0$**

**Ans:** The general circle equation is $x^2 + y^2 + 2gx + 2fy + c = 0$, $r = \sqrt{f^2 + g^2 - c}$

Given curve is a circle $2g = -2,\ 2f = -4,\ c = -4 \quad \therefore \quad g = -1,\ f = -2$

$\therefore$ The radius $r = \sqrt{(-1)^2 + (-2)^2 + 4} = 3$

$\therefore$ The radius of curvature = Radius of the circle = 3

Hence the curvature = $1/r$ = 1/3

**4.** **Find the radius of curvature at any point on $y = c\log\left(\sec\left(\frac{x}{c}\right)\right)$**

**Ans:** $y_1 = c\frac{1}{\sec\left(\frac{x}{c}\right)}\sec\left(\frac{x}{c}\right)\tan\left(\frac{x}{c}\right)\frac{1}{c} = \tan\left(\frac{x}{c}\right)$, $\quad y_2 = \frac{1}{c}\sec^2\left(\frac{x}{c}\right)$

$$\rho = \frac{\left(1+\tan^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\sec^2\left(\frac{x}{c}\right)} = c\sec\left(\frac{x}{c}\right)$$

**5.** **Find the centre of curvature of $y = x^2$ at the origin**

**Ans:** $y = x^2$

$y_1 = 2x,\ \ y_1\big|_{(0,0)} = 0;\qquad y_2 = 2,\ \ y_2\big|_{(0,0)} = 2$

$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right) = 0$; $\quad \bar{y} = y + \frac{1}{y_2}\left(1 + y_1^2\right) = \frac{1}{2}$

$\therefore$ The centre of curvature is $\left(0, \frac{1}{2}\right)$

**6. Define the curvature of a plane curve and what is the curvature of a straight line?**

**Ans:** The rate of bending of a curve is called curvature. Curvature of straight line is zero.

**7. Find the curvature of the curve $2x^2 + 2y^2 + 5x - 2y + 1 = 0$**

**Ans:** Radius of the circle $= \sqrt{f^2 + g^2 - c} = \sqrt{\frac{25}{16} + \frac{1}{4} - \frac{1}{2}} = \frac{\sqrt{21}}{4} =$ radius of curvature

$\therefore \text{Curvature} = \frac{1}{\rho} = \frac{4}{\sqrt{21}}$

**8. Define Evolute.**

**Ans:** The locus of centre of curvature of a curve is called Evolute.

**9. State any two properties of evolute.**

**Ans:**

(i) The normals drawn to a curve become tangents to its evolute.

(ii) The difference between the radii of curvature at two points of a curve is equal to the arc length of the evolute between the two corresponding points.

**10. Find the centre of curvature of the curve $y = x^2$ at the point (1,–1)**

**Ans:** $\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right) = -4$; $\quad \bar{y} = y + \frac{\left(1 + y_1^2\right)}{y_2} = \frac{3}{2}$

$\therefore$ Centre of curvature is $\left(-4, \frac{3}{2}\right)$

**11. Find the evolute of the curve whose centre of curvature of the curve is $\bar{x} = 2a + 3at^2$, $\bar{y} = -2at^3$**

**Ans:** $\left(\frac{\bar{x} - 2a}{3a}\right)^3 = t^6 = \left(\frac{\bar{y}}{2a}\right)^2 \Rightarrow 27a\bar{y}^2 = 4\left(\bar{x} - 2a\right)^3$

$\therefore$ Equation of the evolute is $27ay^2 = 4\left(x - 2a\right)^3$

**12. Find the curvature at (3,–4) for the curve $x^2+y^2 = 25$**

**Ans:** Radius =5, $\therefore$ The curvature at any point is 1/5

**13. Find the envelope of the family of lines $\frac{x}{t} + yt = 2c$ , $t$ being the parameter.**

**Ans:** $\frac{x}{t} + yt = 2c$ --------(1)

Diff. w.r.to '$t$' partially, we get $-\frac{x}{t^2} + y = 0$

$t^2 = \frac{x}{y} \Rightarrow t = \sqrt{\frac{x}{y}}$ --------(2)

Using (2) in (1), we get $xy = c^2$

**14. Find the envelope of $(x–\alpha)^2 + y^2 = 4\alpha$, $\alpha$ being the parameter.**

**Ans:** $(x–\alpha)^2 + y^2 = 4\alpha$ ----------- (1)

Differentiating partially w.r.to $\alpha$, we get

$2(x–\alpha)(–1) = 4 \Rightarrow \alpha = x + 2$

Substituting $\alpha$ in (1), we get

$(–2)^2 + y^2 = 4(x+2) \Rightarrow y^2 = 4(x+1)$ is the required envelope.

**15. Find the envelope of the family of straight lines $x \cos\alpha + y \sin\alpha = a \sec\alpha$, $\alpha$ being the parameter.**

**Ans:** $x \cos\alpha + y \sin\alpha = a \sec\alpha$-----------------(1)

Dividing by, $\cos\alpha$ we get

$x + y \tan\alpha = a( 1+\tan^2\alpha)$

$a(\tan^2\alpha) – y \tan\alpha + a – x = 0$ which is quadratic in $\tan\alpha$

$\therefore$ The envelope is $B^2 – 4AC = 0$

$(– y)^2 – 4a(a – x) = 0 \Rightarrow y^2 = 4a(a – x)$

**16. Find the envelope of the family of lines $(x/a) \cos\theta + (y/b) \sin\theta =1$, $\theta$ being the parameter.**

**Ans:** $(x/a) \cos\theta + (y/b) \sin\theta =1$ --------(1)

Diff. w.r.to '$\theta$' partially, we get

$– (x/a) \sin\theta + (y/b) \cos\theta =0$ --------(2)

$(1)^2 + (2)^2 \Rightarrow \frac{x^2}{a^2} (\cos^2\theta + \sin^2\theta) + \frac{y^2}{b^2} (\sin^2\theta + \cos^2\theta) =1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**17. Find the envelope of the family of straight lines $y = mx + a/m$ , $m$ being the parameter**

**Ans:** Given $y = mx + a/m$--------(1)

$my = m^2x + a$

$m^2x–my + a = 0$ which is quadratic in $m$

$\therefore$ The envelope of (1) is $B^2 – 4AC = 0$ (i.e.) $y^2 = 4ax$

**18.** **Find the radius of curvature of the parabola $y^2 = 4ax$ at $y = 2a$**

**Ans:** $y = 2a \Rightarrow x = a$

$$y_1 = \frac{2a}{y}, \quad y_1\big|_{(a,2a)} = 1 \text{ and } y_2 = \frac{-2ay'}{y^2}, \quad y_2\big|_{(a,2a)} = -\frac{1}{2a}$$

The radius of curvature is $\rho = \dfrac{[1+y_1^2]^{\frac{3}{2}}}{y_2} = \dfrac{[1+1]^{\frac{3}{2}}}{-1/2a} = \dfrac{2^{3/2}}{-1/2a} = -4\sqrt{2}a$

**19.** **For the catenary $y = c\cosh(x/c)$, find the curvature.**

**Ans:** $y_1 = \sinh\left(\dfrac{x}{c}\right), \quad y_2 = \dfrac{1}{c}\cosh\left(\dfrac{x}{c}\right)$

$$\rho = \frac{\left(1+\sinh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = \frac{y^2}{c} \qquad \therefore \frac{1}{\rho} = \frac{c}{y^2}$$

**20.** **Find the envelope of the family of circles $(x-\alpha)^2 + y^2 = r^2$, $\alpha$ being the parameter.**

**Ans:** $(x-\alpha)^2 + y^2 = r^2$--------(1)

Diff. w.r.to 'α' partially, we get $-2(x-\alpha) = 0 \Rightarrow \alpha = x$……….(2)

Substitute (2) in (1) we get $y^2 = r^2 \Rightarrow y = \pm r$ is the required envelope.

## PART – B

**1(a)** **Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 1$ at $\left(\dfrac{1}{4}, \dfrac{1}{4}\right)$**

**Hints:** Given $y = \left(1-\sqrt{x}\right)^2 = 1 + x - 2\sqrt{x}$

$$y_1 = 1 - \frac{1}{\sqrt{x}} \quad y_2 = \frac{1}{2x^{3/2}} \quad . \text{ At } \left(\frac{1}{4}, \frac{1}{4}\right),\ y_1 = -1, \quad y_2 = 4$$

$$\therefore \qquad \rho = \frac{\left(1+y_1{}^2\right)^{\frac{3}{2}}}{y_2} = \frac{1}{\sqrt{2}}$$

**(b)** **Find the radius of curvature of the curve $x^3 + y^3 = 3axy$ at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$**

**Hints:** *At* $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$, $y_1 = -1$; $y_2 = -\dfrac{32}{3a}$ ; $\rho = \dfrac{\left[1+y_1^2\right]^{3/2}}{y_2} = \dfrac{-3\sqrt{2}\,a}{16}$ $\quad \therefore |\rho| = \dfrac{3\sqrt{2}\,a}{16}$

**2(a)** **Find the radius of curvature at the point 't' of the curve**
**$x = a(\cos t + t\sin t);\ y = a(\sin t - t\cos t)$**

**Hints:** $y_1 = \tan t;\ y_2 = \dfrac{1}{at\cos^3 t};\quad \rho = \dfrac{\left[1+y_1^2\right]^{3/2}}{y_2} = at$

**(b) Find the radius of curvature at any point $\theta$ of the cycloid $x = a(\theta+\sin\theta);\ y = a(1-\cos\theta)$**

**Hints:** $\dfrac{dy}{dx} = \tan\left(\dfrac{\theta}{2}\right);\ \dfrac{d^2y}{dx^2} = \dfrac{1}{4a\cos^4\left(\dfrac{\theta}{2}\right)};\quad \rho = \dfrac{\left[1+y_1^2\right]^{3/2}}{y_2} = 4a\cos\left(\dfrac{\theta}{2}\right)$

**3(a) If $\rho$ is the radius of curvature at any point $(x, y)$ on the curve $y = \dfrac{ax}{a+x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$**

**Hints:** $\dfrac{dy}{dx} = \dfrac{a^2}{(a+x)^2};\ \dfrac{d^2y}{dx^2} = \dfrac{-2y^3}{ax^3};\quad \therefore\quad \rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} \Rightarrow \left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$

**(b) Find the points on the curve $y^2 = 4x$ at which the radius of curvature is $4\sqrt{2}$**

**Hints:** $\dfrac{dy}{dx} = \dfrac{2}{y};\quad \dfrac{d^2y}{dx^2} = \dfrac{-4}{y^3};\quad \rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \dfrac{\left(y^2+4\right)^{3/2}}{4}.$

Given $\rho = 4\sqrt{2} \Rightarrow \dfrac{\left(y^2+4\right)^{3/2}}{4} = 4\sqrt{2} \quad \Rightarrow y = \pm 2$ and $x = 1$ $\therefore$ The points are $(1, \pm 2)$

**4(a) Find the equation of the circle of curvature of the curve at $(c, c)$ on $xy = c^2$**

**Hints:** $\left(\dfrac{dy}{dx}\right)_{(c,c)} = -1;\quad \left(\dfrac{d^2y}{dx^2}\right)_{(c,c)} = \dfrac{2}{c};\quad \rho = \dfrac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \sqrt{2}\,c;$

$\bar{x} = x - \dfrac{y_1}{y_2}\left(1+y_1{}^2\right) = 2c;\quad \bar{y} = y + \dfrac{\left(1+y_1{}^2\right)}{y_2} = 2c$

The circle of curvature is $\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2 \quad \Rightarrow \quad (x-2c)^2 + (y-2c)^2 = 2c^2$

**(b) Find the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a/4, a/4)$.**

**Hints:** $\left(\frac{dy}{dx}\right)_{(a/4.a/4)} = -1;\ \left(\frac{d^2y}{dx^2}\right)_{(a/4,a/4)} = \frac{4}{a};\quad \rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{a}{\sqrt{2}}$

$\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1{}^2\right) = \frac{3a}{4};\quad \bar{y} = y + \frac{\left(1+y_1{}^2\right)}{y_2} = \frac{3a}{4}$

The circle of curvature is $\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2 \Rightarrow \left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2$

**5(a) Find the equation of circle of curvature of $y^2 = 12x$ at $(3,6)$**

**Hints:** $\left(\frac{dy}{dx}\right)_{(3,6)} = 1;\ \left(\frac{d^2y}{dx^2}\right)_{(3,6)} = -\frac{1}{6};\quad \rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = -12\sqrt{2} \quad \therefore\ |\rho| = 12\sqrt{2}$

$\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1{}^2\right) = 15;\quad \bar{y} = y + \frac{\left(1+y_1{}^2\right)}{y_2} = -6$

The circle of curvature is $\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2 \Rightarrow (x-15)^2 + (y+6)^2 = \left(12\sqrt{2}\right)^2$

**(b) Find the equation of the circle of curvature of the rectangular hyperbola $xy = 12$ at the point $(3,4)$**

**Hints:** $\left(\frac{dy}{dx}\right)_{(3,4)} = \frac{-4}{3};\ \left(\frac{d^2y}{dx^2}\right)_{(3,4)} = \frac{8}{9};\quad \rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{125}{24}$

$\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1{}^2\right) = \frac{43}{6};\quad \bar{y} = y + \frac{\left(1+y_1{}^2\right)}{y_2} = \frac{57}{8}$

The circle of curvature is $\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2 \Rightarrow \left(x-\frac{43}{6}\right)^2 + \left(y-\frac{57}{8}\right)^2 = \left(\frac{125}{24}\right)^2$

**6(a) Find the equation of the evolute of the parabola $y^2 = 4ax$**

**Hints:** The parametric form of the parabola is $x = at^2,\ y = 2at$

$$y_1 = \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t};\quad y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx} = -\frac{1}{t^2}\,\frac{1}{2at} = -\frac{1}{2at^3}$$

$$X = x - \frac{y_1}{y_2}\left(1 + y_1^{\,2}\right) = 3at^2 + 2a \text{------ } (1)$$

$$Y = y + \frac{\left(1 + y_1^{\,2}\right)}{y_2} = -2at^3 \text{------}(2)$$

Now we have to eliminate $t$ between (1) and (2), we get

$$27aY^2 = 4\left(X - 2a\right)^3$$

Changing X and Y to x and y, the locus of (X,Y) becomes $27ay^2 = 4\left(x - 2a\right)^3$

**(b) Find the equation of the evolute of the curve** $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

**Hints:** The parametric equations of the ellipse are

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{-b}{a^2}\operatorname{cosec}^3\theta$$

$$X = x - \frac{y_1\left(1 + y_1^{\,2}\right)}{y_2} = \left(\frac{a^2 - b^2}{a}\right)\cos^3\theta \text{-----------}(1)$$

$$Y = y + \frac{1 + y_1^{\,2}}{y_2} = \left(\frac{b^2 - a^2}{b}\right)\sin^3\theta \text{-----------}(2)$$

To find the equation of the evolute we have to eliminate $\theta$ between (1) and (2), we have

$$\left(aX\right)^{2/3} + \left(bY\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}$$

$\therefore$ The locus of $(X,Y)$ is $\left(ax\right)^{2/3} + \left(by\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}$ which is the equation of the evolute of the given ellipse.

**7(a) Show that the evolute of the cycloid** $\boldsymbol{x = a\left(\theta - \sin\theta\right), y = a\left(1 - \cos\theta\right)}$ **is another cycloid**

Hints: $y_1 = \cot\left(\frac{\theta}{2}\right)$ $\qquad y_2 = -\frac{1}{4\sin^4\left(\frac{\theta}{2}\right)}$

$$X = x - \frac{y_1\left(1+y_1^2\right)}{y_2} = a\left(\theta+\sin\theta\right)\text{-----------}(1)$$

$$Y = y + \frac{1+y_1^2}{y_2} = -a\left(1-\cos\theta\right)\text{-----------}(2)$$

The locus of (X,Y) is $x = a\left(\theta+\sin\theta\right), y = -a\left(1-\cos\theta\right)$ which represents another cycloid.

**(b) Find the evolute of the rectangular hyperbola $xy = c^2$**

**Hints:** The parametric equation for $xy = c^2$ are $x = ct, \quad y = \frac{c}{t}$

$$y_1 = \frac{-1}{t^2}; \quad y_2 = \frac{2}{ct^3}$$

$$X = x - \frac{y_1}{y_2}\left(1+y_1^2\right) = \frac{3ct}{2} + \frac{c}{2t^3}\text{--------}(1)$$

$$Y = y + \frac{1+y_1^2}{y_2} = \frac{3}{2}\frac{c}{t} + \frac{ct^3}{2}\text{----------}(2)$$

Now we have to eliminate *t* between (1) and (2), we get

$$\left(X+Y\right)^{2/3} - \left(X-Y\right)^{2/3} = \left(4c\right)^{2/3}$$

Changing X and Y to x and y, the locus of (X,Y) becomes

$$\left(x+y\right)^{2/3} - \left(x-y\right)^{2/3} = \left(4c\right)^{2/3}$$

**8(a) Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ where the parameters *a* and *b* are related by $ab = c^2$, where *c* is constant.**

**Hints:** Given $b = \frac{c^2}{a}$, hence the straight line becomes $a^2y - ac^2 + c^2x = 0$ which is a quadratic in '*a*'. Hence the envelope is $B^2 - 4AC = 0 \Rightarrow 4xy = c^2$

**(b) Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$, where *a* and *b* are parameters that are connected by $a^2+b^2=c^2$, *c* being a constant**

Hints: Given $\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow \quad \frac{da}{db} = \frac{-a^2y}{b^2x}$ ------(1)

and $a^2+b^2=c^2 \Rightarrow \frac{da}{db} = -\frac{b}{a}$ ------(2)

From (1) and (2) $\frac{-a^2y}{b^2x} = = -\frac{b}{a} \quad \Rightarrow \; a = \left(xc^2\right)^{1/3}, \; b = \left(yc^2\right)^{1/3}$

Substituting in $a^2 + b^2 = c^2$ we get $x^{2/3} + y^{2/3} = c^{2/3}$

**9(a) Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ given $a^n + b^n = c^n$, where $c$ is a known constant**

**Hints:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{da}{db} = \frac{-a^3y^2}{b^3x^2}$ -----(1)

$a^n + b^n = c^n \Rightarrow \frac{da}{db} = \frac{-b^{n-1}}{a^{n-1}}$ --------(2)

From (1) and (2) $\frac{-a^3y^2}{b^3x^2} = \frac{-b^{n-1}}{a^{n-1}} \Rightarrow a^n = \left(c^n x^2\right)^{\frac{n}{n+2}}; \; b^n = \left(c^n y^2\right)^{\frac{n}{n+2}}$

Substituting in $a^n + b^n = c^n$ we get $x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = c^{\frac{2n}{n+2}}$

**(b) Find the envelope of the system of lines $\frac{x}{l} + \frac{y}{m} = 1$, where $l$ and $m$ are connected by the relation $\frac{l}{a} + \frac{m}{b} = 1$, $l$ and $m$ are the parameters.**

**Hints:** $\frac{x}{l} + \frac{y}{m} = 1$-----(1); $\frac{l}{a} + \frac{m}{b} = 1$,------(2)

Differentiating (1) and (2) w.r.t. t, we get

$-\frac{x}{l^2}\frac{dl}{dt} - \frac{y}{m^2}\frac{dm}{dt} = 0$ ------(3)

$\frac{1}{a}\frac{dl}{dt} + \frac{1}{b}\frac{dm}{dt} = 0$ ------(4)

From (3) and (4) we have

$\frac{x}{l^2/a} = \frac{y}{m^2/b} \Rightarrow l = \sqrt{ax}, \; m = \sqrt{by}$ -----(5)

Using (5) in (1) we get $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$

**10(a) Find the evolute of $y^2 = 4ax$ considering it as the envelope of its normals.**

**Hints:** Given $y^2 = 4ax$

The parametric form is $x = at^2, y = 2at$

$m = \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

We know that the equation of the normal is

$$y - y_1 = \frac{-1}{m}(x - x_1) \quad \Rightarrow \quad y = at^3 - tx + 2at \text{------}(1)$$

Now to find envelope of (1)

Diff. (1), p.w.r. to '$t$', we get $t = \left(\frac{x - 2a}{3a}\right)^{1/2}$

$$(1) \Rightarrow \quad y = a\left(\frac{x-2a}{3a}\right)^{3/2} - \left(\frac{x-2a}{3a}\right)^{1/2} x + 2a\left(\frac{x-2a}{3a}\right)^{1/2}$$

$$\Rightarrow \quad 27ay^2 = 4(x-2a)^3$$

**(b)** **Find the evolute of $x^{2/3} + y^{2/3} = a^{2/3}$ considering it as the envelope of its normals.**

**Hints:** The parametric equation is $x = a\cos^3\theta,\ y = a\sin^3\theta$

$$m = \frac{dy}{dx} = -\tan\theta$$

We know that the equation of the normal is

$$y - y_1 = \frac{-1}{m}(x - x_1) \quad \Rightarrow \quad y\sin\theta - x\cos\theta = -a\cos 2\theta \text{------}(1)$$

Now to find envelope of (1)

Diff. (1), p.w.r. to '$\theta$', we get $y\cos\theta + x\sin\theta = 2a\sin 2\theta$ -----(2)

From (1) and (2), $x = a\cos^3\theta + 3a\cos\theta\sin^2\theta$ --------(3)

$y = a\sin^3\theta + 3a\cos^2\theta\sin\theta$ ----------(4)

Eliminate $\theta$ from (3) and (4) we get $(x+y)^{2/3} + (x-y)^{2/3} = a^{2/3}$ which is the required equation.

## UNIT – IV DIFFERENTIAL CALCULUS OF SEVERAL VARIABLES

### PART – A

**1.** **If $u = \frac{y}{z} + \frac{z}{x}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$**

**Ans:** Given $u = \frac{y}{z} + \frac{z}{x}$, $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x} = 0$

[Note: $\frac{\partial u}{\partial x} = -\frac{z}{x^2}; \frac{\partial u}{\partial y} = \frac{1}{z}; \frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$]

**2.** **If $u = f\left(x-y, y-z, z-x\right)$ find $u_x + u_y + u_z$**

**Ans:** Let $x_1 = x-y,\ x_2 = y-z,\ x_3 = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial x} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3}.$$

Similarly, $\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}$ and $\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}$

$\therefore u_x + u_y + u_z = 0$

**3.** **Find $\frac{du}{dt}$ when $u = \sin\left(x/y\right)$, $x = e^t$, $y = t^2$**

**Ans:** $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$\frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right).e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t$$

**4.** **If $x = r\cos\theta,\ y = r\sin\theta$ find $\frac{\partial(r,\theta)}{\partial(x,y)}$**

**Ans:** $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}} = \frac{1}{r}$$

**5.** **If $x^y + y^x = c$, find $\frac{dy}{dx}$**

**Ans:** Let $f\left(x,y\right) = x^y + y^x - c$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{yx^{y-1} + y^x\log y}{x^y\log x + xy^{x-1}}$$

**6.** **Show that $f\left(z\right) = \frac{x^2 y}{x^4 + y^2}$, $z \neq 0$ and $f\left(0\right) = 0$ is discontinuous at $z = 0$**

**Ans:** Consider the radius $y = mx$

$$\lim_{z\to 0} f(z) = \lim_{\substack{y=mx \\ x\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0}\frac{x^2\left(mx\right)}{x^4 + x^2 m^2} = \lim_{x\to 0}\frac{mx}{x^2 + m^2} = 0$$

Now consider a curve, $y = x^2$ let us take the limit by approaching 0 along this curve

$$\lim_{z\to 0} f(z) = \lim_{\substack{y=x^2 \\ x\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^2\left(x^2\right)}{x^4 + x^4} = \frac{1}{2}$$

Since $\lim_{z\to 0} f(z)$ does not have a unique value, $\lim_{z\to 0} f(z)$ does not exist and hence $f(z)$ is discontinuous at $z = 0$

**7. State any two properties of Jacobian.**

**Ans:**

(i) If $u$ and $v$ are functions of $r$ and $s$ , $r$ and $s$ are functions of $x$ and $y$ then,

$$\frac{\partial(u,v)}{\partial(r,s)} \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

(ii) If $u$ and $v$ are functions of $x$ and $y$ then, $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = 1$ (i.e) $JJ' = 1$

**8. If $u = xy$ and $v = x + y$ find $\frac{\partial(x,y)}{\partial(u,v)}$**

**Ans:**

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix} = y - x$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{y - x}$$

**9. If $x = u(1+v)$ and $y = v(1+u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$**

**Ans:** $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$

**10. Prove that $JJ' = 1$.**

**Ans:** $JJ' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 1$

**11. Find the Jacobian of the transformation $x = r\cos\theta,\ y = r\sin\theta$**

**Ans:** $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

**12. If $u=\dfrac{y^2}{x}$, $v=\dfrac{x^2}{y}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$**

**Ans:** $$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{-y^2}{x^2} & \frac{2y}{x}\\ \frac{2x}{y} & \frac{-x^2}{y^2}\end{vmatrix}=-3$$

**13. If $x+y+z=u$, $y+z=uv$, $z=uvw$ find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$**

**Ans:** $y=uv-z=uv(1-w)$; $x=u-uv=u(1-v)$; $z=uvw$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}1-v & -u & 0\\ v(1-w) & u(1-w) & -uv\\ vw & uw & uv\end{vmatrix}=u^2v$$

**14. Find Taylor's series expansion of $x^y$ near the point (1,1) up to first degree terms.**

**Ans:** $f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]$

$f(x,y)=x^y$, $\quad f(1,1)=1$

$f_x(x,y)=yx^{y-1}$, $\quad f_x(1,1)=1$

$f_y(x,y)=x^y\log x$, $\quad f_y(1,1)=0$

$f(x,y)=1+[(x-1)(1)+(y-1)(0)]=x.$

**15. Find Taylor's series expansion of $e^x\sin y$ near the point $\left(-1,\dfrac{\pi}{4}\right)$ up to first degree terms.**

**Ans:** $f(x,y)=e^x\sin y$, $\quad f_x(x,y)=e^x\sin y$, $\quad f_y(x,y)=e^x\cos y$

$f(-1,\frac{\pi}{4})=\frac{1}{e\sqrt{2}}$, $\quad f_x(-1,\frac{\pi}{4})=\frac{1}{e\sqrt{2}}$, $\quad f_y(-1,\frac{\pi}{4})=\frac{1}{e\sqrt{2}}$ .

$\therefore\ f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]=\dfrac{1}{e\sqrt{2}}\left[1+(x+1)+\left(y-\dfrac{\pi}{4}\right)\right]$

**16. Expand $e^x\cos y$ in Taylor's series in powers of $x$ and $y$ up to terms of first degree.**

**Ans:** $f(x,y)=e^x\cos y$, $\quad f(0,0)=1$

$f_x(x,y)=e^x\cos y$, $\quad f_x(0,0)=1$

$f_y(x,y)=-e^x\sin y$, $\quad f_y(0,0)=0.$

$\therefore f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]=1+x$

**17. Write the sufficient conditions for $f(x,y)$ to have a maximum value at $(a,b)$**

**Ans:** If $f_x(a,b)=0, f_y(a,b)=0$ and $f_{xx}(a,b)=A,\ f_{xy}(a,b)=B,\ f_{yy}(a,b)=C$ then

$f(x,y)$ is maximum value at $(a,b)$ if $AC-B^2>0$ and $A<0$

**18. Find the maxima and minima of $f(x,y) = 3x^2 + y^2 + 12x + 36$**

**Ans:** $f_x = 6x + 12 = 0 \Rightarrow x = -2$; $f_y = 2y = 0 \Rightarrow y = 0$.

The stationary point is $(-2,0)$.

$A = f_{xx} = 6$, $B = f_{xy} = 0$, $C = f_{yy} = 2$, $AC - B^2 = 12 > 0$ and $A > 0$.

$\therefore$ $f$ is minimum at $(-2,0)$ and the minimum value is $f(-2,0) = 24$.

**19. Find the possible extreme point of $f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$**

**Ans:** $f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$

$$\frac{\partial f}{\partial x} = 2x - \frac{2}{x^2};\quad \frac{\partial f}{\partial x} = 0 \Rightarrow x = 1$$
$$\frac{\partial f}{\partial y} = 2y - \frac{2}{y^2};\quad \frac{\partial f}{\partial y} = 0 \Rightarrow y = 1$$

$\therefore$ The possible extreme point is $(1,1)$.

**20. Find the stationary points of $f(x,y) = x^2 - xy + y^2 - 2x + y$**

**Ans:** $f_x = 2x - y - 2 = 0$, $f_y = -x + 2y + 1 = 0$.

$2x - y = 2, -2x + 4y = -2 \Rightarrow y = 0, x = 1.$

$\therefore$ The stationary point is $(1,0)$.

## PART – B

**1(a) If $u$ is a homogeneous function of degree $n$ in $x$ and $y$, show that**

**(i)** $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

**(ii) Given** $u(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

**Find the value of** $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

**Hints:** (i) By Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ ------(1)

Differentiating (1) partially w.r.to $x$,we get

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}\text{ -------(2)}$$

Differentiating (1) partially w.r.to $y$,we get

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial y\partial x}=(n-1)\frac{\partial u}{\partial y}\text{ --------(3)}$$

$(2)\times x+(3)\times y\Rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

(ii) $u(x,y)$ is a homogeneous function of degree 2.

Hence by Euler's extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=2(1)u=2u$$

**(b) If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$**

**Hints:** Let $\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\cos u=\frac{x}{\sqrt{x}}\left(\frac{1+y/x}{1+\sqrt{y}/\sqrt{x}}\right)=x^{1/2}F(y/x)$$

This is a homogenous function of degree $\frac{1}{2}$

By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nf$

$$x\frac{\partial(\cos u)}{\partial x}+y\frac{\partial(\cos u)}{\partial y}=\frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x}+y(-\sin u)\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

**2(a) If $u=x^2+y^2+z^2$ and $x=e^{2t}$, $y=e^{2t}\cos 3t$, $z=e^{2t}\sin 3t$, find $\frac{du}{dt}$**

**Hints:**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x(2e^{2t}) + 2y(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2z(2e^{2t}\sin 3t + 3\,e^{2t}\cos 3t)$$

$$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t\cos 3t$$

$$= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t) = 8e^{4t}$$

**(b) If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, show that**

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

**Hints:** Here $z = f(x,y)$ and $x = x(r,\theta)$, $y = y(r,\theta)$

$$\therefore \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \text{-----}(1)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$$

$$\therefore \quad \frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \cos\theta\frac{\partial z}{\partial y} \text{--------}(2)$$

$$(1)^2 + (2)^2 \Rightarrow \quad \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

**3(a) If $z = f(x,y)$, where $x = u^2 - v^2$, $y = 2uv$, prove that**

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

**Hints:** Given $z = f(x,y)$, $x = u^2 - v^2$, $y = 2uv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = 2u.\frac{\partial z}{\partial x} + 2v.\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}4u^2 + \frac{\partial^2 z}{\partial y \partial x}8uv + 2\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}4v^2 \text{ -------}(1)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = (-2v).\frac{\partial z}{\partial x} + (-2u).\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}4v^2 - \frac{\partial^2 z}{\partial y \partial x}4uv - 2\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial x \partial y}4uv + \frac{\partial^2 z}{\partial y^2}4u^2 \text{ --------}(2)$$

$(1)+(2) \Rightarrow$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\frac{\partial^2 z}{\partial x^2} + 4(u^2+v^2)\frac{\partial^2 z}{\partial y^2} \quad \Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

**(b) Given the transformation $u = e^x \cos y$, $v = e^x \sin y$ and that $\varphi$ is a function of $u$ and $v$ and also $x$ and $y$, prove that $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right)$**

**Hints: Given** $u = e^x \cos y;\ v = e^x \sin y$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial x}=u\frac{\partial\varphi}{\partial u}+v\frac{\partial\varphi}{\partial v}\text{ ------(1)}$$

$$\frac{\partial^2\varphi}{\partial x^2}=u^2\frac{\partial^2\varphi}{\partial u^2}+uv\frac{\partial^2\varphi}{\partial u\partial v}+uv\frac{\partial^2\varphi}{\partial v\partial u}+v^2\frac{\partial^2\varphi}{\partial v^2}+u\frac{\partial\varphi}{\partial u}+v\frac{\partial\varphi}{\partial v}\text{ ------(1)}$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial y}=-v\frac{\partial\varphi}{\partial u}+u\frac{\partial\varphi}{\partial v}$$

$$\frac{\partial^2\varphi}{\partial y^2}=v^2\frac{\partial^2\varphi}{\partial u^2}-uv\frac{\partial^2\varphi}{\partial u\partial v}-uv\frac{\partial^2\varphi}{\partial v\partial u}+u^2\frac{\partial^2\varphi}{\partial v^2}-u\frac{\partial\varphi}{\partial u}-v\frac{\partial\varphi}{\partial v}\text{ ------(2)}$$

$$(1)+(2)\Rightarrow \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right)$$

**4(a) Find the Jacobian of $y_1, y_2, y_3$ with respect to $x_1, x_2, x_3$ if $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_1x_3}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$**

**Hints:** $$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}=\begin{vmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3}\\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3}\end{vmatrix}=\begin{vmatrix}\frac{-x_2x_3}{x_1^{\,2}} & \frac{x_3}{x_1} & \frac{x_2}{x_1}\\ \frac{x_3}{x_2} & \frac{-x_3x_1}{x_2^{\,2}} & \frac{x_1}{x_2}\\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^{\,2}}\end{vmatrix}=4$$

**(b) If $u = 2xy$, $v = x^2 - y^2$, and $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.**

**Hints:** $$\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}$$

$$=\begin{vmatrix}2y & 2x\\ 2x & -2y\end{vmatrix}\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=-4\left(x^2+y^2\right)r=-4r^3\quad\left(\because x^2+y^2=r^2\right)$$

**5(a) If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$. Determine the functional relationship between $u, v, w$**

**Hints:**

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (y+z)(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y)$$
$$= 2[y^2 - z^2 - (x^2 - z^2) + x^2 y^2] = 0$$

$\therefore$ $u, v$ *and* $w$ are functionally dependent.

$w^2 = (x+y+z)^2 = (x^2+y^2+z^2) + 2(xy+yz+zx)$

(i.e.,) $w^2 = v + 2u$ is the required relationship.

**(b) Expand the function $\sin(xy)$ at $\left(1, \frac{\pi}{2}\right)$ as a Taylor series.**

**Hints:**

| **Function** | **Values at $\left(1, \frac{\pi}{2}\right)$** |
|---|---|
| $f(x,y) = \sin xy$ | $f = 1$ |
| $f_x = y\cos(xy)$ | $f_x = 0$ |
| $f_y = x\cos(xy)$ | $f_y = 0$ |
| $f_{xx} = -y^2 \sin(xy)$ | $f_{xx} = -\frac{\pi^2}{4}$ |
| $f_{xy} = -xy\sin(xy) + \cos(xy)$ | $f_{xy} = -\frac{\pi}{2}$ |
| $f_{yy} = -x^2 \sin(xy)$ | $f_{yy} = -1$ |

$$f(x,y) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{\underline{|2}}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \ldots\ldots.$$

$$= 1 + \frac{1}{2}\left[\frac{-\pi^2}{4}(x-1)^2 - \pi(x-1)\left(y - \frac{\pi}{2}\right) - \left(y - \frac{\pi}{2}\right)^2\right] + \cdots$$

**6(a) Expand $x^2 y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ upto 3[rd] degree terms.**

**Hints:**

| | |
|---|---|
| $f(x,y) = x^2 y + 3y - 2$ | $f(1,-2) = -10$ |
| $f_x(x,y) = 2xy$ | $f_x(1,-2) = -4$ |
| $f_y(x,y) = x^2 + 3$ | $f_y(1,-2) = 4$ |
| $f_{xx}(x,y) = 2y$ | $f_{xx}(1,-2) = -4$ |

$f_{xy}(x,y) = 2x$ $\qquad f_{xy}(1,-2) = 2$

$f_{yy}(x,y) = 0$

$f_{xxx}(x,y) = 0$

$f_{xxy}(x,y) = 2$

$f_{xyy}(x,y) = 0$

$f_{yyy}(x,y) = 0$

$$f(x,y) = f(1,-2) + (x-1)f_x(1,-2) + (y+2)f_y(1,-2) + \frac{1}{2!}\{(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2)f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)\}$$

$$+\frac{1}{3!}\left\{\begin{array}{c}(x-1)^3 f_{xxx}(1,-2) + 3(x-1)^2(y+2)f_{xxy}(1,-2) \\ +3(x-1)(y+2)^2 f_{xyy}(1,-2) + (y+2)^3 f_{yyy}(1,-2)\end{array}\right\}$$

$$= -10 - 4(x-1) + 4(y+2) + \tfrac{1}{2!}\{-4(x-1)^2 + 4(x-1)(y+2)\} + \frac{1}{3!}\{6(x-1)^2(y+2)\}$$

**(b)** **Expand $e^x \log(1+y)$ in powers of *x* and *y* up to third degree using Taylor's series.**

**Hints:**

| **Function** | **Values at (0,0)** |
|---|---|
| $f(x,y) = e^x \log(1+y)$ | $f = 0$ |
| $f_x = e^x \log(1+y)$ | $f_x = 0$ |
| $f_y = e^x (1+y)^{-1}$ | $f_y = 1$ |
| $f_{xx} = e^x \log(1+y)$ | $f_{xx} = 0$ |
| $f_{xy} = e^x (1+y)^{-1}$ | $f_{xy} = 1$ |
| $f_{yy} = -e^x (1+y)^{-2}$ | $f_{yy} = -1$ |
| $f_{xxx} = e^x \log(1+y)$ | $f_{xxx} = 0$ |
| $f_{xxy} = e^x (1+y)^{-1}$ | $f_{xxy} = 1$ |
| $f_{xyy} = -e^x (1+y)^{-2}$ | $f_{xyy} = -1$ |
| $f_{yyy} = 2e^x (1+y)^{-3}$ | $f_{yyy} = 2$ |

$$f(x,y) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{\lfloor 2}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \ldots\ldots.$$

$$= \frac{y}{1!} + \frac{2xy - y^2}{2!} + \frac{3x^2 y - 3xy^2 + 2y^3}{3!} + \cdots$$

**7(a) Find the maximum and minimum values of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$**

**Hints:** $f(x,y) = x^3+y^3-3x-12y+20$

$f_x(x,y) = 3x^2 - 3; f_y(x,y) = 3y^2 - 12$

$A = f_{xx}(x,y) = 6x; B = f_{xy}(x,y) = 0; C = f_{yy}(x,y) = 6y$

To find the stationary points:

| $fx = 0$ | $fy = 0$ |
|---|---|
| $\therefore 3x^2 - 3 = 0$<br>$x^2 - 1 = 0$<br>$x = \pm 1$ | $3y^2 - 12 = 0$<br>$y^2 - 4 = 0$<br>$y = \pm 2$ |

$\therefore$

The stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$

| | (1,2) | (1,–2) | (–1,2) | (–1,–2) |
|---|---|---|---|---|
| $A = 6x$ | $6 > 0$ | $6 > 0$ | $-6 < 0$ | $-6 < 0$ |
| $B = 0$ | 0 | 0 | 0 | 0 |
| $AC - B^2$ | $72 > 0$ | $-72 < 0$ | $-72 < 0$ | $72 > 0$ |
| Conclusion | min. point | saddle point | saddle point | max. point |

Maximum value of $f(x,y)$ *is* $f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$

$= -1 - 8 + 3 + 24 + 20 = 38$

Minimum value of $f(x,y)$ *is* $f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$

**(b) Find the extreme values of the function $f(x,y) = x^3 y^2 (1 - x - y)$**

**Hints:** $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_y = 2x^3y - 2x^4y - 3x^3y^2$

$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$

The stationary points are $(0,0)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$

At $(0,0)$, $AC - B^2 = 0$

At $\left(\frac{1}{2},\frac{1}{3}\right)$, $AC-B^2>0$ & $A<0$

Thus $\left(\frac{1}{2},\frac{1}{3}\right)$ is a maximum point and maximum value is $\frac{1}{432}$.

**8(a) Find the maximum value of $x^m y^n z^p$, when $x+y+z=a$**

**Hints:** Let F= $x^m y^n z^p + \lambda(x+y+z-a)$-------(1), where $\lambda$ is Lagrange multiplier

$F_x = mx^{m-1}y^n z^p + \lambda$; $F_y = nx^m y^{n-1} z^p + \lambda$; $F_z = px^m y^n z^{p-1} + \lambda$

For a maximum at $(x,y,z)$ we have

$$F_x = 0 \Rightarrow mx^{m-1}y^n z^p + \lambda = 0 \Rightarrow mx^{m-1}y^n z^p = -\lambda \quad (2)$$

$$F_y = 0 \Rightarrow nx^m y^{n-1} z^p + \lambda = 0 \Rightarrow nx^m y^{n-1} z^p = -\lambda \quad (3)$$

$$F_z = 0 \Rightarrow px^m y^n z^{p-1} + \lambda = 0 \Rightarrow px^m y^n z^{p-1} = -\lambda \quad (4)$$

From(2),(3) and (4)

$$mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$(ie) \quad \frac{mx^m y^n z^p}{x} = \frac{nx^m y^n z^p}{y} = \frac{px^m y^n z^p}{z}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$$

$$\therefore \quad \frac{m}{x} = \frac{m+n+p}{a}, y = \frac{m+n+p}{a}, z = \frac{m+n+p}{a}$$

$$\therefore \quad x = \frac{am}{m+n+p}, y = \frac{an}{m+n+p}, z = \frac{ap}{m+n+p}$$

Thus the maximum value

$$x^m y^n z^p = \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

**(b) A rectangular box open at the top is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction.**

**Hints:** Let $x,y,z$ be the length, breadth and height of the box

surface area $= xy + 2yz + 2zx$

volume $= xyz = 32$

Let the auxiliary function be,

$$F(x,y,z) = (xy + 2yz + 2zx) + \lambda(xyz - 32) \quad (1)$$

where λ is langrange multiplier

$F_x = y + 2z + \lambda(yz); F_y = x + 2z + \lambda(xz); F_z = 2y + 2x + \lambda(xy)$

To find the stationary point:

$$F_x = 0 \Rightarrow y + 2z + \lambda(yz) = 0 \Rightarrow \frac{1}{z} + \frac{2}{y} + \lambda = 0$$

$=> \frac{1}{z}+\frac{2}{y}=-\lambda$ (2)

$F_y = 0 => x+2z+\lambda(xz)=0 \quad => \frac{1}{z}+\frac{2}{x}+\lambda=0$

$=> \frac{1}{z}+\frac{2}{x}=-\lambda$ (3)

$F_z = 0 => 2y+2x+\lambda(xy)=0 => \frac{2}{x}+\frac{2}{y}+\lambda=0$

$=> \frac{2}{x}+\frac{2}{y}=-\lambda$ (4)

From (2) and (3) $\Rightarrow \quad x = y$

From (3) and (4) $\Rightarrow \quad y = 2z$

$\therefore \quad x = y = 2z$

since $xyz = 32$

$(2z)(2z)z = 32$

$z^3 = \frac{32}{4} = 8 \quad z = 2 \quad \therefore x = 4, \quad y = 4.$

Thus the dimension of the box are 4,4,2

**9(a) The temperature $u(x,y,z)$ at any point in space is $u = 400\,xyz^2$. Find the highest temperature on surface of the sphere $x^2+y^2+z^2=1$**

**Hints:**

Let the auxiliary function F be

$F(x,y,z) = (400xyz^2) + \lambda(x^2+y^2+z^2-1)$-----(1)

For maximum or minimum

$F_x = 0 \Rightarrow \frac{200yz^2}{x} = -\lambda$ ---------(2)

$F_y = 0 \Rightarrow \frac{200xz^2}{y} = -\lambda$ -----------(3)

$F_z = 0 \Rightarrow 400xy = -\lambda$ ------------(4)

From (2) and (3) we get $x = y$ and from (3) and (4) we get $z^2 = 2y^2$ and hence we have

$z = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{2}$

$\therefore \; u = 400xyz^2$, we take $x,y,z$ to be positive

$\therefore \; u = 50$.

**(b) Find the volume of the largest rectangular solid which can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$**

**Hints:** Let $2x, 2y, 2z$ be the dimensions of the required rectangular solid.

Let $F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

$$\frac{\partial F}{\partial x} = 8yz + \lambda\frac{2x}{a^2} = 0 \ldots..(1) \quad \& \quad \frac{\partial F}{\partial y} = 8xz + \lambda\frac{2y}{b^2} = 0 \ldots..(2)$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda\frac{2z}{c^2} = 0 \ldots..(3) \quad \& \quad \frac{\partial F}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \ldots..(4)$$

Equating the value of $\lambda$ from (1), (2) and (3)

$$\lambda = \frac{-4yza^2}{x} = \frac{-4xzb^2}{y} = \frac{-4xyc^2}{z}$$

From the first two ratios, we get $\frac{x^2}{a^2} = \frac{y^2}{b^2} \ldots..(5)$

and from the last two ratios, we get $\frac{y^2}{b^2} = \frac{z^2}{c^2} \ldots..(6)$

from (5) and (6) we have $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = k, say$

Substituting this in (4), we get $3k = 1, (i.e.,)\ k = \frac{1}{3}$

$$\therefore\ x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

$\therefore$ Maximum volume = $\frac{8abc}{3\sqrt{3}}$.

**10(a) Find the dimensions of the rectangular box open at the top of maximum capacity whose surface area is 432 square metre.**

**Hints:** Let $x$, $y$, $z$ be the dimensions of the open rectangular box
The surface area is $S = xy + 2xz + 2yz$
Volume $V = xyz$
We have to maximize $f(x,y,z) = xyz$ subject to the constraint

$$\emptyset(x,y,z) = xy + 2xz + 2yz = 432$$

$F = f + \emptyset\lambda = xyz + \lambda(xy + 2xz + 2yz - 432)$ necessary condition are

$$F_x = 0 = F_y = F_z = F_\lambda$$

$$yz + \lambda(y + 2z) = 0$$
$$xz + \lambda(x + 2z) = 0$$
$$yx + \lambda(2x + 2y) = 0$$
$$xy + 2xz + 2yz = 432$$

$$\lambda = \frac{-yz}{y+2z} = \frac{-xz}{x+2z} = \frac{-yx}{2x+2y}$$

Solving we have $x = y = 2z$ hence $12z^2 = 432$ and $z = 6, -6$

$x = 12, \quad y = 12, \quad z = 6$

**(b) Find the shortest and the longest distances from $(1,2,-1)$ to the sphere $x^2 + y^2 + z^2 = 24$, using Lagrange's method of maxima and minima.**

**Hints:**

**Let** $(x, y, z)$ be any point on the sphere, then distance of the point $(x, y, z)$ from $(1,2,-1)$ is given by $d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$; $\quad d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$

Let $F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$, where $\lambda$ is Lagrange multiplier.

The stationary points of F are given by $F_x = 0; F_y = 0; F_z = 0$ and the points are $(2,4,-2)$, $(-2,-4,2)$

The value of $d$ at $(2,4,-2)$ is $\sqrt{6}$ and at $(-2,-4,2)$ is $3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$ respectively.

## UNIT – V MULTIPLE INTEGRALS

## PART – A

1. **Evaluate $\int_1^2 \int_0^{x^2} x \, dy \, dx$.**

   **Ans:** $\int_1^2 \int_0^{x^2} x \, dy \, dx = \int_1^2 x(y)_0^{x^2} dx = \int_1^2 x^3 \, dx = \frac{15}{4}$

2. **Evaluate $\int_{x=1}^{2} \int_{y=0}^{x} \frac{1}{x^2 + y^2} dx \; dy$.**

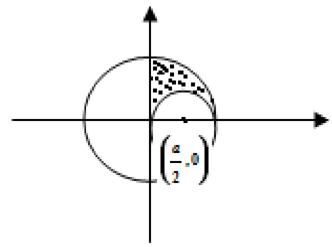
   **Ans:** $I = \int_{x=1}^{2} \int_{y=0}^{x} \frac{1}{x^2 + y^2} dy \, dx = \int_{x=1}^{2} \left[ \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x dx = \frac{\pi}{4} \log 2$

3. **Evaluate $\int_2^a \int_2^b \frac{dx \, dy}{xy}$.**

   **Ans:** $I = \int_2^a \frac{1}{y}\left[\log x\right]_2^b dy = \int_2^a [\log b - \log 2] \frac{dy}{y} = \log\left(\frac{b}{2}\right) \int_2^a \frac{dy}{y}$

   $= \log\left(\frac{b}{2}\right) [\log y]_2^a = \log\left(\frac{b}{2}\right) [\log a - \log 2] = \log\left(\frac{b}{2}\right) \log\left(\frac{a}{2}\right)$

4. **Shade the region of integration in $\int_0^a \int_{\sqrt{ax - x^2}}^{\sqrt{a^2 - x^2}} dx \, dy$.**



**Ans:** $y = \sqrt{ax - x^2} \Rightarrow x^2 + y^2 - ax = 0$ which is a circle

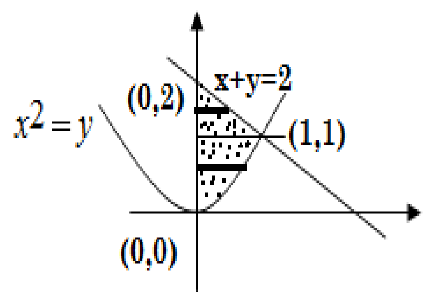
with centre at $(a/2, 0)$ and radius $a/2$

$y = \sqrt{a^2 - x^2} \Rightarrow x^2 + y^2 = a^2$ which is a circle

with centre at (0,0) and radius $a$

**5.** **Change the order of integration in** $\displaystyle\int_0^1 \int_{x^2}^{2-x} \mathbf{f(x,y)\,dy\,dx}$.

**Ans:**

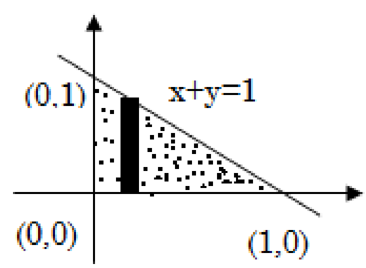



Given, $I = \displaystyle\int_0^1 \int_{x^2}^{2-x} f(x,y)\,dy\,dx$

After changing order of integration

$$I = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy + \int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy$$

**6.** **Evaluate** $\displaystyle\iint_R (x^2 + y^2)\,dy\,dx$ **over the region R for which x ,y ≥ 0, x+y ≤ 1.**




**Ans:** The region of integration is the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$

Limits of y : 0 to 1–x ; Limits of x : 0 to 1

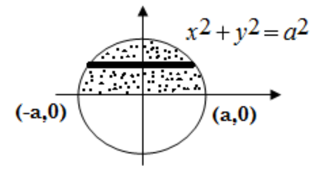
$$\iint_R (x^2 + y^2)\,dy\,dx = \int_0^1 \int_0^{1-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}$$

**7.** **Change the order of integration in** $\displaystyle\int_{-a}^{a} \int_0^{\sqrt{a^2 - x^2}} \mathbf{(x^2 + y^2)\,dx\,dy}$.

**Ans:** $I = \int\limits_{x=-a}^{x=a} \int\limits_{y=0}^{y=\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$ ( Correct Form )

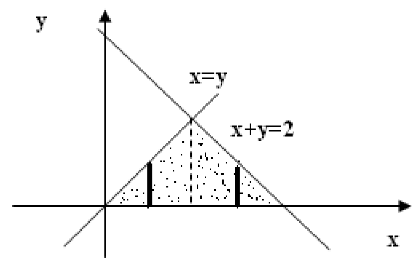



$$\int\limits_{y=0}^{y=a} \int\limits_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy \quad \text{(afterchangingthe order)}$$

**8.** **Change the order of integration in $\int\limits_0^1 \int\limits_y^{2-y} xy\,dx\,dy$ .**

**Ans:** Given, $I = \int\limits_0^1 \int\limits_y^{2-y} xy\ dx\ dy$

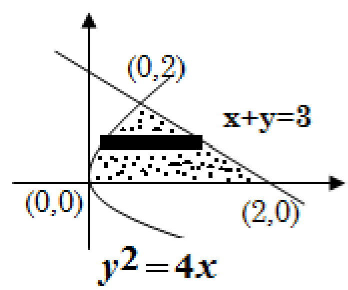
After changing order of integration




$$I = \int\limits_0^1 \int\limits_0^x xy\ dy\ dx + \int\limits_1^2 \int\limits_0^{2-x} xy\ dy\ dx$$

**9.** **Compute the area enclosed by $y^2 = 4x$ , $x + y = 3$ and $y = 0$.**

**Ans:** Area $A = \iint\limits_R dx\,dy = \int\limits_{y=0}^{2} \int\limits_{x=y^2/4}^{3-y} dx\,dy = \int\limits_{y=0}^{2} [x]_{y^2/4}^{3-y}\,dy$




$$= \int\limits_{y=0}^{2} \left[3-y-\frac{y^2}{4}\right] dy = \left[3y-\frac{y^2}{2}-\frac{y^3}{12}\right]_0^2$$

$$= 6-2-\frac{8}{12} = 4-\frac{2}{3} = \frac{10}{3}$$

**10.** **Evaluate $\int\limits_0^a \int\limits_0^{\sin\theta} r dr\, d\theta.$**

**Ans:** $\int\limits_0^a \int\limits_0^{\sin\theta} r dr\, d\theta = \int\limits_0^a \left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta = \int\limits_0^a \left[\frac{\sin^2\theta}{2}\right] d\theta = \frac{1}{4}\left(a - \frac{\sin 2a}{2}\right)$

**11.** **Evaluate $\int\limits_0^{\pi} \int\limits_0^{5} r\sin^2\theta\, dr\, d\theta.$**

**Ans:** $I = \int_0^{\pi} \sin^2\theta \left[\int_0^5 r\,dr\right] d\theta = \int_0^{\pi} \sin^2\theta \left[\frac{r^2}{2}\right]_0^5 d\theta = \frac{25}{2}\int_0^{\pi} \sin^2\theta \; d\theta$

$$= \frac{25}{4}\int_0^{\pi} [1-\cos 2\theta] \; d\theta = \left(\frac{25}{4}\right)\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \left(\frac{25}{4}\right)\left[\left\{\pi - \frac{\sin 2\pi}{2}\right\} - 0\right] = \frac{25\pi}{4}$$

**12. Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$.**

**Ans:** $I = \int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2} = \int_0^{\pi/2} \frac{1}{2}\left(\int_0^{\infty} \frac{d\left(r^2\right)}{(r^2+a^2)^2}\right) d\theta = \int_0^{\pi/2} \frac{1}{2}\left[\frac{-1}{r^2+a^2}\right]_0^{\infty} d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \left[0 + \frac{1}{a^2}\right] d\theta = \frac{1}{2}\left(\frac{1}{a^2}\right) [\theta]_0^{\pi/2} = \frac{\pi}{4a^2}$$

**13. Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$.**

**Ans:**

$$I = \int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta = \int_0^{\pi/2}\left[\frac{\sin^2\theta}{2} - 0\right] d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi}{8}$$

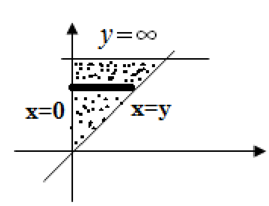
**14. Evaluate $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$.**

**Ans:** $I = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi} \cos^2\theta \; d\theta = \frac{1}{2} \; 2\int_0^{\frac{\pi}{2}} \cos^2\theta \; d\theta = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$

**15. Transform the integration $\int_0^{\infty}\int_0^{y} dx\,dy$ into polar coordinates.**

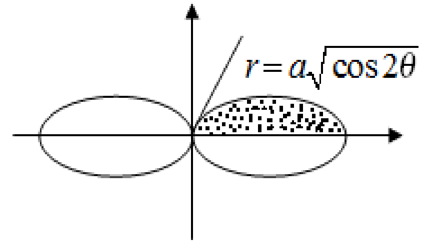
**Ans:** Let x = r cosθ and y = r sinθ , dxdy = r drdθ

$$\int_0^{\infty}\int_0^{y} dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{\infty} r\,dr\,d\theta$$



**16. Compute the entire area bounded by $r^2 = a^2 \cos 2\theta$.**

**Ans:**




$$\text{Area } A = \iint_R r\,dr\,d\theta = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = 4\int_0^{\pi/4}\left[\frac{a^2\cos 2\theta}{2}\right]d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2$$

**17. Transform the integration from Cartesian to polar co-ordinates**

$$\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy.$$

**Ans:** Let $x = r\cos\theta$ and $y = r\sin\theta$, $dxdy = r\,drd\theta$

$$\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

**18. Express the region bounded by $x \ge 0,\ y \ge 0,\ z \ge 0,\ x^2+y^2+z^2 \le 1$ as a triple integral.**

**Ans:** Here $z$ varies from 0 to $\sqrt{1-x^2-y^2}$, $y$ varies from 0 to $\sqrt{1-x^2}$, $x$ varies from 0 to 1

$$\therefore I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

**19. Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dxdydz$.**

**Ans:** $I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dxdydz = \int_0^1\int_0^1\left[e^{1+y+z} - e^{y+z}\right]dydz$

$$= \int_0^1\left(e^{z+2} - 2e^{z+1} + e^z\right)dz = e^3 - 3e^2 + 3e - 1 = (e-1)^3$$

**20. Evaluate $\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dx\,dy\,dz$.**

**Ans:** $I = \int_0^4 \int_0^x \int_0^{\sqrt{x+y}} z \, dz \, dy \, dx = \int_0^4 \int_0^x \left[ \frac{z^2}{2} \right]_0^{\sqrt{x+y}} dydx = \frac{1}{2} \int_0^4 \int_0^x (x+y) dydx$

$$= \frac{1}{2} \int_0^4 \left( xy + \frac{y^2}{2} \right)_0^x dx = \frac{1}{2} \int_0^4 \left( x^2 + \frac{x^2}{2} \right) dx = \frac{3}{4} \int_0^4 x^2 dx = \frac{3}{4} \left( \frac{x^3}{3} \right)_0^4 = 16$$

## PART –B

**1(a) Evaluate $\iint_R \frac{e^{-y}}{y} dx\,dy$, where R is the region bounded by the lines $x = 0,\ x = y,\ and\ y = \infty$**

**Hints:**

We first integrate w.r.to x and then y

$$\therefore I = \int_{y=0}^{\infty} \left[ \int_{x=0}^{y} \frac{e^{-y}}{y} dx \right] dy = \int_0^{\infty} \left[ \frac{e^{-y}}{y} . x \right]_0^y dy = \int_0^{\infty} e^{-y} \, dy = \left[ \frac{e^{-y}}{-1} \right]_0^{\infty}$$

$I = 1$

**(b) Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\,dy$ and hence evaluate it.**

**Hints:**

We change the order of integration the first integration should be w.r.to y and then w.r.to x.

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\,dy = \int_0^a \int_0^x \frac{x}{x^2+y^2} dy\ dx = \int_0^a \left[ \tan^{-1}\left( \frac{y}{x} \right) \right]_{y=0}^{y=x} dx = \int_0^a \left[ \tan^{-1}(1) - \tan^{-1}(0) \right] dx$$

$$= \int_0^a \left[ \frac{\pi}{4} - 0 \right] dx = \frac{\pi}{4} \int_0^a dx = \frac{\pi}{4} [x]_0^a = \frac{\pi}{4} [a - 0]$$

$$\boxed{\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\,dy = \frac{\pi}{4} a}$$

**2(a) Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy \ dy\,dx$ and hence evaluate it.**

**Hints:**

We change the order of integration the first integration should be w.r.to x and then w.r.to y.

$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2\left(4y-4y^2+y^3\right)dy$$

$$= \frac{1}{6}+\frac{1}{2}\left[2y^2-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2 = \frac{1}{6}+\frac{1}{2}\left[\left(8-\frac{32}{3}+4\right)-\left(2-\frac{4}{3}+\frac{1}{4}\right)\right]$$

$$\boxed{\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \frac{3}{8}}$$

**(b) Change the order of integration in $\int_0^\infty\int_0^y y e^{\frac{-y^2}{x}}\,dx\,dy$ and hence evaluate it.**

**Hints:**

We change the order of integration the first integration should be w.r.to y and then w.r.to x.

$$\int_0^\infty\int_0^y y e^{\frac{-y^2}{x}}\,dx\,dy = \int_0^\infty\int_x^\infty y e^{\frac{-y^2}{x}}\,dy\,dx = \frac{1}{2}\int_0^\infty\int_x^\infty 2y e^{\frac{-y^2}{x}}\,dy\,dx = \frac{1}{2}\int_0^\infty\left[\int_x^\infty e^{\frac{-y^2}{x}}\,d\left(y^2\right)\right]dx$$

$$= \frac{1}{2}\int_0^\infty\left[\frac{e^{\frac{-y^2}{x}}}{-\frac{1}{x}}\right]_x^\infty dx = \frac{1}{2}\int_0^\infty\left[-xe^{\frac{-y^2}{x}}\right]_x^\infty dx = \frac{1}{2}\int_0^\infty\left[0-\left(-xe^{-x}\right)\right]dx = \frac{1}{2}\int_0^\infty xe^{-x}\,dx = \frac{1}{2}\left[x\frac{e^{-x}}{-1}-(1)\frac{e^{-x}}{(-1)^2}\right]_0^\infty$$

$$= -\frac{1}{2}\left[xe^{-x}+e^{-x}\right]_0^\infty = \frac{-1}{2}\left[(0+0)-(0+1)\right]$$

$$\boxed{\int_0^\infty\int_0^y y e^{\frac{-y^2}{x}}\,dx\,dy = \frac{1}{2}}$$

**3(a) Transform the integral into polar co-ordinates and hence evaluate $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy$**

**Hints:**

Let us transform this integral in polar co-ordinates by taking $x = r\cos\theta,\ y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

$$\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\frac{1}{2}d\left(r^2\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left[-e^{-r^2}\right]_0^\infty d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$\boxed{\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy = \frac{\pi}{4}}$$

**(b) Transform the integral into polar co-ordinates and hence evaluate $\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dx\,dy$**

**Hints:**

Let us transform this integral in polar co-ordinates by taking $x=r\cos\theta,\ y=r\sin\theta$ and $dx\,dy=r\,dr\,d\theta$

$$\therefore \int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dx\,dy=\int_0^{\pi/4}\int_0^{a\sec\theta}\frac{r^2\cos^2\theta}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}r\,dr\,d\theta=\int_0^{\pi/4}\int_0^{a\sec\theta}\frac{r^3\cos^2\theta}{\sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)}}dr\,d\theta$$

$$=\int_0^{\pi/4}\int_0^{a\sec\theta}\frac{r^3\cos^2\theta}{r}dr\,d\theta=\int_0^{\pi/4}\cos^2\theta\left[\frac{r^3}{3}\right]_0^{a\sec\theta}d\theta=\int_0^{\pi/4}\cos^2\theta\left[\frac{a^3\sec^3\theta}{3}-0\right]d\theta=\frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta$$

$$=\frac{a^3}{3}\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4}=\frac{a^3}{3}\left[\log\left(\sqrt{2}+1\right)-\log(1+0)\right]$$

$$\boxed{\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dx\,dy=\frac{a^3}{3}\log\left(\sqrt{2}+1\right).}$$

**4(a) Evaluate $\iint_R (x+y)^2\,dx\,dy$, where R is the parallelogram in the xy-plane with vertices $(1,0),(3,1),(2,2),(0,1)$ using the transformation $u=x+y$ and $v=x-2y$**

**Hints:**

The vertices $A(1,0),B(3,1),C(2,2),D(0,1)$ of the parallelogram in the xy-plane become $A'(1,1),B'(4,1),C'(4,-2),D'(1,-2)$ in the uv-plane under the given transformation.

The region R in the xy-plane becomes the region $R'$ in the uv-plane which is the rectangle bounded by the lines $u=1,u=4,$ and $v=-2,v=1$

Solving the given equations , we get, $x=\frac{1}{3}(2u+v),\ y=\frac{1}{3}(u-v)$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=-\frac{1}{3}$$

$$\therefore \iint_R (x+y)^2\,dxdy=\iint_{R'}u^2|J|\,dudv=\int_{-2}^{1}\int_1^4 u^2\frac{1}{3}dudv=\int_{-2}^{1}\frac{1}{3}\left(\frac{u^3}{3}\right)_1^4 dv=\int_{-2}^{1}7dv=21$$

**(b) By using the transformation $x+y=u,\ y=uv$, show that $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx=\frac{e-1}{2}$**

**Hints:**

Given $x+y=u,\ \ y=uv,\ \ x=u(1-v)$

$$Now\ \ y=0 \Rightarrow u=0(or)v=0$$
$$y=1-x \Rightarrow u=1, x=0 \Rightarrow u=0(or)v=1$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=u$$

$$\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx=\iint_{R'} e^{\frac{uv}{u}}|J|\,du\,dv=\int_0^1\int_0^1 e^{v}u\,du\,dv=\frac{1}{2}\int_0^1 e^{v}dv=\frac{e-1}{2}$$

**5(a)** **By transforming into polar coordinates, Evaluate $\iint_R \frac{x^2y^2}{\sqrt{x^2+y^2}}dxdy$ over the annular region R**

**between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2, (b>a)$**

**Hints:**

Put $x=r\cos\theta, y=r\sin\theta$, then $x^2+y^2=a^2 \Rightarrow r=a,$ $x^2+y^2=b^2 \Rightarrow r=a$ and $\theta$ varies from 0 to $2\pi$

$$\iint\frac{x^2y^2}{x^2+y^2}dxdy=\int_0^{2\pi}\int_a^b\frac{r^2\cos^2\theta r^2\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}rdr\,d\theta=\int_0^{2\pi}\int_a^b r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$
$$=\int_0^{2\pi}\cos^2\theta\sin^2\theta\left(\frac{r^4}{4}\right)_a^b d\theta=\left(\frac{b^4-a^4}{4}\right)\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta=\left(\frac{b^4-a^4}{4}\right)4\int_0^{\pi/2}\cos^2\theta\sin^2\theta d\theta$$
$$\iint\frac{x^2y^2}{x^2+y^2}dxdy=\frac{\pi}{16}\left(b^4-a^4\right)$$

**(b)** **Find the area of the cardioid $r=a(1+\cos\theta)$**

**Hints:**

The curve is symmetrical about the initial line.

$$\text{The required area }=2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a(1+\cos\theta)} r\ dr\,d\theta=2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}d\theta$$
$$=\int_0^{\pi}\left[a^2(1+\cos\theta)^2-0\right]d\theta=a^2\int_0^{\pi}\left[1+\cos^2\theta+2\cos\theta\right]d\theta=a^2\int_0^{\pi}\left[1+\frac{1+\cos2\theta}{2}+2\cos\theta\right]d\theta$$
$$=a^2\left[\frac{3}{2}\theta+\frac{\sin2\theta}{4}+2\sin\theta\right]_0^{\pi}=a^2\left[\left(\frac{3}{2}\pi+0+0\right)-(0+0+0)\right]=\frac{3}{2}\pi a^2\ \text{Sq.units..}$$

**6(a) Find the smaller of the area bounded by $y=2-x$ and $x^2+y^2=4$**

**Hints:**

The required area $= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx = \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}}\,dx = \int_0^2 \left[\sqrt{4-x^2}-(2-x)\right]dx$

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx = \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - \left[2x-\frac{x^2}{2}\right]_0^2$$

$$= \left[\left(0+2\frac{\pi}{2}\right)-(0+0)\right]-\left[(4-2)-(0-0)\right] = \pi-2 \text{ Square units.}$$

**(b) Find the area lying inside the circle $r=a\sin\theta$ and outside the cardioid $r=a(1-\cos\theta)$**

**Hints:**

$\therefore$ The required area $= \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\sin^2\theta-1-\cos^2\theta+2\cos\theta\right]d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left[-2\cos^2\theta+2\cos\theta\right]d\theta = a^2\int_0^{\pi/2}\left[-\cos^2\theta+\cos\theta\right]d\theta$$

$$= a^2\left[-\frac{1}{2}\frac{\pi}{2}+1\right] = a^2\left(1-\frac{\pi}{4}\right)$$

**7(a) Find the surface area of the sphere $x^2+y^2+z^2=a^2$**

**Hints:**

Given $x^2+y^2+z^2=a^2 \Rightarrow z^2=a^2-x^2-y^2$

Surface Area $S = \iint_{S'} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy = 2\iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$

Let us transform this integral in polar co-ordinates by taking $x=r\cos\theta$, $y=r\sin\theta$ and $dx\,dy=r\,dr\,d\theta$

$$S = 2\int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2}}rdrd\theta = -2a\int_0^{2\pi}\left(\sqrt{a^2-r^2}\right)_0^a d\theta = 4\pi a^2 \text{ sq units}$$

**(b) Find the area of the portion of the cylinder $x^2+z^2=4$ lying inside the cylinder $x^2+y^2=4$**

**Hints:**

Given $x^2 + y^2 = 4 \Rightarrow y = \sqrt{4-x^2}$

Surface Area $S = \iint\limits_{S'} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dzdx = 8\int_0^2 \int_0^{\sqrt{2^2-x^2}} \frac{2}{\sqrt{2^2 - x^2}} dzdx = 16\int_0^2 dx = 32$ sq.units.

**8(a) Find the common area between the curves $y^2 = 4x$ *and* $x^2 = 4y$**

**Hints:**

Given $y^2 = 4x$ *and* $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

$\frac{x^2}{16} = 4x \Rightarrow x^4 - 64x = 0 \Rightarrow x = 0, 4$

$\therefore$ The required area $= \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left(\frac{4}{3}x^{\frac{3}{2}} - \frac{x^3}{12}\right)_0^4 = \frac{16}{3}$ square units.

**(b) Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$**

**Hints:**

$$V = 8\iint\limits_R \text{zdxdy} = 8\int_0^a \left[\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ \text{dy}\right] dx = 8\int_0^a \left[\int_0^{bt} c\sqrt{t^2 - \frac{y^2}{b^2}}\ \text{dy}\right] dx,$$

Where $t = \sqrt{1-\frac{x^2}{a^2}}.$ (keeping $x$ constant) Put $y = bt\sin\theta$. Then we have,

$$V = 8c\int_0^a \left[\int_0^{\pi/2} t\cos\theta . bt\cos\theta\, d\theta\right] dx = 8bc\int_0^a \left[\int_0^{\pi/2} t^2\cos^2\theta\, d\theta\right] dx = 8bc\int_0^a \frac{t^2}{2}\frac{\pi}{2} dx = 2bc\pi\int_0^a \left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi\left(x - \frac{x^3}{3a^2}\right)_0^a = 2bc\pi\left(a - \frac{a}{3}\right) = \frac{4}{3}\pi abc \text{ Cubic units.}$$

**9(a) Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.**

**Hints:**

The volume of the tetrahedron is obtained by integrating the surface $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$ over the region R.

$$\therefore V = \iint_R z\,dx\,dy = \int_0^a \left[ \int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \right] dx = c\int_0^a \left[ y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left[ b(1 - x/a) - \frac{x}{a} b(1 - x/a) - \frac{b^2}{2b}(1 - x/a)^2 \right] dx = cb\int_0^a \left[ (1 - x/a)^2 - \frac{1}{2}(1 - x/a)^2 \right] dx$$

$$= \frac{cb}{2}\int_0^a (1 - x/a)^2 dx = \frac{cb}{2}\left[ \frac{(1 - x/a)^3}{3(-1/a)} \right]_0^a = \frac{abc}{6}$$

**(b)** **Evaluate** $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

**Hints:**

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[ \sin^{-1}\left( \frac{z}{\sqrt{a^2-x^2-y^2}} \right) \right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[ \sin^{-1}(1) - \sin^{-1}(0) \right] dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[ \frac{\pi}{2} - 0 \right] dy\,dx = \frac{\pi}{2}\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi}{2}\left[ \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a = \frac{\pi}{2}\left[ \left(0 + \frac{a^2}{2}\frac{\pi}{2}\right) - (0+0) \right]$$

$$\boxed{\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}} = \frac{\pi^2 a^2}{8}}$$

**10(a)** **Find the volume of the sphere** $x^2 + y^2 + z^2 = a^2$

**Hints:**

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[ \sqrt{a^2-x^2-y^2} - 0 \right] dy\,dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2 - y^2}\,dy\,dx$$

$$= 8\int_0^a \left[ \frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx = 8\int_0^a \left[\left(0+\frac{a^2-x^2}{2}\frac{\pi}{2}\right)-(0+0)\right]dx$$

$$= 2\pi\int_0^a \left(a^2-x^2\right)dx = 2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a = 2\pi\left[\left(a^3-\frac{a^3}{3}\right)-(0-0)\right]$$

$$\boxed{V=\frac{4}{3}\pi a^3}.$$

**(b) Evaluate $\iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes**

**$x=0, y=0,\ z=0, x+y+z=1$**

**Hints:**

$$\iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z+1)^{-3}dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-(x+y+1)^{-2}\right]dy\,dx = -\frac{1}{2}\int_0^1\left[\frac{1}{4}y+(x+y+1)^{-1}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[\left(\frac{1}{4}(1-x)+2^{-1}\right)-\left(0+(x+1)^{-1}\right)\right]dx = -\frac{1}{2}\int_0^1\left[\frac{3}{4}-\frac{x}{4}-\frac{1}{1+x}\right]dx = -\frac{1}{2}\left[\frac{3}{4}x-\frac{x^2}{8}-\log(1+x)\right]_0^1$$

$$= -\frac{1}{2}\left[\left(\frac{3}{4}-\frac{1}{8}-\log 2\right)-(0-0-0)\right]$$

$$\boxed{\iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \frac{1}{2}\log 2-\frac{5}{16}}$$