St. Joseph's College of Engineering
St. Joseph's Institute of Technology
Mathematics II (MA6251)
Assignment – II
UNIT- III LAPLACE TRANFORM
PART A

1. **State under which conditions Laplace transform of $f(t)$ exists.**
2. **If L[f(t)] = F(s), prove that L{f(t/5)}= 5 F(5s).**
3. **Find the Laplace transform of unit step function.**
4. **Does $L\left[\frac{\cos at}{t}\right]$ exist?**
5. **Find $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]$**
6. **If L{f(t)} = $F(S)$, find the value of $\int_0^\infty f(t)dt$**
7. **Find $L^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right)$**
8. **Solve using Laplace transform $\frac{dy}{dt}+y=e^{-t}$ given that y(0)=0.**
9. **Give an example for a function that do not have Laplace transform.**
10. **State the Convolution theorem.**

PART B

1(a) Find $L\left[t^2e^t\sin t\right]$

SOLUTION:

$$L\left[t^2e^t\sin t\right]=(-1)^2\frac{d^2}{ds^2}L\left[e^t\sin t\right] \quad \text{... (1)}$$

Now $L\left[e^t\sin t\right]=\left[L[\sin t]\right]_{s\to(s-1)}=\frac{1}{(s-1)^2+1} \quad \text{... (2)}$

Substituting (2) in (1) we get

$$L\left[t^2e^t\sin t\right]=\frac{d}{ds}\left[\frac{0-2(s-1)}{\left((s-1)^2+1\right)^2}\right]=\frac{d}{ds}\left[\frac{-2(s-1)}{\left(s^2-2s+2\right)^2}\right]$$

$$=\frac{\left(s^2-2s+2\right)^2(-2)+2(s-1)2\left(s^2-2s+2\right)(2s-2)}{\left(s^2-2s+2\right)^4}$$

$$=\frac{2\left(s^2-2s+2\right)\left[-\left(s^2-2s+2\right)+4(s-1)^2\right]}{\left(s^2-2s+2\right)^4}$$

$$=\frac{2\left(s^2-2s+2\right)\left[-s^2+2s-2+4s^2+4-8s\right]}{\left(s^2-2s+2\right)^4}$$

$$\therefore F(s)=\frac{2\left(s^2-2s+2\right)\left[3s^2-6s+2\right]}{\left(s^2-2s+2\right)^4}=\frac{2\left(3s^2-6s+2\right)}{\left(s^2-2s+2\right)^3}$$

(b) Find $L\left[\frac{\sin^2 t}{t}\right]$

SOLUTION:

$$L\left[\frac{\sin^2 t}{t}\right]=L\left[\frac{1-\cos 2t}{2t}\right]=\frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right]=\frac{1}{2}\int_s^\infty L[1-\cos 2t]\ ds$$

$$=\frac{1}{2}\int_s^\infty\left\{L[1]-L[\cos 2t]\right\}ds=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+4}\right]ds$$

$$=\frac{1}{2}\left[\log s-\frac{1}{2}\log\left(s^2+4\right)\right]_s^\infty=\frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty=\frac{1}{2}\left[\log 1-\log\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]=\frac{1}{2}\left[0-\log\frac{s}{\sqrt{s^2+4}}\right]$$

$$\therefore\ F(s)=\frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1}=\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

2(a) Find the Laplace transform of $e^{-4t}\int_0^t t\ \sin 3t dt$

SOLUTION:

$$L[\sin 3t]=\frac{3}{s^2+9}$$

$$L[t\sin 3t]=-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)=-\left(\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right)=\frac{6s}{\left(s^2+9\right)^2}$$

$$L\left(\int_0^t t\sin 3t dt\right)=\frac{L(t\sin 3t)}{s}=\frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = L\left(\int_0^t t\sin 3t dt\right)\Bigg|_{s\to s+4} = \frac{6}{\left((s+4)^2+9\right)^2} = \frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$\therefore L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = \frac{6}{\left(s^2+8s+25\right)^2}$$

(b) Verify initial and final value theorems for the function f(t) = 1 + e^{-t}(sin t + cos t)

SOLUTION:

Initial value theorem states that $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$L\left[f(t)\right] = F(s)$$

$$= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

L.H.S. $= \lim_{t\to 0} f(t) = 1+1 = 2$

R.H.S $= \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to\infty}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right]$

$$= \lim_{s\to\infty}\left[1+\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right] = \lim_{s\to\infty}\left[1+\frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1+1 = 2$$

L.H.S=R.H.S

Initial value theorem verified.

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

L.H.S. $= \lim_{t\to\infty}\left[1+e^{-t}(\sin t + \cos t)\right] = 1+0 = 1$

R.H.S $= \lim_{s\to 0}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right] = 1+0 = 1$

L.H.S.=R.H.S

Hence final value theorem verified.

3(a) Find the Laplace transform of f(t) = $\begin{cases} t, & 0\le t\le a \\ 2a-t, & a\le t\le 2a \end{cases}$ and f(t+2a) = f(t) for all t

SOLUTION:

$$L\left[f(t)\right] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[-t\left(\frac{e^{-st}}{s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[-(2a-t)\left(\frac{e^{-st}}{s}\right)+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\left(-a\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\frac{e^{-2as}}{s^2}-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right] = \frac{\left(1-e^{-sa}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)}$$

$$\therefore\ F(s) = \frac{1-e^{-sa}}{s^2\left(1+e^{-as}\right)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

(b) Find the Laplace transform of the rectangular wave given by $f(t) = \begin{cases} \sin\omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$

SOLUTION:

This function is periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left[-s\sin\omega t-\omega\cos\omega t\right]\right]_0^{\frac{\pi}{\omega}}$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right]$$

$$=\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

$$=\frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

4(a) **Find** $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

SOLUTION:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

Put $s=-1 \Rightarrow \boxed{A=-\frac{1}{3}}$

Equating the coefficients of $s^3 \Rightarrow \boxed{B=\frac{1}{3}}$

Put $s=2 \Rightarrow \boxed{D=-7}$

Put $s=0 \Rightarrow \boxed{C=4}$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1/3}{s+1}+\frac{1/3}{s-2}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=-\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}t-\frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore f(t)=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}t-\frac{7}{2}e^{2t}t^2$$

(b) **Find the inverse Laplace transform of** $\log\left(\frac{1+s}{s^2}\right)$

SOLUTION:

Let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=f(t)$

$\therefore\ L\left[f(t)\right]=\log\left(\frac{1+s}{s^2}\right)$

$$L\left[t\,f(t)\right]=\frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{-d}{ds}\left[\log(1+s)-\log\left(s^2\right)\right]=-\frac{1}{1+s}+\frac{1}{s^2}2s$$

$$L\left[t\,f(t)\right]=\frac{2}{s}-\frac{1}{s+1}$$

$$t\,f(t)=L^{-1}\left[\frac{2}{s}-\frac{1}{s+1}\right]=2L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]=2(1)-e^{-t}$$

$$\therefore\quad f(t)=\frac{2-e^{-t}}{t}$$

$$\therefore\quad L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{2-e^{-t}}{t}$$

5(a) Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.

SOLUTION:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$=\cos at*\cos bt$$

$$=\int_0^t \cos au\ \cos b(t-u)\ du$$

$$=\frac{1}{2}\int_0^t\left[\cos(au+bt-bu)+\cos(au-bt+bu)\right]\ du$$

$$=\frac{1}{2}\int_0^t\left[\cos((a-b)u+bt)+\cos((a+b)u-bt)\right]\ du$$

$$=\frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b}+\frac{\sin((a+b)u-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\left(\frac{\sin(bt+at-bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\sin(at)}{a-b}+\frac{\sin(at)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2a\sin(at)}{a^2-b^2}-\frac{2b\sin(bt)}{a^2-b^2}\right)$$

$$f(t)=\frac{a\sin(at)-b\sin(bt)}{a^2-b^2}$$

(b) Using Convolution theorem find the inverse Laplace transform of $\frac{2}{(s+1)(s^2+4)}$

SOLUTION:

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]=L^{-1}\left[\frac{1}{s+1}\ \frac{2}{s^2+4}\right]=L^{-1}\left[\frac{1}{s+1}\right]*L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=e^{-t}*\sin 2t$$

$$=\int_0^t e^{-u}\ \sin 2(t-u)\,du$$

$$=\int_0^t e^{-u}\ \sin(2t-2u)\,du$$

$$=\int_0^t e^{-u}\ [\sin 2t\ \cos 2u-\cos 2t\ \sin 2u]\,du$$

$$=\int_0^t e^{-u}\ \sin 2t\ \cos 2u\ du-\int_0^t e^{-u}\ \cos 2t\ \sin 2u\ du$$

$$=\sin 2t\int_0^t e^{-u}\ \cos 2u\ du-\cos 2t\int_0^t e^{-u}\ \sin 2u\ du$$

$$=\sin 2t\left[\frac{e^{-u}}{1+4}(-\cos 2u+2\sin 2u)\right]_0^t-\cos 2t\left[\frac{e^{-u}}{1+4}(-\sin 2u-2\cos 2u)\right]_0^t$$

$$=\sin 2t\left[\left(\frac{e^{-t}}{5}(-\cos 2t+2\sin 2t)\right)-\left(\frac{1}{5}(-1)\right)\right]-\cos 2t\left[\frac{e^{-t}}{5}(-\sin 2t-2\cos 2t)-\left(\frac{1}{5}(-2)\right)\right]$$

$$=\sin 2t\left[\frac{e^{-t}}{5}(-\cos 2t+2\sin 2t)+\frac{1}{5}\right]-\cos 2t\left[\frac{e^{-t}}{5}\left(-\sin 2t-2\cos 2t+\frac{2}{5}\right)\right]$$

$$=\frac{e^{-t}}{5}\left[-\sin 2t\cos 2t+2\sin^2 2t+\sin 2t\ \cos 2t+2\cos^2 2t\right]+\frac{1}{5}\sin 2t-\frac{2}{5}\cos 2t$$

$$=\frac{e^{-t}}{5}\left[2(1)\right]+\frac{1}{5}\sin 2t-\frac{2}{5}\cos 2t$$

$$f(t)=\frac{1}{5}\left[2e^{-t}+\sin 2t-2\cos 2t\right]$$

6(a) Using Convolution theorem find $L^{-1}\left[\dfrac{s}{\left(s^2+a^2\right)^2}\right]$

SOLUTION:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos at*\frac{1}{a}\sin at=\frac{1}{a}\left[\cos at*\sin at\right]$$

$$=\frac{1}{a}\int_0^t \cos au\ \ \sin a(t-u)\ du=\frac{1}{a}\int_0^t \sin(at-au)\ \cos au\ \ du$$

$$=\frac{1}{a}\int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\ du$$

$$=\frac{1}{2a}\int_0^t \left[\sin at+\sin a(t-2u)\right]\ du$$

$$=\frac{1}{2a}\left[\sin at\ u+\left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[u\ \sin at+\left(\frac{\cos a(t-2u)}{2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[t\ \sin at+\left(\frac{\cos at}{2a}\right)-\left(0+\frac{\cos at}{2a}\right)\right]$$

$$\therefore\ f(t)=\frac{1}{2a}\left[t\sin at+\frac{\cos at}{2a}-\frac{\cos at}{2a}\right]=\frac{1}{2a}t\sin at$$

(b) Solve the equation y″+ 9y=cos2t with y(0) =1 y $\left(\frac{\pi}{2}\right)$ **= –1**

SOLUTION:

Given $\left(D^2+9\right)y=\cos 2t$

Taking Laplace transforms on both sides

$$L\left[y''(t)\right]+9L\left[y(t)\right]=L\left[\cos 2t\right]$$

$$s^2L\left[y(t)\right]-sy(0)-y'(0)+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

Using the initial conditions

$y(0)=1,$ and taking $y'(0)=k$

We have

$$s^2L\left[y(t)\right]-(s)(1)-k+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

$$\Rightarrow L\left[y(t)\right]=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{s^2+9}$$

$$=\frac{s}{5\left(s^2+4\right)}-\frac{s}{5\left(s^2+9\right)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

$$\therefore\ y(t)=\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right]+L^{-1}\left[\frac{s}{s^2+9}\right]+k\,L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{k}{3}\sin 3t$$

Put $t=\frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right)=\frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{k}{3}(-1)=-\frac{1}{5}-\frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right)=-1$

$$\therefore\ -1=-\frac{1}{5}-\frac{k}{3}$$

$$\Rightarrow k=\frac{12}{5}$$

$$\therefore\ y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 3t$$

$$y(t)=\frac{4}{5}\left[\cos 3t+\sin 3t\right]+\frac{1}{5}\cos 2t$$

7(a) Solve $y''+2y'-3y=\sin t$, given $y(0)=0,\ y'(0)=0$

SOLUTION:

Given $y''+2y'-3y=\sin t$

$$L\left[y''(t)+2y'(t)-3y(t)\right]=L[\sin t]$$

$$L\left[y''(t)\right]+2L\left[y'(t)\right]-3L\left[y(t)\right]=L[\sin t]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]+2\left[sL\left[y(t)\right]-y(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$\left[s^2L\left[y(t)\right]-s(0)-0\right]+2\left[sL\left[y(t)\right]-(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$s^2L\left[y(t)\right]+2sL\left[y(t)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]\left(s^2+2s-3\right)=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}$$

$$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s^2+2s-3)}\right] == L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$$

Now

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{(s^2+1)}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)$$

Put $s = 1 \Rightarrow \boxed{A = \frac{1}{8}}$

Put $s = -3 \Rightarrow \boxed{B = \frac{-1}{40}}$

Equating coeff. of $s^3 \Rightarrow \boxed{C = \frac{-1}{10}}$

Equating the constant terms $\Rightarrow \boxed{D = \frac{-1}{5}}$

$$\therefore \frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{(-1/10)s - 1/5}{(s^2+1)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] = L^{-1}\left[\frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{(-1/10)s-1/5}{(s^2+1)}\right]$$

$$= \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$= \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\left[L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{2}{s^2+1}\right]\right]$$

$$= \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}[\cos t + 2\sin t]$$

(b) **Determine y which satisfies the equation** $\frac{dy}{dt} + 2y + \int_0^t y\,dt = 2\cos t,\ y(0) = 1$

SOLUTION:

Given $y'(t) + 2y(t) + \int_0^t y(t)\,dt = 2\cos t,\ y(0) = 1$

$$L\left[y'(t)\right] + 2L\left[y(t)\right] + L\left[\int_0^t y(t)\,dt\right] = L[2\cos t]$$

$$sL\left[y(t)\right] - y(0) + 2L\left[y(t)\right] + \frac{1}{s}L\left[y(t)\right] = \frac{2s}{s^2+1}$$

$$sL\left[y(t)\right]-1+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$\Rightarrow L\left[y(t)\right]=\frac{s}{s^2+1}$$

$$y(t)=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

UNIT-IV ANALYTIC FUNCTIONS

PART A

1. **Define an analytic function (or) Regular function.**
2. **State the necessary condition for $f(z)$ to be analytic [Cauchy – Riemann Equations].**
3. **Define harmonic function.**
4. **Define conformal mapping.**
5. **Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not?**
6. **Prove that an analytic function whose real part is constant must itself be a constant.**
7. **Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic.**
8. **Obtain the invariant points (fixed points) of the transformation $w = 2 - \frac{2}{z}$**
9. **Define a critical point of the bilinear transformation.**
10. **Find the critical point of the transformation $w^2 = (z-\alpha)(z-\beta)$**

PART B

1(a) Show that the function $f(z) = \left|z^2\right|$ is differentiable at $z = 0$ but not analytic at $z = 0$

SOLUTION:

Let $z = x + iy$

$\bar{z} = x - iy$

$|z|^2 = x^2 + y^2$

$f(z) = |z|^2 = x^2 + y^2$

$u = x^2 + y^2, \quad v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

So, the C-R equations $u_x = v_y$ & $u_y = -v_x$ are not satisfied every where except at z=0

So f(z) may be differentiable only at z=0

Now $u_x = 2x,\ u_y = 2y, v_x = 0\ \&\ v_y = 0$ are continuous everwhere and in particular at (0,0)

Hence the sufficient conditions for differentiability are satisfied by f(z) at z=0

So f(z) is differentiable at z=0 only and not analytic there.

(b) The function f(z) = u + iv is analytic, show that u = constant and v = constant are orthogonal.

SOLUTION:

If $f(z) = u + iv$ is an analytic function of z, then it satisfies C-R equations

$u_x = v_y,\ u_y = -v_x$

Given $u(x, y) = C_1$............(1)

$v(x, y) = C_2$............(2)

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Differentiate equation (1) & (2) we get $du = 0$, $dv = 0$

$$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = m_1 (say)$$

$$\frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} = m_2 (say)$$

$$\therefore m_1\, m_2 = -\frac{-\partial u/\partial x}{\partial u/\partial y} \times \frac{-\partial v/\partial x}{\partial v/\partial y} \quad (\because u_x = v_y \quad u_y = -v_x)$$

$$\therefore m_1\, m_2 = -1$$

The curves $u(x, y) = C_1$ and $v(x, y) = C_2$ cut orthogonally.

2(a) Prove that an analytic function with constant modulus is constant.

SOLUTION:

Let $f(z) = u + iv$ be analytic

By C.R equations satisfied

i.e., $u_x = v_y$, $u_y = -v_x$

$$\therefore f(z) = u + iv$$

$$\Rightarrow |f(z)| = \sqrt{u^2 + v^2} = C \Rightarrow |f(z)|^2 = u^2 + v^2 = C^2$$

$u^2 + v^2 = C^2$..............(1)

Diff (1) with respect to x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$uu_x + vv_x = 0$............(2)

Diff (1) with respect to y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$-uv_x + vu_x = 0$............(3)

$$(2)\times u + (3)\times v \Rightarrow (u^2 + v^2)u_x = 0$$

$$\Rightarrow u_x = 0$$

$$(2)\times v - (3)\times u \Rightarrow (u^2 + v^2)v_x = 0$$

$$\Rightarrow v_x = 0$$

W.K.T $f'(z) = u_x + iv_x = 0$

$f'(z)=0$ Integrate w.r.to z

$f(z)=C$

(b) If f(z) is an analytic function, prove that $\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}|f(z)|^2=4|f'(z)|^2$

SOLUTION:

Let f(z) = u + iv be analytic.

Then $u_x = v_y$ and $u_y = -v_x$ (1)

Also $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Now $|f(z)|^2 = u^2 + v^2$ and $f'(z) = u_x + iv_x$

$$\therefore \frac{\partial}{\partial x}|f(z)|^2 = 2u.u_x + 2v.v_x$$

and $\frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right]$ (3)

Similarly $\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right]$ (4)

Adding (3) and (4)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u_x^2 + u_y^2 + u(u_{xx}+u_{yy}) + v_x^2 + v_y^2 + v(v_{xx}+v_{yy})\right]$$

$$= 2\left[u_x^2 + v_x^2 + u(0) + v_x^2 + u_x^2 + v(0)\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

$$= 4.|f'(z)|^2$$

3(a) Show that the function $u=\frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find f(z).

SOLUTION:

Given $u = \frac{1}{2}\log\left(x^2+y^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}.\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\left(x^2+y^2\right)(1)-2y^2}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2} = 0$$

Hence u is harmonic function

To find conjugate of u

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$$

$$\phi_1(z,o)=\frac{1}{z}$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\phi_2(z,o)=0$$

By Milne Thomson Methods

$$f'(z)=\phi_1(z,o)-i\phi_2(z,o)$$

$$\int f'(z)\,dz=\int\frac{1}{z}dz+0$$

$$=\log z+c$$

$$f(z)=\log\ re^{i\theta}$$

$$f(z)=u+iv=\log r+i\theta$$

$$u=\log r\ ,\ v=\theta$$

$$u=\log\sqrt{x^2+y^2}\qquad\left[\therefore r^2=x^2+y^2,\ \ \theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$v=\tan^{-1}\left(\frac{y}{x}\right)$ $\therefore$ Conjugate of u is $\tan^{-1}\left(\frac{y}{x}\right)$.

(b) Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y-\cos 2x}$

SOLUTION:

$$u=\frac{\sin 2x}{\cosh 2y-\cos 2x}$$

$$\phi_1(x,y)=\frac{\partial u}{\partial x}=\frac{(\cosh 2y-\cos 2x)(2\cos 2x)-\sin 2x(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

$$\phi_1(z,0)=\frac{(1-\cos 2z)(2\cos 2z)-2\sin^2 2z}{(1-\cos 2z)^2}$$

$$=\frac{(1-\cos 2z)(2\cos 2z)-2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$=\frac{(1-\cos 2z)(2\cos 2z)-2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$=\frac{-2}{1-\cos 2z}=-\frac{1}{\sin^2 z}=-\cos ec^2 z$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{(\cosh 2y-\cos 2x)(0)-\sin 2x(2\sinh 2y)}{(\cosh 2y-\cos 2x)^2}$$

$$\phi_2(z,0)=0$$

By Milne's Thomson method,

$$f(z)=\int\phi_1(z,0)\,dz-i\int\phi_2(z,0)\,dz$$

$$=\int-\operatorname{cosec}^2 z\,dz-i0$$

$$=\cot z+c$$

4(a) Find the regular function whose imaginary part is $e^{-x}(x\cos y+y\sin y)$

SOLUTION:

$$v=e^{-x}(x\cos y+y\sin y)$$

$$\phi_2(x,y)=\frac{\partial v}{\partial x}=e^{-x}[\cos y]+(x\cos y+y\sin y)(-e^{-x})$$

$$\phi_2(z,0)=e^{-z}+(z)(-e^{-z})=e^{-z}-ze^{-z}=e^{-z}(1-z)$$

$$\phi_1(x,y)=\frac{\partial u}{\partial y}=e^{-x}[-x\sin y+y\cos y+\sin y(1)]$$

$$\phi_1(z,0)=e^{-z}[0+0+0]=0$$

By Milne's Thomson Method

$$f(z)=\int\phi_1(z,0)\,dz+i\int\phi_2(z,0)\,dz$$

$$=\int 0\,dz+i\int(1-z)e^{-z}dz$$

$$=i\left[(1-z)\left[\frac{e^{-z}}{-1}\right]-(-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right]+C$$

$$=i\left[-(1-z)e^{-z}+e^{-z}\right]+C$$

$$=i\left[-e^{-z}+ze^{-z}+e^{-z}\right]+C=i\left[ze^{-z}\right]+C$$

(b) If $f(z)=u+iv$ is an analytic function and $u-v=e^{x}(\cos y-\sin y)$ find $f(z)$ interms of z

SOLUTION:

$$f(z)=u+iv \qquad (1)$$

$$if(z)=iu-v \qquad (2)$$

$$\therefore\ (1+i)f(z)=(u-v)+i(u+v)$$

$F(z)=U+iV$, where $F(z)=(1+i)f(z)$, $U=u-v$, $V=u+v$

$$\therefore U=u-v=e^{x}(\cos y-\sin y)$$

$$\phi_1(x,y)=\frac{\partial U}{\partial x}=e^{x}[\cos y-\sin y]$$

$$\phi_1(z,0)=e^{z}$$

$$\phi_2(x,y)=\frac{\partial U}{\partial y}=e^{x}[-\sin y-\cos y]$$

$\phi_2(z,0) = e^z(-1) = -e^z$

By Milne's Thomson Method

$$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int e^z dz - i\int -e^z dz = e^z + ie^z$$

$$= (1+i)e^z$$

$$(1+i)f(z) = (1+i)e^z + C_1$$

$$f(z) = e^z + C$$

5(a) Find the image of $|z-2i| = 2$ under the transformation $w = \frac{1}{z}$

SOLUTION:

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

Now $w = u + iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

i.e., $x+iy = \frac{u-iv}{u^2+v^2}$ $\therefore$ $x = \frac{u}{u^2+v^2}$..........(1) $y = \frac{-v}{u^2+v^2}$..........(2)

Given $|z-2i| = 2$

$|x+iy-2i| = 2 \Rightarrow |x+i(y-2)| = 2$

$x^2 + (y-2)^2 = 4 \Rightarrow x^2+y^2-4y = 0$..................(3)

Sub (1) and (2) in (3)

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left[\frac{-v}{u^2+v^2}\right] = 0$$

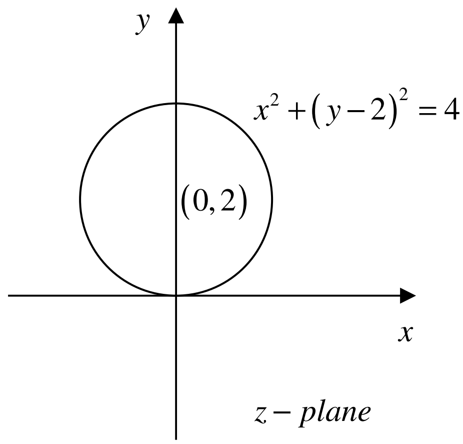
$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \left[\frac{4v}{u^2+v^2}\right] = 0$$

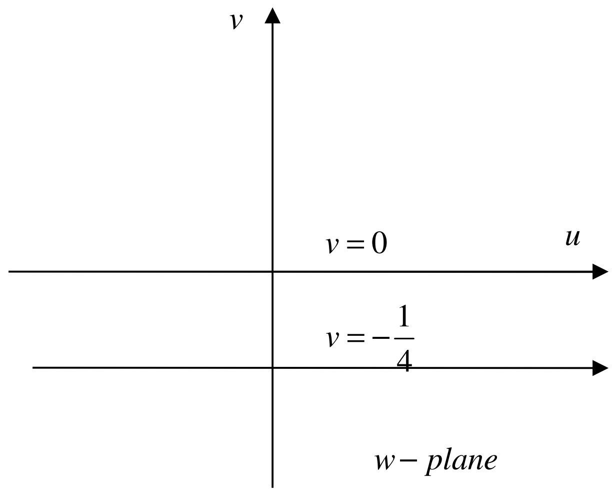
$$\frac{(u^2+v^2)+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\frac{(1+4v)(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$1+4v = 0 \Rightarrow v = -\frac{1}{4}$ $\quad (\therefore u^2+v^2 \neq 0)$

which is a straight line in w-plane.





(b) Find the image of the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ under the transformation $w=\frac{1}{z}$

SOLUTION:

$w=\frac{1}{z}\Rightarrow z=\frac{1}{w}$

$z=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}\Rightarrow x=\frac{u}{u^2+v^2}$........(1) $y=-\frac{v}{u^2+v^2}$........(2)

Given strip is $\frac{1}{4}<y<\frac{1}{2}$ when $y=\frac{1}{4}$

$\frac{1}{4}=-\frac{v}{u^2+v^2}$ $(by\ 2)$

$u^2+(v+2)^2=4$..........(3)

which is a circle whose centre is at $(0,-2)$ in the w-plane and radius 2.

When $y=\frac{1}{2}$

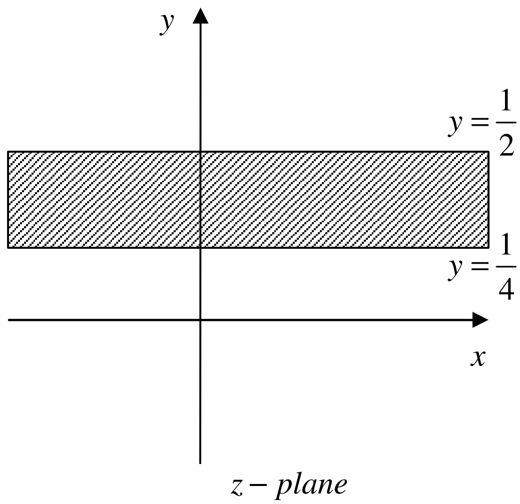
$\frac{1}{2}=\frac{-v}{u^2+v^2}$ $(by\ 2)$

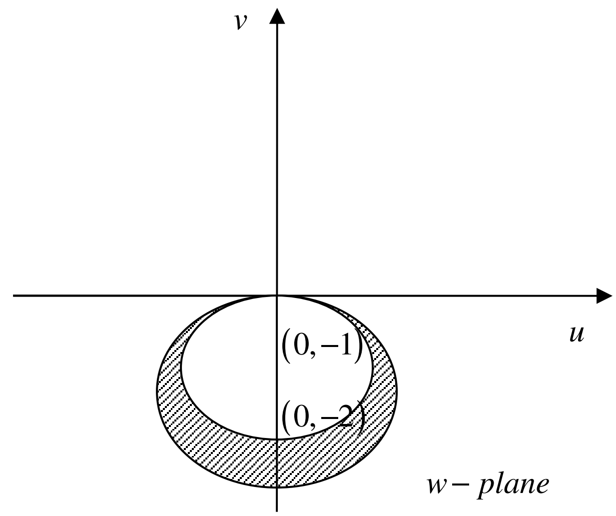
$u^2+v^2+2v=0$

$u^2+(v+1)^2=1$..........(4)

which is a circle whose centre is at $(0,-1)$ and radius is 1 in the w-plane.

Hence the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ is transformed into the region between circles $u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w-plane.





6(a) Find the image of the circle $|z-1|=1$ under the transformation $w=z^2$

SOLUTION:

In polar form $z=r\,e^{i\theta}$, $\quad w=\mathrm{R}\,e^{i\phi}$

Given

$$|z-1|=1$$
$$|re^{i\theta}-1|=1$$
$$|r\cos\theta+i\,r\sin\theta-1|=1$$
$$|(r\cos\theta-1)+i\,r\sin\theta|=1$$
$$(r\cos\theta-1)^2+(r\sin\theta)^2=1^2$$
$$r^2-2\,r\cos\theta=0$$
$$r=2\cos\theta----(1)$$

Now, we have

$$w=z^2$$
$$\mathrm{R}\,e^{i\phi}=\left(r\,e^{i\theta}\right)^2$$
$$\mathrm{R}\,e^{i\phi}=r^2\,e^{i2\theta}$$
$$\mathrm{R}=r^2,\qquad \phi=2\theta$$

$(1)\Rightarrow$

$$r^2=\left(2\cos\theta\right)^2$$
$$=4\cos^2\theta$$
$$=4\left[\frac{1+\cos 2\theta}{2}\right]$$
$$r^2=2(1+\cos 2\theta)$$
$$R=2(1+\cos\phi)$$

(b) Find the bilinear transformation of the points –1,0,1 in z- plane onto the points 0,*i*,3*i* in w-plane.

SOLUTION:

Given $z_1 = -1,\ w_1 = 0$

$z_2 = 0,\ w_2 = i$

$z_3 = i,\ w_3 = 3i$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z-(-1))(0-1)}{(z-1)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2wz - 2w = wz + w - 3iz - 3i$$

$$w(2z-2-z-1) = -3i(z+1)$$

$$w(z-3) = -3i(z+1)$$

$$\therefore w = -3i\frac{(z+1)}{(z-3)}$$

7(a) Find the bilinear transformation which maps the points $0,1,\infty$ in z-plane into itself in w-plane.

SOLUTION:

Given $z_1 = 0,\ w_1 = 0$

$z_2 = 1,\ w_2 = 1$

$z_3 = \infty,\ w_3 = \infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)\,z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)}=\frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$w=z$$

(b) Find the bilinear transformation which maps the points $z=\infty, i, 0$ into $w=0, i, \infty$ respectively.

SOLUTION:

Given $z_1=\infty,\ w_1=0$

$z_2=i,\ w_2=i$

$z_3=0,\ w_3=\infty$

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\,z_1\left(\frac{z_2}{z_1}-z_1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)}=\frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)}=\frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i}=\frac{i}{z}$$

$$w=\frac{i^2}{z}$$

$$\therefore w=-\frac{1}{z}$$

UNIT-V COMPLEX INTEGRATION

PART A

1. **State Cauchy's Integral formula for Complex Integration.**
2. **What is the value of $\int_C e^z dz$, where C is $|z|=1$?**
3. **Evaluate $\int_C \frac{\cos \pi z}{z-1}dz$ where C is $|z|=2$**
4. **Evaluate $\int_C \frac{e^{2z}}{\left(z^2+1\right)}dz$ where C is $|z|=\frac{1}{2}$**
5. **Obtain the Taylor's series expansion of log(1+z) when $|z|=0$**

6. **Obtain the Laurent expansion of the function $\frac{e^z}{z^2}$ in the neighbourhood of its singular point. Hence find the residue at that point.**

7. **Find the Singular points of** $f(z)=\dfrac{\sin z}{(z+1)(z-2)}$

8. **What is the Nature of the singularity at z=0 of the function** $\dfrac{\sin z - z}{z^3}$.

9. **Define essential singularity with an example.**

10. **Find the residue of the function** $f(z)=\dfrac{z^2}{(z-1)(z-2)^2}$ **at a simple pole.**

PART B

1(a) **Using Cauchy's integral formula, find** $\int_C \dfrac{z+4}{z^2+2z+5}\,dz$, **where C is** $|z+1-i|=2$

SOLUTION:

$|z+1-i|=2$

$|x+iy+1-i|=2$

$|x+1+i(y-1)|=2$

$\sqrt{(x+1)^2+(y-1)^2}=2$

Squaring on both sides,

$(x+1)^2+(y-1)^2=4$

This is equation of circle with centre $(-1,1)$ and radius 2.

$z^2+2z+5=0$

$$z=\frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

-1+2*i* lies inside the circle c.

-1-2*i* lies outside the circle c.

$a=-1+2i$

By Cauchy's integral formula, $f(a)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-a}\,dz$

Substituting for a, $f(-1+2i)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-(-1+2i)}\,dz$ ......(1)

Comparing equation (1) with given problem,

$$f(z)=\frac{z+4}{z-(-1-2i)}$$

$$f(-1+2i)=\frac{-1+2i+4}{-1+2i-(-1-2i)}=\frac{2i+3}{-1+2i+1+2i}=\frac{2i+3}{4i}$$

Substituting for $f(-1+2i)$ in (1)

$$\frac{2i+3}{4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}\,dz$$

Cross multiplying

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\frac{(2i+3)(2\pi i)}{4i}=\frac{\pi}{2}(3+2i)$$

(b) **Using Cauchy's integral formula, evaluate** $\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)\ (z-1)}dz$ **, where C is** $|z|=3$

SOLUTION:

We know that, Cauchy's integral formula is $f(a)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-a}\,dz$

$$(i.e)\ 2\pi i\, f(a)=\int_C \frac{f(z)}{z-a}\,dz$$

Given: $\displaystyle\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ Here, $f(z)=\sin\pi z^2+\cos\pi z^2$

The points $a_1=1, a_2=2$ lies inside $|z|=3$

Now, $\dfrac{1}{(z-1)(z-2)}=\dfrac{-1}{(z-1)}+\dfrac{1}{(z-2)}$ (by Partial fraction method)

$$\therefore \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=-\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}dz+\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}dz$$

$$=-2\pi i\, f(1)+2\pi i\, f(2)$$

$$\mathrm{f(z)}=\sin\pi \mathrm{z}^2+\cos\pi \mathrm{z}^2$$

$$\mathrm{f}(1)=\sin\pi+\cos\pi=-1 \text{ and } \mathrm{f}(2)=\sin 4\pi+\cos 4\pi=1$$

$$\therefore \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=-2\pi i(-1)+2\pi i(1)=4\pi i$$

2(a) **Using Cauchy's integral formula, evaluate** $\displaystyle\int_C \frac{1}{(z-2)\ (z+1)^2}dz$ **, where C is** $|z|=\dfrac{3}{2}$

SOLUTION:

Here $z=-1$ is a pole lies inside the circle

$z=2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{(z+1)^2(z-2)}=\int \frac{\frac{1}{z-2}}{(z+1)^2}\,dz$$

Here $\mathrm{f}(\mathrm{z}) = \dfrac{1}{z-2}$

$f'(z) = -\dfrac{1}{(z-2)^2}$

Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^n(a)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2}dz$$

$$= \frac{2\pi i}{1!}f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \quad \left(\because f'(z) = \frac{-1}{(z-2)^2}\right)$$

$$= 2\pi i\left[\frac{-1}{9}\right]$$

$$\int_C \frac{1}{(z-2)\ (z+1)^2}dz = \frac{-2}{9}\pi i.$$

(b) **Find the Taylor's series expansion of f(z) = $\dfrac{z}{(z+1)(z-3)}$, about z = 0**

SOLUTION:

Splitting f(z) into partial fractions, we have

$$\mathrm{f}(\mathrm{z}) = \frac{\mathrm{z}}{(\mathrm{z}+1)(\mathrm{z}-3)} = \frac{\mathrm{A}}{(\mathrm{z}+1)} + \frac{\mathrm{B}}{(\mathrm{z}-3)}$$

$\Rightarrow \mathrm{z} = \mathrm{A}(\mathrm{z}-3) + \mathrm{B}(\mathrm{z}+1)$

put $\mathrm{z} = -1$, we get

$\mathrm{A} = \dfrac{1}{4}$

put $\mathrm{z} = 3$, we get

$\mathrm{B} = \dfrac{3}{4}$

$$f(z) = \frac{1}{4}\left(\frac{1}{z+1}\right) + \frac{3}{4}\left(\frac{1}{z-3}\right) = \frac{1}{4}\left(\frac{1}{1+z}\right) + \frac{3}{4}\left(\frac{1}{-3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$= \frac{1}{4}\left[(1+z)^{-1} - \left(1-\frac{z}{3}\right)^{-1}\right]$$

$$=\frac{1}{4}\left[\left(1-z+z^2-....\right)-\left(1+\frac{z}{3}+\frac{z^2}{9}+...\right)\right]$$

$$=\frac{1}{4}\left[\left((-1)-\frac{1}{3}\right)z+\left((-1)^2-\left(\frac{1}{3}\right)^2\right)z^2+...\right]$$

$$\therefore f(z)=\frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n-\left(\frac{1}{3}\right)^n\right)z^n$$

3(a) Expand f(z) = $\frac{z^2-1}{z^2+5z+6}$ in a Laurent's series expansion for $|z|>3$ and $2<|z|<3$

SOLUTION:

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

Consider $\frac{-5z-7}{(z+3)(z+2)}$

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{A}{z+2}+\frac{B}{z+3}=\frac{A(z+3)+B(z+2)}{(z+3)(z+2)}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-2$ then $A=3$

Put $z=-3$ then $B=-8$

Substituting we get, $\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

(i) **Given** $|z|>3 \quad \Rightarrow \quad \frac{3}{|z|}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-...\right)-\frac{8}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-...\right)$$

(ii) **Given** $2<|z|<3 \quad \Rightarrow \quad \frac{2}{|z|}<1$ and $\frac{|z|}{3}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-...\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

(b) Obtain the Laurent's series expansion for the function f(z) = $\frac{4z}{\left(z^2-1\right)\left(z-4\right)}$ in

$|z-1|>4$ **and** $2<|z-1|<3$

SOLUTION:

Put $z-1=u \Rightarrow z=u+1$

Now, $f(z)=\frac{4z}{\left(z^2-1\right)(z-4)}=\frac{4z}{(z-1)(z+1)(z-4)}$

Hence $f(u)=\frac{4(u+1)}{u(u+2)(u-3)}$

$$\frac{4(u+1)}{u(u+2)(u-3)}=\frac{A}{u}+\frac{B}{u+2}+\frac{C}{u-3}=\frac{A(u+2)(u-3)+Bu(u-3)+Cu(u+2)}{u(u+2)(u-3)}$$

$$4(u+1)=A(u+2)(u-3)+Bu(u-3)+Cu(u+2)$$

Put $u=0$ then $A=\frac{-2}{3}$

Put $u=-2$ then $B=\frac{-2}{5}$

Put $u=3$ then $C=\frac{16}{15}$

$$f(u)=\frac{4(u+1)}{u(u+2)(u-3)}=\frac{-2/3}{u}+\frac{-2/5}{u+2}+\frac{16/15}{u-3}$$

(i) $|u|>4 \quad \Rightarrow \quad \frac{4}{|u|}<1$

$$f(u)=\frac{-2/3}{u}-\frac{2/5}{u+2}+\frac{16/15}{u-3}$$

$$f(u)=-\frac{2}{3}\left(\frac{1}{u}\right)-\frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right)+\frac{16}{15}\left(\frac{1}{u\left(1-\frac{3}{u}\right)}\right)$$

$$=-\frac{2}{3}\left(\frac{1}{u}\right)-\frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1}+\frac{16}{15}\left(\frac{1}{u}\right)\left(1-\frac{3}{u}\right)^{-1}$$

$$=\frac{1}{u}\left[-\frac{2}{3}-\frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right)+\frac{16}{15}\left(1+\frac{3}{u}+\frac{9}{u^2}+....\right)\right]$$

$$\therefore f(z)=\frac{1}{(z-1)}\left[-\frac{2}{3}-\frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right)+\frac{16}{15}\left(1+\frac{3}{(z-1)}+\frac{9}{(z-1)^2}+....\right)\right]$$

(ii) $2<|u|<3 \quad \Rightarrow \quad \frac{2}{|u|}<1$ and $\frac{|u|}{3}<1$

$$f(u)=-\frac{2}{3}\left(\frac{1}{u}\right)-\frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right)+\frac{16}{15}\left(\frac{1}{-3\left(1-\frac{u}{3}\right)}\right)$$

$$=-\frac{2}{3}\left(\frac{1}{u}\right)-\frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1}-\frac{16}{45}\left(1-\frac{u}{3}\right)^{-1}$$

$$=\frac{1}{u}\left[-\frac{2}{3}-\frac{2}{5}\left(1-\frac{2}{u}+\frac{4}{u^2}-....\right)-\frac{16}{45}\left(1+\frac{u}{3}+\frac{u^2}{9}+....\right)\right]$$

$$\therefore f(z)=\frac{1}{(z-1)}\left[-\frac{2}{3}-\frac{2}{5}\left(1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-....\right)-\frac{16}{45}\left(1+\frac{(z-1)}{3}+\frac{(z-1)^2}{9}+....\right)\right]$$

4(a) Using Cauchy's residue theorem evaluate $\int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz$ **, where C is** $|z|=2$

SOLUTION:

$|z|=2$ is the equation of the circle with centre at origin and radius 2.

$\left(z^2-1\right)(z-3)=0$

$\left(z^2-1\right)=0, \qquad (z-3)=0$

$z^2=1, \qquad z=3$

$z=\pm1, \qquad z=3$

$z=1,-1$ lies inside the circle and $z=3$ lies outside the circle

Residue at $z=1$ is

$$Lt_{z\to1}\left((z-1)f(z)\right)=Lt_{z\to1}\left((z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$=Lt_{z\to1}\left(\frac{3z^2+z-1}{(z+1)(z-3)}\right)$$

$$=-\frac{3}{4}$$

Similarly **Residue at** $z=-1$ is

$$Lt_{z\to-1}\left((z+1)f(z)\right)=Lt_{z\to-1}\left((z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to -1}\left(\frac{3z^2+z-1}{(z-1)(z-3)}\right)$$

$$= \frac{1}{8}$$

Residue at $z=3$ is **Zero**

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\{\text{Sum of Residues}\}$$

$$\therefore \int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz = 2\pi i\left(\frac{1}{8}-\frac{3}{4}+0\right) = -\frac{5\pi i}{4}$$

(b) **Evaluate** $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ **, where C is** $|z-i|=2$ **using Cauchy's residue theorem**

SOLUTION:

Let $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

poles of $f(z)$ are z = - 1 (pole of order 2) and z = 2 (simple pole)

Given: $|z-i|=2$

$$|x+iy-i| = 2 \Rightarrow |x+i(y-1)| = 2$$

Squaring on both sides $\sqrt{x^2+(y-1)^2} = 2 \Rightarrow x^2+(y-1)^2 = 4$

This is equation of circle with centre $(0,1)$ and radius 2

Hence, The pole $z=2$ lies outside C and $z=-1$ lies inside C

Therefore, **Residue of f(z) at** $z=2$ is **Zero**

Residue of f(z) at $z=-1$ is $Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left((z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right)$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left(\frac{(z-1)}{(z-2)}\right) = Lt_{z\to -1}\left(\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right)$$

$$= Lt_{z\to -1}\left(\frac{-1}{(z-2)^2}\right) = -\frac{1}{9}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\{\text{Sum of Residues}\}$$

$$\therefore \int_C \frac{(z-1)}{(z+1)^2(z-2)}dz = 2\pi i\left(0-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

5. **Evaluate** $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+4\sin\theta}$ **, using contour integration.**

SOLUTION:

Consider |z| = 1

Put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}, \quad \sin\theta = \frac{z^2-1}{2iz}$$

$$\int_0^{2\pi} \frac{d\theta}{13+4\sin\theta} = \int_C \frac{dz/iz}{13+4\left(\dfrac{z^2-1}{2iz}\right)} = \int_C \frac{dz/iz}{\dfrac{13iz+2z^2-2}{iz}} = \int_C \frac{dz}{2z^2+13iz-2}$$

The poles are at $2z^2+13iz-2=0$

$$z = \frac{-13i \pm \sqrt{-169-4(2)(-2)}}{2(2)} = \frac{-13i \pm 3i\sqrt{17}}{4}$$

The poles are at $\dfrac{-13\mathrm{i}+3\mathrm{i}\sqrt{17}}{4}$ and $\dfrac{-13\mathrm{i}-3\mathrm{i}\sqrt{17}}{4}$

$\dfrac{-13\mathrm{i}+3\mathrm{i}\sqrt{17}}{4}$ lies inside the circle and $\dfrac{-13\mathrm{i}-3\mathrm{i}\sqrt{17}}{4}$ lies outside the circle |z| = 1

<u>Residue at</u> $\dfrac{-13i+3i\sqrt{17}}{4}$:

$$Lt_{z \to \frac{-13i+3i\sqrt{17}}{4}} \left(z - \left(\frac{-13i+3i\sqrt{17}}{4}\right)\right) f(z) = Lt_{z \to \frac{-13i+3i\sqrt{17}}{4}} \left(z - \frac{-13i+3i\sqrt{17}}{4}\right) \frac{1}{2z^2+13iz-2}$$

$$= Lt_{z \to \frac{-13i+3i\sqrt{17}}{4}} \left(z - \left(\frac{-13i+3i\sqrt{17}}{4}\right)\right) \frac{1}{\left(z - \left(\dfrac{-13i+3i\sqrt{17}}{4}\right)\right)\left(z - \left(\dfrac{-13i-3i\sqrt{17}}{4}\right)\right)}$$

$$= Lt_{z \to \frac{-13i+3i\sqrt{17}}{4}} \frac{1}{\left(z - \left(\dfrac{-13i-3i\sqrt{17}}{4}\right)\right)} = \frac{2}{3i\sqrt{17}}$$

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{13+4\sin\theta} = 2\pi\mathrm{i}\{\text{Sum of Residues}\} = 2\pi i \times \frac{2}{3i\sqrt{17}} = \frac{4\pi}{3\sqrt{17}}$$

6(a) **Evaluate** $\displaystyle\int_0^{\infty} \frac{\cos ax\,dx}{x^2+1}$**,** $a>0$**, using contour integration.**

SOLUTION:

$$\int_0^{\infty} \frac{\cos ax\ dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax\ dx}{1+x^2}$$

Now $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax\ dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{\text{RP of } e^{iax}}{1+x^2} dx \qquad \{\because e^{i\theta} = \cos\theta + i\sin\theta\}$

Consider $\int_c f(z)\,dz = \text{R.P}\int_c \frac{e^{iaz}}{1+z^2}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [–R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

The poles of f(z) are at $1+z^2=0$

$z^2=-1 \Rightarrow z=\pm i$

The point $z=i$ lies inside the semi-circle and the point $z=-i$ lies outside the semi-circle

Residue at $z=i$ is given by

$$Lt_{z\to i}(z-i)f(z) = Lt_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$=Lt_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{ia(i)}}{i+i} = \frac{e^{ai^2}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy Residue theorem,

$$R.P\int_c \frac{e^{iaz}\,dz}{1+z^2} = \text{R.P of } 2\pi i\left(\frac{e^{-a}}{2i}\right) = \text{R.P of } \pi e^{-a} = \pi e^{-a}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \pi e^{-a}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$

$$\int_0^{\infty}\frac{\cos ax\,dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^2} = \frac{\pi e^{-a}}{2}$$

(b) Evaluate $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$, using contour integration.

SOLUTION:

Let $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$

Consider $\int_c f(z)\,dz = \int_c \frac{z^2-z+2}{z^4+10z^2+9}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

The poles f(z) are at $z^4+10z^2+9=0$

$\left(z^2+1\right)\left(z^2+9\right)=0$

$z^2=-1; \qquad z^2=-9$

$z=\pm i; \qquad z=\pm 3i$

The poles are at $3i,\ -3i,\ i,\ -i$

Here the poles $3i$ and i lie inside the semi-circle.

Residue at $z=3i$ is given by

$$Lt_{z\to 3i}\left(z-3i\right)f\left(z\right)=Lt_{z\to 3i}\left(z-3i\right)\frac{z^2-z+2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$=Lt_{z\to 3i}\left(z-3i\right)\frac{z^2-z+2}{\left(z-3i\right)\left(z+3i\right)\left(z^2+1\right)}$$

$$=Lt_{z\to 3i}\frac{z^2-z+2}{\left(z+3i\right)\left(z^2+1\right)}=\frac{7+3i}{48i}$$

Residue at $z=i$ is given by

$$Lt_{z\to i}\left(z-i\right)f\left(z\right)=Lt_{z\to i}\left(z-i\right)\frac{z^2-z+2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$=Lt_{z\to i}\left(z-i\right)\frac{z^2-z+2}{\left(z-i\right)\left(z+i\right)\left(z^2+9\right)}$$

$$=Lt_{z\to i}\frac{z^2-z+2}{\left(z+i\right)\left(z^2+9\right)}=\frac{1-i}{16i}$$

By Cauchy Residue theorem,

$$\int_c \frac{z^2-z+2}{z^4+10z^2+9}dz=2\pi i\left[\frac{7+3i}{48i}+\frac{1-i}{16i}\right]=2\pi i\left[\frac{7+3i+3-3i}{48i}\right]=2\pi i\left[\frac{10}{48i}\right]=\frac{5\pi}{12}$$

$$\therefore \int_{-R}^{R} f(x)\,dx+\int_{\Gamma} f(z)\,dz=\frac{5\pi}{12}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz\to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx=\dfrac{5\pi}{12}$

$$\therefore \int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx=\frac{5\pi}{12}$$