# UNIT 2 - SEQUENCES AND SERIES

## Sequence:

If every number 'n' of a series of natural numbers 1, 2, 3, 4, ... n is put into correspondence with a certain real number $x_n$, then the set of the numbered real quantities $x_1, x_2, \dots x_n, \dots$ is called a number sequence or simply a sequence.

Representation of Sequence: $\{x_n\} = \{x_1, x_2, \dots x_n, \dots\}$

or

$\{s_n\} = \{s_1, s_2, s_3 \dots s_n, \dots\}$

Examples: $\{1+(-1)^n\}$, $\{1/n\}$

* Constant Sequence: $\{3, 3, 3, \dots 3\}$
* Null Sequence: $\{0, 0, \dots 0, 0, \dots\}$
* Infinite Sequence: It is a sequence in which the number of terms is infinite and is denoted by $\{s_n\}_{n=1}^{\infty}$

Operation on Sequences: - If $\{s_n\}$, $\{t_n\}$ are sequences then

* Sum of Sequence is $\{s_n + t_n\} = \{s_n\} + \{t_n\}$
* Product of Sequence is $\{s_n \cdot t_n\} = \{s_n\}\{t_n\}$
* If $k \in R$, then $k\{s_n\} = \{ks_n\}$
* $\left\{\frac{1}{s_n}\right\}$ is called the reciprocal sequence of $\{s_n\}$
* $\left\{\frac{s_n}{t_n}\right\}$ is the quotient of sequences. $(t_n \neq 0)$

## Bounded Sequence:

A sequence $\{s_n\}$ is said to be bounded if there exist numbers $m$ and $M$ such that

$$m < a_n < M \text{ for every } n.$$

Otherwise it is said to be unbounded.

**Example:** $\{1/n\}$ is bounded

$\{2^n\}$ is unbounded.

## Monotonic Sequence:

A sequence $\{s_n\}$ is said to be

(i) Monotonically increasing if $s_{n+1} \geq s_n$ for every $n$.

(ii) Monotonically decreasing if $s_{n+1} \leq s_n$ for every $n$.

(iii) Monotonic if it is either monotonically increasing or monotonically decreasing.

## Limit of a Sequence:

Let $\{s_n\}$ be a sequence. '$l$' is said to be limit of the sequence $\{s_n\}$, if to each $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|s_n - l| < \epsilon$, $\forall n \geq m$

i.e., $\lim_{n \to \infty} s_n = l$

## Convergence, Divergence and oscillation of a sequence:

* A sequence $\{S_n\}$ is said to be convergent if it has a finite limit.

i.e., $\lim_{n\to\infty} S_n = l$. Example: $\{\frac{1}{n^2}\}$

* If $\lim_{n\to\infty} S_n = \infty$, $\{S_n\}$ is divergent. Example: $\{n\}$

* If $\lim_{n\to\infty} S_n$ is not unique (oscillates finitely) or $\pm\infty$ (oscillates infinitely) then $\{S_n\}$ is oscillatory sequence. Example: $\{(-1)^n . n^2\}$.

**To find the nature of the sequence whose $n^{th}$ term is $a_n$:-**

(1) $a_n = \dfrac{2n+1}{1-3n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{1-3n} = \lim_{n\to\infty} \frac{\cancel{n}[2+1/n]}{\cancel{n}[1/n-3]}$$

$$= \lim_{n\to\infty} \frac{2+1/n}{1/n-3}$$

$$= \frac{2+1/\infty}{1/\infty-3}$$

$$= \frac{2+0}{0-3}$$

$$= -2/3 \quad (\text{finite})$$

$\therefore \{a_n\}$ is convergent.

(ii) $a_n = e^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^n$$
$$= e^{\infty}$$
$$= \infty$$

$\therefore \{a_n\}$ is divergent.

(iii) $a_n = e^{-n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{-n}$$
$$= e^{-\infty}$$
$$= 0 \quad \text{(finite)}$$

$\therefore \{a_n\}$ is convergent.

(iv) $a_n = 2 + (-1)^n$. putting $n = 1, 2, 3, \ldots$

$\{a_n\} = \{1, 3, 1, 3, \ldots\}$

i.e., $\lim_{n\to\infty} a_{2n} = 3$ & $\lim_{n\to\infty} a_{2n-1} = 1$

Sequence oscillates finitely since it has more than one finite limits.

<u>Numerical Series:</u>

Suppose we have an infinite sequence of numbers $u_1, u_2, u_3, \ldots u_n, \ldots$, the expression

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is called a numerical series.

**Finite Series:** If the number of terms are finite, then the series is called as finite series.

**Infinite Series:** If the number of terms are infinite, then the series is called an infinite series.

**Definition of $n^{th}$ partial sum of series: ($S_n$)**

The sum of a finite number of terms of a series is called the $n^{th}$ partial sum of the series.

i.e., $S_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum u_n$

**Convergence, Divergence and oscillation of Series.**

* If $\lim_{n\to\infty} S_n = S$ (finite), then the series $\sum u_n$ converges.
* If $\lim_{n\to\infty} S_n = \pm\infty$, then the series $\sum u_n$ diverges.
* If $\lim_{n\to\infty} S_n \to$ more than one limit (or) $\pm\infty$, the $\sum u_n$ is said to be oscillatory or non-convergent.

**Example :-**

1). Test the convergence of the series $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots$

Solution:-

Let $S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \left(\frac{1}{3}\right)^{n-1}$

$$= \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{3^n}}{\frac{2}{3}}$$

$$= \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2}\left(\frac{1}{3^{n-1}}\right)\right] = \frac{3}{2}-\frac{1}{2}\left(\frac{1}{3^{\infty-1}}\right) = \frac{3}{2}-\frac{1}{2}\cdot\frac{1}{\infty}$$

$$= \frac{3}{2} \quad (\text{finite})$$

Hence the given series is convergent & the sum of the series is $3/2$.

(2) Test the convergence of $1+2+3+\cdots\infty$

Solution:-

Let $S_n = 1+2+3+\cdots n$

$$= \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n(n+1)}{2} = \frac{\infty(\infty+1)}{2} \to \infty$$

Hence the given series is divergent.

(3) Test the convergence of the series $7-4-3+7-4+3+\cdots$

Solution:- $S_1 = 7$, $S_2 = 7-4 = 3$, $S_3 = 7-4-3 = 0$, $S_4 = 7-4-3+7 = 7$ ----

$$\lim_{n\to\infty} S_n = 0 \text{ or } 7 \text{ or } 3$$

Since the limit is not unique, the series oscillates (finitely)

Properties of Series:

* The convergence of a series is not affected by the suppression of a finite number of its terms.

* If a series $u_1+u_2+u_3+\cdots$ converges to $s$ then the series $cu_1+cu_2+cu_3+\cdots$ converges to $cs$.
* If the series $u_1+u_2+u_3+\cdots$ converges to $s_1$, and $v_1+v_2+v_3+\cdots$ converges to $s_2$ then the series $(u_1+v_1)+(u_2+v_2)+(u_3+v_3)+\cdots$ converges to $s_1+s_2$.

Necessary condition for convergence of a Series:

If a series converges, its $n^{th}$ term approaches zero as $n$ tends to infinity.

i.e., $\lim_{n\to\infty} u_n = 0$.

Example: Consider $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\cdots$

Here $u_n = \frac{n}{2n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(2+1/n)}$$

$$= \lim_{n\to\infty} \frac{1}{2+1/n}$$

$$= \frac{1}{2+1/\infty}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2} \neq 0.$$

$\therefore$ The given series is divergent.

## Series with positive terms:-

Consider the series $u_1 + u_2 + u_3 + \cdots \infty$.
If all terms of the above series are positive then the series is called an infinite series with positive terms. Ex: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

## Alternating Series:-

An infinite series whose terms are alternately positive and negative is called an alternating series. Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

## Available Tests for convergence:-

Series with positive terms
- Comparison Test
- Integral Test
- D' Alembert's ratio test.

Alternating Series.
- Leibnitz's Test

## Comparison Test: (only for positive term series)

Consider the two +ve term series.

$\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots$ (given Series)

$\sum v_n = v_1 + v_2 + \cdots + v_n + \cdots$ (Auxiliary Series)

* If $u_n \leq v_n$ for every $n$ (or) $u_n \geq v_n$ for every $n$ then, "if $\Sigma v_n$ converges then $\Sigma u_n$ also converges".

**Limit comparison test:**

If $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite quantity, then $\Sigma u_n$ and $\Sigma v_n$ either both converge or both diverge together.

**Note:**

* The geometric series $1+x+x^2+\cdots$ converges if $x<1$ & diverges if $x\geq 1$.

* The K-series $\frac{1}{1^K}+\frac{1}{2^K}+\frac{1}{3^K}+\cdots$ converges if $K>1$ and diverges if $K\leq 1$.

(i.e.) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty$ converges.

$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\infty$ converges

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But $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\infty$ diverges

$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots\infty$ diverges.

**How to choose $v_n$=?**

If $u_n = \frac{f(n)}{g(n)}$

then $v_n = \frac{\text{highest power term of } n \text{ in the Numerator}}{\text{highest power term of } n \text{ in the Denominator}}$.

Example :

$$u_n = \frac{2n+3}{n(n+1)(n+3)} = \frac{2n+3}{n^3+4n^2+3n}$$

$$v_n = \frac{n}{n^3} = 1/n^2 .$$

Problems on comparison Test.

1) Test the convergence of the series

$$1 + 1/2^2 + 1/3^3 + \cdots + 1/n^n + \cdots \infty$$

Solution :

Let $\sum u_n = 1 + 1/2^2 + 1/3^3 + \cdots + 1/n^n + \cdots \infty$

Let the auxiliary series be

$$\sum v_n = 1 + 1/2^2 + 1/2^3 + 1/2^4 + \cdots + 1/2^n + \cdots \infty$$

$u_1 = 1$, $u_2 = 1/2^2$, $u_3 = 1/3^3$, $u_4 = 1/4^4$, ---

$v_1 = 1$, $v_2 = 1/2^2$, $v_3 = 1/2^3$, $v_4 = 1/2^4$, ---

Cleary $u_1 = v_1$, $u_2 = v_2$, $u_3 \leq v_3$ $u_4 < v_4$ ---

(ie) clearly $u_n \leq v_n$ for every $n$.

∴ we can apply comparison test.

Now $\sum v_n = 1 + 1/2^2 + 1/2^3 + \cdots$

W.K.T. $1 + x + x^2 + \cdots$ is convergent if $x < 1$.

Here $x = 1/2 < 1$. ∴ $\sum v_n$ Converges.

∴ By comparison test, $\sum u_n$ also converges.

(2) Test the convergence of the series

$1 + 1/\sqrt{2} + 1/\sqrt{3} + \dots + 1/\sqrt{n} + \dots \infty$

Solution:

Let $\sum u_n = 1 + 1/\sqrt{2} + 1/\sqrt{3} + \dots$

consider the auxiliary series,

$\sum v_n = 1 + 1/2 + 1/3 + \dots$

$1 = 1$

$\frac{1}{\sqrt{2}} > 1/2$

$\frac{1}{\sqrt{3}} > 1/3$

$\dots$

$u_n \geq v_n$

Clearly $u_n \geq v_n$ for every $n$.

$\therefore$ We can apply comparison test

Now $\sum v_n = 1 + 1/2 + 1/3 + \dots$ is divergent.

$\therefore$ By comparison test, $\sum u_n$ is also a divergent series.

3) Test the convergence of the series

$\frac{1}{1\cdot 2} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \dots$

Solution:

$1, 2, 3 \dots n$

$1, 3, 5 \dots 2n-1$

$2, 5, 7 \dots 2n+1$

Let $u_n = \frac{n}{(2n-1)(2n+1)}$

$v_n = \frac{n}{n^2}$

$v_n = 1/n$

Cleary $u_n > v_n$

Also $\Sigma v_n = \Sigma 1/n = 1 + 1/2 + 1/3 + \cdots$ is a divergent series.

$\therefore$ By comparison test, $\Sigma u_n$ also diverges.

4). Test the convergence of the series $\sum_{n=1}^{\infty} \left[\sqrt{n^4+1} - n^2\right]$

Solution :

$$\text{Here } u_n = \sqrt{n^4+1} - n^2$$

$$= \left(\sqrt{n^4+1} - n^2\right) \frac{\left(\sqrt{n^4+1} + n^2\right)}{\sqrt{n^4+1} + n^2}$$

$$= \frac{\sqrt{(n^4+1)}^2 - (n^2)^2}{\sqrt{n^4+1} + n^2}$$

$$= \frac{(n^4+1) - n^4}{\sqrt{n^4+1} + n^2}$$

$$= \frac{1}{\sqrt{n^4+1} + n^2}$$

$$\text{Let } v_n = \frac{1}{n^2}$$

$$\left(\because \frac{1}{\sqrt{n^4} + n^2} = \frac{1}{n^2 + n^2} = \frac{1}{2n^2}\right)$$

W.K.T. $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^4+1}+n^2}}{1/n^2} = \lim_{n\to\infty} \frac{1}{\sqrt{n^4+1}+n^2} \times \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4(1+1/n^4)} + n^2} = \lim_{n\to\infty} \frac{n^2}{n^2\sqrt{1+1/n^4} + n^2} \Rightarrow$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{1+1/n^4}+1}$$

$$= \frac{1}{\sqrt{1+\frac{1}{\infty^4}}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2} \text{ (finite)}$$

$\because$ $\sum v_n$ is convergent & $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite,

by Limit comparison test, $\sum u_n$ is also convergent.

<u>Integral Test :</u> (only for +ve term series)

Let $\sum u_n = u_1 + u_2 + \ldots + u_n + \ldots$ be a series

with positive and decreasing terms.

$$u_1 \geq u_2 \geq u_3 \geq \ldots$$

Let $f$ be a non-negative decreasing function

in $[1, \infty)$ such that $f(1) = u_1$, $f(2) = u_2$, $f(3) = u_3 \ldots f(n) = u_n$

Then the improper integral

$$\int_1^{\infty} f(x)\,dx \text{ and the series } \sum_{n=1}^{\infty} u_n$$

are both finite (in this case $\sum u_n$ is convergent) or

both infinite (in this case $\sum u_n$ is divergent)

## Problems on Integral Test.

1) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

**Solution:**

Let $f(x) = \frac{1}{x^2+1}$

$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) > 0$ and $f(x)$ is decreasing in $[1, \infty)$

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{dx}{x^2+1}$$

$$= \left[\tan^{-1} x\right]_1^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}(1)$$

$$= \pi/2 - \pi/4$$

$$= \pi/4 \text{ (finite)}$$

So $\int_1^{\infty} f(x)\, dx$ converges to $\pi/4$.

$\therefore$ By Integral test, $\sum f(n)$ also converges.

2) Using integral test, test the convergence of the series $1 + 1/3 + 1/5 + \cdots + \frac{1}{2n-1} + \cdots$

**Solution:**

Let $f(x) = \frac{1}{2x-1}$

$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{2n-1}$

$f(x) > 0$ and $f(x)$ is decreasing in $[1, \infty)$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{2x-1}\,dx$$

put $2x-1=t$
$2\,dx = dt$
$x=1, t=1,$
$x=\infty, t=\infty$

$$= \frac{1}{2}\int_1^\infty \frac{dt}{t}$$

$$= \frac{1}{2}\left[\log t\right]_1^\infty$$

$$= \frac{1}{2}\left[\log\infty - \log 1\right]$$

$$= \infty$$

$\therefore \int_1^\infty f(x)\,dx$ diverges in $[1,\infty)$.

By integral test, $\sum_{n=1}^{\infty} f(n)$ also diverges.

## D'Alembert's Ratio Test:

In a series with positive terms

$u_1+u_2+u_3+ \cdots + u_n + \cdots$

if (i) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, then the series is convergent.

(ii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$, then the series is divergent.

(iii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the test fails and in this

we can use comparison test.

Examples:-

1) Test the convergence of the series $\sum \frac{n!\,2^n}{n^n}$

Solution:

Here $u_n = \frac{n!\,2^n}{n^n}$

$$\therefore u_{n+1} = \frac{(n+1)!\,2^{n+1}}{[(n+1)]^{n+1}} = \frac{(n+1)\,n!\,2^n\,2^1}{(n+1)^n\,(n+1)}$$

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{(n+1)!\, 2^{n+1}}{(n+1)^{n+1}}}{\dfrac{n!\, 2^n}{n^n}} = \frac{(n+1)!\, 2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!\, 2^n}$$

$$= \frac{\cancel{(n+1)}\, \cancel{n!}\, \cancel{2^n} . 2^1}{(n+1)^n . \cancel{(n+1)}^1} \times \frac{n^n}{\cancel{n!}\, \cancel{2^n}}$$

$$= 2\left[\frac{(n\cancel{+1})}{(n+1)}\right]^n$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} 2\left(\frac{n\cancel{+1}}{n+1}\right)^n$$

$$= \lim_{n\to\infty} 2\left[\frac{1}{1+1/n}\right]^n$$

$$= 2(1/e) \qquad \left(\because \lim_{n\to\infty}\left(1+1/n\right)^n = e\right]$$

$$= \frac{2}{e} < 1$$

$\therefore$ The series is convergent.

2) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)} x^n$.

Solution:

Let $u_n = \frac{n+1}{n(n+2)} x^n$

$$\therefore \quad u_{n+1} = \frac{n+1+1}{(n+1)(n+1+2)} x^{n+1} = \frac{(n+2)}{(n+1)(n+3)} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)}{(n+1)(n+3)} x^{n+1} \Big/ \frac{(n+1)}{n(n+2)} x^n$$

$$= \frac{(n+2)\cancel{x^n} . x^1}{(n+1)(n+3)} . \frac{n(n+2)}{(n+1)\cancel{x^n}}$$

$$= \frac{n(n+2)}{(n+1)(n+3)} x .$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n\,(n+2)}{(n+1)\,(n+3)}\, x$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\cdot n\,(1+2/n)}{\cancel{n}(1+1/n)\cdot n\,(1+3/n)}\, x$$

$$= \frac{1+2/\infty}{(1+1/\infty)(1+3/\infty)}\, x$$

$$= \frac{1+0}{(1+0)\,(1+0)}\, x$$

$$= x.$$

$\therefore$ By D'Alembert's Test,

$\sum u_n$ is convergent if $x<1$

$\sum u_n$ is divergent if $x>1$

Test fails, if $x=1$

$\therefore$ If $x=1$, $\quad u_n = \sum \frac{n+1}{n\,(n+2)}\,(1)^n$

$$u_n = \sum \frac{n+1}{n\,(n+2)}$$

Let $v_n = \frac{n}{n^2} = 1/n.$

$\sum v_n = \sum 1/n = 1 + 1/2 + 1/3 + \dots$ is a divergent series.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n+1}{n\,(n+2)} \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\cdot n(1+1/n)}{\cancel{n}\cdot \cancel{n}(1+2/n)}$$

$$= \frac{1+1/\infty}{1+2/\infty}$$

$$= 1 \quad (\text{finite } \& \neq 0)$$

$\therefore$ By Limit Comparison Test,

$\Sigma u_n$ is also divergent.

**Leibnitz theorem** (for Alternating Series):-

If in the alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$ the terms are such that

(i) $u_1 > u_2 > u_3 > \cdots$ (Numerically)

(ie) $u_n > u_{n+1}$ (or) $u_n - u_{n+1} > 0$

and (ii) $\lim_{n\to\infty} u_n = 0$

then the given series is convergent.

**Note:-** If $\lim_{n\to\infty} u_n \neq 0$, then the series is oscillatory.

$\lim_{n\to\infty} x^n = 0$ if $x < 1$

**Examples:-**

1) Show that $1 - 1/2 + 1/3 - 1/4 + \cdots$ is a convergent series.

**Solution:-**

The terms of the given series are alternately positive and negative.

Clearly (1) $1 > 1/2 > 1/3 > \cdots$ (Numerically)

(2) $u_n = 1/n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} 1/n = 0$$

Hence by Leibnitz rule, the given series is convergent.

2) Test the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots \quad (0<x<1)$$

Solution:

The terms of the series are alternatively +ve and −ve.

Let $u_n = \dfrac{x^n}{1+x^n}$

$u_{n+1} = \dfrac{x^{n+1}}{1+x^{n+1}}$

$$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= x^n\left[\frac{1}{1+x^n} - \frac{x}{1+x^{n+1}}\right]$$

$$= x^n\left[\frac{1+x^{n+1} - x(1+x^n)}{1+x^{n+1}}\right]$$

$$= x^n\left\{\frac{1+\cancel{x^{n+1}} - x - \cancel{x^{n+1}}}{1+x^{n+1}}\right\}$$

$$u_n - u_{n+1} = \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0$$

$\left[ \because 0 < x < 1, \right.$ let $x = 0.1$

$\left( \frac{(0.1)^n (0.9)}{1+0.1^n)(1+0.1)^{n+1}} \right)$

$\Rightarrow u_n - u_{n+1} > 0$ [when $x$ is +ve & less than 1, we have $1-x<1$]

$\Rightarrow u_n > u_{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0$$

$\therefore$ By Leibnitz's rule, the given series is convergent.

## Absolute convergence and Conditional Convergence.

### Absolute convergence:

The alternating series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

Ex: $1 - \frac{1}{2!} + \frac{1}{3!} - \cdots$ is absolutely convergent.

### Conditional convergence:

If $\sum u_n$ is convergent while $\sum |u_n|$ is divergent then $\sum u_n$ is said to be conditionally convergent.

Ex: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is conditionally convergent.

**Theorem:-**

(1). Every absolutely convergent series is necessarily a convergent series. But the converse is not true.

(2). Any convergent series of positive terms is also absolutely convergent.

**Examples:-**

(1) Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}}$ is absolutely convergent.

**Solution:-**

Given Series : $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$

Here $u_n = \frac{1}{2^{n-1}}$

$u_1 = 1$
$u_2 = \frac{1}{2}$
$u_3 = \frac{1}{2^2}$
$\cdots$

$1 > \frac{1}{2} > \frac{1}{2^2} > \cdots$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$

By Leibnitz test, $\sum (-1)^{n-1} u_n$ is Convergent.

But $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| (-1)^{n-1} \cdot \frac{1}{2^{n-1}} \right|$

$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

which is a geometric series with common ratio $\frac{1}{2}$ $(<1)$ $\therefore \sum |u_n|$ converges.

(2) Show that $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$ is conditionally convergent.

Solution:-

Let $u_n = \frac{1}{\sqrt{n}}$

(i) Clearly $1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \cdots$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\therefore$ By Leibnitz test, given series is convergent.

But $\sum |u_n| = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots$

$= \sum \frac{1}{\sqrt{n}}$

It is a k-series with $k=\frac{1}{2}<1$

$\therefore \sum \frac{1}{\sqrt{n}}$ is divergent.

Hence $\sum u_n$ is convergent, but

$\sum |u_n|$ is divergent.

$\therefore$ The given series is conditionally convergent.