Unit-V
COMPLEX INTEGRATION

Cauchy's Integral Theorem

If f(z) is analytic inside and on a simple closed curve C then $\int_C f(z)\,dz$ **= 0**

1. Cauchy's Integral Formula

If f(z) is analytic inside and on a simple closed curve C and 'a' is any point inside C then

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\ f(a)$$

2. Cauchy's Integral Formula for derivatives

(i) $\displaystyle\int_C \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a)$ **(ii)** $\displaystyle\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$

(iii) $\displaystyle\int_C \frac{f(z)}{(z-a)^4}\,dz = \frac{2\pi i}{3!} f'''(a)$ **(iv)** $\displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$

3. Taylor's expansion:

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \ldots\ldots$$

4. Some important expansions:

i) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots.. ,\ |z| < 1$

ii) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \ldots.. ,\ |z| < 1$

iii) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \ldots..,\ |z| < 1$

iv) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \ldots..,\ |z| < 1$

v) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots..$ **vi)** $\cos z = 1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots..$

vii) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots..$ **viii)** $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \ldots..$

5. A point at which f(z) = 0 is called zero of f(z)

6. A point at which f(z) is not analytic is called singular point.

7. Residue of f(z) at z = a is the coefficient of $\frac{1}{z-a}$ **in the Laurent's series expansion of f(z) about z =a.**

8. **Residue of f(z) at a simple pole z =a :**

$$\left[\operatorname{Res} of \ f(z)\right]_{z=a} = \lim_{z\to a} (z-a) \ f(z)$$

9. **Residue of f(z) at a pole z =a of order m :**

$$\left[\operatorname{Res} of \ f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m \ f(z) \right\} \right]$$

10. **If f(z) =** $\dfrac{\phi(z)}{\psi(z)}$ **then Residue of f(z) at a simple pole z =a is** $\dfrac{\phi(a)}{\psi\,'(a)}$ **with** $\phi(a) \neq$ **0 and**
$\psi\ (a) = 0$

11. **Cauchy Residue Theorem: If f(z) is analytic within and on a simple closed curve C except at a finite number of poles then** $\int_C f(z)\,dz = 2\pi i \{R_1 + R_2 + \ldots + R_n\}$ **where** $R_1, R_2, \ldots, R_n$ **are the residues of f(z) at its poles lying inside C.**

Contour Integration:

Type I : Integration around Unit circle $|z| = 1$

Integration of the Form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ **where *f* is a rational function**

Put $z = e^{i\theta}$**,** $d\theta = \dfrac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z} \ and \ \sin\theta = \frac{z^2-1}{2iz}$$

Type II: Integration around the upper semi circle of $|z| = R$

Integration of the Form $\int_{-\infty}^{\infty} f(x)\ dx$ **where** $f(x) = \dfrac{P(x)}{Q(x)}$

Use $\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \int_C f(z)\,dz$

By Jordan's lemma $\int_{\Gamma} f(z)\,dz \to 0$ **as R**$\to \infty$ $\quad \therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{C} f(z)\,dz$

Note: $\int_{0}^{\infty} f(x)\ dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\ dx$ **, if f(x) is even.**

Type III : Integration around the upper semi circle of $|z| = R$

Integration of the Form $\int_{-\infty}^{\infty} f(x)\ dx$ **where** $f(x)=\frac{P(x)}{Q(x)}\cos mx$ **or** $\frac{P(x)}{Q(x)}\sin mx$

Use $\cos mx = R.P\ of\ e^{imx}$ **and** $\sin mx = I.P\ of\ e^{imx}$