Unit - IV

Differential Calculus of Several Variables

PARTIAL DIFFERENTIATION

1). If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Given:-

$$u = (x-y)^4 + (y-z)^4 + (z-x)^4$$

$$\frac{\partial u}{\partial x} = 4(x-y)^3 + 4(z-x)^3(-1) \quad (1)$$

$$\frac{\partial u}{\partial y} = 4(x-y)^3(-1) + 4(y-z)^3 \quad (2)$$

$$\frac{\partial u}{\partial z} = 4(y-z)^3(-1) + 4(z-x)^3 \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 4\left[(x-y)^3 - (z-x)^3 - (x-y)^3 + (y-z)^3 - (y-z)^3 + (z-x)^3\right]$$

$$= 0$$

2). If $u = \log(x^2+y^2+z^2)$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2+y^2+z^2}$

Given:-

$$u = \log(x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x)$$

$$= \frac{2x}{(x^2+y^2+z^2)}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\frac{(x^2+y^2+z^2)(1)-(x)(2x)}{(x^2+y^2+z^2)^2}\right]$$

$$= 2\left[\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}\right] \text{ ——— } (1)$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = 2\frac{[x^2+z^2-y^2]}{(x^2+y^2+z^2)^2} \text{ ——— } (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 2\frac{[x^2+y^2-z^2]}{(x^2+y^2+z^2)^2} \text{ ——— } (3)$$

Adding (1), (2), (3) we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{2[y^2+z^2-\cancel{x^2}+\cancel{x^2}+\cancel{z^2}-\cancel{y^2}+x^2+\cancel{y^2}-\cancel{z^2}]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2\left[x^2+y^2+z^2\right]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2}{x^2+y^2+z^2} \quad //$$

3). If $r^2 = x^2+y^2$ then show that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

Given:-

$$r^2 = x^2+y^2 \quad \text{——— ①}$$

Differentiating ① partially w.r.t 'x' we get

$$\therefore 2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \cdots (A)$$

Now, $$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \frac{r \cdot 1 - x\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x \cdot \frac{x}{r}}{r^2} \quad [\text{Using}(A)]$$

$$= \frac{r^2-x^2}{r^3} \quad ②$$

Similarly $$\frac{\partial^2 r}{\partial y^2} = \frac{r^2 - y^2}{r^3} \qquad (3)$$

$$(2)+(3) \Rightarrow \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3}$$

$$= \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{2r^2 - r^2}{r^3} \qquad \left[\because x^2 + y^2 = r^2\right]$$

$$= \frac{1}{r} \qquad \ldots \qquad (A)$$

Now $$\left(\frac{\partial r}{\partial x}\right)^2 = \frac{x^2}{r^2}$$

$$\left(\frac{\partial r}{\partial y}\right)^2 = \frac{y^2}{r^2}$$

$$\therefore \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{x^2}{r^2}$$

$$= \frac{x^2 + y^2}{r^2}$$

$$= \frac{r^2}{r^2}$$

$$= 1$$

$$\therefore \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r} \qquad \ldots \qquad (B)$$

From (A) and (B).

$$\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right) = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

3) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

Given:-

$$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} \quad \text{--- (1)}.$$

$\cos u$ is the homogeneous function in x and y of degree $1/2$.

By Euler's Theorem. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$$x\frac{\partial(\cos u)}{\partial x} + y\frac{\partial(\cos u)}{\partial y} = \frac{1}{2}\cos u.$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u.$$

$$-\sin u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\frac{\cos u}{\sin u}.$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\cot u}$$

2 (i). If $\log u = \dfrac{x^3+y^3}{3x+4y}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} =$

$2u \log u$

$$\log u = \frac{x^3+y^3}{3x+4y}$$

$$= \frac{x^3 \left[1+(y/x)^3\right]}{x\left[3+4(y/x)\right]}$$

$$= \cancel{x^3} \; x^2 f(y/x)$$

$\therefore$ $\log u$ is a homogeneous function of 2.

By Euler's Theorem.

$$x\frac{\partial(\log u)}{\partial x} + y\frac{\partial(\log u)}{\partial y} = 2\log u.$$

$$x\cdot\frac{1}{u}\cdot\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\cdot\frac{\partial u}{\partial y} = 2\log u$$

$$\frac{1}{u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\log u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$$

3). If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ then prove $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$= \frac{x^3\left[1+(y/x)^3\right]}{x\left[1-(y/x)\right]}$$

$$= x^2 f(y/x)$$

$\therefore$ tan u is a homogeneous function of degree 2.

By euler Theorem

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2\tan u.$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= \frac{2\sin u}{\cos u} \times \cos^2 u$$

$$= 2\sin u\cos u$$

$$= \sin 2u$$

4). Given $u(x,y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(y/x)$ find

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}.$$

$u(x,y)$ is a homogeneous function of degree 2.

By euler extension Theorem

$$x^2 u_{xx} + 2xy.u_{xy} + y^2 u_{yy} = n(n-1)u$$
$$= 2(2-1)u \quad (n=2$$
$$= 2u$$

5). If $u = f(x-y, y-z, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let

$x - y = x_1$
$y - z = x_2$
$z - x = x_3$

$x_1 = x - y$	$x_2 = y - z$	$x_3 = z - x$
$\frac{\partial x_1}{\partial x} = 1$	$\frac{\partial x_2}{\partial x} = 0$	$\frac{\partial x_3}{\partial x} = -1$
$\frac{\partial x_1}{\partial y} = -1$	$\frac{\partial x_2}{\partial y} = 1$	$\frac{\partial x_3}{\partial y} = 0$
$\frac{\partial x_1}{\partial z} = 0$	$\frac{\partial x_2}{\partial z} = -1$	$\frac{\partial x_3}{\partial z} = 1$

$$u = f(x-y,\ y-z,\ z-x) = f(x_1, x_2, x_3)$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3} \qquad [\text{using (A)}] \quad \ldots (1)$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial y}$$

$$= \frac{\partial u}{\partial x_1}(-1) + \frac{\partial u}{\partial x_2}(1) + \frac{\partial u}{\partial x_3}(0) \quad [\text{using(A)}]$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \qquad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial z}$$

$$= \frac{\partial u}{\partial x_1}(0) + \frac{\partial u}{\partial x_2}(-1) + \frac{\partial u}{\partial x_3}(1)$$

[using (A)]

$$= -\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} \qquad (3)$$

Adding (1), (2), (3) we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

6). If $z = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$ show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad \text{[Using Table]} \quad (1)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta \quad (2)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta) \quad \text{Using Table}$$

$$\therefore \frac{1}{r}\cdot\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta.$$

$$\left(\frac{1}{r}\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial y}\cos\theta - \frac{\partial z}{\partial x}\sin\theta\right)^2$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta \quad (2)$$

(2) + (3)

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2(\cos^2\theta+\sin^2\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

7) If $z = u(x,y)$ where $x = e^u\cos v$ and $y = e^u \sin v$

show that $y\dfrac{\partial z}{\partial u} + x\dfrac{\partial z}{\partial v} = e^{2u}\dfrac{\partial z}{\partial y}$

$x = e^u \cos v$ $\qquad$ $y = e^u \sin v$

$\dfrac{\partial x}{\partial u} = e^u\cos v$ $\qquad$ $\dfrac{\partial y}{\partial u} = e^u \sin v$

$\dfrac{\partial x}{\partial v} = -e^u \sin v$ $\qquad$ $\dfrac{\partial y}{\partial v} = e^u\cos v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} \cdot e^{u}\cos v + \frac{\partial z}{\partial y} \cdot e^{u}\sin v \quad \text{[Using table]}$$

(or)

$$\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \quad \text{--- (1)}$$

Also

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x} \cdot (-e^{u}\sin v) + \frac{\partial z}{\partial y} \cdot e^{u}\cos v \quad \text{[using Table]}$$

(or)

$$\frac{\partial z}{\partial v} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \quad \text{--- (2)}$$

$(1) \times y$

$$y\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \quad \text{--- (3)}$$

$(2) \times x$

$$x\frac{\partial z}{\partial v} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \quad \text{--- (4)}$$

Adding (3) + (4)

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y}$$

$$= (y^2 + x^2)\frac{\partial z}{\partial y}$$

$$= (e^{2u}\sin^2 v + e^{2u}\cos^2 v)\frac{\partial z}{\partial u}$$

$$= e^{2u}(\sin^2 v + \cos^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}$$

8) If $z = u^2 + v^2$, $x = u^2 - v^2$, $y = uv$ find $\frac{\partial z}{\partial x}$

$z = u^2 + v^2$

$$\boxed{\frac{\partial z}{\partial u} = 2u \qquad \frac{\partial z}{\partial v} = 2v}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \quad \text{[see table]} \quad (1)$$

Given:- $x = u^2 - v^2$, partially differentiating w.r.t 'x'

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \longleftarrow (2)$$

$y = uv$, Partially differentiating w.r.t 'x'

$$0 = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} \longleftarrow (3)$$

From (2) + (3).

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} - 1 = 0 \longleftarrow (4)$$

$$v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} = 0 \longleftarrow (5)$$

Solving (4) & (5)

$$\frac{\partial u}{\partial x} = \frac{u}{2(u^2+v^2)}$$

$$\frac{\partial v}{\partial x} = \frac{-v}{2(u^2+v^2)} \qquad (6)$$

Sub (6) in (1)

$$\frac{\partial z}{\partial x} = \frac{2u^2}{2(u^2+v^2)} - \frac{2v^2}{2(u^2+v^2)}$$

$$= \frac{u^2-v^2}{u^2+v^2}$$

$$= \frac{x}{z}$$

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9) If $x = u(1+v)$ $y = v(1+u)$ find $\frac{\partial(x,y)}{\partial(u,v)}$.

$x = u(1+v)$ $\qquad$ $y = v(1+u)$

$\frac{\partial x}{\partial u} = (1+v)$ $\qquad \frac{\partial x}{\partial v} = u$ $\qquad \frac{\partial y}{\partial u} = v$ $\qquad \frac{\partial y}{\partial v} = 1+u$ ... (A)

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} \qquad \text{[Using (A)]}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + v + u + \cancel{uv} - \cancel{uv} \qquad = 1+u+v$$

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10). If $y_1 = \frac{x_2 x_3}{x_1}$ $y_2 = \frac{x_3 x_1}{x_2}$ $y_3 = \frac{x_1 x_2}{x_3}$. Show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

$y_1 = \frac{x_2 x_3}{x_1}$ $y_2 = \frac{x_3 x_1}{x_2}$ $y_3 = \frac{x_1 x_2}{x_3}$

$\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}$ $\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_1}$ $\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$

$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$ $\frac{\partial y_2}{\partial x_2} = \frac{-x_3 x_1}{x_2^2}$ $\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$

$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$ $\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$ $\frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{vmatrix}$$

$$= \begin{vmatrix} -x_2 x_3/x_1^2 & x_3/x_1 & x_2/x_1 \\ x_3/x_2 & -x_3 x_1/x_2^2 & x_1/x_2 \\ x_2/x_3 & x_1/x_3 & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

[using(

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}.$$

$$= \frac{\cancel{x_1^2 x_2^2 x_3^2}}{\cancel{x_1^2 x_2^2 x_3^2}} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) -1(-1-1) +1(1+1)$$

$$= 0 + 2 + 2$$

$$= 4 //.$$

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10). If $u = x+y+z$ $\quad uv = y+z$ $\quad uvw = z$ then show

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$$

$z = uvw$ (1)

$uv = y + z$

$y = uv - z$ (2)

$= uv - uvw$ [using (1)]

$u = x + y + z$

$x = u - y - z$ [using (1) & (2)].

$= u - uv + \cancel{uvw} - \cancel{uvw}$

$= u - uv$

$x = u - uv \qquad y = uv - uvw \qquad z = uvw$

$\dfrac{\partial x}{\partial u} = 1 - v \qquad \dfrac{\partial y}{\partial u} = v - vw \qquad \dfrac{\partial z}{\partial u} = vw$

$\dfrac{\partial x}{\partial v} = -u \qquad \dfrac{\partial y}{\partial v} = u - uw \qquad \dfrac{\partial z}{\partial v} = uw$

$\dfrac{\partial x}{\partial w} = 0 \qquad \dfrac{\partial y}{\partial w} = -uv \qquad \dfrac{\partial z}{\partial w} = uv \qquad (A)$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \dfrac{\partial x}{\partial w} \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \dfrac{\partial z}{\partial v} & \partial z/\partial w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} \quad \text{using } (A)$$

$$= (1-v)\left[(u-uw)(uv) + u^2vw\right] + u\left[uv(v-vw) + uv \cdot vw\right]$$

$$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v.$$

12). If $x = u\cos v$, $y = u\sin v$, show that $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

$x = u\cos v$	$y = u\sin v$
$\frac{\partial x}{\partial u} = \cos v$	$\frac{\partial y}{\partial u} = \sin v$
$\frac{\partial x}{\partial v} = -u\sin v$	$\frac{\partial y}{\partial v} = u\cos v.$

... (A),

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} \quad \text{[using (A)]}$$

$$= u\cos^2 v + u\sin^2 v$$

$$= u(\cos^2 v + \sin^2 v)$$

$$J = u.$$

Now we have to express u and v interms of x, y

$u = \frac{x}{\cos v}$

v

$x = u\cos v$, $y = u\sin v$

$\therefore x^2+y^2 = u^2(\cos^2 v+\sin^2 v)$

$= u^2.$

$u = \sqrt{x^2+y^2}.$

$\frac{y}{x} = \frac{u \sin v}{u \cos v} = \tan v \Rightarrow v = \tan^{-1}(y/x).$

$u = \sqrt{x^2+y^2}$	$v = \tan^{-1}(y/x)$
$\frac{\partial u}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$ $= \frac{x}{\sqrt{x^2+y^2}}.$ $\frac{\partial u}{\partial y} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$ $= \frac{y}{\sqrt{x^2+y^2}}$	$\frac{\partial v}{\partial x} = \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)$ $= \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$ $= \frac{-y}{x^2+y^2}.$ $\frac{\partial v}{\partial y} = \frac{1}{1+y^2/x^2}\cdot\left(\frac{1}{x}\right)$ $= \frac{x^2}{x^2+y^2}\cdot\left(\frac{1}{x}\right)$ $= \frac{x}{x^2+y^2}$

(B)

$\frac{y}{x} = u$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{vmatrix} \quad (3)$$

$$J' = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

$$J' = \frac{1}{u} \quad \left[\because u = \sqrt{x^2+y^2}\right] \quad (2)$$

$$\therefore JJ' = \cancel{u} \times \frac{1}{\cancel{u}} = 1. \quad [\text{Using (1) and (2)}].$$

---

13). Find the Value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$, where

$= x^2 - y^2 \qquad v = 2xy$ and $x = r\cos\theta \quad y = r\sin\theta$

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 2y$

(A)

$\frac{\partial u}{\partial y} = -2y$ $\qquad$ $\frac{\partial v}{\partial y} = 2x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \quad [\text{using (A)}]$$

$$= 4(x^2+y^2)$$

$$= 4r^2 \quad (1)$$

$x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

(B).

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$ [unit (B)]

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} \quad \cdots (2)$$

$$= 4r^2 \cdot r$$ [From (1) & (2)]

$$= 4r^3$$

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14). If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$ determine whether is a functional relationship between u, v, w and if so, find it?

$u = xy + yz + zx$	$v = x^2 + y^2 + z^2$	$w = x + y + z$
$\frac{\partial u}{\partial x} = y + z$	$\frac{\partial v}{\partial x} = 2x$	$\frac{\partial w}{\partial x} = 1.$
$\frac{\partial u}{\partial y} = z + x$	$\frac{\partial v}{\partial y} = 2y$	$\frac{\partial w}{\partial y} = 1$
$\frac{\partial u}{\partial z} = x + y$	$\frac{\partial v}{\partial z} = 2z$	$\frac{\partial w}{\partial z} = 1.$

..(A)

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$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} \quad (A).$$

$$= 2 \begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\left[(y+z)(y-z) - (z+x)(x-z) + (x+y)(x-y)\right]$$

$$= 2\left[y^2 - yz + zy - z^2 - zx + z^2 - x^2 + xz + x^2 - xy + xy - y^2\right]$$

$$= 0.$$

Hence the functional relationship exists between u, v and w.

$$w^2 = (x+y+z)^2$$

$$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$w^2 = v + 2u$$

$w^2 - v - 2u = 0$ which is the required relationship.

TAYLER'S THEOREM:-

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)]$$

$$+ \frac{1}{2}! [(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \dots$$

15). Expand $e^x \cos y$ near the point $(1, \pi/4)$ by Taylor's series as far as quadratic terms.

Function	Value at $(1, \pi/4)$.
$f(x,y) = e^x \cos y$	$f(1, \pi/4) = e \cdot \frac{1}{\sqrt{2}} = \frac{e}{\sqrt{2}}$
$f_x(x,y) = e^x \cos y$	$f_x(1, \pi/4) = e/\sqrt{2}$.
$f_{xx}(x,y) = e^x \cos y$	$f_{xx}(1, \pi/4) = e/\sqrt{2}$.
$f_y(x,y) = -e^x \sin y$	$f_y = (1, \pi/4) = -e/\sqrt{2}$
$f_{yy}(x,y) = -e^x \cos y$	$f_{yy}(1, \pi/4) = -e/\sqrt{2}$
$f_{xy}(x,y) = -e^x \sin y$	$f_{xy}(1, \pi/4) = -e/\sqrt{2}$.

We k.t that

$$f(x,y) = f(a,b) + [(x-a) f_x (a,b) + (y-b) f_y (a,b)]$$
$$+ \frac{1}{2!} [(x-a)^2 f_{xx} (a,b) + 2(x-a)(y-b) f_{xy} (a,b) + (y-b)^2 f_{yy} (a,b)] + \cdots$$

Here $a=1$ $b=\pi/4$

$$f(x,y) = f(1,\pi/4) + [(x-1) f_x (1,\pi/4) + (y-\pi/4) f_y (1,\pi/4)]$$
$$+ \frac{1}{2!} [(x-1)^2 f_{xx} (1,\pi/4) + 2(x-1)(y-\pi/4) f_{xy}(1,\pi/4)$$
$$+ (y-\pi/4)^2 f_{yy} (1,\pi/4)] + \cdots \quad (B)$$

Sub (A) in (B) we get.

$$e^x \cos y = \frac{e}{\sqrt{2}} + \left[(x-1) \frac{e}{\sqrt{2}} + (y-\pi/4) \left(\frac{-e}{\sqrt{2}}\right) \right]$$
$$+ \frac{1}{2} \left[(x-1)^2 \left(\frac{e}{\sqrt{2}}\right) + 2(x-1)(y-\pi/4) \left(-\frac{e}{\sqrt{2}}\right) + (y-\pi/4)^2 \left(\frac{-e}{\sqrt{2}}\right) \right] + \cdots$$

$$\therefore e^x \cos y = \frac{e}{\sqrt{2}} \left[1 + (x-1) - (y-\pi/4) + \frac{(x-1)^2}{2} - (x-1)(y-\pi/4) - \frac{(y-\pi/4)^2}{2} \right] + \cdots$$

16). Expand $\tan^{-1} y/x$ in the neighbourhood of $(1,1)$.

Function	Value at $(1,1)$.
Let $f(x,y) = \tan^{-1} y/x$	$f(1,1) = \tan^{-1} 1$ $= \pi/4$
$f_x(x,y) = \frac{1}{1+y^2/x^2} \cdot \left(-\frac{y}{x^2}\right)$ $= \frac{-y}{x^2+y^2}$ $= -y(x^2+y^2)^{-1}$	$f_x(1,1) = -1/2$
$f_y(x,y) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$ $= \frac{x}{x^2+y^2}$ $= x(x^2+y^2)^{-1}$.	$f_y(1,1) = 1/2$
$f_{xx}(x,y) = (-y)(-1)(x^2+y^2)^{-2} \cdot 2x$ $= \frac{2xy}{(x^2+y^2)^2}$	$f_{xx}(1,1) = 1/2$

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$$f_{xy}(x,y) = \frac{(x^2+y^2)\,1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$f_{xy}(1,1) = 0.$

$$f_{yy}(x,y) = x(-1)(x^2+y^2)^{-2}\cdot 2y$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$f_{yy}(1,1) = -\frac{1}{2}$

We know that.

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Put $a=1$, $b=1$

$$\therefore \tan^{-1}\frac{y}{x} = f(x,y)$$

$$= f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$= \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2!}\left[(x-1)^2\cdot\frac{1}{2} + 2(x-1)(y-1)(0) + (y-1)^2\left(\frac{-1}{2}\right)\right] + \cdots$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \cdots$$

MAXIMA AND MINIMA

1. Let $f(x,y)$ be the given function.
2. Find $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ and find the stationary points.
3. Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$.
4. Calculate $rt - s^2$.

Rule 1:

If r (or) $t > 0$ and $rt - s^2 > 0$, then f has a minimum value.

Rule 2:

If r (or) $t < 0$, and $rt - s^2 > 0$, then f has a Maximum value.

Rule 3: If $rt - s^2 < 0$, then f has neither maximum nor minimum. Such a point is called a saddle point.

Rule 4: If $rt - s^2 = 0$, then further investigation is required.

1) Find the maximum (or) minimum values of $3x^2 - y^2 + x^3$.

$f(x,y) = 3x^2 - y^2 + x^3$.

$\frac{\partial f}{\partial x} = 6x + 3x^2$	$\frac{\partial f}{\partial y} = -2y$
$\frac{\partial f}{\partial x} = 0$	$\frac{\partial f}{\partial y} = 0$
$\Rightarrow 6x + 3x^2 = 0$ $3x(2+x) = 0$ $x = 0 \quad x = -2$	$\Rightarrow -2y = 0$ $y = 0$.

Turning points are $(0,0)$ $(-2,0)$.

	At $(0,0)$	At $(-2,0)$
$r = \partial^2 f/\partial x^2 = 6 + 6x$	$6 > 0$	$6 - 12 = -6 < 0$
$s = \frac{\partial^2 f}{\partial x \partial f} = 0$	0	0.
$t = \frac{\partial^2 f}{\partial y^2} = -2$	$-2 < 0$	$-2 < 0$.
$rt - s^2$	$-12 < 0$	$12 > 0$
Result:	$r > 0$, $rt - s^2 < 0$ $\therefore (0,0)$ - Saddle point	$r < 0$, $rt - s^2 > 0$ $(-2,0)$ - maximum point

Maximum Value of 'f' is $= 3(-2)^2 - (0) + (-2)^3$.

$= 12 - 8$

$= 4$

2). Find the maximum (or) minimum values of the function

$$f(x,y) = x^3+y^3-3axy$$

$f(x,y) = x^3+y^3-3axy$	
$\frac{\partial f}{\partial x} = 3x^2-3ay$	$\frac{\partial f}{\partial y} = 3y^2-3ax$
$\frac{\partial f}{\partial x} = 0$	$\frac{\partial f}{\partial y} = 0$
$3x^2-3ay = 0$	$3y^2-3ax=0$
$x^2-ay=0$ — (1)	$y^2-ax=0$ — (2)

To find turning point:

To solve (1) and (2)

$(1) \Rightarrow \quad y = x^2/a$ —— (3)

Sub (3) in (1) we get

$$\frac{x^4}{a^2} - ax = 0$$

$$x^4 - a^3x = 0$$

$$x(x^3-a^3) = 0$$

$$x = 0 \quad x = a$$

When $x=0$, $y=0$
$x=a$ $y=a$.

Turning pts are $(0,0)$, (a,a)

	At $(0,0)$	At (a,a)
$r=\frac{\partial^2 f}{\partial x^2}=6x$	0	$6a$
$s=\frac{\partial^2 f}{\partial x \partial y}=-3a$	$-3a$	$-3a$
$t=\frac{\partial^2 f}{\partial y^2}=6y$	0	$6a$
$rt-s^2$	$-9a^2<0$	$36a^2-9a^2>0$
Result:	Saddle point	1). If $a<0, r<0$ $rt-s^2>0$ $\therefore$ maximum pt 2). If $a>0, r>0$ $rt-s^2>0$ $\therefore$ minimum pt

when $a<0$,
Max. Value $=a^3+a^3-3a^3=-a^3$.
when $a<0$,
Minimum Value $=-a^3-a^3-3a^3$
$=-5a^3$.

3) Examine $f(x,y)=x^3+y^3-12x-3y+20$ for its extreme value

$$f(x,y)=x^3+y^3-12x-3y+20$$

$\frac{\partial f}{\partial x}=3x^2-12$ $\qquad$ $\frac{\partial f}{\partial y}=3y^2-3$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$3x^2 - 12 = 0 \qquad 3y^2 - 3 = 0$$

$$x^2 - 4 = 0 \qquad y^2 - 1 = 0$$

$$x = \pm 2 \qquad y = \pm 1$$

Turning points are (2,1) (2,-1), (-2,1), (-2,-1)

	At (2,1)	At (2,-1)	At (-2,1)	At (-2,-1).
$r = \frac{\partial^2 f}{\partial x^2}$	12	12	-12	-12
$s = \frac{\partial^2 f}{\partial x \partial y}$	0	0	0	0
$t = \frac{\partial^2 f}{\partial y^2}$	6	-6	6	-6
$rt - s^2 = 36xy$	72	-72	-72	72
Result:-	$r > 0$, $rt - s^2 > 0$, (2,1) is a minimum point	$r > 0$, $rt - s^2 < 0$, (2,-1) is a Saddle point	$r < 0$, $rt - s^2 < 0$, (-2,1) is a Saddle point	$r < 0$, $rt - s^2 > 0$, (-2,-1) is a maximum point

The minimum value of 'f' is

$$f(2,1) = 2^3 + (1)^3 - 12(2) - 3(1) + 20$$

$$= 2$$

The minimum value of 'f' is

$$f(-2,-1) = (-2)^3 + (-1)^3 - 12(-2) - 3(-1) + 20$$

$$= 38$$

4). Investigate the maxima of the function

$$f(x,y) = x^3y^2(1-x-y)$$

$$f(x,y) = x^3y^2(1-x-y)$$

$$= x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$= x^2y^2(3-4x-3y)$$

$$\frac{\partial f}{\partial x} = 0$$

$$\Rightarrow x^2y^2(3-4x-3y) = 0$$

$x=0 \quad y=0$

$$3-4x-3y=0$$

or

$$4x+3y=3 \quad ①$$

$$\frac{\partial f}{\partial y} = 2yx^3 - 2x^4y - 3y^2x^3$$

$$= x^3y(2-2x-3y)$$

$$\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow x^3y(2-2x-3y) = 0$$

$x=0 \quad y=0$

$$2-2x-3y=0$$

or

$$2x+3y=2 \quad ②$$

To find turning points :-

① $\Rightarrow \quad 4x + 3y = 3 \qquad ③$

② $\Rightarrow \quad 2x + 3y = 2 \qquad ④$

③ − ④ $\quad 2x = 1$

$x = 1/2$

Put $x = \frac{1}{2}$ in (3)

$2 + 3y = 3$

$3y = 1$

$y = 1/3$

Turning points $(0,0)$ $\left(\frac{1}{2}, \frac{1}{3}\right)$.

	At $(0,0)$	At $(1/2, 1/3)$
$r = \dfrac{\partial^2 f}{\partial x^2}$ $= x^2y^2(-4) + (3 - 4x - 3y) \cdot 2xy^2$ $= 6xy^2(1 - 2x - y)$.	0	$6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2\left(1 - 1 - \frac{1}{3}\right)$ $= -1/9$
$s = \dfrac{\partial^2 f}{\partial x \partial y}$ $= y\{x^3(-2) + (2 - 2x - 3y) \cdot 3x^2\}$ $= x^2y(6 - 8x - 9y)$	0	$= \left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)\left(6 - 4 - 8/3\right)$ $= \frac{1}{12}\left(2 - 8/3\right)$ $= \frac{1}{12}\left(-2/3\right)$ $= -1/18 < 0$

$t = \frac{\partial^2 f}{\partial y^2}$ $= x^3\{y(-3) + (2-2x-3y)\cdot 1\}$ $= x^3(2-2x-6y)$	0	$\left(\frac{1}{2}\right)^2(2-1-2)$ $= -1/8 < 0$
$rt - s^2$	0	$\left(\frac{-1}{9}\right)\left(\frac{-1}{8}\right) - \left(\frac{-1}{18}\right)^2$ $= \frac{1}{72} + \frac{1}{324} > 0$
Result:-	$r = 0$ $rt - s^2 = 0$ Further investigation is needed	$r < 0$ $rt - s^2 > 0$ $\left(\frac{1}{2}, \frac{1}{3}\right)$ is a maximum point $\boxed{\text{Max. value} = 1/432}$

Lagrange's Multiplier Method:-

Let $u = f(x, y, z)$.

be a given function for which the extremum values to be determined, subject to the condition

$$g(x, y, z) = 0.$$

* $F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$
* Find extreme value

1). Find the minimum value of $x^2+y^2+z^2$ given that $ax+by+cz=p$

$$f=x^2+y^2+z^2 \quad \text{——} \quad (1)$$

$$g=ax+by+cz-p \quad \text{——} \quad (2)$$

$$F(x,y,z)=(x^2+y^2+z^2)+\lambda(ax+by+cz-p)$$

$\frac{\partial f}{\partial x}=0$	$\frac{\partial f}{\partial y}=0$	$\frac{\partial f}{\partial z}=0$
$2x+\lambda\cdot a=0$	$2y+\lambda b=0$	$2z+\lambda\cdot c=0$
$x=\frac{-\lambda a}{2}$ —— (3)	$y=\frac{-\lambda b}{2}$ —— (4)	$z=\frac{-\lambda c}{2}$ —— (5)

$$\frac{\partial f}{\partial \lambda}=0$$

$$ax+by+cz=p \quad \text{——} \quad (6)$$

Sub (3), (4), (5) in (6).

We get $$a\left(\frac{-\lambda a}{2}\right)+b\left(\frac{-\lambda b}{2}\right)+c\left(\frac{-\lambda c}{2}\right)=p$$

$$-\lambda\left[a^2+b^2+c^2\right]=2p$$

$$\lambda=\frac{-2p}{a^2+b^2+c^2} \quad \text{——} \quad (7)$$

Sub (7) in (3), (4), (5) we get

$$x = \frac{ap}{a^2+b^2+c^2}, \quad y = \frac{bp}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2}$$

The minimum value of the function $f = x^2+y^2+z^2$ is attained at the point $\left(\frac{ap}{\Sigma a^2}, \frac{bp}{\Sigma a^2}, \frac{cp}{\Sigma a^2}\right)$.

The minimum value is obtained by sub these values in (1) we get

$$f = \frac{a^2p^2+b^2p^2+c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2}{a^2+b^2+c^2}$$

This is the minimum value

2). Find the maximum value of $x^m y^n z^p$ when $x+y+z=a$.

Let $f = x^m y^n z^p$ and $g = x+y+z-a$

$$F(x,y,z) = x^m y^n z^p + \lambda(x+y+z-a) \quad \text{—— (1)}$$

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$mx^{m-1}y^n z^p + \lambda = 0$	$ny^{n-1}x^m z^p + \lambda = 0$	$px^m y^n z^{p-1} + \lambda = 0$

$-\lambda = m x^{m-1} y^n z^p$ — (2)	$\Rightarrow -\lambda = n y^{n-1} x^m z^p$ — (3)	$\Rightarrow -\lambda = p x^m y^n z^{p-1}$ — (4)

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x+y+z-a=0 \qquad (5)$$

From (2), (3), (4) we get

$$m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$$

$\div$ by $x^m y^n z^p$

$$\frac{m x^{m-1} y^n z^p}{x^m y^n z^p} = \frac{n x^m y^{n-1} z^p}{x^m y^n z^p} = \frac{p x^m y^n z^{p-1}}{x^m y^n z^p}$$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$= \frac{m+n+p}{x+y+z}$$

$$= \frac{m+n+p}{a}$$

$\therefore$ Hence Max value of f occurs when

$x = \dfrac{am}{m+n+p}$ [Taking 1st and last] (6)

$y = \dfrac{an}{m+n+p}$ [Taking 2nd and last] (7)

$z = \dfrac{ap}{m+n+p}$ [Taking 3rd and last] (8)

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Sub (6), (7), (8) in $f = x^m y^n z^p$ the max value

$$f = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$f = a^{m+n+p} \frac{m^n n^n p^p}{(m+n+p)^{m+n+p}}$$

3). A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box, that requires the least material for its construction.

Let x, y, z be the length, breadth and height of the box resp. When it requires least material, the surface area of the box should be least.

The surface area $S = xy + 2yz + 2zx$. Hence we have to minimize S subject to the condition that the volume of the box $xyz = 32$

$$F = (xy + 2yz + 2zx) + \lambda(xyz - 32) \quad \text{--- (1)}$$

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$

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$y + 2z + \lambda z y = 0$ $y + 2z = -\lambda y z$ $-\lambda = \frac{1}{z} + \frac{2}{y}$ —— (2)	$x + 2y + \lambda z x = 0$ $x + 2z = -\lambda z x$ $-\lambda = \frac{1}{z} + \frac{2}{x}$ —— (3)	$2x + 2y + \lambda xy = 0$ $2x + 2y = -\lambda xy$ $-\lambda = \frac{2}{y} + \frac{2}{x}$ —— (4)

$$\frac{\partial F}{\partial \lambda} = 0$$

$$xyz - 32 = 0 \quad \text{—— (5)}$$

From (2) & (3)

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$2/y = 2/x$$

$$x = y \quad \text{—— (6)}$$

From (2) & (4)

$$\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

$$1/z = 2/y$$

$$y = 2z \quad \text{—— (7)}$$

$$x = y = 2z \quad \text{—— (8)}$$

Sub (8) in (5)

$$(2z)(2z)(z) = 32$$

$$4z^3 = 32$$

$$z^3 = 32/4 = 8$$

$$\boxed{\begin{array}{c} z = 2 \\ y = 4 \qquad x = 4 \end{array}}$$

[From (8)]

∴ The dimensions of box are 4, 4, 2