TRANSFORMS & PDE (MA6351)

Question Bank with Answers

	UNIT I PARTIAL DIFFERENTIAL EQUATIONS **PART-A**
1.	**Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from $(x-a)^2+(y-b)^2=z$** **A.U May/June2013**
	Solution: Given $(x-a)^2+(y-b)^2=z$ -----(1) Diff Partially w.r.t x and y $2(x-a)=p \Rightarrow x-a=p/2$ $2(y-b)=q \Rightarrow y-b=q/2$ Substituting in (1), we get $p^2+q^2=4z$
2.	**Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from $z=(x^2+a)(y^2+b)$** **A.U April/May 2011**
	Solution: Given $z=(x^2+a)(y^2+b)$ -----(1) Diff partially w.r.t x and y $\frac{\partial z}{\partial x}=p=2x(y^2+b),\ \frac{\partial z}{\partial y}=q=2y(x^2+a),$ $\frac{p}{2x}=y^2+b,\ \frac{q}{2y}=x^2+a$ Substituting in (1) we get pq= 4xyz
3.	**Find the PDE of all planes having equal intercepts on the x and y axis.**
	Solution: The equation of plane is $\frac{x}{a}+\frac{y}{a}+\frac{z}{b}=1$ ....(1) Diff. w.r.t. x and y partially respectively $\frac{1}{a}+\frac{p}{b}=0$ ....(2) $\frac{1}{a}+\frac{q}{b}=0$ ....(3) Solving (2) & (3) we get, p=q
4.	**Obtain the partial differential equation by eliminating arbitrary constants 'a' and 'b' from $(x-a)^2+(y-b)^2+z^2=1$.**
	Solution: $(x-a)^2+(y-b)^2+z^2=1$ - (1) Differentiating (1) partially w.r. t x and y we get $2(x-a)+2zp=0 \Rightarrow x-a=-zp$ - (2) $2(y-b)+2zq=0 \Rightarrow y-b=-zq$ - (3) Substituting (2) & (3) in (1) we get $z^2p^2+z^2q^2+z^2=1$ i.e., $z^2(p^2+q^2+1)=1$

5.	**Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ and form the PDE. (Nov/Dec 2012)**
	Solution: $\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$ ....(1) $\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$ ....(2) From (1) & (2) $\frac{p}{q} = \frac{-y}{x} \Rightarrow px + qy = 0$
6.	**Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from z= ax+by**
	Solution: Given z= ax+by -----(1) Diff partially w.r.t x and y $\frac{\partial z}{\partial x} = p = a, \ \frac{\partial z}{\partial y} = q = b,$ Substituting in (1) we get z = px+qy.
7.	**Form the partial differential equation by eliminating f from $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$.**
	Solution: Let $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$ - (1) Differentiate (1) partially w.r. t x and y $\frac{\partial z}{\partial x} = p = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right)$ - (2) $\frac{\partial z}{\partial y} = q = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right)$ - (3) Eliminating f' from (2) & (3) $\therefore \frac{p-2x}{q} = \frac{-1}{x}\left(y^2\right) \Rightarrow px - 2x^2 = -qy^2 \Rightarrow px + qy^2 = 2x^2$
8.	**From the partial differential equation by eliminating g from $g\left(x^2 + y^2 + z^2, x + y + z\right) = 0$.**
	Solution: We know that if g(u, v) = 0 then u = f(v) $\therefore x^2 + y^2 + z^2 = f\left(x + y + z\right)$ - ---- (1) Differentiating (1) partially w.r. t x and y We get

	$2x+2zp = f'(x+y+z)(1+p)$ ----- (2) $2y+2zq = f'(x+y+z)(1+q)$ …..(3) Divide (2) & (3) $\frac{x+zp}{y+zq} = \frac{1+p}{1+q} \Rightarrow x+qx+zp+zpq = y+py+zq+zpq$ $(z-y)p+(x-z)q = y-x$
9.	**Find the complete integral of p+q = pq**
	Solution: Given p+q = pq -----(1) This is of the type f(p,q)=0 Let z= ax+by+c be the solution for (1) Partially Diff w.r.t 'x ' and 'y' we get p= a and q = b => a+b = ab => b= a/(a–1) Therefore z= ax+ (a/a–1)y+c is the complete solution.
10.	**Find the singular integral of** $z = px + qy + 2\sqrt{pq}$
	Solution: This of Clairaut's type and the complete solution is $z = ax + by + 2\sqrt{ab}$ Diff w.r.t 'a ' and 'b' we get $\frac{\partial z}{\partial a} = x + \frac{2\sqrt{b}}{2\sqrt{a}} = 0, \ \frac{\partial z}{\partial b} = x + \frac{2\sqrt{a}}{2\sqrt{b}} = 0$ ⇨ $x = -\frac{\sqrt{b}}{\sqrt{a}}$ & $y = -\frac{\sqrt{a}}{\sqrt{b}} \Rightarrow xy = \frac{\sqrt{b}}{\sqrt{a}} \frac{\sqrt{a}}{\sqrt{b}} = 1$ xy= 1 is singular integral.
11.	**Find the complete integral of p+q=0**
	Solution: Given p +q = 0 -----(1) This is of the type f(p,q)=0 Let z= ax+by+c be the solution for (1) Partially Diff w.r.t 'x ' and 'y' we get p=a and q=b => a+b = 0 => b= –a Therefore z= ax– ay +c be the complete solution.
12.	**Solve** $p(1+q) = qz$ **.**
	Solution: $p(1+q) = qz$ - (1) This equation is of the form $f(z,p,q) = 0$ $z = g(x+ay)$ be the solution Let $x+ay = u$ $z = g(u)$ $p = \frac{dz}{du}$ $q = \frac{adz}{du}$ (1) reduces to $\frac{dz}{du}\left(1+a\frac{dz}{du}\right) = a\frac{dz}{du}z \Rightarrow 1+a\frac{dz}{du} = az \Rightarrow a\frac{dz}{du} = az-1 \Rightarrow \frac{dz}{du} = z-\frac{1}{a} \Rightarrow \frac{dz}{z-\frac{1}{a}} = du$

	Integrating $\log\left(z-\frac{1}{a}\right)=u+b$ i.e., $\log\left(z-\frac{1}{a}\right)=x+ay+b$ is the complete solution.
13.	**Solve yp= 2yx+log q**
	Solution: The equation can be reduced to the type $F_1(x,p)= F_2(y,q)$ ⇨ $y(p-2x) = \log q$ ⇨ $p-2x = (\log q)/y = k$ ⇨ $p= 2x+k$; $q = e^{ky}$ ⇨ $z = \int pdx+\int qdy$ ⇨ $z= \int(2x+k)\,dx+\int e^{ky}dy$ $z = (x^2+kx)+\frac{e^{ky}}{k}+c$
14.	**Solve** $(D-1)(D-D'+1)z=0$ **(Nov/Dec 2012)**
	Solution: $(D-1)(D-D'+1)z=0$ Here $c_1=1$, $c_2=-1$, $m_1=0$, $m_1=1$ General solution $z=e^{x}\ f_1(y)+e^{-x}\ f_2(y+x)$
15.	**Solve** $\sqrt{p}+\sqrt{q}=1.$
	Solution: This is of the form $f(p,q)=0$. The solution is given by $z=ax+by+c$ where $\sqrt{a}+\sqrt{b}=1 \Rightarrow \sqrt{b}=1-\sqrt{a} \Rightarrow b=\left(1-\sqrt{a}\right)^2$ $\therefore$ The complete solution is $z=ax+\left(1-\sqrt{a}\right)^2 y+c$
16.	**Solve** $\left(D-D'-1\right)\left(D-D'-2\right)z=e^{2x-y}$.
	Solution: This is of the form $\left(D-m_1D'-C_1\right)\left(D-m_2D'-C_2\right)....\left(D-m_nD'-C_n\right)Z=0$ Hence $m_1=1\quad c_1=1\quad m_2=1\quad c_1=1\ c_2=2$ Hence the C.F. is $z=e^{x}\phi_1\left(y+x\right)+e^{2x}\phi_2\left(y+x\right)$ P.I. $=\frac{e^{2x-y}}{\left(D-D'-1\right)\left(D-D'-2\right)}=\frac{e^{2x-y}}{\left(2+1-1\right)\left(2+1-2\right)}=\frac{1}{2}e^{2x-y}$ Hence, the complete solution is $Z=e^{x}\phi_1\left(y+x\right)+e^{2x}\phi_2\left(y+x\right)+\frac{1}{2}e^{2x-y}$
17.	**Solve** $\left(D^3-3DD'^2+2D'^3\right)Z=0$.
	Solution:

	Substituting D = m, & $D^1 = 1$ The Auxiliary equation is $m^3 - 3m + 2 = 0$ m = 1, 1, -2 Complimentary function is $\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-2x)$ i.e., $z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-2x)$
18.	**Find the particular integral of** $(D^2+2DD'+5D'^2)z = e^{x-y}$
	Solution: P.I = $\dfrac{1}{D^2+2DD'+5D'^2}e^{x-y}$ Replace D by 1 and D' by 1 ,we get $= \dfrac{1}{1-2+1}e^{x-y} = \dfrac{x}{2D+2D'}e^{x-y} = \dfrac{x^2}{2}e^{x-y}$
19.	**Find the general solution of** $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} - 4\dfrac{\partial^3 z}{\partial x \partial y^2} + 8\dfrac{\partial^3 z}{\partial y^3} = 0$
	Solution: Auxiliary Equations is $m^3 - 2m^2 - 4m+8=0$ ⇨ m= 2,2,–2 General solution z= $f_1(y+2x)+xf_2(y+2x)+f_3(y-2x)$
20.	**Find the Particular integral of** $(D^2+D'^2+2DD'+2D+2D'+1)z = e^{2x-y}$
	Solution: P.I = $\dfrac{1}{(D^2+D'+1)^2}e^{2x-y}$ Replace D by 2 and D' by –1 ,we get P.I $= \dfrac{1}{(2-1+1)^2}e^{2x-y} = \dfrac{1}{4}e^{2x-y}$
	PART - B

1.	**a)**	**Find the general solution of** $x(y^2 - z^2)p + y(x^2 - z^2)q = z(x^2 - y^2)$ [A.U Apr/ May 2008]
		Solution: The equation is of the form Pp+Qq=R where P= $x(y^2 - z^2)$,Q= $y(x^2 - z^2)$,R= $z(x^2 - y^2)$ The auxiliary equation is $\dfrac{dx}{x(z^2-y^2)} = \dfrac{dy}{y(x^2-z^2)} = \dfrac{dz}{z(x^2-y^2)}$ Using the multipliers x, y, z , we get each ratio is equal to $\dfrac{xdx+ydy+zdz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(x^2-y^2)} = 0$ i.e. $xdx+ydy+zdz=0$ Integrating we get $x^2+y^2+z^2=c_1$

using the another set of multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$

we get each of the above ratio is equal to $\dfrac{\frac{1}{x}dx+\frac{1}{y}ydy+\frac{1}{z}dz}{(z^2-y^2)+(x^2-z^2)+(x^2-y^2)}=0$

i.e. $\frac{1}{x}dx+\frac{1}{y}ydy+\frac{1}{z}dz=0$

Integrating we get $\log(xyz)=\log c_2$

The solution of the linear p.d.e is $\varphi(u,v)=0$

i.e. $\varphi(x^2+y^2+z^2, xyz)=0$

b) **Solve** $(D^2+DD'-6D'^2)z=x^2y+e^{3x+y}$

Solution:

The auxiliary equation is $m^2+m-6=0$

$m=2,-3$

C.F= $\phi_1(y+2x)+\phi_2(y-3x)$

P.I= $\dfrac{1}{D^2+DD'-6D'^2}(x^2y+e^{3x+y})$

$$=\frac{1}{D^2\left(1+\frac{D}{D'}-6\frac{D'^2}{D^2}\right)}x^2y+\frac{1}{D^2+DD'-6D'^2}e^{3x+y}$$

$$=\frac{1}{D^2}\left(1+\frac{D}{D'}-6\frac{D'^2}{D^2}\right)^{-1}x^2y+\frac{1}{9+3(1)-6(1)}e^{3x+y}$$

$$=\frac{1}{2}\left[x^4y-\frac{x^5}{5}\right]+\frac{e^{3x+y}}{6}$$

The general Solution is $y=\phi_1(y+2x)+\phi_2(y-3x)+=\frac{1}{2}\left[x^4y-\frac{x^5}{5}\right]+\frac{e^{3x+y}}{6}$

2. a) **Solve** $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy$

This is a Lagrange's linear equation of the form $Pp+Qq=R$

The subsidiary equations are $\dfrac{dx}{x^2-yz}=\dfrac{dy}{y^2-zx}=\dfrac{dz}{z^2-xy}$ - (1)

$$\frac{dx-dy}{\left(x^2-yz\right)-\left(y^2-zx\right)}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dx-dz}{x^2-yz-z^2+xy}$$

i.e., $\dfrac{d\left(x-y\right)}{\left(x^2-y^2\right)+z\left(x-y\right)}=\dfrac{d\left(y-z\right)}{y^2-z^2+x\left(y-z\right)}=\dfrac{d\left(x-z\right)}{x^2-z^2+y\left(1-z\right)}$

i.e., $\dfrac{d\left(x-y\right)}{\left(x-y\right)\left(x+y+z\right)}=\dfrac{d\left(y-z\right)}{\left(y-z\right)\left(x+y+z\right)}=\dfrac{d\left(x-z\right)}{\left(x-z\right)\left(x+y+z\right)}$ - (2)

i.e., $\dfrac{d\left(x-y\right)}{\left(x-y\right)}=\dfrac{d\left(y-z\right)}{y-z}=\dfrac{d\left(x-z\right)}{x-z}$ - (3)

Taking the first two ratios, and integrating $\log\left(x-y\right)=\log\left(y-z\right)+\log a$

$\therefore \dfrac{x-y}{y-z}=a$ - (4)

Similarly taking the last two ratios of (3) we get,

$\therefore \dfrac{y-z}{x-z}=b$ - (5)

But $\dfrac{x-y}{y-z}$ and $\dfrac{y-z}{x-z}$ are not independent solutions for $\dfrac{x-y}{y-z}+1$ gives $\dfrac{x-z}{y-z}$ which is the reciprocal of the second solution.

Therefore solution given by (4) and (5) are not independent. Hence we have to search for another independent solution.

Using multipliers x, y, z in equation (1) each ratio is $=\dfrac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$

Using multipliers 1, 1, 1 each ratio is $=\dfrac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$

$$\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{\frac{1}{2}d\left(x^2+y^2+z^2\right)}{\left(x+y+z\right)\left(x^2+y^2+z^2-xy-yz-zx\right)}=\frac{d\left(x+y+z\right)}{x^2+y^2+z^2-xy-yz-zx}$$

Hence $\dfrac{1}{2}d\left(x^2+y^2+z^2\right)=\left(x+y+z\right)d\left(x+y+z\right)$

Integrating $\dfrac{1}{2}\left(x^2+y^2+z^2\right)=\dfrac{\left(x+y+z\right)^2}{2}+k$

$\therefore \quad \left(x^2+y^2+z^2\right)=\left(x+y+z\right)^2+2k$

$x^2+y^2+z^2=x^2+y^2+z^2+2xy+2yz+2zx+2k$

		i.e., $xy+yz+zx=b$ $\therefore$ the general solution is $\phi\left(xy+yz+zx, \frac{x-y}{y-z}\right)=0$
	b)	**Solve** $z^2(p^2+q^2)=x^2+y^2$ **[**A.U Nov/ Dec 2007]
		Solution: The given equation can be written as $(zp)^2+(zq)^2=x^2+y^2$ ....(1) This is of the type of f (x, z^mp)=f(y, z^mq) $Put\ z^2 = Z\ \ and\ P = 2zp\ ,Q = 2zq \Rightarrow zp = \frac{P}{2}\ \&\ zq = \frac{Q}{2}$ Equation (1) can be written as $P^2+Q^2=4(x^2+y^2)$ ⇨ $P^2-4x^2=Q^2-4y^2$ The equation is of the form f(x,p) =f (y,q) ⇨ $P^2-4x^2=Q^2-4y^2=4a^2$; $P^2-4x^2=4a^2$ and $Q^2-4y^2=4a^2$ ⇨ $P=2\sqrt{x^2+a^2}$ and $Q=2\sqrt{y^2-a^2}$ We know that dZ= Pdx + Qdy $dZ=\left(2\sqrt{x^2+a^2}\right)dx+\left(2\sqrt{y^2-a^2}\right)dy$ Integrating we get Z = $x\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+y\sqrt{y^2-a^2}-\cosh^{-1}\left(\frac{y}{a}\right)$ $z^2=x\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+y\sqrt{y^2-a^2}-\cosh^{-1}\left(\frac{y}{a}\right)$
3.	**a)**	**Find the singular solution of the equation** $z=px+qy+p^2q^2$
		Solution: The equation is of the form z= px+qy+f(p,q) For complete integral replace ‘p’ by ‘a’ and ‘q’ by b ,we get $z=ax+by+a^2b^2$ ⇨ Complete integral = $z=ax+by+a^2b^2$ -----(1) Diff w.r.t ‘a’ and ‘b’ we get $\frac{\partial z}{\partial a}=x+2ab^2=0$......(2) $\frac{\partial z}{\partial b}=y+2ba^2=0$......(3) Solving we get $x=-2ab^2$......(4) & $y=-2ba^2$......(5)

The equation (4) can be written as $a = \frac{-x}{2b^2}$ ......(6)

Substituting (6) in (5) we get $b = \left(\frac{-x^2}{2y}\right)^{1/3}$ ......(7) and $a = \left(\frac{-y^2}{2x}\right)^{1/3}$ ......(8)

Substituting (7) & (8) in (1) we get $16z^3 + 27x^2y^2 = 0$ which is singular integral.

b) **Solve** $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + e^{-x+y}$

Solution:

The auxiliary equation is $2m^2 - 5m + 2 = 0$

$m = 2, \frac{1}{2}$

C.F= $\phi_1(y+2x) + \phi_2\left(y + \frac{1}{2}x\right)$

P.I= $\frac{1}{2D^2 - 5DD' + 2D'^2}\left(5\sin(2x+y) + e^{-x+y}\right)$

$$= \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x+y) + \frac{1}{2D^2 - 5DD' + 2D'^2} e^{-x+y}$$

$$= 5\frac{1}{-8+10-2}\sin(2x+y) + \frac{1}{2(1)-5(-1)+2(1)} e^{-x+y}$$

$$= 5\frac{x}{4D-5D'}\sin(2x+y) + \frac{1}{9}e^{-x+y}$$

$$= 5\frac{x(4D+5D')}{4D-5D'}\frac{4D+5D'}{4D+5D'}\sin(2x+y) + \frac{1}{9}e^{-x+y}$$

$$= 5\frac{x(4D+5D')}{16D^2 - 25D'^2}\sin(2x+y) + \frac{1}{9}e^{-x+y}$$

$$= \frac{5}{39}\left[8x\cos(2x+y) - 5x\cos(2x+y)\right] + \frac{1}{9}e^{-x+y}$$

The General Solution is

$$y = \phi_1(y+2x) + \phi_2\left(y + \frac{1}{2}x\right) + \frac{5}{39}\left[8x\cos(2x+y) - 5x\cos(2x+y)\right] + \frac{1}{9}e^{-x+y}$$

4. **a)** **Solve** $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

Solution:

This is a Lagrange's linear equation of the form $Pp + Qq = R$

The subsidiary equations are $\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$

Taking x, y, z as multipliers, we have each fraction $= \frac{xdx + ydy + zdz}{0}$

$\therefore xdx + ydy + zdz = 0$

		Integrating $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c$ i.e., $x^2+y^2+z^2=C_1$ - (1) Again, taking the last two members, we have $\frac{dy}{x(y+z)}=\frac{dz}{x(y-z)}$ i.e., $\frac{dy}{y+z}=\frac{dz}{y-z}$ $(y-z)dy=(y+z)dz$ $ydy-zdy=ydz+zdz$ $ydy-(zdy+ydz)-zdz=0$ $ydy-d(yz)-zdz=0z$ Integrating we get $y^2-2yz-z^2=C_2$ - (2) From (1) & (2) the general solution is $\phi\left(x^2+y^2+z^2, y^2-2yz-z^2\right)=0$
	b)	**Solve** $(D^2-D'^2-3D+3D')z=xy+7$ (Nov/ Dec 2009)
		Solution: The given equation is $(D^2-D'^2-3D+3D')z=0$ This can be factorized as $(D-D'-0)(D-(-1)D'-3)z=0$ It is of the form $(D-m_1D'-c_1)(D-m_2D'-c_2)=0$ where $m_1=1$ $c_1=0$, $m_2=-1$ $c_2=3$. C.F= $f_1(y+x)+e^{3x}f_2(y-x)$ P.I = $\frac{xy+7}{(D-D')(D+D'-3)}=\frac{1}{(D-D')}\left[\frac{1}{-3\left(1-\frac{D+D'}{3}\right)}\right]xy+7$ $=\frac{-1}{3(D-D')}\left(1-\frac{D+D'}{3}\right)^{-1}(xy+7)=\frac{-1}{3(D-D')}\left(1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\ldots\right)(xy+7)$ $=\frac{-1}{3(D-D')}\left(1+\frac{D+D'}{3}+\frac{2}{9}DD'\right)(xy+7)$ $=\frac{-1}{3(D-D')}\left((xy+7)+\frac{1}{3}D(xy+7)+\frac{1}{3}D'(xy+7)+\frac{2}{9}DD'(xy+7)\right)$

		$$= \frac{-1}{3(D-D')}\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}\right) = = \frac{-1}{3D\left(1-\frac{D'}{D}\right)}\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}\right)$$ $$= \frac{-1}{3D}\left[\left(1-\frac{D'}{D}\right)^{-1}\right]\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}\right)$$ $$= \frac{-1}{3D}\left[\left(1+\frac{D'}{D}+\left(\frac{D'}{D}\right)^2+....\right)\right]\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}\right)$$ $$= \frac{-1}{3D}\left[\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}+\frac{D'}{D}\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}\right)\right)\right]$$ $$= \frac{-1}{3D}\left[\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}+\frac{1}{D}\left(x+\frac{1}{3}\right)\right)\right]$$ $$= \frac{-1}{3D}\left[\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}+\int\left(x+\frac{1}{3}\right)dx\right)\right]$$ $$= \frac{-1}{3D}\left[\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}+\left(\frac{x^2}{2}+\frac{x}{3}\right)\right)\right] =$$ $$\frac{-1}{3}\int\left((xy+7)+\frac{1}{3}(x+y)+\frac{2}{9}+\left((\frac{x^2}{2}+\frac{x}{3}\right)\right)dx$$ $$= \frac{-1}{3}\left[\frac{x^2y}{2}+7x)+\frac{1}{3}\left(\frac{x^2}{2}+xy\right)+\frac{2}{9}x+\left(\frac{x^3}{6}+\frac{x^2}{6}\right)\right] = \frac{-1}{3}\left[\frac{x^2y}{2}+\frac{65x}{9}+\frac{x^2}{3}+\frac{xy}{3}+\frac{x^3}{6}\right]$$ $$z = f_1(y+x)+e^{3x}f_2(y-x)-\frac{1}{3}\left[\frac{x^2y}{2}+\frac{65x}{9}+\frac{x^2}{3}+\frac{xy}{3}+\frac{x^3}{6}\right]$$
5.	**a)**	**Find the singular solution of the equation** $z = px+qy+p^2+pq+q^2$
		Solution: The equation is of the form z = px +qy + f(p,q) For complete integral replace 'p' by 'a' and 'q' by b ,we get $z = ax+by+a^2+ab+b^2$ ⇨ Complete integral = $z = ax+by+a^2+ab+b^2$ -----(1) Diff w.r.t 'a' and 'b' we get $\frac{\partial z}{\partial a} = x+2a+b = 0$......(2) $\frac{\partial z}{\partial b} = y+a+2b = 0$......(3) Solving we get $a = \frac{-(x+b)}{2}$......(4) substituting in (3) we get $b = \frac{(x-2y)}{3}$......(5) Substituting (5) in (4) we get $a = \frac{(y-2x)}{3}$......(6) Substituting (5) & (6) in (1) we get $3z - xy - x^2 - y^2 = 0$ which is singular integral.

	b)	**Solve** $(D^2+2DD'+D'^2-2D-2D')z=e^{3x+y}+4$
		Solution: $(D^2+2DD'+D'^2-2D-2D')z=e^{3x+y}+4$ $\left((D+D')^2-2(D+D')\right)z=e^{3x+y}+4$ $\left((D+D')(D+D'-2)\right)z=e^{3x+y}+4$ C.F= $\phi_1(y-x)+e^{2x}\phi_2(y-x)$ P.I= $\frac{1}{D^2+2DD'+D'^2-2D-2D'}\left(e^{3x+y}+4e^{0x+0y}\right)$ $=\frac{1}{D^2+2DD'+D'^2-2D-2D'}e^{3x+y}+\frac{1}{D^2+2DD'+D'^2-2D-2D'}4e^{0x+0y}$ $=\frac{1}{9+6+1-6-2}e^{3x+y}+\frac{x}{2D+2D'-2}4e^{0x+0y}$ $=\frac{1}{8}e^{3x+y}+\frac{x}{-2}4e^{0x+0y}$ $=\frac{1}{8}e^{3x+y}-2x$
	UNIT II FOURIER SERIES **PART-A**	
1.	**State Dirichlet's conditions for the existence of Fourier series of f(x) in the interval $(0,2\pi)$.** **(Nov./Dec.2011,Reg 2008)**	
	A function f(x) can be expanded as a Fourier series in the interval $(0,2\pi)$ if the following conditions are satisfied. (i) f(x) is periodic, single valued and finite in $(0,2\pi)$ (ii) f(x) has finite number of finite discontinuities and no infinite discontinuity in $(0,2\pi)$ (iii) f(x) has a finite number of maxima and minima in $(0,2\pi)$.	
2.	**Does $f(x)=\tan x$ possess a Fourier expansion?**	
	$\tan x$ does not possess a Fourier expansion because the function $f(x)=\tan x=\frac{\sin x}{\cos x}$ has the infinite discontinuity at the point $x=\frac{\pi}{2}$.	
3.	**If** $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$,deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$ *in* $[-\pi,\pi]$	
	Put $x=\pi$ then $\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$ $\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$	
4.	**The cosine series for f(x) = x sin x in 0 < x < π is given as** **x sin x** = $1-\frac{1}{2}\cos x-2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$ **Deduce that** $1+2\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots\right]=\frac{\pi}{2}$.	

	Given x sin x = $1-\frac{1}{2}\cos x-2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$ Put x = $\frac{\pi}{2}$, a point of continuity => $\frac{\pi}{2}\sin\frac{\pi}{2}=1-\frac{1}{2}\cos\left(\frac{\pi}{2}\right)-2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos\left(\frac{n\pi}{2}\right)$ i.e. $\frac{\pi}{2}=1-2\sum_{2}^{\infty}\frac{(-1)^n}{n^2-1}\cos\left(\frac{n\pi}{2}\right)$ i.e. $\mathbf{1+2\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots\right]=\frac{\pi}{2}}$
5.	**Find the value of the Fourier series for f(x) =** $\begin{cases} x, & 0\le x<1 \\ 2 & 1<x<2 \end{cases}$ **at x=1. (Nov./Dec.2011,Reg 2010).**
	x=1 is a point of discontinuity, value of Fourier series of f(x) is $\frac{f(1-)+f(1+)}{2}$. f(1-) = $\lim_{h\to 0} f(1-h)=\lim_{h\to 0}(1-h)=1$. f(1+) = $\lim_{h\to 0} f(1+h)=\lim_{h\to 0} 2=2$. $\therefore$ f(x) at x = 1 is $\frac{1+2}{3}$=1.5
6.	**Find the constant term of the Fourier series for the function f(x) = x^2 , – π < x < π**
	Since f(x) is an even function in **– π < x < π** $a_0=\frac{2}{\ell}\int_0^{\ell} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} x^2\,dx=\frac{2}{3}\pi^2$ $\therefore$ Constant term = $\frac{a_0}{2}=\frac{\pi^2}{3}$
7.	**To which value, the half range sine series corresponding to f(x) = x^2 expressed in the interval (0, 5) converges at x =5 ?**
	At the end points half range sine series converges to zero. $\therefore$ The half range sine series corresponding to f(x) = x^2 converges to zero at x = 5.
8.	**What you meant by Harmonic analysis ?**
	The process of finding the harmonics in the Fourier series expansion of a function numerically is known as harmonic analysis.
9.	**Find the Fourier constant b_n for x sin x in $-\pi < x < \pi$, when expressed as a Fourier series.**
	Solution: $f(x)=x\sin x \Rightarrow f(-x)=-x\sin(-x)\Rightarrow x\sin x=f(x)$ Here $f(x)$ is an even function $\therefore b_n=0$
10.	**Find the root mean square value of the function f(x) = x in $(0,l)$.**
	RMS value = RMS value = $\bar{y}^2=\sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}=\frac{1}{l}\int_0^l x^2dx=\frac{l^2}{3}$.
11.	**Obtain the constant term of the Fourier cosine series of y = f(x) in (0,6) using the following table**
	x: **0** **1** **2** **3** **4** **5** **y:** **4** **8** **15** **7** **6** **2** a_o = 2 x Mean value of y =2(Σy)/6= 14 $\therefore$ Constant term = $\frac{a_0}{2}=7$
12.	**Find the value of a_n in the cosine series expansion of f(x) = K in the interval (0,10)**
	$a_n=\frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{5}\int_0^{10} K\cos\frac{n\pi x}{10}dx=0$

13.	**If f(x) = x^2+x is expressed as a Fourier series in the interval (-2,2) to which value this series converges at x = 2?**
	Since x = 2 is an end point then f(x) converges to $\frac{f(-2)+f(2)}{2}=\frac{6+2}{2}=4$
14.	**State Parseval's theorem in the interval *(c,c+2l)*.**
	If the Fourier series corresponding to f(x) converges uniformly to f(x) in (c,c+2*l*) then $\frac{1}{\ell}\int_{c}^{c+2\ell}[f(x)]^2\,dx = \frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}\left(a_n^{\,2}+b_n^{\,2}\right)$
15.	**Write down the complex form of Fourier Series in (c,c+2*l*)**
	$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{\ell}}$ where $c_n=\frac{1}{2\ell}\int_{c}^{c+2\ell}f(x)e^{\frac{-in\pi x}{\ell}}dx$
16.	**If f(x) = e^x in $-\pi<x<\pi$, find a_n.**
	Solution: $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\cos nx dx=\frac{1}{\pi}\left\{\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right\}_{-\pi}^{\pi}=\frac{1}{\pi}\left\{\frac{e^{\pi}}{1+n^2}(-1)^n-\frac{e^{-\pi}}{1+n^2}(-1)^n\right\}$ $a_n=\frac{(-1)^n}{\pi\left(1+n^2\right)}\left\{e^{\pi}-e^{-\pi}\right\}.$
17.	**Determine b_n in the Fourier series expansion of $f(x)=\frac{\pi-x}{2}$ in 0< x < 2π.**
	$b_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\sin nx\,dx$ $=\frac{1}{2\pi}\left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}=\frac{1}{2\pi}\frac{2\pi}{n}=\frac{1}{n}$
18.	**Find the constant term in the expression of $\cos^2 x$ as a Fourier series in the interval $(-\pi,\pi)$.**
	Solution: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+\cos 2x}{2}dx=\frac{1}{2\pi}\left[x+\frac{\sin 2x}{2}\right]_{-\pi}^{\pi}=1$ $\therefore$ **Constant term =** $\frac{a_0}{2}=\frac{1}{2}$
19.	**Give the expression for the Fourier series coefficient b_n for the function f(x) defined in $-2\le x\le 2$.**
	Solution: $b_n=\frac{1}{2}\int_{-2}^{2}f(x)\sin\frac{n\pi x}{2}dx$
20.	**Determine the value of a_n & a_0 in the Fourier series expansion of $f(x)=x^3$ in $-\pi<x<\pi$**
	Solution: $f(x)=x^3\Rightarrow f(-x)=(-x)^3=x^3=f(x)\Rightarrow f(x)$ is an odd function $\therefore a_n=a_0=0$
	PART – B
1.	**a)** If $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in $0<x<2\pi$. Hence show that

		(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+............=\frac{\pi^2}{6}$. (ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...=\frac{\pi^2}{12}$.
		Solution: We know that $f(x)=\frac{a_0}{2}+\sum_1^{\infty}(a_n\cos nx+b_n\sin nx)$ $a_0=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2dx$ $=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2dx$ $=\frac{1}{4\pi}\left(\frac{(\pi-x)^3}{(-3)}\right)$ $=\frac{1}{4\pi}\left(\frac{-\pi^3}{-3}+\frac{\pi^3}{3}\right)=\frac{\pi^2}{6}$ $a_n=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{\alpha}\right)^2\cos nxdx$ $=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nxdx$ $=\frac{1}{4\pi}\left\{(\pi-x)^2\frac{\sin nx}{n}-\left(-2(\pi-x)\left(\frac{-\cos nx}{n^2}\right)\right)+\left(2\left(\frac{-\sin nx}{n^3}\right)\right)\right\}_0^{2\pi}$ $=\frac{1}{4\pi}\left[0+2\pi\frac{\cos 2\pi n}{n^2}+0-0+\frac{2\pi}{n^2}\right]$ $=\frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right]=\frac{1}{n^2}$ $b_n=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nxdx$ $=\frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right)-\left(-2(\pi-x)\left(\frac{-\sin nx}{n^2}\right)\right)+\left((2)\left(\frac{\cos nx}{n^3}\right)\right)\right]_0^{2\pi}$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right] = 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{1}^{\infty} \frac{1}{n^2} \cos \ n x.$$

$$= \frac{\pi^2}{12} + \left(\frac{1}{1^2} \cos x + \frac{1}{2^2} \cos x + ... \right) - (1)$$

$Put \ \ x = 0$

$$f(0) = \frac{\pi^2}{12} + \left(\frac{1}{1^2} + \frac{1}{2^2} + ... \right) - (2)$$

$x = 0$ is a pt of discontinuity.

$$\therefore \ f(0) = \frac{1}{2}\left(\frac{\pi^2}{4} + \frac{\pi^2}{4} \right) = \frac{\pi^2}{4}$$

$$\therefore (2) => \frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1^2} + \frac{1}{2^2} + ... \right)$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + ...\infty$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + ...\infty = \frac{\pi^2}{6}$$

Put $x = \pi$ in (1)

$$f(x) = \frac{\pi^2}{12} + \sum_{1}^{\infty} \frac{(-1)^n}{n^2} - (3)$$

Here π is a pt of continuity.

$$\therefore \ f(\pi) = 0.$$

$$(3) => 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ...\infty$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ...\infty$$

$$-\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+.....\infty$$

b) **Find the Fourier series for $f(x)=|\sin x|$ in $-\pi< x<\pi$.**

Solution:

Given $f(x)=|\sin x|$

Since $f(x)$ is an even function $b_n=0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad \dots\dots\dots(1)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx \quad =\frac{2}{\pi}\int_0^{\pi}|\sin x|\,dx \quad =\frac{2}{\pi}\int_0^{\pi}\sin x\,dx$$

$$=\frac{2}{\pi}[-\cos x]_0^{\pi} \quad =-\frac{2}{\pi}[\cos\pi-\cos 0] \quad =\frac{2}{\pi}[-1-1]=\frac{4}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx \quad =\frac{2}{\pi}\int_0^{\pi}|\sin x|\cos nx\,dx \quad =\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$=\frac{2}{\pi}\frac{1}{2}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]\,dx \quad =\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n}{n+1}-\frac{(-1)^n}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right] \quad =\frac{1}{\pi}\left[\frac{(-1)^n+1}{n+1}-\frac{(-1)^n+1}{n-1}\right]$$

$$=\frac{(-1)^n+1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{(-1)^n+1}{\pi}\left[\frac{n-1-n-1}{n^2-1}\right]=\frac{(-1)^n+1}{\pi}\left[\frac{-2}{n^2-1}\right]$$

$$=\frac{-2[(-1)^n+1]}{(n^2-1)\pi} \quad if\ n\neq 1$$

$$a_1=\frac{2}{\pi}\int_0^{\pi}f(x)\cos x\,dx \quad =\frac{2}{\pi}\int_0^{\pi}|\sin x|\cos x\,dx \quad =\frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx$$

$$=\frac{2}{\pi}\frac{1}{2}\int_0^{\pi}\sin 2x\,dx \quad =\frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} \quad =\frac{1}{\pi}\left[-\frac{1}{2}+\frac{1}{2}\right]=0$$

		Substitute in equation (1) we get $f(x)=\frac{2}{\pi}+\sum_{n=2}^{\infty}\frac{-2\,[(-1)^{n}+1]}{(n^{2}-1)\pi}\cos nx$
2.	**a)**	**Obtain the Fourier series to represent the function f(x) = | x | is $-\pi<x<\pi$ and deduce that** $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots=\frac{\pi^2}{8}$**.**
		Solution: Given $f(x)=\lvert x\rvert$ $f(-x)=f(x)$ $\therefore$ The given function is an even function. Hence $b_n=0$ $\therefore f(x)=\frac{a_0}{2}+\sum_{1}^{\infty}a_n\cos nx$ $a_n=\frac{2}{\pi}\int_0^{\pi}x\,dx$ $=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\pi$ $a_n=\frac{2}{\pi}\int_0^{\pi}x\cos nxdx$ $=\frac{2}{\pi}\left\{x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right\}_0^{\pi}=\frac{2}{\pi}\left\{\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right\}=\frac{2}{n^2\pi}\left\{(-1)^n-1\right\}$ $\therefore a_n=0$ if n is even $a_n=-\frac{4}{n^2\pi}$ if n is odd $\therefore f(x)=\frac{\pi}{2}+\sum_{1,3}^{\infty}\frac{-4}{n^2\pi}\cos nx$ $=\frac{\pi}{2}-\frac{-4}{\pi}\left\{\cos x+\frac{\cos 3x}{3^2}+\ldots\right\}$ $Put\ x=0$

$$f(0)=\frac{\pi}{2}-\frac{4}{\pi}\left\{1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right\}$$

Here 0 is a pt of continuity

$$\therefore f(0)=0$$

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left\{1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right\}$$

$$\frac{\pi}{2}=\frac{4}{\pi}\left\{1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right\}$$

$$\therefore \frac{\pi^2}{8}=\left\{1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right\}.$$

b) **Find the first two harmonic of the Fourier series of f (x) given by**

x	**0**	**1**	**2**	**3**	**4**	**5**
f (x)	**9**	**18**	**24**	**28**	**26**	**20**

Solution:

Here the length of the in level is $2l=6,\ l=3$

x	$\frac{\pi x}{3}$	$\frac{2\pi x}{3}$	y	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	0	0	9	9	0	9	0
1	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	18	9	15.7	-9	15.6
2	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	24	-12	20.9	-12	-20.8
3	π	2π	28	-28	0	28	0
4	$\frac{4\pi}{3}$	$\frac{8\pi}{3}$	26	-13	-22.6	-13	22.6
5	$\frac{5\pi}{3}$	$\frac{10\pi}{3}$	20	10	17.6	-10	-17.4
			125	-25	-3.4	-7	0

$$f(x)=\frac{a_0}{2}+\left(a_1\cos\frac{\pi x}{3}+b_1\sin\frac{\pi x}{3}\right)+\left(a_2\cos\frac{2\pi x}{3}+b_2\sin\frac{2\pi x}{3}\right)$$

$$a_0=2\frac{\sum y}{6}=\frac{2(125)}{6}=41.66$$

$$a_1 = \frac{2}{6}\left\{\sum y\cos\frac{\pi x}{3}\right\} = -8.33$$

$$b_1 = \frac{2}{6}\sum y\sin\frac{\pi x}{3} = -1.15$$

$$a_2 = \frac{2}{6}\left\{\sum y \quad \cos\frac{2\pi x}{3}\right\} = -2.33$$

$$b_2 = \frac{2}{6}\sum y \quad \sin\frac{2\pi x}{3} = 0$$

$$\therefore f(x) = \frac{41.66}{2} - 8.33\cos\frac{\pi x}{3} - 2.33\cos\frac{2\pi x}{3} - 1.15\sin\frac{\pi x}{3}.$$

3. a) **Find the Fourier series for the function** $f(x) = \begin{cases} x-1, & -\pi < x < 0 \\ x+1, & 0 < x < \pi \end{cases}$

Solution:

Given $f(x) = \begin{cases} x-1, & -\pi < x < 0 \\ x+1, & 0 < x < \pi \end{cases}$

$$f(-x) = \begin{cases} -x-1, & 0 < x < \pi \\ -x+1, & -\pi < x < 0 \end{cases}$$
$$= \begin{cases} -(x+1), & 0 < x < \pi \\ -(x-1), & -\pi < x < 0 \end{cases}$$
$$= -f(x)$$

Therefore $f(x)$ is an odd function. Therefore $a_0 = 0,\ a_n = 0$

$$Hence\ \ f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots(1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad = \frac{2}{\pi}\int_0^{\pi}(x+1)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x+1)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(\pi+1)\left(\frac{\cos n\pi}{n}\right) + \frac{1}{n}\right] \qquad = \frac{2}{n\pi}\left[-(\pi+1)(-1)^n + 1\right]$$

$$= \frac{2}{n\pi}\left[1 - (1+\pi)(-1)^n\right]$$

Substitute in (1) we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (1+\pi)(-1)^n\right]\sin nx$$

b) **Find the first two harmonic of the Fourier series of f (x). Given by**

x	**0**	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)	**1**	**1.4**	**1.9**	**1.7**	**1.5**	**1.2**	**1.0**

		Solution:

$\because$ The last value of y is a repetition of the first; only the first six values will be used

The values of $y\cos x,\ y\cos 2x,\ y\sin x,\ y\sin 2x$ as tabulated

x	$f(x)$	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$
0	1.0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.5	0.866	-0.5	0.866
$\frac{2\pi}{3}$	1.9	-0.5	0.866	-0.5	0.866
π	1.7	-1	0	1	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.866	-0.5	-0.866
$\frac{5\pi}{3}$	1.2	0.5	-0.866	-0.5	-0.866

$$a_0 = 2\frac{\sum y}{6} = 2.9,\ a_1 = 2\frac{\sum y\cos x}{6} = -0.37,\ a_2 = 2\frac{\sum y\cos 2x}{6} = -0.1$$

$$b_1 = 2\frac{\sum y\sin x}{6} = 0.17,\ b_2 = 2\frac{\sum y\sin 2x}{6} = -0.06$$

4. a) **Obtain the Fourier series for $f(x)$ of period $2l$ and defined as follows**

$$f(x) = \begin{cases} l - x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$$ **Hence deduce that**

$$(i)\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \qquad (ii)\ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Solution:

Given $f(x) = \begin{cases} l - x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx \quad = \frac{1}{l}\int_0^{l}(l-x)dx = \frac{1}{l}\left[\frac{(l-x)^2}{-2}\right]_0^l \quad = \frac{1}{l}\left[\frac{l^2}{2}\right] = \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx \quad = \frac{1}{l}\int_0^{l} (l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)\left(\sin\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\cos\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^l$$

$$= \frac{1}{l}\left[-\frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right]$$

$$= -\frac{l}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{2l}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \quad = \frac{1}{l}\int_0^{l} (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^l$$

$$= \frac{1}{l}\left[l\cos 0\,\frac{l}{n\pi}\right] = \frac{l}{n\pi}$$

$$f(x) = \frac{l}{4} + \sum_{n=1,3,5\ldots}^{\infty}\left[\frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l}\right] + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l}$$

Deduction (i)

Put $x = \frac{l}{2}$ $\qquad f(x) = \frac{l}{2}$

$$\frac{l}{2} = \frac{l}{4} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{l}{n}\sin\frac{n\pi}{l} \qquad \left[\text{since } \cos\frac{n\pi}{2} = 0 \text{ if } n \text{ is odd}\right]$$

$$\frac{l}{4} = \frac{l}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \cdots\right] \Rightarrow \frac{l}{4} = \frac{l}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} + \cdots\right]$$

$$1 - \frac{1}{3} + \frac{1}{5} + \cdots = \frac{\pi}{4}$$

Deduction (ii)

Put $x = l$ $\qquad f(x) = 0$

$$0 = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{l}{n^2}\cos n\pi \qquad [\text{since } \sin n\pi = 0]$$

$$-\frac{l}{4} = \frac{2l}{\pi^2}\left[\frac{\cos\pi}{1^2} + \frac{\cos 3\pi}{3^2} + \frac{\cos 5\pi}{5^2} + \cdots\right] \Rightarrow \frac{-\pi^2}{8} = -\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \cdots$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

b) **Expand $f(x) = x$ as a cosine series in $0 < x < l$ and deduce the value of**

		$(i)\quad \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}\qquad (ii)\quad \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$
		Solution: Given $f(x)=x$ The cosine series is $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\qquad \ldots\ldots(1)$$ $$a_0=\frac{2}{l}\int_0^l f(x)dx\quad =\frac{2}{l}\int_0^l x\,dx=\frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$ $$=\frac{1}{l}[l^2]=\ l$$ $$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx\quad =\frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}dx$$ $$=\frac{2}{l}\left[x\left(\sin\frac{n\pi x}{l}\right)\frac{l}{n\pi}-\left(-\cos\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^l\quad =\frac{2}{l}\left[\frac{l^2}{n^2\pi^2}(-1)^n-\frac{l^2}{n^2\pi^2}\right]$$ $$=\frac{2l}{n^2\pi^2}[(-1)^n-1]$$ $$=\begin{cases}0 & if\ \ n\ is\ even\\ \frac{-4l}{n^2\pi^2} & if\ \ n\ is\ odd\end{cases}$$ Substituting in equation (1) we get $$f(x)=\frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=odd}\frac{1}{n^2}\cos\frac{n\pi x}{l}\qquad \ldots\ldots(2)$$ **<u>Deduction (i)</u>** By Parseval's identity $$\frac{2}{l}\int_0^l[f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2\quad \Rightarrow\frac{2}{l}\int_0^l x^2\,dx=\frac{l^2}{2}+\sum_{n=odd}\left(\frac{-4l}{n^2\pi^2}\right)^2$$ $$\Rightarrow\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l=\frac{l^2}{2}+\sum_{n=odd}\frac{16\,l^2}{n^4\pi^4}\quad \Rightarrow\frac{2}{l}\left[\frac{l^3}{3}\right]=\frac{l^2}{2}+\frac{16\,l^2}{\pi^4}\sum_{n=odd}\frac{1}{n^4}$$ $$\Rightarrow\frac{2l^2}{3}=\frac{l^2}{2}+\frac{16\,l^2}{\pi^4}\sum_{n=odd}\frac{1}{n^4}\quad \Rightarrow\frac{16\,l^2}{\pi^4}\sum_{n=odd}\frac{1}{n^4}=\frac{2l^2}{3}-\frac{l^2}{2}\ =\ l^2\left[\frac{2}{3}-\frac{1}{2}\right]\ =\ \frac{l^2}{6}$$

$$\sum_{n=odd} \frac{1}{n^4} = \frac{l^2}{6}\left[\frac{\pi^4}{16\, l^2}\right] = \frac{\pi^4}{96}$$

$$(i.e.)\ \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

<u>Deduction (ii)</u>

$$Let\ S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$= \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] + \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right]$$

$$= \frac{\pi^4}{96} + \frac{1}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right] \qquad by\ (i)$$

$$(i.e.)\ S = \frac{\pi^4}{96} + \frac{1}{2^4} S = \frac{\pi^4}{96} + \frac{1}{16} S$$

$$S\left(1 - \frac{1}{16}\right) = \frac{\pi^4}{96} \Rightarrow \quad S\left(\frac{15}{16}\right) = \frac{\pi^4}{96}$$

$$S = \frac{\pi^4}{96}\left(\frac{16}{15}\right) = \frac{\pi^4}{90}$$

$$(i.e.)\ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

5. a) Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$

Solution:

The complex form of the Fourier series in $(-1, 1)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x} \qquad \ldots\ldots..(1)$$

Where

$$c_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-x}\, e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)} = \frac{e(-1)^n - e^{-1}(-1)^n}{2(1+in\pi)}$$

$$c_n = \frac{(e - e^{-1})(-1)^n}{2}\left(\frac{1 - in\pi}{1 + n^2\pi^2}\right) = \frac{(-1)^n(1 - in\pi)}{1 + n^2\pi^2}\sinh 1$$

Hence (1) becomes

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1 - in\pi)}{1 + n^2\pi^2}\sinh 1\ e^{in\pi x}$$

	b)	**Find the half range Fourier sine series for $f(x)=x(\pi-x)$ in the interval $(0,\pi)$ and deduce that** $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-......\infty$
		Solution: **The sine series is** $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$ $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ $=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx\;dx$ $=\frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$ $=\frac{4}{\pi n^3}\left[1-(-1)^n\right]=\begin{cases}0 & if\quad n\ is\,even\\ \frac{8}{\pi n^3} & if\quad n\ is\,odd\end{cases}$ $f(x)=\sum_{n=1,2,3,....}^{\infty}\frac{8}{\pi n^3}\sin nx$ **Deduction:** Put $x=\frac{\pi}{2}$ is point of continuity $f\left(\frac{\pi}{2}\right)=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-...\right]$ $\frac{\pi^2}{4}=\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-...\right]$ $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-...=\frac{\pi^2}{32}$
	UNIT- III - APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS	
	PART – A	
1.	**State any two assumptions made in the derivation of the one dimensional wave equation.**	
	i. The string is perfectly elastic and does not offer any resistance to bending. ii. Deflection y and slope dy/dx at every point of the string are small, so that their higher powers may be neglected.	

2.	**Write down the partial differential equation governing one dimensional wave equation?**
	$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$ where y(x,t) is the displacement of the string **.**
3.	**In the wave equation** $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$ **, what does *C^2* stands for?** $C^2 = T/m$, where T is the tension and m is the mass of the string.
4.	**Write all possible solutions of the transverse vibration of the string in one dimension.** (i) $y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$ (ii) $y(x,t) = (A\cos px + B\sin px)(C\cos pat + \sin pat)$. (iii) $y(x,t) = (Ax + B)(Ct + D)$
5.	**Write down the appropriate solution of the vibration of string equation. How is it chosen?** The appropriate solution of the vibration of string equation is y(x,t) = (A cos px+ B sin px) (C cos pat+ D sin pat) . When we deal with the vibration of an elastic string y(x,t) representing the displacement of the string at any point x, it must be periodic in ‘t’. Hence the above solution which consists of periodic functions in‘t’ is the proper solution of the problems on vibration of strings.
6.	**A tightly stretched string with fixed end points x=0 and x=*l* is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity x(*l*-x), write down the boundary and initial conditions of the above problem.** $y(0,t) = 0 \;\; and \;\; y(l,t) = 0 \quad for \;\; t \geq 0$ $y(x,0) = 0 \;\; and \;\; \frac{\partial y(x,0)}{\partial t} = \lambda x(l - x) \quad for \;\; 0 \leq x \leq l$
7.	**State Fourier law of heat conduction.** The rate at which heat flows through an area is jointly proportional to the area and to the temperature gradient normal to the area.
8.	**Write down the partial differential equation that represents one dimensional heat equation.** $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, **where** $\alpha^2 = \frac{k}{\rho c}$
9.	**What are the various possible solutions of one dimensional diffusion equation?** (i) $u(x,t) = (Ae^{px} + Be^{-px})Ce^{\alpha^2 p^2 t}$ (ii) $u(x,t) = (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t}$ (iii) $u(x,t) = Ax + B$
10.	**Why can’t u(x ,t) =** $(Ae^{px} + Be^{-px})Ce^{\alpha^2 p^2 t}$ **be the correct solution in solving one dimension heat equation?** As $t \to \infty$, $u \to \infty$. It is not possible.
11.	**In steady state condition derive the solution of one dimension heat flow equation.**

	The one dimension heat flow equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$. In steady state condition, $\frac{\partial u}{\partial t}=0$. $\therefore\ \frac{\partial^2 u}{\partial x^2}=0 \Rightarrow \frac{\partial u}{\partial x}=A \Rightarrow u=Ax+B$.
12.	**An insulated rod of length 60 cm has its ends A and B maintained at) ℃ and) ℃ respectively. Find the steady state temperature in the rod. (Nov. 2012)** Steady state heat equation is $\frac{d^2u}{dx^2}=0$. The solution is $u(x)=Ax+B$ ……..(1) Given conditions are $u(0)=20, u(60)=80$. Substituting in (1) we get $u=x+20$, which is the steady state temperature in the rod.
13.	**Write down the partial differential equation that represents variable heat flow in two dimensions? Deduce the equations of steady state heat flow in two dimensions.** The heat flow equation in two dimensional Cartesian co-ordinate system is $\frac{\partial u}{\partial t}=\alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$ When steady state conditions prevail in the plate the temperature at any point of the plate does not depend on ‘t’, but depends on x and y only (i.e.) $\frac{\partial u}{\partial t}=0$. Thus steady state temperature distribution in a two dimensional plate is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (Laplace equation).
14.	**Write any two solutions of the Laplace equation obtained by the method of separation of variables involving exponential terms in x and y.** $u(x,y)=(Ae^{px}+Be^{-px})(C\cos py+\sin py)$ & $u(x,y)=(A\cos px+B\sin px)(Ce^{py}+e^{-py})$
15.	**Given the boundary conditions on a square or rectangular plate, how will you identify the proper solution?** If the non-zero temperature is prescribed either on x=0 or on x=a (vertical edges), the proper solution will be $u(x,y)=(Ae^{px}+Be^{-px})(C\cos py+\sin py)$ If the non-zero temperature is prescribed either on y=0 or y=b (horizontal edges), the proper solution will be $u(x,y)=(A\cos px+B\sin px)(Ce^{py}+e^{-py})$
16.	**An infinitely long plate is bounded by two parallel edges and an end at right angles to them. The breath of the edge y = 0 is π and it is maintained at constant temperature u_0 at all points and the other edges are kept at zero temperatures. Formulate the boundary value problem to determine the steady state temperature.** $\nabla^2 u=0$ **subject to** $u(0,y)=0,\ u(\pi,y)=0 \quad for \quad y\ge 0$ $u(x,\infty)=0 \ \ and \ \ u(x,0)=u_0 \quad for \ \ 0\le x\le \pi$
17.	**A plate is bounded by the lines x=0, y=0, x=*l*, and y=*l*. Its faces are insulated. The edge coinciding with x-axis is kept at)0 ℃ . The edge coinciding with y-axis is kept at) ℃ . The other two**

edges are kept at C . Write the boundary conditions that are needed for solving two dimensional heat flow equation. (Nov. 2012)

Boundary conditions are

(i) $u(x,0)=100^\circ C;\ 0<x<l$. (ii) $u(0,y)=50^\circ C; 0<y<l$.

(ii) $u(x,l)=0^\circ C; 0<x<l$. (iv) $u(l,y)=0^\circ C;\ 0<y<l$.

18. The boundary value problem governing the steady state temperature distribution in a flat thin plate is given by $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ **,** $u(x,0)=0$ **,** $u(0,y)=0$, $u(a,y)=0$ **and** $u(x,a)=4\sin^3\left[\frac{\pi x}{a}\right]$**. Find C_n when the most general solution is** $u(x,y)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$.

$$u(x,a)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\sinh n\pi = 4\sin^3\left[\frac{\pi x}{a}\right]=4\left[\frac{3\sin\frac{\pi x}{a}-\sin\frac{3\pi x}{a}}{4}\right]$$

$$c_1\sin\frac{\pi x}{a}\sinh\pi+c_2\sin\frac{2\pi x}{a}\sinh 2\pi+c_3\sin\frac{3\pi x}{a}\sinh 3\pi+........=3\sin\frac{\pi x}{a}-\sin\frac{3\pi x}{a}$$

$\therefore\ c_1\sinh\pi=3,$ $\quad c_3\sinh 3\pi=1,$

$\Rightarrow c_1=\frac{3}{\sinh\pi}$ $\quad \Rightarrow c_3=\frac{1}{\sinh 3\pi}$

$c_2=c_4=c_5=c_6=.........=0$

19. Define thermally insulated ends.

The end at which the temperature gradient is zero is called thermally insulated ends. (i.e) there is no heat flow throws the ends.

20. When the heat flow is called two dimensional?

When the heat flow is along curves instead of along straight lines, all the curves lying in parallel planes, then the flow is called two dimensional.

PART B

1. A uniform string is stretched and fastened to two points $'l'$ apart. Motion is started by displacing the string into the form of the curve $y=kx(l-x)$ and then releasing it from this position at time $t=0$. Find the displacement of the point of the string at a distance x from one end at time t.

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

Boundary conditions are (i) $y(0,t)=0$ (ii) $y(l,t)=0$ (iii) $\frac{\partial y}{\partial t}(x,0)=0$ (iv) $y(x,0)=f(x)=kx(l-x)$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \qquad (1)$$

Put $x = 0$ and apply the boundary condition (i) in (1)

$$y(0,t) = (A)(B\cos pat + D\sin pat) = 0 \quad \Rightarrow \boxed{A = 0}$$

Equation (1) becomes $y(x,t) = B\sin px(C\cos pat + D\sin pat)$ (2)

Put $x = l$ and apply the boundary condition (ii) in (2)

$$y(l,t) = B\sin pl(C\cos pat + D\sin pat) = 0 \quad \Rightarrow \sin pl = 0 \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Equation (2) becomes $y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$ (3)

Differentiate Partially (3) w. r. to t

$$\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{l}\left(-C\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + D\frac{n\pi a}{l}\cos\frac{n\pi at}{l}\right) \qquad (4)$$

Put t = 0 and apply the boundary condition (iii) in (4)

$$\frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}D\frac{n\pi a}{l} = 0 \quad \Rightarrow \boxed{D = 0}$$

Equation (3) becomes

$$y(x,t) = B\sin\frac{n\pi x}{l}C\cos\frac{n\pi at}{l}$$

$$= B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

where $B_n = BC$

By the superposition principle, the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad (5)$$

Put t = 0 and apply the boundary condition (iv) in (5)

	$$y(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}.1 = f(x) \qquad (6)$$ By half range Fourier sine series $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ (7) By comparing (6) and (7),we get $\boxed{b_n = B_n}$ $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$ $$=\frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}dx$$ $$=\frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$ $$=\frac{2k}{l}\left[0+0-2.\frac{l^3}{n^3\pi^3}\cos n\pi+0-0+2.\frac{l^3}{n^3\pi^3}.1\right]$$ $$=\frac{2k}{l}\frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$ $$=\frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$ $$=\begin{cases}0 & if\ n\ is\ even\\ \frac{8kl^2}{n^3\pi^3} & if\ n\ is\ odd\end{cases}$$ $$=B_n$$ $\therefore$ The equation (5) becomes $y(x,t)=\sum_{n=1,3,5}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$
2.	**If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that** $v=\begin{cases}cx;\ 0<x<\frac{l}{2}\\ c(l-x);\ \frac{l}{2}<x<l\end{cases}$ **, find the displacement at any time** t **.**
	Solution: One dimensional wave equation $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$ The boundary conditions are

(i) $y(0,t) = 0$ (ii) $y(l,t) = 0$ (iii) $y(x,0) = 0$

(iv) $$\frac{\partial y}{\partial t}(x,0) = \begin{cases} cx; & 0 < x < \frac{l}{2} \\ c(l-x); & \frac{l}{2} < x < l \end{cases}$$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad (1)$$

Put $x = 0$ and apply the boundary condition (i) in (1)

$$y(0,t) = (A)(B\cos pat + D\sin pat) = 0 \quad \Rightarrow \boxed{A = 0}$$

Equation (1) becomes $y(x,t) = B\sin px(C\cos pat + D\sin pat)$ (2)

Put $x = l$ and apply the boundary condition (ii) in (2)

$$y(l,t) = B\sin pl(C\cos pat + D\sin pat) = 0 \quad \Rightarrow \sin pl = 0 \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Equation (2) becomes $y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$ (3)

Put t = 0 and apply the boundary condition (iii) in (3)

$$y(x,0) = B\sin\frac{n\pi x}{l}C = 0 \quad \Rightarrow \boxed{C = 0}$$

Equation (3) becomes

$$\begin{aligned} y(x,t) &= B\sin\frac{n\pi x}{l}D\sin\frac{n\pi at}{l} \\ &= B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \end{aligned} \qquad \text{where } B_n = BD$$

By the superposition principle, the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad (4)$$

Differentiate Partially (4) w.r.to t

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\frac{n\pi a}{l} \qquad (5)$$

Apply the boundary condition (iv) in (5)

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}\frac{n\pi a}{l}=\begin{cases} cx; & 0<x<\frac{l}{2} \\ c(l-x); & \frac{l}{2}<x<l \end{cases}=f(x) \qquad (6)$$

By half range Fourier sine series $f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$ (7)

By comparing (6) and (7), we get $b_n=B_n\frac{n\pi a}{l} \Rightarrow B_n=\frac{l}{n\pi a}b_n$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2c}{l}\left[\int_0^{l/2}x\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$=\frac{2c}{l}\left\{\left(x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right)_0^{l/2}\right.$$

$$\left.+\left((l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right)_{l/2}^{l}\right\}$$

$$=\frac{2c}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n=\frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore B_n=\frac{l}{n\pi a}\frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2}=\frac{4cl^2}{n^3\pi^3a}\sin\frac{n\pi}{2}$$

$\therefore$ The equation (4) becomes

	$$y(x,t)=\sum_{n=1}^{\infty}\frac{4cl^2}{n^3\pi^3a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$
3.	**The ends A and B of a rod *l* cm long have their temperatures kept at $30^\circ C$ and $80^\circ C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60^\circ C$ and that of A is increased to $40^\circ C$. Find the steady state temperature distribution in the rod after time t.**

Solution:

One dimensional heat equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ (1)

In the steady state condition the solution is $u(x)=ax+b$ (2)

The boundary conditions are $(i)\ u(0)=30 \quad (ii)\ u(l)=80$

Apply (i) in (2), $u(0)=b=30$

Then the equation (2) becomes $u(x)=ax+30$ (3)

Apply (ii) in (3) $u(l)=al+30=80 \Rightarrow a=\frac{50}{l}$

Then the equation (3) becomes $u(x)=\frac{50x}{l}+30$

Now consider the unsteady state condition.

In unsteady state the correct solution is

$$u(x,t)=(A\cos px+B\sin px)\,Ce^{-a^2p^2t} \quad (4)$$

The boundary conditions are

$$(iii)\ u(0,t)=40 \qquad (iv)\ u(l,t)=60 \qquad (v)\ u(x,0)=u(x)=\frac{50x}{l}+30$$

Since we have all non zero boundary conditions, we write the temperature distribution function

$$u(x,t)=u_s(x)+u_t(x,t) \quad (5)$$

$$\Rightarrow \quad u_t(x,t)=u(x,t)-u_s(x)$$

To find $u_s(x)$

The solution is $u_s(x) = ax + b$ (6)

The boundary conditions are $u_s(0) = 60$, $u_s(l) = 40$

$\therefore \quad u_s(0) = b = 40,$

$$u_s(l) = al + b = 60$$
$$\Rightarrow al + 40 = 60$$
$$\Rightarrow a = \frac{20}{l}$$

$\therefore$ the equation (6) becomes $u_s(x) = \dfrac{20x}{l} + 40$ (7)

To find $u_t(x,t)$

Given the boundary conditions are

$$(vi)\, u_t(x,t) = u(0,t) - u_s(0) = 40 - 40 = 0$$
$$(vii)\, u_t(l,t) = u(l,t) - u_s(l) = 60 - 60 = 0$$
$$(viii)\, u_t(x,0) = u(x,0) - u_s(x) = \frac{50x}{l} + 30 - \left(\frac{20x}{l} + 40\right)$$
$$= \frac{30x}{l} - 10$$

In unsteady state, the correct solution is $u(x,t) = (A\cos px + B\sin px)\, Ce^{-a^2p^2t}$ (8)

Apply the boundary condition (vi) in (8)

$$u_t(0,t) = ACe^{-a^2p^2t} = 0$$

here $A = 0$

$\therefore$ the equation (8) becomes $u_t(x,t) = B\sin px\, Ce^{-a^2p^2t}$ (9)

Apply the boundary condition (vii) in (9)

$$u_t(l,t) = BC\sin\, ple^{-a^2p^2t} = 0$$

$$\sin\, pl = 0$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$\therefore$ the equation (9) becomes $u_t(x,t) = B\sin\frac{n\pi x}{l}\, xCe^{\frac{-a^2n^2\pi^2}{l^2}t}$

$$u_t(x,t) = BC\sin\frac{n\pi x}{l}e^{\frac{-a^2n^2\pi^2}{l^2}t}$$

$$= B_n\sin\frac{n\pi x}{l}e^{\frac{-a^2n^2\pi^2}{l^2}t}$$

where $BC = B_n$

By the super position principle, the most general solution is $u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{\frac{-a^2n^2\pi^2}{l^2}t}$ (10)

Apply the boundary condition (viii) in (10)

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}.1 = \frac{30x}{l} - 10 = f(x)$$

Half range Fourier sine series of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ (11)

From the equations (10) & (11) we get $b_n = B_n$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l \left(\frac{30x}{l} - 10\right)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[\left(\frac{30x}{l} - 10\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{30}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{10l}{n\pi}.1\right]$$

	$= \frac{-20}{n\pi}\left[1+2(-1)^n\right]$ $= B_n$ $\therefore$ the equation (10) becomes $u_t(x,t) = \sum_{n=1}^{\infty} \frac{-20}{n\pi}\left[1+2(-1)^n\right]\sin\frac{n\pi x}{l} e^{\frac{-a^2n^2\pi^2}{l^2}t}$ Then the required temperature distribution function is $u(x,t) = \frac{20x}{l} + 40 - \sum_{n=1}^{\infty} \frac{20}{n\pi}\left[1+2(-1)^n\right]\sin\frac{n\pi x}{l} e^{\frac{-a^2n^2\pi^2}{l^2}t}$
4.	**A rectangular plate is bounded by the lines** $x=0, y=0, x=a, y=b$**. Its surfaces are insulated. The temperature along** $x=0$ **and** $y=0$ **are kept at** $0^\circ C$ **and the others at** $100^\circ C$**. Find the steady state temperature at any point of the plate.**
	Solution: We know that two dimensional heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Let l be the length of the square plate Given the boundary conditions are $(i)\ u(0,y)=0 \quad (ii)\ u(x,0)=0 \quad (iii)\ u(a,y)=100$ $(iv)\ u(x,b)=100$ We write the temperature function as $u(x,y) = u_1(x,y) + u_2(x,y)$ (A) To find $u_1(x,y)$ Consider the boundary conditions $(v)\ u_1(0,y)=0 \quad (vi)\ u_1(x,0)=0 \quad (vii)\ u_1(x,b)=0$ $(viii)\ u_1(a,y)=100$ The suitable solution which satisfying the given boundary conditions is

$$u_1(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \quad (1)$$

Apply the boundary condition (v) in (1)

$$u_1(0,y) = (A+B)(C\cos py + D\sin py) = 0$$

here $A+B=0$

$$B = -A$$

then the equation (2) becomes

$$\begin{aligned} u_1(x,y) &= (Ae^{px} - Ae^{-px})(C\cos py + D\sin py) \\ &= A(e^{px} - e^{-px})(C\cos py + D\sin py) \end{aligned} \quad (2)$$

Apply the boundary condition (vi) in (2)

$$u_1(x,0) = A(e^{px} - e^{-px})C = 0$$

here $C = 0$

then the equation (3) becomes

$$u_1(x,y) = A(e^{px} - e^{-px})\, D\sin py \quad (3)$$

Apply the boundary condition (vii) in (3)

$$u_1(x,b) = A(e^{px} - e^{-px})\, D\sin pb = 0$$

here $\sin pb = 0 \quad \Rightarrow \quad pb = n\pi \Rightarrow \quad p = \frac{n\pi}{b}$

then the equation (4) becomes

$$\begin{aligned} u_1(x,y) &= A(e^{\frac{n\pi x}{b}} + e^{-\frac{n\pi x}{b}})\, D\sin\frac{n\pi y}{b} \\ &= AD\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \qquad \text{where } B_n = 2AD \\ &= B_n\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \end{aligned}$$

The most general solution is

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \quad (4)$$

Apply the boundary condition (viii) in (4)

$$u_1(a,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b} = 100 = f(y) \quad (5)$$

To find B_n, expand $f(x)$ in half range sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b} \quad (6)$$

From the equations (5) & (6) we get

$$b_n = B_n \sinh\frac{n\pi a}{b} \Rightarrow B_n = \frac{b_n}{\sinh\frac{n\pi a}{b}}$$

$$\begin{aligned}
b_n &= \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy \\
&= \frac{2}{b}\int_0^b 100\sin\frac{n\pi y}{b}\,dy \\
&= \frac{200}{b}\left[\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right)\right]_0^b \\
&= \frac{200}{b}\left[\frac{b}{n\pi}(-\cos n\pi + 1)\right] \\
&= \frac{200}{n\pi}\left[1+(-1)^n\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

$$\therefore\ B_n = \frac{400}{n\pi \sinh n\pi} \qquad \text{if n is odd}$$

then the equation (6) becomes

$$u_1(x,y) = \sum_{n=1,3,5}^{\infty} \frac{400}{n\pi \sinh \frac{n\pi a}{b}} \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

To find $u_2(x,y)$

Consider the boundary conditions

$$(ix)\ u_2(0,y)=0 \qquad (x)\ u_2(x,0)=0 \qquad (xi)\quad u_2(a,y)=0$$
$$(xii)\quad u_2(x,b)=100$$

and the suitable solution in this case is

$$u_2(x,y) = (E\cos px + F\sin px)(Ge^{py} + He^{-py}) \qquad (7)$$

Apply the boundary condition (ix) in (7)

$$u_2(0,y) = E(Fe^{py} + He^{-py}) = 0$$

here $\quad E = 0$

then the equation (8) becomes

$$u_2(x,y) = F\sin px(Ge^{py} + Ge^{-py}) \qquad (8)$$

Apply the boundary condition (x) in (8)

$$u_2(x,0) = F\sin px(G+H) = 0$$

here $\ G + H = 0 \qquad \Rightarrow H = -G$

then the equation (9) becomes

$$\begin{aligned} u_2(x,y) &= F\sin px(Ge^{py} - Ge^{-py}) \\ &= FG\sin px(e^{py} - e^{-py}) \end{aligned} \qquad (9)$$

Apply the boundary condition (xi) in (9)

$$u_2(a, y) = FG \sin pa(e^{py} - e^{-py}) = 0$$

here $\sin pa = 0 \Rightarrow pa = n\pi \Rightarrow p = \dfrac{n\pi}{a}$

then the equation (10) becomes

$$u_2(x, y) = FG \sin \frac{n\pi x}{a}(e^{\frac{n\pi y}{a}} - e^{\frac{n\pi y}{a}})$$
$$= FG\, 2\sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a} \qquad \text{where } B_n = 2FG$$
$$= C_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

The most general solution is

$$u_2(x, y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a} \qquad (11)$$

Apply the boundary condition (xii) in (11)

$$u_2(x, b) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = 100 = f(x)$$

To find C_n, expand $f(x)$ in half range sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi x}{a} \qquad (12)$$

From the equations (11) & (12) we get

	$$d_n = C_n \sinh\frac{n\pi b}{a} \Rightarrow C_n = \frac{d_n}{\sinh\frac{n\pi b}{a}}$$ $$d_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$ $$= \frac{2}{a}\int_0^a 100\sin\frac{n\pi x}{a}dx$$ $$= \frac{200}{a}\left[\left(\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)\right]_0^a$$ $$= \frac{200}{a}\left[\frac{a}{n\pi}\left(-\cos n\pi + 1\right)\right]$$ $$= \frac{200}{n\pi}\left[1+(-1)^n\right]$$ $$= \begin{cases} 0 & if\ n\ is\ even \\ \frac{400}{n\pi} & if\ n\ is\ odd \end{cases}$$ $$\therefore C_n = \frac{400}{n\pi\sinh\frac{n\pi b}{a}} \qquad \text{if n is odd}$$ then the equation (10) becomes $$u_2(x,y) = \sum_{n=1,3,5}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$ Then the equation (A) becomes $$u(x,y) = u_1(x,y) + u_2(x,y)$$ $$= \sum_{n=1,3,5}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} + \sum_{n=1,3,5}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$
5.	**A rectangular plate with insulated surface is 10cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge** $y = 0$ **is given by** $u = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$ **and all the other three edges are kept**

	at $0^\circ C$. Find the steady state temperature at any point in the plate.

Solution:

We know that two dimensional heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Given the boundary conditions are

$(i)\ u(0, y) = 0 \qquad (ii)\ u(10, y) = 0 \qquad (iii)\quad u(x, \infty) = 0$

$$(iv)\ u(x,0) = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10 - x), & 5 \le x \le 10 \end{cases}$$

The suitable solution which satisfying the given boundary conditions is

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \qquad (1)$$

Apply the boundary condition (i) in (1)

$$u(0, y) = A\left(Ce^{py} + De^{-py}\right) = 0$$

here $A = 0$

Then the equation (1) becomes

$$u(x, y) = B\sin px\left(Ce^{py} + De^{-py}\right) \qquad (2)$$

Apply the boundary condition (ii) in (2)

$$u(10, y) = B\sin p10\left(Ce^{py} + De^{-py}\right) = 0$$

here $\quad \sin p10 = 0 \quad \Rightarrow \quad p10 = n\pi \Rightarrow \quad p = \frac{n\pi}{10}$

Then the equation (2) becomes

$$u(x, y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{10}} + De^{\frac{-n\pi y}{10}}\right) \qquad (3)$$

Apply the boundary condition (iii) in (3)

$$u(x,\infty) = B\sin\frac{n\pi x}{10}\left(Ce^{\infty} + De^{-\infty}\right) = 0$$

here $C = 0$

Then the equation (3) becomes

$$u(x,y) = B\sin\frac{n\pi x}{a}e^{\frac{-n\pi y}{10}}$$

$$= B_n \sin\frac{n\pi x}{a}e^{\frac{-n\pi y}{10}}$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}e^{\frac{-n\pi y}{10}} \qquad (4)$$

Apply the boundary condition (iv) in (4)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}.1 = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$$

To find B_n, expand $f(x)$ in Half range Fourier sine series.

We know that Half range Fourier Cosine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \qquad (5)$$

From (4) and (5) we get $B_n = b_n$

$$b_n = \frac{2}{10}\int_0^{10} f(y)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[\int_0^5 20x\sin\frac{n\pi x}{10}\,dx + \int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$= \frac{20}{5}\left[\left(x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right)_0^5+\left((10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right)_5^{10}\right]$$

$$b_n = 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Then the equation (4) becomes

$$u(x,y)=\sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}e^{\frac{-n\pi y}{10}}\sin\frac{n\pi x}{10}$$

UNIT IV – FOURIER TRANSFORMS

PART - A

1. Write the Fourier transform pair. **[Nov/Dec 2011, 2010]**

The Fourier transform pair is defined as

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)dx \quad = F(s) \quad : F^{-1}\left[F\left[f(x)\right]\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}F(s)\,ds = f(x)$$

2. State Fourier integral theorem. **[Nov/Dec 2011]**

If $f(x)$ is piecewise continuous, continuously differentiable and absolutely integrable in $(-\infty,\infty)$ then

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\,ds$$

3. Find the Fourier transform of $\mathbf{f(x)}=\begin{cases}\mathbf{1;\ for\ |x|<2}\\ \mathbf{0;\ for\ |x|>2}\end{cases}$

The Fourier transform of the function $f(x)$ is

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$\therefore F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-2}^{2} 1.e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-2}^{2}=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i2s}}{is}-\frac{e^{-i2s}}{is}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i2s}-e^{-i2s}}{is}\right]=\frac{1}{\sqrt{2\pi}}\frac{1}{is}(2i\sin 2s) \quad \therefore F(s) \ =\sqrt{\frac{2}{\pi}}\frac{\sin 2s}{s}$$

4. Find $f(x)$ **from the integral equation** $\int_0^{\infty} \mathrm{f(x)}\cos s\,\mathrm{x}=\mathrm{e}^{-\mathrm{s}}$

	$given\, that \int_0^{\infty} f(x)\cos sx dx = e^{-s}$, $\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-s}\cos sx\ dx = \sqrt{\frac{2}{\pi}}e^{-s}$ $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos sx\ ds = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-s}\cos sx\, ds = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$
5.	**Define Self-reciprocal Fourier transform and give an example.** **[Nov / Dec 2013]** If $f(s)$ is the Fourier transform of $f(x)$, then $f(x)$ is said to be self-reciprocal under Fourier transform. $F\left(e^{-x^2/2}\right) = e^{-s^2/2}$.
6.	**Find the Fourier transform of $e^{-a\|x\|}$, if a 0** **[Nov / Dec 2012]** $\therefore F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\|x\|}e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\|x\|}(\cos sx + i\sin sx)dx$ $= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{-a\|x\|}\cos sxdx + i\int_{-\infty}^{\infty} e^{-a\|x\|}\sin sxdx\right\} = \frac{1}{\sqrt{2\pi}}\left\{2\int_0^{\infty} e^{-ax}\cos sxdx\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{(s^2+a^2)}\right)$
7.	**State Convolution theorem in Fourier Transform.** **[Nov / Dec 2012]** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms i.e. $F\{f(x)*g(x)\} = F\{f(x)\}F\{g(x)\}$
8.	**State Parseval's identity on Fourier Transform.** **[Nov/Dec 2011]** $\int_{-\infty}^{\infty}\|F(s)\|^2 ds = \int_{-\infty}^{\infty}\|f(x)\|^2 dx, \quad where \quad F[f(x)] = F(s)$
9.	**Find** $\mathrm{F}\{\mathrm{xf}(\mathrm{x})\}$ **from** $\mathrm{F}(\mathrm{f}(\mathrm{x})) = \sqrt{\frac{2}{\pi}}\left(\frac{\mathrm{a}}{\mathrm{s}^2+\mathrm{a}^2}\right)$**.** By property $F[xf(x)] = -i\frac{d}{ds}F\left[f(x)\right] \therefore F\left[xe^{-a\|x\|}\right] = -i\frac{d}{ds}F\left[e^{-a\|x\|}\right]$ $= -i\frac{d}{ds}\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2} = = -i\sqrt{\frac{2}{\pi}}\frac{-a.2s}{(s^2+a^2)^2} == i\sqrt{\frac{2}{\pi}}\left(\frac{2as}{(s^2+a^2)^2}\right)$
10.	**Find the Fourier sine transform of** $\frac{e^{-ax}}{x}$ **where** $a>0$ **.** $F_S\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sxdx$. $F_S\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}\sin sxdx$ Differentiate both sides with respect to s ,

	$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right]=\frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sxdx=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}\sin sxdx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}x\cos sxdx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sxdx=\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$ Now integrating both sides w. r. to s, $F_S\left[\frac{e^{-ax}}{x}\right] \quad =\int\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}ds=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$
11.	**Find the Fourier sine transform of** $\mathrm{e}^{-\mathrm{ax}}$ **,** $a>0$ **. Hence find** $\mathrm{F_s}\left[\mathrm{xe}^{-\mathrm{a\,x}}\right]$**.** **[May / June 2012]** $F_S[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sxdx$. $F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sxdx=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$ We know that by property $F_S[xf(x)]=-\frac{d}{ds}F_C[f(x)]$ $\therefore F_s\left[xe^{-ax}\right] \quad =-\frac{d}{ds}\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}=-\sqrt{\frac{2}{\pi}}\frac{s^2+a^2-s.(2s)}{\left(s^2+a^2\right)^2}=\sqrt{\frac{2}{\pi}}\left(\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)$
12.	**Write the Fourier sine transform pair and Fourier Cosine transform pair.** The Fourier sine transform pair is defined as $F_S[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sxdx=F_S(s)$: $\qquad F^{-1}\left[F_s[s]\right]=f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\,F_s(s)\,ds$ The Fourier cosine transform pair is defined as $F_C[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sxdx=F_C(s)$: $\qquad F^{-1}\left[F_c[s]\right]=f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,F_c(s)\,ds$
13.	**Prove that** $\mathrm{F_c}(\mathrm{f}(\mathrm{ax}))=\frac{1}{\mathrm{a}}\mathrm{F_c}\left(\frac{\mathrm{s}}{\mathrm{a}}\right), a\neq 0$ $F_C[f(ax)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sxdx$, $\quad Put\ ax=t,\ adx=dt, dx=\frac{dt}{a}$ $when\ x=0, t=0\ \ and\ \ x=\infty, t=\infty\ \ F_c\left[f(ax)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{st}{a}\right)\frac{dt}{a}$ $F_c\left[f(ax)\right]=\frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{s}{a}\right)t\,dt=\frac{1}{a}F_C\left(\frac{s}{a}\right).$
14.	**State Parseval's identity in Fourier sine and cosine Transform.** $\int_0^\infty \lvert F_C(s)\rvert^2 ds=\int_0^\infty \lvert f(x)\rvert^2 dx$ & $\int_0^\infty \lvert F_S(s)\rvert^2 ds=\int_0^\infty \lvert f(x)\rvert^2 dx$

<table>
<tr><td>15.</td><td>Find the Fourier Sine transform of $\frac{1}{x}$. [Nov/Dec 2011, Nov / Dec 2009]
$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}dx=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}dt=\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}}$</td></tr>
<tr><td>16.</td><td>If $F_C(f(x))=F_C(s)\ and\ F_S(f(x))=F_S(s)$, prove that $F_c(f(x)\sin ax)=\frac{1}{2}\left[F_s(a+s)+F_s(a-s)\right]$
[Nov/Dec 2011, MA1201]
Solution:
$F_c(f(x)\sin ax)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\ \sin ax\ \cos sx\ \ dx=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\sin(s+a)x+\sin(a-s)x]\ dx$
$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\sin(a+s)x]\ dx+\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)[\sin(a-s)x]\ dx\right]$
$F_c(f(x)\sin ax)=\frac{1}{2}\left[F_s(a+s)+F_s(a-s)\right]$.</td></tr>
<tr><td>17.</td><td>State and prove the change of scale property of Fourier Transform. [Apr/May 2011,May/June 2013]
Statement:
If F [f(x)] = F(s) then $F(f(ax))=\frac{1}{|a|}F\left(\frac{s}{a}\right),\ a\neq 0$
Proof:
$F\left[f(ax)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\ e^{isx}\ \ dx$,
If a > 0 $\ Put\ ax=t,\ \ adx=dt, dx=\frac{dt}{a}$
$when\ \ x=-\infty\Rightarrow t=-\infty\ \ and\ \ x=\infty\Rightarrow t=\infty\ \Rightarrow F\left[f(ax)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}=\frac{1}{a}F\left(\frac{s}{a}\right)$. –(1)
If a < 0 $\ Put\ ax=t,\ \ adx=dt, dx=\frac{dt}{a}$
$when\ x=-\infty\Rightarrow t=\infty\ \ and\ \ x=\infty\Rightarrow t=-\infty$
$\Rightarrow F\left[f(ax)\right]=\frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}=\frac{1}{a}F\left(\frac{s}{a}\right)$. ---(2)
From (1) & (2) we get $F(f(ax))=\frac{1}{|a|}F\left(\frac{s}{a}\right),\ a\neq 0$</td></tr>
<tr><td>18.</td><td>Find the Fourier cosine transform of $f(x)=\begin{cases}1-x^2\ ;\ 0<x<1\\ 0\ ,\ otherwise\end{cases}$.</td></tr>
</table>

	$$\therefore F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^1 (1-x^2)\cos sx dx$$ $$F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx dx = \sqrt{\frac{2}{\pi}} \left\{(1-x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right\}_0^1$$ $$= \sqrt{\frac{2}{\pi}} \left(\frac{-2\cos s}{s^2} + \frac{2\sin s}{s^3}\right) = 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)$$
19.	**Find the Fourier Sine Transform of** $f(x) = e^{-x},\ x > 0$ **[Apr/May2011, MA1201]** We know that, $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin sx dx$. $$F_s\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx dx = \sqrt{\frac{2}{\pi}} \left\{\frac{e^{-x}}{s^2+1}(-\sin sx - s\cos sx)\right\}_0^\infty = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+1}$$
20	**If F(s) is the Fourier transform of f (x), then show that** $F\{f(x-a)\} = e^{ias} F(s)$ **[Nov/Dec 2013, 2010 ,May/June 2012]** F {f(x-a)}= $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i(a+t)s} dt$ where x - a = t $$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{its} dt = F(s) = e^{ias} F\{f(x)\}$$
	PART - B
1	**Show that the Fourier transforms of** $f(x) = \begin{cases} a^2 - x^2; & \lvert x\rvert < a \\ 0; & \lvert x\rvert > a > 0 \end{cases}$ **is** $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^3}\right]$**. Hence deduce that** $\int_0^\infty \frac{\sin t - t\cos t}{t^3} dt = \frac{\pi}{4}$**. Using Parseval's Identity, show that** $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$**.**
	The Fourier transform of the function $f(x)$ is $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$. Given $f(x) = a^2 - x^2$ in $-a < x < a$ and 0 otherwise.

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-a}^{a}(a^2-x^2)\cos sxdx + i\int_{-a}^{a}(a^2-x^2)\sin sxdx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{2\int_{0}^{a}(a^2-x^2)\cos sxdx + 0\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{(a^2-x^2)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right\}_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[0-\frac{2a\cos sa}{s^2}+\frac{2\sin sa}{s^3}\right]-[0-0+0]\right\}$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}$$

By inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}e^{-isx}ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}(\cos sx - i\sin sx)ds$$

$$= \frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}\cos sxds - i\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}\sin sxds\right\}$$

$$f(x) = \frac{2}{\pi}\left\{2\int_{0}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}\cos sxds - 0\right\}$$

Put $x = 0$ & $a = 1$, we get

$$1 = \frac{4}{\pi}\int_0^{\infty}\left\{\frac{\sin s - s\cos s}{s^3}\right\}ds$$

$$i.e \quad \int_0^{\infty}\left\{\frac{\sin s - s\cos s}{s^3}\right\}ds = \frac{\pi}{4}$$

Replace s by t, we get $\int_0^{\infty}\left\{\frac{\sin t - t\cos t}{t^3}\right\}dt = \frac{\pi}{4}$.

By Parseval's Identity,

		$\int_{-\infty}^{\infty}\lvert F(s)\rvert^2 ds = \int_{-\infty}^{\infty}\lvert f(x)\rvert^2 dx, \quad where \quad F[f(x)] = F(s)$ $\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}^2 ds = \int_{-a}^{a}(a^2-x^2)^2\ dx$ $\frac{8}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}^2 ds = \int_{-a}^{a}(a^4-2a^2x^2+x^4)\ dx$ $2\times\frac{8}{\pi}\int_{0}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}^2 ds = 2\int_{0}^{a}(a^4-2a^2x^2+x^4)\ dx$ $\frac{8}{\pi}\int_{0}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}^2 ds = \left[a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]_0^a$ $\int_{0}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}^2 ds = \frac{\pi}{8}\left[a^5-\frac{2}{3}a^5+\frac{a^5}{5}\right] = \frac{\pi}{8}\left[\frac{15a^5-10a^5+3a^5}{15}\right]$ $\int_{0}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}^2 ds = \frac{\pi a^5}{15}$ Put $as = t \Rightarrow s = \frac{t}{a},\ ds = \frac{dt}{a},\quad$ when $s=\infty,\ t=\infty$ and $s=0,\ t=0$ $\int_{0}^{\infty}\left\{\frac{\sin t - t\cos t}{\left(\frac{t^3}{a^3}\right)}\right\}^2 \frac{dt}{a} = \frac{\pi a^5}{15} \Rightarrow \int_{0}^{\infty}\left\{\frac{\sin t - t\cos t}{t^3}\right\}^2 dt = \frac{\pi}{15}.$
2	**a)**	**Find the Fourier sine transform of** $\frac{e^{-ax}}{x}$ **where** $a>0$**. [Nov/Dec 2011, 181301]**
		$F_S(f(x)) = \frac{e^{-ax}}{x}$ $\therefore f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-ax}}{x}\sin sx\ ds$ Differentiating both sides w.r.to ‘s’ $\frac{df}{ds} = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-ax}}{x}\cos sx\ (x)\ ds = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-ax}\cos sx\ ds$ $\frac{df}{ds} = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^{\infty} \Rightarrow \sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+s^2}$

		Integrating w.r.to 's' $f(x)=\sqrt{\frac{2}{\pi}}\, a\int \left(\frac{ds}{a^2+s^2}\right) \Rightarrow \sqrt{\frac{2}{\pi}}\, a.\frac{1}{a}\,\tan^{-1}\left(\frac{s}{a}\right)+c \Rightarrow \sqrt{\frac{2}{\pi}}\,\tan^{-1}\left(\frac{s}{a}\right)+c$ $Put\ \ x=0,\ \ f(0)=c \Rightarrow c=0$ $\therefore\ f(x)=\sqrt{\frac{2}{\pi}}\,\tan^{-1}\left(\frac{s}{a}\right)$
	b)	**Show that $e^{-\frac{x^2}{2}}$ is self-reciprocal with respect to the Fourier cosine Transform.**
		The Fourier cosine transform of the function $f(x)$ is $F_C[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$. $\therefore F_C\left[e^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{x^2}{2}}\cos sxdx = \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\cos sxdx = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{x^2}{2}}\text{R.P of }(e^{isx})dx$ $=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{x^2}{2}}e^{isx}dx=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{x^2}{2}+isx}dx$ $=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(\frac{x^2}{2}-isx\right)}dx=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left[\left(\frac{x}{\sqrt{2}}-\frac{is}{\sqrt{2}}\right)^2-\left(\frac{is}{\sqrt{2}}\right)^2\right]}dx$ $=R.P of\ \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\int_{-\infty}^\infty e^{-\left[\left(\frac{x-is}{\sqrt{2}}\right)^2\right]}dx$ Put $\frac{x-is}{\sqrt{2}}=y$, then $dx=\sqrt{2}dy$, $x=-\infty \Rightarrow y=-\infty\ \ and\ \ x=\infty \Rightarrow y=\infty$ $=\text{R.P of }\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\int_{-\infty}^\infty e^{-y^2}\sqrt{2}dy=\text{R.P of }\frac{1}{\sqrt{\pi}}e^{-\frac{s^2}{2}}\int_{-\infty}^\infty e^{-y^2}dy$ $=\text{R.P of }\frac{1}{\sqrt{\pi}}e^{-\frac{s^2}{2}}\sqrt{\pi}\ \Rightarrow F_C[f(x)]=e^{-\frac{s^2}{2}}$ $\therefore\ e^{-\frac{x^2}{2}}$ is self-reciprocal with respect to Fourier transform
3	a)	**Find the Fourier transform of $f(x)=\begin{cases}1-\lvert x\rvert\ ;\ if\ \ \lvert x\rvert<1\\ 0\ ;\ if\ \ \lvert x\rvert\ge 1\end{cases}$. Hence Evaluate $\int_0^\infty \frac{\sin^4 t}{t^4}dt$.**
		The Fourier transform of the function $f(x)$ is $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{isx}dx$. Given $f(x)=1-\lvert x\rvert$ in $-1<x<1$ and 0 otherwise.

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{1}(1-|x|)\cos sxdx + i\int_{-1}^{1}(1-|x|)\sin sxdx\right\} = \frac{1}{\sqrt{2\pi}}\left\{2\int_{0}^{1}(1-x)\cos sxdx + 0\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{(1-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right\}_0^1 = \sqrt{\frac{2}{\pi}}\left\{\left[0-\frac{\cos s}{s^2}\right]-\left[0-\frac{1}{s^2}\right]\right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos s}{s^2}\right\}$$

By inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos s}{s^2}\right\}e^{-isx}ds = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}(\cos sx - i\sin sx)ds$$

$$= \frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}\cos sxds - i\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}\sin sxds\right\}$$

$$f(x) = \frac{1}{\pi}\left\{2\int_{0}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}\cos sxds - 0\right\}$$

Put $x = 0$, we get

$$1 = \frac{2}{\pi}\int_{0}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}ds$$

$$i.e \quad \int_{0}^{\infty}\left\{\frac{2\sin^2\left(s/2\right)}{s^2}\right\}ds = \frac{\pi}{2}.1$$

Let $\frac{s}{2} = t$, then $ds = 2dt$ $\;s = 0 \Rightarrow t = 0$ and $\;s = \infty \Rightarrow t = \infty$

$$\int_{0}^{\infty}\left\{\frac{2\sin^2 t}{(2t)^2}\right\}2dt = \frac{\pi}{2} \Rightarrow \int_{0}^{\infty}\left\{\frac{\sin^2 t}{t^2}\right\}dt = \frac{\pi}{2}$$

By Parseval's identity $\int_{-\infty}^{\infty}|F(s)|^2 ds = \int_{-\infty}^{\infty}|f(x)|^2 dx$

$$\therefore \quad \int_{-\infty}^{\infty}\left|\sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos s}{s^2}\right\}\right|^2 ds = \int_{-1}^{1}\left|(1-|x|)\right|^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left|\left\{\frac{1-\cos s}{s^2}\right\}\right|^2 ds = 2\int_{0}^{1}(1-|x|)^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}^2 ds = 2\int_{0}^{1}(1-x)^2 dx$$

$$\frac{2}{\pi}2\int_0^\infty \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = 2\int_0^1 \left(1+x^2-2x\right)dx$$

$$\frac{2}{\pi}\int_0^\infty \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = \left\{x+\frac{x^3}{3}-\frac{2x^2}{2}\right\}_0^1$$

$$\frac{2}{\pi}\int_0^\infty \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = \left\{\left[1+\frac{1}{3}-1\right]-\left[0+0-0\right]\right\} \Rightarrow \quad 2\int_0^\infty \left\{\frac{2\sin^2\left(s/2\right)}{s^2}\right\}^2 ds = \frac{\pi}{3}$$

Let $\frac{s}{2}=t$, then $ds = 2dt \quad s=0 \Rightarrow t=0 \ \ and \ \ s=\infty \Rightarrow t=\infty$

$$2\int_0^\infty \left\{\frac{2\sin^2 t}{(2t)^2}\right\}^2 2dt = \frac{\pi}{3} \Rightarrow \int_0^\infty \left\{\frac{\sin^2 t}{t^2}\right\}^2 dt = \frac{\pi}{3} \Rightarrow \int_0^\infty \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}$$

b) **Evaluate** $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ **using Fourier Cosine Transforms of** e^{-ax} *and* e^{-bx}, $a,b>0$ **.**

The Fourier cosine transform of the function $f(x)$ is $F_C\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$.

$$\therefore F_C\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx dx = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

and

$$\therefore F_C\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos sx dx = \sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}$$

Let $f(x)=e^{-ax}$ and $g(x)=e^{-bx}$

By Convolution theorem, $\int_0^\infty F_c\left[f(x)\right]F_c\left[g(x)\right]ds = \int_0^\infty f(x)\, g(x)dx$

$$\therefore \int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\sqrt{\frac{2}{\pi}}\frac{b}{s^2+a^2}ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$i.e. \frac{2}{\pi}\int_0^\infty \frac{ab}{\left(s^2+a^2\right)\left(s^2+a^2\right)}ds = \int_0^\infty e^{-(a+b)x}dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{\left(s^2+a^2\right)\left(s^2+a^2\right)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{\left(s^2+a^2\right)\left(s^2+a^2\right)} = \left[\frac{e^{-(a+b)\infty}-e^{-(a+b)0}}{-(a+b)}\right]$$

		$\frac{2ab}{\pi}\int_{0}^{\infty}\frac{ds}{(s^2+a^2)(s^2+a^2)}=\left[\frac{-1}{-(a+b)}\right]$ $\int_{0}^{\infty}\frac{ds}{(s^2+a^2)(s^2+a^2)}=\frac{\pi}{2ab(a+b)}$ Replace s by x , we get $\int_{0}^{\infty}\frac{dx}{(x^2+a^2)(x^2+a^2)}=\frac{\pi}{2ab(a+b)}$
4	**a)**	**Find the Fourier transforms of $f(x)$ defined by $f(x)=\begin{cases}1, & \lvert x\rvert<a\\ 0, & \lvert x\rvert>a\end{cases}$ and hence find the value of $\int_{0}^{\infty}\frac{\sin t}{t}dt$ also prove that $\int_{0}^{\infty}\frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$.**
		The Fourier transform of the function $f(x)$ is $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$. Given $f(x)=1$ in $-a<x<a$ and 0 otherwise. $\therefore F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}1.e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a}=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{ias}}{is}-\frac{e^{-ias}}{is}\right]=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{ias}-e^{-ias}}{is}\right]$ $=\frac{1}{\sqrt{2\pi}}\frac{1}{is}2i\sin as\Rightarrow F(s)=\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}$ By inverse Fourier transform $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$ $\therefore f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}e^{-isx}ds=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin sa}{s}(\cos sx-i\sin sx)ds$ $=\frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\frac{\sin sa}{s}\cos sxds-i\int_{-\infty}^{\infty}\frac{\sin sa}{s}\sin sxds\right\}=\frac{1}{\pi}\left\{2\int_{0}^{\infty}\frac{\sin sa}{s}\cos sxds-0\right\}$ $f(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin sa}{s}\cos sxds$ Put $x=0$ & $a=1$ we get $1=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}ds \quad \Rightarrow \int_{0}^{\infty}\frac{\sin s}{s}ds=\frac{\pi}{2}$, Replace s by t, we get $\int_{0}^{\infty}\frac{\sin t}{t}dt=\frac{\pi}{2}$. By Parseval's identity $\int_{-\infty}^{\infty}\lvert F(s)\rvert^2 ds=\int_{-\infty}^{\infty}\lvert f(x)\rvert^2 dx$

		$\therefore \int_{-\infty}^{\infty}\left\|\sqrt{\frac{2}{\pi}}\frac{\sin s}{s}\right\|^2 ds = \int_{-1}^{1}\|1\|^2 dx \Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty}\left\|\frac{\sin s}{s}\right\|^2 ds = [x]_{-1}^{1}$ $\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = [1-(-1)] \Rightarrow \frac{2}{\pi}2\int_{0}^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = 2 \Rightarrow \int_{0}^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2}$ Replace s by t, we get $\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} \Rightarrow \int_{0}^{\infty}\frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.
	b)	**Find Fourier Sine and Cosine transforms of e^{-ax} and hence find the Fourier sine transforms of $\frac{x}{x^2+a^2}$ and Fourier cosine transforms of $\frac{1}{x^2+a^2}$.**
		The Fourier sine transform of the function $f(x)$ is $F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx dx$. $F_S\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\sin sx dx = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$ The Fourier cosine transform of the function $f(x)$ is $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx dx$. $\therefore F_C\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\cos sx dx = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$ $\therefore F_C\left[\frac{1}{x^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{1}{x^2+a^2}\cos sx dx \Rightarrow \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2a}e^{-as}\right) \Rightarrow \sqrt{\frac{\pi}{2}}\frac{e^{-as}}{a}$. $F_S\left[\frac{x}{x^2+a^2}\right] = -\frac{d}{ds}\left(F_C\left[\frac{1}{x^2+a^2}\right]\right) = -\frac{d}{ds}\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2a}e^{-as}\right) \Rightarrow -\sqrt{\frac{\pi}{2}}\frac{e^{-as}}{a}(-a) = \sqrt{\frac{\pi}{2}}e^{-as}$
5	**a)**	**Find the function f(x) if its sine transform is** $\frac{e^{-as}}{s}$ [Nov / Dec 2013]
		Let $F_s(f(x)) = \frac{e^{-as}}{s}$ Then f(x) = $\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-as}}{s}\sin sx\ ds \qquad -----(1)$ Differentiating both side with respect to 'x' we get $\frac{df}{dx} = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-as}}{s}(\cos sx)(s) ds = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-as}\cos sx ds$ $\frac{df}{dx} == \sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+x^2}(-a\cos sx + x\sin sx)\right]_{0}^{\infty} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{x^2+a^2}\right)$

		Integrating w.r.t ' x ' f(x) = $a\sqrt{\frac{2}{\pi}}\int\frac{dx}{x^2+a^2} = a\sqrt{\frac{2}{\pi}}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)+c$ --------(2) put x= 0, f(0) = 0 from (1) from (2) , we get f(0) = c =0, $\therefore f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$
	b)	**Express** $f(x)=\begin{cases}1 & ,\|x\|<1\\ 0 & ,\|x\|>1\end{cases}$ **as Fourier integral, hence evaluate** $\int_0^{\infty}\frac{\sin s\ \cos sx}{s}ds$ **and find the values of** $\int_0^{\infty}\frac{\sin s}{s}ds$
		We know that the Fourier integral formula for f(x)= $\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(t)\ \cos s(t-x)\ dt\ ds$ f(x)= $\frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1}\cos s(t-x)\ dt\ ds = \frac{1}{\pi}\int_0^{\infty}\left(\frac{\sin\ s(t-x)}{s}\right)_{-1}^{1}$ $\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{\sin s(1-x)+\sin s(1+x)}{s}\right)ds = \frac{2}{\pi}\int_0^{\infty}\frac{\sin s\ \cos\ sx}{s}ds$ ....(1) This is fourier integral representation of f(x). $\Rightarrow \frac{2}{\pi}\int_0^{\infty}\frac{\sin s\ \cos\ sx}{s}ds = f(x)$ From (1) $\Rightarrow \int_0^{\infty}\frac{\sin s\ \cos\ sx}{s}ds = \frac{\pi}{2}f(x) = \begin{cases}\frac{\pi}{2} & ,\ \|x\|<1\\ 0 & ,\ \|x\|>1\end{cases}$ Put x= 0, $\int_0^{\infty}\frac{\sin s}{s}ds = \frac{\pi}{2}$
	UNIT V Z – TRANSFORMS AND DIFFERENCE EQUATIONS	
	PART - A	
1.	**What is the Z- transform of discrete unit step function**	
	Solution: Discrete Unit step function is $u(n)=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$ $Z\left[u(n)\right]=\sum_{n=0}^{\infty}1z^{-n} = 1+z^{-1}+z^{-2}+.. = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$ if $\|z\|>1$	

2.	**Find** $Z\left(\frac{1}{2^n}\right)$**.**
	Solution: $Z\left(\frac{1}{2^n}\right)=\sum_0^{\infty}\frac{1}{2^n}z^{-n}=1+\frac{1}{2}z^{-1}+\frac{1}{2^2}z^{-2}+\frac{1}{2^3}z^{-3}+\dots$ $=1+\frac{1}{2z}+\frac{1}{4z^2}+\frac{1}{8z^3}+\dots \quad =\frac{z}{z-\frac{1}{2}}=\frac{2z}{2z-1},\lvert z\rvert>2$
3.	**Find** $Z\left[u(n-1)\right]$
	Solution: $Z\left[u(n-1)\right]=\sum_{n=0}^{\infty}u(n-1)z^{-n}=\sum_{n=1}^{\infty}u(n-1)z^{-n}$ $=\sum_{n=1}^{\infty}z^{-n}=\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots. \qquad =\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\dots.\right]$ $=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(\frac{z-1}{z}\right)^{-1}=\frac{1}{z}\left(\frac{z}{z-1}\right)=\frac{1}{z-1}$ if $\lvert z\rvert>1$
4.	**Find the Z- transform of unit impulse function**
	Solution: Unit impulse function is $\delta(n)=1,\quad n=0$ $\quad\;=0,\quad n\neq 0$ $Z\left[\delta(n)\right]=\sum_{n=0}^{\infty}\delta(n)z^{-n}=z^{-0}=1.$
5.	**If** $Z\left[f(n)\right]=U(z)$ **then find** $Z\left[a^n f(n)\right]$
	Solution: $Z\left[a^n f(n)\right]=\sum_{n=0}^{\infty}a^n f(n)z^{-n}=\sum_{n=0}^{\infty}f(n)(a/z)^n=U(z/a).$
6.	**If** $Z\left[f(n)\right]=U(z)$**, then show that** $Z\left[f(n+k)\right]=z^kU(z)$
	Solution: $Z\left[f(n+k)\right]=\sum_0^{\infty}f(n+k)z^{-n}=z^{+k}\sum_0^{\infty}f(n+k)z^{-(n+k)}$ $=z^{+k}\sum_0^{\infty}f(r)z^{-r}=z^{+k}U(z)$

7.	**If** $z\left[f(n)\right]=U(z)$**, then** $Z\left(\frac{f(n)}{n}\right)=-\int z^{-1}U(z)\,dz$**.**
	Solution: $Z\left[\frac{f(n)}{n}\right]=\sum_{n=0}^{\infty}\frac{f(n)}{n}z^{-n}=-\sum_{n=0}^{\infty}f(n)\int z^{-n-1}dz$, since $\frac{z^{-n}}{n}=-\int z^{-n-1}dz$ $=-\sum_{n=0}^{\infty}f(n)\int z^{-n-1}dz=-\int\left(z^{-1}\sum f(n)z^{-n}\right)dz=-\int z^{-1}\,U(z)\,dz$.
8.	**If** $Z\left[f(n)\right]=U(z)$**, then find** $Z\left[n\,f(n)\right]$
	Solution: $Z\left[nf(n)\right]=\sum nf(n)z^{-n}=-z\sum -nf(n)z^{-n-1}$ $=-z\sum f(n)\frac{d}{dz}\left(z^{-n}\right)=-z\frac{d}{dz}\sum f(n)z^{-n}=-z\frac{d}{dz}U(z)$.
9.	**Find** $Z\left[a^{n+3}\right]$**.**
	Solution: $Z\left[a^{n+3}\right]=\sum_{n=0}^{\infty}a^{n+3}z^{-n}=a^3Z\left(a^n\right)\;=a^3\frac{z}{z-a}$
10.	**Find** $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$
	Solution: Let $\left[\frac{z}{(z-1)(z-2)}\right]=U(z)$ Then $\frac{U(z)}{z}=\frac{1}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$ $z=2,\quad 1=B$ $z=1,\quad 1=-A$ $\therefore\frac{U(z)}{z}=\frac{-1}{z-1}+\frac{1}{z-2}$

	$\therefore U(z)=\frac{-z}{z-1}+\frac{z}{z-2} \Rightarrow \therefore u(n)=-1+2^n=2^n-1$
11.	**Find $Z(n)$.**
	Solution: $Z\{n\} = \sum_{n=1}^{\infty} n\ z^{-n} = 1\cdot\frac{1}{z} + 2\cdot\left(\frac{1}{z}\right)^2 + 3\cdot\left(\frac{1}{z}\right)^3 ....$ $= \frac{1}{z}\left(1+\frac{2}{z}+3\cdot\frac{1}{z^2}+\cdots\right)=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^2=\frac{z}{(z-1)^2}$.
12.	**Find Z-Transform of $\frac{1}{n}$.**
	Solution: $Z\left(\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{n}z^{-n} = \frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^2+\frac{1}{3}\left(\frac{1}{z}\right)^3+....$ $=-log\left(1-\frac{1}{z}\right)=-\log\left(\frac{z-1}{z}\right)=\log\left(\frac{z}{z-1}\right)$
13.	**Find Z-Transform of a^n.**
	Solution: $Z\{a^n\}=\sum_{n=0}^{\infty}a^n z^{-n}=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n=\frac{z}{z-a}$
14.	**Find Z-Transform of $Z\left\{\frac{1}{n!}\right\}$.**
	Solution: $Z\left\{\frac{1}{n!}\right\} = \sum_{n=1}^{\infty}\frac{1}{n!}z^{-n} = 1+\frac{1}{1!\ z}+\frac{1}{2!\ z^2}\ .... = e^{\frac{1}{z}}$
15.	**Solve $y_{n+1}-y_n=2^n$, given that $y(0)=1$.**
	Solution:

	$Z\{y_{n+1}-y_n\}=Z\{2^n\}$ $zY(z)-y(0)-Y(z)=\frac{z}{z-2}$, $Y(z)=Z\{y_n\}$ $Y(z)(z-1)=z+\frac{z}{z-2} \Rightarrow Y(z)=\frac{z(z-2)+z}{(z-2)(z-1)} \Rightarrow Y(z)=\frac{z(z-1)}{(z-2)(z-1)}$ $y_n=Z^{-1}\left(\frac{z}{(z-2)}\right)=2^n$
16.	**Using Convolution theorem , evaluate** $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$
	Solution: We know that $Z^{-1}\left[\frac{z}{(z-1)}\right]=z(1)$ and $Z^{-1}\left[\frac{z}{(z-3)}\right]=3^n$ Now $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]=Z^{-1}\left[\frac{z}{z-1}.\frac{z}{z-3}\right]$ $=1*3^n=\sum_{m=0}^{n}1.3^{n-m}=3^n\sum_{m=0}^{n}3^{-m}$ $=3^n\sum_{m=0}^{n}\left(\frac{1}{3}\right)^m=3^n\frac{\left(\frac{1}{3}\right)^{n+1}-1}{\left(\frac{1}{3}\right)-1}=\frac{3^n\left(1-3^{n+1}\right)/3^{n+1}}{(1-3)/3}=\frac{1-3^{n+1}}{-2}=\frac{3^{n+1}-1}{2}.$
17.	**Find** $Z\left\{e^{-2t}\ \sin 2t\right\}$
	Solution: $Z\left\{e^{-2t}\ \sin 2t\right\}=\left(Z\{\sin 2t\}\right)_{z\to ze^{2T}}=\left(\frac{z\ \sin\ 2T}{z^2-2z\cos 2T+1}\right)_{z\to ze^{2T}}$ $=\frac{ze^{2T}\ \sin\ 2T}{z^2e^{4T}-2ze^{2T}\ \cos 2T+1}$

18.	**Find** $Z\left\{\sin at\right\}$
	Solution: $Z\left\{\sin at\right\} = \sum_{n=0}^{\infty} \sin\left(anT\right)z^{-n}$ $= \left[Z\left(\sin n\theta\right)\right]_{\theta=aT} = \left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]_{\theta=aT} = \frac{z\ \sin aT}{z^2-2z\ \cos\ aT+1}$
19.	**State initial and final value theorem of Z - transform.**
	Solution: Initial value Theorem $If\ \ Z\left[f\left(n\right)\right]=U\left(z\right), n\geq 0$ then $\underset{n\to 0}{Limit} f\left(n\right)= f\left(0\right)=\underset{z\to\infty}{Limit} U\left(z\right)$ **Final value Theorem** $If\ Z\left[f\left(n\right)\right]=U\left(z\right), n\geq 0$ then $\underset{n\to\infty}{Limit} f\left(n\right)=\underset{z\to 1}{Limit}\left(z-1\right)U\left(z\right)$
20.	**Form the difference equation from** $y_n = A.2^n + B.3^n$
	Solution: Given $y_n = A.2^n + B.3^n$, $y_{n+1} = 2A.2^n + 3B.3^n$, $y_{n+2} = 4A.2^n + 9B.3^n$ Elimination of A and B forms the difference equation $\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & 3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$ $y_{n+2} - 5y_{n+1} + 6y_n = 0$.

PART - B

1.	**a)**	**Solve** $y_{n+2} - 5y_{n+1} + 6y_n = 5^n, y\left(0\right)=0, y\left(1\right)=0$ **using Z-transform.**
		Solution: $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ .Taking Z-transform on both sides $z^2\left(Y\left(z\right)-0-z^{-1}\times 0\right)-5z\left(Y\left(z\right)-0\right)+6Y\left(z\right)=\frac{z}{z-5}$

		$\left[z^2-5z+6\right]Y(z)=\dfrac{z}{z-5}$ $\dfrac{Y(z)}{z}=\dfrac{1}{(z-5)(z^2-5z+6)}=\dfrac{1}{(z-5)(z-2)(z-3)}=\dfrac{A}{z-5}+\dfrac{B}{z-2}+\dfrac{C}{z-3}$ $1=A(z-2)(z-3)+B(z-5)(z-3)+C(z-5)(z-2)$ $z=5,\quad 1=6\Rightarrow A=\dfrac{1}{6}$ $z=3,\quad 1=-2C\Rightarrow C=-1/2$ $z=2,\quad 1=3B\Rightarrow B=1/3$ $\therefore\dfrac{Y(z)}{z}=\dfrac{1}{6}\dfrac{1}{z-5}+\dfrac{1}{3}\dfrac{1}{z-2}-\dfrac{1}{2}\dfrac{1}{z-3}$ $Y(z)=\dfrac{1}{6}\dfrac{z}{z-5}+\dfrac{1}{3}\dfrac{z}{z-2}-\dfrac{1}{2}\dfrac{z}{z-3}$ $\therefore y_n=\dfrac{1}{6}5^n+\dfrac{1}{3}2^n-\dfrac{1}{2}3^n.$
	b)	**State and Prove Second Shifting Property of Z-Transform.**
		Statement: Let $Z(u_n)=U(z)$ and $k>0$ then $Z(u_{n+k})=Z^k\left[U(z)-u_0-u_1z^{-1}-u_2z^{-2}-\ldots-u_{k-1}z^{-(k-1)}\right]$ Proof $Z(u_{n+k})=\sum_{n=0}^{\infty}u_{n+k}z^{-n}=z^k\sum_{n=0}^{\infty}u_{n+k}z^{-(n+k)}$ $=z^k\left[\sum_{n=0}^{\infty}u_nz^{-n}-\sum_{n=0}^{k-1}u_nz^{-n}\right]$ $=z^k\left[U(z)-u_0-u_1z^{-1}-u_2z^{-2}-\ldots-u_{k-1}z^{-(k-1)}\right]$
2.	**a)**	**Solve** $y(n+3)-3y(n+1)+2y(n)=0,$ **given that** $y(0)=4, y(1)=0, y(2)=8$ **Using Z-transform.**
		Solution: $y(n+3)-3y(n+1)+2y(n)=0$. Taking z-transform on both sides $Z[y_{n+3}]-3Z(y_{n+1})+2Z(y_n)=0$

		$z^3\left[y(z)-y_0-y_1z^{-1}-y_2z^{-2}\right]-3z\left[y(z)-y_0\right]+2Y(z)=0$ $z^3\left(Y(z)-4-0-8z^{-2}\right)-3z\left(Y(z)-4\right)+2Y(z)=0$ $Y(z)\left[z^3-3z+2\right]=8z+4z^3-12z=4z^3-4z$ $\frac{Y(z)}{z}=\frac{4z^2-4}{z^3-3z+2}=\frac{4z^2-4}{(z-1)^2(z+2)}$ $\therefore z^3-3z+2=(z-1)^2(z+2)$ $\therefore \frac{Y(z)}{z}=\frac{4z^2-4}{(z-1)^2(z+2)}=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z+2}$ $\therefore 4z^2-4=A(z-1)(z+2)+B(z+2)+C(z-1)^2$ $z=1,\quad 0=3B\Rightarrow B=0$ $z=-2\quad 12=9c\Rightarrow c=12/9=4/3$ $z^2,\qquad 4=A+C\Rightarrow A=4-4/3=8/3$ $\therefore \frac{Y(z)}{z}=\frac{8}{3}\frac{1}{z-1}+0+\frac{4}{3}+\frac{1}{z+2}$ $\therefore Y(z)=\frac{8}{3}\frac{1}{z-1}+\frac{4}{3}+\frac{1}{z+2}$ $y_n=\frac{8}{3}+\frac{4}{3}(-2)^n.$
	b)	**If** $U(z)=\frac{2z^2+3z+12}{(z-1)^4}$**, find** u_2 **and** u_3**.**
		Solution: $u_0=\lim_{z\to\infty}it\, U(z)=\lim_{z\to\infty}it\,\frac{2z^2+3z+12}{(z-1)^4}=0$ $u_1=\lim_{z\to\infty}it\left[z\left[U(z)\right]-u_0\right]=\lim_{z\to\infty}it\left[\frac{2z^2+3z+12}{(z-1)^4}-0\right]=0$

		$u_2 = \lim_{z \to \infty} it\, z^2 \left[U(z) - u_0 - u_1 z^{-1} \right]$ $= \lim_{z \to \infty} it\, z^2 \left[\frac{2z^2+3z+12}{(z-1)^4} - 0 - 0 \right] = 2$ $u_3 = \lim_{z \to \infty} it\, z^3 \left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} \right]$ $= \lim_{z \to \infty} it\, z^3 \left[\frac{2z^2+3z+12}{(z-1)^4} - 0 - 0 - \frac{2}{z^2} \right]$ $= \lim_{z \to \infty} it\, z^3 \left[\frac{2z^4+3z+12z^2-2\left(z^4-4z^3+6z^2-4z+1\right)}{z^2(z-1)^4} \right]$ $= \lim_{z \to \infty} it\, \frac{z}{(z-1)^4}\left[11z^3+8z-2 \right] = 11.$
3.	**a)**	**Find** $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$ **by convolution theorem.**
		Solution: $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = Z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}\right]$ $= Z^{-1}\left[\frac{z}{(z-\frac{1}{2})}\right] * Z^{-1}\left[\frac{z}{(z-\frac{1}{4})}\right] \quad by \quad C.T$ $= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$ $= \sum_{r=0}^{n} \left(\frac{1}{2}\right)^{n-r} \left(\frac{1}{4}\right)^r = \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{1}{2}\right)^{-r} \left(\frac{1}{4}\right)^r$ $= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{\frac{1}{4}}{\frac{1}{2}}\right)^r = \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^n \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \ldots. + \left(\frac{1}{2}\right)^n \right]$

		$=\left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\left(\frac{1}{2}\right)}\right]=2\left(\frac{1}{2}\right)^n\left[1-\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)\right]$ $=2\left[\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)\right]=2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n$
	b)	**Find** $Z\left[n(n-1)(n-2)\right]$
		Solution: $Z\left[n(n-1)(n-2)\right]=Z\left\{n^3-3n^2+2n\right\}$ $=Z\left\{n^3\right\}-3Z\left\{n^2\right\}+2Z\left\{n\right\}$ $=\frac{z^3+4z^2+z}{(z-1)^4}-\frac{3z(z+1)}{(z-1)^3}+\frac{2z}{(z-1)^2}$ $=\frac{z^3+4z^2+z-3z(z+1)(z-1)+2z(z-1)^2}{(z-1)^4}=\frac{6z}{(z-1)^4}$
4.	**a)**	**Find** $Z\left(r^n\cos n\theta\right)$ **and hence deduce** $Z\left(cos\frac{n\pi}{2}\right)$
		Solution: $\therefore Z\left(e^{-in\theta}\right)=Z\left[\left(e^{-i\theta}\right)^n\right]=\frac{z}{z-e^{-i\theta}}$ $\therefore Z\left(e^{-in\theta}\right)=\frac{z}{z-e^{-i\theta}}=\frac{z\left(z-e^{i\theta}\right)}{\left(z-e^{-i\theta}\right)\left(z-e^{i\theta}\right)}$ $=\frac{z\left[z-(\cos\theta+i\sin\theta)\right]}{z^2-z\left(e^{i\theta}+e^{-i\theta}\right)+1}=\frac{z(z-\cos\theta)-iz\sin\theta}{z^2-(2z\cos\theta)+1}$ $\therefore z(\cos n\theta+i\sin n\theta)=\frac{z(z-\cos\theta)}{z^2-(2z\cos\theta)+1}-i\frac{z\sin\theta}{z^2-(2z\cos\theta)+1}$ equating real and imaginary parts,

$$Z\left(\cos n\theta\right)=\frac{z\left(z-\cos\theta\right)}{z^2-2z\cos\theta+1}$$

$$Z\left\{r^n\cos n\theta\right\}=\left[Z\left\{\cos n\theta\right\}\right]_{z\to\frac{z}{r}}=\left[\frac{z\left(z-r\cos\theta\right)}{z^2-2rz\cos\theta+r^2}\right]$$

$$Z\left\{\cos\frac{n\pi}{2}\right\}=\left[Z\left\{\cos n\theta\right\}\right]_{\theta=\frac{\pi}{2}}=\left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right]_{\theta=\frac{\pi}{2}}=\frac{z(z-\cos\frac{\pi}{2})}{z^2-2z\cos\frac{\pi}{2}+1}=\frac{z^2}{z^2+1}$$

b) **Find the inverse Z-Transform of $\frac{z\left(z+1\right)}{\left(z-1\right)^3}$ by residue method.**

Solution:

$$\text{F (z)} = \frac{z^2+z}{(z-1)^3}$$

$$\therefore \quad Z^{n-1}F(z) = \frac{z^{n+1}+z^n}{(z-1)^3}$$

Here z = 1 is a pole of order 3

$$\left[RES\right]_{z=1} = \underset{z\to1}{Lt}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3\times\frac{z^{n+1}+z^n}{(z-1)^3}\right]$$

$$=\underset{z\to1}{Lt}\frac{1}{2!}\frac{d^2}{dz^2}\left[z^{n+1}+z^n\right]=\underset{z\to1}{Lt}\frac{1}{2!}\frac{d}{dz}\left[(n+1)z^n+nz^{n-1}\right]$$

$$=\frac{1}{2}\underset{z\to1}{Lt}\left[n(n+1)z^{n-1}+n(n-1)z^{n-2}\right]=\frac{1}{2}\left[n(n+1)+n(n-1)\right]$$

$$=\frac{1}{2}\left[(n^2+n+n^2-n\right]=n^2$$

5. a) **Form the difference equation corresponding to the family of curves $y_x=ax+b2^x$.**

Solution:

$$y_x=ax+b2^x \qquad -\ (1)$$

$$y_{x+1}=a\left(x+1\right)+b2^{x+1}$$

$$\Delta y=a+b\left(2^{x+1}-2^x\right)=a+b2^x \qquad -(2)$$

		$\Delta^2 y = \left(a + b2^{x+1}\right) - \left(a + b2^x\right) = b2^x \qquad -(3)$ $from(3),\ b = \frac{\Delta^2 y}{2^x} \quad sub\ \ in\ (2)\ \ \Delta y = a + \frac{\Delta^2 y}{2^x} 2^x$ $\therefore a = \Delta y - \Delta^2 y$ $sub\ in\ (1)\ y_x = \left(\Delta y - \Delta^2 y\right)^x + \frac{\Delta^2 y}{2^x} 2^x.$ $= (1-x)\Delta^2 y + x\Delta y\ \ or\,(1-x)\left(y_{x+2} - 2y_{x+1} + y_x\right) + x\left(y_{x+1} - y_x\right) - y_x = 0$ $ie\ (x-1)\,y_{x+2} - (3x-2)\,y_{x+1} + 2xy_x = 0$
	b)	**Find** $Z^{-1}\left[\frac{z}{(z+1)(z-1)^2}\right]$ **using the method of partial fraction.**
		Solution: Let $F(z) = \frac{z}{(z+1)(z-1)^2}$ $\frac{F(z)}{z} = \frac{1}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$ $= \frac{A(z-1)^2 + B(z-1)(z+1) + C(z+1)}{(z+1)(z-1)^2}$ $1 = A(z-1)^2 + B(z-1)(z+1) + C(z+1)$ Putting $z = 1$ and $z = -1$ we get $A = \frac{1}{4}$ and $c = \frac{1}{2}$ Equating the coefficient of Z^2 on both side 0=A+B $B = -A = -\frac{1}{4}$ $\frac{F(z)}{z} = \frac{\frac{1}{4}}{(z+1)} - \frac{\frac{1}{4}}{(z-1)} + \frac{\frac{1}{2}}{(z-1)^2}$

		$F(z)=\frac{1}{4}\frac{z}{z+1}-\frac{1}{4}\frac{z}{z-1}+\frac{1}{2}\frac{z}{(z-1)^2}$ $Z^{-1}\left[F(z)\right]=\frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right]+\frac{1}{2}Z^{-1}\left[\frac{z}{(z-1)^2}\right]$ $=\frac{1}{4}(-1)^n-\frac{1}{4}+\frac{n}{2}$