Z-transform:

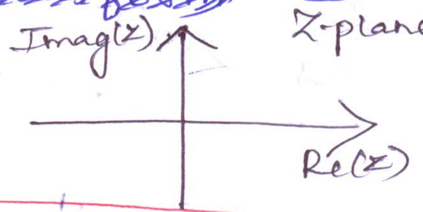
①

* The z-transform of a discrete time signal $x(n)$ is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}} \rightarrow ①$$

where z is a complex variable.

* In polar form Z-~~transform~~ can be expressed as



$$z = re^{j\omega}$$

$$\boxed{X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}}$$

* The relationship between $x(n)$ & $X(z)$ is given as

$$X(z) = Z\{x(n)\}.$$

Region of Convergence:

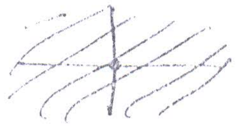
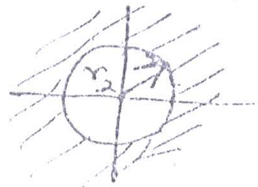
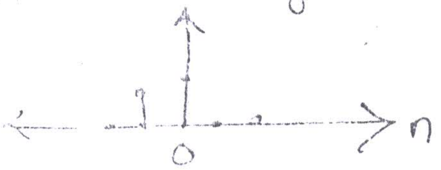
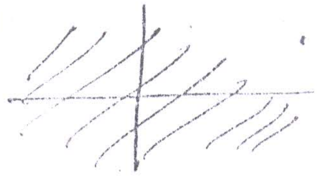
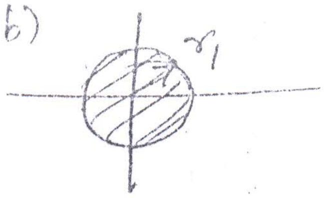
The Region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Z-transform and ROC:

(i) Finite Duration Signal:

(a) Causal Signal: $\{x(n) = 0, n < 0\}$

eg: $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$.

Signal	ROC
Finite Duration Signal	
Causal 	a)  entire z-plane except $z=0$.
Infinite duration	
"	b)  $\lvert z\rvert > r_2$
Anticausal signal 	a)  entire z-plane except $z=\infty$
	b)  $\lvert z\rvert < r_1$

Inverse z-transform:

Inverse z-transform of $x(n)$ is given by

$$\boxed{x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz}$$

Properties of z-transform:

1. Linearity:

if

$$x_1(n) \xleftrightarrow{Z} X_1(z)$$

&

$$x_2(n) \xleftrightarrow{Z} X_2(z)$$

then

$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} X(z) = a_1 X_1(z) + a_2 X_2(z)$.

ROC: intersection of individual transform.

eg:

Determine Z-transform and ROC of

$x(n) = 3(2^n)u(n) - 4(3^n)u(n)$.

Soln:

$x_1(n) = 2^n u(n)$

$x_2(n) = 3^n u(n)$.

$$\therefore X_1(z) = \frac{1}{1-2z^{-1}} \quad |z| > 2$$

$$X_2(z) = \frac{1}{1-3z^{-1}} \quad |z| > 3$$

if $(x(n)) = \alpha^n u(n)$

$$\therefore X(z) = \frac{1}{1-\alpha z^{-1}} \quad \text{ROC: } |z| > |\alpha|$$

$$\therefore X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}} \quad |z| > 3.$$

---

2 Time Shifting:

If

$x(n) \xleftrightarrow{Z} X(z)$

then

$x(n-k) \xleftrightarrow{Z} z^{-k} X(z)$

ROC: same as $X(z)$ ~~to~~ except for $z=0$ if $k>0$ & $z=\infty$ if $k<0$.

eg:

Determine z-transform of $x(n+2)$ where

$$x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$$

Soln:

$$x_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$x(n+2) \overset{z}{\longleftrightarrow} z^2 X(z)$$

$$X(z) = z^2 (X_1(z))$$

$$= z^2 + 2z + 5 + 7z^{-1} + z^{-3}.$$

ROC: entire z-plane except $z = \infty$.

3 Scaling in z-domain:

If

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad \text{ROC}: r_1 < |z| < r_2$$

then

$$\alpha^n x(n) \overset{z}{\longleftrightarrow} X(\alpha^{-1} z) \quad \text{ROC}: |\alpha| r_1 < |z| < |\alpha| r_2$$

eg:

Determine z-transform of

$$x(n) = \alpha^n u(n)$$

Soln:

$$x_1(n) = u(n)$$

$$X_1(z) = \frac{1}{1 - z^{-1}} \quad |z| > 1$$

$$X(z) = X(\alpha^{-1} z)$$

$$= \frac{1}{1 - \alpha z^{-1}} \quad |z| > |\alpha|$$

Proof:

$$Z\{\alpha^n x(n)\} = \sum_{n=-\infty}^{\infty} \alpha^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (\alpha^{-1} z)^{-n}$$

$$= X(\alpha^{-1} z).$$

ROC of $x(n)$: $r_1 < |z| < r_2$

$$r_1 < |\alpha^{-1} z| < r_2$$

$$|\alpha| r_1 < |z| < |\alpha| r_2.$$

(4) Time reversal:

If

$$x(n) \overset{Z}{\longleftrightarrow} X(z) \quad ROC: r_1 < |z| < r_2$$

then

$$x(-n) \overset{Z}{\longleftrightarrow} X(z^{-1}) \quad ROC: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l} \quad l = -n$$

$$= \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$$

$$= X(z^{-1})$$

ROC: $r_1 < |z^{-1}| < r_2$

$$\frac{1}{r_2} < |z| < \frac{1}{r_1}$$

eg:

Determine Z-transform of

$$x(n) = u(-n).$$

Solution:

$$x_1(n) = u(n)$$

$$X_1(z) = \frac{1}{1-z^{-1}} \quad |z| > 1$$

$$X(z) = X_1(z^{-1})$$

$$= \frac{1}{1-z} \quad |z| < 1.$$

(6) Convolution theorem:

If $x_1(n) \overset{Z}{\longleftrightarrow} X_1(z)$

$x_2(n) \overset{Z}{\longleftrightarrow} X_2(z)$

ROC: Intersection of ROC of $X_1(z)$ & $X_2(z)$.

$$x(n) = x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X(z) = X_1(z) X_2(z).$$

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k).$$

Proof:

Let

$$x(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} X_2(z)$$

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1(k) z^{-k}$$

$$= X_2(z) X_1(z)$$

(7) Correlation of 2 sequences:

$$x_1(n) \overset{Z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \overset{Z}{\longleftrightarrow} X_2(z)$$

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l) \xleftrightarrow{Z}$$

Proof

$r_{x_1x_2}(l)$

$R_{x_1x_2}(z) = x_1(l) * x_2(-l)$

$= X_1(z) X_2(z^{-1})$

$$\boxed{r_{x_1x_2}(l) = x_1(l) * x_2(-l)}$$

$R_{x_1x_2}(z) = Z\{x_1(l)\}\, Z\{x_2(-l)\}$

$= X_1(z) X_2(z^{-1})$

ROC: intersection of ROC of $X_1(z)$ & $X_2(z)$.

Multiplication of 2 sequences:

$x_1(n) \xleftrightarrow{Z} X_1(z)$

$x_2(n) \xleftrightarrow{Z} X_2(z)$

$$x(n) = x_1(n)\, x_2(n) \xleftrightarrow{Z} X(z) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

ROC: $r_{1l}\, r_{2l} < |z| < r_{1u}\, r_{2u}$

Parseval's relation:

$x_1(n)$ & $x_2(n) \rightarrow$ complex valued sequence.

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv.$$

Initial value theorem:

If $x(n)$ is causal, then

$$x(0) = \lim_{z \to \infty} X(z)$$

Proof: $x(n) = x_1(n)\, x_2(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, z^{-n}$$

$$x_1(n) = \frac{1}{2\pi j} \oint_C X_1(v)\, v^{n-1}\, dv.$$

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi j} \oint_C X_1(v)\, x_2(n)\, z^{-n} v^{n-1}\, dv$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{z}{v}\right)^{-n} v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint_C X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

X(

$* \Sigma$

Y(

H(z)

Scaling in z-domain:

$$x(n) \xleftrightarrow{z} X(z) \quad ROC: r_1 < |z| < r_2$$

then

$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z) \quad ROC: |a| r_1 < |z| < |a| r_2.$$

Eg:

Determine z-transform of signal

$x(n) = a^n (\cos \omega_0 n) u(n)$.

Soln:

$$\cos(\omega_0 n) u(n) \xleftrightarrow{z} \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}. \quad ROC: |z| > 1$$

$$a^n \cos(\omega_0 n) u(n) \xleftrightarrow{z} \frac{1 - az^{-1}\cos\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$$

ROC:

$|z| > |a|$.

Time reversal:

$$x(n) \xleftrightarrow{z} X(z) \quad ROC: r_1 < |z| < r_2$$

then

$$x(-n) \xleftrightarrow{z} X(z^{-1}) \quad ROC: \frac{1}{r_2} < |z| < \frac{1}{r_1}.$$

Eg.

Determine z-transform of the signal

$x(n) = u(-n)$

Soln:

$$u(n) \xleftrightarrow{z} \frac{1}{1 - z^{-1}}. \quad ROC: |z| > 1$$

$$u(-n) \xleftrightarrow{z} \frac{1}{1-z} \quad ROC: |z| < 1$$

⑥ Differentiation in z-domain:

$$\cancel{x}(n) \xleftrightarrow{z} X(z)$$

then

$$n\,x(n) \xleftrightarrow{z} -z\frac{d\,X(z)}{dz} \quad \text{same ROC}$$

Ex:)

Determine z-transform of the signal

$$x(n) = n a^n u(n)$$

Soln:

$$x_1(n) = a^n u(n) \xleftrightarrow{z} X_1(z) = \frac{1}{1-az^{-1}} \quad ROC: |z| > |a|$$

$$n a^n u(n) \xleftrightarrow{z} X(z) = -z\frac{d}{dz}X_1(z)$$

$$= +\, z\frac{a z^{-2}}{(1-az^{-1})^2}$$

$$= \frac{a z^{-1}}{(1-az^{-1})^2} \quad ROC: |z| > |a|.$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$\frac{d\,X(z)}{dz} = \sum_{n=-\infty}^{\infty} -n\,x(n) z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty} (n\,x(n)) z^{-n}.$$

$$= -z^{-1}\, Z\{n x(n)\}$$

$$-z\frac{d\,X(z)}{dz} = Z\{n x(n)\}.$$

Problem:

Determine z-transform and ROC of the signal (4)

$x(n) = \cos(\omega n)\, u(n)$.

Solution:

$x(n) = \cos(\omega n)\, u(n)$

$\because e^{j\theta} = \cos\theta + j\sin\theta$

$\therefore \cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$.

$\therefore x(n) = \frac{1}{2} e^{j\omega n} u(n) + \frac{1}{2} e^{-j\omega n} u(n)$.

~~[if (x(n) = a^n x_1(n) then X(z)~~

$$X(z) = \frac{1}{2} Z\{e^{j\omega_0 n} u(n)\} + \frac{1}{2} Z\{e^{-j\omega_0 n} u(n)\}$$

$$Z\{a^n u(n)\} = \frac{1}{1 - a z^{-1}} \quad \text{ROC: } |z| > |\alpha|$$

$$X(z) = \frac{1}{2}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}}\right]$$

$$\text{ROC: } |z| > |e^{j\omega_0}| = 1.$$

$$X(z) = \frac{1 - e^{-j\omega_0} z^{-1} + 1 - e^{j\omega_0} z^{-1}}{2(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}$$

$$= \frac{2 - (\cos \omega_0) z^{-1} + j(\sin \omega_0) z^{-1} - (\cos \omega_0) z^{-1} - j(\sin \omega_0) z^{-1}}{2(1 - e^{-j\omega_0} z^{-1} - e^{j\omega_0} z^{-1} + z^{-2}}$$

$$= \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \qquad \text{ROC: } |z| > 1$$

Common Z-transform pairs:

Signal $x(n)$	Z-transform $X(z)$	ROC
$\delta(n)$	1	All Z
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(\cos \omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(\sin \omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$(a^n \cos \omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n \sin \omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$