UNIT 5 ADVANCED QUEUEING MODELS
PART A

1 When do you say a series queue is blocked?

Solution:

When a customer in any service station has to wait in that station itself, until the next service station is free of customer, that service queue is said to be blocked

2 State the steady state probabilities of the Finite source queuing model represented by (M/M/R):(GD/K/K)

Solution:

Assume *m=K and c=R*

The steady state probabilities

$$P_n = \begin{cases} {}^{m}C_n \left(\frac{\lambda}{\mu}\right)^n P_0 & , n = 0,1,2,\ldots,c \\ \frac{m!}{(m-n)!c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n P_0 & , n = c+1, c+2, \ldots, m \end{cases}$$

$$P_0 = \left[1 + \sum_{n=1}^{c-1} {}^{m}C_n \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{m} \frac{m!}{(m-n)!c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

3 When a M/G/1 queuing model will become classic M/M/1 model?

Solution:

In the M/G/1 queuing model, G stands for the general service time distribution.

If the general service time distribution is replaced by exponential service time distribution then the the M/G/1 model becomes the classical M/M/1 model.

4 Define series queues (Tandem queues) with an example.

Solution:

In series queues , there are a series of service stations through which each calling unit must progress prior leaving the system. Ex. A physical examination for a patient where the patient under goes a series of stages, lab tests, ECG , X ray etc.

5 Consider a service facility with two sequential stations with respective service rates of 3/min and 4/min. The arrival rate is 2/min. What is the average waiting time of the system if the system could be approximated by a two series Tandem queue?

Solution:

$\mu_1 = 3/\text{min.}, \mu_2 = 4/\text{min.}, \lambda = 2/\text{min.}$

Average waiting time of customers in the system =W_s(station 1) + W_s(station 2)

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{3-2} + \frac{1}{4-2} = 1.5\text{min.}$$

6 Write down the flow balance equation of open Jackson network.

Solution:

$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}, j = 1,2,\ldots k$, where r_j is the arrival rate to node j , λ_j is the total arrival rate of customers to node j and P_{ij} is the probability that a departure from server i joins the queue at server j.

7 For an M/G/1 model if λ = 5 min, μ = 6 min and σ = 1/20, find the length of the queue?

Solution:

$\mu_1 = 3/\text{min.}, \mu_2 = 4/\text{min.}, \lambda = 2/\text{min.}$

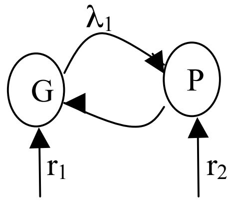
Average waiting time of customers in the system =W_s(station 1) + W_s(station 2)

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{3-2} + \frac{1}{4-2} = 1.5\text{min.}$$

8 In a departmental store there are 2 sections, namely, grocery section and perishable (vegetables and fruits) section. Customers from outside arrive at the G-Section according to a Poisson process at a

mean rate of 10/hour and they reach the P-section at a mean rate of 2/hour. The service times at both the sections are exponentially distributed with parameters 15 and 12 respectively. On finishing the job in the G-section, a customer is equally likely to go the P-section or to leave the store, whereas a customer on finishing his job in the P-section will go to the G-section with probability 0.25 and leave the store otherwise. Assuming that there is only one salesman in each section, write the equations concerned with the arrival rates to each section.

Solution:



Arrival rate to G-section = $r_1 = 10/\text{hr}$

Arrival rate to P-section = $r_2 = 2/\text{hr}$

Service rate of G-section = $\mu_1 = \frac{1}{15}/\text{hr}$

Service rate of P-section = $\mu_2 = \frac{1}{12}/\text{hr}$

Let $\lambda_1 =$ total arrival rate to G-section

$\lambda_2 =$ total arrival rate to P-section

$P_{GP} = \frac{1}{2}$ & $P_{PG} = 0.25$

$\lambda_1 = r_1 + \lambda_2 P_{PG} \Rightarrow \lambda_1 = 10 + \frac{1}{2}\lambda_2$

$\lambda_2 = r_2 + \lambda_1 P_{GP} \Rightarrow \lambda_2 = 2 + 0.25\lambda_1$

PART B

1. Derive the expected steady state system size for the single server with poisson input and general service. Hence deduce the same for i) (M/M/1):(∞/FIFO) model ii) (M/D/1):(∞/FIFO) model iii) (M/Ek/1):(∞/FIFO) model

Solution:

Pollaczek –Khinchine Formula : $(M/G/1);(\infty/GD)$ Model:

Let as assume the arrivals follow poisson process with arrival rate λ.

Let $X_n = N(t_n)$ represents the number of customer in the system when the nth customer departs.

Also, the sequence of random variables $\{X_n : n = 1,2,3,...\}$ is a markov chain.

Hence, we have,

$$X_{n+1} = \begin{cases} X_n - 1 + A, & if\ X_n > 0 \\ A & , if\ X_n = 0 \end{cases} \quad ------(1)$$

Where A is the number of customer who arrived during the service time T of the $(n+1)^{th}$customer.

To express X_{n+1} as single equation, we introduce unit step function.

$$U(X_n) = \begin{cases} 1 & if\ X_n > 0 \\ 0 & f\ X_n = 0 \end{cases}$$

$\therefore$ equation (1) can be written as

$$X_{n+1} = X_n - U(X_n) + A$$

Squaring on both sides and taking expectation, we have,

$$X_{n+1}^2 = \left(X_n - U(X_n) + A\right)^2$$

$$= X_n^2 + U^2(X_n) + A^2 - 2X_n + U(X_n) - 2U(X_n)A + 2AX_n$$

Taking Expectations on both sides

$$E\left(X_{n+1}^2\right) = E\left(X_n^2\right) + E\left(U^2(X_n)\right) + E\left(A^2\right) - 2E\left(X_n U(X_n)\right) - 2E\left(U(X_n)A\right) + 2E\left(AX_n\right) \quad ----(2)$$

Under steady state, we have the expectation of

$$E\left(X_{n+1}\right) = E\left(X_n\right) = E(X)$$

$(2) \Rightarrow\ E\left(X^2\right) = E\left(X^2\right) + E\left(U^2\left(X\right)\right) + E\left(A^2\right) - 2E\left(XU\left(X\right)\right) - 2E\left(U\left(X\right)A\right) + 2E\left(AX\right)$

$\Rightarrow E\left(U^2\left(X\right)\right) + E\left(A^2\right) - 2E\left(XU\left(X\right)\right) - 2E\left(U\left(X\right)A\right) + 2E\left(AX\right) = 0 \ -----(3)$

To find $E\left[U^2\left(X\right)\right]$:

$U^2\left(X\right) = U\left(X\right)$

$E\left[U^2\left(X\right)\right] = E\left[U\left(X\right)\right]$

But $E\left[U\left(X\right)\right] = \sum_{x=0}^{\infty} U\left(X\right).P\left(X = x\right) = \sum_{x=0}^{\infty} P\left(X = x\right) = P(X \geq 1) = \rho$

$E\left[U^2\left(X\right)\right] = E\left[U\left(X\right)\right] = \rho$

To find $E\left[X.U\left(X\right)\right]$:

It is clear that $X.U\left(X\right) = X$ so we can say that $2E\left[X.U\left(X\right)\right] = 2E\left(X\right)$

To find $E\left[U\left(X\right)A\right]$:

The number of customers arrive in the system between the n^{th} and $(n+1)^{th}$ departure instant is A and is independent of number of customers in the queueing system immediately after the n^{th} departure instant, X_n.

Thus $E\left[U\left(X\right).A\right] = E\left[U\left(X\right)\right].E\left(A\right) = \rho E\left(A\right) ----(4)$ $\quad\quad \because E\left[U\left(X\right)\right] = \rho$

Where $E\left(A\right)$ is to find from (1)

$(1) \Rightarrow X_{n+1} = X_n - U\left(X_n\right) + A$

Taking expectations on bothsides

$E\left[X_{n+1}\right] = E\left[X_n\right] - E\left[U\left(X_n\right)\right] + E\left(A\right)$

$E\left[X\right] = E\left[X\right] - E\left[U\left(X\right)\right] + E\left(A\right)$ since the system is in steady state

$E\left(A\right) = E\left[U\left(X\right)\right] = \rho$

$(4) \Rightarrow E\left[U\left(X\right)A\right] = \rho.\rho = \rho^2$

To find $E\left(AX\right)$:

Let X and A are two random variables and its independent, so

$E\left(AX\right) = E\left(X\right) + E\left(A\right) = \rho E\left(X\right)$

Substituting all these values in (3)

$(3) \Rightarrow \rho + E\left(A^2\right) - 2E\left(X\right) - 2\rho^2 + 2E\left(X\right)\rho = 0$

$\rho + E\left(A^2\right) - 2\rho^2 - 2(1-\rho)E\left(X\right) = 0$

$\rho + E\left(A^2\right) - 2\rho^2 - 2(1-\rho)E\left(X\right) = 0$

$2(1-\rho)E\left(X\right) = \rho + E\left(A^2\right) - 2\rho^2$

$$E\left(X\right) = \frac{\rho + E\left(A^2\right) - 2\rho^2}{2(1-\rho)} \ -----(5)$$

To find $E\left(A^2\right)$:

Let T be the service time random variable, then $E\left[T\right] = \frac{1}{\mu}$

$\therefore E\left[A^2\right] = E\left[E\left(A^2/T\right)\right] = E\left(\lambda^2T^2 + \lambda T\right) = E\left(\lambda^2T^2\right) + E\left(\lambda T\right)$

$$= \lambda^2 E(T^2) + \lambda E(T) = \lambda^2 E(T^2) + \frac{\lambda}{\mu} \qquad \because \text{ If X \& Y are independent random variables}$$

$$= \lambda^2 E(T^2) + \rho \qquad \text{in same probability space then } \mathrm{E(X)} = \mathrm{E}\left[\mathrm{E}\left(\mathrm{X/Y}\right)\right]$$

$$(5) \Rightarrow E(X) = \frac{\rho + \lambda^2 E(T^2) + \rho - 2\rho^2}{2(1-\rho)} = \frac{2\rho - 2\rho^2 + \lambda^2 E(T^2)}{2(1-\rho)} = \frac{2\rho(1-\rho)}{2(1-\rho)} + \frac{\lambda^2 E(T^2)}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2 E(T^2)}{2(1-\rho)}$$

$$= \rho + \frac{\lambda^2\left[\mathrm{var}(T) + [E(T)]^2\right]}{2(1-\rho)} \qquad \because \mathrm{var}(T) = E(T^2) - [E(T)]^2 \Rightarrow E(T^2) = \mathrm{var}(T) + [E(T)]^2$$

$$E(X_n) = L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left[\sigma^2 + \left(\frac{1}{\mu}\right)^2\right]}{2\left(1 - \frac{\lambda}{\mu}\right)} \qquad \because \mathrm{var}(T) = \sigma^2 \ \& \ E(T) = \frac{1}{\mu}$$

$$E(X_n) = L_s = \rho + \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} \qquad \text{where } \rho = \frac{\lambda}{\mu}$$

Deduction 1: (M/M/1):(∞/FIFO)

M/G/1 queuing model will become classic M/M/1 model if the service time follows exponential distribution with mean $E(T) = \frac{1}{\mu}$ and variance $\mathrm{var}(T) = \frac{1}{\mu^2}$

By P-K formula,

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\mathrm{var}(T) + \frac{1}{\mu^2}\right)}{2\left(1 - \frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\frac{1}{\mu^2} + \frac{1}{\mu^2}\right)}{2\left(1 - \frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu} + \frac{2\frac{\lambda^2}{\mu^2}}{2\left(1 - \frac{\lambda}{\mu}\right)} = \frac{\frac{\lambda}{\mu} - \frac{\lambda^2}{\mu^2} + \frac{\lambda^2}{\mu^2}}{\left(1 - \frac{\lambda}{\mu}\right)} = \frac{\frac{\lambda}{\mu}}{\left(1 - \frac{\lambda}{\mu}\right)}$$

$$L_s = \frac{\lambda}{\mu - \lambda}$$

Deduction 2: (M/D/1):(∞/FIFO) model

In this queueing system arrivals are poisson, service time is deterministic or constant and single service channel.

Since service time T is constant. ∴ Var(T) $=\sigma^2=0$

By P-K formula, we get

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\mathrm{var}(T) + \frac{1}{\mu^2}\right)}{2\left(1 - \frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu} + \frac{\lambda^2\left(0 + \frac{1}{\mu^2}\right)}{2\left(1 - \frac{\lambda}{\mu}\right)}$$

$$L_s = \rho + \frac{\rho^2}{2(1-\rho)}$$

Deduction 3: (M/E$_k$/1):(∞/FIFO) model

In this queueing system arrivals are poisson, service time follows Erlang distribution with K phases and its mean $E(T) = \frac{k}{\mu}$ and variance $\mathrm{var}(T) = \frac{k}{\mu^2}$ and single service channel.

By P-K formula, we get

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\mathrm{var}(T) + \left(E(T)\right)^2\right)}{2\left(1-\frac{\lambda}{\mu}\right)} = \frac{k\lambda}{\mu} + \frac{\lambda^2\left(\frac{k}{\mu^2} + \frac{k^2}{\mu^2}\right)}{2\left(1-\frac{k\lambda}{\mu}\right)} = k\rho + \frac{k\rho^2(1+k)}{2(1-k\rho)}$$

2 A car wash facility operates with only one bay. Cars arrive according to a Poisson fashion with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. The parking lot is large enough to accommodate any number of cars. Find L_s, L_q, W_s, and W_q,
(i) if the time for washing and cleaning a car is constant and equal to 10 minutes.
(ii) if the time for washing and cleaning a car follows a uniform distribution between 8 and 12 minutes
(iii) if the time for washing and cleaning a car follows a discrete distribution with values equal to 4,8,15 minutes and corresponding probabilities 0.2, 0.6 and 0.2.
Solution:

Arrival rate $\lambda = 4/hr \Rightarrow \lambda = \frac{4}{60} = \frac{1}{15}/\min$

(i) Given that the time for washing and cleaning a car is constant and equal to 10 minutes

(i.e) service time is constant $\therefore$ Var (T)=0 $E(T) = 10 \quad \Rightarrow \mu = \frac{1}{E(T)} = \frac{1}{10}$ minutes

By P-K formula,

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left[\mathrm{var}(T) + \left[E(T)\right]^2\right]}{2\left(1-\frac{\lambda}{\mu}\right)}$$

$$L_s = \frac{1/15}{1/10} + \frac{\lambda^2\left[0+10^2\right]}{2\left(1-\frac{1/15}{1/10}\right)} = \frac{2}{3} + \frac{\frac{100}{225}}{2\left(1-\frac{2}{3}\right)} = 0.6667 + \left(\frac{0.4444}{0.6667}\right) = 1.33$$

$$L_q = L_s - \frac{\lambda}{\mu} = 1.33 - \frac{1/15}{1/10} = 1.33 - 0.667 = 0.663$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.663}{1/15} = 0.663(15) = 9.945\,\text{min}$$

$$W_s = \frac{L_s}{\lambda} = \frac{1.33}{1/15} = 1.33(15) = 19.95\,\text{min}$$

ii) Let T be a continuous random variables and it follows uniform distribution in (8,12)

$E(T) =$ Mean of uniform distribution in (8,12)

$$E(T) = \frac{1}{\mu} = \frac{1}{2}(b+a) = \frac{1}{2}(12+8) = 10 \Rightarrow \mu = \frac{1}{E(T)} = \frac{1}{10}$$

$$Var(T) = \frac{1}{2}(b-a)^2 = \frac{1}{2}(12-8)^2 = \frac{4}{3}$$

By P-K formula,

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left[\mathrm{var}(T) + \left[E(T)\right]^2\right]}{2\left(1-\frac{\lambda}{\mu}\right)} = 0.667 + \frac{\frac{1}{225}\left[\frac{4}{3}+10^2\right]}{0.667} = 0.667 + \frac{0.4504}{0.667} = 0.667 + 0.6753 = 1.342$$

$$L_q = L_s - \frac{\lambda}{\mu} = 1.342 - \frac{1/15}{1/10} = 1.342 - 0.667 = 0.675$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.675}{1/15} = 0.675(15) = 10.13\,\text{min}$$

$$W_s = \frac{L_s}{\lambda} = \frac{1.342}{1/15} = 1.342(15) = 20.13\,\text{min}$$

iii) The service time T is discrete random variable and its probability mass function is given by

T	4	8	15
$P(T)$	0.2	0.6	0.2

$E(T) = \sum tP(T=t) = (4)(0.2) + 8(0.6) + 15(0.2) = 8.6\,\text{min}$

$E\left(T^2\right) = \sum t^2 P(T=t) = (4^2)(0.2) + 8^2(0.6) + 15^2(0.2) = 86.6\,\text{min}$

$Var\left(T\right) = E\left(T^2\right) - \left[E\left(T\right)\right]^2 = 86.6 - 8.6^2 = 12.64$

$$L_s = \lambda E(T) + \frac{\lambda^2\left[\text{var}(T) + \left[E(T)\right]^2\right]}{2\left(1-\lambda E(T)\right)} = \frac{8.6}{15} + \frac{\frac{1}{225}\left[12.64 + 8.6^2\right]}{2\left(1-\frac{8.6}{15}\right)} = 0.573 + 0.451 = 1.024\ ;\ 1\,car$$

$$L_q = L_s - \frac{\lambda}{\mu} = 1.024 - \frac{8.6}{15} = 1.024 - 0.573 = 0.451$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.451}{1/15} = 0.451(15) = 6.77\,\text{min}$$

$$W_s = \frac{L_s}{\lambda} = \frac{1.024}{1/15} = 1.024(15) = 15.36\ \text{min}$$

3(a) An average of 120 students arrive each hour (inter-arrival times are exponential) at the controller office to get their hall tickets. To complete the process, a candidate must pass through three counters. Each counter consists of a single server; service times at each counter are exponential with the following mean times: counter 1, 20 seconds; counter 2, 15 seconds and counter 3, 12 seconds. On the average how many students will be present in the controller's office.

Solution:

This problem comes under series queues with 3 counters

Given arrival rate $= \lambda = 120$ /hr

Service rate at counter1 = $\mu_1 = \frac{1}{20}/\text{sec} \Rightarrow \mu_1 = \frac{1}{20} \times 60 \times 60 = 180/hr$

Service rate at counter 2 = $\mu_2 = \frac{1}{15}/\text{sec} \Rightarrow \mu_2 = \frac{1}{15} \times 60 \times 60 = 240/hr$

Service rate at counter 3 = $\mu_1 = \frac{1}{12}/\text{sec} \Rightarrow \mu_3 = \frac{1}{12} \times 60 \times 60 = 300/hr$

The average number of students will be present in the counter 1is

$$L_{s_1} = \frac{\lambda}{\mu_1 - \lambda} = \frac{120}{180-120} = \frac{120}{60} = 2$$

The average number of students will be present in the counter 2 is

$$L_{s_2} = \frac{\lambda}{\mu_2 - \lambda} = \frac{120}{240-120} = \frac{120}{120} = 1$$

The average number of students will be present in the counter 3 is

$$L_{s_3} = \frac{\lambda}{\mu_3 - \lambda} = \frac{120}{300-120} = \frac{120}{180} = \frac{2}{3}$$

The average number of total students will be present in the controller's office is

$$L_s = L_{s_1} + L_{s_2} + L_{s_3} = 2 + 1 + \frac{2}{3} = \frac{11}{3}.$$

3(b) The police department of a city has 5 patrol cars. A patrol car breaks down and requires service once every 30 days. The police department has two repair workers, each of whom takes an average of 3 days to repair a car. Break down time and repair time are exponential. Determine

(i) the average number of cars in good condition.

(ii) the average down time for a car that needs repairs.

(iii) the fraction of the time a particular repair worker is idle.

Solution:

This problem comes under the machine inference model

Number of machine = $m=5$

Number of server = $c=2$

Arrival rate = $\lambda = \frac{1}{30}$ /day

Service rate = $\mu = \frac{1}{3}$/day

First we find steady state probabilities

$$P_n = \begin{cases} {}^{m}C_n \left(\frac{\lambda}{\mu}\right)^n P_0 & , n = 0,1,2,\ldots,c \\ \frac{m!}{(m-n)!c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0 & , n = c+1, c+2,\ldots,m \end{cases}$$

$$P_n = {}^{m}C_n \left(\frac{\lambda}{\mu}\right)^n P_0 \quad , n = 0,1,2$$

$$P_1 = {}^{5}C_1 \left(\frac{1}{10}\right)^1 P_0 = 0.5P_0$$

$$P_2 = {}^{5}C_2 \left(\frac{1}{10}\right)^2 P_0 = 0.1P_0$$

$$P_n = \frac{m!}{(m-n)!c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad , n = 3,4,5$$

$$P_3 = \frac{5!}{(5-3)!\,2!2^{3-2}}\left(\frac{1}{10}\right)^3 P_0 = 0.015P_0$$

$$P_4 = \frac{5!}{(5-4)!\,2!2^{4-2}}\left(\frac{1}{10}\right)^4 P_0 = 0.0015P_0$$

$$P_5 = \frac{5!}{(5-5)!\,2!2^{5-2}}\left(\frac{1}{10}\right)^5 P_0 = 0.000075P_0$$

w.k.t $P_1 + P_2 + P_3 + P_4 + P_5 = 1$

$\therefore P_0(1 + 0.5 + 0.1 + 0.015 + 0.0015 + 0.000075) = 1$

$\therefore P_0 = 0.619$

and $P_1 = 0.310$, $P_2 = 0.062$, $P_3 = 0.009$, $P_4 = 0.001$, $P_5 = 0$

i) The expected number of cars in good condition

= Total cars – Expected Number of cars in bad condition

$$= m - \sum_{n=0}^{m} nP_n = 5 - \sum_{n=0}^{5} nP_n$$

$$= 5 - \left[0 + 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5\right]$$

$$= 5 - \left[0 + 1(0.310) + 2(0.062) + 3(0.009) + 4(0.001) + 5(0)\right] = 5 - \left[0.456\right]$$

$\therefore$ The expected number of cars in good condition =4.535 cars

ii) The average down time for a car that needs repairs is

$$W_s = \frac{L_s}{\lambda'} \text{ where } \lambda' = \lambda(m - L_s) = \frac{1}{30}\left(5 - \sum_{n=0}^{5} nP_n\right) = \frac{1}{30}(5 - 0.456) = 0.151$$

$$W_s = \frac{0.456}{0.151} = 3.019 \text{ days}$$

iii) The fraction of time that a particular repair worker will be idle is

$$= P_0 + \frac{c-1}{c}P_1 + \frac{c-2}{c}P_2 + \frac{c-3}{c}P_3 + ...$$

$$= P_0 + \frac{2-1}{2}P_1 + \frac{2-2}{c}P_2 = P_0 + \frac{1}{2}P_1 = 0.619 + \frac{0.310}{2} = 0.774$$

4(a) Consider an open queuing network with parameter values shown below:

Facility j	s_j	μ_j	α_j	i = 1	i = 2	i =3
j = 1	1	10	1	0	0.1	0.4
j = 2	2	10	4	0.6	0	0.4
j = 3	1	10	3	0.3	0.3	0

(i) Find the steady state distribution of the number of customers at facility 1, facility 2 and facility 3.

(ii) Find the expected total number of customers in the system.

(iii) Find the expected total waiting time for a customer.

Solution:

Given an open Jackson network with 3 facilities with 1,2,3 servers respectively.

Assume $s_j = c_j$ $\therefore$ $c_1 = 1, c_2 = 2, c_3 = 1$

Arrival rate from outside $r_1 = 1, r_2 = 4, r_3 = 3$

Service rate $\mu_1 = 10, \mu_2 = 10, \mu_3 = 10$

Also given $P_{11} = 0;\ P_{12} = 0.6;\ P_{13} = 0.3;\ P_{21} = 0.1;\ P_{22} = 0;\ P_{23} = 0.3;\ P_{31} = 0.4;\ P_{32} = 0.4;\ P_{33} = 0$

Let $\lambda_1, \lambda_2, \lambda_3$ be the total arrival rates to stations 1,2,3 respectively.

The flow balance equations are given by

$$\lambda_j = r_j + \sum_{i-1}^{c} \lambda_i r_{ij} \quad , j = 1,2,3...c$$

$$\lambda_1 = r_1 + \lambda_2 P_{21} + \lambda_3 P_{31} \Rightarrow \lambda_1 = 1 + 0.1\lambda_2 + 0.4\lambda_3 \quad ------(1)$$

$$\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_3 P_{32} \Rightarrow \lambda_2 = 4 + 0.6\lambda_1 + 0.4\lambda_3 \quad ------(2)$$

$$\lambda_3 = r_3 + \lambda_1 P_{13} + \lambda_2 P_{23} \Rightarrow \lambda_3 = 3 + 0.3\lambda_1 + 0.3\lambda_3 \quad ------(3)$$

Solving (1),(2) and (3) we get, $\lambda_1 = 5, \lambda_2 = 10, \lambda_3 = \frac{15}{2}$

For facility 1:

i) To find the steady state distribution of the number of customers:

Facility 1 contain only one server so, treating as *(M/M/1);(∞/FCFS)*

$$P_{n_1} = \left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1 - \frac{\lambda_1}{\mu_1}\right) = \left(\frac{1}{2}\right)^{n_1}\left(1 - \frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1}\left(\frac{1}{2}\right)$$

$$P_0 = \left(1 - \frac{\lambda_1}{\mu_1}\right) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

For facility 2:

i) To find the steady state distribution of the number of customers:

Facility 1 contains 2 server so, treating as *(M/M/c);(∞/FCFS)*

$$P_{n_2} = \begin{cases} \dfrac{1}{c_2!c_2^{\,n-c}}\left(\dfrac{\lambda_2}{\mu_2}\right)^{n_2} P_0 & ,n \ge c_2 \\ \dfrac{1}{n_2!}\left(\dfrac{\lambda_2}{\mu_2}\right)^{n_2} P_0 & ,0 \le n < c_2 \end{cases}$$

$$P_0 = \left[\sum_{n=0}^{c_2-1}\frac{1}{n!}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2} + \frac{1}{c_2!}\left(\frac{\lambda_2}{\mu_2}\right)^{c_2}\left(\frac{\mu_2 c_2}{\mu_2 c_2 - \lambda_2}\right)\right]^{-1}$$

$$P_0 = \left[\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{10}{10}\right)^{n_2} + \frac{1}{2!}\left(\frac{10}{10}\right)^{c_2}\left(\frac{10\times 2}{10\times 2-10}\right)\right]^{-1} = \frac{1}{3}$$

$$P_{n_2} = \begin{cases} \dfrac{1}{2!2^{n_2-2}}\left(\dfrac{10}{10}\right)^{n_2}\dfrac{1}{3} & ,n \ge 2 \\ \dfrac{1}{n_2!}\left(\dfrac{10}{10}\right)^{n_2}\dfrac{1}{3} & ,n = 0,1 \end{cases}$$

$$P_{n_2} = \begin{cases} \dfrac{1}{3}\dfrac{1}{2^{n_2-1}} & ,n \ge 2 \\ \dfrac{1}{3} & ,n = 0,1 \end{cases}$$

For facility 3:

i) To find the steady state distribution of the number of customers:

Facility 1 contain only one server so, treating as *(M/M/1);(∞/FCFS)*

$$P_{n_3} = \left(\frac{\lambda_3}{\mu_3}\right)^{n_3}\left(1-\frac{\lambda_3}{\mu_2}\right) = \left(\frac{15/2}{10}\right)^{n_3}\left(1-\frac{15/2}{10}\right) = \left(\frac{3}{4}\right)^{n_3}\left(\frac{1}{4}\right)$$

$$P_0 = 1-\frac{\lambda_1}{\mu_1} = 1-\frac{15/2}{10} = \frac{1}{4}$$

a) The expected number of customers in the facility 1 = $L_{s_1} = \dfrac{\lambda_1}{\mu_1-\lambda_1} = \dfrac{5}{10-5} = 1$

b) The expected number of customers in the facility 2 = $L_{s_2} = L_{q_2} + \dfrac{\lambda_2}{\mu_2}$

c) The expected number of customers in the facility 3 = $L_{s_3} = \dfrac{\lambda_3}{\mu_3-\lambda_3} = \dfrac{15/2}{10-\frac{15}{2}} = 3$

Where

$$L_{q_2} = \frac{1}{c_2.c_2!}\left(\frac{\lambda_2}{\mu_2}\right)^{c+1}\left(1-\frac{\lambda_2}{\mu_2 c_2}\right)^{-2} P_0$$

$$= \frac{1}{(2)2!}(1)^3\left(1-\frac{10}{10\times 2}\right)^{-2}(0.333)$$

$$= \frac{1}{4}\left(\frac{1}{2}\right)^{-2}(0.333) = 0.333$$

$$\therefore\ L_{s_2} = 0.333 + \frac{10}{10} = 1.333$$

$\therefore$ The expected total number of customers in the system $L_s = L_{s_1} + L_{s_2} + L_{s_3} = 1 + 1.333 + 3 = 5.333$

$\therefore$ The expected total waiting time for a customer $W_s = \dfrac{L_s}{\lambda} = \dfrac{L_s}{r_1 + r_2 + r_3} = \dfrac{5.333}{1+4+3} = 0.67$

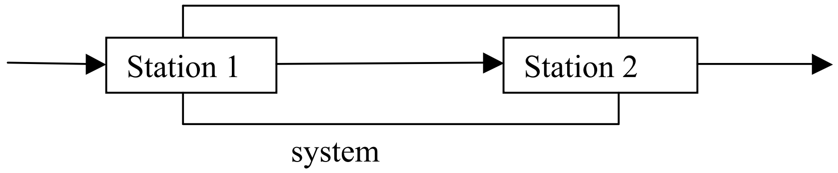
Since λ=Total number of customers in the system=$r_1 + r_2 + r_3$

4(b) (b) Explain series queue with blocking stating the governing equations related to system probabilities

Solution:

<u>Series queues with blocking (series queues where queue is not allowed):</u>

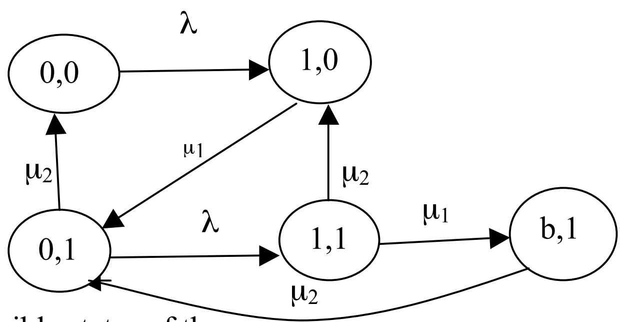
We consider a queueing system with two stations in series, one server at each station and no queue allowed to form at either station. A customer entering for service has to go through station 1 and then station 2. No queues are allowed infront of station1 and station2.



The following points have to be noted.

1. If a customer is being served at station 1, then arriving cutomers are turned away even if station2 is empty.
2. If a customer is in station2 and service is completed in station1, the station 1 customer must wait there until the service for station2 customer is completed. In this case we say that the system is blocked. When this happens, arrivals at station1 are turned away.

The state diagram for the two stage series queue with blocking:



Possible states of the system:

P_{00}=Probability that there is no customer in the system (or) system is empty

P_{10} = Probability that there is customer in station1 only

P_{01} = Probability that there is customer in station2 only

P_{11} = Probability that there is customer in both stations

P_{b0} = Probability that customer has finished getting service at station1 but is waiting since there is a customer in station2. (i.e the system is blocked)

The flow balance equations are,

$\therefore$ i) $\lambda P_{00} - \mu_2 P_{01} = 0$

ii) $\lambda P_{00} + \mu_2 P_{11} - \mu_1 P_{10} = 0$

iii) $\mu_1 P_{10} + \mu_2 P_{b1} - \lambda P_{01} - \mu_2 P_{01} = 0$

iv) $\mu_2 P_{11} + \mu_1 P_{11} - \lambda P_{01} = 0$

v) $\mu_1 P_{11} - \mu_2 P_{b1} = 0$

Also vi) $P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$

Solving the above equations we get,

$$P_{00} = \frac{\lambda^2 \mu_1 (\mu_1 + \mu_2)}{\delta}$$

$$P_{10} = \frac{\lambda \mu_2^2 (\lambda + \mu_1 + \mu_2)}{\delta}$$

$$P_{01} = \frac{\lambda \mu_1 \mu_2 (\mu_1 + \mu_2)}{\delta}$$

$$P_{11} = \frac{\lambda^2 \mu_1 \mu_2}{\delta}$$

$$P_{b1} = \frac{\lambda^2 \mu_1^2}{\delta}$$

Where $\delta = \mu_1(\mu_1 + \mu_2)(\lambda^2 + \lambda\mu_2 + \mu_2)^2 + \lambda(\lambda + \mu_2 + \mu_1)\mu_2^2$