MA6453 - PROBABILITY AND QUEUEING THEORY
(Common to II year CSE & IT – IV sem)
UNIT I - RANDOM VARIABLES
PART - A

1. **Let the random variable *X* denotes the sum obtained in rolling a pair of fair dice. Determine the Probability mass function.**

Solution:

$X=x$	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{7}{36}$	$\frac{8}{36}$	$\frac{9}{36}$	$\frac{10}{36}$	$\frac{11}{36}$

2. **A fair coin is tossed three times. Let *X* be the number of tail appearing. Find the mean and variance.**

Solution:

A coin is tossed three times then the sample space is

$$S=\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let *X* be an event getting tail, The p.m.f is

$X=x$	0	1	2	3
$P(X=x)$	1/8	3/8	3/8	1/8

Mean = $E(X)$ $=\sum x_i P(X=x_i)=(0)(1/8)+(1)(3/8)+(2)(3/8)+(3)(1/8)=3/2$

$$E(X^2)=(0)\left(\frac{1}{8}\right)+\left(1^2\right)\left(\frac{3}{8}\right)+\left(2^2\right)\left(\frac{3}{8}\right)+\left(3^2\right)\left(\frac{1}{8}\right)=\left(\frac{21}{4}\right)$$

Variance $=E(X^2)-[E(X)]^2 \;=\frac{21}{4}-\frac{9}{4}=\frac{12}{4}=3$

3. **The number of hardware failures of a computer system in a week of operations has the following p.m.f**

Number of failures	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Probability	**0.018**	**0.28**	**0.25**	**0.18**	**0.06**	**0.04**	**0.01**

Find the mean failures in a week.

Solution:

Mean = $E(X)$

$=\sum x_i P(X=x_i)$

$=(0)(0.18)+(1)(0.28)+(2)(0.25)+(3)(0.18)+(4)(0.06)+(5)(0.04)+(6)(0.01)$

$=1.82$

4 **A random variable X has the p.d.f $f(x)$ given by** $f(x)=\begin{cases} Cxe^{-x}; & if\ x>0 \\ 0 & ; if\ x\le 0 \end{cases}$ **Find the value of C and cumulative distribution function of X.**

Since *f(x)* is a continuous random variable $f(x)\ge 0, \forall x$ and $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\Rightarrow \int_0^{\infty} Cxe^{-x}dx=1 \quad \Rightarrow \quad C\left[x\left(-e^{-x}\right)-\left(e^{-x}\right)\right]_0^{\infty}=1 \Rightarrow C=1$$

$$\therefore f(x) = \begin{cases} xe^{-x}; x > 0 \\ 0 \quad ; x \le 0 \end{cases}$$

CDF is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0 \text{ . } x \le 0$$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{x} te^{-t}dt = 0 + \left[-te^{-t} - e^{-t}\right]_0^x = -xe^{-x} - e^{-x} + 1 = 1 - (1+x)e^{-x} \quad , x > 0$$

5 **The cumulative distribution function of the random variable *X* is given by** $F_X(x) = \begin{cases} 0 & , \quad x < 0 \\ x + \frac{1}{2}, & 0 \le x \le \frac{1}{2} \\ 1 & , \quad x > \frac{1}{2} \end{cases}$ **. Compute** $P\left(X > \frac{1}{4}\right)$.

Solution:

$$P\left(X > \frac{1}{4}\right) = 1 - P\left(X \le \frac{1}{4}\right) = 1 - F\left(\frac{1}{4}\right) = 1 - \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{1}{4}$$

6 **The CDF of a continuous random variable is given by** $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{5}}, & 0 \le x < \infty \end{cases}$

Find the PDF of *X* and mean of *X*.

Solution:

PDF = $f(x) = \frac{d}{dx}\left[F(x)\right] = \begin{cases} 0, & x < 0 \\ \frac{1}{5}e^{-\frac{x}{5}}, & x \ge 0 \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} \frac{1}{5} xe^{-\frac{x}{5}}\,dx = \frac{1}{5}\left[(x)\left(\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right) - (1)\left(\frac{e^{-\frac{x}{5}}}{\frac{1}{25}}\right)\right]_0^{\infty} = 5$$

7. **A Continuous random variable** X **has a probability density function** $f(x) = 3x^2;\ 0 \le x \le 1.$ **Find '*a*' such that** $P(X \le a) = P(X > a)$

Solution:

We know that the total probability =1

Given $P(X \le a) = P(X > a) = K(say)$

Then $K + K = 1 \Rightarrow K = \frac{1}{2}$

ie $P(X \le a) = \frac{1}{2}$ & $P(X > a) = \frac{1}{2}$

Consider $P(X \le a) = \frac{1}{2}$ i.e. $\int_{-\infty}^{a} f(x)\,dx = \frac{1}{2}$

$$\int_0^a 3x^2 dx = \frac{1}{2} \Rightarrow 3\left(\frac{x^3}{3}\right)_0^a = \frac{1}{2} \Rightarrow a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{1/3}.$$

8. **If a random variable X has the p.d.f** $f(x) = \begin{cases} \frac{1}{2}(x+1); -1 < x < 1 \\ 0 \quad ; otherwise \end{cases}$**. Find the mean and variance of X.**

Solution:

$$\text{Mean} = \int_{-\infty}^{\infty} xf(x)\,dx = \frac{1}{2}\int_{-1}^{1} x(x+1)\,dx = \frac{1}{2}\int_{-1}^{1}\left(x^2+x\right)dx = \frac{1}{2}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{3}$$

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{2}\int_{-1}^{1}\left(x^3+x^2\right)dx = \frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\right] = \frac{1}{2}.\frac{2}{3} = \frac{1}{3}$$

$$Variance = \mu_2' - \left(\mu_1'\right)^2 = \frac{1}{3}-\frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}.$$

9. **Let** $M_x(t) = \frac{1}{1-t}$ **such that t≠ 1, be the mgf of r.v X. Find the mgf of Y = 2X +1.**

$$M_Y(t) = M_{2X+1}(t) = e^t M_X(2t) = e^t\left[\frac{1}{1-t}\right]_{t\to 2t} = \frac{e^t}{1-2t}.$$

10. **If the RV has the mgf** $M_x(t) = \frac{2}{2-t}$**, determine the variance of X.**

$$M_x(t) = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1+\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2+\left(\frac{t}{2}\right)^3+\cdots$$

$$= 1+\frac{1}{2}\left(\frac{t}{1!}\right)+\frac{1}{2}\left(\frac{t^2}{2!}\right)+\frac{3}{4}\left(\frac{t^3}{3!}\right)+\cdots$$

$$\mu_r' = \text{coefficient of } \left(\frac{t^r}{r!}\right) \text{ in } M_X(t), \qquad \therefore \mu_1' = \frac{1}{2},\ \mu_2' = \frac{1}{2}$$

$$\text{Var}(X) = \mu_2' - \left(\mu_1'\right)^2 = \frac{1}{2}-\frac{1}{4} = \frac{1}{4}.$$

11. **A random variable X has density function given by** $f(x) = \begin{cases} 2e^{-2x}; x \ge 0 \\ 0 \quad ; x < 0 \end{cases}$**. Find m.g.f of X**

Solution:

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}2e^{-2x}dx = 2\int_0^{\infty} e^{-(2-t)x}dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{2}{2-t}, t < 2.$$

12. **A continuous RV X has the pdf** $f(x)=\dfrac{x^2e^{-x}}{2},\ x\geq 0$. **Find the rth moment of X about the origin.**

$$\mu_r' = E\left[X^r\right] = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^{\infty} x^r\left(\frac{x^2e^{-x}}{2}\right)dx = \frac{1}{2}\int_0^{\infty} e^{-x}x^{r+2}\,dx$$

$$= \frac{1}{2}\Gamma(r+3) = \frac{1}{2}(r+2)!$$

13. **The mean of a Binomial distribution is 20 and standard deviation is 4. Find the parameters of the distribution**.

$\therefore np = 20$ and $\sqrt{npq} = 4 \Rightarrow npq = 16 \Rightarrow (20)q = 16 \Rightarrow q = \dfrac{4}{5}$

$p = 1-q = 1-\dfrac{4}{5} = \dfrac{1}{5}.\ \Rightarrow np = 20 \Rightarrow n = 100 \ \therefore$ 100 and $\dfrac{1}{5}$ are the parameters.

14. **If** X **and** Y **are independent binomial variates following** $B\left(5,\dfrac{1}{2}\right)$ **and** $B\left(7,\dfrac{1}{2}\right)$ **respectively find** $P\left[X+Y=3\right]$.

Solution:

By additive property, $X+Y$ is also a binomial variate with parameters $n_1+n_2=12$ & $p=\dfrac{1}{2}$

$$\therefore P\left[X+Y=3\right] = 12C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^9 = \frac{55}{2^{10}}$$

15. **One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core-size limitations. Find the probability that among a sample of 200 jobs there are no jobs that have to wait until weekends.**

Solution:

X be the Random variable denoting the no. of jobs that have to wait

p = 0.01, n = 200, $\lambda = np = 2$,

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!},\quad x = 0,1,2,.....$$

$$P(X=0) = \frac{e^{-2}2^0}{0!} = e^{-2} = 0.1353$$

16. **If the probability that a target is destroyed on any one shot is 0.5, find the probability that it would be destroyed on 6th attempt.**

Solution:

Given that, the probability that a target is destroyed on any one shot is 0.5

$p = 0.5 \Rightarrow q = 1-p = 1-0.5 = 0.5$

By Geometric Distribution, $P(X=x) = q^{x-1}p,\ x = 1,2,3,....$; $P(X=6) = (0.5)^{6-1}(0.5) = (0.5)^6 = 0.0156$

17. **If X is a Uniformly distributed R.V with mean 1 and variance** $\dfrac{4}{3}$**, find P(X<0).**

Mean = $\dfrac{a+b}{2} = 1 \Rightarrow a+b = 2$ and variance = $\dfrac{(b-a)^2}{12} = \dfrac{4}{3} \Rightarrow b - a = 4$

By solving the above eqns. We get a = -1 and b = 3

$$f(x)=\begin{cases}\dfrac{1}{b-a}, & a<x<b\\ 0 & \text{, otherwise}\end{cases}$$

$$f(x)=\begin{cases}\dfrac{1}{4} & , -1<x<3\\ 0 & \text{, otherwise}\end{cases}$$

$$P(X<0)=\int_{-1}^{0} f(x)dx = \int_{-1}^{0}\frac{1}{4}dx=\frac{1}{4}[x]_{-1}^{0}=\frac{1}{4}.$$

18 **X is a normal variate with mean $=30$ and S.D = 5. Find $P[26\le X\le 40]$**

Solution:

X follows N(30, 5) $\therefore \mu=30$ & $\sigma=5$

Let $Z=\dfrac{X-\mu}{\sigma}$ be the standard normal variate

$$P[26\le X\le 40]=P\left[\frac{26-30}{5}\le Z\le\frac{40-30}{5}\right]$$

$$=P[-0.8\le Z\le 2]=P[-0.8\le Z\le 0]+P[0\le Z\le 2]$$

$$=P[0\le Z\le 0.8]+[0\le Z\le 2]=0.2881+0.4772=0.7653.$$

19. **If X follows $N(2,3)$ Find $P\left[Y\ge\frac{3}{2}\right]$ where $Y+1=X$.**

Solution:

$$P\left[Y\ge\frac{3}{2}\right]=P\left[X-1\ge\frac{3}{2}\right]=P[X\ge 2.5]=P[Z\ge 0.17]=0.5-P[0\le Z\le 0.17]$$

= 0.5- 0.0675= 0.4325

20. **Suppose the length of life of an appliance has an exponential distribution with mean 10 years. What is the probability that the average life time of a random sample of the appliances is atleast 10.5?**

$$\lambda=\frac{1}{10},\ f(x)=\lambda e^{-\lambda x},\ x>0\Rightarrow f(x)=\frac{1}{10}e^{-\frac{x}{10}},x>0$$

$$P(X>10.5)=\int_{10.5}^{\infty} f(x)dx=\int_{10.5}^{\infty}\frac{1}{10}e^{-\frac{x}{10}}dx = e^{-1.05} = 0.3499$$

PART - B

1. i) A random variable X has the following probability function:

X	:	0	1	2	3	4	5	6	7
$P(X)$	:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) K , (ii) Evaluate $P(X<6), P(X\ge 6)$ and $P(0<X<5)$ (iii) Determine the distribution function of X . (iv) $P(1.5<X<4.5/X>2)$ (v) $E(3X-4)$, $\text{Var}(3X-4)$ (vi) If

$P\left[X \le C\right] > \frac{1}{2}$**, find the minimum value of** C

Solution:

(i) We know that $\sum_i P(X = x_i) = 1$

$$\Rightarrow \sum_{x=0}^{7} P\left(X = x\right) = 1, \ \Rightarrow K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$\Rightarrow$ 10K^2+9K−1=0 $\Rightarrow K = \frac{1}{10}$ *or* $K = -1$ (here $K = -1$ is impossible, since $P(X = x) \ge 0$)

$$\therefore K = \frac{1}{10}$$

$\therefore$ The probability mass function is

$X = x$	0	1	2	3	4	5	6	7
$P(X = x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(ii) $P\left(X < 6\right) = P\left(X = 0\right) + P\left(X = 1\right) + \ldots + P\left(X = 5\right) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$

$$P\left(X \ge 6\right) = 1 - P\left(X < 6\right) = 1 - \frac{81}{100} = \frac{19}{100}$$

$$P\left(0 < X < 5\right) = P\left(X = 1\right) + P\left(X = 2\right) + P\left(X = 3\right) + P\left(X = 4\right) = K + 2K + 2K + 3K$$

$$= 8K = \frac{8}{10} = \frac{4}{5}$$

(iii) The distribution of X is given by $F_X\left(x\right)$ defined by $F_X\left(x\right) = P\left(X \le x\right)$

$X = x$	$P(X = x)$	$F_X\left(x\right) = P\left(X \le x\right)$
0	0	0 , $x < 1$
1	$\frac{1}{10}$	$0+\frac{1}{10} = \frac{1}{10}$, $1 \le x < 2$
2	$\frac{2}{10}$	$0+\frac{1}{10}+\frac{2}{10} = \frac{3}{10}$, $2 \le x < 3$
3	$\frac{2}{10}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10} = \frac{5}{10}$, $3 \le x < 4$
4	$\frac{3}{10}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10} = \frac{8}{10} = \frac{4}{5}$, $4 \le x < 5$
5	$\frac{1}{100}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100} = \frac{81}{100}$, $5 \le x < 6$
6	$\frac{2}{100}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100} = \frac{83}{100}$, $6 \le x < 7$
7	$\frac{17}{100}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}+\frac{17}{100} = \frac{100}{100} = 1$, $x \le 7$

(iv) $P(1.5 < X < 4.5 / X > 2) = \dfrac{P(X = 2,3,4 \cap X = 3,4,5,6,7)}{P(X = 3,4,5,6,7)} = \dfrac{P(X = 3,4)}{P(X = 3,4,5,6,7)}$

$$= \frac{P(X=3)+P(X=4)}{P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)}$$

$$= \frac{2K+3K}{2K+3K+K^2+2K^2+7K^2+K} = \frac{5K}{6K+10K^2} = \frac{5}{6+10K} = \frac{5}{7}$$

(v) To find $E(3X-4)$, $\text{Var}(3X-4)$

$$E(3X-4) = 3E(X) - E(4) = 3E(X) - 4 \quad ------(1)$$

$$Var(3X-4) = 3^2 Var(X) - Var(4) = 9Var(X) - 0 = 9Var(X)$$

$$Var(3X-4) = 9Var(X) \quad ---------- (2)$$

$$E(X) = \sum x P(X=x)$$

$$= 0\times P(X=0) + 1\times P(X=1) + 2\times P(X=2) + 3\times P(X=3) + 4\times P(X=4) + 5\times P(X=5) + 6\times P(X=6) + 7\times P(X=7)$$

$$= 0 + 1\times K + 2\times 2K + 3\times 2K + 4\times 3K + 5\times K^2 + 6\times 2K^2 + 7\times(7K^2+K)$$

$$= K + 4K + 6K + 12K + 5K^2 + 12K^2 + 49K^2 + 7K \quad = 30K + 66K^2 = \frac{30}{10} + \frac{66}{100} \quad = \frac{366}{100}$$

$$E(X^2) = \sum x^2 P(X=x)$$

$$= 0\times P(X=0) + 1^2\times P(X=1) + 2^2\times P(X=2) + 3^2\times P(X=3) + 4^2\times P(X=4) + 5^2\times P(X=5) + 6^2\times P(X=6) + 7^2\times P(X=7)$$

$$= 0 + 1^2\times K + 2^2\times 2K + 3^2\times 2K + 4^2\times 3K + 5^2\times K^2 + 6^2\times 2K^2 + 7^2\times(7K^2+K)$$

$$= K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K$$

$$= 124K + 440K^2 = \frac{124}{10} + \frac{440}{100} = \frac{1240+440}{100} = \frac{1680}{100} = \frac{168}{10} = \frac{84}{5}$$

(1)$\Rightarrow$ $E(3X-4) = \dfrac{3\times 366}{100} - 4 = \dfrac{1098-400}{100} = \dfrac{698}{100} = 69.8$

(2) $\Rightarrow$ $Var(3X-4) = 9Var(X) = 9\left[E(X^2) - \left(E(X)\right)^2\right] = 9\left[16.8 - 13.3956\right] = 30.6396$

(vi) <u>To find the minimum value of C</u> if $P\left[X \le C\right] > \dfrac{1}{2}$

$X = x$	$P(X=x)$	$P(X \le x)$
0	0	$0 < \frac{1}{2}$
1	$\frac{1}{10}$	$0 + \frac{1}{10} = \frac{1}{10} < \frac{1}{2}$
2	$\frac{2}{10}$	$0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$
3	$\frac{2}{10}$	$0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$
4	$\frac{3}{10}$	$0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$

5	$\frac{1}{100}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}=\frac{81}{100}>\frac{1}{2}$
6	$\frac{2}{100}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}=\frac{83}{100}>\frac{1}{2}$
7	$\frac{17}{100}$	$0+\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+\frac{2}{100}+\frac{17}{100}=\frac{100}{100}=1>\frac{1}{2}$

$\therefore$ the minimum value of C is 4

ii) 2 percent of the fuses manufactured by a firm are expected to be defective, Find the probability that a box of 200 fuses contains (i) defective fuses (ii) 3 or more defective fuses.

Solutions:

Let X denotes the number of defective fuses in the box of 200 fuses.

2 % of the fuses are defective.

$\therefore$, probability that a fuse is defective is p=2/100 = 0.02

Here, p is very small and n is very large.

Therefore, X can be treated as Poisson variate with parameter λ=np = 200 * 0.02 = 4.

The p.m.f. is $p(x)=\dfrac{e^{-4}4^x}{x!}, x=0,1,2,3,\ldots$

P[defective fuses] = 1-P[no defective fuses]

$=1-P(X=0)$

$= 1 - e^{-4}[(4^0)/0!] = 1 - 0.0183 = 0.9817$

P[3 or more defective fuses] = 1-P[less than 3 defective fuses]

$=1-\left[P(X=0)+P(X=1)+P(X=2)\right]$

$= 1 - e^{-4}[1+4+8]$

$= 1 - 0.0183x13= 1-0.2379 = 0.7621$

2. i) The probability mass function of random variable X is defined as $P(X=0)=3C^2$, $P(X=1)=4C-10C^2$, $P(X=2)=5C-1$, where $C>0$, and $P(X=r)=0$ if $r\neq 0,1,2$. Find (i) The value of C. (ii) $P(0<X<2/X>0)$. (iii) The distribution function of X. (iv) The largest value of X for which $F(x)<\frac{1}{2}$.

Solution:

(i) Given $P[x=0]=3C^2, P[X=1]=4C-10C^2, P[X=2]=5C-1$

$X=x$	:	0	1	2
$P(X=x)$	:	$3C^2$	$4C-10C^2$	$5C-1$

We Know that $\sum P(X=x)=1$

$\Rightarrow 3C^2+4C-10C^2+5C-1=1$

$\Rightarrow -7C^2+9C-1=1$

$\Rightarrow 7C^2-9C+2=0 \Rightarrow (7C-2)(C-1)=0 \Rightarrow 7C-2=0$ or $C-1=0$

$C=\frac{2}{7}, C=1 \Rightarrow C=1$ is not permissible as the probability values are greater than 1 $\therefore C=\frac{2}{7}$

The Probability distribution is

$$X = x \quad : \quad 0 \quad 1 \quad 2$$

$$P(X = x) \quad : \quad \frac{12}{49} \quad \frac{16}{49} \quad \frac{21}{49}$$

$$P\left[0 < X < 2/X > 0\right] = \frac{P\left[(0 < X < 2) \cap X > 0\right]}{P\left[X > 0\right]} = \frac{P\left[0 < X < 2\right]}{P\left[X > 0\right]} = \frac{P\left[X = 1\right]}{P\left[X = 1\right] + P\left[X = 2\right]}$$

$$= \frac{\frac{16}{49}}{\frac{16}{49} + \frac{21}{49}} = \frac{16}{37}$$

(iii). The distribution function of X is

X	$P(X = x)$	$F(x) = P(X \le x)$
0	$\frac{12}{49}$	$F(0) = P(X \le 0) = \frac{12}{49} = 0.24$
1	$\frac{16}{49}$	$F(1) = P(X \le 1) = P(X = 0) + P(X = 1) = \frac{12}{49} + \frac{16}{49} = 0.57$
2	$\frac{21}{49}$	$F(2) = P(X \le 2)$ $= P(X = 0) + P(X = 1) + P(X = 2) = \frac{12}{49} + \frac{16}{49} + \frac{21}{49} = 1$

(iv). The Largest value of X for which $F(X) < \frac{1}{2}$ is 0

ii) A continuous random variable X has the p.d.f $f(x) = kx^3 e^{-x},\ x \ge 0.$ Find the r$^{\text{th}}$ order moment of X about the origin. Hence find m.g.f, mean and variance of X.

Solution:

Since $\int_0^\infty kx^3 e^{-x} dx = 1 \Rightarrow k\left[x^3\left(\frac{e^{-x}}{-1}\right) - (3x^2)\left(\frac{e^{-x}}{1}\right) + (6x)\left(\frac{e^{-x}}{-1}\right) - (6)\left(\frac{e^{-x}}{1}\right)\right]_0^\infty = 1$

$$k\left[-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6\right]_0^\infty = 1 \Rightarrow k\left[(0) - (-6)\right] = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}.$$

$$E(X^r) = \mu_r' = \int_0^\infty x^r f(x) dx = \frac{1}{6}\int_0^\infty x^r x^3 e^{-x} dx = \frac{1}{6}\int_0^\infty x^{r+3} e^{-x} dx \qquad \because \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx,\ n > 0$$

here $n = r + 4$ $\qquad = \frac{1}{6}\int_0^\infty e^{-x} x^{(r+3+1)-1} dx = \frac{1}{6}\Gamma(r+4) = \frac{(r+3)!}{6} \qquad \because \Gamma n = (n-1)!$

Putting $r = 1,\ E(X) = \mu_1' = \frac{4!}{6} = \frac{24}{6} = 4$

$$r = 2,\ E(X^2) = \mu_2' = \frac{5!}{6} = \frac{120}{6} = 20$$

$\therefore$ Mean = $E(X) = \mu_1' = 4$; Variance = $E(X^2) - \left[E(X)\right]^2 = \mu_2' - \left(\mu_1'\right)^2$

$$\mu_2 = 20-(4)^2 = 20-16 = 4$$

To find M.G.F

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{6}x^3e^{-x}dx$$

$$= \frac{1}{6}\int_0^{\infty} x^3e^{tx-x}dx = \frac{1}{6}\int_0^{\infty} x^3e^{-(1-t)x}dx$$

$$= \frac{1}{6}\left[\left(x^3\right)\left(\frac{e^{-(1-t)x}}{-(1-t)}\right)-\left(3x^2\right)\left(\frac{e^{-(1-t)x}}{(1-t)^2}\right)+\left(6x\right)\left(\frac{e^{-(1-t)x}}{-(1-t)^3}\right)-\left(6\right)\left(\frac{e^{-(1-t)x}}{(1-t)^4}\right)\right]_0^{\infty}$$

$$= \frac{1}{6}\left[-x^3\frac{e^{-(1-t)x}}{(1-t)}-3x^2\frac{e^{-(1-t)x}}{(1-t)^2}-6x\frac{e^{-(1-t)x}}{(1-t)^3}-6\frac{e^{-(1-t)x}}{(1-t)^4}\right]_0^{\infty}$$

$$= \frac{1}{6}\left[(0)-\left(\frac{-6}{(1-t)^4}\right)\right]$$

$$\therefore M_X(t) = \frac{1}{(1-t)^4}$$

3. i) A continuous random variable has the p.d.f $f(x) = \begin{cases} \dfrac{K}{1+x^2}, & -\infty < x < \infty \\ 0, & otherwise \end{cases}$

(i) Find the value of *K*. (ii) Find the c.d.f of *f(x)* (iii) Find *P(X>0)*.

Solution:

(i) Since $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty}\frac{K}{1+x^2}dx = 1 \Rightarrow K\left(\tan^{-1}x\right)_{-\infty}^{\infty} = 1$

$$K\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 1 \Rightarrow K\pi = 1 \Rightarrow K = \frac{1}{\pi}$$

(ii) $F_X(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x}\frac{K}{1+x^2}dx = \frac{1}{\pi}\left(\tan^{-1}x\right)_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1}x-\left(-\frac{\pi}{2}\right)\right]$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}+\tan^{-1}x\right], -\infty < x < \infty$$

(iii) $P(X>0) = \frac{1}{\pi}\int_0^{\infty}\frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1}x\right)_0^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2}-\tan^{-1}0\right) = \frac{1}{2}.$

ii) A component has an exponential time to failure distribution with mean of 10,000 hours.

(i) The component has already been in operation for its mean life. What is the probability that it will fail by 15,000 hours?

(ii) At 15,000 hours the component is still in operation. What is the probability

that it will operate for another 5000 hours?

Solution:

Let X be the random variable denoting the time to failure of the component following exponential distribution with $\text{Mean} = 10000$ hours. $\therefore \frac{1}{\lambda} = 10,000 \Rightarrow \lambda = \frac{1}{10,000}$

The p.d.f. of X is $f(x) = \begin{cases} \frac{1}{10,000} e^{-\frac{x}{10,000}} & , x \geq 0 \\ 0 & , otherwise \end{cases}$

(i) Probability that the component will fail by 15,000 hours given that it has already been in operation for its mean life $= P[X < 15,000 / X > 10,000]$

$$= \frac{P[10,000 < X < 15,000]}{P[X > 10,000]} \quad ---------(1)$$

$$P[10,000 < X < 15,000] = \int_{10,000}^{15,000} \frac{1}{10000} e^{-\frac{x}{10000}} dx$$

$$= \frac{1}{10000} \left[\frac{e^{-\frac{x}{10000}}}{\frac{-1}{10000}} \right]_{10000}^{15000} = -\left[e^{-\frac{x}{10000}} \right]_{10000}^{15000}$$

$$= -\left[e^{-\frac{15000}{10000}} - e^{-\frac{10000}{10000}} \right] = -\left[e^{-\frac{3}{2}} - e^{-1} \right] = e^{-1} - e^{-1.5} \quad ----(2)$$

$$P[X > 10,000] = \int_{10,000}^{\infty} \frac{1}{10000} e^{-\frac{x}{10000}} dx = \frac{1}{10000} \left[\frac{e^{-\frac{x}{10000}}}{\frac{-1}{10000}} \right]_{10000}^{\infty} = -\left[e^{-\frac{x}{10000}} \right]_{10000}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1} \right] = e^{-1} \quad -------(3)$$

Sub (2) & (3) in (1)

(1) $\Rightarrow$ $P[X < 15,000 / X > 10,000] = \frac{e^{-1} - e^{-1.5}}{e^{-1}} = \frac{0.3679 - 0.2231}{0.3679} = 0.3936$.

(ii) Probability that the component will operate for another 5000 hours given that it is in operation 15,000 hours $= P[X > 20,000 / X > 15,000]$

$$= P[X > 5000] \qquad \text{[By memoryless prop]}$$

$$= \int_{5000}^{\infty} f(x) dx$$

$$= \int_{5000}^{\infty} \frac{1}{10000} e^{-\frac{x}{10000}} dx = \frac{1}{10000} \left[\frac{e^{-\frac{x}{10000}}}{\frac{-1}{10000}} \right]_{5000}^{\infty} = -\left[e^{-\frac{x}{10000}} \right]_{5000}^{\infty} = e^{-0.5} = 0.6065$$

4. i) (A) Six dice are thrown 729 times. How many times do you expect atleast 3 dice to show 5

or 6?

Solution:

(A) $P[5 \; or \; 6] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Let X be the number of times the dice shown 5 or 6. Here $n = 6$.

Then X follows binomial distribution with n = 6 and $p = \frac{1}{3}$

By Binomial theorem, $P[X = r] = 6C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r}$ where $r = 0,1,2...6$.

$$P[X \geq 3] = P(3) + P(4) + P(5) + P(6)$$

$$= 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + 6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + 6C_6 \left(\frac{1}{3}\right)^6 = 0.3196$$

$\therefore$ Expected number of times atleast 3 dice to show 5 or 6 $= N \times P[X \geq 3]$ =729 X 0.3196 ≅ 233.

(B) Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads 10 times?

Probability of getting six heads in one toss of six coins is $p = \left(\frac{1}{2}\right)^6$ $\lambda = np = 6400 \times \left(\frac{1}{2}\right)^6$ =100

Let X be the number of times getting 6 heads $P(X = 10) = \frac{e^{-100}(100)^{10}}{10!} = 1.025 \times 10^{-30}$

ii) State and prove the memoryless property of geometric distribution.

Statement:

If X is a random variable with geometric distribution, then X lacks memory, in the sense that $P[X > s + t / X > s] = P[X > t] \quad \forall s, t > 0$.

Proof:

The p.m.f of the geometric random variable X is $P(X = x) = q^{x-1} p$, $\quad x = 1,2,3,....$

$$P[X > s + t / X > s] = \frac{P[X > s + t \cap X > s]}{P[X > s]} = \frac{P[X > s + t]}{P[X > s]} \text{ ------(1)}$$

$$\therefore P[X > t] = \sum_{x=t+1}^{\infty} q^{x-1} p = q^t p + q^{t+1} p + q^{t+2} p + ... = q^t p\left[1 + q + q^2 + q^3 + ...\right]$$

$$= q^t p(1-q)^{-1} = q^t p(p)^{-1} = q^t$$

Hence $P[X > s + t] = q^{s+t}$ and $P[X > s] = q^s$

(1) $\Rightarrow P[X > s + t / X > s] = \frac{q^{s+t}}{q^s} = q^t = P[X > t] \Rightarrow P[X > s + t / X > s] = P[X > t]$

5. i) The average percentage of marks of candidates in an examination is 42 with a standard deviation of 10. If the minimum mark for pass is 50% and 1000 candidates appear for the examination, how many candidates can be expected to get the pass mark? If it is required, that double the number of the candidates should pass, what should be the minimum mark for pass?

Solution:

Let X denote the marks of the candidates, then $X \sim N\left(42,10^2\right)$

Let $z = \dfrac{X-42}{10}$, $P\left[X \ge 50\right] = P\left[z \ge 0.8\right] = 0.5 - P\left[0 < z < 0.8\right] = 0.5 - 0.2881 = 0.2119$

If 1000 students write the test, $1000P\left[X \ge 50\right] \cong 212$ students would pass the examination.

If double that number should pass, then the no of passes should be 424.

We have to find z_1, such that $P\left[z \ge z_1\right] = 0.424$

$$\therefore P\left[0 < z < z_1\right] = 0.5 - 0.424 = 0.076$$

From tables, z_1= 0.19 , $\therefore z_1 = \dfrac{50-x_1}{10} \Rightarrow x_1 = 50 - 10z_1 = 50 - 1.9 = 48.1$

The pass mark should be 48 nearly.

ii) A random variable X has a uniform distribution over (-3, 3). Compute (i) P(X < 2) (ii) P(|X|< 2) (iii) Find k for which P(X > k) = 1/3

If X is uniformly distributed over *(a, b)*, then $f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0 & \text{, otherwise} \end{cases}$

$\Rightarrow$ X is uniformly distributed over (-3,3), then $f(x) = \begin{cases} \dfrac{1}{6}, & -3 < x < 3 \\ 0 & otherwise \end{cases}$

(1) $P(X < 2) = \int_{-3}^{2} f(x)dx = \int_{-3}^{2} \dfrac{1}{6}dx = \left[\dfrac{1}{6}x\right]_{-3}^{2} = \left[\dfrac{2}{6} + \dfrac{3}{6}\right] = \dfrac{5}{6}$

(2) $P\left(|X| < 2\right) = P(-2 < X < 2) = \int_{-2}^{2} \dfrac{1}{6}dx = \dfrac{1}{6}\left[x\right]_{-2}^{2} = \dfrac{1}{6}\left[2+2\right]_{-2}^{2} = \dfrac{4}{6} = \dfrac{2}{3}$

(3) Given $\int_{k}^{3} \dfrac{1}{6}dx = \dfrac{1}{3} \Rightarrow \dfrac{1}{6}\left[x\right]_{k}^{3} = \dfrac{1}{3} \Rightarrow 3 - k = 2 \Rightarrow k = 1$

UNIT – II TWO DIMENSIONAL RANDOM VARIABLES

PART – A

1 **Given the joint density function of X and Y** $f(x,y) = \begin{cases} \dfrac{x}{4}\left(1+3y^2\right), 0 < x < 2, 0 < y < 1 \\ 0 \quad , otherwise \end{cases}$.

Find the marginal probability density function of Y.

Solution:

The marginal density function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 \frac{x}{4}\left(1+3y^2\right)dx = \left[\frac{x^2}{8}\left(1+3y^2\right)\right]_0^2 = \frac{1}{2}\left(1+3y^2\right),\ 0 < y < 1$$

2 **The joint probability density functions of a bivariate random variable (X , Y) is** $f_{XY}(x,y)=\begin{cases} k(x+y), 0<x<2, 0<y<2 \\ 0 \quad , elsewhere \end{cases}$. **Find k.**

Solution:

Given the joint pdf of the continuous random variables (X , Y) is

$f_{XY}(x,y)=k(x+y), 0<x<2, 0<y<2$

$$\therefore \int_0^2\int_0^2 f(x,y)dxdy=1 \Rightarrow \int_0^2\int_0^2 k(x+y)dxdy=1 \Rightarrow k\int_0^2\left[\frac{x^2}{2}+yx\right]_0^2 dy=1 \Rightarrow k\int_0^2(2+2y)dy=1$$

$$\Rightarrow k\left[2y+2\frac{y^2}{2}\right]_0^2=1 \Rightarrow k(4+4)=1 \Rightarrow 8k=1 \Rightarrow k=\frac{1}{8}$$

3 **The joint probability density function of bivariate random variable (X , Y) is given by** $f(x,y)=\begin{cases} 4xy, 0<x<1, 0<y<1 \\ 0 \quad , elsewhere \end{cases}$. **Find P (X + Y < 1)**

Solution:

Given the joint pdf of (X , Y) is $f(x,y)=\begin{cases} 4xy, 0<x<1, 0<y<1 \\ 0 \quad , elsewhere \end{cases}$.

$$\therefore P(X+Y<1)=\int_0^1\int_0^{1-x} 4xydydx = 4\int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = 2\int_0^1 x(1-x)^2dx = 2\int_0^1 x(1-2x+x^2)dx$$

$$=2\int_0^1(x-2x^2+x^3)dx = 2\left[\frac{x^2}{2}-2\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1 = 2\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]=\frac{1}{6}$$

4 **If** $f(x,y)=\begin{cases} 8xy, 0<x<1, 0<y<x \\ 0 \quad , elsewhere \end{cases}$ **is the joint probability density function of X and Y, find f(y/x).**

Solution:

$$f_X(x)=\int_y f(x,y)dy = \int_{y=0}^{x} 8xydy = \left[8x\frac{y^2}{2}\right]_0^x = 4x^3,\ 0<x<1$$

$$f(y/x)=\frac{f(x,y)}{f_X(x)}, = \frac{8xy}{4x^3}=\frac{2y}{x^2},\ 0<y<x, 0<x<1$$

5 **Two random variables are defined as Y = 4X + 9. Find the correlation coefficient between X and Y.**

Solution: Given Y = 4X + 9;Var(X) = σ_X^2; Var(Y) = Var(4X + 9) = 4^2Var(X)

$$\Rightarrow \sigma_Y^2 = 16\sigma_X^2 \Rightarrow \sigma_Y = 4\sigma_X$$

$\therefore$ Cov(X , Y) = Cov (X , 4X + 9) = 4 Cov(X , X) = $4\sigma_X^2$

$$\therefore r_{XY}=\frac{\operatorname{cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{4\sigma_x^2}{\sigma_X.4\sigma_X}=1$$

6 The following table gives the joint probability distribution of X and Y, find the marginal distribution function of X and Y.

Y \ X	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Solution:

Y \ X	1	2	3	p(y)
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
p(x)	0.3	0.4	0.3	1

The marginal distribution of X is

X	1	2	3
p(x)	0.3	0.4	0.3

The marginal distribution of Y is

Y	1	2
p(y)	0.4	0.6

7 The regression lines between two random variables X and Y is given by 3x + y = 10, 3x + 4y = 12. Find the mean values of X and Y.

Solution:

Regression lines pass through the mean values of X and Y. Solving the two equations we get the mean values.

Multiply equation 1 by 4 and subtract equation 2

$$\begin{aligned} 12X + 4Y &= 40 \\ -3X - 4Y &= -12 \\ \hline 9X &= 28 \end{aligned}$$

$X = 28/9$

Substitute in equation 1

$3(28/9) + Y = 10 \Rightarrow Y = 10 - 28/3 = 2/3.$

Thus mean value of X = 28/9 and mean value of Y = 2/3.

8 If Y = -2X + 3, find Cov (X , Y).

Solution:

$$\begin{aligned} \text{Cov}(X,Y) &= E(XY) - E(X)E(Y) = E(X(-2X+3)) - E(X)\{E(-2X+3)\} \\ &= [E(-2X^2 + 3X) - E(X)]\{-2E(X) + 3\} \\ &= -2E(X^2) + 3E(X) + 2(E(X))^2 - 3E(X) \\ &= 2(E(X))^2 - 2E(X^2) = -2\,\text{var}(X) \end{aligned}$$

9 The joint density of X and Y is given by f(x, y) = $\begin{cases} e^{-(x+y)}, & x \ge 0, y \ge 0 \\ 0 & ,\text{otherwise} \end{cases}$

i) Check whether X and Y are independent. ii) P(0 < X < 1 / Y ≠ 2)

Solution:

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_0^{\infty} e^{-(x+y)}dy \;=\left[-e^{-(x+y)}\right]_0^{\infty}=e^{-x},\; x\ge 0$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx=\int_0^{\infty} e^{-(x+y)}dx=\left[-e^{-(x+y)}\right]_0^{\infty} \;=e^{-y},\; y\ge 0$$

$f_X(x)\,.\,f_Y(y) = e^{-x}.e^{-y} = e^{-(x+y)}, x \ge 0, y \ge 0$

$f_X(x)\,.\,f_Y(y) = f(x, y) \Rightarrow$ X and Y are independent.

$P(0<X<1/Y\neq 2)=P(0<X<1)$ $\quad [\because X \text{ and } Y \text{ are independent}]$

$$=\int_0^1 f_X(x)dx=\int_0^1 e^{-x}dx \;=\left[-e^{-x}\right]_0^1=1-e^{-1}=1-\frac{1}{e}=\frac{e-1}{e}$$

10 If X&Y are independent random variables with variance 2 & 3, find the variance of 3X +4Y.

Solution:

Since X and Y are independent random variables Var (X + Y) = Var (X) + Var (Y)

$\therefore$ Var (3X + 4Y) = Var (3X) + Var (4Y) = 3^2 Var (X) + 4^2 Var (Y) = 9X2 + 16X3= 66.

11 Define covariance and correlation between the random variable X and Y.

Let X and Y are two random variables defined on the same sample space. The covariance of X and Y is denoted by Cov(X, Y) and is defined by

$$\text{Cov}(X, Y) = E([X - E(X)]\,[Y - E(Y)])$$

The coefficient of correlation between X and Y is denoted by ρ_{XY} or r_{XY} and is defined by

$$\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y},\; \sigma_X \neq 0, \sigma_Y \neq 0$$

12 Let X and Y be two random variables having joint density function

$f(x,y)=\frac{3}{2}(x^2+y^2), 0\le x\le 1, 0\le y\le 1.$ **Determine** $P\left(X<\frac{1}{2}, Y>\frac{1}{2}\right)$

Solution:

$$P\left(X<\frac{1}{2},Y>\frac{1}{2}\right)=\int_{x=-\infty}^{\frac{1}{2}}\int_{y=\frac{1}{2}}^{\infty} f(x,y)dydx=\int_{x=0}^{\frac{1}{2}}\int_{y=\frac{1}{2}}^{1}\frac{3}{2}\left(x^2+y^2\right)dydx=\frac{3}{2}\int_0^{\frac{1}{2}}\left[x^2y+\frac{y^3}{3}\right]_{\frac{1}{2}}^{1}dx$$

$$=\frac{3}{2}\int_0^{\frac{1}{2}}\left[x^2\left(1-\frac{1}{2}\right)+\frac{1}{3}\left(1-\frac{1}{8}\right)\right]dx=\frac{3}{2}\int_0^{\frac{1}{2}}\left(\frac{x^2}{2}+\frac{7}{24}\right)dx=\frac{3}{2}\left[\frac{x^3}{6}+\frac{7x}{24}\right]_0^{\frac{1}{2}}$$

$$=\frac{3}{2}\left[\frac{1}{6}.\frac{1}{8}+\frac{7}{24}.\frac{1}{2}\right]=\frac{3}{2}\left[\frac{8}{48}\right]=\frac{1}{4}$$

13 Let (X, Y) be two dimensional random variable. If X and Y are independent, what will be the covariance of (X, Y)?

Solution:

If X & Y are independent, then E(XY) = E(X) E(Y) $\therefore$ Cov(XY) = E(XY) – E(X)E(Y)= 0

14 Write any two properties of regression coefficients.

Solution:

1. Correlation coefficient is the geometric mean of regression coefficients
2. If one of the regression coefficients is greater than unity then the other should be less than unity.

15 **The joint probability mass function of X and Y is**

p(x,y)	0	1	2
0	0.1	0.04	0.02
1	0.08	0.2	0.06
2	0.06	0.14	0.3

(rows: X; columns: Y)

Check if X and Y are independent.

Solution:

p(x,y)	0	1	2	$p_X(x)$
0	0.1	0.04	0.02	0.16
1	0.08	0.2	0.06	0.34
2	0.06	0.14	0.3	0.5
$p_Y(y)$	0.24	0.38	0.38	1

(rows: X; columns: Y)

$p_X(0) \cdot p_Y(0) = (0.16)(0.24) \neq 0.1 = p(0,0)$ $\therefore$ X and Y are not independent.

16 **The correlation coefficient of two random variables X and Y is $-\frac{1}{4}$ while their variances are 3 and 5. Find the covariance.**

Solution:

Given $r_{XY} = -\frac{1}{4}$, $\sigma_X^2 = 3, \sigma_Y^2 = 5$ $\quad r_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}, \sigma_X \neq 0, \sigma_Y \neq 0$

$Cov(X, Y) = r_{XY}\, \sigma_X \sigma_Y = -\frac{1}{4}\sqrt{3} \cdot \sqrt{5} = -0.968$

17 **If the tangent of the angle between the lines of regression y on x and x on y is 0.6 and $\sigma_x = \frac{1}{2}\sigma_y$. Find the correlation coefficient.**

Solution:

$$\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left[\frac{1-r^2}{r}\right] \quad \Rightarrow 0.6 = \frac{\frac{1}{2}\sigma_x \sigma_y}{\frac{1}{4}\sigma_x^2 + \sigma_y^2}\left[\frac{1-r^2}{r}\right] = \frac{\frac{1}{2}}{\frac{5}{4}}\left[\frac{1-r^2}{r}\right]$$

$$\Rightarrow 1.5r = 1 - r^2 \qquad \Rightarrow r = -2, 0.5 \qquad \boxed{\because |-2| > 1,\ r = 0.5}$$

18 **Given the joint density function of X and Y as f(x , y) = $\begin{cases} xe^{-y}, 0 < x < 2, y > 0 \\ 0 \quad , elsewhere \end{cases}$. Find the range space for the transformation X + Y.**

Solution:

Let the auxillary random variable be V = Y

The transformation functions are u = x + y, v = y

$y > 0 \Rightarrow v > 0$ and $0 < x < 2 \Rightarrow 0 < u - v < 2 \Rightarrow v < u < v + 2$

19 The joint probability mass function of the discrete random variable (X , Y) is given by the table

Y \ X	2	4
1	1/10	1.5/10
3	2/10	3/10
5	1/10	1.5/10

Find the conditional probability P (X = 2 / Y = 3)

Solution:

Y \ X	2	4	$P_Y(y)$
1	1/10	1.5/10	2.5/10
3	2/10	3/10	5/10
5	1/10	1.5/10	2.5/10
$P_X(x)$	4/10	6/10	1

$$P(X = 2 / Y = 3) = \frac{P(X=2,Y=3)}{P_Y(3)} = \frac{2/10}{5/10} = \frac{2}{5}$$

20 The two lines of regression are 4x – 5y + 33 = 0 and 20x – 9y = 107. Calculate the coefficient of correlation between X and Y.

Solution:

$4x - 5y + 33 = 0 \quad .....(1) \qquad 20x - 9y = 107 \quad ......(2)$

Let (1) be the regression line of Y on X and let (2) be the regression line of X on Y.

$$\therefore y = \frac{4}{5}x + \frac{33}{5} \Rightarrow b_1 = \frac{4}{5}$$

$$x = \frac{9}{20}y + \frac{107}{20} \Rightarrow b_2 = \frac{9}{20}$$

$$\therefore r = \sqrt{b_1 b_2} = \sqrt{\frac{4}{5}\cdot\frac{9}{20}} = \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6$$

PART – B

1 i) Two random variables X and Y have the joint probability density function $f(x,y) = \begin{cases} c(4-x-y), 0 \le x \le 2, 0 \le y \le 2 \\ 0 \quad , elsewhere \end{cases}$ **. Find the equations of two lines of regression.**

Solution:

To find the value of c:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^2\int_0^2 c(4-x-y)\,dx\,dy = 1 \Rightarrow \int_0^2 c\left(4x - \frac{x^2}{2} - xy\right)_0^2 dy = 1$$

$$\Rightarrow c\int_0^2 (8-2-2y)dy = 1 \Rightarrow c\int_0^2 (6-2y)dy = 1 \Rightarrow c\left[6y - y^2\right]_0^2 = 1 \Rightarrow c(8) = 1 \Rightarrow c = \frac{1}{8}$$

$$\therefore f(x,y) = \begin{cases} \frac{1}{8}(4-x-y), 0 \le x \le 2, 0 \le y \le 2 \\ 0 \quad , elsewhere \end{cases}$$

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \frac{1}{8}(4-x-y)\,dy = \frac{1}{8}\left[4y - xy - \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{8}[8-2x-2] = \frac{1}{8}(6-2x) = \frac{1}{4}(3-x),\ 0 \le x \le 2$$

The marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{1}{8}(4-x-y)\,dx = \frac{1}{8}\left[4x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}[8-2-2y] = \frac{1}{8}(6-2y) = \frac{1}{4}(3-y),\ 0 \le y \le 2$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^2 x.\frac{1}{4}(3-x)\,dx = \frac{1}{4}\int_0^2 (3x - x^2)\,dx = \frac{1}{4}\left[3.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 = \frac{1}{4}\left[6 - \frac{8}{3}\right] = \frac{5}{6}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^2 x^2.\frac{1}{4}(3-x)\,dx = \frac{1}{4}\int_0^2 (3x^2 - x^3)\,dx = \frac{1}{4}\left[3.\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{1}{4}[8-4] = 1$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y.\frac{1}{4}(3-y)\,dy = \frac{1}{4}\int_0^2 (3y - y^2)\,dy = \frac{1}{4}\left[3.\frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 = \frac{1}{4}\left[6 - \frac{8}{3}\right] = \frac{5}{6}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(x)\,dx = \int_0^2 y^2.\frac{1}{4}(3-y)\,dy = \frac{1}{4}\int_0^2 (3y^2 - y^3)\,dx = \frac{1}{4}\left[3.\frac{y^3}{3} - \frac{y^4}{4}\right]_0^2 = \frac{1}{4}[8-4] = 1$$

$$\therefore \sigma_X^2 = E(X^2) - (E(X))^2 = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36} \Rightarrow \sigma_X = \frac{\sqrt{11}}{6}$$

$$\therefore \sigma_Y^2 = E(Y^2) - (E(Y))^2 = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36} \Rightarrow \sigma_Y = \frac{\sqrt{11}}{6}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy = \int_0^2\int_0^2 xy\frac{1}{8}(4-x-y)dx\,dy = \frac{1}{8}\int_0^2 y\left(\int_0^2 (4x - x^2 - xy)dx\right)dy$$

$$= \frac{1}{8}\int_0^2 y\left[4\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^2}{2}y\right]_0^2 dy = \frac{1}{8}\int_0^2 y\left(8 - \frac{8}{3} - 2y\right)dy = \frac{1}{8}\int_0^2\left(\frac{16}{3} - 2y\right)dy$$

$$= \frac{1}{8}\left[\frac{16}{3}y - y^2\right]_0^2 = \frac{1}{8}\left[\frac{32}{3} - 4\right] = \frac{5}{6}$$

Cov(X , Y) = $= E(XY) - E(X)E(Y) = \frac{5}{6} - \frac{5}{6}.\frac{5}{6} = \frac{5}{36}$ $\qquad \therefore r_{XY} = \frac{\frac{5}{36}}{\frac{\sqrt{11}}{6}.\frac{\sqrt{11}}{6}} = \frac{5}{11}$

The regression line of X on Y is

$$X - \overline{X} = \frac{r\sigma_X}{\sigma_Y}\left(Y - \overline{Y}\right) \Rightarrow X - \frac{5}{6} = \frac{\frac{5}{11}.\frac{\sqrt{11}}{6}}{\frac{\sqrt{11}}{6}}\left(Y - \frac{5}{6}\right) \Rightarrow X = \frac{5}{11}Y + \frac{5}{11}$$

The regression line of Y on X is

$$Y-\overline{Y}=\frac{r\sigma_Y}{\sigma_X}\left(X-\overline{X}\right)\Rightarrow Y-\frac{5}{6}=\frac{\frac{5}{11}\cdot\frac{\sqrt{11}}{6}}{\frac{\sqrt{11}}{6}}\left(X-\frac{5}{6}\right)\Rightarrow Y=\frac{5}{11}X+\frac{5}{11}$$

ii) The joint distribution of x and Y is given by f(x , y) = $\frac{x+y}{21}, x=1,2,3, y=1,2$. Find the marginal distributions and conditional distributions.

Solution:

Y \ X	1	2	3	p(y)
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
p(x)	5/21	7/21	9/21	1

The marginal distribution of X is

X	1	2	3
p(x)	5/21	7/21	9/21

The marginal distribution of Y is

Y	1	2
p(y)	9/21	12/21

The conditional distributions of X given Y are

P(x = 1 / y = 1) = $\frac{2/21}{9/21}=\frac{2}{9}$ P(x = 1 / y = 2) = $\frac{3/21}{12/21}=\frac{1}{4}$

P(x = 2 / y = 1) = $\frac{3/21}{9/21}=\frac{1}{3}$ P(x = 2 / y = 2) = $\frac{4/21}{12/21}=\frac{1}{3}$

P(x = 3 / y = 1) = $\frac{4/21}{9/21}=\frac{4}{9}$ P(x = 3 / y = 2) = $\frac{5/21}{12/21}=\frac{5}{12}$

The conditional distributions of Y given X are

P(y = 1 / x = 1) = $\frac{2/21}{5/21}=\frac{2}{5}$ P(y = 1 / x = 2) = $\frac{3/21}{7/21}=\frac{3}{7}$ P(y = 1 / x = 3) = $\frac{4/21}{9/21}=\frac{4}{9}$

P(y = 2 / x = 1) = $\frac{3/21}{5/21}=\frac{3}{5}$ P(y = 2 / x = 2) = $\frac{4/21}{7/21}=\frac{4}{7}$ P(y = 2 / x = 3) = $\frac{5/21}{9/21}=\frac{5}{9}$

2 **i) Two dimensional random variable (X, Y) have the joint probability density function $f(x,y)=\begin{cases}8xy, 0<x<y<1\\ 0 \ , elsewhere\end{cases}$. Find (i) $P\left(X<\frac{1}{2}\cap Y<\frac{1}{4}\right)$ (ii) the marginal and conditional distributions. (iii) Are X and Y independent?**

Solution:

(i) $P\left(X<\frac{1}{2}\cap Y<\frac{1}{4}\right)=\int_0^{\frac{1}{4}}\int_0^{y}8xy\,dx\,dy=8\int_0^{\frac{1}{4}}\left[\frac{x^2}{2}y\right]_0^{y}dy=4\int_0^{\frac{1}{4}}y^3dy=\left[y^4\right]_0^{\frac{1}{4}}=\frac{1}{256}$

(ii) The marginal pdf of X is

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_x^1 8xy\,dy=8x\left[\frac{y^2}{2}\right]_x^1=8x\left[\frac{1}{2}-\frac{x^2}{2}\right]=4x(1-x^2),\ 0<x<1$$

The marginal pdf of Y is

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^y 8xy\,dx=8y\left[\frac{x^2}{2}\right]_0^y=8y\left[\frac{y^2}{2}\right]=4y^3,\ 0<y<1$$

The conditional probability density function of X given Y is

$$f(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{8xy}{4y^3}=\frac{2x}{y^2}\ ,\ 0<x<y\ ,\ 0<y<1$$

The conditional probability density function of Y given X is

$$f(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{8xy}{4x(1-x^2)}=\frac{2y}{(1-x^2)}\ ,\ x<y<1\ ,\ 0<x<1$$

(iii) To check whether X and Y are independent.

$f_X(x).\,f_Y(y)=4x(1-x^2).4y^3=16xy^3(1-x^2)\neq f(x,y)$ $\therefore$ X and Y are not independent.

ii) Let X and Y be random variables having joint density function $f(x,y)=\begin{cases}\frac{3}{2}\left(x^2+y^2\right), & 0\le x\le 1 \quad 0\le y\le 1\\ 0 & ,elsewhere\end{cases}$ **. Find the correlation coefficient r_{xy}.**

Solution:

The marginal pdf of X is

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^1\frac{3}{2}\left(x^2+y^2\right)dy=\frac{3}{2}\left[x^2y+\frac{y^3}{3}\right]_0^1=\frac{3}{2}\left[x^2+\frac{1}{3}\right]=\frac{1}{2}(3x^2+1),\ 0\le x\le 1$$

The marginal pdf of Y is

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1\frac{3}{2}\left(x^2+y^2\right)dx=\frac{3}{2}\left[\frac{x^3}{3}+y^2x\right]_0^1=\frac{3}{2}\left[\frac{1}{3}+y^2\right]=\frac{1}{2}(3y^2+1),\ 0\le y\le 1$$

$$E(X)=\int_{-\infty}^{\infty} xf_X(x)\,dx=\int_0^1 x.\frac{1}{2}(3x^2+1)\,dx=\frac{1}{2}\int_0^1(3x^3+x)\,dx=\frac{1}{2}\left[3.\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1=\frac{1}{2}\left[\frac{3}{4}+\frac{1}{2}\right]=\frac{5}{8}$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx=\int_0^1 x^2.\frac{1}{2}(3x^2+1)\,dx=\frac{1}{2}\int_0^1(3x^4+x^2)\,dx=\frac{1}{2}\left[3.\frac{x^5}{5}+\frac{x^3}{3}\right]_0^1=\frac{1}{2}\left[\frac{3}{5}+\frac{1}{3}\right]=\frac{7}{15}$$

$$E(Y)=\int_{-\infty}^{\infty} yf_Y(y)\,dy=\int_0^1 y.\frac{1}{2}(3y^2+1)\,dy=\frac{1}{2}\int_0^1(3y^3+y)\,dx=\frac{1}{2}\left[3.\frac{y^4}{4}+\frac{y^2}{2}\right]_0^1=\frac{1}{2}\left[\frac{3}{4}+\frac{1}{2}\right]=\frac{5}{8}$$

$$E(Y^2)=\int_{-\infty}^{\infty} y^2 f_Y(y)\,dy=\int_0^1 y^2.\frac{1}{2}(3y^2+1)\,dy=\frac{1}{2}\int_0^1(3y^4+y^2)\,dx=\frac{1}{2}\left[3.\frac{y^5}{5}+\frac{y^3}{3}\right]_0^1=\frac{1}{2}\left[\frac{3}{5}+\frac{1}{3}\right]=\frac{7}{15}$$

$$\therefore\sigma_X^2=E(X^2)-(E(X))^2=\frac{7}{15}-\left(\frac{5}{8}\right)^2=\frac{73}{15\times 64}\Rightarrow\sigma_X=\frac{1}{8}\sqrt{\frac{73}{15}}$$

$$\therefore \sigma_Y^2 = E(Y^2) - (E(Y))^2 = \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{73}{15\times 64} \Rightarrow \sigma_Y = \frac{1}{8}\sqrt{\frac{73}{15}}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy = \int_0^1\int_0^1 xy\frac{3}{2}(x^2+y^2)dx\,dy = \frac{3}{2}\int_0^1 y\left(\int_0^1 (x^3+xy^2)dx\right)dy$$

$$= \frac{3}{2}\int_0^1 y\left[\frac{x^4}{4} + \frac{x^2}{2}y^2\right]_0^1 dy = \frac{3}{2}\int_0^1 y\left(\frac{1}{4} + \frac{1}{2}y^2\right)dy$$

$$= \frac{3}{2}\int_0^1 \left(\frac{y}{4} + \frac{y^3}{2}\right)dy = \frac{3}{2}\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1 = \frac{3}{2}\left[\frac{1}{8} + \frac{1}{8}\right] = \frac{3}{8}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{3}{8} - \frac{5}{8}.\frac{5}{8} = -\frac{1}{64}$$

$$\therefore r_{XY} = \frac{-\frac{1}{64}}{\frac{1}{8}\sqrt{\frac{73}{15}}.\frac{1}{8}\sqrt{\frac{73}{15}}} = -0.21$$

3 i) **If the independent random variables X and Y have variances 36 and 16 respectively, find the correlation coefficient between X + Y and X – Y.**

Solution:

Given X and Y are independent random variables, $\therefore$ Cov (X , Y) = 0

Given $\sigma_X^2 = 36$, $\sigma_Y^2 = 16$. Let U = X + Y and V = X – Y

Cov(U , V) = Cov(X + Y , X – Y) = Cov(X + Y , X) – Cov(X + Y , Y)

= Cov(X , X) + Cov(X , Y) – Cov(X , Y) – Cov(Y , Y)= Var (X) – Var (Y)= $\sigma_X^2 - \sigma_Y^2$ = 36 – 16 = 20

$\sigma_U^2 =$ Var (X + Y) = Var (X) + Var (Y) = 36 + 16 = 52

$\sigma_V^2 =$ Var (X – Y) = Var (X) + Var (Y) = 36 + 16 = 52

$$\therefore \rho_{UV} = \frac{Cov(U,V)}{\sigma_U \sigma_V} = \frac{20}{\sqrt{52}.\sqrt{52}} = \frac{20}{52} = 0.3$$

ii) **If X and Y are independent random variables with probability density functions $e^{-x}, x \ge 0$ and $e^{-y}, y \ge 0$ respectively, find the density functions of $U = \frac{X}{X+Y}$ and V = X + Y. Are U and V independent?**

Solution:

Given $f_X(x) = e^{-x}, \ x \ge 0$ and $f_Y(y) = e^{-y}, \ y \ge 0$ Also given that X and Y are independent, $\therefore$ the joint pdf $f_{XY}(x,y) = f_X(x).f_Y(y) = e^{-x}.e^{-y} = e^{-(x+y)}, \ x \ge 0, \ y \ge 0$

The transformation functions are $u = \frac{x}{x+y}$ *and* $v = x + y$

Solving for x and y, we get $u = \frac{x}{v} \Rightarrow x = uv$; v = x + y = uv + y $\Rightarrow$ y = v(1 – u)

The Jacobian of the transformation is J = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v$

The joint density of U and V is $f_{UV}(u,v) = |J|\, f_{XY}(x,y) = |v| e^{-(x+y)} = ve^{-v}$

The range space of (U , V) is obtained from the range space of (X , Y) and the transformations

x = uv, y = v(1 – u)

$\therefore x \ge 0 \;\&\; y \ge 0 \Rightarrow uv \ge 0 \;\&\; v(1-u) \ge 0 \Rightarrow v \ge 0 \;;\; u \ge 0 \;\&\; u \le 1$

$$f_{UV}(u,v) = \begin{cases} ve^{-v}, \; v \ge 0, 0 \le u \le 1 \\ 0 \;,\; \text{elsewhere,} \end{cases}$$

The pdf of U is the marginal density function of U,

$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)dv = \int_0^{\infty} ve^{-v}\, dv = \left[v.\frac{e^{-v}}{-1} - 1.\frac{e^{-v}}{(-1)^2} \right]_0^{\infty} = 0 + 1 = 1 \;,\; 0 \le u \le 1$$

The pdf of V is the marginal density function of V,

$$f_V(v) = \int_{-\infty}^{\infty} f(u,v)du = \int_0^1 v.e^{-v}du = ve^{-v} \;, v \ge 0$$

Now $f_U(u).\, f_V(v) = 1.ve^{-v} = f_{UV}(u,v)$ $\therefore$ U and V are independent.

4 i) Obtain the equations of regression lines from the following data.

X	**10**	**14**	**18**	**22**	**26**	**30**
Y	**18**	**12**	**24**	**6**	**30**	**36**

Also estimate the value of Y when X = 20 and the value of X when Y = 28.

Solution:

X	Y	$U = \frac{X-22}{4}$	$V = \frac{Y-24}{6}$	U^2	V^2	UV
10	18	–3	–1	9	1	3
14	12	–2	–2	4	4	4
18	24	–1	0	1	0	0
22	6	0	–3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
		–3	**–3**	**19**	**19**	**12**

$$n = 6, \sum U = -3, \sum V = -3, \sum U^2 = 19, \sum V^2 = 19, \sum UV = 12$$

$$\overline{U} = \frac{\sum U}{n} = \frac{-3}{6} = -0.5;\; \overline{V} = \frac{\sum V}{n} = \frac{-3}{6} = -0.5,$$

$$\sigma_U^2 = \frac{\sum U^2}{n} - \left(\overline{U}\right)^2 = \frac{19}{6} - (-0.5)^2 = 2.916\,;\; \sigma_U = 1.708,$$

$$\sigma_V^2 = \frac{\sum V^2}{n} - \left(\overline{V}\right)^2 = \frac{19}{6} - (-0.5)^2 = 2.916\,;\; \sigma_V = 1.708$$

Cov(U , V) = $\frac{\sum UV}{n} - \overline{U}\,\overline{V} = \frac{12}{6} - (-0.5)(-0.5) = 1.75$

$$\therefore r_{UV} = \frac{Cov(U,V)}{\sigma_U . \sigma_V} = \frac{1.75}{(1.708).(1.708)} = 0.6 \quad \therefore r_{XY} = 0.6$$

$$\overline{X} = 4\overline{U} + 22 \Rightarrow \overline{X} = 4(-0.5) + 22 = 20$$

$$\overline{Y} = 6\overline{V} + 24 \Rightarrow \overline{Y} = 6(-0.5) + 24 = 21$$

$$\sigma_X = 4\sigma_U \Rightarrow \sigma_X = 4(1.708) = 6.832$$

$$\sigma_Y = 6\sigma_V \Rightarrow \sigma_Y = 6(1.708) = 10.248$$

The regression line of X on Y is

$$X - \overline{X} = \frac{r\,\sigma_X}{\sigma_Y}\left(Y - \overline{Y}\right) \Rightarrow X - 20 = \frac{(0.6).(6.832)}{10.248}\left(Y - 21\right) \Rightarrow X = 0.4Y + 11.6$$

The regression line of Y on X is

$$Y - \overline{Y} = \frac{r\,\sigma_Y}{\sigma_X}\left(X - \overline{X}\right) \Rightarrow Y - 21 = \frac{(0.6).(10.248)}{(6.832)}\left(X - 20\right) \Rightarrow Y = 0.9X + 3$$

ii) If X and Y are two continuous random variables with joint density function $f(x,y) = \begin{cases} kxy, 0 < x < 1, 0 < y < 1 \\ 0 \quad , otherwise \end{cases}$ **. Find the constant k and the joint probability distribution of V = X^2 and W = XY.**

Solution:

Given $f(x,y) = \begin{cases} kxy, 0 < x < 1, 0 < y < 1 \\ 0 \quad , otherwise \end{cases}$ is the pdf of X and Y.

To find the value of k:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1\int_0^1 kxy\,dx\,dy = 1 \Rightarrow k\int_0^1 x\,dx . \int_0^1 y\,dy = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^1 . \left[\frac{y^2}{2}\right]_0^1 = 1 \Rightarrow k . \frac{1}{2} . \frac{1}{2} = 1 \Rightarrow k = 4$$

$$\therefore f(x,y) = \begin{cases} 4xy, 0 < x < 1, 0 < y < 1 \\ 0 \quad , otherwise \end{cases}$$

The transformation equations are $v = x^2$, $w = xy$, $0 < x < 1$, $0 < y < 1$

$$\therefore x = \sqrt{v}, \quad y = \frac{w}{x} = \frac{w}{\sqrt{v}}$$

$$\therefore 0 < x < 1 \Rightarrow 0 < \sqrt{v} < 1 \quad \text{and} \quad 0 < y < 1 \Rightarrow 0 < \frac{w}{\sqrt{v}} < 1 \Rightarrow 0 < w < \sqrt{v}$$

The Jacobian of transformation is $J = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{v}} & 0 \\ w\left(\frac{-1}{2}\right)v^{\frac{-3}{2}} & \frac{1}{\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$

$\therefore$ the joint pdf of V and W is given by $f_{VW}(v,w) = |J|\, f(x,y) = \left|\frac{1}{2v}\right| 4xy = \frac{1}{2v} 4w = 2\frac{w}{v}$

$$\therefore\ f(u,v) = \begin{cases} 2\dfrac{w}{v}, 0 < v < 1, 0 < w < \sqrt{v} \\ 0\ ,\ elsewhere \end{cases}$$

5 i) The equations of two regression lines are given by 3x + 12y = 19, 3y + 9x = 46. Obtain i) the correlation coefficient ii) the mean value of X and Y iii) If the variance of X is 16, find the standard deviation of Y.

Solution:

Given 3x + 12y = 19 ……(1)

3y + 9x = 46 ……(2)

Let (1) be the regression line of y on x and (2) be the regression line of x on y.

$\therefore$ 12y = –3x + 19 $\Rightarrow y = -\dfrac{3x}{12} + \dfrac{19}{12}$

$\therefore$ the regression coefficient of y on x is $b_1 = -\dfrac{3}{12} = -\dfrac{1}{4}$

$\therefore$ 9x = –3y + 46 $\Rightarrow x = -\dfrac{3y}{9} + \dfrac{46}{9}$

$\therefore$ the regression coefficient of x on y is $b_2 = -\dfrac{3}{9} = -\dfrac{1}{3}$

$\therefore$ b_1b_2 = $\left(-\dfrac{1}{4}\right)\left(-\dfrac{1}{3}\right) = \dfrac{1}{12} < 1$

$\therefore\ r = -\sqrt{b_1b_2} = -\sqrt{\dfrac{1}{12}} = -0.29$ [Since both regression coefficients are negative, r is negative]

The two regression lines pass through $(\bar{x}, \bar{y})$. $\therefore$ Solving (1) and (2) we get $\bar{x}, \bar{y}$

3x + 12y = 19

4 X (2) $\Rightarrow$ 36x + 12y = 184

Subtracting –33x = –165 $\therefore$ x = 5

Now, 3x + 12y = 19 $\Rightarrow$ 3 X 5 + 12y = 19 $\Rightarrow$ 12y = 19 – 15 $\Rightarrow$ 12y = 4 $\Rightarrow$ y = $\dfrac{1}{3}$

$\therefore\ \bar{x} = 5; \bar{y} = \dfrac{1}{3}$

We know, $\dfrac{\sigma_X^2}{\sigma_Y^2} = \dfrac{b_2}{b_1} \Rightarrow \sigma_Y^2 = \dfrac{b_1}{b_2}\sigma_X^2 \Rightarrow \sigma_Y^2 = \dfrac{-\frac{1}{4}}{-\frac{1}{3}}.(16) = 12 \Rightarrow \sigma_Y = \sqrt{12} = 2\sqrt{3}$

$\therefore$ Standard deviation of Y is $2\sqrt{3}$

ii) The joint probability density function of a two dimensional random variable (X , Y) is given by $f(x,y) = xy^2 + \dfrac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$. Compute P(X > 1), P(Y < ½), P(X > 1 / Y < ½), P(X < Y), P(X + Y ≤ 1).

Solution:

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \left(xy^2 + \frac{x^2}{8} \right) dy = \left[x\frac{y^3}{3} + \frac{x^2}{8}y \right]_0^1$$

$$= \frac{x}{3} + \frac{x^2}{8} = \frac{x}{24}(8+3x), 0 \le x \le 2$$

The marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \left(xy^2 + \frac{x^2}{8} \right) dx = \left[\frac{x^2}{2}y^2 + \frac{x^3}{24} \right]_0^2 = 2y^2 + \frac{1}{3}, 0 \le y \le 1$$

P(X > 1) = $\int_1^x f_X(x)\,dx = \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8} \right) dx = \left[\frac{x^2}{6} + \frac{x^3}{24} \right]_1^2$

$$= \frac{1}{6}(4-1) + \frac{1}{24}(8-1) = \frac{3}{6} + \frac{7}{24} = \frac{19}{24}$$

$$P\left(Y < \frac{1}{2} \right) = \int_{-\infty}^{1/2} f_Y(y)\,dy = \int_0^{1/2} \left(2y^2 + \frac{1}{3} \right) dy = \left[\frac{2y^3}{3} + \frac{y}{3} \right]_0^{1/2} = \frac{2}{3}.\frac{1}{8} + \frac{1}{3}.\frac{1}{2} = \frac{3}{12} = \frac{1}{4}$$

$$P(X > 1, Y < 1/2) = \int_1^2 \int_0^{1/2} f(x,y)\,dy\,dx = \int_1^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8} \right) dy\,dx = \int_1^2 \left[x\frac{y^3}{3} + \frac{x^2}{8}y \right]_0^{1/2}$$

$$= \int_1^2 \left(\frac{x}{3}.\frac{1}{8} + \frac{x^2}{8}.\frac{1}{2} \right) dx = \frac{1}{8}\int_1^2 \left(\frac{x}{3} + \frac{x^2}{2} \right) dx = \frac{1}{8}\left[\frac{x^2}{6} + \frac{x^3}{6} \right]_1^2$$

$$= \frac{1}{8.6}\left[(4-1) + (8-1) \right] = \frac{5}{24}$$

$$\therefore P(X > 1 \ / \ Y < 1/2) = \frac{P(X > 1 \ / \ Y < 1/2)}{P(Y < 1/2)} = \frac{\frac{5}{24}}{\frac{1}{4}} = \frac{5}{6}$$

$$P(X < Y) = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8} \right) dx\,dy = \int_0^1 \left[\frac{x^2}{2}y^2 + \frac{x^3}{24} \right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24} \right] dy = \left[\frac{y^5}{10} + \frac{y^4}{24.4} \right]_0^1 = \frac{1}{10} + \frac{1}{24.4} = \frac{53}{480}$$

$$P(X + Y \le 1) = \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dx\,dy = \int_0^1 \left[\frac{x^2}{2}y^2 + \frac{x^3}{24} \right]_0^{1-y} dy = \int_0^1 \left[\frac{(1-y)^2}{2}y^2 + \frac{(1-y)^3}{24} \right] dy$$

$$= \int_0^1 \left[\frac{y^2}{2}(1 - 2y + y^2) + \frac{1}{24}(1-y)^3 \right] dy = \frac{1}{2}\left[\frac{y^3}{3} - 2\frac{y^4}{4} + \frac{y^5}{5} \right]_0^1 + \frac{1}{24}\left[\frac{(1-y)^4}{-4} \right]_0^1$$

$$=\frac{1}{2}\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right]-\frac{1}{24.4}\left[0-1\right]=\frac{1}{60}+\frac{1}{96}=\frac{13}{480}$$

UNIT – III RANDOM PROCESSES

PART – A

1 **Define (a) Continuous-time random process (b) Discrete state random process.(May 2011)**

Solution:

Consider a random process $\{X(t), t \in T\}$, where T is the index set or parameter set. The values assumed by X(t) are called the states, and the set of all possible values of the states forms the state space E of the random process.

(a) If the state space E and index set T are both continuous, then the random process is called continuous-time random process.

(b) If the state space E is discrete and the index set T is continuous, then the random process is called discrete state random process

2 **Define Wide sense stationary process**. **(MAY 2012, MAY 2013)**

Solution:

A random process {X(t)} is called wide-sense stationary if the following conditions hold:

(i) $E\left[X(t)\right]=a \ \ constant$

(ii) $R_{XX}(t_1,t_2)=E\left[X(t_1)X(t_2)\right]=R_{XX}(t_1-t_2)$ = function of time difference.

3 **If the initial state probability distribution of a Markov chain is $P^{(0)}=\begin{pmatrix}\frac{5}{6} & \frac{1}{6}\end{pmatrix}$ and the transition probability matrix of the chain is $\begin{pmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$, find the probability distribution of the chain after 2 steps.** **(MAY 2012)**

Solution:

Probability distribution after 2 steps = $P^{(2)} = P^{(1)}P = \{P^{(0)}P\}P$

Now $P^{(1)}=P^{(0)}P=\begin{pmatrix}\frac{5}{6} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}1/12 & 11/12\end{pmatrix}$

$\therefore P^{(2)}=P^{(1)}P=\begin{pmatrix}\frac{1}{12} & \frac{11}{12}\end{pmatrix}\begin{pmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}11/24 & 23/24\end{pmatrix}$

4 **If the transition probability matrix of a Markov chain is $\begin{pmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$, find the steady state distribution of the chain.** **(MAY 2013)**

Solution:

The steady state distribution of the chain is given by $A=\left[a \quad b\right]$, where A.P = A

$$A.P = A \Rightarrow \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

$$\Rightarrow b/2 = a \quad \cdots (1) \quad and \quad a + b/2 = b \quad \cdots \ (2)$$

$$We\ know that\ a + b = 1 \qquad \cdots (3)$$

$$substituting\ (1)\ in\ (3), we\ get\ 3b/2 = 1 \Rightarrow b = 2/3$$

$$\therefore \ (1) \Rightarrow a = 1/3$$

$$\therefore \ Steady\ state\ distribution\ of\ the\ chain\ = \ A \ = \ \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

5 **Examine whether the Poisson process {X(t)} is stationary or not. (DEC 2010, 2012)**

Solution:

A random process to be stationary in any sense, its mean must be a constant. We know that the mean of a Poisson process with rate λ is given by $\mathbf{E\{X(t)\} = \lambda t}$ which depends on the time t. Thus the Poisson process is not a stationary process.

6 **When is a Markov chain, called homogeneous? (DEC 2010)**

Solution:

If the one-step transition probability is independent of n, i.e., $p_{ij}(n, n+1) = p_{ij}(m, m+1)$

The Markov chain is said to have stationary transition probabilities and the process is called as homogeneous Markov chain.

7 **Is a Poisson process a continuous time Markov chain? Justify your answer (MAY2010)**

Solution:

We know that Poisson process has the Markovian property. Therefore, it is a Markov chain as the states of Poisson process are discrete. Also, the time'tʼ in a Poisson process is continuous. Therefore, the Poisson process a continuous time Markov chain.

8 **Define transition probability matrix**. **(DEC 2011)**

Solution:

The transition probability matrix (TPM) of the process $\{X_n, n \geq 0\}$ is defined by

$$P = \left[p_{ij}\right] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Where the transition probabilities (elements of P) satisfy $p_{ij} \geq 0, \& \sum_{j=1}^{\infty} p_{ij} = 1, \ i = 1,2,3,\cdots$

9 **Define Markov process**. **(DEC 2011)**

Solution:

A random process or Stochastic process X(t) is said to be a Markov process if given the value of X(t), the value of X(v) for v > t does not depend on values of X(u) for u < t. In other words, the future behavior of the process depends only on the present value and not on the past value.

10 **Prove that first order stationary random process has a constant mean. (DEC 2013)**

Solution:

Let X(t) be a first-order stationary process. Then the first-order probability density function of X(t) satisfies $f_X(x_1 ; t_1) = f_X(x_1 ; t_1 + \varepsilon)$ …..(A) for all t_1 and ε.

Now, consider any two time instants t_1 and t_2, and define the random variable $X_1 = X(t_1)$ and $X_2 = X(t_2)$. By definition, the mean values of X_1 and X_2 are given by

$$E(X_1) = E[X(t_1)] = \int_{x_1=-\infty}^{\infty} x_1 f_X(x_1;t_1)dx_1 \quad \cdots(1)$$

$$E(X_2) = E[X(t_2)] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_2)dx_2 \quad \cdots(2)$$

$$Let\, t_2 = t_1 + \varepsilon \quad (2) \Rightarrow E[X(t_1+\varepsilon)] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_1+\varepsilon)dx_2$$

Using (A), $E[X(t_1+\varepsilon)] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_1)dx_2 = \int_{x_1=-\infty}^{\infty} x_1 f_X(x_1;t_1)dx_1 = E[X(t_1)]$

which shows first order stationary random process has a constant mean.

11 **Consider the random process $X(t) = \cos(t+\phi)$ where ϕ is a random variable with density function $f(\varphi) = \frac{1}{\pi}, \frac{-\pi}{2} < \phi < \frac{\pi}{2}$. Check whether the process is wide sense stationary or not.**

Given $X(t) = \cos(t+\phi)$ and $f(\varphi) = \frac{1}{\pi}, \frac{-\pi}{2} < \phi < \frac{\pi}{2}$

$$\mathrm{E}[\mathrm{X(t)}] = \int_{-\infty}^{\infty} \mathrm{X(t)f}(\varphi)\mathrm{d}\varphi = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(\mathrm{t}+\varphi)\mathrm{d}\varphi = \frac{1}{\pi}\{\sin(\mathrm{t}+\varphi\}_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\{\cos\mathrm{t} - (-\cos\mathrm{t})\} = \frac{1}{\pi}2\cos\mathrm{t} \neq \text{constant}$$

So, X(t) is not WSS.

12 **Consider a random Process $X(t) = \cos(wt+\theta)$, where w is a real constant and θ is a uniform variable in $\left(0, \pi/2\right)$. Show that X(t) is not wide sense stationary.**

Given $X(t) = \cos(wt+\theta)$. Since θ is uniformly distributed in $(0, \pi/2)$, $f(\theta) = \frac{2}{\pi}, 0 < \theta < \frac{\pi}{2}$

$$\mathrm{E}[\mathrm{X(t)}] = \int_{-\infty}^{\infty} \mathrm{X(t)f}(\theta)\mathrm{d}\theta = \frac{2}{\pi}\int_{0}^{\pi/2} \cos(\mathrm{wt}+\theta)\mathrm{d}\theta = \frac{2}{\pi}\{\sin(\mathrm{wt}+\theta\}_0^{\pi/2} = \frac{1}{\pi}\{\cos\mathrm{wt} - \sin\mathrm{wt})\} \neq \text{CONSTANT}$$

So, X(t) is not WSS.

13 **Define Markov Chain.**

A Markov process is called Markov chain if the states $\{X_i\}$ are discrete no matter whether 't' is discrete or continuous.

14 **Define Chapman-Kolmogrov Equation**

The Chapman-Kolmogrov equation provides a method to compute the n-step transition probabilities. The equation can be represented as $P_{ij}^{n+m} = \sum_{k=0}^{\infty} p_{ik}^{n} p_{kj}^{m} \ \forall n,m \geq 0$.

15 **When do you say that a Markov chain is irreducible?**

The Markov chain is irreducible if all states communicate with each other at some time.

16 **When do you say the Markov chain is regular?**

A regular Markov chain is defined as a chain having a transition matrix P such that for some power of P, it has only non-zero positive probability values.

17 **When do you say that state 'i' is periodic and aperiodic?**

Let A be the set of all positive integers n such that $p_{ii}^{(n)} > 0$ and 'd' be the Greatest Common Divisor of the set A. We say state 'i' is periodic if d >1 and aperiodic if d = 1.

18 **What are the properties of Poisson process?**

(a)The poisson process is not a stationary process. It is vivid from the expressions of moments of poisson process that they are time dependent.

(b)The poisson process is a Markov process.

19 **Determine whether the given matrix is irreducible or not.** $P = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

$P = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}; P^2 = \begin{bmatrix} 0.16 & 0.49 & 0.35 \\ 0.07 & 0.33 & 0.60 \\ 0.02 & 0.24 & 0.74 \end{bmatrix} \therefore P_{ij}^{(n)} > 0, \forall i, j$. So, P is irreducible.

20 **The number of particles emitted by a radioactive source is Poisson distributed. The source emits particles at the rate of 6 per minute. Each emitted particle has a probability of 0.7 of being counted. Find the probability that 11 particles are counted in 4 minutes.**

The number of particles N(t) emitted is poisson with parameter $\lambda = np = 6(0.7) = 4.2$

$$P(N(t) = m) = \frac{e^{-4.2t}(4.2t)^m}{m!} \Rightarrow P(N(4) = 11) = \frac{e^{-4.2(4)}(4.2(4))^{11}}{11!} = 0.038.$$

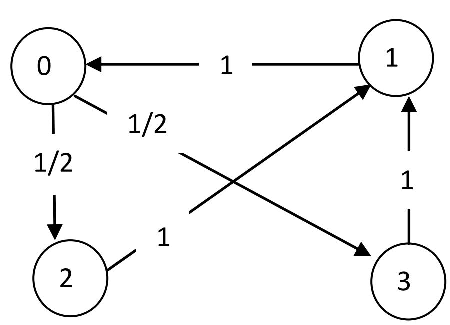
PART – B

1 i) **Let the Markov Chain consisting of the states 0, 1, 2, 3 have the transition probability matrix** $P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. **Determine which states are transient and which are recurrent by defining transient and recurrent states.** **(MAY 2010)**

Solution:

Transient state: A state 'a' is transient if $F_{aa} < 1$. Recurrent state: A state 'a' is recurrent if $F_{aa}=1$.

Here $F_{aa} = \sum_{n=0}^{\infty} f_{aa}^{(n)}$, where $f_{aa}^{(n)} =$ first time return probability of state 'a' after n steps.



$$\text{Here } P_{00}^{3} > 0,\ P_{01}^{2} > 0,\ P_{02}^{1} > 0,\ P_{03}^{1} > 0$$
$$P_{10}^{1} > 0,\ P_{11}^{3} > 0,\ P_{12}^{2} > 0,\ P_{13}^{2} > 0$$
$$P_{20}^{2} > 0,\ P_{21}^{1} > 0,\ P_{22}^{3} > 0,\ P_{23}^{3} > 0$$
$$P_{30}^{2} > 0,\ P_{31}^{1} > 0,\ P_{32}^{3} > 0,\ P_{33}^{3} > 0$$

Therefore, the Markov chain is irreducible. And also it is finite.
So, all the states are of same nature.
Consider the state '0'

$$f_{00}^{1} = 0;\ \ f_{00}^{2} = 0;\ \ f_{00}^{3} = \frac{1}{2} + \frac{1}{2} = 1;\ \ f_{00}^{4} = 0 \text{ and so on.}$$

Therefore, the state '0' is recurrent.
Since, the chain is irreducible, all the states are recurrent.

1 ii) **The transition probability matrix of a Markov chain {X(t)}, n = 1,2,3,….} having three states 1, 2 and 3 is** $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ **and the initial distribution is** $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$**.**

Find $(i)\ P[X_2 = 3]$ $\ (ii)\ P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$**.**

(MAY 2012, 2014, DEC 2013)

Solution:

We have $P^2 = P.P = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$

(I) $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3 / X_0 = i) P(X_0 = i)$

$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2) \ + P(X_2 = 3/X_0 = 3)P(X_0 = 3)$

$= P_{13}^{2}P(X_0 = 1) + P_{23}^{2}P(X_0 = 2) + P_{33}^{2}P(X_0 = 3) = 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$

(II) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$= P[X_0 = 2, X_1 = 3, X_2 = 3] P[X_3 = 2 / X_0 = 2, X_1 = 3, X_2 = 3]$

$= P[X_0 = 2, X_1 = 3, X_2 = 3] P[X_3 = 2 / X_2 = 3]$

$= P[X_0 = 2, X_1 = 3] P[X_2 = 3 / X_0 = 2, X_1 = 3] P[X_3 = 2 / X_2 = 3]$

$= P[X_0 = 2, X_1 = 3] P[X_2 = 3 / X_1 = 3] P[X_3 = 2 / X_2 = 3]$

$= P[X_0 = 2] P[X_1 = 3 / X_0 = 2] P[X_2 = 3 / X_1 = 3] P[X_3 = 2 / X_2 = 3]$

$= (0.2)(0.2)(0.3)(0.4) = 0.0048$

2 i) **Suppose that customers arrive at a bank according to Poisson process with mean rate of 3 per minute. Find the probability that during a time interval of two minutes i) exactly 4 customers arrive ii) greater than 4 customers arrive iii) fewer than 4 customers arrive. (MAY 2012, DEC 2013)**

Solution:

Mean of the Poisson process = λt.

Mean arrival rate = mean number of arrivals per minute (unit time) = λ

Given λ = 3. $P\{X(t)=k\}=\dfrac{e^{-\lambda t}(\lambda t)^k}{\lfloor k}$

(i) Probability that exactly 4 customers arrive $P\{X(2)=4\}=\dfrac{e^{-6}6^4}{\lfloor 4}=0.133$

(ii) Probability that greater than 4 customers arrive

$$P\{X(2)>4\}=1-\{P[X(2)=0]+P[X(2)=1]+P[X(2)=2]+P[X(2)=3]+P[X(2)=4]\}$$

$$=1-\left\{\frac{e^{-6}6^0}{\lfloor 0}+\frac{e^{-6}6^1}{\lfloor 1}+\frac{e^{-6}6^2}{\lfloor 2}+\frac{e^{-6}6^3}{\lfloor 3}+\frac{e^{-6}6^4}{\lfloor 4}\right\}=0.715$$

(iii) Probability that fewer than 4 customers arrive

$$P\{X(2)<4\}=P[X(2)=0]+P[X(2)=1]+P[X(2)=2]+P[X(2)=3]$$

$$=\frac{e^{-6}6^0}{\lfloor 0}+\frac{e^{-6}6^1}{\lfloor 1}+\frac{e^{-6}6^2}{\lfloor 2}+\frac{e^{-6}6^3}{\lfloor 3}=0.151$$

2 ii) **Prove that (i) difference of two independent Poisson processes is not a Poisson process and (ii) Poisson process is a Markov process. (MAY 2013)**

Solution:

(i) Let $X(t)=X_1(t)-X_2(t)$ where $X_1(t)$ and $X_2(t)$ are poisson processes with λ_1 and λ_2 as the parameters

$$E[X(t)]=E[X_1(t)]-E[X_2(t)]=(\lambda_1-\lambda_2)t$$

$$E[X^2(t)]=E\{[X_1(t)-X_2(t)]^2\}=E[X_1^2(t)]+E[X_2^2(t)]-2E[X_1(t)X_2(t)]$$

$$=E[X_1^2(t)]+E[X_2^2(t)]-2E[X_1(t)]E[X_2(t)]$$

$$=(\lambda_1^2t^2+\lambda_1t)+(\lambda_2^2t^2+\lambda_2t)-2(\lambda_1t)(\lambda_2t)=(\lambda_1+\lambda_2)t+(\lambda_1^2+\lambda_2^2)t^2-2\lambda_1\lambda_2t^2$$

$$=(\lambda_1+\lambda_2)t+(\lambda_1-\lambda_2)^2t^2\neq(\lambda_1-\lambda_2)t+(\lambda_1-\lambda_2)^2t^2$$

$\therefore X_1(t)-X_2(t)$ is not a Poisson process.

(ii) Consider $P[X(t_3)=n_3/X(t_2)=n_2,X(t_1)=n_1)]=\dfrac{P[X(t_1)=n_1,X(t_2)=n_2,X(t_3)=n_3]}{P[X(t_1)=n_1,X(t_2)=n_2]}$

$$=\frac{\dfrac{e^{-\lambda t_3}\lambda^{n_3}t_1^{n_1}(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}}{\dfrac{e^{-\lambda t_2}\lambda^{n_2}t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}}=\frac{e^{-\lambda(t_3-t_2)}\lambda^{n_3-n_2}(t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!}=P[X(t_3)=n_3/X(t_2)=n_2]$$

$$\therefore P[X(t_3=n_3/X(t_2)=n_2,X(t_1)=n_1)]=P[X(t_3)=n_3/X(t_2)=n_2]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1)= n_1, X(t_2)= n_2$ depends only on the most recent values $X(t_2)= n_2$.

i.e., The Poisson process possesses Markov property.

3 i) **A salesman territory consists of three cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However if he sells in either city B or city C, the next day he is twice as likely to sell in city A as in the other city. In the long run how often does he sell in each of the cities? (MAY 2012,DEC 2013)**

Solution:

States: A, B and C

The transition probability matrix is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$

The long run probability is given by $\pi = \begin{bmatrix} a & b & c \end{bmatrix}$, where $\pi\text{P} = \pi$.

Now $\pi\text{P} = \pi \Rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$

$\therefore\ 0.\text{a} + \frac{2}{3}b + \frac{2}{3}c = \text{a} \Rightarrow -\text{a} + \frac{2}{3}b + \frac{2}{3}c = 0 \cdots (1)$

$1.\text{a} + 0.\text{b} + \frac{1}{3}c = b \Rightarrow \text{a - b} + \frac{1}{3}c = 0 \cdots (2)$

$0.\text{a} + \frac{1}{3}\text{b} + 0.\text{c} = \text{c} \Rightarrow \frac{1}{3}\text{b - c} = 0 \cdots (3)$

Also, we know that a + b + c = 1 $\cdots (4)$

From (3), c = b/3

From (2), a - b + b/9 = 0 $\Rightarrow$ a = 8b/9

From (4), 8b/9 + b + b/3 = 1 $\Rightarrow$ 20b/9 = 1 $\Rightarrow$ $\boxed{\text{b = 9/20}}$

$\therefore \boxed{\text{c = 3/20 and a = 8/20}}$

$\therefore$Long run probability = $\begin{bmatrix} 8/20 & 9/20 & 3/20 \end{bmatrix}$

3 ii) **A gambler has Rs. 2. He bets Re. 1 at a time and wins Re. 1 with probability ½. He stops playing if he loses Rs. 2 or wins Rs. 4. i) What is the tpm of the related Markov chain? ii) What is the probability that he has lost his money at the end of 5 plays? (MAY 2013)**

Solution:

Let X_n denote the amount with the player at the end of the n^{th} round of the play.

The possible values of X_n = State space = {0, 1, 2, 3, 4, 5, 6}

Initial probability distribution= $\text{P}^{(0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(i) The transition probability matrix is =

$$\begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(ii) Probability that he has lost his money at the end of 5 plays = $\mathrm{P}\left[X_5 = 0\right]$

To find this we need $\mathrm{P}^{(5)}$

$$\mathrm{P}^{(1)} = \mathrm{P}^{(0)} P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{P}^{(2)} = \mathrm{P}^{(1)} P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}$$

$$P^{(4)} = P^{(3)}P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix}$$

$$P^{(5)} = P^{(4)}P = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{32} & 0 & \frac{9}{32} & 0 & \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$

$\therefore \boxed{P(X_5 = 0) = 3/8}$

4 i) **Show that the random process $X(t) = A\cos(w_0 t + \theta)$ is wide-sense stationary, if A and w_0 are constants and θ is uniformly distributed random variable in $(0,2\pi)$ (DEC 2011, MAY 2014)**

Solution:

Since θ is uniformly distributed in $(0,2\pi)$ p.d.f. is $f(\theta) = \frac{1}{2\pi}\ (0 \le 0 \le 2\pi)$

$$E[X(t)] = \int_0^{2\pi} \frac{1}{2\pi} A\cos(\omega_0 t + \theta)\,d\theta = \frac{A}{2\pi}\left[\sin(\omega_0 t + \theta)\right]_0^{\pi} = \frac{A}{2\pi}\left[\sin(\omega_0 t + 2\pi) - \sin(\omega_0 t)\right]$$

$$= \frac{A}{2\pi}\left[\sin\omega_0 t - \sin\omega_0 t\right] = 0$$

$$R(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = E\left[\mathrm{A}\cos(\omega_0 t_1 + \theta)\mathrm{A}\cos(\omega_0 t_2 + \theta)\right]$$

$$= E\left[A^2\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta)\right] = \frac{A^2}{2}E\left[\cos\left[\omega_0(t_1 + t_2) + 2\theta\right] + \cos\omega_0(t_1 - t_2)\right]$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos\left(\omega_0(t_1+t_2) + 2\theta\right) + \cos\left(\omega_0(t_1 - t_2)\right)\right]d\theta$$

$$= \frac{A^2}{4\pi}\left\{\left[\frac{\sin\left(\omega_0(t_1+t_2)+2\theta\right)}{2}\right]_0^{2\pi} + \cos\left(\omega_0(t_1-t_2)\right)\left[\theta\right]_0^{2\pi}\right\}$$

$$= \frac{A^2}{4\pi}\left\{\frac{\sin\left(\omega_0(t_1+t_2)+4\pi\right)}{2} - \frac{\sin\left(\omega_0(t_1+t_2)\right)}{2} + \cos\left(\omega_0(t_1-t_2)\right)(2\pi)\right\}$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin\omega_0(t_1+t_2)}{2} - \frac{\sin\omega_0(t_1+t_2)}{2} + 2\pi\cos\omega_0(t_1-t_2)\right]$$

$$= \frac{A^2}{4\pi}.2\pi\cos\omega_0(t_1-t_2) = \frac{A^2}{2}\cos\omega_0(t_1-t_2)$$ =a function of $t_1 - t_2$

$\therefore$ The process $X(t)$ is W.S.S.

4 ii) **Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ is wide sense stationary, if E(AB) =0; E(A)= E(B) = 0, & E(A^2) = E(B^2), where A &B are random variables. (MAY 2013)**

Solution:

Given $X(t) = A\cos\lambda t + B\sin\lambda t$, E(A) = E(B) = E(AB) = 0, E(A^2) = E(B^2) = k(say)

To prove X(t) is wide sense stationary, we have to show

(i) E{X(t)} = constant

(ii) $R_{XX}(t,t+\tau)$ = function of time difference = function of τ

Now, $E\left[X(t)\right] = E\left[A\cos\lambda t + B\sin\lambda t\right] = \cos\lambda t E(A) + \sin\lambda t E(B) = 0 \therefore$ $E\left[X(t)\right]$ = constant

$$R_{XX}(t,t+\tau) = E\left\{X(t)X(t+\tau)\right\}$$

$$= E\left\{\left[A\cos\lambda t + B\sin\lambda t\right]\left[A\cos(\lambda t + \lambda\tau) + B\sin(\lambda t + \lambda\tau)\right]\right\}$$

$$= \mathrm{E}\left[\begin{array}{r} A^2\cos\lambda t\cos(\lambda t+\lambda\tau)+AB\cos\lambda t\sin(\lambda t+\lambda\tau)+AB\sin\lambda t\cos(\lambda t+\lambda\tau) \\ +B^2\sin\lambda t\sin(\lambda t+\lambda\tau)\end{array}\right]$$

$$= \cos\lambda t\cos(\lambda t+\lambda\tau)E(A^2)+\cos\lambda t\sin(\lambda t+\lambda\tau)E(AB)+\sin\lambda t\cos(\lambda t+\lambda\tau)E(AB)$$
$$+\sin\lambda t\sin(\lambda t+\lambda\tau)E(B^2)$$

$$= \cos\lambda t\cos(\lambda t+\lambda\tau)k+\sin\lambda t\sin(\lambda t+\lambda\tau)k$$

$$\left[\because E(AB)=0\ \&\ E(A^2)=E(B^2)=k(say)\right]$$

$$= \mathrm{k}\left[\cos\lambda t\cos(\lambda t+\lambda\tau)+\sin\lambda t\sin(\lambda t+\lambda\tau)\right]$$

$= \mathrm{k}\cos(\lambda t+\lambda\tau-\lambda t) = \mathrm{k}\cos\lambda\tau = a$ function of time difference.

$\therefore$ X(t) is wide sense stationary.

5 i) **A man either drives a car (or) catches a train to go to office each day. He never goes two days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose than on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. Find (1) the probability that he takes a train on the third day and (2) the probability that he drives to work in the long run. (DEC 2011)**

Solution:

i) Here train (T) and car(C) are the states. The tpm is $P=\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$

Initial state probability distribution $P^{(1)}=\left[\frac{5}{6},\frac{1}{6}\right]$,

as P[traveling by car]=P[getting 6]=$\frac{1}{6}$; P[traveling by train]= $\frac{5}{6}$

$$P^{(2)}=P^1P=\left[\frac{5}{6},\frac{1}{6}\right]\begin{bmatrix}0 & 1\\ 1/2 & 1/2\end{bmatrix}=\left[\frac{1}{12},\frac{11}{12}\right];P^{(3)}=P^2P=\left[\frac{1}{12},\frac{11}{12}\right]\begin{bmatrix}0 & 1\\ 1/2 & 1/2\end{bmatrix}=\left[\frac{11}{24},\frac{13}{24}\right]$$

Probability that the man travels by train on 3rd day = $\frac{11}{24}$

ii) Let $\pi=(\pi_1,\pi_2)$ be the stationary state distribution of the Markov chain. By property of $\pi P=\pi$

$$(\pi_1,\pi_2)\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=(\pi_1,\pi_2) \Rightarrow \quad \frac{1}{2}\pi_2=\pi_1 \ and\ \pi_1+\frac{1}{2}\pi_2=\pi_2 \Rightarrow\ 2\pi_1=\pi_2$$

Also $\pi_1+\pi_2=1$ ($\because$ Since π is the probability distribution)

$\therefore \pi_1+2\pi_1=1 \Rightarrow \pi_1=\frac{1}{3}\ \therefore \pi_2=\frac{2}{3}$ $\quad\therefore$ Probability that he travels by car in the long run= $\frac{2}{3}$.

5 ii) The process {X(t)} whose probability distribution under certain condition is given by

$$P[X(t)=n]=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n+1}}, n=1,2,3,\cdots\\[2mm] \dfrac{at}{1+at}, \qquad n=0.\end{cases}.$$

Show that {X(t)} is not stationary. (MAY 2012, DEC 2013)

Solution:

Given $P[X(t)=n]=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n+1}}, n=1,2,3,\cdots\\[2mm] \dfrac{at}{1+at}, \qquad n=0.\end{cases}$

$$E\left[X(t)\right]=\sum_{n=0}^{\infty}nP\left\{X(t)=n\right\} \quad =0+1P\left\{X(t)=1\right\}+2P\left\{X(t)=2\right\}+3P\left\{X(t)=3\right\}+\cdots$$

$$=1\left(\frac{1}{(1+at)^2}\right)+2\left(\frac{at}{(1+at)^3}\right)+3\left(\frac{(at)^2}{(1+at)^4}\right)+\cdots$$

$$=\frac{1}{(1+at)^2}\left[1+2\frac{at}{1+at}+3\left(\frac{at}{1+at}\right)^2+\cdots\right] = \frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}=1=a\,\text{constant}$$

$$E\left[X^2(t)\right]=\sum_{n=0}^{\infty}n^2P\left\{X(t)=n\right\} = \sum_{n=0}^{\infty}\left\{n(n+1)-n\right\}P\left\{X(t)=n\right\}$$

$$=\sum_{n=0}^{\infty}n(n+1)P\left\{X(t)=n\right\}-nP\left\{X(t)=n\right\}$$

$$=\sum_{n=0}^{\infty}n(n+1)P\left\{X(t)=n\right\}-1 \qquad \left\{\because \sum_{n=0}^{\infty}nP\left\{X(t)=n\right\}=1\right\}$$

$$=\left[0+1.2P\left\{X(t)=1\right\}+2.3P\left\{X(t)=2\right\}+3.4P\left\{X(t)=3\right\}+\cdots\right]-1$$

$$=\left\{2\left(\frac{1}{(1+at)^2}\right)+6\left(\frac{at}{(1+at)^3}\right)+12\left(\frac{(at)^2}{(1+at)^4}\right)+\cdots\right\}-1$$

$$=\frac{2}{(1+at)^2}\left[1+3\frac{at}{1+at}+4\left(\frac{at}{1+at}\right)^2+\cdots\right]-1=\frac{2}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-3}-1$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-3}-1 \;=1+at-1=at,\text{not a constant.}$$

So, X(t) is not a stationary process.

UNIT – IV: QUEUEING MODELS

PART – A

1 **What are the basic characteristics of queueing system**?
The basic characteristics of queueing system are Arrival time pattern, Service time pattern, Number of service channels, Capacity of the system, Service discipline.

2 **What do you mean by transient state and steady state queueing system?**
Transient state: System depends on time t, Steady state: System is independent of time t.

3 **What do you mean by traffic intensity?**
The traffic intensity $\rho = \dfrac{\text{Arrival rate}}{\text{Service rate}}$

4 **Write down Little's formula**.

$$L_S = \lambda W_S \;,\quad L_q = \lambda W_q,\; W_s = W_q + \frac{1}{\mu},\; W_q = \frac{L_q}{\lambda}.$$

5 **Consider an (M/M/1) queuing system. If $\lambda = 6$ & $\mu = 8$, find the probability of more than 10 customers in the system.**

$$P(n > 10) = \left(\frac{\lambda}{\mu}\right)^{10+1} = \left(\frac{6}{8}\right)^{11} = (0.75)^{11}$$

6 **In a given $(M/M/1):(\infty/FCFS)$ queue $\rho = 0.6$, what is the probability that the queue contains 5 or more customers**?

$$P(n \ge 5) = \left(\frac{\lambda}{\mu}\right)^5 = \rho^5 = (0.6)^5$$

7 **What is the probability that a customer has to wait more than 15 min. to get his service Completed in $(M/M/1):(\infty/FIFO)$ queue system if $\lambda = 6/hr$ and $\mu = 10/hr$**?

$P(w_S > 15\,\text{min}) = P(w_S > \frac{1}{4}\text{ hr})$, w_s is exponential random variable with parameter μ - λ

$$= \int_{1/4}^{\infty} (\mu-\lambda)e^{-(\mu-\lambda)\omega} d\omega = \left[-e^{-(\mu-\lambda)\omega}\right]_{1/4}^{\infty} = e^{\frac{-(\mu-\lambda)}{4}} = e^{\frac{-(10-6)}{4}} = e^{-1}$$

8 **A duplicating machine maintained for office use is operated by an office assistant. If jobs arrive at a rate of 5 per hour and the time to complete each job varies according to an exponential distribution with mean 6 min., find the percentage of idle time of the machine in a day.**

$(M/M/1):(\infty/FIFO)$ Model. $\lambda = 5/\text{hr.}$, $\mu = \frac{1}{6}\text{min.} = 10/\text{hr.}$

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{5}{10} = 0.5$$, 50% of the time machine is idle.

9 **Derive the average no. of customers in the system for** $(M/M/1):(\infty/FIFO)$

$$L_S = \sum_{n=0}^{\infty} n\,p_n = \sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n p_0 = p_0\left[\frac{\lambda}{\mu} + 2\left(\frac{\lambda}{\mu}\right)^2 + 3\left(\frac{\lambda}{\mu}\right)^3 + \ldots\right]$$

$$= p_0\frac{\lambda}{\mu}\left[1 + 2\left(\frac{\lambda}{\mu}\right) + 3\left(\frac{\lambda}{\mu}\right)^2 + \ldots\right] = p_0\,\frac{\lambda}{\mu}\,\frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2} \qquad \therefore L_s = \left(1-\frac{\lambda}{\mu}\right).\frac{\lambda}{\mu}\frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2} = \frac{\lambda}{\mu-\lambda}$$

10 **Write down the Kendall's notations for queueing model**.

(a/b/c):(d/e) is the Kendall's notation where (a) is Arrival Pattern, (b) Service pattern, (c) No. of servers, (d) Capacity of the system, (e) Service discipline.

11 **A one person barber shop has six chairs to accommodate people waiting for a haircut. Assume customers who when all six chairs are full leave without entering the barber shop, customer arrive at the average rate 3/hr and customer spend an average of 15 minutes in the barber chair, what is the effective arrival rate?**

$$\lambda = 3/\text{hr} \ \& \ \mu = 4/\text{hr}, K = 6+1 = 7, P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = 0.2778$$

The Effective arrival rate $= \lambda' = \mu(1-P_0) = 4(1-0.2778) = 2.888$

12 **In a railway marshaling yard, goods trains arrive at a rate of 30 trains per day: Assume that the inter arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 36 minutes. Calculate the probability that the yard is empty.**

$$\boldsymbol{\lambda = \frac{1}{48}/\text{min}, \ \mu = \frac{1}{36}/\text{min} \ \ \rho = \frac{\lambda}{\mu} = 0.75 \ , \ P_0 = 1-\rho = 0.25}$$

13 **The local one person barber shop can accommodate a maximum of five people at a time (4 waiting and 1 getting haircut). Customers arrive according to a poisson distribution with mean 5/ hr. The barber cuts hair at an average rate of 4/hr. (Exponential service time).What fraction of the potential customers are turned away?**

$$\lambda = 5, \mu = 4, K = 5 \quad P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{K+1}} = 0.0888, \quad P_5 = \left(\frac{\lambda}{\mu}\right)^5 P_0 = 0.2711$$

14 **If $\lambda = 4/\text{hr} \ \& \ \mu = 12/\text{hr}$ in an $(M/M/1:4/FIFO)$ model, Find the probability that there is no customer in the system. If $\lambda = \mu$, what is the value of this probability?**

$$p_0 = \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right], \text{ if } \lambda \neq \mu \quad \text{i.e.} \quad = \frac{1-(4/12)}{1-(4/12)^5} = 0.669$$

If $\lambda = \mu$, $p_0 = \frac{1}{k+1} = \frac{1}{5}$

15 **What is the effective arrival rate for $(M/M/1):(4/FCFS)$ queueing model?**

The effective arrival rate is $\lambda' = \mu(1-p_0)$, where $p_0 = \begin{cases} \frac{1}{4+1} & \text{if } \lambda = \mu \\ \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{4+1}} & \text{if } \lambda \neq \mu \end{cases}$

16 **For $(M/M/S):(N/FIFO)$ Model, write down the formula for L_s, W_s.**

$$L_s = L_q + \frac{\lambda'}{\mu} \quad W_s = \frac{L_s}{\lambda'}, \text{ Where } L_q = p_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left(1-\rho^{k-s}-(k-s)(1-\rho)\rho^{k-s}\right)$$

$$\lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)p_n\right] \text{ and } \rho = \frac{\lambda}{s\mu}$$

17 **Find the probability that an arriving customer is forced to join the queue of M/M/C model.**

Arriving customers have to join the queue if No. of customers in the system $\geq$ no. of servers =C

$$P(n \geq c) = 1 - [p_0 + p_1 + p_2 + \dots + p_{c-1}]$$

$$= 1 - \left[p_0 + \frac{1}{1!}\frac{\lambda}{\mu}p_0 + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 p_0 + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 p_0 + \dots + \frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^{c-1} p_0\right]$$

$$= 1 - p_0\left[1 + \frac{1}{1!}\frac{\lambda}{\mu} + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \dots + \frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^{c-1}\right]$$

18 **What is the probability that an arrival to an infinite capacity 3 server poisson queue, with $\frac{\lambda}{c\mu} = \frac{2}{3}$ *and* $p_0 = \frac{1}{9}$ enters the service without waiting?**

Arriving customer shall enter the service without waiting if

No. of customer in the system $\leq$ No. of server (=3)

$$\therefore P(n < 3) = p_0 + p_1 + p_2 = p_0 + \frac{1}{1}\frac{\lambda}{\mu}p_0 + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 p_0 \qquad \because P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n p_0$$

Given $\frac{\lambda}{c\mu} = \frac{2}{3} \Rightarrow \frac{\lambda}{3\mu} = \frac{2}{3} \Rightarrow \frac{\lambda}{\mu} = 2$ and $c = 3$ $\therefore P(n < 3) = \frac{1}{9} + 2.\frac{1}{9} + \frac{1}{2}4\frac{1}{9} = \frac{5}{9}$

19 **A two person barber shop has 5 chairs to accommodate waiting customers, Potential Customers, who arrive when all 5 chairs are full, leave without entering the shop. Customers arrive at an average rate of 4 / hr and spend an average of 12 min. in the barber chair. Find ρ**

(M/M/2) with infinite capacity : Accommodation for 5 customers to wait + 2 chair = 7

$$\lambda = 4\ /\text{hr},\ \mu = \frac{1}{12}/\text{min.} = 5\ /\text{hr},\ s = 2, k = 7,\ \frac{\lambda}{\mu} = \frac{4}{5} = 0.8,\ \rho = \frac{\lambda}{s\mu} = \frac{4}{(2)(5)} = 0.4$$

20 **Explain the terms: Balking, Reneging.**

Balking: A customer who refuses to enter queueing system because the queue is too long is said to be balking.

Reneging: A customer who leaves the queue without receiving service because of too much waiting (or due to impatience) is said to have reneged.

PART - B

1.(i) **Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.**

(a) Find the average number of persons waiting in the system.

(b) What is the probability that a person arriving at the booth will have to wait in the queue?

(c) What is the probability that it will take him more than 10 min. altogether to wait for the phone and complete his call?

(d) Estimate the fraction of the day the phone will be in use.

This is an infinite queueing model with single server

Mean inter arrival time $=\frac{1}{\lambda}=12\,\text{min}$, therefore Arrival rate $\lambda = \frac{1}{12}/\text{min}$.

Mean service time $=\frac{1}{\mu}=4\,\text{min}$, therefore Service rate $\mu = \frac{1}{4}/\text{min}$.

(a) $L_s = \frac{\lambda}{\mu-\lambda} = \frac{\frac{1}{12}}{\frac{1}{4}-\frac{1}{12}} = 0.5$ customer

(b) $P(n > 0) = 1 - P(n = 0) = 1 - P$ (no customer in the system) = 1- p_0 $= 1-\left(1-\frac{\lambda}{\mu}\right)=\frac{\lambda}{\mu}=\frac{\frac{1}{12}}{\frac{1}{4}}=\frac{1}{3}$

(c) $P(W_s > 10) = e^{-(\mu-\lambda)10} = e^{-5/3} = 0.1889$

(d) P(phone will be idle) = P(n = 0) = Po $=1-\frac{\lambda}{\mu}=\frac{2}{3}$ $\therefore$ P (Phone will be in use) $=1-\frac{2}{3}=\frac{1}{3}$

(ii) **There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour,**
(a) What fraction of the time all the typists will be busy?
(b) What is the average number of letters waiting to be typed?
(c) What is the average time a letter has to spend for waiting and for being typed. Assume that arrival and service rates follow poisson distribution.

This is an $(M/M/3) : (\infty/FIFO)$ model.

$\lambda = 15 \ / \ \text{hr.}\ \&\ \mu = 6/\text{hr},\ s = 3,\ \therefore\ \frac{\lambda}{\mu} = 2.5\ \&\ \frac{\lambda}{s\mu} = \frac{2.5}{3} = 0.833$

(a) All the typists will be busy if there are at least 3 customers (letters) in the system

$p(n \ge 3) = p(3)+p(4)+p(5)+\ldots.. = 1-[p_0+p_1+p_2]$

$$p_0 = \left(\sum_{n=0}^{s-1}\frac{1}{n\ !}\ \frac{\lambda^n}{\mu^n} + \frac{\lambda^s}{\mu^s\ s\ !}\frac{1}{\left(1-\frac{\lambda}{s\mu}\right)}\right)^{-1} = \left[1+2.5+\frac{2.5^2}{2}+\frac{(2.5)^3}{6\times\left(1-\frac{5}{6}\right)}\right]^{-1} = \frac{1}{22.25} = 0.0449$$

$$p_1 = \frac{\lambda}{1!\ \mu}\ p_0 = 2.5p_0, \quad p_2 = \frac{\lambda^2}{2!\ \mu^2}\ p_0 = \frac{1}{2}(2.5)^2\ p_0$$

$P(n\ge 3) = 1 - [1 + 2.5 + (2.5^2/2)].(.0449)= 1-0.2974625= 0.7025375= 0.7025$

(b) Waiting to be typed (queue)

$$L_q = \frac{\lambda^{s+1}}{(s)(s!)\mu^{s+1}}\ p_0\ \frac{1}{\left(1-\frac{\lambda}{s\mu}\right)^2} = \frac{1}{3\ \times 6}(2.5)^4\ p_0 = 3.5078$$

(c) $W_s = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right] = \frac{1}{15}\left[3.5078+ \ \ 2.5\right] = 0.4005 \ \ \text{hr.}$

2(i) **Customers arrive at a one man barber shop according to a poisson process with a mean inter arrival time of 20 min. Customers spend an average of 15 min. in the barber's chair, then (a) What is the probability that a customer need not wait for a hair cut ?**
(b) What is the expected no. of customers in the barber shop and in the queue?
(c) How much time can a customer expect to spend in the barber shop?
(d) Find the average time that the customers spend in the queue.
(e) What is the probability that there will be 6 or more customers waiting for service?

This is an infinite queueing model with single server $\lambda = \frac{1}{20}/\text{min.}$ and $\mu = \frac{1}{15}/\text{min.}$

$$\text{(a)}\ \ p(n=0) = 1-\frac{\lambda}{\mu} = 1-\frac{\frac{1}{20}}{\frac{1}{15}} = \frac{1}{4},$$

$$\text{(b)}\ \ L_s = \frac{\lambda}{\mu-\lambda} = \frac{1/20}{1/15 - 1/20} = 3,\ L_q = L_s - \frac{\lambda}{\mu} = 3 - \frac{3}{4} = \frac{9}{4} = 2.25$$

$\therefore L_s = 3 \ \ \& \ \ L_q = 2.25$

$$\text{(c)}\ W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{15}-\frac{1}{20}} = 60\,\text{min}.$$

$$\text{(d)}\quad W_q = \frac{\lambda}{\mu\,(\mu-\lambda)} = \frac{\frac{1}{20}}{\frac{1}{15}\left(\frac{1}{15}-\frac{1}{20}\right)} = 45\ \text{min}.$$

$$\text{(e)}\quad p(n\geq 6) = \left(\frac{\lambda}{\mu}\right)^6 = \left(\frac{1/20}{1/15}\right)^6 = 0.1779$$

(ii) **A two person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full, leave without entering the barber shop. Customers arrive at average rate of 4 / hr and spend average of 12 min. in the barber's chair. Compute** p_0, p_1, p_7.

This is an $(M/M/2)\ :\ (7/FIFO)$

$\lambda = 4/\text{hr},\ \mu = \frac{1}{12}/\text{min.} = 5/\text{hr},\ \ s=2,\ k=7,\ \frac{\lambda}{\mu} = \frac{4}{5} = 0.8,\ \ \rho = \frac{\lambda}{s\mu} = \frac{4}{2(5)} = 0.4$

$$\text{(a)}\ \frac{1}{p_0} = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\frac{\lambda^n}{\mu^n} + \frac{\lambda^s}{\mu^s\, s!}\sum_{n=s}^{k}\left(\frac{\lambda}{s\mu}\right)^{n-s}\right] = \left[\sum_{n=0}^{1}\frac{1}{n!}\frac{\lambda^n}{\mu^n} + \frac{\lambda^2}{\mu^2\, 2!}\sum_{n=2}^{7}\left(\frac{\lambda}{s\mu}\right)^{n-2}\right]$$

$$= \left[1 + (0.8) + \frac{1}{2}(0.8)^2\left((0.4)^0 + (0.4)^1 + (0.4)^2 + (0.4)^3 + (0.4)^4 + (0.4)^5\right)\right] = 2.311488$$

$$\therefore p_0 = \frac{1}{2.311488} = 0.4289$$

(b) $p_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu}\right)^1 p_0 = 0.3431$

(c) $p_7 = \frac{1}{s^{n-s}\, s!}\left(\frac{\lambda}{\mu}\right)^7 p_0 = \frac{1}{2^5\, 2!}(0.8)^7(0.4289) = 0.0014$

3.(i) **A bank has two tellers working on savings accounts. The first teller handles withdrawals only while the second teller handles deposits only. It has been found that the service time distribution for the deposits and withdrawals both is exponential with mean service time 3 minutes per customer. Depositors are found to arrive in a Poisson fashion throughout the day with mean arrival rate 16 per hour. Withdrawer's also arrive in a Poisson fashion with mean arrival rate of 14 per hour.**
i) What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits?
ii) What would be the effect if this could only be accomplished by increasing the service time to 3.5 minutes?

Case(i) :Given $\frac{1}{\mu} = 3\min \Rightarrow \mu = 20/hr$

Average waiting time in the queue	Depositors	Withdrawers
When there is a separate channel then for the (i)depositors $\lambda_1 = 16/hr$ (ii)withdrawers $\lambda_2 = 14/\text{hr}$	$W_q = \frac{\lambda_1}{\mu(\mu-\lambda_1)} = \frac{16}{20(20-16)}$ $= \frac{1}{5} hr\,(or)\; 12\min$	$W_q = \frac{\lambda_2}{\mu(\mu-\lambda_2)} = \frac{14}{20(20-14)}$ $= \frac{7}{60} hr\,(or)\; 7\min$
If both tellers do the service then s=2, $\mu = 20/hr$ $\lambda = \lambda_1 + \lambda_2 = 30/hr$	$W_q = \frac{1}{\mu}.\frac{1}{s.s!}.\frac{\left(\frac{\lambda}{\mu}\right)^s.P_0}{\left(1-\frac{\lambda}{\mu s}\right)^2} = \frac{1}{20}.\frac{1}{2.2}.\frac{(1.5)^2}{(1-.75)^2}.\frac{1}{7} = \frac{0.45}{7}\text{h}$ or $3.86\min$ Where $p_0 = \left(\sum_{n=0}^{s-1}\frac{1}{n!}\frac{\lambda^n}{\mu^n} + \frac{\lambda^s}{\mu^s\, s!}\frac{1}{\left(1-\frac{\lambda}{s\mu}\right)}\right)^{-1} = \left[1 + 1.5 + \frac{(1.5)^2}{2\times 0.25}\right]^{-1} = \frac{1}{7}$	

Hence if both tellers do both types of service, the customers get benefited as their waiting time is considerably reduced.

Case (ii): If both tellers do both types of service with increased service time then s=2, $\mu = \frac{60}{3.5} = \frac{120}{7} / hr$, $\lambda = \lambda_1 + \lambda_2 = 30 / hr$

$$P_0 = \left[1 + 1.75 + \frac{(1.75)^2}{2x\frac{1}{8}}\right]^{-1} = \frac{1}{15}$$

$$\mathrm{W_q} = \frac{1}{\mu}.\frac{1}{\mathrm{s.s!}}.\frac{\left(\frac{\lambda}{\mu}\right)^{\mathrm{s}}.\mathrm{P_0}}{\left(1-\frac{\lambda}{\mu \mathrm{s}}\right)^2} = \frac{7}{120}.\frac{1}{2.2}.\frac{(1.75)^2}{\left(1-\frac{7}{8}\right)^2}.\frac{1}{15} = \frac{2.86}{15}\mathrm{h} \quad \text{or } 11.44\,\mathrm{min}$$

So, if this arrangement is adopted, withdrawers stand to lose as their waiting time is increased considerably and depositors get slightly benefited.

4(i) **A car service station has 2 bays offering service simultaneously. Because of space constraints, only 4 cars are accepted for servicing. The arrival pattern is Poisson with 12 cars per day. The service time in both the bays is exponentially distributed with μ = 8 cars per day per bay. Find the average number of cars in the service station, the average number of cars waiting for service and the average time a car spends in the system.**

This is a multiple server model with finite capacity. λ=12/day and μ=8/day, S=2, K=4

$$p_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu}\right)^{n-s}\right]^{-1} = \left[1 + \frac{1.5}{1} + \frac{1}{2}(1.5)^2\left[1 + (0.75) + (0.75)^2\right]\right]^{-1}$$

$P_0 = 0.1960$,

$$L_q = P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left[1-\rho^{k-s} - (k-s)(1-\rho)\rho^{k-s}\right] where\ \rho = \frac{\lambda}{s\mu}$$

$$\mathrm{L_q} = 0.1960(1.5)^2\left(\frac{0.75}{2(0.25)^2}\right)\left[1\text{-}(0.75)^2\text{-}2(0.25)(0.75)^2\right] = 0.4134\,\text{car.}$$

$$\mathrm{L_s} = \mathrm{L_q} + s\sum_{n=0}^{s-1}(s-n)p_n = 0.4134 + 2\sum_{n=0}^{1}(2-n)p_n = 2.4134 - 2(2p_0 + p_1) = 1.73\,\text{cars.}$$

$$W_s = \frac{\mathrm{L_s}}{\lambda'} \text{ where } \lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)\,p_n\right] = 8\left[2 - (2p_0 + p_1)\right] = 10.512$$

$$W_s = \frac{1.73}{10.512} = 0.1646 \text{ day}$$

Average time that a car has to spend in the system = 0.16

(ii) **Patients arrive at a clinic according to Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with a mean rate of 20 per hour.**

(a) Find the effective arrival rate at the clinic.
(b)What is the probability that an arriving patient will not wait?
(c)What is the expected waiting time until a patient is discharged from the clinic?

Arrival rate $\lambda = 30/\text{hr.}$, Service rate $\mu = 20/\text{hr.}$, $M/M/1:K/FIFO$ Model.

$$\because \lambda \neq \mu,\ p_0 = \frac{1-\lambda/\mu}{1-\left(\lambda/\mu\right)^{k+1}} = \frac{1-3/2}{1-\left(3/2\right)^{16}} = 0.00076$$

(a) The effective arrival rate is $\lambda' = \mu(1-p_0) = 20(1-0.00076) = 19.98/\text{hr.}$

(b) $P(\text{a patient will not wait}) = p(n=0) = p_0 = 0.00076$

(c) $$L_s = \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} = (-3) - \frac{16\left(\frac{3}{2}\right)^{16}}{1-\left(\frac{3}{2}\right)^{16}} = 13 \text{ patients nearly}$$

$$W_s = \frac{L_s}{\lambda'} = \frac{13}{19.98} = 0.65 \text{ hr or } 39 \text{ min.}$$

5(i) **A Telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service of these calls is approximately exponentially distributed with mean length 5 minutes. (1) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day? (2) If the subscribers will wait and are serviced in turn, what is the expected waiting time?**

Soln:

$\lambda = \frac{15}{60} = \frac{1}{4}$, $\mu = \frac{1}{5}$, $s = 2$ and $\rho = \frac{\lambda}{s\mu} = \frac{5}{8}$

$$P(w>0) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)} \times P_0$$, where P_0 is

$$P_0 = \left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1} = \left[\sum_{n=0}^{1}\frac{\left(\frac{5}{4}\right)^n}{n!} + \frac{\left(\frac{5}{4}\right)^2}{2!\left(1-\frac{5}{8}\right)}\right]^{-1} = \left[\sum_{n=0}^{2}\frac{\left(\frac{5}{3}\right)^n}{n!} + \frac{\left(\frac{5}{3}\right)^3}{6\times 4}\right]^{-1}$$

$$= \left[1 + \frac{5}{3} + \frac{25}{9}\times\frac{1}{2} + \frac{75}{27}\times\frac{9}{24}\right]^{-1} = \left[\frac{13}{3}\right]^{-1} = \frac{3}{13}$$

(1) $$P(w>0) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)} \times P_0 \Rightarrow \frac{\left(\frac{5}{4}\right)^2 \frac{3}{13}}{2!\left(1-5/8\right)} = \frac{25}{52} = 0.48$$

(2) $W_q = \frac{L_q}{\lambda} = \frac{\rho(s\rho)^2}{s!\,(1-\rho)^2} P_0 \Rightarrow \frac{4\times\frac{5}{8}\left(\frac{5}{4}\right)^2}{2!\left(1-\frac{5}{8}\right)} \times \frac{3}{13} = \frac{125}{39} = 3.2 \text{ min}$

(ii) **People arrive at a theatre ticket booth in Poisson distributed arrival rate of 25 per hour. Average service time is 2 minutes following exponential distribution. Calculate (1) the mean number in waiting line (2) the mean waiting time (3) the utilization factor.**

Soln. $\lambda = 25 \ \textit{per hour}, \ \mu = \frac{1}{2}\times 60 = 30 \ \textit{per hour}, \ \rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833$

(1) $\textit{Length of queue } L_q = \frac{\rho^2}{1-\rho} = 4 \ (\textit{appr.})$

(2) $\textit{Mean waiting time} = \frac{L_q}{\lambda} = 9.6 \text{ min}$

(3) $\textit{Utilisation factor} = \rho = 0.833$

UNIT V ADVANCED QUEUEING MODELS

PART - A

1 **Explain M/G/1 model.**

It is a non Markovian queuing model. Where the arrival pattern M is Poisson, the Service time distribution G follows any general distribution and the number of servers is one.

2 **What are the differences between Markovian and non Markovian queueing model?**

Markovian queueing model: Service time distribution follows Poisson distribution.

Non Markovian queueing model: Service time distribution follows any general distribution.

3 **Write down Pollaczek-khintchin formula.** **[June 2014]**

$$L_s = \lambda E(T) + \frac{\lambda^2\left[\text{Var}(T) + E(T)^2\right]}{2\left[1-\lambda E(T)\right]}$$

4 **Write down Little's formula relation in M/G/1 model.**

$$L_s = \lambda W_s \ , \quad L_q = \lambda W_q \ , \quad W_s = W_q + \frac{\lambda}{\mu}, \quad W_q = \frac{L_q}{\lambda}.$$

5 **Write down Pollaczek-khintchin formula for the case when service time distribution is Erlang distribution with k phases.** **[May 2011]**

$$E(T) = \frac{1}{\mu}, Var(T) = \frac{1}{k\mu^2} \ , \quad L_s = \frac{\lambda}{\mu} + \frac{\lambda^2[1+k]}{2k\mu(\mu-\lambda)}$$

6 **A one man barber shop takes exactly 25 minutes to complete one hair cut. If customers arrive at the barber shop in a Poisson at an average rate of one in every 40 minutes, how long on the average a customer spends in the shop?**

E(T) = 25, Var (T) = 0, $\lambda = \frac{1}{40}/\text{min}$, $L_s = \frac{25}{40} + \frac{625}{(1600)2(1-25/40)} = \frac{55}{48}$.

7 **Write short notes on queue networks.**

Network of queues can be described as a group of nodes, where each of the nodes represents a Service facility of the same type. The customers may enter the system at some node, can traverse from node to node in the system and finally can leave the system from any node. Also customers can return to the nodes already visited.

8 **Define series queues (Tandem queues) with an example.**

In series queues , there are a series of service stations through which each calling unit must progress prior to leaving the system. Ex. A physical examination for a patient where the patient under goes a series of stages, lab tests, ECG , X ray etc.

9 **What is the joint probability that there are m customers at station 1 and n customers at station 2 for a 2 stage series queue?**

$$P_{mn} = \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^n \left(1-\frac{\lambda}{\mu_2}\right)$$

10 **Consider a service facility with two sequential stations with respective service rates of 3/min and 4/min. The arrival rate is 2/min. What is the average waiting time of the system if the system could be approximated by a two series Tandem queue?**

$\mu_1 = 3/\text{min.}$, $\mu_2 = 4/\text{min.}$, $\lambda = 2/\text{min.}$

Average waiting time of customers in the system $=W_s$(station 1) + W_s(station 2)

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{3-2} + \frac{1}{4-2} = 1.5\,\text{min.}$$

11 **A TVS company in Chennai containing a repair station shared by a large number of machines has 2 sequential stations with respective service rates of 3 per hour and 4/hr. the failure rate of all the machines is 1 per hour. Assuming that the system behavior can be approximated by a 2 stage tandem queue find the probability that the service stations are idle.**

$$\mu_1 = 3/\text{hr.}\ \&\ \mu_2 = 4/\text{hr.}, \lambda = 1/\text{hr.},\quad P_{mn} = \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^n \left(1-\frac{\lambda}{\mu_2}\right)$$

$$P_{00} = \left(\frac{1}{3}\right)^0 \left(1-\frac{1}{3}\right) \left(\frac{1}{4}\right)^0 \left(1-\frac{1}{4}\right) = 0.5$$

12 **What do you mean by bottle neck of a network?**

The service station for which the utilization factor is maximum among all the other service stations of the network is called the bottle neck of a network.

13 **A company's repair section has 2 sequential stations with respective service rates of 4/hour and 5/ hour. The cumulative failure rate of all the machines is 1/hour. Assuming the system behavior can be approximated by the two stage tandem queue find the bottle neck of the repair facility.**

$\frac{\lambda}{\mu_1} = \frac{1}{4}, \frac{\lambda}{\mu_2} = \frac{1}{5}$, since $\frac{\lambda}{\mu_1} > \frac{\lambda}{\mu_2}$, the service station 1 is the bottle neck of the repair facility.

14 **Write the steady state equations for 2 stage series queues with blocking.**

Let λ be the arrival rate at station 1 which follows Poisson distribution and service time is exponential with parameters μ_1 and μ_2 respectively. $P_{n_1 n_2}$ represents the probability that there are n_1 customers in station 1 and n_2 customers in station 2.

$$-\lambda P_{0,0} + \mu_2 P_{0,1} = 0,\quad -\mu_1 P_{1,0} + \mu_2 P_{1,1} + \lambda P_{0,0} = 0, -(\lambda + \mu_2)P_{0,1} + \mu_1 P_{1,0} + \mu_2 P_{b,1} = 0$$

$$-(\mu_1 + \mu_2)P_{1,1} + \lambda P_{0,1} = 0,\ -\mu_2 P_{b,1} + \mu_1 P_{1,1} = 0$$

15 **Define Jackson networks.**

Networks preserving the following characteristics are called Jackson networks.

1. Arrivals from outside through node i follow a Poisson process with mean arrival rate r_i.
2. Service times at node i are independent and exponentially distributed with parameter μ_i
3. The probability that a customer who has completed service at node i will go to next node j is P_{ij} , i = 1,2,...k , j= 0,1,2,...k.

16 **Define open Jackson networks.**
In open Jackson networks, the arrivals from outside to the node i is allowed and once a customer gets the service completed at node i, joins the queue at node j with probability P_{ij} or leaves the system with probability P_{i0}.

17 **Write a short note on Finite resource model.**
Models in which arrivals are drawn from a small population are called Finite resource model.
In this case the arrival rate may greatly depend on the state of the system. For example the case of machine repairman model.

18 **Write down the flow balance equation of open Jackson network.**
$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$, $j = 1, 2, \ldots k$, where r_j is the arrival rate to node j , λ_j is the total arrival rate of customers to node j and P_{ij} is the probability that a departure from server i joins the queue at server j.

19 **A police department has 5 patrol cars. A patrol car breaks down and requires service once every 30 days. The department has two repair workers, each of whom takes an average of 3 days to repair a car. Breakdown and repair times are exponential. Identify the queuing model of the problem.**
Since the server here is the repair man and the customer is the cars [only 5 cars] that require service from the repairman, this problem comes under machine interference model which is one of the finite resource model.

20. **Write down the applications of network queues.**
Telecommunication field, Bioinformatics, biomedical Industry.

PART - B

1 **A Laundromat has 5 washing machines. A typical machine breaks down once every 5 days. A repairer takes an average of 2.5 days to repair a machine. Currently, there are three repair workers on duty. The owner has the option of replacing them with a super worker, who can repair a machine in an average of (5/6) day. The salary of the super worker equals the pay of the three repair workers. Break down time and repair time are exponential. Should the Laundromat replace the three repairers with a super worker?**

Sol: This problem comes under the machine interference problem and let K = No. of machines; R = No. of repairman, the steady state probabilities.

$$P_j = \begin{cases} KC_j \rho^j P_0, & j = 0,1,2,3\ldots R \\ \dfrac{KC_j \rho^j j! P_0}{R! R^{j-R}} & j = R+1, R+2, \ldots. K \end{cases}$$

Part 1 : Three repair Workers K = 5 and R = 3:

$$\lambda = (1/5)\,\text{mch}/\text{day}, \mu = (1/2.5)\,\text{mch}/\text{day}, \ \rho = 1/2$$

$$P_j = \begin{cases} 5C_j (1/2)^j P_0, & j = 0,1,2,3 \\ \dfrac{5C_j (1/2)^j j! P_0}{3! 3^{j-3}} & j = 4,5 \end{cases}$$

$$L_S = \sum_{j=0}^{5} jP_j = 1.7149 \quad \text{with} \quad P_0 = 0.1292$$

Part 2 : One Super Workers K = 5 and R =1 :

$\lambda = (1/5)\, mch/day, \mu = (6/5)\, mch/day,\ \rho = 1/6$

$$P_j = \begin{cases} 5C_j (1/6)^j P_0, & j = 0,1, \\ \dfrac{5C_j (1/6)^j\, j!P_0}{1!1^{j-1}} & j = 2,3,4,5 \end{cases}$$

$$L_S = \sum_{j=0}^{5} jP_j = 1.162 \quad with \quad P_0 = 0.3604$$

Therefore One Super Worker is better option than the three repair workers as Ls is lesser in this case.

2(i) In a heavy machine shop, the overhead crane is 75% utilized. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes. What is the average calling rate for the services of the crane and what is the average delay in getting service? If the average service time is cut to 8.0 minutes with a standard deviation of 6.0 minutes, how much reduction will occur on average in the delay of getting served?

The service time is not following exponential distribution. Hence it is a $(M/G/1)$ queuing model

$$\rho = 0.75,\ E(T) = 10.5,\ \mu = \frac{1}{E(T)} = \frac{1}{10.5}$$

Average calling rate for the services of the crane = arrival rate = $\lambda = \rho\mu = 0.0714 \min. or\, 4.286\, hr.$

The average delay in getting service = Waiting time in the queue = $W_q = \dfrac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}$

$$= \frac{(4.29)^2\left(\frac{8.8}{60}\right)^2 + (0.75)^2}{2\times 4.29(1-0.75)} = 0.449 \text{ hrs. or } 26.8\text{min.}$$

If the service time is cut to 8 minutes with $\sigma = 6\,\text{min., then } \mu = \dfrac{60}{8} = 7.5/\text{hr. and } \rho = \dfrac{4.29}{7.5} = 0.571$

Now average delay in getting service = $\dfrac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} = \dfrac{\rho^2\mu^2\sigma^2 + \rho^2}{2\rho\mu(1-\rho)} = \dfrac{\rho\left(1+\mu^2\sigma^2\right)}{2\mu(1-\rho)}$

$$= \frac{0.571\left[1 + (0.75)^2\left(6/60\right)^2\right]}{2\times 7.5(1-0.751)} = 0.1386 \text{ hrs. or } 8.32\,\text{min.}$$

Reduction of time is $26.8 - 8.32 = 18.5\,\text{min}$.

(ii) A one-man barber shop takes exactly 25 minutes to complete one hair-cut. If customers arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer in the spends in the shop. Also, find the average time a customer must wait for service?

The service time is constant. Hence it is a $(M/D/1)$ queuing model.

$E(T)=25$, $Var(T)=0$ Arrival rate $\lambda=\frac{1}{40}$/min.

$$L_s = \lambda E(T) + \frac{\lambda^2\left[E^2(t)+Var(T)\right]}{2\left[1-\lambda E(T)\right]} = \frac{1}{40}(25) + \frac{\left(\frac{1}{40}\right)^2\left[25^2+0\right]}{2\left[1-\frac{1}{40}(25)\right]} = \frac{55}{48}\text{customers.}$$

$$W_s = \frac{L_s}{\lambda} = \frac{\left(\frac{55}{48}\right)}{\left(\frac{1}{40}\right)} = 45.8\text{min.},\quad W_q = W_s - \frac{1}{\mu} = W_s - \frac{1}{\frac{1}{E(T)}} = 45.8-25 = 20.8 \text{ min.}$$

3(i) A patient goes to a single doctor clinic for a general check up has to go through 4 phases. The doctor takes on the average 4 minutes for each phase of the check up and the time taken for each phase is exponentially distributed. If the arrivals of the patients at the clinic are approximately Poisson at the average rate of 3 per hour, what is the average time spent by a patient (i) in the examination

(ii) waiting in the clinic?

The clinic has 4 phases (each having different service nature) in series as follows:
Considering all the phases (each with exponential service time) together as "one server" we shall take it as a server with Erlang service time. So this is a $(M/E_k/1)$ model with

$$E(T) = \frac{k}{\theta} = \frac{4}{\left(\frac{1}{4}\right)} = 16\text{min. and } Var(T) = \frac{k}{\theta^2} = \frac{4}{\left(\frac{1}{16}\right)} = 64\text{min.}$$

The arrival rate $\lambda = 3\, per\,\text{hr.} = \frac{3}{60}\, per\, \text{min} = \frac{1}{20}\, per\, \text{min.}$

$$L_s = \lambda E(T) + \frac{\lambda^2\left[E^2(t)+Var(T)\right]}{2\left[1-\lambda E(T)\right]} = \frac{1}{20}(16) + \frac{\left(\frac{1}{20}\right)^2\left[16^2+64\right]}{2\left[1-\frac{1}{20}(16)\right]} = \frac{14}{5}$$

$$W_s = \frac{L_s}{\lambda} = \frac{\left(\frac{14}{5}\right)}{\left(\frac{1}{20}\right)} = 56\text{min},\qquad W_q = W_s - \frac{1}{\mu} = W_s - \frac{1}{\frac{1}{E(T)}} = 56-16 = 40\text{min.}$$

(ii) Derive the Pollaczek-Khinchine formula. [Repeated UQ]

Let N and N' be the number of customers in the system at times t and t+T, when two consecutive customers have just left the system after getting service. Thus T is the random service time, which is a continuous random variable.

Let $f(t), E(T), V(T)$ be the probability density function, mean and variance of the service time T. Let M be the number of customers arriving in the system during service time T, then

$$N' = \begin{cases} M, & if \quad N = 0 \\ N-1+M, & if \quad N > 0 \end{cases}$$

where M is a discrete random variable taking the values 0,1,2,…
The same can be written as

$$N' = N-1+M+\delta .... (1), \text{ where } \delta = \begin{cases} 1, & \text{if} \quad N=0 \\ 0, & \text{if} \quad N > 0 \end{cases}$$

Note that by the definition of δ, $\delta^2 = \delta$ and $N\delta = 0$
Squaring both sides of 1, we have,

$$N'^2 = N^2 + (M-1)^2 + \delta^2 + 2N(M-1) + 2N\delta + 2(M-1)\delta = N^2 + (M-1)^2 + \delta + 2N(M-1) + 2(M-1)\delta$$

$$= N^2 + M^2 - 2M + 1 + 2N(M-1) + (2M-2+1)\delta$$

$$2N(1-M) = N^2 - N'^2 + M^2 - 2M + 1 + (2M-1)\delta ....(2)$$

Taking Expectations on the both sides of (1) we get,

$$E(N') = E(N) - 1 + E(M) + E(\delta)$$

In steady state the probability that the number of customers in the system is constant

$$E(N') = E(N) \qquad and \qquad E(N^2) = E(N'^2)........(3)$$

and substituting in previous equation we have $E(\delta) = 1 - E(M).....(4)$

Taking Expectations on the both sides of (2) we get

$$2E[N(1-M)] = E(N^2) - E(N'^2) + E(M^2) - 2E(M) + 1 + (2E(M)-1)E(\delta)$$

using (3) we have

$$2E[N(1-M)] = E(M^2) - 2E(M) + 1 + (2E(M)-1)E(\delta)..$$

using (4) we have

$$2E[N(1-M)] = E(M^2) - 2E(M) + 1 + (2E(M)-1)(1-E(M))$$

$$= E(M^2) - 2E(M) + 1 + 2E(M) - 2E(M)^2 - 1 + E(M) = E(M^2) - 2E(M)^2 + E(M)$$

Since the number of arrivals (M) to a system is independent of the number of customers already in the system (N) we have

$$2E[N]E[(1-M)] = E(M^2) - 2E(M)^2 + E(M) = E(M^2) - 2E^2(M) + E(M)$$

$$E(N) = \frac{E(M^2) - 2E^2(M) + E(M)}{2[1-E(M)]} ......(5)$$

Since the arrivals in service time T follows Poisson process with parameter λ .

$$E(M/T) = \lambda T \ , E(M^2/T) = \lambda^2 T^2 + \lambda T \ , \sigma^2_{M/T} = \lambda T$$

$$E(M) = E(\lambda T) = \lambda E(T)$$

$$E(M^2) = E(\lambda^2 T^2 + \lambda T) = \lambda^2 E(T^2) + \lambda E(T) = \lambda^2 [\text{var}(T) + E^2(T)] + \lambda E(T)$$

$$E^2(M) = \lambda^2 E^2(T)$$

substituting in (5) we have,

$$E(N) = \frac{\lambda^2\left[\text{var}(T)+E^2(T)\right]+\lambda E(T)-2\lambda^2E^2(T)+\lambda E(T)}{2\left[1-\lambda E(T)\right]}$$

$$= \frac{\lambda^2\left[\text{var}(T)+E^2(T)\right]+2\lambda E(T)-2\lambda^2E^2(T)}{2\left[1-\lambda E(T)\right]}$$

$$= \frac{\lambda^2\left[\text{var}(T)+E^2(T)\right]+2\lambda E(T)\left(1-E(T)\right)}{2\left[1-\lambda E(T)\right]}$$

$$= \frac{\lambda^2\left[\text{var}(T)+E^2(T)\right]}{2\left[1-\lambda E(T)\right]}+\frac{2\lambda E(T)\left(1-E(T)\right)}{2\left[1-\lambda E(T)\right]}$$

$$= \frac{\lambda^2\left[\text{var}(T)+E^2(T)\right]}{2\left[1-\lambda E(T)\right]}+\lambda E(T)$$

4(i) A repair facility by a large number of machines has two sequential stations with respective rates one per hour and two per hour. The cumulative failure rate of all the machines is 0.5 per hour. Assuming that the system behavior may be approximated by the two-stage tandem queue, determine the average repair time.

$\lambda = 0.5,\ \mu_1 = 1,\ \ \mu_2 = 2,\ \rho_1 = 0.5,\ \rho_2 = 0.25$

Note that each station is a (M/M/1) queue model

The average length of the queue at station 1

$$L_{S1} = \frac{\rho_1}{1-\rho_1} = \frac{0.5}{1-0.5} = 1$$

$$W_{S_1} = \frac{L_{S_1}}{\lambda} = \frac{1}{0.5} = 2$$

The average length of the queue at station 2

$$L_{S2} = \frac{\rho_2}{1-\rho_2} = \frac{0.25}{1-0.25} = \frac{1}{3}$$

$$W_{S_2} = \frac{L_{S2}}{\lambda} = \frac{1/3}{0.5} = \frac{2}{3}$$

The total repair time of the network is $= W_{s1} + W_{s2} = 2 + \frac{2}{3} = \frac{8}{3}$ hours

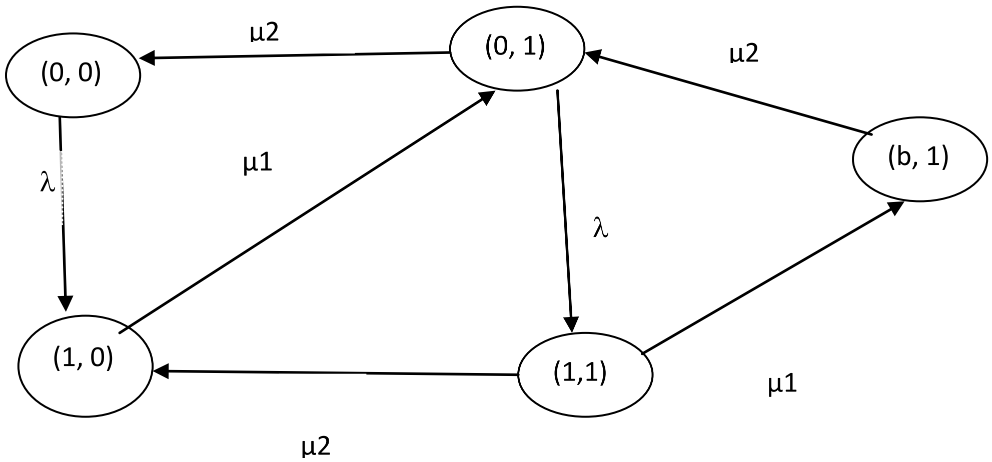
(ii) There are 2 salesmen in a super market. Out of the 2 salesman, one is in charge of billing and receiving payment while other salesman is in charge of weighing and delivering the items. Due to lack of space, Only one customer is allowed to enter the shop, only if the billing clerk is free. The customer who has finished is billing job has to wait until the delivery section becomes free. If customer arrive according to Poisson process at a 1/hour and the service times of 2 clerks are independent and have exponential rates of 3 and 2/hour.

Find (i) The proportion of customers who enter the supermarket.

(ii)The average number of customer in the system.

(iii)The average amount of time a customer spends in the shop.

Soln: This problem is 2 stage sequential queuing model with blocking



From the balance equations with $\lambda=1$; $\mu_1 = 3$; $\mu_2 = 2$

we have $p_{0,0} = 2p_{0,1}$ .....(1)

$3p_{1,0} = p_{0,0} + 2p_{1,1}$ ....(2)

$3p_{0,1} = 3p_{1,0} + 2p_{b,1}$ ....(3)

$5p_{1,1} = p_{0,1}$ ....(4)

$2p_{b,1} = 3p_{1,1}$ .. ..(5)

From the boundary condition

$p_{0,0} + p_{1,0} + p_{0,1} + p_{1,1} + p_{b,1} = 1$.....(6)

$\Rightarrow$ $P_{0,1} = (1\backslash 2)P_{0,0}$; $P_{1,0} = (2\backslash 5)P_{0,0}$; $P_{1,1} = (1\backslash 10)P_{0,0}$; $P_{b,1} = (3\backslash 20)P_{0,0}$

Solve the above equations we get $p_{0,0} = \frac{20}{43}, p_{1,0} = \frac{8}{43}, p_{0,1} = \frac{10}{43}, p_{1,1} = \frac{2}{43}, p_{b,1} = \frac{3}{43}$

(i)Proportion of customers entering the shop is equal to $p_{0,0} + p_{0,1} = \frac{20}{43} + \frac{10}{43} = \frac{30}{43}$

(ii) $L_s = \sum_m \sum_n (m+n) p_{m,n} = 1(p_{1,0} + p_{0,1}) + 2(p_{1,1} + p_{b,1}) = \frac{28}{43}$

(iii) $W_s = \frac{L_s}{\lambda_A}, \lambda_A =$ Average rate entering of super market $\lambda_A = \frac{30}{43}(1)$,

$$W_s = \frac{28/43}{30/43} = \frac{14}{15}\text{hrs.} = 56\text{min.}$$

5(i) For an open queuing network with three nodes 1, 2 and 3., let customers arrive from outside the system to node j according to a Poisson input process with parameters r_j and let Pij denote the proportion of customers departing from facility i to facility j . Given $(r_1, r_2, r_3) = (1, 4, 3)$ and

$$P_{ij} = \begin{bmatrix} 0 & 0.6 & 0.3 \\ 0.1 & 0 & 0.3 \\ 0.4 & 0.4 & 0 \end{bmatrix}$$ **determine the average arrival rate λj to the node j for j = 1,2,3**

[June2012]

Solution:

The traffic equations for the arrival rates are $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$, $j = 1, 2, ...k$

$\Rightarrow \lambda_1 = 1 + 0.1\lambda_2 + 0.4\lambda_3; \lambda_2 = 4 + 0.6\lambda_1 + 0.4\lambda_3; \lambda_3 = 3 + 0.3\lambda_1 + 0.3\lambda_2$

$\lambda_1 - 0.1\lambda_2 - 0.4\lambda_3 = 1$ ------------------------(1)

$-0.6\lambda_1 + \lambda_2 - 0.4\lambda_3 = 4$ ---------------------- (2)

$-0.3\lambda_1 - 0.3\lambda_2 + \lambda_3 = 3$ ---------------------(3)

(1) – (2) $\Rightarrow 1.6\lambda_1 - 1.1\lambda_2 = -3$ ---------------------(4)

(2) + 0.4(3) $\Rightarrow -0.72\lambda_1 - 0.88\lambda_2 = 2.8$ ---------------------(5)

0.8(4) $\Rightarrow 1.28\lambda_1 - 0.88\lambda_2 = -2.64$ ---------------------(6)

(4) + (6) $\Rightarrow 0.56\lambda_1 = 0.16 \Rightarrow \lambda_1 = 0.2857$

From (4) $\lambda_2 = 3.1428$ and from (3) $\lambda_3 = 4.0286$

(ii) Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates for server 1 and 2 are 8 and 10 respectively. A customer upon completion of service at server 1 is likely to go server 2 or leave the system; whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise. Determine the limiting probabilities, L_s and W_s

[June 2013]

Solution:

The traffic equations for the arrival rates are $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$, $j = 1, 2, ...k$

$\Rightarrow \lambda_1 = 4 + \frac{1}{4}\lambda_2; \lambda_2 = 5 + \frac{1}{2}\lambda_1$

$4\lambda_1 - \lambda_2 = 16$ ------------------------(1)

$-\lambda_1 + 2\lambda_2 = 10$ ------------------ (2)

Solving $\Rightarrow \lambda_1 = 6$ and $\lambda_2 = 8$

$$P(n_1, n_2) = (1-\rho_1)(1-\rho_2)\rho_{1_1}^{n_1} \quad \rho_2^{n_2} = \left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)^{n_1}\left(\frac{4}{5}\right)^{n_2}$$

$$L_{s_1} = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{6}{2}, \quad L_{s_2} = \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{8}{2}, \ L_s = 3 + 4 = 7$$

$$W_s = \frac{L_s}{\lambda} = \frac{7}{9}, \ \lambda = 4 + 5 = 9$$