Mathematical Modeling of Shot Put Throwing

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Abstract

There are multiple impacts on the result of the Shot Put competition, e.g. the speed of the ball, the throwing degrees, height, etc. Different factors can cause different results. This article used the kinetic principle and the mathematics model in the 2017 Athletics World Championships Women's Shot Put Final which includes Strike Speed, Strike Height, Strike Height Relative to Body Height, Distance Beyond Toe, Strike Tilt, etc., and created a regression model to analyze key points on the result of the Shot Put competition.

Keywords

Shot Put Competition, Regression, Kinetic Trace, Mathematics Model

Contents

1 Introduction

The ideal routine of the throwing trace is a parabola (as projectile motion). However, the trace isn't a parabola because of the air residence.

Without air residence, the trace will be like in 1.

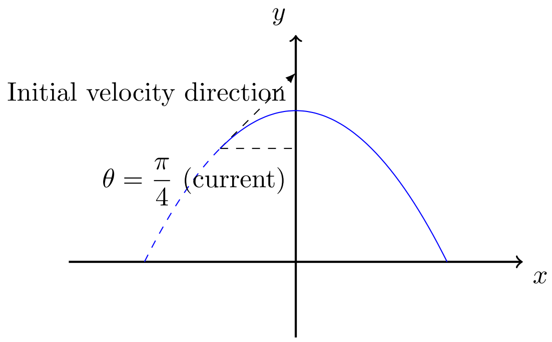

Figure 1: Ideal trace

However, because of the following reasons,

- Air residence,
- Distortion of the throwing direction,
- Start position,
- ...

The trace in 1 is inaccurate.

Therefore, the article will develop a mathematical model for the trajectory of the shot put that includes the effect of air resistance. As for the distortion of the throwing direction, the article may use the 3D transforms with Euler angles to adjust the result.

2 Parameters

Some parameters are included, while some parameters are ignored.

2.1 Parameters in Kinetic Model

In the kinetic analysis, we don't bring the exact data. Some definitions are following the table 1.

explain	**parameter**	**e.g. value**
gravitational acceleration	g	$9.787\mathrm{m/s^2}$
shot put mass	m	4kg
atmospheric drag coefficient	k	N/A
throw speed	$\boldsymbol{v}$	N/A [1]
throwing angle	θ	$37.0°$
throwing height	h	2.08m
tilt angle left and right	φ	$-21°$
horizontal displacement of the shot put [2]	s	19.94m

[1] Actually, the speed is a vector, and the v_x, v_y not only depends on the value of the absolute value, but also depends on the throwing angle. Expressing with complex number is also OK, one of the example is $13.24\mathrm{e}^{\cos 37.0° + \mathrm{i}\sin 37.0°}\mathrm{m/s}$, whose v_y is 7.944m/s, and v_x is 10.592m/s.

[2] including exceeds toe board distance.

Table 1: Parameters of the Mathematical Model

We also assume the following conditions:

1. The velocity is small, so $F = -kv$ should be used instead of $F = -kv^2$.

2. The horizontal velocity is not reduced to 0 due to air resistance.
3. There is no influence of field forces other than the gravity field, which is assumed to be uniformly strong.

2.2 Parameters in the Regression Model

2.3 Ignore

Because some parameters are useless, we ignore following parameters:

1. Front and rear tilt angles,
2. Strike height relative to body height,
3. Best round,
4. ...

3 Mathematical Model with Kinetic Analysis

3.1 Force Analysis

During the movement of a shot put ball, it is subjected to 2 forces: gravity and air resistance. The force of gravity is always vertically downward, however, air resistance changes with the direction of velocity.

The exciting thing is that we can break down air resistance in 2.

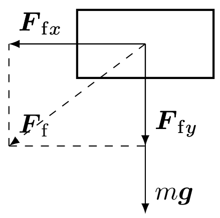

Figure 2: Force Situation

We define air resistance as $\boldsymbol{F}_{\mathrm{f}} = -k\boldsymbol{v}$. So, the residence vertically is $\boldsymbol{F}_{\mathrm{f}y} = -k\boldsymbol{v}_y$, and horizontal direction is the same.

3.2 Vertical Motion

There are 2 parts of the vertical motion, **ascend** and **descend**. However, we can regard them as a directed motion with the help of vectors.

Defining the direction of vertically downward is the positive one, so $\boldsymbol{v}_0$, which is the primal velocity, is negative horizontally and vertically. Meanwhile, the acceleration of gravity, g, is positive vertically.

Via Newton's Second Theorem, we can know the force situation vertically:

$$ma = mg - kv_y,$$

That is to say,

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = mg - kv_y \tag{1}$$

Separate parametric variables and take integrals on both sides.

$$\int \frac{\mathrm{d}v_y}{mg - kv_y} = \int \frac{\mathrm{d}t}{m} \tag{2}$$

Then we can get

$$-\frac{1}{k}\ln\left(mg-kv_y\right)=\frac{t}{m}+C \tag{3}$$

Also bring in the initial velocity (vertical): $v_y\big|_{t=0}=v_0\sin\theta$ into the C, we can get the equation of vertical velocity:

$$v_y=\frac{mg}{k}-\left(\frac{mg}{k}-v_0\sin\theta\right)\mathrm{e}^{-\frac{t}{m}k} \tag{4}$$

Since we have defined the positive direction, we can simply integral the velocity to calculate the value of displacement:

$$\int_0^t v_y\mathrm{d}t=h \tag{5}$$

We can solve this equation by methods such as Newton's method or Newton-like methods. After receiving t, we can get the horizontal displacement.

3.3 Horizontal Motion

The horizontal motion of the shot put is decelerate linear motion. However, the deceleration changes with the velocity.

Through Newton's Second Theorem, we can also get the horizontal force situation:

$$ma=kv_x$$

And the initial velocity (horizontal) is $v_0\cos\theta$. The methodology is consistent with the above, so it will not be repeated. After solving this differential equation, we can get the velocity:

$$v_x=v_0\cos\theta\mathrm{e}^{-\frac{k}{m}t} \tag{6}$$

Then we can use the integral to get the horizontal displacement.

$$\int_0^t v_x\mathrm{d}t=s \tag{7}$$

It is important to note that since the magnitude of the air resistance is related to the velocity, differential equations must be used instead of the ordinary integral method. Since the equations in the vertical direction cannot be solved algebraically and directly, no simple and explicit algebraic relation among parameters above can be given.

So, the trace of the shot put is in the figure 3. That's a bit of an exaggeration.

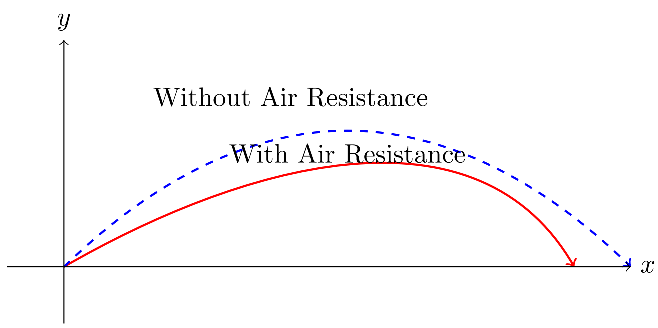

Figure 3: The Trace of the Shot Put with Air Residence

4 Model Analysis

The mathematical model uses differential equations to solve the question.

4.1 Data Brought into the Analysis

Using the formula 4, we can get the vertical velocity of the shot put. Using `Geogebra`, we can get the $v_y - t$ graph in the figure 4. Data is used following the table 4.1.

Parameter	Value	Unit
v_0	-12	m/s
θ	37.0	$^\circ$
h	2.08	m
k	0.00215 [1]	$\mathrm{N \cdot s/m}$
m	4	kg

[1] The value of k is following the formula: $k = \frac{1}{2} C_D A \rho$, where C_D is the drag coefficient, A is the cross-sectional area, and ρ is the air density.
We regard the shot put as a sphere whose r is 0.10m, and the air density is $1.225\mathrm{kg/m^3}$. So, the value of k is $0.00215\mathrm{N \cdot s/m}$.
The value of k is not accurate, but it is enough for the model.

Table 2: Parameters of the Mathematical Model

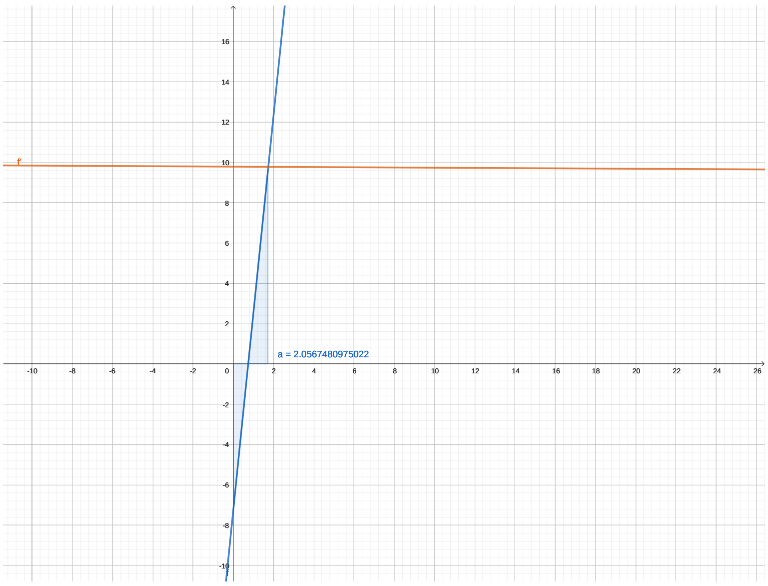

Figure 4: The $v_y - t$ Graph

The blue one is the graph, and the orange one is the derivative of the blue one. It seems that the velocity is decreasing little by little.

When $t \approx 1.76$s, the integral of the velocity, which is the displacement, is 2.08m, which is the throwing height.

Then, we can bring t into the horizontal equation to get the horizontal displacement, which is the score.

4.2 Example: Gong's Shot Put in 2017

We selected Gong Lijiao's shot put in 2017 as an example. The data is in the attachment.

We set the $k = 0.00215$, and received the result of the horizontal displacement with the model, which returns 19.62599m, which is close to the real data.

Through `calculate/newton.py` provided in the attachment, we use `scipy` to calculate the approx value of t, which is 1.856988167816502s.

Then we can get the horizontal displacement, which is 19.62599m, which is close to the real data (1.6% error).

If we add *FB Truck Lean* into the release angle, the error will be reduced to 0.8% (19.78m).

4.3 Features

The mathematical model considers the air residence, which improves the accuracy of the model significantly. Ease of calculation through decomposition of motion.

5 Optimization

The optimization of the shot put can be implemented with the gradient descent method.

Because the air resistance is quite small, and the solution of t needs Newton's Method. So it is hard to implement the algorithm. In this part, we ignore the air resistance temporarily.

5.1 Optimization Formula

We regard impacts as the following:

1. The initial velocity of the ball (negative because of its direction) v,
2. The release angle θ,
3. The release height h,
4. NOTE: Other related parameters (except distance) are not included in the model, we ignore them.

The final formula of the optimization is the following:

$$d = \frac{v_0 \cos\theta \sqrt{v_0^2 \sin^2\theta + 2hg} - v_0 \sin\theta}{g} \tag{8}$$

We can use the gradient descent method to optimize the value of v_0, θ, and h, to get the maximum value of d.

NOTE: The d includes the distance beyond the toe board.

5.2 Optimization Result

Through the code `optimize/grads.py` with the auto differentiation of `PyTorch`, we can get the result of the optimization.

Be caution that the optimization includes the limit, which is list in the table 3 (lower and upper limit depends on the absolute value):

Parameter	Lower Limit	Upper Limit	Optimized Value
v_0	-10.0m/s	-13.5m/s	13.5m/s
θ	0°	60.0°	36.2°
h	1.6m	2.1m	2.1m

Table 3: Optimization Result

And the result distance is 20.25m.

It is easy to find out that the h and the v_0 is the max in the range.

5.3 Extra Ranges

Except the θ, v_0 and h touch the max of the range. If the range is larger, the result will be different.

So we set the h and v_0 bigger to see the result (0.1 each), and the result is following the table 4 (unit: m):

Parameter	**Upper Result**	**Lower Result**	**Upper Diff**	**Lower Diff**
h	20.36m	20.14m	0.11m	−0.11m
v_0	20.52m	19.99m	0.27m	−0.27m

Table 4: Optimization Result Extend

It is easy to find out that the v_0 is more useful in this case.