Mathematical Modeling of Shot Put Throwing Distance

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Abstract

There are multiple impacts on the result of the Shot Put competition, e.g. the speed of the ball, the throwing degrees, height, etc. Different factors can cause different results. This article used the kinetic principle and the mathematics model in the 2017 Athletics World Championships Women's Shot Put Final which includes Strike Speed, Strike Height, Strike Height Relative to Body Height, Distance Beyond Toe, Strike Tilt, etc., and created a model which uses gradient descent algorithm to analyze key points on the result of the Shot Put competition.

Keywords

Shot Put Competition, Gradient Descent, Kinetic Trace, Mathematics Model

Contents

1 Preface

1.1 Introduction of the Scenario

The ideal routine of the throwing trace is a parabola (as projectile motion). However, the trace isn't a parabola because of the air residence.

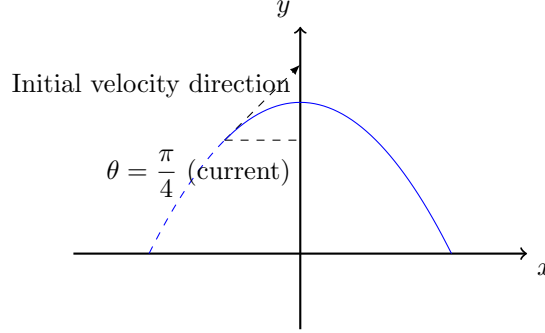Without air residence, the trace will be like in 1.

Figure 1: Ideal trace

However, because of the following reasons,

- Air residence,

- Distortion of the throwing direction,

- Start position,

- …

The trace in 1 is inaccurate.

It is easy to figure out that the release distance (horizontal) is the most important implicit variable, which reflects the overall performance of the candidate. Release velocity (v_0), angle (θ), and release height (h) are the most important independent variable, which determines the final score.

1.2 Introduction to the Structure of the Paper

Through the mathematical model of the throwing distance, which includes v_0, θ, and h, we can through kinetic analysis create the model that returns the ideal distance of the throwing turn. With the help of the differential equation, we can consider the air resistance and evaluate the degree of the impact. Then we can create a final equation to reflect the trace, which is used to adjust and optimize parameters with gradient descent algorithm.

This article is divided into the following parts:

1. In section 2, we introduced variables in the mathematics model and provided some of example values of each.

2. In section 3, we created the mathematical model with kinetic analysis, including cases with air resistance with the help of differential equations.

3. In section 4, we analyzed the model and set examples to check the accuracy.

4. In section 5, we used the gradient descend algorithm to optimize parameters, and then displayed the best parameters in a specific range.

5. In section 6, we evaluated the mathematics model.

6. In section 7, we solve problems provided in B with refers to the content.

2 Parameters

Some parameters are included, while some parameters are ignored.

2.1 Parameters in Kinetic Model

In the kinetic analysis, we don't bring the exact data. Some definitions are following the table 1.

Explanation	Parameter	Example
gravitational acceleration	g	9.787m/s^2
shot put mass	m	4kg
atmospheric drag coefficient	k	0.0215Ns/m
throw speed	$\boldsymbol{v}$	N/A [1]
throwing angle	θ	$37.0°$
throwing height	h	2.08m
tilt angle left and right	φ	$-21°$
horizontal displacement of the shot put [2]	s	19.94m

[1] Actually, the speed is a vector, and the v_x, v_y not only depend on the value of the absolute value but also on the throwing angle. Expressing with complex numbers is also OK, one of the examples is $13.24\text{e}^{\cos 37.0° + \text{i} \sin 37.0°}\text{m/s}$, whose v_y is 7.944m/s, and v_x is 10.592m/s.

[2] including exceeds toe board distance.

Table 1: Parameters of the Mathematical Model

We also assume the following conditions:

1. The velocity is small, so $F = -kv$ should be used instead of $F = -kv^2$.

2. The horizontal velocity is not reduced to 0 due to air resistance.

3. There is no influence of field forces other than the gravity field, which is assumed to be uniformly strong.

2.2 Parameters in the Regression Model

2.3 Ignore

Because some parameters are useless, we ignore the following parameters:

1. Front and rear tilt angles,

2. Strike height relative to body height,

3. Best round,

4. . . .

3 Mathematical Model with Kinetic Analysis

3.1 Force Analysis

During the movement of a shot put ball, it is subjected to 2 forces: gravity and air resistance. The force of gravity is always vertically downward, however, air resistance changes with the direction of velocity.

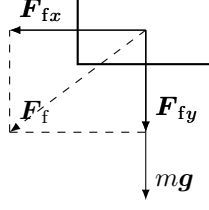The exciting thing is that we can break down air resistance in 2.

Figure 2: Force Situation

We define air resistance as $\boldsymbol{F}_{\mathrm{f}} = -k\boldsymbol{v}$. So, the residence vertically is $\boldsymbol{F}_{\mathrm{f}y} = -k\boldsymbol{v}_y$, and horizontal direction is the same.

3.2 Vertical Motion

There are 2 parts of the vertical motion, **ascend** and **descend**. However, we can regard them as a directed motion with the help of vectors.

3.2.1 With Air Resistance

Defining the direction vertically downward is the positive one, so $\boldsymbol{v}_0$, which is the primal velocity, is negative horizontally and vertically. Meanwhile, the acceleration of gravity, g, is positive vertically.

Via Newton's Second Theorem, we can know the force situation vertically:

$$ma = mg - kv_y,$$

That is to say,

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = mg - kv_y \tag{1}$$

Separate parametric variables and take integrals on both sides.

$$\int \frac{\mathrm{d}v_y}{mg - kv_y} = \int \frac{\mathrm{d}t}{m} \tag{2}$$

Then we can get

$$-\frac{1}{k}\ln\left(mg - kv_y\right) = \frac{t}{m} + C \tag{3}$$

Also bring in the initial velocity (vertical): $v_y\big|_{t=0} = v_0\sin\theta$ into the C, we can get the equation of vertical velocity:

$$v_y = \frac{mg}{k} - \left(\frac{mg}{k} - v_0\sin\theta\right)\mathrm{e}^{-\frac{t}{m}k} \tag{4}$$

Since we have defined the positive direction, we can simply integral the velocity to calculate the value of displacement:

$$\int_0^t v_y\mathrm{d}t = h \tag{5}$$

We can solve this equation by methods such as Newton's method or Newton-like methods. After receiving t, we can get the horizontal displacement.

3.2.2 Without Air Resistance

If there is no air resistance, it is an ideal motion with gravity only. So the acceleration is simply g.

The motion time can be calculated with the height h:

$$h = v_0\sin\theta t + \frac{1}{2}gt^2 \tag{6}$$

Then we can get the horizontal displacement after solving the equation whose result is t.

3.3 Horizontal Motion

The horizontal motion of the shot put is decelerated linear motion. However, the deceleration changes with the velocity.

3.3.1 With Air Resistance

Through Newton's Second Theorem, we can also get the horizontal force situation:

$$ma = kv_x$$

And the initial velocity (horizontal) is $v_0 \cos \theta$. The methodology is consistent with the above, so it will not be repeated. After solving this differential equation, we can get the velocity:

$$v_x = v_0 \cos \theta \mathrm{e}^{-\frac{k}{m}t} \tag{7}$$

Then we can use the integral to get the horizontal displacement.

$$\int_0^t v_x \mathrm{d}t = s \tag{8}$$

It is important to note that since the magnitude of the air resistance is related to the velocity, differential equations must be used instead of the ordinary integral method. Since the equations in the vertical direction cannot be solved algebraically and directly, no simple and explicit algebraic relation among the parameters above can be given.

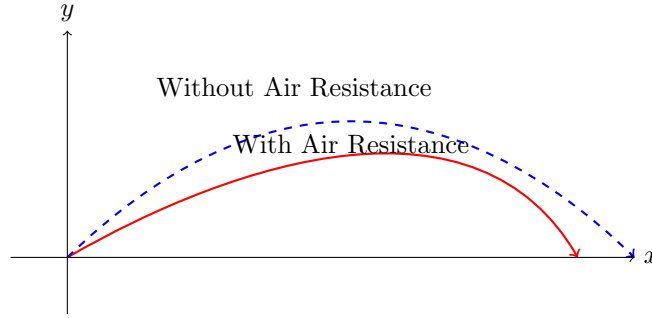So, the trace of the shot put is in the figure 3. That's a bit of an exaggeration.

Figure 3: The Trace of the Shot Put with Air Residence

3.3.2 Without Air Resistance

The horizontal motion without air resistance is a linear motion without any acceleration. When time is t, so the speed is $v_0 \cos \theta$ all the time. So the displacement is:

$$s = v_0 \cos \theta t \tag{9}$$

The full equation is used in the optimization, which doesn't consider the air resistance.

4 Model Analysis

The mathematical model uses differential equations to solve the question.

4.1 Data Brought into the Analysis

Using the formula 4, we can get the vertical velocity of the shot put. Using `Geogebra`, we can get the $v_y - t$ graph in the figure 4. Data is used following the table 4.1.

Parameter	Value	Unit
v_0	-12	m/s
θ	37.0	°
h	2.08	m
k	0.00215 [1]	$N \cdot s/m$
m	4	kg

[1] The value of k is following the formula: $k = \dfrac{1}{2}C_D A \rho$, where C_D is the drag coefficient, A is the cross-sectional area, and ρ is the air density.

We regard the shot put as a sphere whose r is 0.10m, and the air density is 1.225kg/m^3. So, the value of k is $0.00215\text{N} \cdot \text{s/m}$.

The value of k is not accurate, but it is enough for the model.

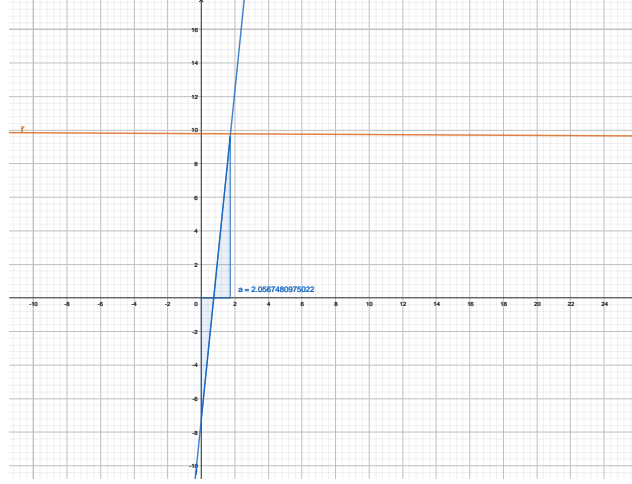Table 2: Parameters of the Mathematical Model

Figure 4: The $v_y - t$ Graph

The blue one is the graph, and the orange one is the derivative of the blue one. It seems that the velocity is decreasing little by little.

When $t \approx 1.76$s, the integral of the velocity, which is the displacement, is 2.08m, which is the throwing height.

Then, we can bring t into the horizontal equation to get the horizontal displacement, which is the score.

4.2 Example: Gong's Shot Put in 2017

We selected Gong Lijiao's shot put in 2017 as an example. The data is in the attachment.

We set the $k = 0.00215$ and received the result of the horizontal displacement with the model, which returns 19.62599m, which is close to the real data.

Through `calculate/newton.py` provided in the attachment, we use `scipy` to calculate the approx value of t, which is 1.856988167816502s.

Then we can get the horizontal displacement, which is 19.62599m, which is close to the real data (1.6% error).

If we add *FB Truck Lean* into the release angle, the error will be reduced to 0.8% (19.78m).

4.3 Features

The mathematical model considers the air residence, which improves the accuracy of the model significantly. Ease of calculation through decomposition of motion.

5 Optimization

The optimization of the shot put can be implemented with the gradient descent method.

Because the air resistance is quite small, the solution of t needs Newton's Method. So it is hard to implement the algorithm. In this part, we ignore the air resistance temporarily.

5.1 Optimization Formula

We regard impacts as the following:

1. The initial velocity of the ball (negative because of its direction) v,

2. The release angle θ,

3. The release height h,

4. NOTE: Other related parameters (except distance) are not included in the model, we ignore them.

The final formula of the optimization is the following:

$$d = \frac{v_0 \cos\theta \sqrt{v_0^2 \sin^2\theta + 2hg} - v_0 \sin\theta}{g} \tag{10}$$

We can use the gradient descent method to optimize the value of v_0, θ, and h, to get the maximum value of d.

NOTE: The d includes the distance beyond the toe board.

5.2 Optimization Result

We can get the optimization result through the code `optimize/grads.py` with the auto differentiation of `PyTorch`.

Be cautious that the optimization includes the limit, which is listed in the table 3 (lower and upper limit depends on the absolute value):

Parameter	Lower Limit	Upper Limit	Optimized Value
v_0	-10.0m/s	-13.5m/s	13.5m/s
θ	$0°$	$60.0°$	$36.2°$
h	1.6m	2.1m	2.1m

Table 3: Optimization Result

And the result distance is 20.25m.

It is easy to find out that the h and the v_0 are the max in the range.

5.3 Extra Ranges

Except the θ, v_0, and h touch the max of the range. If the range is larger, the result will be different.

So we set the h and v_0 bigger to see the result (0.1 each), and the result is following the table 4 (unit: m):

Parameter	Upper Result	Lower Result	Upper Diff	Lower Diff
h	20.36m	20.14m	0.11m	-0.11m
v_0	20.52m	19.99m	0.27m	-0.27m

Table 4: Optimization Result Extend

It is easy to find out that the v_0 is more useful in this case.

6 Evaluation

This model uses the differential equation, gradient descent, etc. algorithms and successfully fits the real situation, which reduces the loss to about 1% (distance). The air resistance is so tiny that we often ignore this factor, and it is subtle whether to use $f = -mv$ or $f = -mv^2$ in this situation.

6.1 Advantages

This model combines mathematics and computer technology which uses `PyTorch`, `scipy`, etc. that can reduce the calculation. We can get benefits from these techniques, which can automatically calculate the differential, and find the best value.

This module evaluated the impact of the tiny factor *air resistance* which can be just ignored in the shot put competition for approx calculations.

The result of the model is good, which is 0.15m per line and 1% among the full data.

6.2 Disadvantages

The model that used the differential equation needs to be solved with Newton's Method, which is hard to optimize. So the training of the target used the equation that doesn't include the air resistance and then brought the result into the equation that considers the air resistance. It's not very rigorous. Fortunately, we've proved that it is the optimal result in the near range, whether air resistance is considered.

The calculation of the model is complex so the author spent about 2 nights solving the differential equation. However, with the help of the computer, we can get the answer easily.

7 Problematic endings

The modeling question raises several issues, and in this one section, the paper will present the introduction.

7.1 Model Creating

The mathematics model is introduced in 3, and you can refer to the section to find the model. We used the data from the IAAF Championship which is contained in the attachments and bibliography part.

7.2 Optimization

The optimization result is declared in the section 5.2. It considered v_0, h, and θ.

8 Acknowledgement

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Appendix

A   Repository

The repository of this paper is in GitHub: `https://github.com/7086cmd/shot-put-essay.git`. Data and programs are included.

B   Origin Questions

The questions are displayed here, which are translated into English from Chinese.

Shot Put (athletics event)

The shot put throw is a familiar sport. In general, the athlete first holds the lead ball in the correct position, then slides with the ball, when the left side of the body is close to perpendicular to the ground for a moment, with the left shoulder as the axis, the right leg is quickly straightened, the body is turned to the direction of the throw, the chest is straightened, the head is raised, and the right shoulder is pushed forward with force, at the same time, the right arm is straightened quickly to push the ball forward and upwards.

1. Please build a suitable mathematical model to study how to throw to get a better result.

2. The following are the strike parameters for the best-throwing rounds of the 12 competitors in the Women's Shot Put final at the 2017 World Championships in Athletics. Analyze and improve your model by comparing your findings to a real-world situation.   *The data is included in [4].*