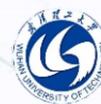


熵的性质 (1)

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熵的性质

熵公式:
$$H(X) = \sum_{k=1}^K p_k \log \frac{1}{p_k} = -\sum_k p_k \log p_k$$

1 对称性:

$$H(p_1, p_2, \dots, p_K) = H(p_{m(1)}, p_{m(2)}, \dots, p_{m(K)})$$

其中 $\{m(1), m(2), \dots, m(K)\}$ 是 $\{1, 2, \dots, K\}$ 的任意置换。

2 可扩展性: 加入零概率事件不会改变熵。

$$H(p_1, p_2, \dots, p_K) = H(p_1, \dots, p_i, 0, p_{i+1}, \dots, p_K) \quad \because 0 \log 0 = 0$$
$$i = 1, 2, \dots, K-1$$

熵的性质

3 非负性:

熵公式:
$$H(X) = \sum_{k=1}^K p_k \log \frac{1}{p_k} = -\sum_k p_k \log p_k$$

$$H(p_1, p_2, \dots, p_K) = H(P) \geq 0$$

确定性概率分布: $P_s = \{1, 0, \dots, 0\}$

$$H(P_s) = 0$$

熵的性质

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强可加性:

$$\begin{aligned} & H(p_1 q_{11}, \dots, p_1 q_{1J}, \dots, p_K q_{K1}, \dots, p_K q_{KJ}) \\ &= H(p_1, p_2, \dots, p_K) + \sum_{k=1}^K p_k H(q_{k1}, \dots, q_{kJ}) \end{aligned}$$

$$X = \{x_1, x_2, \dots, x_K\}, \quad PX = \{p_1, p_2, \dots, p_K\}$$

$$Y = \{y_1, y_2, \dots, y_J\}, \quad PY = \{q_1, q_2, \dots, q_J\}$$

q_{kj} 为转移概率 $q_{kj} = P(y_j | x_k)$

$$0 \leq p_k \leq 1 \quad k = 1, 2, \dots, K \quad \sum_{k=1}^K p_k = 1$$

$$0 \leq q_{kj} \leq 1 \quad j = 1, 2, \dots, J \quad \sum_{j=1}^J q_{kj} = 1 \quad k = 1, 2, \dots, K$$



“强可加性” 证明

定义新函数:

$$L(x) = -x \log x$$

则 $L(xy) = -xy \log(xy) = -xy \log x - xy \log y = yL(x) + xL(y)$

于是

$$H(p_1 q_{11}, \dots, p_1 q_{1J}, \dots, p_K q_{K1}, \dots, p_K q_{KJ})$$

$$= \sum_{k=1}^K \sum_{j=1}^J L(p_k q_{kj}) = \sum_{k=1}^K \sum_{j=1}^J [q_{kj} L(p_k) + p_k L(q_{kj})]$$

$$= \sum_{k=1}^K \left(L(p_k) \sum_{j=1}^J q_{kj} \right) + \sum_{k=1}^K \left(p_k \sum_{j=1}^J L(q_{kj}) \right)$$

$$= H(p_1, p_2, \dots, p_M) + \sum_{k=1}^K p_k H(q_{k1}, q_{k2}, \dots, q_{kJ})$$

$$H(XY) = H(X) + H(Y|X)$$

$$\sum_{j=1}^J q_{kj} = 1$$

$$k = 1, 2, \dots, K$$



“强可加性” 证明



可加性:

$$\begin{aligned} & H(p_1q_1, \dots, p_1q_J, \dots, p_Kq_1, \dots, p_Kq_J) \\ &= H(p_1, p_2, \dots, p_K) + H(q_1, q_2, \dots, q_J) \end{aligned}$$

$$0 \leq p_k \leq 1 \quad k = 1, 2, \dots, K \quad \sum_{k=1}^K p_k = 1$$

$$0 \leq q_j \leq 1 \quad j = 1, 2, \dots, J \quad \sum_{j=1}^J q_j = 1$$

“可加性” 是 “强可加性” 的特殊情况，在 “强可加性” 中，令

$$q_{1j} = q_{2j} = \dots = q_{Kj} = q_j, j = 1, \dots, J$$

就可得出可加性。

»»» “可加性” 证明

强可加性:

$$H(p_1 q_{11}, \dots, p_1 q_{1J}, \dots, p_K q_{K1}, \dots, p_K q_{KJ}) \\ = H(p_1, p_2, \dots, p_K) + \sum_{k=1}^K p_k H(q_{k1}, \dots, q_{kJ})$$

$$0 \leq p_k \leq 1 \quad k = 1, 2, \dots, K \quad \sum_{k=1}^K p_k = 1$$

$$0 \leq q_{kj} \leq 1 \quad j = 1, 2, \dots, J \quad \sum_{j=1}^J q_{kj} = 1 \quad k = 1, 2, \dots, K$$

令 $q_{1j} = q_{2j} = \dots = q_{Kj} = q_j, j = 1, \dots, J$

则
$$H(p_1 q_1, \dots, p_1 q_J, \dots, p_K q_1, \dots, p_K q_J) \\ = H(p_1, p_2, \dots, p_K) + \sum_{k=1}^K p_k H(q_1, \dots, q_J) \\ = H(p_1, p_2, \dots, p_K) + H(q_1, \dots, q_J) \sum_{k=1}^K p_k \\ = H(p_1, p_2, \dots, p_K) + H(q_1, \dots, q_J)$$

$$H(XY) = H(X) + H(Y)$$

感谢观看!

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