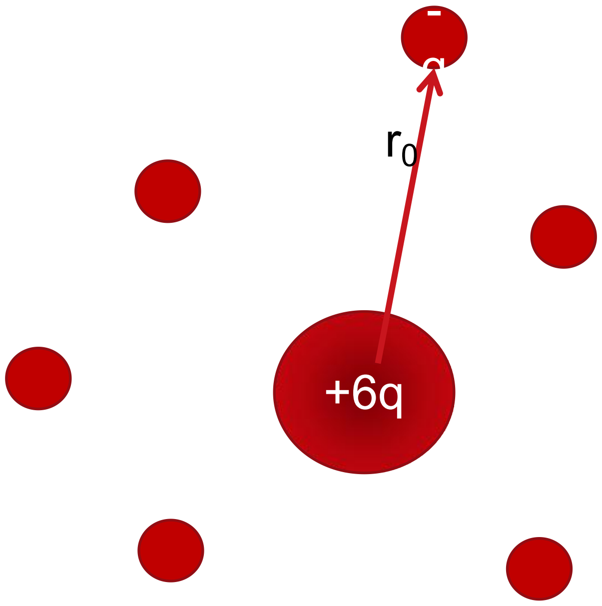
第六章：双态系统

§1.离散能级系统和双态系统

§2.拉比模型动力学

§3. 微波激射

§4. 磁共振

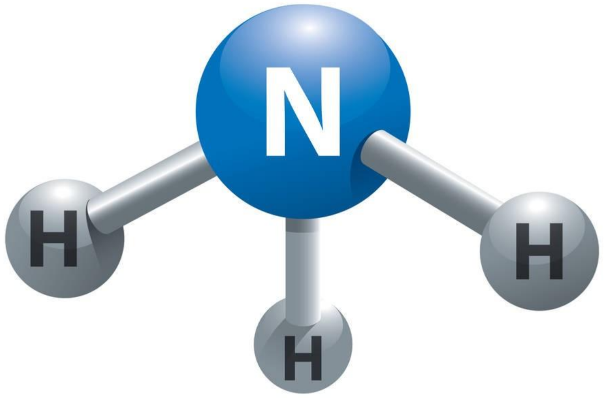


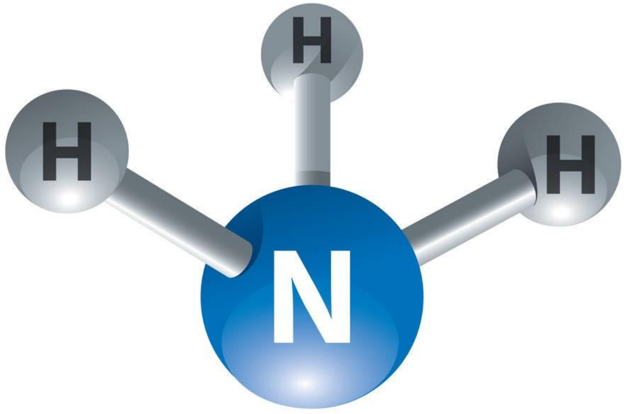
-q
r_0
+6q

6.1.离散能级系统和双态系统

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=\hat{H}\psi(x,t)\rightarrow\begin{cases}i\hbar\dfrac{\partial}{\partial t}\varphi(t)=E\varphi(t)\\ \hat{H}\psi(x)=E\psi(x)\end{cases}$$

$$\rightarrow\psi(x,t)=\psi(x)\varphi(t)$$





$$\rightarrow\Phi(x,t)=C_1\psi_1(x,t)+C_2\psi_2(x,t)$$

C1和C2描述的是两种状态的概率振幅，与之前的c_p类似

薛定锷方程

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=\hat{H}\psi(x,t) \quad \rightarrow \begin{cases} \psi(r,t)=\sum_{p} c(p,t)\psi_{p}(r) \\ c_1(p_1,t)=\int \psi_{p_1}^{*}\psi(r,t)dr \end{cases}$$

$$\psi(x,t)=\sum_{m} a_m(t)u_m(x) \quad \rightarrow i\hbar\frac{\partial}{\partial t}\sum_{m} a_m(t)u_m(x)=\hat{H}\sum_{m} a_m(t)u_m(x)$$

两边左乘 u*$_m$(x)并对 x 积分

$$\rightarrow i\hbar\frac{\partial}{\partial t}u_m^{*}(x)\sum_{m} a_m(t)u_m(x)=u_m^{*}(x)\hat{H}\sum_{m} a_m(t)u_m(x)$$

$$i\hbar\frac{\partial}{\partial t}\sum_{m} a_m(t)\int u_m^{*}(x)u_m(x)dx=\int u_m^{*}(x)\hat{H}u_m(x)dx\sum_{m} a_m(t)$$

$$u_1^{*}(x)\rightarrow i\hbar\frac{\partial}{\partial t}a_1(t)\int u_1^{*}(x)u_1(x)dx=\int u_1^{*}(x)\hat{H}u_1(x)dxa_1(t)$$

$$\begin{cases} u_1^{*}(x)\rightarrow i\hbar\frac{\partial}{\partial t}a_1(t)=\int u_1^{*}(x)\hat{H}u_1(x)dxa_1(t)+\int u_1^{*}(x)\hat{H}u_2(x)dxa_2(t)+\ldots \\ u_2^{*}(x)\rightarrow i\hbar\frac{\partial}{\partial t}a_2(t)=\int u_2^{*}(x)\hat{H}u_1(x)dxa_1(t)+\int u_2^{*}(x)\hat{H}u_2(x)dxa_2(t)+\ldots \\ \ldots. \\ u_m^{*}(x)\rightarrow i\hbar\frac{\partial}{\partial t}a_2(t)=\int u_m^{*}(x)\hat{H}u_1(x)dxa_1(t)+\int u_m^{*}(x)\hat{H}u_m(x)dxa_2(t) \end{cases}$$

$$
\begin{cases}
u_1^*(x) \rightarrow i\hbar \dfrac{\partial}{\partial t} a_1(t) = \int u_1^*(x)\hat{H}u_1(x)dx a_1(t) + \int u_1^*(x)\hat{H}u_2(x)dx a_2(t) + \ldots \\
u_2^*(x) \rightarrow i\hbar \dfrac{\partial}{\partial t} a_2(t) = \int u_2^*(x)\hat{H}u_1(x)dx a_1(t) + \int u_2^*(x)\hat{H}u_2(x)dx a_2(t) + \ldots \\
\ldots. \\
u_m^*(x) \rightarrow i\hbar \dfrac{\partial}{\partial t} a_2(t) = \int u_m^*(x)\hat{H}u_1(x)dx a_1(t) + \int u_m^*(x)\hat{H}u_m(x)dx a_2(\mathrm{t})
\end{cases}
$$

$$
\rightarrow i\hbar \frac{\partial}{\partial t} \sum_m a_m(t) = \int u_m^*(x)\hat{H}u_m(x)dx \sum_m a_m(t)
$$

令 $H_{mn} = \int u_m^*(x)\hat{H}(x, \frac{\hbar}{i}\frac{\partial}{\partial x})u_n(x)dx$

$$
i\hbar \frac{d}{dt}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \ldots \\ a_m(t) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & & H_{1m} \\ H_{21} & H_{22} & & H_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ H_{n1} & H_{n2} & & H_{nm} \end{pmatrix}\begin{pmatrix} a_1(t) \\ a_2(t) \\ \ldots \\ a_m(t) \end{pmatrix}
\qquad
i\hbar \frac{d}{dt} C(t) = HC(t)
$$

为了进一步说明，我们考虑一个量子系统，它的态空间是两维的

$$i\hbar\frac{\partial}{\partial t}C_j(t)=\sum_j H_{ij}C_j(t)\quad\rightarrow i\hbar\frac{\partial}{\partial t}\begin{bmatrix}C_1(t)\\C_2(t)\end{bmatrix}=\begin{vmatrix}H_{11} & H_{12}\\H_{21} & H_{22}\end{vmatrix}\begin{bmatrix}C_1(t)\\C_2(t)\end{bmatrix}$$

$$H_{ij}=\rightarrow\begin{cases}H_{11}=\int u_1(x)\hat{H}u_1(x)\,dx=E_{11}\\H_{12}=\int u_1(x)\hat{H}u_2(x)\,dx=0\\H_{21}=\int u_2(x)\hat{H}u_1(x)\,dx=0\\H_{22}=\int u_2(x)\hat{H}u_2(x)\,dx=E_{22}\end{cases}$$

薛定谔方程+正交归一化

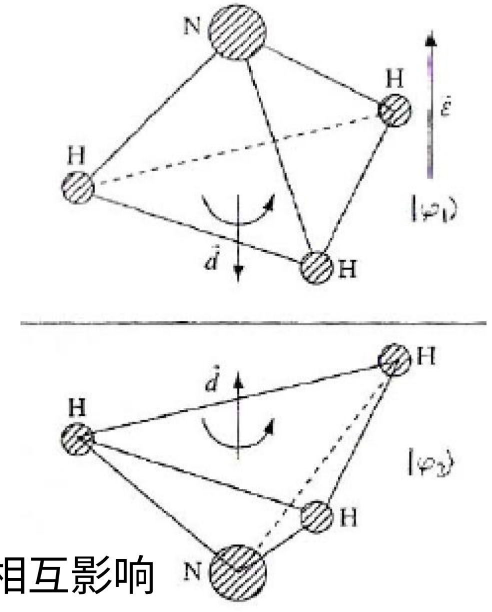
$$\rightarrow i\hbar\frac{\partial}{\partial t}\begin{bmatrix}C_1(t)\\C_2(t)\end{bmatrix}=\begin{vmatrix}H_{11} & H_{12}\\H_{21} & H_{22}\end{vmatrix}\begin{bmatrix}C_1(t)\\C_2(t)\end{bmatrix}=\begin{vmatrix}E_{11} & 0\\0 & E_{22}\end{vmatrix}\begin{bmatrix}C_1(t)\\C_2(t)\end{bmatrix}$$

这里考虑的是两个状态相互独立，互相不影响

$$\begin{cases}i\hbar\dfrac{\partial C_1(t)}{\partial t}=E_{11}C_1(t)\\i\hbar\dfrac{\partial C_2(t)}{\partial t}=E_{22}C_2(t)\end{cases}$$

在能量表象哈密顿矩阵是对角的，薛定谔方程有

$$\begin{cases} i\hbar \dfrac{\partial C_1(t)}{\partial t} = E_{11}C_1(t) \\ i\hbar \dfrac{\partial C_2(t)}{\partial t} = E_{22}C_2(t) \end{cases} \rightarrow \begin{cases} C_1(t) = Ae^{-\frac{iE_{11}t}{\hbar}} \\ C_2(t) = Be^{-\frac{iE_{22}t}{\hbar}} \end{cases}$$



2. 氨分子的双态模型:两种状态存在的概率（1）

如果两个本征态之间存在量子隧道效应

可以理解为两个状态之间相互影响

$$\begin{cases} i\hbar \dfrac{\partial C_1(t)}{\partial t} = E_{11}C_1(t) - AC_2(t) \\ i\hbar \dfrac{\partial C_2(t)}{\partial t} = E_{22}C_2(t) - AC_1(t) \end{cases} \rightarrow \begin{cases} i\hbar \dfrac{\partial\left[C_1(t) + C_2(t)\right]}{\partial t} = \left[E_{11} - A\right]C_1(t) + \left[E_{22} - A\right]C_2(t) \\ i\hbar \dfrac{\partial\left[C_1(t) - C_2(t)\right]}{\partial t} = \left[E_{11} + A\right]C_1(t) - \left[E_{22} + A\right]C_2(t) \end{cases}$$

对于氨分子$E_{11}=E_{22}=E_0$

可以理解为，氨分子的两种状态只是区分了其所处的形态(尖端朝上或朝下)，但是在相互独立的情况下，能量应该是相等的并且不受影响的。

$$\begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=\left[E_{11}-A\right]C_1(t)+\left[E_{22}-A\right]C_2(t) \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=\left[E_{11}+A\right]C_1(t)-\left[E_{22}+A\right]C_2(t) \end{cases} \to \begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=\left[E_0-A\right]\left[C_1(t)+C_2(t)\right] \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=\left[E_0+A\right]\left[C_1(t)-C_2(t)\right] \end{cases}$$

$$\to \begin{cases} \left[C_1(t)+C_2(t)\right]=ae^{-\frac{i(E_0-A)t}{\hbar}} \\ \left[C_1(t)-C_2(t)\right]=be^{-\frac{i(E_0+A)t}{\hbar}} \end{cases} \to \begin{cases} C_1(t)=\dfrac{a}{2}e^{-\frac{i(E_0-A)t}{\hbar}}+\dfrac{b}{2}e^{-\frac{i(E_0+A)t}{\hbar}} \\ C_2(t)=\dfrac{a}{2}e^{-\frac{i(E_0-A)t}{\hbar}}-\dfrac{b}{2}e^{-\frac{i(E_0+A)t}{\hbar}} \end{cases}$$

假如t=0是在1状态，2状态为0 边界条件

$$\to \begin{cases} C_1(0)=\dfrac{a}{2}e^{-\frac{i(E_0-A)0}{\hbar}}+\dfrac{b}{2}e^{-\frac{i(E_0+A)0}{\hbar}}=\dfrac{a+b}{2}=1 \\ C_2(0)=\dfrac{a}{2}e^{-\frac{i(E_0-A)0}{\hbar}}-\dfrac{b}{2}e^{-\frac{i(E_0+A)0}{\hbar}}=\dfrac{a-b}{2}=0 \end{cases} \to \{a=b=1$$

$$\rightarrow \left\{ a = b = 1 \right.$$

$$\rightarrow \begin{cases} C_1(t) = \dfrac{a}{2} e^{-\frac{i(E_0 - A)t}{\hbar}} + \dfrac{b}{2} e^{-\frac{i(E_0 + A)t}{\hbar}} = \dfrac{1}{2} e^{-\frac{iE_0 t}{\hbar}} \left(e^{\frac{iAt}{\hbar}} + e^{-\frac{iAt}{\hbar}} \right) \\ C_2(t) = \dfrac{a}{2} e^{-\frac{i(E_0 - A)t}{\hbar}} - \dfrac{b}{2} e^{-\frac{i(E_0 + A)t}{\hbar}} = \dfrac{1}{2} e^{-\frac{iE_0 t}{\hbar}} \left(e^{\frac{iAt}{\hbar}} - e^{-\frac{iAt}{\hbar}} \right) \end{cases}$$

$$\rightarrow \begin{cases} C_1(t) = \dfrac{1}{2} e^{-\frac{iE_0 t}{\hbar}} \left(e^{\frac{iAt}{\hbar}} + e^{-\frac{iAt}{\hbar}} \right) = e^{-\frac{iE_0 t}{\hbar}} \cos \dfrac{At}{\hbar} \\ C_2(t) = \dfrac{1}{2} e^{-\frac{iE_0 t}{\hbar}} \left(e^{\frac{iAt}{\hbar}} - e^{-\frac{iAt}{\hbar}} \right) = ie^{-\frac{iE_0 t}{\hbar}} \sin \dfrac{At}{\hbar} \end{cases}$$

则两种状态存在的概率 $\rightarrow \begin{cases} P_1 = C_1(t) C_1(t)^* = \cos^2 \dfrac{At}{\hbar} \\ P_2 = C_2(t) C_2(t)^* = \sin^2 \dfrac{At}{\hbar} \end{cases}$

2. 氨分子的双态模型:两种状态对应的能级（2）

将之前的解代入原来的薛定谔方程

$$\begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=\left[E_0-A\right]\left[C_1(t)+C_2(t)\right] \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=\left[E_0+A\right]\left[C_1(t)-C_2(t)\right] \end{cases} \rightarrow \frac{i\hbar\partial}{\partial t}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}=\begin{vmatrix} E_0-A & 0 \\ 0 & E_0+A \end{vmatrix}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}$$

$$\rightarrow \frac{i\hbar\partial}{\partial t}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}=\begin{vmatrix} E_0-A & 0 \\ 0 & E_0+A \end{vmatrix}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}$$

$$\begin{cases} \varphi_1(t)=C_1(t)+C_2(t) \\ \varphi_2(t)=C_1(t)-C_2(t) \end{cases} \rightarrow \frac{i\hbar\partial}{\partial t}\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}=\begin{vmatrix} E_0-A & 0 \\ 0 & E_0+A \end{vmatrix}\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}$$

说明存在双重能级

与波尔的理论:相邻两能级能量差为hw相符合

$$\rightarrow \begin{cases} E_1=E_0-A \\ E_2=E_0+A \end{cases} \quad \rightarrow E_2-E_1=2A \quad \rightarrow E_2-E_1=2A=\hbar\omega_0$$

6.2. 微波激射

1.氨分子在静电场中：对应波函数能量本征值

$$\hat{H}\psi(x)=E\psi(x) \qquad \hat{H}=-\frac{\hbar^2}{2m}\frac{\partial}{\partial r^2}$$

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial C_1(t)}{\partial t}=E_{11}C_1(t) \\ i\hbar\dfrac{\partial C_2(t)}{\partial t}=E_{22}C_2(t) \end{cases} \qquad \rightarrow\begin{cases} i\hbar\dfrac{\partial C_1(t)}{\partial t}=E_{11}C_1(t)-AC_2(t) \\ i\hbar\dfrac{\partial C_2(t)}{\partial t}=E_{22}C_2(t)-AC_1(t) \end{cases}$$

注意这里的符号要相反，因为电偶极矩方向变化了

ε 对应的电场大小

$$\hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}+\varepsilon p(\text{电偶极矩}) \qquad \rightarrow\begin{cases} i\hbar\dfrac{\partial C_1(t)}{\partial t}=(E_{11}-\varepsilon p)C_1(t)-AC_2(t) \\ i\hbar\dfrac{\partial C_2(t)}{\partial t}=(E_{22}+\varepsilon p)C_2(t)-AC_1(t) \end{cases}$$

$$E_{11}=E_{22}=E_0\rightarrow\begin{cases} i\hbar\dfrac{\partial[C_1(t)+C_2(t)]}{\partial t}=(E_0-A)[C_1(t)+C_2(t)]-\varepsilon p[C_1(t)-C_2(t)] \\ i\hbar\dfrac{\partial[C_1(t)-C_2(t)]}{\partial t}=(E_0+A)[C_1(t)-C_2(t)]-\varepsilon p[C_1(t)+C_2(t)] \end{cases}$$

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=(E_0-A)\left[C_1(t)+C_2(t)\right]-\varepsilon p\left[C_1(t)-C_2(t)\right] \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=(E_0+A)\left[C_1(t)-C_2(t)\right]-\varepsilon p\left[C_1(t)+C_2(t)\right] \end{cases}$$

$$\rightarrow\begin{bmatrix} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t} \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t} \end{bmatrix}=\begin{bmatrix} (E_0-A) & -\varepsilon p \\ -\varepsilon p & (E_0+A) \end{bmatrix}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}$$

这里的正交归一化基指的是：
对于两个相互独立的波函数，
其归一化之后的本征值矩阵一定
是一个对角矩阵(但是不一定是单位阵的
E倍)

正交归一化基 $\rightarrow\begin{bmatrix} (E_0-A) & -\varepsilon p \\ -\varepsilon p & (E_0+A) \end{bmatrix}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}=E\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}$

$$\rightarrow\begin{bmatrix} (E_0-A)-E & -\varepsilon p \\ -\varepsilon p & (E_0+A)-E \end{bmatrix}\begin{bmatrix} C_1(t)+C_2(t) \\ C_1(t)-C_2(t) \end{bmatrix}=0$$

$$\rightarrow\begin{bmatrix}(E_0-A)-E & -\varepsilon p\\ -\varepsilon p & (E_0+A)-E\end{bmatrix}\begin{bmatrix}C_1(t)+C_2(t)\\ C_1(t)-C_2(t)\end{bmatrix}=0$$

$$\rightarrow\left[(E_0-A)-E\right]\left[(E_0+A)-E\right]-(\varepsilon p)^2=0$$

$$\rightarrow E_0^2-A^2-E_0E-EA-E_0E+EA+E^2-(\varepsilon p)^2=0$$

$$\rightarrow E_0^2-A^2-2E_0E+E^2-(\varepsilon p)^2=0$$

$$\rightarrow E=\frac{2E_0\pm\sqrt{4E_0^2-4\left(E_0^2-A^2-(\varepsilon p)^2\right)}}{2}=E_0\pm\sqrt{A^2+(\varepsilon p)^2}$$

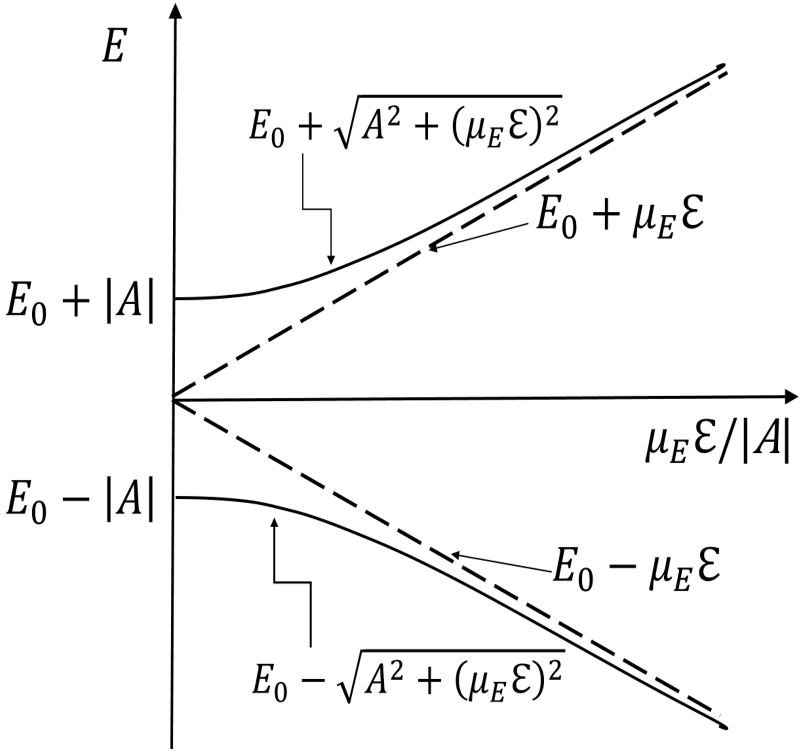
$$E = E_0 \pm \sqrt{A^2 + (\varepsilon p)^2}$$

当 $\varepsilon p \ll A$

$$\text{对应能量的本征值} \rightarrow E_{1,2} = E_0 \pm A + \frac{(\varepsilon p)^2}{2A^2}$$

当电场非常大时，EE趋向与电场成正比，

$$\text{对应能量的本征值} \rightarrow E_{1,2} = E_0 \pm \varepsilon p$$



能量的本征值：$E=E_0\pm\sqrt{A^2+(\varepsilon p)^2}$ 对应的波函数为：：

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=\left(E_0+\sqrt{A^2+(\varepsilon p)^2}\right)\left[C_1(t)+C_2(t)\right] \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=\left(E_0-\sqrt{A^2+(\varepsilon p)^2}\right)\left[C_1(t)-C_2(t)\right] \end{cases}$$

$$\rightarrow\begin{cases} \left[C_1(t)+C_2(t)\right]=e^{-i\frac{\left[E_0+\sqrt{A^2+(\varepsilon p)^2}\right]t}{\hbar}} \\ \left[C_1(t)-C_2(t)\right]=e^{-i\frac{\left[E_0-\sqrt{A^2+(\varepsilon p)^2}\right]t}{\hbar}} \end{cases}$$

$$\rightarrow\begin{cases} C_1(t)=\dfrac{1}{2}e^{-\frac{iE_0t}{\hbar}}\left(e^{-\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}}+e^{+\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}}\right) \\ C_2(t)=\dfrac{1}{2}e^{-\frac{iE_0t}{\hbar}}\left(e^{-\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}}-e^{+\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}}\right) \end{cases}$$

$$\rightarrow \begin{cases} C_1(t) = \frac{1}{2} e^{-\frac{iE_0 t}{\hbar}} \left(e^{-\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}} + e^{+\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}} \right) \\ C_2(t) = \frac{1}{2} e^{-\frac{iE_0 t}{\hbar}} \left(e^{-\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}} - e^{+\frac{i\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar}} \right) \end{cases}$$

$$\rightarrow \begin{cases} C_1(t) = e^{-\frac{iE_0 t}{\hbar}} \cos \frac{\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar} \\ C_2(t) = -ie^{-\frac{iE_0 t}{\hbar}} \sin \frac{\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar} \end{cases}$$

氨分子在外场中两种状态的概率$\rightarrow \begin{cases} P_1 = C_1(t) C_1(t)^* = \cos^2 \frac{\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar} \\ P_2 = C_2(t) C_2(t)^* = \sin^2 \frac{\sqrt{A^2+(\varepsilon p)^2}\,t}{\hbar} \end{cases}$

2.交变电场中的氨分子的状态和对应能级-拉比模型动力学的典型应用

$$\begin{bmatrix}(E_0-A) & -\varepsilon p \\ -\varepsilon p & (E_0+A)\end{bmatrix}\begin{bmatrix}C_1(t)+C_2(t) \\ C_1(t)-C_2(t)\end{bmatrix}=E\begin{bmatrix}C_1(t)+C_2(t) \\ C_1(t)-C_2(t)\end{bmatrix}$$

$$\varepsilon_{1,2}=2\varepsilon_0\cos\omega t=\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)$$

$$\rightarrow\begin{bmatrix}(E_0-A) & -\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p \\ -\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p & (E_0+A)\end{bmatrix}\begin{bmatrix}C_1(t)+C_2(t) \\ C_1(t)-C_2(t)\end{bmatrix}=E\begin{bmatrix}C_1(t)+C_2(t) \\ C_1(t)-C_2(t)\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}(E_0-A)-E & -\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p \\ -\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p & (E_0+A)-E\end{bmatrix}\begin{bmatrix}C_1(t)+C_2(t) \\ C_1(t)-C_2(t)\end{bmatrix}=0$$

$$\rightarrow\begin{bmatrix}(E_0-A)-E & -\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p \\ -\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p & (E_0+A)-E\end{bmatrix}=0 \qquad \rightarrow E_{1,2}=E_0\pm\sqrt{A^2+\left(\frac{\varepsilon_0}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)p\right)^2}$$

交变电场中的氨分子的状态

交变电场中的氨分子的状态

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial C_1(t)}{\partial t}=(E_0-\varepsilon p)C_1(t)-AC_2(t) \\ i\hbar\dfrac{\partial C_2(t)}{\partial t}=(E_0+\varepsilon p)C_2(t)-AC_1(t) \end{cases}$$

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=(E_0-A)\left[C_1(t)+C_2(t)\right]-\varepsilon p\left[C_1(t)-C_2(t)\right] \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=(E_0+A)\left[C_1(t)-C_2(t)\right]-\varepsilon p\left[C_1(t)+C_2(t)\right] \end{cases}$$

这里的C_+和C_-表示的都是氨分子可能出现的状态，即也是可能的本征态

$$\rightarrow\begin{cases} C_+=\left[C_1(t)+C_2(t)\right] \\ C_-=\left[C_1(t)-C_2(t)\right] \end{cases}\rightarrow\begin{cases} i\hbar\dfrac{\partial C_+(t)}{\partial t}=(E_0-A)C_+(t)-\varepsilon pC_-(t) \\ i\hbar\dfrac{\partial C_-(t)}{\partial t}=(E_0+A)C_-(t)-\varepsilon pC_+(t) \end{cases}$$

$$\varepsilon=2\varepsilon_0\cos\omega t=\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)\rightarrow\begin{cases} i\hbar\dfrac{\partial C_+(t)}{\partial t}=(E_0-A)C_+(t)-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)pC_-(t) \\ i\hbar\dfrac{\partial C_-(t)}{\partial t}=(E_0+A)C_-(t)-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)pC_+(t) \end{cases}$$

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial C_+(t)}{\partial t}=(E_0-A)C_+(t)-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)pC_-(t)\\ i\hbar\dfrac{\partial C_-(t)}{\partial t}=(E_0+A)C_-(t)-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)pC_+(t)\end{cases}$$

试探解 $\rightarrow\begin{cases} C_+(t)=\gamma_1 e^{-\frac{i(E_0+A)t}{\hbar}}\\ C_-(t)=\gamma_2 e^{-\frac{i(E_0-A)t}{\hbar}}\end{cases}$

此处的两个系数是用来表征对应状态出现的概率

$$\rightarrow\begin{cases}(E_0-A)e^{-\frac{i(E_0+A)t}{\hbar}}\gamma_1+i\hbar e^{-\frac{i(E_0+A)t}{\hbar}}\dfrac{\partial\gamma_1}{\partial t}=(E_0-A)\gamma_1 e^{-\frac{i(E_0+A)t}{\hbar}}-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_2 e^{-\frac{i(E_0-A)t}{\hbar}}\\ (E_0+A)e^{-\frac{i(E_0-A)t}{\hbar}}\gamma_2+i\hbar e^{-\frac{i(E_0-A)t}{\hbar}}\dfrac{\partial\gamma_2}{\partial t}=(E_0+A)\gamma_2 e^{-\frac{i(E_0-A)t}{\hbar}}-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_1 e^{-\frac{i(E_0+A)t}{\hbar}}\end{cases}$$

化简 $\rightarrow\begin{cases} i\hbar e^{-\frac{i(E_0+A)t}{\hbar}}\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_2 e^{-\frac{i(E_0-A)t}{\hbar}}\\ i\hbar e^{-\frac{i(E_0-A)t}{\hbar}}\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_1 e^{-\frac{i(E_0+A)t}{\hbar}}\end{cases}\rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_2 e^{\frac{i2At}{\hbar}}\\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_1 e^{-\frac{i2At}{\hbar}}\end{cases}$

利用 $E_2-E_1=(E_0+A)-(E_0-A)=2A=\hbar\omega_0$

相邻2个能级的能量差

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_2 e^{i\omega_0 t}\\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_1 e^{-i\omega_0 t}\end{cases}$$

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_2 e^{i\omega_0 t} \\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0\left(e^{i\omega t}+e^{-i\omega t}\right)p\gamma_1 e^{-i\omega_0 t} \end{cases} \Rightarrow\rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0\left(e^{i(\omega+\omega_0)t}+e^{-i(\omega-\omega_0)t}\right)p\gamma_2 \\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0\left(e^{i(\omega-\omega_0)t}+e^{-i(\omega+\omega_0)t}\right)p\gamma_1 \end{cases}$$

$\omega+\omega_0 \gg \omega-\omega_0$，前者对应项平均值近似为0 $\rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0 p e^{-i(\omega-\omega_0)t}\gamma_2 \\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0 p e^{i(\omega-\omega_0)t}\gamma_1 \end{cases}$

可以类比于简谐振动进行考虑，即在外加频率与故有频率相差很小的时候可能共振，且此时能量应该最大。

第一种情况：$\omega-\omega_0=0 \rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0 p\gamma_2 \\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0 p\gamma_1 \end{cases} \rightarrow\begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0 p\gamma_2 \\ -i\hbar\dfrac{\partial\gamma_2}{\varepsilon_0 p\partial t}=\gamma_1 \end{cases}$

刚好共振

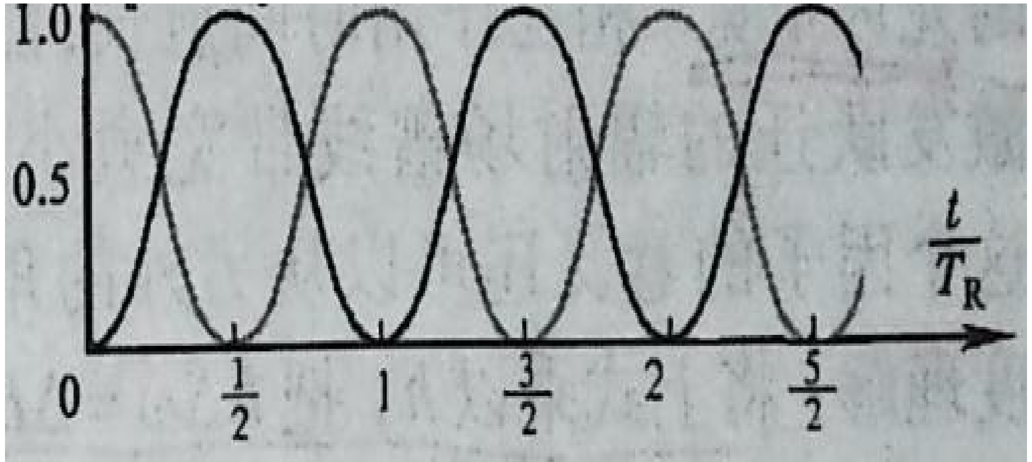
$$\rightarrow\left\{ i\hbar\frac{\partial}{\partial t}\left(-i\hbar\frac{\partial\gamma_2}{\varepsilon_0 p\partial t}\right)=-\varepsilon_0 p\gamma_2 \Rightarrow\rightarrow\left\{ \frac{\partial^2\gamma_2}{\partial t^2}=-\left(\frac{\varepsilon_0 p}{\hbar}\right)^2\gamma_2 \right.\right.$$

$$\rightarrow \left\{ \frac{\partial^2 \gamma_2}{\partial t^2} = -\left(\frac{\varepsilon_0 p}{\hbar}\right)^2 \gamma_2 \Rightarrow \rightarrow \left\{ \frac{\partial^2 \gamma_2}{\partial t^2} + \left(\frac{\varepsilon_0 p}{\hbar}\right)^2 \gamma_2 = 0 \right.\right. \qquad \rightarrow \left\{ \lambda^2 + \left(\frac{\varepsilon_0 p}{\hbar}\right)^2 \lambda = 0 \right.$$

$$\rightarrow \begin{cases} \lambda_1 = i\dfrac{\varepsilon_0 p}{\hbar} \\ \lambda_2 = -i\dfrac{\varepsilon_0 p}{\hbar} \end{cases} \text{通解} \rightarrow \begin{cases} \gamma_1 = a\cos\dfrac{\varepsilon_0 p}{\hbar}t + b\sin\dfrac{\varepsilon_0 p}{\hbar}t \\ \gamma_2 = ib\cos\dfrac{\varepsilon_0 p}{\hbar}t - ia\sin\dfrac{\varepsilon_0 p}{\hbar}t \end{cases}$$

$$t = 0 \rightarrow \begin{cases} \gamma_1 = a\cos\dfrac{\varepsilon_0 p0}{2\hbar} + b\sin\dfrac{\varepsilon_0 p0}{2\hbar} = 1 \\ \gamma_2 = -ib\cos\dfrac{\varepsilon_0 p0}{2\hbar} - ia\sin\dfrac{\varepsilon_0 p0}{2\hbar} = 0 \end{cases} \qquad \rightarrow \begin{cases} a = 1 \\ b = 0 \end{cases}$$

$$\begin{cases} P_1 = (\gamma_1)^2 = \cos^2\dfrac{\varepsilon_0 pt}{2\hbar} \\ P_2 = (\gamma_2)^2 = \sin^2\dfrac{\varepsilon_0 pt}{2\hbar} \end{cases}$$



$\omega+\omega_0 \gg \omega-\omega_0$,前者对应项平均值近似为0 $\rightarrow \begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0 p e^{-i(\omega-\omega_0)t}\gamma_2 \\ i\hbar\dfrac{\partial\gamma_2}{\partial t}=-\varepsilon_0 p e^{i(\omega-\omega_0)t}\gamma_1 \end{cases}$

第二种情况:$(\omega-\omega_0)\neq 0$

$$\rightarrow \begin{cases} i\hbar\dfrac{\partial\gamma_1}{\partial t}=-\varepsilon_0 p e^{-i(\omega-\omega_0)t}\gamma_2 \\ \gamma_1=-\dfrac{i\hbar}{\varepsilon_0 p}\dfrac{\partial\gamma_2}{\partial t}e^{-i(\omega-\omega_0)t} \end{cases} \quad \rightarrow \left\{ i\hbar\frac{\partial}{\partial t}\left[-\frac{i\hbar}{\varepsilon_0 p}\frac{\partial\gamma_2}{\partial t}e^{-i(\omega-\omega_0)t}\right]=-\varepsilon_0 p e^{-i(\omega-\omega_0)t}\gamma_2 \right.$$

$$\rightarrow \left\{ \frac{\hbar^2}{\varepsilon_0 p}\frac{\partial^2\gamma_2}{\partial t^2}e^{-i(\omega-\omega_0)t}-\frac{\hbar^2 i(\omega-\omega_0)}{\varepsilon_0 p}\frac{\partial\gamma_2}{\partial t}e^{-i(\omega-\omega_0)t}=-\varepsilon_0 p e^{-i(\omega-\omega_0)t}\gamma_2 \right.$$

$$\rightarrow \left\{ \frac{\partial^2\gamma_2}{\partial t^2}-i(\omega-\omega_0)\frac{\partial\gamma_2}{\partial t}+\left(\frac{\varepsilon_0 p}{\hbar}\right)^2\gamma_2=0 \right.$$

$$\to \left\{ \frac{\partial^2 \gamma_2}{\partial t^2} - i(\omega - \omega_0)\frac{\partial \gamma_2}{\partial t} + \left(\frac{\varepsilon_0 p}{\hbar}\right)^2 \gamma_2 = 0 \right. \qquad \to \left\{ \lambda^2 - i(\omega - \omega_0)\lambda + \frac{(\varepsilon_0 p)^2}{\hbar^2} = 0 \right.$$

$$\to \lambda_{1,2} = i\frac{(\omega - \omega_0) \pm \sqrt{(\omega - \omega_0)^2 + 4\frac{(\varepsilon_0 p)^2}{\hbar^2}}}{2}$$

令 $\to \omega_{\mathrm{r}} = \sqrt{(\omega - \omega_0)^2 + \frac{4(\varepsilon_0 p)^2}{\hbar^2}} \to$ 定义为拉比振荡频率

$$\to \lambda_{1,2} = i\frac{(\omega - \omega_0) \pm \omega_r}{2} \qquad \gamma_2 = A\left(e^{i\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right]t} - e^{i\left(\frac{(\omega-\omega_0)-\omega_r}{2}\right)t} \right)$$

$$\gamma_2 = A\left(e^{i\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right]t} - e^{i\left(\frac{(\omega-\omega_0)-\omega_r}{2}\right)t} \right)$$

$t = 0 \to \gamma_2 = A\left(e^{i\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right]t} - e^{i\left(\frac{(\omega-\omega_0)-\omega_r}{2}\right)t} \right) = 0 \to$ 导致 $\gamma_1 = 1$

$$\gamma_2 = A\left(e^{i\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right]t} - e^{i\left(\frac{(\omega-\omega_0)-\omega_r}{2}\right)t} \right)$$

$$\rightarrow \frac{d\gamma_2}{dt} = A\left(\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right] e^{i\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right]t} - \left[\frac{(\omega-\omega_0)-\omega_r}{2}\right] e^{i\left[\frac{(\omega-\omega_0)-\omega_r}{2}\right]t} \right)$$

$$t = 0 \rightarrow \frac{d\gamma_2}{dt} = A\left(\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right] - \left[\frac{(\omega-\omega_0)-\omega_r}{2}\right] \right) = A\,\omega_r$$

$$t = 0 \begin{cases} \gamma_1 = 1 \\ \dfrac{\partial\gamma_2}{\partial t} = A\omega_r \end{cases} \rightarrow i\hbar\frac{\partial\gamma_2}{\partial t} = -\varepsilon_0 p e^{i(\omega-\omega_0)t}\gamma_1 \rightarrow A = -\frac{i\varepsilon_0 p}{\hbar\omega_r}$$

$$\rightarrow \gamma_2 = A\left(e^{i\left[\frac{(\omega-\omega_0)+\omega_r}{2}\right]t} - e^{i\left(\frac{(\omega-\omega_0)-\omega_r}{2}\right)t}\right) = \frac{2\varepsilon_0 p}{\hbar\omega_r} e^{i\frac{(\omega-\omega_0)t}{2}} \sin\frac{\omega_r t}{2}$$

跃迁几率 $\rightarrow P = \gamma_2\gamma_2 * = \left(\frac{2\varepsilon_0 p}{\hbar\omega_r}\right)^2 \sin^2\frac{\omega_r t}{2}$

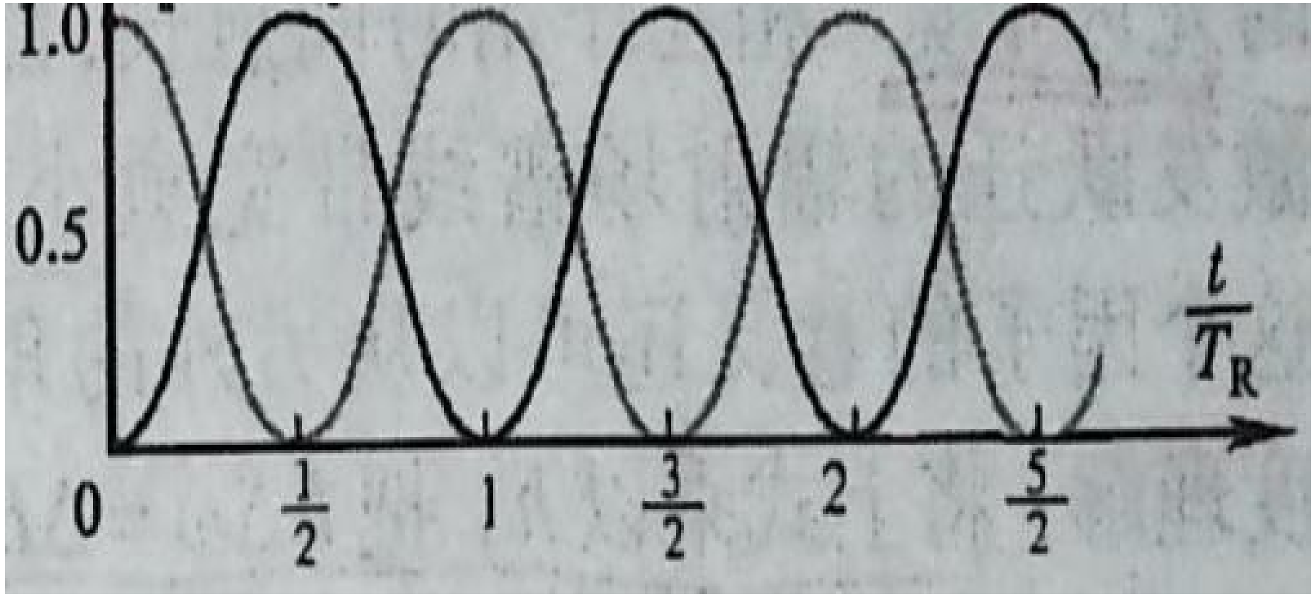
这里是给定的初始状态为状态1，所以跃迁几率说的是从状态1变为状态2的概率，即可以理解为氨分子在初始时刻有多大的概率会以状态2的形式出现

跃迁几率 $\rightarrow P = \gamma_2\gamma_2 * = \left(\frac{2\varepsilon_0 p}{\hbar\omega_r}\right)^2 \sin^2\frac{\omega_r t}{2}$

基态几率 $\rightarrow P_0 = 1 - P = 1 - \left(\frac{2\varepsilon_0 p}{\hbar\omega_r}\right)^2 \sin^2\frac{\omega_r t}{2}$

$$跃迁几率 \to P = \gamma_2 \gamma_2{}^* = \left(\frac{2\varepsilon_0 p}{\hbar \omega_r} \right)^2 \sin^2 \frac{\omega_r t}{2}$$

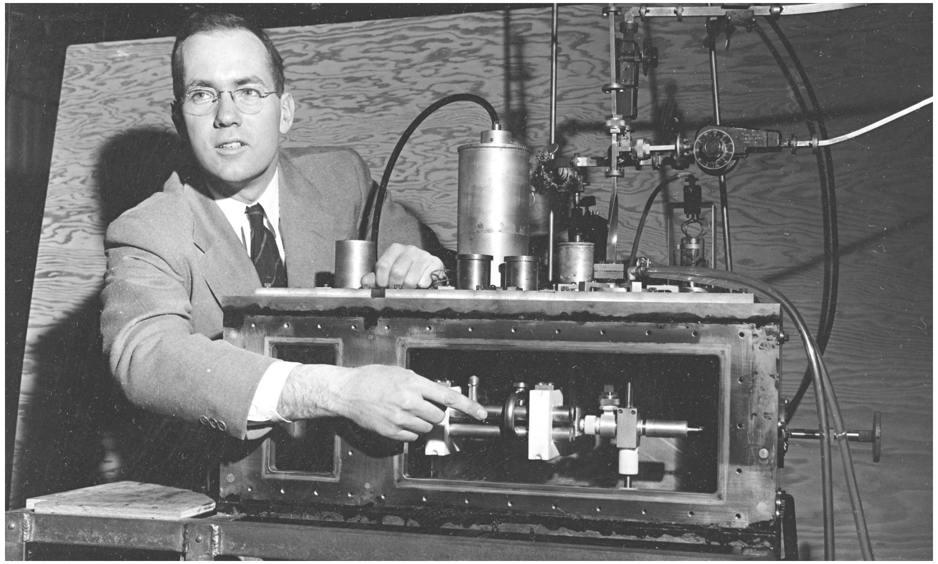
$$基态几率 \to P_0 = 1 - P = 1 - \left(\frac{2\varepsilon_0 p}{\hbar \omega_r} \right)^2 \sin^2 \frac{\omega_r t}{2}$$



3. 微波激射

氨分子在振荡过程中，会放出频率为24GHz的电磁波。这种电磁波的波长为1.25厘米，属于微波范围。假设氨分子占据两个不同能级中的一个能级，两能级差等于波长1.25厘米的光子的能量。若氨分子从高能级跃迁到低能级时，就会发射出上述波长的光子。反之，若处于低能级的分子吸收了这一波长的光子，便能跃迁至高能级。1953年，美国物理学家汤斯(Charles Townes)研制出一种方法，获得高能级的氨分子，再利用适当波长的微波光子去激励它们。只要有少量的光子射入，便能放射出大量相同的光子，也就是相当于入射的微波被放大了许多倍。这个过程被称为微波激射放大，这种仪器被称为微波激射器。

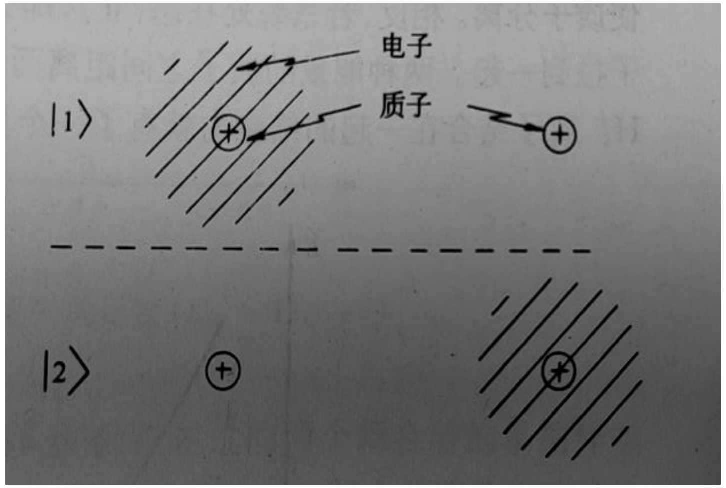
3. 微波激射

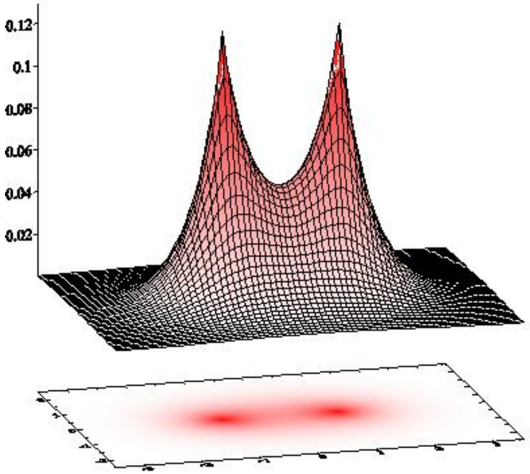


查尔斯•汤斯与氨束微波激射器，该设备的外侧板已被移走以显示出其内部结构

汤斯、巴索夫(Nikolai Basov)和普罗霍洛夫(Aleksandr Prokhorov)分享了1964年诺贝尔物理奖

6.3. 氢分子离子的双态模型





$$\begin{cases} i\hbar \dfrac{\partial C_1(t)}{\partial t} = E_{11} C_1(t) \\ i\hbar \dfrac{\partial C_2(t)}{\partial t} = E_{22} C_2(t) \end{cases}$$

$$\begin{cases} i\hbar \dfrac{\partial C_1(t)}{\partial t} = E_{11} C_1(t) \\ i\hbar \dfrac{\partial C_2(t)}{\partial t} = E_{22} C_2(t) \end{cases} \rightarrow \begin{cases} i\hbar \dfrac{\partial C_1(t)}{\partial t} = E_{11} C_1(t) + E_{12} C_2(t) \\ i\hbar \dfrac{\partial C_2(t)}{\partial t} = E_{21} C_1(t) + E_{22} C_2(t) \end{cases}$$

$$\rightarrow \begin{cases} i\hbar \dfrac{\partial \left[C_1(t) + C_2(t) \right]}{\partial t} = E_{11} C_1(t) + E_{12} C_2(t) + E_{21} C_1(t) + E_{22} C_2(t) \\ i\hbar \dfrac{\partial \left[C_1(t) - C_2(t) \right]}{\partial t} = E_{21} C_1(t) + E_{22} C_2(t) - E_{11} C_1(t) - E_{12} C_2(t) \end{cases}$$

$$E_{12} = E_{21} = U \rightarrow \begin{cases} i\hbar \dfrac{\partial \left[C_1(t) + C_2(t) \right]}{\partial t} = E_{11} C_1(t) + U C_2(t) + U C_1(t) + E_{22} C_2(t) \\ i\hbar \dfrac{\partial \left[C_1(t) - C_2(t) \right]}{\partial t} = U C_1(t) + E_{22} C_2(t) - E_{11} C_1(t) - U C_2(t) \end{cases}$$

$$\begin{cases} E_{11} = E_0 - \Delta \\ E_{22} = E_0 + \Delta \end{cases} \rightarrow \begin{cases} i\hbar \dfrac{\partial \left[C_1(t) + C_2(t) \right]}{\partial t} = (E_0 - \Delta) C_1(t) + U C_2(t) + U C_1(t) + (E_0 + \Delta) C_2(t) \\ i\hbar \dfrac{\partial \left[C_1(t) - C_2(t) \right]}{\partial t} = U C_1(t) + (E_0 + \Delta) C_2(t) - (E_0 - \Delta) C_1(t) - U C_2(t) \end{cases}$$

$$\begin{cases} E_{11} = E_0 - \Delta \\ E_{22} = E_0 + \Delta \end{cases} \rightarrow \begin{cases} i\hbar \dfrac{\partial\left[C_1(t) + C_2(t)\right]}{\partial t} = \left(E_0 - \Delta\right)C_1(t) + UC_2(t) + UC_1(t) + \left(E_0 + \Delta\right)C_2(t) \\ i\hbar \dfrac{\partial\left[C_1(t) - C_2(t)\right]}{\partial t} = \left(E_0 - \Delta\right)C_1(t) + UC_2(t) - UC_1(t) - \left(E_0 + \Delta\right)C_2(t) \end{cases}$$

$$\rightarrow \begin{cases} i\hbar \dfrac{\partial\left[C_1(t) + C_2(t)\right]}{\partial t} = \left(E_0 + U\right)\left[C_1(t) + C_2(t)\right] - \left[C_1(t) - C_2(t)\right]\Delta \\ i\hbar \dfrac{\partial\left[C_1(t) - C_2(t)\right]}{\partial t} = \left[E_0 - U\right]\left[C_1(t) - C_2(t)\right] - \left[C_1(t) + C_2(t)\right]\Delta \end{cases}$$

$$\rightarrow i\hbar \frac{\partial}{\partial} \begin{bmatrix} C_1(t) + C_2(t) \\ \left[C_1(t) - C_2(t)\right] \end{bmatrix} = \begin{bmatrix} \left(E_0 + U\right) & -\Delta \\ -\Delta & \left[E_0 - U\right] \end{bmatrix} \begin{bmatrix} \left[C_1(t) + C_2(t)\right] \\ \left[C_1(t) - C_2(t)\right] \end{bmatrix}$$

有解的条件 $\rightarrow \begin{bmatrix} \left(E_0 + U\right) & -\Delta \\ -\Delta & \left[E_0 - U\right] \end{bmatrix} \begin{bmatrix} \left[C_1(t) + C_2(t)\right] \\ \left[C_1(t) - C_2(t)\right] \end{bmatrix} = E \begin{bmatrix} \left[C_1(t) + C_2(t)\right] \\ \left[C_1(t) - C_2(t)\right] \end{bmatrix}$

有解的条件 $\rightarrow \begin{bmatrix} (E_0+U) & -\Delta \\ -\Delta & [E_0-U] \end{bmatrix} \begin{bmatrix} [C_1(t)+C_2(t)] \\ [C_1(t)-C_2(t)] \end{bmatrix} = E \begin{bmatrix} [C_1(t)+C_2(t)] \\ [C_1(t)-C_2(t)] \end{bmatrix}$

$$\rightarrow \begin{bmatrix} (E_0+U)-E & -\Delta \\ -\Delta & [E_0-U]-E \end{bmatrix} \begin{bmatrix} [C_1(t)+C_2(t)] \\ [C_1(t)-C_2(t)] \end{bmatrix} = 0 \quad \rightarrow [(E_0+U)-E][[E_0-U]-E]-\Delta^2 = 0$$

$$\rightarrow (E_0+U)[E_0-U]-E(E_0+U)-E(E_0-U)+E^2-\Delta^2=0$$

$$\rightarrow E_0^2-U^2-EE_0-EU-EE_0+EU+E^2-\Delta^2=0$$

$$\rightarrow E^2-2EE_0+E_0^2-U^2-\Delta^2=0$$

$$\rightarrow E_{1,2}=\frac{2E_0\pm\sqrt{4E_0^2-4\left(E_0^2-U^2-\Delta^2\right)}}{2}$$

$$\rightarrow E_{1,2}=\frac{2E_0\pm\sqrt{4E_0^2-4\left(E_0^2-U^2-\Delta^2\right)}}{2}$$

$$\begin{cases} E_0=\dfrac{E_{11}+E_{22}}{2} \\ \Delta=\dfrac{E_{11}-E_{22}}{2} \end{cases} \rightarrow E_{1,2}=\frac{E_{11}+E_{22}\pm\sqrt{\left(E_{11}+E_{22}\right)^2-\left(\left(E_{11}+E_{22}\right)^2-4U^2-\left(E_{11}-E_{22}\right)^2\right)}}{2}$$

$$\text{化简}\rightarrow E_{1,2}=\frac{E_{11}+E_{22}\pm\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}$$

$$\rightarrow\begin{cases} i\hbar\dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t}=\dfrac{E_{11}+E_{22}\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}\left[C_1(t)+C_2(t)\right] \\ i\hbar\dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t}=\dfrac{E_{11}+E_{22}\pm\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}\left[C_1(t)-C_2(t)\right] \end{cases}$$

$$\rightarrow \begin{cases} i\hbar \dfrac{\partial\left[C_1(t)+C_2(t)\right]}{\partial t} = \dfrac{E_{11}+E_{22}\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}\left[C_1(t)+C_2(t)\right] \\ i\hbar \dfrac{\partial\left[C_1(t)-C_2(t)\right]}{\partial t} = \dfrac{E_{11}+E_{22}\pm\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}\left[C_1(t)-C_2(t)\right] \end{cases}$$

$$\rightarrow \begin{cases} \left[C_1(t)+C_2(t)\right] = e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}t} \\ \left[C_1(t)-C_2(t)\right] = e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}t} \end{cases}$$

$$\rightarrow \begin{cases} C_1(t) = \dfrac{1}{2}\left(e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}t} + e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}t}\right) \\ C_2(t) = \dfrac{1}{2}\left(e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}t} - e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+\left(E_{11}-E_{22}\right)^2}}{2}t}\right) \end{cases}$$

$$\rightarrow \begin{cases} C_1(t) = \dfrac{1}{2}\left(e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} + e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} \right) \\ C_2(t) = \dfrac{1}{2}\left(e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} - e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} \right) \end{cases}$$

$$\rightarrow \begin{cases} C_1(t) = e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}}{2}t} \cos\left(\dfrac{\sqrt{4U^2+(E_{11}-E_{22})^2}}{2\hbar} t \right) \\ C_2(t) = i e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}}{2}t} \sin\left(\dfrac{\sqrt{4U^2+(E_{11}-E_{22})^2}}{2\hbar} t \right) \end{cases}$$

$$\begin{cases}\left[C_1(t)+C_2(t)\right]=e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}\\ \left[C_1(t)-C_2(t)\right]=e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}\end{cases}\rightarrow\begin{cases}C_+=e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}\\ C_-=e^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}\end{cases}$$

$$\rightarrow C = AC_+ + BC_- = Ae^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} + Be^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}$$

$$\rightarrow P = CC^* = \left(Ae^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} + Be^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}\right)$$

$$\left(Ae^{\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t} + Be^{\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^2+(E_{11}-E_{22})^2}}{2}t}\right)$$

$$\rightarrow P = CC^{*} = \left(Ae^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}}{2}t} + Be^{-\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}}{2}t} \right)$$

$$\left(Ae^{\frac{i}{\hbar}\frac{E_{11}+E_{22}+\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}}{2}t} + Be^{\frac{i}{\hbar}\frac{E_{11}+E_{22}-\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}}{2}t} \right)$$

$$\rightarrow P = \left(A^{2} + B^{2} + ABe^{-\frac{i}{\hbar}\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}t} + ABe^{\frac{i}{\hbar}\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}t} \right) = 1$$

$$\rightarrow P = A^{2} + B^{2} + 2AB\cos\frac{\sqrt{4U^{2}+\left(E_{11}-E_{22}\right)^{2}}}{\hbar}t = 1$$

$$\rightarrow A^{2} + B^{2} = 1$$

§ 4. 磁共振

磁矩在外磁场中的能量

$$\begin{cases} E_1 = \dfrac{1}{2} M_{磁矩} \bullet \vec{B}_{外磁场的磁感应强度} = \dfrac{1}{2} M_{磁矩} B \cos\theta = \dfrac{1}{2} M_{磁矩} B \to \theta = 0 \\ E_2 = \dfrac{1}{2} M_{磁矩} \bullet \vec{B}_{外磁场的磁感应强度} = \dfrac{1}{2} M_{磁矩} B \cos\theta = -\dfrac{1}{2} M_{磁矩} B \to \theta = \pi \end{cases}$$

构建薛定锷方程

$$\begin{cases} i\hbar \dfrac{\partial C_1(t)}{\partial t} = \dfrac{1}{2} M B C_1(t) \\ i\hbar \dfrac{\partial C_2(t)}{\partial t} = -\dfrac{1}{2} M B C_2(t) \end{cases}$$

如果在垂直于磁矩方向上存在外磁场B_0分量

$$E_0 = \frac{1}{2} \vec{M}_{磁矩} \bullet \vec{B}_{x0外磁场的磁感应强度} + \frac{1}{2} \vec{M}_{磁矩} \bullet \vec{B}_{y0外磁场的磁感应强度} = \frac{1}{2} M_{磁矩} B_0 \cos\omega t + \frac{1}{2} M_{磁矩} B_0 \sin\omega t$$

$$H^2 = \left[\frac{1}{2} M_{磁矩} B_{0x} \cos\omega t\right]^2 + \left[\frac{1}{2} M_{磁矩} B_{0y} \sin\omega t\right]^2$$

$$H^2=\left[\frac{1}{2}M_{磁矩}B_{0x}\cos\omega t\right]^2+\left[\frac{1}{2}M_{磁矩}B_{0y}\sin\omega t\right]^2$$

$$\to H\bullet H^*=\frac{1}{2}\left\{M_{磁矩}B_{0x}\cos\omega t+iM_{磁矩}B_{0y}\sin\omega t\right\}\frac{1}{2}\left\{M_{磁矩}B_{0x}\cos\omega t-iM_{磁矩}B_{0y}\sin\omega t\right\}$$

$$\to\begin{cases}H_1=\frac{1}{2}\left(M_{磁矩}B_{0x}\cos\omega t+iM_{磁矩}B_{0y}\sin\omega t\right)=\frac{1}{2}M_{磁矩}B_{0x}e^{i\omega t}\\ H_2=\frac{1}{2}\left(M_{磁矩}B_{0x}\cos\omega t-iM_{磁矩}B_{0y}\sin\omega t\right)=\frac{1}{2}M_{磁矩}B_{0x}e^{-i\omega t}\end{cases}$$

则存在如下方程 $\to\begin{cases}i\hbar\frac{\partial C_1(t)}{\partial t}=\frac{1}{2}MBC_1(t)+\frac{1}{2}MB_0e^{-i\omega t}C_2(t)\\ i\hbar\frac{\partial C_2(t)}{\partial t}=\frac{1}{2}MB_0e^{i\omega t}C_1(t)-\frac{1}{2}MBC_2(t)\end{cases}$

令 $b_1=C_1e^{\frac{i\omega t}{2}},b_2=C_2e^{-\frac{i\omega t}{2}}\to\begin{cases}i\hbar b_1\left(-i\frac{\omega}{2}\right)e^{-\frac{i\omega t}{2}}+i\hbar e^{-\frac{i\omega t}{2}}\frac{db_1}{dt}=\frac{1}{2}MBb_1e^{-\frac{i\omega t}{2}}+\frac{1}{2}MB_0e^{-i\omega t}b_2e^{\frac{i\omega t}{2}}\\ i\hbar b_2\left(i\frac{\omega}{2}\right)e^{\frac{i\omega t}{2}}+i\hbar e^{\frac{i\omega t}{2}}\frac{db_2}{dt}=\frac{1}{2}MB_0e^{i\omega t}b_1e^{-\frac{i\omega t}{2}}-\frac{1}{2}MBb_2e^{\frac{i\omega t}{2}}\end{cases}$

$$\begin{cases} i\hbar b_1\left(-i\frac{\omega}{2}\right)e^{-\frac{i\omega t}{2}}+i\hbar e^{-\frac{i\omega t}{2}}\frac{db_1}{dt}=\frac{1}{2}MBb_1e^{-\frac{i\omega t}{2}}+\frac{1}{2}MB_0e^{-i\omega t}b_2e^{\frac{i\omega t}{2}} \\ i\hbar b_2\left(i\frac{\omega}{2}\right)e^{\frac{i\omega t}{2}}+i\hbar e^{\frac{i\omega t}{2}}\frac{db_2}{dt}=\frac{1}{2}MB_0e^{i\omega t}b_1e^{-\frac{i\omega t}{2}}-\frac{1}{2}MBb_2e^{\frac{i\omega t}{2}} \end{cases}$$

$$\begin{cases} i\hbar b_1\left(-i\frac{\omega}{2}\right)+i\hbar\frac{db_1}{dt}=\frac{1}{2}MBb_1+\frac{1}{2}MB_0b_2 \\ i\hbar b_2\left(i\frac{\omega}{2}\right)+i\hbar\frac{db_2}{dt}=\frac{1}{2}MB_0b_1-\frac{1}{2}MBb_2 \end{cases} \quad 令\omega_0=-\frac{MB}{\hbar},\omega_1=-\frac{MB_1}{\hbar}\rightarrow\begin{cases} i\frac{db_1}{dt}=\frac{1}{2}\omega_0b_1-\frac{1}{2}\omega b_1+\frac{1}{2}\omega_1b_2 \\ i\frac{db_2}{dt}=\frac{1}{2}\omega_1b_1-\frac{1}{2}\omega_0b_2+\frac{\omega}{2}b_2 \end{cases}$$

$$令\Delta\omega=\omega_0-\omega\rightarrow\begin{cases} i\frac{db_1}{dt}=\frac{\Delta\omega}{2}b_1+\frac{\omega_1}{2}b_2 \\ i\frac{db_2}{dt}=\frac{\omega_1}{2}b_1-\frac{\Delta\omega}{2}b_2 \end{cases} \qquad \begin{bmatrix} i\hbar\frac{db_1}{dt} \\ i\hbar\frac{db_2}{dt} \end{bmatrix}=\frac{\hbar}{2}\begin{bmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\rightarrow\frac{\hbar}{2}\begin{bmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}=\omega_r\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\rightarrow \frac{\hbar}{2}\begin{bmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \omega_r \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \rightarrow \begin{bmatrix} \Delta\omega - \omega_r & \omega_1 \\ \omega_1 & -\Delta\omega - \omega_r \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0$$

$$-\Delta\omega^2 + \omega_r^{\ 2} - \omega_1^2 = 0 \rightarrow \omega_r = \sqrt{\Delta\omega^2 + \omega_1^2} = \sqrt{\left(\omega_0 - \omega\right)^2 + \left(\frac{MB_0}{\hbar}\right)^2}$$

$$\omega_r = \sqrt{\Delta\omega^2 + \omega_1^2} = \sqrt{\left(\omega_0 - \omega\right)^2 + \left(\frac{MB_0}{\hbar}\right)^2}$$

$$\begin{bmatrix} i\hbar\dfrac{db_1}{dt} \\ i\hbar\dfrac{db_2}{dt} \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} \Delta\omega & \omega_1 \\ \omega_1 & -\Delta\omega \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{\hbar}{2}\omega_r \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \begin{bmatrix} i\hbar\dfrac{db_1}{dt} \\ i\hbar\dfrac{db_2}{dt} \end{bmatrix} = \frac{\hbar}{2}\omega_r \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{cases} i\hbar\dfrac{db_1}{dt} = \dfrac{\hbar}{2} b_1 \omega_r \\ i\hbar\dfrac{db_2}{dt} = -\dfrac{\hbar}{2} b_2 \omega_r \end{cases} \rightarrow \begin{cases} b_1 = A_1 e^{-i\frac{\omega_r t}{2}} \\ b_2 = A_2 e^{i\frac{\omega_r t}{2}} \end{cases}$$

$$\rightarrow\begin{cases} b_1 = A_1 e^{-i\frac{\omega_r t}{2}} \\ b_2 = A_2 e^{i\frac{\omega_r t}{2}} \end{cases} \quad \rightarrow \text{其解集：} \; b_+ = b_1 + b_2 = \left(A_1 e^{-i\frac{\omega_r t}{2}} + A_2 e^{i\frac{\omega_r t}{2}} \right)$$

$$t = 0 \rightarrow b_+ = \left(A_1 e^{-i\frac{\omega_r t}{2}} + A_2 e^{i\frac{\omega_r t}{2}} \right) = 0, A_1 = -A_2 = A \qquad \rightarrow b_- = b_1 - b_2$$

$$t=0 \rightarrow b_+ = b_2 + b_1 = A\left(e^{-i\frac{\omega_r t}{2}} - e^{i\frac{\omega_r t}{2}} \right) = 0 \text{则} b_- = b_1 - b_2 = 1$$

$$t = 0 \rightarrow \frac{db_+}{dt} = A\left(-i\frac{\omega_r}{2} e^{-i\frac{\omega_r t}{2}} - i\frac{\omega_r}{2} e^{i\frac{\omega_r t}{2}} \right) = -iA\omega_r$$

$$\begin{cases} i\dfrac{db_1}{dt} = \dfrac{\Delta\omega}{2} b_1 + \dfrac{\omega_1}{2} b_2 \\ i\dfrac{db_2}{dt} = \dfrac{\omega_1}{2} b_1 - \dfrac{\Delta\omega}{2} b_2 \end{cases} \quad \rightarrow \begin{cases} i\dfrac{db_1}{dt} = \dfrac{\omega_1}{2} b_2 \\ i\dfrac{db_2}{dt} = \dfrac{\omega_1}{2} b_1 \end{cases} \quad \rightarrow A = -i\frac{\omega_1}{2\omega_r} = -i\frac{MB_0}{2\hbar\omega_r}$$

$$\rightarrow b_+ = A\left(e^{-i\frac{\omega_r t}{2}} - e^{i\frac{\omega_r t}{2}} \right) = 2Ai\sin\left(\frac{\omega_r t}{2}\right) = \frac{MB_0}{\hbar\omega_r}\sin\left(\frac{\omega_r t}{2}\right) \qquad \rightarrow P = b_+^2 = \left(\frac{MB_0}{\hbar\omega_r}\right)^2 \sin^2\frac{\omega_r t}{2}$$

自旋翻转几率

$$P_{\downarrow} = \left|\frac{\omega_1}{\omega_\mathrm{R}}\right|^2 \sin^2\frac{\omega_\mathrm{R} t}{2} = \left|\frac{2\mu_\mathrm{M} B_1}{\hbar\omega_\mathrm{R}}\right|^2 \sin^2\frac{\omega_\mathrm{R} t}{2}$$

其中拉比频率

$$\omega_\mathrm{R} = \sqrt{(\omega-\omega_0)^2 + \left|\frac{2\mu_\mathrm{M} B_1}{\hbar}\right|^2}$$

翻转几率$P_{\downarrow}$在$t=0$等于0，然后在0和$\left|\frac{\omega_1}{\omega_\mathrm{R}}\right|^2$之间随着时间按照正旋规律变化。

有三种频率ω_0、ω_1和ω，前面是有静磁场决定的，后面两者是旋转磁场的强度和旋转角速度决定的，这些可以在实验中控制。

讨论两种情况：

（1）旋转磁场的频率ω等于或接近拉莫尔角频率ω_0，即$\omega \sim \omega_0$，那么$\omega_R \sim \omega_1$，$P_\downarrow$在0和1时间振荡。严格共振时且$t = (2n+1)\pi/\omega_1$，$P_\downarrow = 1$，这是共振现象。在共振时候，很弱的旋转磁场能够翻转自旋的方向。

（2）$|\omega - \omega_0| \gg \omega_1$，$P_\downarrow(t)$都是接近零，即测量自旋角动量时几乎不变。

4. 核磁共振

电子的磁矩是以玻尔磁子为量子化单位来衡量的，$\boldsymbol{\mu}_s = -2\mu_B \boldsymbol{s}/\hbar$

$$\mu_B = e\hbar/2m_e$$

而核磁矩以核磁子μ_N来衡量，与玻尔磁子差别在于电子质量换成质子质量

$$\mu_N = \frac{e\hbar}{2m_p} = \frac{1}{1836}\mu_B$$

核磁共振技术应用