Algorithm Design X

Dynamic Programming II

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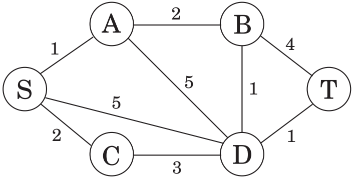

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Dynamic programming will work!

Dynamic Programming

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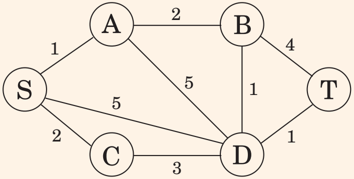
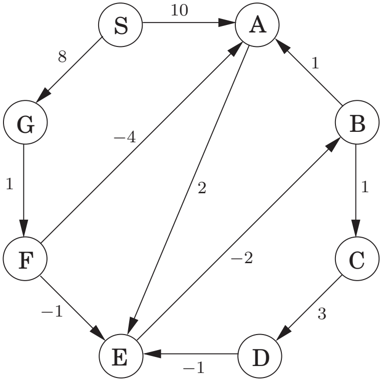
$$dist(v, i) = \min_{(u,v) \in E} \{dist(u, i-1) + l(u, v)\}$$

Shortest Reliable Paths

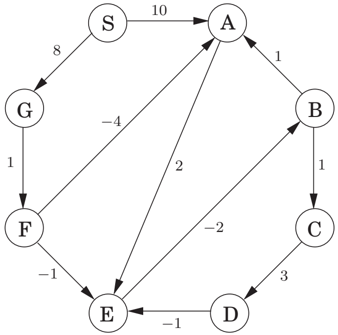
Find out the shortest reliable path from S to T, when $k = 3$.



An Example



Bellman-Ford Algorithm



	Iteration							
Node	0	1	2	3	4	5	6	7
S	0	0	0	0	0	0	0	0
A	∞	10	10	5	5	5	5	5
B	∞	∞	∞	10	6	5	5	5
C	∞	∞	∞	∞	11	7	6	6
D	∞	∞	∞	∞	∞	14	10	9
E	∞	∞	12	8	7	7	7	7
F	∞	∞	9	9	9	9	9	9
G	∞	8	8	8	8	8	8	8

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The total running time would then be $O(|V|^2|E|)$.

We'll now see a better alternative, the $O(|V|^3)$, named Floyd-Warshall algorithm.

Floyd-Warshall Algorithm

Dynamic programming again!

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For $k \geq 1$

$$dist(i, j, k) = \min\{dist(i, j, k-1), dist(i, k, k-1) + dist(k, j, k-1)\}$$

The Program


```
for i = 1 to n do
    for j = 1 to n do
        dist(i, j, 0) = ∞;
    end
end
for all (i, j) ∈ E do
    dist(i, j, 0) = l(i, j);
end
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            dist(i, j, k) = min{dist(i, j, k − 1), dist(i, k, k − 1) + dist(k, j, k − 1)};
        end
    end
end
```

Traveling Salesman Problem

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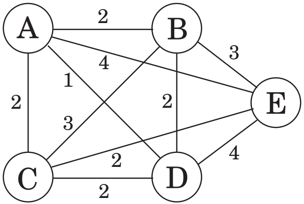
A traveling salesman is getting ready for a big sales tour. Starting at his hometown, he will conduct a journey in which each of his target cities is visited exactly once before he returns home.

Q: Given the pairwise distances between cities, what is the best order in which to visit them, so as to minimize the overall distance traveled?

The brute-force approach is to evaluate every possible tour and return the best one. Since there are $(n-1)!$ possibilities, this strategy takes $O(n!)$ time.



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The goal is to design a tour that starts and ends at 1, includes all other cities exactly once, and has minimum total length.

The Subproblems

For a subset of cities $S \subseteq \{1, 2, \ldots, n\}$ that includes 1, and $j \in S$, let $C(S, j)$ be the length of the shortest path visiting each node in S exactly once, starting at 1 and ending at j.

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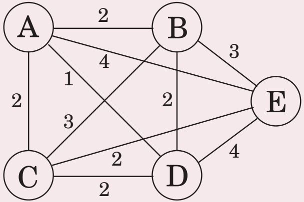
For $j \neq 1$ with $j \in S$ we have

$$C(S, j) = \min_{i \in S : i \neq j} C(S \backslash \{j\}, i) + d_{ij}$$

Exercise



The Program

$C(1,1) = 0$;
for $s = 2$ *to* n **do**
  for *all subsets* $S \subseteq \{1,2,\ldots,n\}$ **do**
    $C(S,1) = \infty$;
    for *all* $j \in S$ *and* $j \neq 1$ **do**
      $C(S,j) = \min_{i \in S: i \neq j} C(S \backslash \{j\}, i) + d_{ij}$;
    end
  end
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`return`($\min_j C(\{1,2,\ldots,n\}, j) + d_{j1}$);

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There are at most $2^n \cdot n$ subproblems, and each one takes linear time.

The total running time is therefore $O(n^2 \cdot 2^n)$.

Independent Sets in Trees

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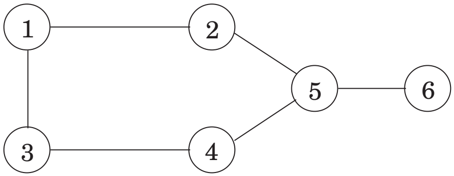
However, when the graph happens to be a tree, the problem can be solved in linear time, using dynamic programming.



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$$I(u) = \max\{1 + \sum_{\text{grandchildren } w \text{ of } u} I(w), \sum_{\text{children } w \text{ of } u} I(w)\}$$

Homework

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Assignment 4. Exercises 6.17, 6.20, 6.21, and 6.22.