Algorithm Design XI

Linear Programming I

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An Introduction to Linear Programming

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1. satisfy a set of linear equations and/or linear inequalities involving these variables, and
2. maximize or minimize a given linear objective function.

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- and the more decadent and deluxe Pyramide Nuit.

Q: How much of each should it produce to maximize profits?

- Every box of Pyramide has a a profit of $\$1$.
- Every box of Nuit has a profit of $\$6$.
- The daily demand is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
- The current workforce can produce a total of at most 400 boxes of chocolate per day.

LP Formulation

`Objective function`	$\max x_1 + 6x_2$
`Constraints`	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$

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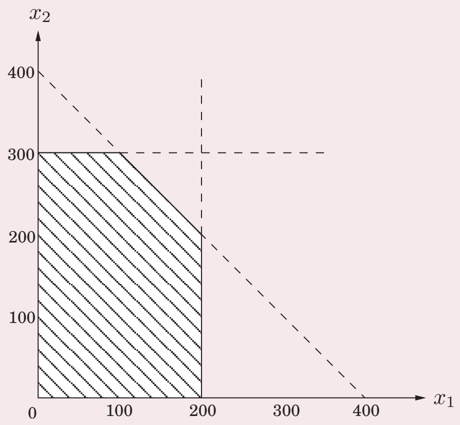
It is a convex polygon.
凸边界

The Convex Polygon



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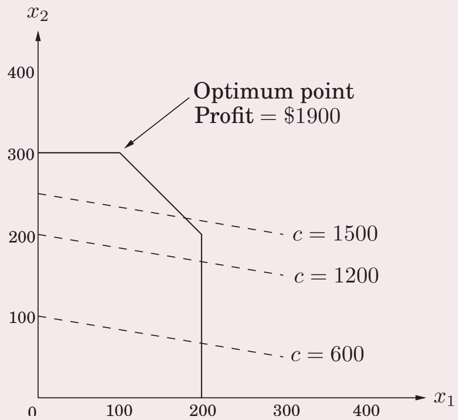
The optimum solution will be the very last feasible point that the profit line sees and must therefore be a vertex of the polygon.

The Convex Polygon



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   - For instance, $\max x_1 + x_2$
   - $x_1, x_2 \geq 0$

一般都在顶点处取到，但是有特例，如果约束条件没有解或者不会构成一个边界那么就不会在顶点处取到最优解。

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This algorithm starts at a vertex, and repeatedly looks for an adjacent vertex of better objective value.

It does hill-climbing on the vertices of the polygon, walking from neighbor to neighbor so as to steadily increase profit along the way.

Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.

Solving Linear Programs

Q: Why does this local test imply global optimality?

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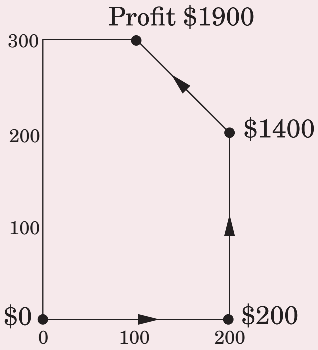
By simple geometry. Since all the vertex's neighbors lie below the line, the rest of the feasible polygon must also lie below this line.

The Example



More Products

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Nuit and Luxe require the same packaging machinery. Luxe uses it three times as much, which imposes another constraint $x_2 + 3x_3 \leq 600$.

LP

$$\max x_1 + 6x_2 + 13x_3$$
$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 + x_3 \le 400$$
$$x_2 + 3x_3 \le 600$$
$$x_1, x_2, x_3 \ge 0$$

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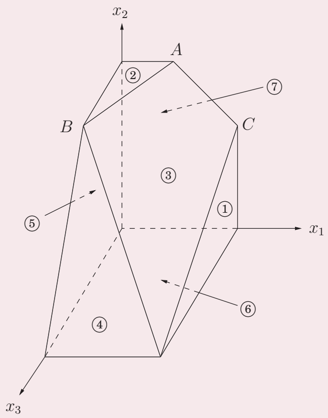
A profit of c corresponds to the plane $x_1 + 6x_2 + 13x_3 = c$.

As c increases, this profit-plane moves parallel to itself, further into the positive orthant until it no longer touches the feasible region.

The Example



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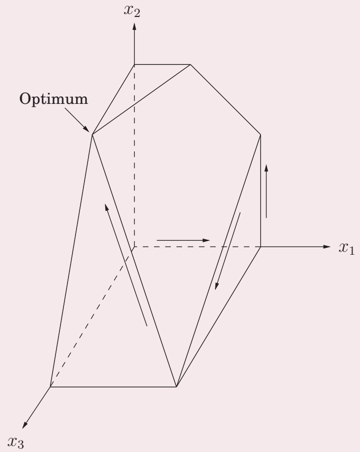
A possible trajectory

$$\frac{(0,0,0)}{\$0} \rightarrow \frac{(200,0,0)}{\$200} \rightarrow \frac{(200,200,0)}{\$1400} \rightarrow \frac{(200,0,200)}{\$2800} \rightarrow \frac{(0,300,100)}{\$3100}$$

The Example



Integer Linear Programming and Rounding

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Currently with 30 employees, each of whom makes 20 carpets per month and gets a monthly salary of $\$2000$.

With no initial surplus of carpets.

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1. Overtime. Overtime pay is 80% more than regular pay. Workers can put in at most 30% overtime.
2. Hiring and firing, costing $\$320$ and $\$400$, respectively, per worker.
3. Storing surplus production, costing $\$8$ per carpet per month. Currently without stored carpets on hand, and without any carpets stored at the end of year.

$$
\begin{aligned}
w_i &= \text{number of workers during } i\text{-th month; } w_0 = 30. \\
x_i &= \text{number of carpets made during } i\text{-th month.} \\
o_i &= \text{number of carpets made by overtime in month } i. \\
h_i, f_i &= \text{number of workers hired and fired, respectively,} \\
&\quad \text{at beginning of month } i. \\
s_i &= \text{number of carpets stored at end of month } i\text{; } s_0 = 0.
\end{aligned}
$$

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All variables must be nonnegative:

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The number of workers can potentially change at the start of each month:

$$w_i = w_{i-1} + h_i - f_i$$

LP Formulation

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And overtime is limited:

$$o_i \leq 6w_i$$

LP Formulation

The objective function is to minimize the total cost:

$$\min 2000\sum_i w_i + 320\sum_i h_i + 400\sum_i f_i + 8\sum_i s_i + 180\sum_i o_i$$

180是怎么来的？
一天正常工时为200/3元，则加班时一天的工资为120元，平均每一条毛毯生产时间为2/3天，所以生产Oi条加班毛毯需要花费：120*Oi/(2/3)=180*Oi。

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This number would have to be rounded to either 10 or 11 in order to make sense, and the overall cost would then increase correspondingly.

In the example, most of the variables take on fairly large values, and thus rounding is unlikely to affect things too much.

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In NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.

Duality

对偶性

Product Planning Revisit

Recall:

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We take the first inequality and add it to six times the second inequality:

$$x_1 + 6x_2 \leq 2000$$

Product Planning Revisit

Multiplying the three inequalities by 0, 5, and 1, respectively, and adding them up yields

$$x_1 + 6x_2 \leq 1900$$

Multipliers

Let's investigate the issue by describing what we expect of these three multipliers, call them y_1, y_2, y_3.

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We want the left-hand side to look like the objective function $x_1 + 6x_2$ so that the right-hand side is an upper bound on the optimum solution.

Multipliers

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$$

if

$$\begin{aligned} y_1, y_2, y_3 &\geq 0 \\ y_1 + y_3 &\geq 1 \\ y_2 + y_3 &\geq 6 \end{aligned}$$

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What we want is a bound as tight as possible, so we minimize

$$200y_1 + 300y_2 + 400y_3$$

subject to the preceding inequalities. This is a new linear program!

The Dual Program

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\min 200y_1 + 300y_2 + 400y_3 \\
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Here is just such a pair:

- Primal: $(x_1, x_2) = (100, 300)$;
- Dual: $(y_1, y_2, y_3) = (0, 5, 1)$.

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They both have value 1900 and certify each other's optimality.

Matrix-Vector Form and Its Dual

关系：原问题与对偶问题的最值类型相反；原问题每有一个约束条件，对偶问题就有一个优化变量；原问题有一个优化变量，对偶问题就有一个约束条件。

Primal LP	Dual LP
$\max c^T\mathbf{x}$	$\min \mathbf{y}^T b$
$A\mathbf{x} \leq b$	$\mathbf{y}^T A \geq c^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Primal LP:

$$
\begin{aligned}
\max\ & c_1x_1 + \cdots + c_nx_n \\
& a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i \quad \text{for } i \in I \\
& a_{i1}x_1 + \cdots + a_{in}x_n = b_i \quad \text{for } i \in E \\
& x_j \geq 0 \quad \text{for } j \in N
\end{aligned}
$$

Dual LP:

$$
\begin{aligned}
\min\ & b_1y_1 + \cdots + b_my_m \\
& a_{1j}y_1 + \cdots + a_{mj}y_m \geq c_j \quad \text{for } j \in N \\
& a_{1j}y_1 + \cdots + a_{mj}y_m = c_j \quad \text{for } j \notin N \\
& y_i \geq 0 \quad \text{for } i \in I
\end{aligned}
$$

Matrix-Vector Form and Its Dual

$$
\begin{aligned}
\max\ & x_1 + 6x_2 \\
& x_1 \le 200 \\
& x_2 \le 300 \\
& x_1 + x_2 \le 400 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\min\ & 200y_1 + 300y_2 + 400y_3 \\
& y_1 + y_3 \ge 1 \\
& y_2 + y_3 \ge 6 \\
& y_1, y_2, y_3 \ge 0
\end{aligned}
$$

Matrix-Vector Form and Its Dual

Theorem (Duality)

If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide.

若一个线性规划问题有界，那么他的对偶问题也有界，并且两者的最优解对应的最优数值应该相同。

Complementary Slackness

The number of variables in the dual is equal to that of constraints in the primal and the number of constraints in the dual is equal to that of variables in the primal.

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An inequality constraint has slack if the slack variable is positive.

The complementary slackness refers to a relationship between the slackness in a primal constraint and the associated dual variable.

LP and Its Dual

$$\begin{aligned} \max\ x_1 + 6x_2 \\ x_1 \le 200 \\ x_2 \le 300 \\ x_1 + x_2 \le 400 \\ x_1, x_2 \ge 0 \end{aligned}$$

$$x_1 = 100, x_2 = 300$$

$$\begin{aligned} \min\ 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \ge 1 \\ y_2 + y_3 \ge 6 \\ y_1, y_2, y_3 \ge 0 \end{aligned}$$

$$y_1 = 0, y_2 = 5, y_3 = 1$$

Complementary Slackness

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Theorem

Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^*.

1. *If $x_j^* > 0$, then the j-th constraint in (D) is binding.*
2. *If the j-th constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i-th constraint in (P) is binding.*
4. *If the i-th constraint in (P) is not binding, then $y_i^* = 0$.*

e.g.: 6x_1 + 12x_2 + 6
x_1,x_2>=0
x1+x2<=2
x1+2x2<=3
x1-3x2<=5
的对偶？
由于对偶实质上就是在找极值
等价的等价问题，所以只需要先将+6
移除，之后在对偶之后的问题上面在加6就可以了。