# Mathematical Foundation of Computer Sciences II

Context-Free Languages and Pushdown Automata

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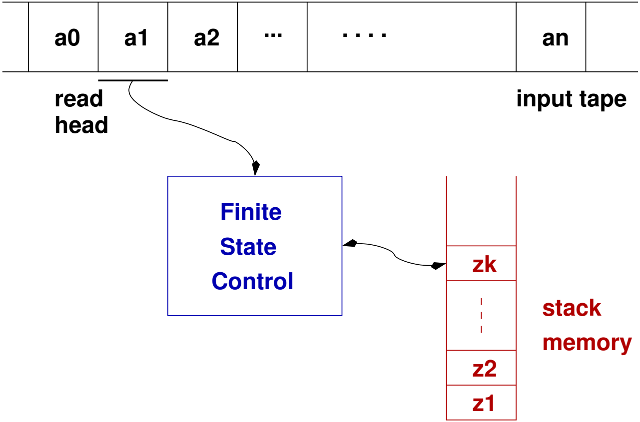
# A Program Example

```
void m() {
    if (?) {
        s(); right();
        if (?) m();
    } else {
        up(); m(); down();
    }
}
```

```
void s() {
    if (?) return;
    up(); m(); down();
}

main() {
  s();
}
```

# A Program Example



# Context Free Languages

# An Example

The grammar

$$\begin{array}{rcl} A & \rightarrow & 0A1 \\ A & \rightarrow & B \\ B & \rightarrow & \# \end{array}$$

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A derivation:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000\#111.$$

# An Example

$$
\begin{aligned}
\langle \text{SENTENCE} \rangle &\rightarrow \langle \text{NOUN-PHRASE} \rangle \langle \text{VERB-PHRASE} \rangle \\
\langle \text{NOUN-PHRASE} \rangle &\rightarrow \langle \text{CMPLX-NOUN} \rangle \mid \langle \text{CMPLX-NOUN} \rangle \langle \text{PREP-PHRASE} \rangle \\
\langle \text{VERB-PHRASE} \rangle &\rightarrow \langle \text{CMPLX-VERB} \rangle \mid \langle \text{CMPLX-VERB} \rangle \langle \text{PREP-PHRASE} \rangle \\
\langle \text{PREP-PHRASE} \rangle &\rightarrow \langle \text{PREP} \rangle \langle \text{CMPLX-NOUN} \rangle \\
\langle \text{CMPLX-NOUN} \rangle &\rightarrow \langle \text{ARTICLE} \rangle \langle \text{NOUN} \rangle \\
\langle \text{CMPLX-VERB} \rangle &\rightarrow \langle \text{VERB} \rangle \mid \langle \text{VERB} \rangle \langle \text{NOUN-PHRASE} \rangle \\
\langle \text{ARTICLE} \rangle &\rightarrow \text{a} \mid \text{the} \\
\langle \text{NOUN} \rangle &\rightarrow \text{boy} \mid \text{girl} \mid \text{flower} \\
\langle \text{VERB} \rangle &\rightarrow \text{touches} \mid \text{likes} \mid \text{sees} \\
\langle \text{prep} \rangle &\rightarrow \text{with}
\end{aligned}
$$

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&\Rightarrow \langle\text{ARTICLE}\rangle\langle\text{NOUN}\rangle\langle\text{VERB-PHRASE}\rangle\\
&\Rightarrow \text{a }\langle\text{NOUN}\rangle\langle\text{VERB-PHRASE}\rangle\\
&\Rightarrow \text{a boy}\langle\text{VERB-PHRASE}\rangle\\
&\Rightarrow \text{a boy}\langle\text{CMPLX-VERB}\rangle\\
&\Rightarrow \text{a boy}\langle\text{VERB}\rangle\\
&\Rightarrow \text{a boy sees.}
\end{aligned}
$$

# Context-Free Grammar

**Definition**

A context-free grammar (CFG) is a 4-tuple $(V, \Sigma, R, S)$, where

1. $V$ is a finite set called the variables,
2. $\Sigma$ is a finite set, disjoint from $V$, called the terminals,
3. $R$ is a finite set of rules, with each rule being a variable and a string of variables and terminals,
4. $S \in V$ is the start variable.

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Let $u, v, w$ be strings of variables and terminals, and

$$A \to w \quad \in R$$

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Which is a context-free language(CFL).

# Examples

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$$\begin{array}{rcl} S & \rightarrow & S_1 \mid S_2 \\ S_1 & \rightarrow & 0S_11 \mid \epsilon \\ S_2 & \rightarrow & 1S_20 \mid \epsilon. \end{array}$$

# Ambiguity

$$\langle EXPR \rangle \rightarrow \langle EXPR \rangle + \langle EXPR \rangle \mid \langle EXPR \rangle \times \langle EXPR \rangle \mid (\langle EXPR \rangle) \mid a$$

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1. $\langle EXPR \rangle \rightarrow \langle EXPR \rangle \times \langle EXPR \rangle \Rightarrow \langle EXPR \rangle + \langle EXPR \rangle \times \langle EXPR \rangle \overset{*}{\Rightarrow} a+a\times a$
2. $\langle EXPR \rangle \rightarrow \langle EXPR \rangle + \langle EXPR \rangle \Rightarrow \langle EXPR \rangle + \langle EXPR \rangle \times \langle EXPR \rangle \overset{*}{\Rightarrow} a+a\times a$

# Leftmost derivations

A derivation of a sting $w$ in a grammar $G$ is a leftmost derivation if at every step the leftmost remaining variable is the one replaced.

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$\{a^i b^j c^k \mid i = j \ \vee \ j = k\}$ is inherently ambiguous,i.e., its every grammar is ambiguous.

# Chomsky Normal Form

A context-free grammar is in Chomsky normal form if every rule is of the form

$$\begin{array}{rcl} A & \rightarrow & BC \\ A & \rightarrow & a \end{array}$$

where $a$ is any terminal and $A$, $B$ and $C$ are any variables, except that $B$ and $C$ may be not the start variable.

In addition, we permit the rule $S \rightarrow \epsilon$, where $S$ is the start variable.

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In addition, we permit the rule $S \rightarrow \epsilon$, where $S$ is the start variable.

**Theorem**

Any context-free language is generated by a context-free grammar in Chomsky normal form.

# Proof of the Theorem

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2. Remove every $A \rightarrow \epsilon$, where $A \neq S_0$.
   For each occurrence of $A$ on the right-hand side of a rule, we add a new rule with that occurrence deleted.
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   b) Do the above operation for each occurrence of $A$: e.g. $R \to uAvAw$, will be replaced by $R \to uvAw \mid uAvw \mid uvw$.
   c) For $R \to A$, we add $R \to \epsilon$ unless we had previously removed $R \to \epsilon$.
3. Remove every $A \to B$.
   Whenever a rule $B \to u$ appears, where $u$ is a string of variables and terminals, we add the rule $A \to u$ unless this was previously removed.

# Proof of the Theorem (cont.)

1. New start variable $S_0$.
2. Remove every $A \to \epsilon$.
3. Remove every $A \to B$.

# Proof of the Theorem (cont.)

1. New start variable $S_0$.
2. Remove every $A \to \epsilon$.
3. Remove every $A \to B$.
4. Replace each rule $A \to u_1 u_2 \cdots u_k$ with $k \geq 3$ and each $u_i$ is a variable or terminal with the rules

$$A \to u_1 A_1, A_1 \to u_2 A_2, A_2 \to u_2 A_3, \cdots, \text{ and } A_{k-2} \to u_{k-1} u_k.$$

   The $A_i$s are new variables. We replace any terminal $u_i$ with the new variable $U_i$ and add $U_i \to u_i$.

# An Example

Applying the first step to make a new start variable appears on the right.

$$\begin{aligned} S &\rightarrow ASA \mid aB \\ A &\rightarrow B \mid S \\ B &\rightarrow b \mid \varepsilon \end{aligned}$$

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# An Example

Remove $\varepsilon$-rules $B \to \varepsilon$ on the left, and $A \to \varepsilon$ on the right.

$$\begin{aligned} S_0 &\to S \\ S &\to ASA \mid aB \mid a \\ A &\to B \mid S \mid \varepsilon \\ B &\to b \mid \varepsilon \end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
S_0 &\to S \\
S &\to ASA \mid aB \mid a \mid AS \mid SA \mid S \\
A &\to B \mid S \mid \varepsilon \\
B &\to b
\end{aligned}$$

# An Example

Remove unit rules $S \to S$ on the left, and $S_0 \to S$ on the right.

$$
\begin{aligned}
S_0 &\to S \\
S &\to ASA \mid aB \mid a \mid AS \mid SA \\
A &\to B \mid S \\
B &\to b
\end{aligned}
$$

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Remove unit rules $S \to S$ on the left, and $S_0 \to S$ on the right.

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$$\begin{aligned} S_0 &\to ASA \mid aB \mid a \mid AS \mid SA \\ S &\to ASA \mid aB \mid a \mid AS \mid SA \\ A &\to S \mid b \mid ASA \mid aB \mid a \mid AS \mid SA \\ B &\to b \end{aligned}$$

# An Example

Convert the remaining rules into the proper form by adding additional variables and rules.

$$
\begin{aligned}
S0 &\rightarrow AA_1 \mid UB \mid a \mid SA \mid AS \\
S &\rightarrow AA_1 \mid UB \mid a \mid SA \mid AS \\
A &\rightarrow b \mid AA_1 \mid UB \mid a \mid SA \mid AS \\
A_1 &\rightarrow SA \\
U &\rightarrow a \\
B &\rightarrow b
\end{aligned}
$$

# Efficient Derivation

**Theorem**

*If $G$ is a context-free grammar in Chomsky normal form then any $w \in L(G)$ such that $w \neq \varepsilon$ can be derived from the start state in exactly $2|w| - 1$ steps.*

# Pushdown automata

# Pushdown Automata

**Definition**

A pushdown automata (PDA) is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where

1. $Q$ is a finite set of states,
2. $\Sigma$ is a finite set of input alphabet,
3. $\Gamma$ is a finite set of stack alphabet,
4. $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $F \subseteq Q$ is the set of accept states.

# Formal Definition of Computation

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a pushdown automaton. $M$ accepts input $w$ if $w$ can be written as $w = w_1 \ldots w_m$, and sequences of states $r_0, r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots, s_m \in \Gamma^*$ exist that satisfy the following three conditions.

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1. $r_0 = q_0$ and $s_0 = \epsilon$.
2. For $i = 0, \dots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$.
3. $r_m \in F$.

# PDA for $\{0^n 1^n \mid n \geq 0\}$

$$
\begin{aligned}
Q &= \{q_1, q_2, q_3, q_4\}, \\
\Sigma &= \{0, 1\}, \\
\Gamma &= \{0, \$\}, \\
q_1 & \quad \text{is the start state} \\
F &= \{q_1, q_4\}
\end{aligned}
$$

The transition function is defined by the following table, wherein blank entries signify $\emptyset$

| Input: | 0 | | | 1 | | | $\epsilon$ | | |
|---|---|---|---|---|---|---|---|---|---|
| Stack: | 0 | \$ | $\epsilon$ | 0 | \$ | $\epsilon$ | 0 | \$ | $\epsilon$ |
| $q_1$ | | | | | | | | | $\{(q_2, \$)\}$ |
| $q_2$ | | | $\{(q_2, 0)\}$ | $\{(q_3, \epsilon)\}$ | | | | | |
| $q_3$ | | | | $\{(q_3, \epsilon)\}$ | | | | $\{(q_4, \epsilon)\}$ | |
| $q_4$ | | | | | | | | | |

# Equivalence of CFL and PDA

**Theorem**

*A language is context free if and only if some pushdown automaton recognizes it.*

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# Every Context-Free Language Can Be Recognized by a PDA

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   2.2 If the top of stack is a terminal symbol $a$, read the next symbol from the input and compare it to $a$. If they match, repeat.If they do not match, reject on this branch of the nondeterminism.
   2.3 If the top of stack is the symbol \$, enter the accept state. Doing so accepts the input if it has all been read.

# Push a long string in "one step"

Let $q$ and $r$ be states of the PDA and let $a \in \Sigma_\varepsilon$ and $s \in \Gamma_\varepsilon$.

We want the PDA to go from $q$ to $r$ when it reads $a$ and pops $s$.
Furthermore, we want it to push the entire string $u = u_1 \dots u_l$ on the stack at the same time.

$$
\begin{aligned}
&(q_1, u_l) \in \delta(q, a, s) \\
&\delta(q_1, \varepsilon, \varepsilon) = \{(q_2, u_{l-1})\} \\
&\delta(q_2, \varepsilon, \varepsilon) = \{(q_3, u_{l-2})\} \\
&\vdots \\
&\delta(q_{l-1}, \varepsilon, \varepsilon) = \{(r, u_1)\}
\end{aligned}
$$

We use the abbreviation

$$(r, u) \in \delta(q, a, s)$$

# Proof

We construct a pushdown automaton $P$ as follows.

The states of $P$ are

$$Q = \{q_{start}, q_{loop}, q_{accept}\} \cup E$$

where $E$ is the set of states we need for the construction in the previous slide.

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For the transition function,

- $\delta(q_{start}, \varepsilon, \varepsilon) = \{(q_{loop}, S\$)\}$
- $\delta(q_{loop}, \varepsilon, A) = \{(q_{loop}, w) \mid A \rightarrow w \text{ is a rule in the given grammar}\}$
- $\delta(q_{loop}, a, a) = \{(q_{loop}, \varepsilon)\}$
- $\delta(q_{loop}, \varepsilon, \$) = \{(q_{accept}, \varepsilon)\}$

# Every Language Recognized by a PDA is Context Free

Let $P$ be a PDA. For each pair of states $p$ and $q$, the grammar has a variable $A_{pq}$ which generates

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We modify $P$ such that:

1. It has a single accept state $q_{accept}$.
2. It empties its stack before accepting.
3. Each transition either pushes a symbol onto the stack or pops one off the stack, but it does not do both at the same time.

# Inductive Definition of $A_{pq}$

Two possibilities occur during $P$'s computation on an input string $x$.

1. The symbol popped at the end is the symbol that was pushed at the beginning. Then, we have a rule $A_{pq} \to aA_{rs}b$.
2. Otherwise, we have a rule $A_{pq} \to A_{pr}A_{rq}$.

# Proof (1)

Assume $P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{accept}\})$.

The variables of the desired context-free grammar $G$ are

$$\{A_{pq} \mid p, q \in Q\}$$

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Induction step: The derivation has $k+1$ step with $A_{pq} \Rightarrow^* x$. Thus, either $A_{pq} \Rightarrow aA_{rs}b$ or $A_{pq} \Rightarrow A_{pr}A_{rq}$.

In case $A_{pq} \Rightarrow aA_{rs}b$ the claim follows from (R1) and the induction hypothesis.

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For $A_{pq} \Rightarrow A_{pr}A_{rq}$, there exist $y$ and $z$ with $x = yz$ such that $A_{pr} \Rightarrow^* y$ and $A_{qr} \Rightarrow^* z$ both in at most $k$ steps. The claim then again follows from the induction hypothesis.

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Induction step:

If the stack is always non-empty in the middle of the computation, then:

- There is a $u$ which is pushed in the first move and popped in the last move.
- In the first move, let $a$ be the input and $r$ be the state after; in the last move let $b$ be the input and $s$ be the state before.
- We deduce $(r, u) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, u)$. Hence, $G$ has the rule $A_{pq} \to aA_{rs}b$.

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If the stack becomes empty in the middle of the computation, the claim then again follows from the induction hypothesis.

# Closure Properties

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The context-free languages are closed under union, concatenation, and kleene star.

# Closure Properties - Union

Proof

$N_1 = (V_1, \Sigma_1, R_1, S_1)$ recognize $A_1$,
$N_2 = (V_2, \Sigma_2, R_2, S_2)$ recognize $A_2$. w.l.o.g. $V_1 \cap V_2 = \emptyset$.

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- **Union.** $S$ is a new symbol. Let $N = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R, S)$, where $R = R_1 \cup R_2 \cup \{S \to S_1, S \to S_2\}$.

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# Pumping Lemma

**Lemma (Pumping Lemma)**

*If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided as $s = uvxyz$ satisfying the conditions*

1. *for each* $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$,
3. $|vxy| < p$.

# Proof

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For any string $s \in A$ with $|s| \geq p$, any of its parse trees must be at least $|V| + 1$ high.

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Let $\tau$ be one parse tree of $s$ with smallest number of nodes, whose height is at least $|V| + 1$. So $\tau$ has a path from the root to a leaf of length $|V| + 1$ with $|V| + 2$ nodes.

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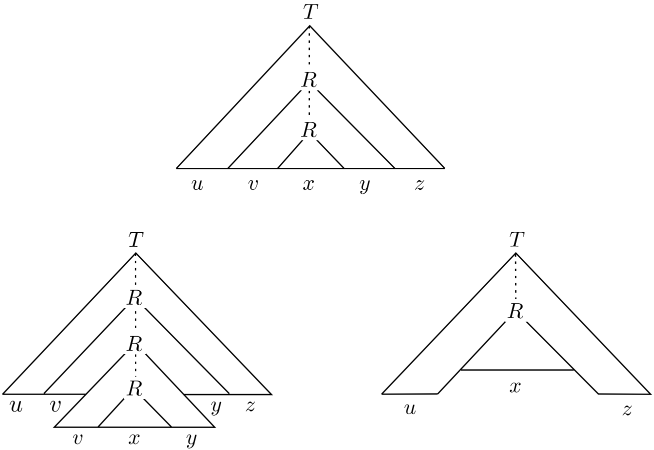
We divide $s$ into $uvxyz$:

- $u$ from the leftmost leaf of $\tau$ to the leaf left next to the leftmost leaf of the subtree hanging on the first $R$,
- $v$ from the leftmost leaf of the subtree hanging on the first $R$ to the leaf left next to the leftmost leaf of the subtree hanging on the second $R$,
- $x$ for all the leaves of the subtree hanging on the second $R$,
- $y$ from the leaf right next to the rightmost leaf of the subtree hanging on the second $R$ to the rightmost leaf of the subtree hanging on the first $R$,
- $z$ from the leaf right next to the rightmost leaf of the subtree hanging on the first $R$ to the rightmost leaf of $\tau$.

# Pumping Lemma



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**Condition 3.** To see $|vxy| \leq p = b^{|V|+1}$, note that $vxy$ is generated by the first $R$. We can always choose $R$ so that its last two occurrences fall within the bottom $|V| + 1$ high. A tree of this height can generate a string of length at most $b^{|V|+1} = p$.

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- When both $v$ and $y$ contain only one type of symbols, i.e., one of $a, b, c$, then $uv^2xy^2z$ cannot contain equal number of $a$'s, $b$'s, and $c$'s.

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- If either $v$ or $y$ contains more than one type of symbols, then $uv^2xy^2z$ would have symbols interleaved.

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- Similarly $vxy$ cannot occur only in the second half of $s$.
- If $vxy$ straddles the midpoint of $s$, then pumping $s$ to the form $0^p1^i0^j1^p$ cannot ensure $i = j = p$.

# Other Computations

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To the second part of the statement,

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

rules out the closure under complementation.