# 0. Review

These chapters should be review from MATH 1ZA3 (or equivalent). You should know everything within by heart.

## 7.5: Strategies for Integration

1. $\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \frac{1}{x} dx = \ln|x|$
3. $\int e^x dx = e^x$
4. $\int b^x dx = \frac{b^x}{\ln b}$
5. $\int \sin x \; dx = -\cos x$
6. $\int \cos x \; dx = \sin x$
7. $\int \sec^2 x \; dx = \tan x$
8. $\int \csc^2 x \; dx = -\cot x$
9. $\int \sec x \tan x \; dx = \sec x$
10. $\int \csc x \cot x \; dx = -\csc x$
11. $\int \sec x \; dx = \ln|\sec x + \tan x|$
12. $\int \csc x \; dx = \ln|\csc x - \cot x|$
13. $\int \tan x \; dx = \ln|\sec x|$
14. $\int \cot x \; dx = \ln|\sin x|$
15. $\int \sinh x \; dx = \cosh x$
16. $\int \cosh x \; dx = \sinh x$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right); \quad a > 0$

*19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$ *This one is easily avoided by using partial fractions.
*20. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right|$ *This one can be avoided with trig. substitution.

Tips for integration:

- Manipulate integrand.
  - Expand. Simplify. Factor. Substitute. Rewrite radicals as powers.
  - Try to work it into a form with an obvious solution.
- Classify according to form.
  - Trig Functions
    - If the function is made up entirely of trig functions, like $f(x) = \sin^n x \cos^m x$, consider a **trigonometric substitution**.
  - Rational Functions
    - If the function is a rational function, like $f(x) = \frac{A+Bx}{C+Dx}$, consider **partial fraction decomposition**.
  - Radicals

        - For the common identities $\sqrt{x^2+a^2}$, $\sqrt{x^2+a^2}$, $\sqrt{a^2-x^2}$, use a **trigonometric substitution**. For radicals of form $\sqrt[n]{g(x)}$, use a **substitution**.
- If all else fails, try the above again.
    - There are two main methods of integration: **substitution** and **parts**. Try to make any substitution. Remember that you can use the differential term ($dx$, $dy$, $dt$, etc.) as the second term for integration by parts.
        - Try algebraic manipulation.
            - You can split terms separated by $+/-$ into multiple integrals: try to coax the integral into this form.
            - Try multiplying the integral by a term that cancels to zero: for example, multiplying by $\frac{\cos x+1}{\cos x+1}$ to create a trigonometric identity.
            - If you don't know how to, learn how to rationalize denominators.

# 1. Series, Sequences, and Infinity

This 'section' of the course will span from around its start to the first midterm.

## 7.8: Improper Integrals

**Improper integrals** are definite integrals with an infinite interval or a discontinuous integrand. We will refer to the former as **Type 1** and the latter as **Type 2**.

### 7.8.1: Integrals with Infinite Intervals

This is **Type 1**. Integrals of this type will appear as:

$$\int_b^a f(x)\,dx$$

where either $a$ or $b$ are $\infty$.

In order to solve such an integral, we abstract the infinite end to a variable, then take the integral's limit as that variable approaches infinity. Thus,

$$\int_b^\infty f(x)\,dx = \lim_{a\to\infty}\int_b^a f(x)\,dx$$

and similar for integrals bounded from $-\infty$.

If the limit exists, we call that integral **convergent**. If not, we call it **divergent**.

Formally, we define

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a,$ then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b,$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

The following important fact is also presented:

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.}$$

We call a function of this kind a **p-series**.

## 7.8.2: Integrals with Discontinuities

This is **Type 2**. Integrals of this type will appear as:

$$\int_a^t f(x)\,dx$$

where the function $f(x)$ has a discontinuity at $t$, or like this:

$$\int_a^b f(x)\,dx$$

where there is a discontinuity at $x = c$ where $a < c < b$.

In the first case, you replace $t$ with the limit as the bound approaches the discontinuity, as in

$$\int_a^t f(x)\,dx = \lim_{t\to b^-} \int_a^b f(x)\,dx \qquad \text{or} \qquad \int_t^b f(x)\,dx = \lim_{t\to a^+} \int_a^b f(x)\,dx$$

We call the subsequent improper integral **convergent** if the limit exists and **divergent** if it does not.

Formally, we define

(a) If $f$ is continuous on $[a, b)$ and is discontinuous at $b,$ then
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$
if this limit exists (as a finite number).

(b) If $f$ is continuous on $(a, b]$ and is discontinuous at $a,$ then
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$
if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If $f$ has a discontinuity at $c,$ where $a < c < b,$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are converger
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The Type II p-series acts differently than the Type I. It is one of the biggest mistakes students make (McLean said that).

$$\boxed{\int_0^1 \frac{1}{x^p}\,dx \text{ is convergent if } p < 1 \text{ and divergent if } p \geq 1.}$$

or the exact opposite of the Type I.

## 7.8.3: The Comparison Test

This is a way to find out if an improper integral converges or diverges without finding its exact value.

If $f, g$ are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a,$

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Basically, if you know a function $f$ to be either convergent or divergent, you can guarantee another function $g$ to be one of them based on the relationship between their $y$-values. If $f$ converges and $g$ is underneath $f$ for all $x$ (at least, all relevant $x$ values in the integral), it will also converge. If $f$ diverges and $g$ is above $f$ for all $x$, it will also diverge.

# 11.1: Sequences

Sequences are lists of numbers following some specific pattern. As such, they can be represented as functions or in sigma notation. (**Note: sequences and series are NOT the same! This is a common mistake! Most of the theorems in 11 will only work for one or the other so do not mix them up.**)

## 11.1.1: Infinite Sequences

An infinite sequence is a sequence where there is a term $a_n$ for every positive integer $n$.

If we graph a lot of values of an infinite sequence, it may appear to begin to converge at a specific value.

$$\boxed{\begin{array}{c} \text{A sequence } \{a_n\} \text{ has the limit } L \text{ and we write} \\ \lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty \\ \text{if we can make the terms } a_n \text{ as close to } L \text{ as we like by taking } n \text{ sufficiently large.} \\ \text{If } L \text{ exists, we say the sequence } \textbf{converges} \text{ (or is } \textbf{convergent}\text{).} \\ \text{Otherwise, we say the sequence } \textbf{diverges} \text{ (or is } \textbf{divergent}\text{).} \end{array}}$$

Again, do not confuse this with the convergence of a series.

If we define the general term $a_n$ to be expressed by a function $f(n) = a_n$, we can say that $\lim_{n\to\infty} a_n = L$. Find below some useful sequence laws. These are the same as the limit laws you already know but in a slightly different format.

$$\boxed{\begin{array}{c} \text{Suppose that } \{a_n\} \text{ and } \{b_n\} \text{ are convergent sequences and } c \text{ is a constant. Then:} \\ \begin{array}{ll} 1. & \lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \\ 2. & \lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \\ 3. & \lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \\ 4. & \lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \\ 5. & \lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0 \\ 6. & \lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p \text{ if } p > 0 \text{ and } a_n > 0 \end{array} \end{array}}$$

The squeeze theorem is also helpful. It is like the comparison test, but you determine the exact value of a limit. Essentially, if a sequence is 'squished' between two other sequences above and on the bottom that both converge to the same value, the sequence in between will also converge to that value.

$$\boxed{\text{If } a_n \leq b_n \leq c_n \text{ for } n \geq n_0 \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{then } \lim_{n\to\infty} b_n = L.}$$

The following formula can also prove nichely useful:

$$\boxed{\text{If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0}$$

A sequence's convergability cannot be removed by function composition, that is if you pipe every term of a convergent sequence through a function $f(a_n)$, it will still be convergent. Formally,

$$\boxed{\begin{gathered}\text{If } \lim_{n\to\infty} a_n = L \text{ and the function } f \text{ is continuous at } L, \text{ then} \\ \lim_{n\to\infty} f(a_n) = f(L)\end{gathered}}$$

**This will be important.** Sequences of the form $r^n$ are convergent/divergent if:

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{div.} & \text{if } r \geq 1 \text{ or } r < -1 \end{cases}}$$

### 11.1.2: Monotonic/Bounded Sequences

A sequence is called **increasing** if $a_{n+1} > a_n$ for all terms $n$. A sequence is **decreasing** if $a_{n+1} < a_n$ for all terms $n$. If a sequence is either increasing or decreasing, it is **monotonic**.

$$\boxed{\begin{gathered}\text{A sequence } a_n \text{ is } \textbf{bounded above} \text{ by } M \text{ if:} \\ \\ a_n \leq M \text{ for all } n \geq 1 \\ \\ \text{Or it is } \textbf{bounded below} \text{ by } m \text{ if:} \\ \\ m \leq a_n \text{ for all } n \geq 1\end{gathered}}$$

If a sequence is bound above and below, then it is called a **bounded sequence**.

The **Monotonic Sequence Theorem** states that

$$\boxed{\begin{gathered}\text{Every bounded, montononic sequence is convergent.} \\ \ldots \\ \text{In particular, an increasing and bounded above sequence converges} \\ \text{and a decreasing and bounded below sequence converges.}\end{gathered}}$$

## 11.2: Series

A series is the sum of a sequence's terms. Please do remember not to confuse the two.

### 11.2.1: Infinite Series

Sigma notation is most commonly used for series. An infinite series, where all the infinite terms of an infinite sequence are summed, is notated like such:

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

A partial sum is a sum of the terms of a sequence up to a term $n$, such that $s_n = a_1 + a_2 + \cdots + a_n$.

Given the series $\sum_{n=1}^{\infty} a_n$ and its $n$-th partial sum

$$s_n = a_1 + a_2 + \cdots + a_n,$$

If the sequence of partial sums $\{s_n\}$ converges and $\lim_{n\to\infty} s_n = L$ where $L$ is a real number, $\{a_n\}$ is **convergent** and L is the **sum** of the series, expressed as

$$\sum_{n=1}^{\infty} a_n = L$$

But if $\{s_n\}$ **diverges**, the series is **divergent**.

Something you should take away from this is that a series is just **the limit of the sequence of partial sums of a sequence**. This can be helpful to visualize the concept.

## 11.2.2: Sum of a Geometric Series

The geometric series is a series of form

$$\sum_{n=1}^{\infty} ar^{n-1} \quad \text{where} \quad a \neq 0$$

This series is **very common**, and all you must know about it is that

A geometric series is convergent if

$$|r| < 1$$

If a geometric series is convergent, its sum is

$$s_n = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| > 1$, the series is divergent.

## 11.2.3: Tests for Convergence

Remember a series is divergent if its sequence of partial sums is divergent.

$$\boxed{\text{If the series } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} a_n = 0}$$

Essentially, if a series is convergent, its sequence will converge to 0. **Common mistake:** this is not true in reverse! This is a very very very common mistake to make. Kindly do not make it. All convergent series will have a 0-sequence, but not all 0-sequences have a convergent series. It's the square/rectangle thing.

$$\boxed{\text{If } \lim_{n\to\infty} a_n \text{ does not exist or } \neq 0, \text{ the series is divergent.}}$$

Remember the common mistake, this does not mean that the series will be convergent if the limit is 0. Here is a helpful graphic:

$$\lim_{n\to\infty} a_n = \begin{cases} \text{D.N.E}: & \text{Series is guaranteed divergent.} \\ \neq 0: & \text{Series is guaranteed divergent.} \\ 0: & \text{We have no idea. (The series could be convergent, but we don't know based o} \end{cases}$$

The first three limit laws listed above for sequences are also correct for series. In particular, these are the addition, subtraction, and multiplication laws.

If $\Sigma a_n$ and $\Sigma b_n$ are convergent series, then:

$$\text{(a)} \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$
$$\text{(b)} \sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
$$\text{(c)} \sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

# 11.3: The Integral Test and Estimates of Sums

## 11.3.1: The Integral Test

Suppose $f$ is a continuous, decreasing, positive function on $[1, \infty)$ and let $a_n = f(n)$. Then the series

converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

$$\text{(i)} \quad \text{If } \int_1^{\infty} f(x)\,dx \text{ is convergent, then } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$
$$\text{(ii)} \quad \text{If } \int_1^{\infty} f(x)\,dx \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

There are caveats to the integral test.

- The function must be ultimately decreasing for $x > N$ for some number N.
- The function must be entirely positive.
- The function must be completely continuous.
- You can start the improper integral with other bounds like $\int_n^{\infty}$.

The p-series was discussed back in **7.8.1**, but only its role in an improper integral. That rule can also be applied to the p-series as a series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\,dx \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

**Common mistake:** Please remember that an improper integral is not the same thing as a series summation. All the improper integral does is prove the sum of the area underneath the sequence exists as $x$ gets large.

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$$

## 11.3.2: Estimating the Sum of a Series

The partial sum at any term $n$ can be used as a approximation of the actual sum, $s$. The difference between an approximation of the sum and the sum itself, $s - s_n$ is known as $R_n$ or the **remainder** of the approximation.

$$\boxed{\begin{array}{c}\text{Suppose } f(k) = a_k, \text{ where } f \text{ is a continuous, positive, decreasing function} \\ \text{for } x \geq n \text{ and } \Sigma_{n=1}^{\infty} a_n \text{ is convergent. If } R_n = s - s_n, \\ \displaystyle\int_{n+1}^{\infty} f(x) \ \leq R_n \leq \int_n^{\infty} f(x)\,dx\end{array}}$$

Or in simpler terms, iterating through partial sums of each term, each successive approximation will be more precise than the one before. This is useful when a question asks for accuracy to certain decimal point. Note that the improper integrals are NOT the approximations themselves. Here is some explanation on what these integrals actually are:

- Recall that $R_n$ is not an integral, but a sigma infinite summation of the sequence.
- Since $f$ is decreasing, the sum of values at integer points can be compared to integrals:
  - The sum from $k = n + 1$ to $\infty$ is roughly comparable to the integral of $f(x)$ from $x = n + 1$ to $\infty$.
    - This is a **common mistake**: the sum of a series from $n \to \infty$ is not the same as the integral on the same range. Why?
      - On sigma notation, we add each term for **every positive integer** value of $n$.
      - For the integral, we add each term for **every positive** value of $n$.
      - Don't confuse all of this with the convergence of the sequence!
- The integral $\int_{n+1}^{\infty} f(x)\,dx$ underestimates the sum because the rectangle heights (like $f(k)$) are greater than or equal to $f(x)$ for $x \in [k, k+1]$.
  - But if the integral is summing an infinite number of values and the sum is summing only values from each positive integer, how is this possible? If the integral is summing $f(59.999), f(60), f(60.001)$ while the summation is summing only $f(60)$, how is the summation larger than the integral?
    - Well, remember the condition of the integral test: the function **must be decreasing**. If we create a rectangle spanning width 1, it will EXPAND PAST THE SLOPE OF THE DECREASING LINE, since we only take a sample every positive integer.
    - In contrast, the integral is much smoother, since we take infinite samples, and thus is always smaller than the partial sum.
- Conversely, $\int_n^{\infty} f(x)\,dx$ overestimates the sum because the rectangles at integer points can be thought of as "covering" the area under the curve, but the value at $k$ is greater than the function's value on the interval $[k, k+1)$, so the integral from $n$ is an upper bound.

By adding $s_n$ to all sides, we get

$$\boxed{s_n + \int_{n+1}^{\infty} f(x) \ \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx}$$

This can be used to provide upper/lower bounds for $s$.

# 11.4: The Comparison Tests

The comparison tests work similarly for series as they do for sequences.

## 11.4.1: Direct Comparison Test

$$\boxed{\begin{array}{c} \text{Supppose that } \Sigma a_n, \Sigma b_n \text{ are series with positive terms.} \\ \\ \text{(i)} \quad \text{If } \Sigma a_n \text{ is convergent and } b_n \leq a_n \text{ for all } n,\ \Sigma b_n \text{ is also convergent.} \\ \text{(ii)} \quad \text{If } \Sigma a_n \text{ is divergent and } b_n \geq a_n \text{ for all } n,\ \Sigma b_n \text{ is also divergent.} \end{array}}$$

- Since you need a known convergent/divergent series to use the direct comparison tests, these two families are most common.
  - P-series
    - Converges when $p > 1$, diverges when $p \leq 1$
  - Geometric series
    - Converges when $|r| < 1$, diverges when $|r| \geq 1$

## 11.4.2: Limit Comparison Test

The direct comparison test is useless in the case where your known series is divergent and larger than your target series, or when your known series is convergent and smaller than your target series. In these cases where you still want to use a comparison (like if the series is very similar to one of the good families above), you use this.

$$\boxed{\begin{array}{c} \text{Supppose that } \Sigma a_n, \Sigma b_n \text{ are series with positive terms. If} \\ \lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = c \\ \text{where } c \text{ is finite and } c > 0, \text{ then either both series converge or both diverge.} \end{array}}$$

## 11.4.3: Estimating Sums

If we can say that every term of the series $\{a_n\}$ is smaller or equal to than every corresponding term of the sequence $\{b_n\}$, then the remainder of $\{a_n\}$, $R_n$, will be smaller or equal to the remainder of $\{b_n\}$, $T_n$.

If we look at the remainder estimate for the integral test, we have

$$\int_{n+1}^{\infty} f(x) \ \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

which we can adapt to this method of sum estimation:

$$R_n \leq T_n \leq \int_n^\infty f(x)\,dx$$

We can use this to approximate the sum of a series given $n$. Remember all these methods to estimate sums do not give the sum itself, unless you are given the partial sum $s_n$. You estimate the sum by finding $R_n$, or how far the partial sum at $n$ is from the real sum.

# 11.5: Alternating Series and Absolute Convergence

## 11.5.1: The Alternating Series

An **alternating series** is a series of form $a_n = (-1)^{(n-1\,|\,n)} b_n$ where $b_n$ is positive.

$$\boxed{\begin{array}{c}\text{If the alternating series}\\ a_n = (-1)^{(n-1\,|\,n)} b_n \quad (b_n > 0)\\ \text{satisfies the conditions:}\\ \begin{array}{ll}\text{(i)} & b_{n+1} \leq b_n \quad \text{for all } n\\ \text{(ii)} & \lim\limits_{n\to\infty} b_n = 0\end{array}\\ \text{Then the series is guaranteed to be convergent.}\end{array}}$$

This is obvious if you think about it.

## 11.5.2: Estimating Sums of Alternating Series

$$\boxed{\begin{array}{c}\text{If } s = \sum (-1)^{n-1} b_n,\ \text{where } b_n > 0,\ \text{is the sum of an alternating series that satisfies}\\ \text{(i)} \quad b_{n+1} \leq b_n \quad \text{and} \quad \text{(ii)} \quad \lim\limits_{n\to\infty} b_n = 0\\ \text{then:}\\ |R_n| = |s - s_n| \leq b_{n+1}\end{array}}$$

Essentially, the remainder is always smaller than the **value** of the first neglected term. Note this does not apply to other kinds of series. Do not try.

## 11.5.3: Absolute and Conditional Convergence

$$\boxed{\begin{array}{c}\text{A series } \Sigma a_n \text{ is called } \textbf{absolutely convergent} \text{ if the series}\\ \text{of absolute values } \Sigma |a_n| \text{ is convergent.}\end{array}}$$

Note that if a convergent series only has positive terms, it will be absolutely convergent too.

$$\boxed{\begin{array}{c}\text{A series } \Sigma a_n \text{ is called } \textbf{conditionally convergent} \text{ if it is}\\ \text{convergent but not absolutely convergent.}\end{array}}$$

The following is evident:

$$\boxed{\text{If a series } \Sigma a_n \text{ is absolutely convergent, it is also convergent.}}$$

# 11.6: The Ratio and Root Tests

## 11.6.1: Ratio Test

$$\text{(i) If } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1, \quad \text{then the series } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent (and th}$$

$$\text{(ii) If } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \quad \text{or} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty, \quad \text{then the series } \sum_{n=1}^{\infty} a_n$$

$$\text{(iii) If } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1, \quad \text{the Ratio Test is inconclusive; that is, no conclusion can be drawn about th}$$

The ratio test usually will fail for certain types of sequences. More on this in 11.7.

## 11.6.2: Root Test

This is most conveniently used when there are $n$-th powers involved.

$$\text{(i) If } \lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1, \quad \text{then the series } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent (and therefore convergent}$$

$$\text{(ii) If } \lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1 \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty, \quad \text{then the series } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

$$\text{(iii) If } \lim_{n\to\infty}\sqrt[n]{|a_n|} = 1, \quad \text{the Root Test is inconclusive.}$$

If one of the root/ratio test fails, the other will also fail so don't bother.

# 11.7: Strategy for Testing Series

The strategy around testing a series for convergence/divergence is based on its form.

1. **Test for Divergence**: If you can see that $\lim_{n\to\infty} a_n$ may be different from 0, then apply the Test for Divergence.
2. **p-Series**: If the series is of the form $\sum 1/n^p$, then it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
3. **Geometric Series**: If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, then it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. You may need to do some algebraic manipulation.
4. **Comparison Tests**: If the series has a form that is similar to a p-series or a geometric series, then a comparison test is a good choice.
   - In particular, if $a_n$ is a rational function or an algebraic function of $n$ (involving roots of polynomials), then use a p-series.

    - Break it down to a p-series form, then compare. You should keep the highest power of p.
  - The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply a comparison test to $\sum |a_n|$ and test for absolute convergence instead.
5. **Alternating Series Test**: If the series takes the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is a strong candidate.
   - Note that if $\sum b_n$ converges, the original series is absolutely convergent and thus converges.
6. **Ratio Test**: This test is often effective for series containing factorials or other products, including terms with a base raised to the $n$-th power.
   - Be aware that for p-series and rational/algebraic functions of $n$, the limit of $|a_{n+1}/a_n|$ approaches 1 as $n \to \infty$. Therefore, the Ratio Test is generally not useful for these types of series.
     - Use a Comparison Test instead.
7. **Root Test**: If the $n$-th term $a_n$ can be expressed in the form $(b_n)^n$, then the Root Test can be a helpful tool.
8. **Integral Test**: When $a_n = f(n)$ and the integral $\int_1^\infty f(x)dx$ is straightforward to evaluate, the Integral Test can be applied effectively, provided its hypotheses are met.
   - Remember the function must be ultimately decreasing, positive, and continuous on the interval.

## 11.8: Power Series

### 11.8.1: Power Series

A **power series** is a series of form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

where the $c$s are constants and the $x$s are variables. For values of $x$, the series may either converge or diverge. The goal of solving a power series is usually to find for what values of $x$ the series will converge, the **interval of convergence**.

A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is called a **power series centered at** $a$.

To solve a power series, you usually use either the **ratio test** or the **root test**.

### 11.8.2: Interval of Convergence

For a power series $c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only at $x = a$.
(ii) The series converges for all $x$.
(iii) There is a positive number $R$ for which the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

$R$ is the **radius of convergence** of the power series. In case (i), $R = 0$, and in (ii), $R = \infty$. The **interval of convergence** is all values for which the power series converges. In case (i), the interval of convergence is a single point $x = a$, and in (ii), the radius of convergence is $x = (-\infty, \infty)$. In case (iii), the inequality can be transformed:

$$|x-a| < R \longrightarrow a - R < x < a + R$$

to form the interval of convergence.

In case (iii), when $x = a \pm R$, the power series may converge or diverge. Thus, if you find yourself in this situation, **you must test both endpoints** to make sure they either converge or diverge. Eg. On the interval $2-5$, you must test both 2 and 5 to figure out if it is $[2,5], (2,5], \ldots$

# Appendix E: Sigma Notation

You should probably know what sigma notation is before starting the course, especially if you read the prior chapters (sorry but I'm not putting the chapters in strictly learning order), but contained are some formulas and facts that may be important.

If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and $m, n$ are integers such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m, a_{m+1}, \ldots, a_n$$

This notation means the following:

- Starting from $i = m$, for every integer value of $i$, the term $a_i$ is added to a cumulative sum.
- The summing stops when $i = n$, thus the last value is $a_n$.

Several rules similar to those from the Limit Laws hold.

If $c$ is any constant (not dependent on $i$), then:

$$\text{(a)} \quad \sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\text{(b)} \quad \sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\text{(b)} \quad \sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

The following formulas are the most useful part of this section.

Let $c$ be a constant and $n$ a positive integer.

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$ (b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

At least memorize the last three, since the first two are at least somewhat derivable on the fly.

## Mathematical Induction

The textbook is [redacted] when it comes to induction, so I recommend THIS VIDEO by my mathematics GOAT Professor Leonard to learn about it. Here is what the textbook has to say:

Let $S_n$ be a statement about the positive integer $n$. If we prove that:
- $S_1$ is true.
- $S_{k+1}$ is true whenever $S_k$ is true.

Then $S_k$ is true for all positive integers $n$.

Just do a lot of practice problems and you'll probably be fine.

# 2. Power Series, Differentials, and Parametrics

The following will cover around the material from Test 2.

## 11.9: Representations of Functions as Power Series

### 11.9.1: Representations of Functions using Geometric Series

Similar to how we used trigonometric identities to break down integrals into more easily solved forms, we can do the same to help convert functions into power series using a geometric series.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

The idea is to turn a function into this form, where we replace $x$ by something else, and turn it into a power series. Remember that not necessarily all of the $x$ terms must be removed in the power series: $x^3\Sigma_{n=0}^{\infty} n$ is perfectly valid. This means you can reshape the sequence, factor out $x$ terms, do whatever to get it in the format required.

### 11.9.2: Differentiation and Integration of Power Series

If a function can be a power series, we might like to integrate or differentiate it. A way to do this is to differentiate/integrate each term of the power series separately.

If the power series $\Sigma c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval

$$(a-R, a+R)$$

and the following is true.

$$\text{(i)} \quad f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\text{(ii)} \quad \int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Here is an alternate phrasing of the two true statements above:

$$\text{(i.alt)} \quad \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$$

$$\text{(ii.alt)} \quad \int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int [c_n(x-a)^n]\,dx$$

There are a few considerations you should keep in mind:

- These only work with **power series**. Do not rely on this for other types of series.
- While the **radius of convergence** $R$ is unchanged after differentiation/integration, its **interval of convergence** may change.
  - Specifically, the new differentiated/integrated function may now diverge/converge at one of the interval's endpoints.

## 11.10: Taylor and McLaurin Series

In the last chapter, we used a geometric series to turn a function into a power series. Using what we will learn in this chapter, we can express some more functions as power series.

### 11.10.1: Definitions of Taylor and McLaurin Series

$$\boxed{\begin{array}{c}\text{If } f \text{ has a power series expansion at } a, \text{ or} \\ f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \quad |x-a| < R \\ \text{then its coefficients are given by} \\ c_n = \dfrac{f^{(n)}(a)}{n!}\end{array}}$$

In other words,

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}}$$

This series is the **Taylor expansion** of $f(x)$ about $a$. When $a = 0$,

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \end{aligned}}$$

This series is called the **McLaurin series**.

Note that a Taylor series's sum is not necessarily equal to $f$, like with a geometric series. If the function does not have a valid power series representation at $a$, then its Taylor series will just be an approximation. Additionally the power series expansion of $f$ at $a$ is unique.

## 11.10.2: When is a Function Represented by its Taylor Series?

We call the sum of polynomials up to the $n$-th term of a Taylor series $\frac{f^{(n)}(a)}{n!}(x-a)^n$ by the name $T_n(x)$, or the $n$-th degree Taylor polynomial of $f$ at $a$. Let the quantity $R_n(x)$ be the remainder of a Taylor series's sum and the function itself at $n$ terms, i.e $f(x) = T_n(x) + R_n(x)$. If we can prove that $R(x)$ approaches 0 at infinity, we can say that the Taylor series represents the function. Formally,

$$\boxed{\begin{array}{c}\text{If } f(x) = T_n(x) + R_n(x), \text{ where } T_n \text{ is the } n\text{-th degree Taylor polynomial of } f \text{ at } a, \text{ and} \\ \lim_{n\to\infty} R_n(x) = 0 \\ \text{for } |x-a| < R, \text{ then } f \text{ is equal to the sum of the Taylor series on } |x-a| < R.\end{array}}$$

The following is called **Taylor's Inequality** and is useful to prove a function is equal to the sum of its Taylor series on an interval.

$$\boxed{\begin{array}{c}\text{If } |f^{(n+1)}(x)| \le M \text{ for } |x-a| \le d, \text{ then the remainder } R_n(x) \text{ of the Taylor series satisfies} \\ |R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \le d\end{array}}$$

This fact may also come up when solving these kinds of problems, as well as other limit problems:

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$$

It can be found that $e^x$ is equal to the sum of its Maclaurin series:

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x}$$

Fun fact: this is a way to find an expression for the constant $e$, as if we set $x = 1$:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

## 11.10.3: Taylor Series of Important Functions

You should try to derive these yourself as an exercise, but here they are in one block for reference.

| | | |
|---|---|---|
| $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ | for $\lvert x\rvert < 1$ | $F$ |
| $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ | for all $x$ | $F$ |
| $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | for all $x$ | $F$ |
| $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ | for all $x$ | $F$ |
| $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ | for $\lvert x\rvert < 1$ | $F$ |
| $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ | for $\lvert x\rvert < 1$ | $F$ |
| $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ | for $\lvert x\rvert < 1, k \in \mathbb{R}$ | $F$ |

## 11.10.4: New Taylor Series from Old

If we know the Taylor series expression of a function, we can reverse-engineer a function from its expression by manipulating the series expansion we know. The previous table of known Maclaurin series expansions is useful for doing this.

Idk man that's all there is to it this is another reason I recommend you get the textbook. The textbook, the Cengage Early Transcendentals (ISBN: 978-1-337-61392-7), has examples and stuff in this section you should read. It can be expensive, which is why I definitely don't recommend GETTING IT FOR FREE from a SHADOW LIBRARY like ANNAS-ARCHIVE DOT ORG, since we should support the authors of the textbook. In fact, I would never suggest PIRACY or DOWNLOADING A FREE PDF OF THE TEXTBOOK from a website like the example I gave, since it GIVES YOU THE ENTIRE TEXTBOOK for free and that's stealing! So you totally shouldn't! You're taking money out of the pockets of the multi-billion dollar company! That's morally wrong!

### 11.10.5: Multiplication and Division of Power Series

Since series expansions are polynomials, we can multiply and divide them. So if you get asked to find the series expansion for a function that looks like a multiplication/division of two functions whose expansions you know (again, refer to the table above), you can easily find the series expansion that way.

## 11.11: Applications of Taylor Polynomials

### 11.11.1: Approximating Functions by Polynomials

The main use for Taylor polynomials is to approximate a function $f$ around a point $a$ with polynomials. The more terms you add, the better your approximation becomes. The remainder $R_n(x) = f(x) - T_n(x)$ comes to mind again, since it defines how 'good' of an approximation your $n$-term Taylor series is to the function itself. You can also use Taylor's inequality to estimate your error.

### 11.11.2: Applications to Physics

If you want to simplify an equation in physics, you can use a Taylor series to get a simpler approximation, especially if the interval of interest is small. That's it.

## 8.2: Area of a Surface of Revolution

## 8.3

## 9.1

## 9.3

## 3.8

## 9.5

## 10.1

## 10.2

# 3. The Polar System, Partial Derivatives, and Multi-Integrals

### 10.3

### 10.4

### 14.1

### 14.2

### 14.3

### 14.4

### 14.5

### 14.6

### 15.1

### 15.2

# Extra

## Regex Scripts

Since Obsidian is a janky program especially when it comes to LaTeX, I had to use some regex scripts to format everything properly. For example, since Obsidian is an Electron app (you can even open the dev console like a webpage using `Ctrl` - `Shift` - `I`) and MathJax is also a janky mess sometimes, having lone double dollar signs $$ (which escape blocks of LaTeX-formatted math) on lines will break the compiler.

Too bad that my Latex Suite shortcuts do that automatically.

So, I had to write this Python script to remove each of them using a neat little method and some regex.

```
import re

def fix(text):
    pattern = re.compile(r'\$\$(.*?)\$\$', re.DOTALL)

    def repl(match):
        content = match.group(1)
        lines = content.splitlines()

        while lines and lines[0].strip() == '':
            lines.pop(0)
        while lines and lines[-1].strip() == '':
            lines.pop()
```

```
        new_content = ' '.join(line.strip() for line in lines)
        return f'$${new_content}$$'

    fixed_text = pattern.sub(repl, text)
    return fixed_text

with open("note.txt", "r") as f:
    a = f.read()
    with open("finish.txt", "w+") as f1:
        f1.write(fix(a))
```

todo: table showing behaviours of various functions as x increases/decreases
todo: label boxed equations