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# The behaviour of supersonic flow past a body of revolution, far from the axis

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A theory is developed of the supersonic flow past a body of revolution at large distances from the axis, where a linearized approximation is valueless owing to the divergence of the characteristics at infinity. It is used to find the asymptotic forms of the equations of the shocks which are formed from the neighbourhoods of the nose and tail. In the special case of a slender pointed body, the general theory at large distances is used to modify the linearized approximation to give a theory which is uniformly valid at all distances from the axis. The results which are of physical importance are summarized in the conclusion (§ 9) and compared with the results of experimental observations.

## 1. INTRODUCTION

In the supersonic flow of a compressible fluid past a body of revolution, curved shocks are formed from the neighbourhood of the nose and tail. The characteristics from the surface of the body intersect the shocks, which curve in towards each other and become weaker as a consequence. The present paper describes how the asymptotic shapes and strengths at large distances from the body may be found. However, to do this, a general theory of axisymmetrical flow at large distances from the axis must be developed and it is this topic which receives most attention.

In linearized theory the characteristics are straight and parallel, whereas in fact they are neither. Since the characteristics diverge, the linearized theory becomes less accurate as the distance from the body increases and is, therefore, quite inadequate away from the body. The correct results are established by a new method of attacking non-linear partial differential equations of the hyperbolic type, where a linearized approximation exists and an asymptotic solution with respect to one of the independent variables is required. The same technique may be applied to certain analogous problems of unsteady one-dimensional wave motion, such as the flow behind an expanding spherical shock, as will be described in a later paper. The method

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was discovered in collaboration with Mr M. J. Lighthill, who has extended it into a general technique for rendering approximate solutions to physical problems uniformly valid.

The corresponding problem in two dimensions for flow past an aerofoil is comparatively easy, because the details of the exact two-dimensional 'simple-wave' flow of a fluid are known. Lighthill (1944) has shown by a geometrical argument, that the shocks must become parabolae at large distances. Neglecting changes of entropy, Friedrichs (1948) has found an exact expression for the shape of the shock, which gives the asymptotic behaviour deduced by Lighthill. The basis of the method is that the characteristics can be used to connect conditions on the surface of the aerofoil (which are known) with conditions at the shock. This is *only* possible because the characteristics are known to be straight lines with flow variables constant along them. The result, for large $y$, is

$$x = y \cot \mu_0 - a y^{\frac{1}{2}} - \ldots, \tag{1}$$

where $a$ is a constant depending on the aerofoil and $\mu_0$ is the Mach angle of the undisturbed flow. Clearly, such a method cannot be used for the body of revolution, since it would require a detailed knowledge of the characteristics and the flow near the body. However, if only the asymptotic behaviour at large distances is considered, the problem becomes easier.

In any case, the predictions of two-dimensional theory can give approximations to flow past a wing, even of large aspect ratio, only in the region near the surface of the body; its predictions of occurrences at large distances would be wrong. The results for the case of a body of revolution, however, should give an estimate of the truth for flow past any finite body, provided the distance from it is sufficient. Therefore, apart from its scientific interest, the solution may be of practical value; for example, in detecting the passage of a supersonic projectile.

The method of determining the shapes of the shocks requires the asymptotic behaviour of the characteristics. The necessary results are deduced in § 2 by a short unrigorous method, leaving the full discussion and higher approximations until § 5. This presentation is adopted because these first results suggest the approach used later and also they are sufficient for the work of §§ 3 and 4. In § 3, the equation of the front shock is shown to be

$$x = r \cot \mu_0 - b r^{\frac{1}{4}} - \ldots, \tag{2}$$

using a geometrical result of the shock conditions; and in § 4 some properties of the flow between the shocks are considered. In § 5, the exact equations for irrotational flow are solved using certain expansions of the velocities, for large distances from the axis, in which the form of the characteristics plays an important role. With these full results, further approximations to the results of §§ 3 and 4 are found; and in § 7, this accurate theory which holds at large distances is incorporated with the linearized theory which is valid near the body, for the special case in which the body is slender. In § 8, the form of the rear shock and the flow behind it, topics which have not been discussed in sections mentioned above, are considered, but no complete results can be established. Finally, in § 9 are collected those results which are of importance from a practical point of view.

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## 2. The characteristics in axisymmetrical flow

If the velocity of the undisturbed stream is $(U, 0)$ and at a general point $(x, r)$, in cylindrical co-ordinates, the velocity is $(U+Uu, Uv)$, then the equations of motion for irrotational flow are

$$u_r - v_x = 0, \tag{3}$$

$$\frac{a^2}{U^2}\left(u_x + v_r + \frac{v}{r}\right) - (1+u)^2 u_x - 2v(1+u)\,v_x - v^2 v_r = 0, \tag{4}$$

where $a$ is the local velocity of sound, given by

$$\frac{2a^2}{\gamma-1} + U^2[(1+u)^2+v^2] = \frac{2a_0^2}{\gamma-1} + U^2, \tag{5}$$

and $a_0$ is the velocity of sound in the undisturbed stream.

Neglecting terms of the third order in $u$ and $v$, since these must be small at large distances, the equations become

$$u_r - v_x = 0, \tag{6}$$

$$[\alpha^2 u_x - v_r - v/r] + M^2 u[(\gamma+1)\,u_x + (\gamma-1)\,(v_r + v/r)] + 2M^2 v v_x = 0, \tag{7}$$

where $M = U/a_0$ and $\alpha = \surd(M^2-1)$. The linearized theory makes a further approximation by neglecting second-order terms in (7), to give $u_r - v_x = 0$, $\alpha^2 u_x - v_r - v/r = 0$, $\alpha^2 u_{xx} = v_{rx} + v_x/r = u_{rr} + u_r/r$. This equation for $u$ is the equation of cylindrical wave motion, with solution (see Lighthill 1945)

$$u = \int_0^{x-\alpha r} \frac{\lambda(t)\,dt}{\surd\{(x-t)^2 - \alpha^2 r^2\}} = \frac{1}{\surd(2\alpha r)}\int_0^{\xi} \frac{\lambda(t)}{\surd(\xi - t)}\left\{1 - \frac{\xi - t}{4\alpha r} + \dots\right\} dt \tag{8}$$

$$= \frac{F(\xi)}{r^{\frac{1}{2}}} + \frac{F_1(\xi)}{r^{\frac{3}{2}}} + \dots, \tag{9}$$

say, where $\xi = x - \alpha r$. For large $r$, $u \simeq F(x-\alpha r)\, r^{-\frac{1}{2}}$ and from (6), $v \simeq -\alpha u$.

The form of the linearized solution suggests that there is some advantage in using co-ordinates $(\xi, r)$, where $\xi = x - \alpha r$. With this transformation,

$$\partial/\partial x = \partial/\partial\xi, \quad (\partial/\partial r)_x = (\partial/\partial r)_\xi - \alpha\partial/\partial\xi,$$

so that equations (6) and (7) become

$$u_r - \alpha u_\xi = v_\xi, \tag{10}$$

$$[\alpha^2 u_\xi - v_r + \alpha v_\xi - v/r] + M^2 u[(\gamma+1)\,u_\xi + (\gamma-1)\,(v_r - \alpha v_\xi + v/r)] + 2M^2 v v_\xi = 0. \tag{11}$$

Substituting for $v_\xi$ from (10), (11) becomes

$$[\alpha u_r - v_r - v/r] + [(\gamma-1)\,M^2(v_r - \alpha u_r + v/r)\,u + 2M^2 v u_r] + M^2[\{(\gamma+1) + (\gamma-1)\,\alpha^2\}\,u - 2\alpha v]\,u_\xi = 0. \tag{12}$$

In this equation, the coefficient of $u_\xi$ is $O(u)$ as $u \to 0$, whereas those of $u_r$ and $v_r$ are $O(1)$. Therefore, probably the only quadratic terms which have a marked effect on the solution are those involving $u_\xi$, since they form the *only* terms in $u_\xi$. Thus the second

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term in (12) is omitted and $v$ is replaced therein by the linearized approximation $-\alpha u$, to obtain

$$2u_r + \frac{u}{r} + kuu_\xi = 0, \tag{13}$$

where $k = M^4(\gamma+1)\alpha^{-1}$. Now equation (13) can be written as

$$\frac{\partial}{\partial r}(u\sqrt{r}) + \frac{k(u\sqrt{r})}{2\sqrt{r}}\frac{\partial}{\partial \xi}(u\sqrt{r}) = 0, \tag{14}$$

or as the vanishing of the Jacobian

$$\frac{\partial}{\partial r}(u\sqrt{r})\frac{\partial}{\partial \xi}(\xi - kur) - \frac{\partial}{\partial \xi}(u\sqrt{r})\frac{\partial}{\partial r}(\xi - kur) = 0. \tag{15}$$

This implies a functional relation between $u\sqrt{r}$ and $\xi - kur$, so that, if $z$ denotes $-ku\sqrt{r}$, the solution is $u = -z/(kr^{\frac{1}{2}})$, $v = -\alpha u$ and $x = \alpha r - zr^{\frac{1}{2}} - h(z)$, where $h(z)$ is an arbitrary function. In other words, there is a set of curves $z$ = constant, each a parabola with axis in the Mach direction, on each of which $u\sqrt{r}$ is constant. In linearized theory, $u\sqrt{r}$ is constant along characteristics $x - \alpha r$ = constant, which suggests that on the more accurate theory the curves $z$ = constant are characteristics; that this is so can be shown as follows.

Along a characteristic, $dx/dr = \cot(\mu+\theta)$, where $\mu$ is the local Mach angle and $\theta$ is the direction of flow. To the first order in $u$, using $M = \operatorname{cosec}\mu_0$, $\alpha = \cot\mu_0$,

$$\mu = \sin^{-1}\frac{a}{U(1+u)} = \mu_0 - \left(1 + \frac{\gamma-1}{2}M^2\right)\alpha^{-1}u, \tag{16}$$

and $\theta = \tan^{-1}[v/(1+u)] = -\alpha u$, hence

$$\frac{dx}{dr} = \cot(\mu+\theta) = \alpha + \tfrac{1}{2}ku \tag{17}$$

along a characteristic. But this is also true along $z$ = constant, since

$$dx/dr = \alpha - \tfrac{1}{2}zr^{-\frac{1}{2}} = \alpha + \tfrac{1}{2}ku.$$

This completes the results required for §§ 3 and 4: *asymptotically the characteristics are parabolae* $x = \alpha r - zr^{\frac{1}{2}} +$ *constant, with* $u\sqrt{r}$ *and* $v\sqrt{r}$ *constant along them: in fact* $u\sqrt{r} = -z/k$, $v\sqrt{r} = \alpha z/k$.

## 3. The shapes of the shocks

The determination of the shapes of the shocks, as in the two-dimensional case, depends upon the following geometrical property, which is an easy consequence of the shock conditions. *If two regions of supersonic flow are separated by a shock, then, to the first order in the strength, the direction of the shock bisects the Mach directions of the two regions of flow.* (The Mach direction at a point is here understood to mean the *outward* direction making the local Mach angle with the direction of flow at the point.)

Only bodies of revolution of finite length are considered here, but otherwise the arguments apply quite generally, that is, shell shapes are included and the bow shock

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may be detached. The shocks will be formed from the neighbourhoods of the nose and tail, with the characteristics from the body intersecting them. At large distances from the axis, the shocks are weak and approximately at the undisturbed Mach angle, $\mu_0$, to the stream. Therefore, the asymptotic equation of the front shock is expressed in the form

$$x = r\cot\mu_0 - br^{n+1} = \alpha r - br^{n+1}, \tag{18}$$

where $b$ and $n$ are constants, with $n < 0$ since the shock is weak, and $b > 0$ since the shock is ahead of the Mach cone. The value of $n$ will now be determined.

The characteristics through points on $r = R$ are considered and, in particular, a characteristic $C$ through the point $(X, R)$ which meets the front shock $S$ in $(x_1 r_1)$, say (figure 1). From the results of § 2 the asymptotic equation of $C$ is

$$x = \alpha r - zr^{\frac{1}{2}} + X - \alpha R + zR^{\frac{1}{2}}. \tag{19}$$

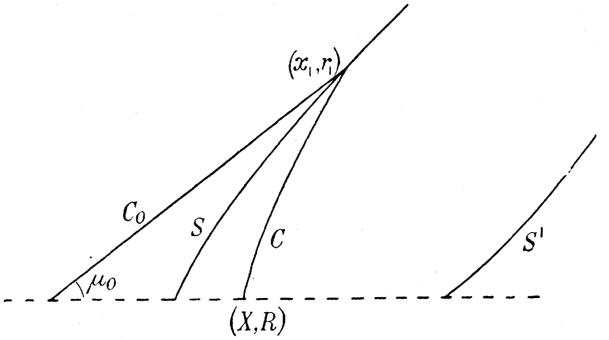


FIGURE 1

The slope of $S$ is given by $$\frac{dr}{dx} = \frac{1}{\alpha-(n+1)\,br^n}, \tag{20}$$

and the slope of $C$ by $$\frac{dr}{dx} = \frac{1}{\alpha-\frac{1}{2}zr^{-\frac{1}{2}}}. \tag{21}$$

At $(x_1 r_1)$, let the angle of $S$ to the stream be $\mu_0 + \epsilon_1$ and the angle of $C$ $\mu_0 + \epsilon_2$, then, from (20) and (21),

$$\epsilon_1 = b(n+1)\,r_1^n \sin^2\mu_0, \quad \epsilon_2 = \tfrac{1}{2}zr_1^{-\frac{1}{2}}\sin^2\mu_0. \tag{22}$$

If $C_0$ is a characteristic of the undisturbed flow ahead of the shock $S$, the angles of $C_0$, $S$ and $C$, to the stream, are $\mu_0$, $\mu_0+\epsilon_1$ and $\mu_0+\epsilon_2$ respectively. Therefore $\epsilon_2 = 2\epsilon_1$, giving

$$z = 4(n+1)\,br_1^{n+\frac{1}{2}}. \tag{23}$$

Finally $(x_1 r_1)$ lies on $C$ and $S$; hence eliminating $x_1$ from (18) and (19),

$$(4n+3)\,br_1^{n+1} - 4(n+1)\,br_1^{n+\frac{1}{2}}R^{\frac{1}{2}} + \alpha R - X = 0. \tag{24}$$

This equation expresses the condition that the point $(x_1 r_1)$ of the shock and the point $(X, R)$ of the line $r = R$, lie on the same characteristic.

Consider the characteristics meeting the front shock such that $r_1 \gg R$. Such characteristics exist since through any point of the shock there is a characteristic and

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it must ultimately meet the line $r = R$. (It cannot meet the rear shock since this must be at an angle to the local flow larger than the local Mach angle.) In this case the second term in (24) can be neglected, if it is assumed that $4n+3 \neq 0$. *With this assumption*, (24) becomes

$$(4n+3)\,br_1^{n+1} = X - \alpha R. \tag{25}$$

But $\epsilon_1 > 0$, since a shock is always at a greater angle than the local Mach angle to the flow ahead, therefore $(n+1) > 0$. Hence, if $r_1 \to \infty$ in (25) keeping $R$ fixed, $X \to \infty$. This is impossible because any characteristic meeting the front shock must always lie between the two shocks. Therefore the assumption $(4n+3) \neq 0$ is false and hence $n = -\frac{3}{4}$.

Therefore, the equation of the shock is

$$x = \alpha r - br^{\frac{1}{4}}, \tag{26}$$

where $b$ is a constant depending on the body, and the quantity $\epsilon_1$, which is a measure of the strength of the shock, falls off like $r^{-\frac{3}{4}}$.

If the flow behind the rear shock is assumed to return sufficiently near to the undisturbed state that the perturbation may be neglected in considering a first approximation to the equation of the shock, then the above argument may be applied to show that the equation of the rear shock is

$$x = \alpha r + b_1 r^{\frac{1}{4}}, \tag{27}$$

where $b_1$ is a constant depending on the body. This will be discussed in detail in § 8; though no conclusive result will be obtained, further grounds will be given for believing that (27) is the correct form. This will be assumed in the next section which would otherwise require modification.

## 4. Properties of the flow between the shocks

From equation (24), the characteristic through the point $(X, R)$ meets the shock in $(x_1, r_1)$, where

$$br_1^{-\frac{1}{4}} R^{\frac{1}{2}} = \alpha R - X. \tag{28}$$

(Only points $(X, R)$ on characteristics meeting the front shock are considered, since the corresponding results for the rear shock follow in the same way.) From (23), with $n = -\frac{3}{4}$,

$$z = br_1^{-\frac{1}{4}}, \tag{29}$$

therefore along a characteristic meeting the shock at $(x_1 r_1)$,

$$u\sqrt{r} = -zk^{-1} = -k^{-1}\,br_1^{-\frac{1}{4}}, \tag{30}$$

and, using (28), this gives

$$u(X, R) = -k^{-1}\frac{\alpha R - X}{R}, \tag{31}$$

$$v(X, R) = \alpha k^{-1}\frac{\alpha R - X}{R}. \tag{32}$$

These expressions still apply for a point $(X, R)$ on a characteristic meeting the rear shock.

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To the first order, the excess pressure at $(X, R)$ between the shocks is given by

$$p = -\rho_0 U^2 u = \rho_0 U^2 k^{-1} \frac{\alpha R - X}{R}, \tag{33}$$

so that the variation of pressure along a line $r = R$ is as shown in figure 2. There is a jump $\rho_0 U^2 k^{-1} b R^{-\frac{3}{4}}$ at the front shock, followed by a *linear* decrease to $-\rho_0 U^2 k^{-1} b_1 R^{-\frac{3}{4}}$ at the rear shock, after which the form is as yet unknown. The slope of the pressure curve for points between the two shocks is $\rho_0 U^2/kR$, *and this does not depend upon the particular shape of body considered.* The result is, in fact, independent of the equations of the shocks, as will be seen in § 6.

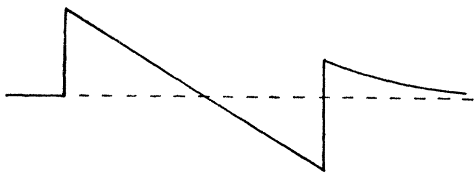

FIGURE 2

The aerodynamic forces on a body can be found from the rate of transfer of momentum through a 'control surface' enclosing the body. Since the shapes of the shocks are now known, together with some knowledge of the velocities between them, it may be inquired whether it is possible to calculate the drag. This is not so: *the drag on the body cannot be determined solely from the rate of transfer of momentum to infinity between the shocks.*

The drag is given by

$$D = -2\pi \int_\Gamma pr\,dr - 2\pi \int_\Gamma \rho U^2 u(1+u)\,r\,dr + 2\pi \int_\Gamma \rho U^2 uvR\,dx, \tag{34}$$

where the control surface is a cylinder radius $R$ with axis in the direction of the main stream and the integration is along the path $\Gamma$ in the meridian plane, as shown in figure 3. Since $r$ is constant on $AB$, the contribution from the region between the two shocks is

$$D_1 = -2\pi\rho_0 U^2 R \int_A^B uv\,dx \tag{35}$$

$$= \frac{2\pi\rho_0 U^2}{3} \frac{\alpha}{k^2} (b^3 + b_1^3)\, R^{-\frac{1}{4}}, \tag{36}$$

from (26), (27), (31) and (32). This tends to zero, as $R$ tends to infinity, therefore the above statement follows. The drag, which is not zero, must, therefore, be connected with loss of momentum in the wake due to gain in entropy at the shock wave.

For the two-dimensional case, the same result is found for the drag, although the lift can be determined correctly by the above method.

It seems surprising, since the solution of the flow contains an arbitrary *function* $h(z)$, depending on the body, that the pressure signature (between the shocks) is the same for all bodies, apart from two arbitrary constants. However, a particular value

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of $z$ specifies a particular characteristic so that $h(z)$ gives a measure of the distance between any two characteristics on $r = R$, say. This depends on the body considered, but since it is a finite distance it does not affect asymptotic considerations.

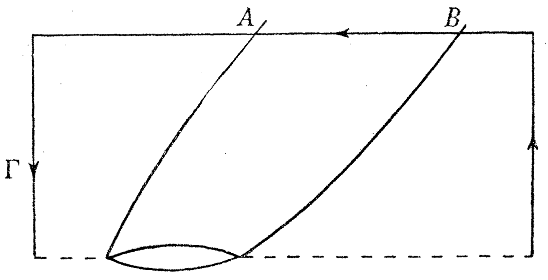


FIGURE 3

## 5. Full expansions of the physical quantities for large $r$

In § 2 certain results were found which open the way for a detailed investigation of axisymmetrical flow, at large distances from the axis. In terms of a variable $z$ the results were $u = -z/(kr^{\frac{1}{2}})$ and $x = \alpha r - zr^{\frac{1}{4}} - h(z)$, but they can be put in a form which compares more directly with linearized theory by taking a new variable $y$, such that $h(z) = -y$ and $z = -kF(y)$, say. Then

$$u = \frac{F(y)}{r^{\frac{1}{2}}} + \dots, \tag{37}$$

where $y$ is determined by $y = x - \alpha r - kF(y)\, r^{\frac{1}{4}}$ and $y =$ constant is a characteristic. The solution given by linearized theory is of the form

$$u = \frac{F(x-\alpha r)}{r^{\frac{1}{2}}} + \frac{F_1(x-\alpha r)}{r^{\frac{3}{2}}} + \dots, \tag{38}$$

$$v = \frac{G(x-\alpha r)}{r^{\frac{1}{2}}} + \frac{G_1(x-\alpha r)}{r^{\frac{3}{2}}} + \dots. \tag{39}$$

Comparison of (37) and (38) suggests the substitution of expansions

$$u = \frac{F(y)}{r^{\frac{1}{2}}} + \frac{F_1(y)}{r^{\frac{3}{2}}} + \dots, \tag{40}$$

$$v = \frac{G(y)}{r^{\frac{1}{2}}} + \frac{G_1(y)}{r^{\frac{3}{2}}} + \dots, \tag{41}$$

in the exact equations of motion, from which conditions on $y$ for such solutions to exist and relations between the functions $F, F_1, \dots, G, G_1, \dots$ should be found.

This approach to the solution of non-linear partial differential equations is similar, in some respects, to a method of Poincaré for non-linear ordinary differential equations. For instance, consider the equation for non-linear damped oscillations

$$\frac{d^2x}{dt^2} + \mu f\left(x, \frac{dx}{dt}\right) + x = 0, \tag{42}$$

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where $\mu$ is a small parameter. When $\mu = 0$, the equation has solution $A \sin t$, periodic with period $2\pi$. To find periodic solutions of (42) for small values of $\mu$, the possibility of a solution $x = A \sin \omega t + O(\mu)$, uniformly in $t$ (i.e. even for large $t$), is investigated. It is found that this is so only for a certain value of $\omega$ depending on $\mu$, and for certain values of the constant $A$. These restrictions are the conditions of periodicity and take the form

$$\int_0^{2\pi} f(A \sin t, A \cos t) \cos t\, dt = 0, \quad \omega = 1 - \frac{\mu}{A} \int_0^{2\pi} f(A \sin t, A \cos t) \sin t\, dt + O(\mu^2). \tag{43}$$

In the same way, (40) and (41) have the form of the linearized solution, and conditions on $y$ are found such that this solution is possible. Later, $y$ is restricted such that $y =$ constant is a characteristic.

This is the principle of the approach, but in fact it requires modifications. It is found that the expansions (40) and (41) are inadequate and

$$u = \frac{F(y)}{r^{\frac{1}{2}}} + \frac{F_1(y)}{r} + \frac{F_2(y)}{r^{\frac{3}{2}}} + \dots, \tag{44}$$

$$v = \frac{G(y)}{r^{\frac{1}{2}}} + \frac{G_1(y)}{r} + \frac{G_2(y)}{r^{\frac{3}{2}}} + \dots, \tag{45}$$

must be taken. This is reasonable because the presence of $u^2$, etc., in the exact equations of motion, immediately introduces terms in $r^{-1}$. Expansions (44) and (45) are still not quite the correct forms to take, but they will be used since the necessary modifications, which are only slight, can be carried out later. Actually § 2 suggests $y$ is of the form

$$y = x - \alpha r - H(y)\, r^{\frac{1}{2}} - \dots; \tag{46}$$

in other words $x$, like $u$ and $v$, can be expanded in powers of $r$ with coefficients functions of $y$. Since it is found that the equations of motion can be solved assuming three expansions of this type, it is unnecessary to go through the procedure of finding the behaviour of $y$ *ab initio*.

This introduction shows how the method of solution which will be used is suggested by linearized theory and analogy with Poincaré's method, but in fact it is much more convenient in all this work to use the original variable $z$ instead of $y$, and write

$$x = \alpha r - z r^{\frac{1}{2}} - m(z) - \dots, \tag{47}$$

$$u = f(z)\, r^{-\frac{1}{2}} + g(z)\, r^{-1} + \dots, \tag{48}$$

$$v = -\alpha u + a(z)\, r^{-1} + b(z)\, r^{-\frac{3}{2}} + \dots, \tag{49}$$

in place of (44), (45) and (46). To keep the argument general a term in $r^{-\frac{1}{2}}$ should be included in (49), but it would be found to be unnecessary and so is omitted.

The exact equations of steady irrotational flow, by (3), (4) and (5), are

$$u_r - v_x = 0, \tag{50}$$

$$[\alpha^2 u_x - v_r - v/r] + M^2 u[(\gamma+1)\, u_x + (\gamma-1)\,(v_r + v/r)] + 2M^2 v v_x$$
$$+ M^2[\tfrac{1}{2}(\gamma-1)\,(u^2+v^2)\,(u_x + v_r + v/r) + u^2 u_x + 2uv v_x + v^2 v_r] = 0. \tag{51}$$

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From (47), using primes to denote differentiation with respect to $z$,

$$z_x = \frac{-1+\dots}{r^{\frac{1}{2}}+m'+\dots} = -r^{-\frac{1}{2}}+m'r^{-1}+O(r^{-\frac{3}{2}}), \tag{52}$$

$$z_r = \frac{\alpha-\frac{1}{2}zr^{-\frac{1}{2}}-\dots}{r^{\frac{1}{2}}+m'+\dots} = -\alpha z_x-\tfrac{1}{2}zr^{-1}+\tfrac{1}{2}zm'r^{-\frac{3}{2}}+O(r^{-2}). \tag{53}$$

From (48) and (49)

$$\begin{aligned} u_x = \frac{\partial u}{\partial z}z_x &= (f'r^{-\frac{1}{2}}+g'r^{-1}+\dots)(-r^{-\frac{1}{2}}+m'r^{-1}+\dots) \\ &= -f'r^{-1}+(f'm'-g')r^{-\frac{3}{2}}+O(r^{-2}), \end{aligned} \tag{54}$$

$$\begin{aligned} v_x = \frac{\partial v}{\partial z}z_x &= -\alpha u_x+(-r^{-\frac{1}{2}}+m'r^{-1}+\dots)(a'r^{-1}+b'r^{-\frac{3}{2}}+\dots) \\ &= -\alpha u_x-a'r^{-\frac{3}{2}}+(a'm'-b')r^{-2}+O(r^{-\frac{5}{2}}), \end{aligned} \tag{55}$$

$$\begin{aligned} u_r = z_r\frac{\partial u}{\partial z}+\left(\frac{\partial u}{\partial r}\right)_z &= -\alpha z_x\frac{\partial u}{\partial z}+(-\tfrac{1}{2}zr^{-1}+\tfrac{1}{2}zm'r^{-\frac{3}{2}}+\dots) \\ &\qquad \times(f'r^{-\frac{1}{2}}+g'r^{-1}+\dots)-\tfrac{1}{2}fr^{-\frac{3}{2}}-gr^{-2}+\dots \\ &= -\alpha u_x-\tfrac{1}{2}(zf'+f)r^{-\frac{3}{2}}+(\tfrac{1}{2}zm'f'-\tfrac{1}{2}zg'-g)r^{-2}+O(r^{-\frac{5}{2}}). \end{aligned} \tag{56}$$

It is convenient, before finding $v_r$, to use equation (50). From (55) and (56), equating coefficients of powers of $r$,

$$a' = \tfrac{1}{2}(zf'+f), \tag{57}$$

$$b' = a'm'-\tfrac{1}{2}zm'f'+\tfrac{1}{2}zg'+g = \tfrac{1}{2}fm'+\tfrac{1}{2}zg'+g. \tag{58}$$

Hence

$$\begin{aligned} v_r &= -\alpha u_r+\left(\frac{\partial}{\partial r}+z_r\frac{\partial}{\partial z}\right)(ar^{-1}+br^{-\frac{3}{2}}+\dots) \\ &= \alpha^2u_x+\tfrac{1}{2}\alpha(zf'+f)r^{-\frac{3}{2}}-\alpha(\tfrac{1}{2}zm'f'-\tfrac{1}{2}zg'-g)r^{-2}-ar^{-2} \\ &\qquad +(\alpha r^{-\frac{1}{2}}-(\alpha m'+\tfrac{1}{2}z)r^{-1})(a'r^{-1}+b'r^{-\frac{3}{2}})+O(r^{-\frac{5}{2}}) \\ &= \alpha^2u_x+w, \end{aligned} \tag{59}$$

where

$$w = 2\alpha a'r^{-\frac{3}{2}}-[\alpha(zm'f'-zg'-2g)+a+\tfrac{1}{2}za']r^{-2}+O(r^{-\frac{5}{2}}), \tag{60}$$

using (58). Substituting in equation (51) and neglecting terms $O(r^{-\frac{5}{2}})$,

$$\begin{aligned} &[-w-(-\alpha fr^{-\frac{3}{2}}-\alpha gr^{-2}+ar^{-2})]+M^2u[(\gamma+1)u_x+(\gamma-1)(\alpha^2u_x+w-\alpha fr^{-\frac{3}{2}})] \\ &\quad +2M^2(-\alpha u+ar^{-1})(-\alpha u_x-a'r^{-\frac{3}{2}})+M^2[\tfrac{1}{2}(\gamma-1)(1+\alpha^2)^2+1+2\alpha^2+\alpha^4]u^2u_x = 0, \end{aligned} \tag{61}$$

or

$$\begin{aligned} &[-w+\alpha fr^{-\frac{3}{2}}+(\alpha g-a)r^{-2}]+(\gamma+1)M^4uu_x+(\gamma-1)M^2(w-\alpha fr^{-\frac{3}{2}})u \\ &\quad -2M^2\alpha au_xr^{-1}+2M^2\alpha a'ur^{-\frac{3}{2}}+\tfrac{1}{2}(\gamma+1)M^6u^2u_x = 0. \end{aligned} \tag{62}$$

The coefficient of $r^{-\frac{3}{2}}$ using (57) gives $zf'+kff' = 0$, where $k = M^4(\gamma+1)/\alpha$. Hence if $f' \neq 0$ (a hypothesis which will be justified later)

$$f = -z/k. \tag{63}$$

From (57) $a' = -z/k$, hence

$$a = -\frac{z^2}{2k}-A, \tag{64}$$

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where $A$ is an arbitrary constant. Next the coefficient of $r^{-2}$ gives that

$$\left[\alpha\left(-\frac{zm'}{k}-zg'-2g\right)+\alpha g-\frac{z^2}{2k}\right]+\alpha k\left[\frac{g}{k}+\frac{z}{k}\left(\frac{m'}{k}+g'\right)\right]+(\gamma-1)M^2\left(-\frac{z}{k}\right)\left(-\frac{2\alpha z}{k}+\frac{\alpha z}{k}\right)$$
$$+2M^2\alpha\left(\frac{z^2}{2k^2}+\frac{A}{k}\right)+\frac{2M^2\alpha z^2}{k^2}+\frac{M^2\alpha z^2}{2k^2}=0, \qquad (65)$$

that is $$\frac{2\alpha M^2 A}{k}+\frac{z^2M^2}{2\alpha k^2}[M^2(\gamma+4)-(2\gamma+5)]=0. \qquad (66)$$

Excluding the trivial result $z=$ constant, this is only true if

$$A=0 \quad \text{and} \quad M=\sqrt{\{(2\gamma+5)/(\gamma+4)\}}=1{\cdot}202$$

for air. For other values of $M$, this contradiction can be eliminated by modifying the original expansions as follows. In the above, let $m(z)$, $b(z)$ stand for

$$m_1(z)\log r+m_2(z), \quad b_1(z)\log r+b_2(z),$$

respectively. ($a(z)$ and $g(z)$ could be assumed to be of this form also, but the equations give $a_1=0$ and $g_1=0$ making it unnecessary.) This change in $m$ and $b$ only affects terms involving derivatives with respect to $r$. In (53)

$$z_r=\frac{\alpha-\frac{1}{2}zr^{-\frac{1}{2}}-m_1r^{-1}-\ldots}{r^{\frac{1}{2}}+m'+\ldots}=-\alpha z_x-\tfrac{1}{2}zr^{-1}+\tfrac{1}{2}zm'r^{-\frac{3}{2}}-m_1r^{-\frac{3}{2}}+\ldots, \qquad (67)$$

hence $z_r$ has an extra term $-m_1r^{-\frac{3}{2}}$. Similarly in (56) $u_r$ has an extra term $-m_1f'r^{-2}$; in (58) $b'$ has an extra term $m_1f'$; and in (60) $w$ has an extra term $2\alpha m_1f'r^{-2}$. Now, from the coefficient of $r^{-2}$ in the equation of motion (62), (66) is replaced by

$$\frac{2\alpha M^2A}{k}+\frac{z^2M^2}{2\alpha k^2}[M^2(\gamma+4)-(2\gamma+5)]-2\alpha m_1f'=0, \qquad (68)$$

hence, from (63), $$m_1=-kKz^2-M^2A, \qquad (69)$$

where $$K=\frac{M^2}{4\alpha^2k^2}[M^2(\gamma+4)-(2\gamma+5)]. \qquad (70)$$

The equations of motion do not determine the functions $g(z)$ and $b(z)$ unless the condition $z=$ constant is a characteristic is stipulated. The characteristics of the equations

$$\left.\begin{aligned}A_1u_x+B_1u_r+C_1v_x+D_1v_r+E_1&=0,\\ A_2u_x+B_2u_r+C_2v_x+D_2v_r+E_2&=0,\end{aligned}\right\} \qquad (71)$$

are given by Courant & Friedrichs (1948):

$$[AC]\,dr^2-([AD]+[BC])\,dx\,dr+[BD]\,dx^2=0, \qquad (72)$$

where $[XY]$ denotes $X_1Y_2-X_2Y_1$. For equations (50) and (51),

$$A_1=D_1=0,\ B_1=1,\ C_1=-1,$$

and
$$A_2=\alpha^2+(\gamma+1)M^2u+M^2\{\tfrac{1}{2}M^2(\gamma-1)+1\}u^2+O(r^{-\frac{3}{2}}),$$
$$B_2=0,$$
$$C_2=2M^2(-\alpha u+ar^{-1})(1+u)+O(r^{-\frac{3}{2}}),$$
$$D_2=-1+(\gamma-1)M^2u+M^2\{\tfrac{1}{2}(\gamma-1)M^2+\alpha^2\}u^2+O(r^{-\frac{3}{2}}).$$

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If $z$ = constant is a characteristic, $dx/dr = \alpha - \frac{1}{2}zr^{-\frac{1}{2}} - m_1 r^{-1} + O(r^{-\frac{3}{2}})$; hence substituting in (72),

$$\alpha^2 + (\gamma+1) M^2 u + M^2\{\tfrac{1}{2}(\gamma-1) M^2 + 1\} u^2 - 2M^2(-\alpha u - \alpha u^2 + ar^{-1})(\alpha - \tfrac{1}{2}zr^{-\frac{1}{2}})$$
$$+ \{-1 + (\gamma-1) M^2 u + M^2(\tfrac{1}{2}(\gamma-1) M^2 + \alpha^2) u^2\}\{\alpha^2 - \alpha zr^{-\frac{1}{2}} + \tfrac{1}{4}z^2 r^{-1} - 2\alpha m_1 r^{-1}\}$$
$$+ O(r^{-\frac{3}{2}}) = 0, \tag{73}$$

or

$$(\gamma+1) M^4 u + \alpha zr^{-\frac{1}{2}} + M^2\{\tfrac{1}{2}M^2(\gamma-1)+1\} u^2 - 2M^2\{-\alpha^2 u^2 + \alpha ar^{-1} + \tfrac{1}{2}\alpha zur^{-\frac{1}{2}}\}$$
$$+ M^2\{\tfrac{1}{2}(\gamma-1) M^2 + \alpha^2\} \alpha^2 u^2 - (\gamma-1) M^2 \alpha zur^{-\frac{1}{2}} - \tfrac{1}{4}z^2 r^{-1} + 2\alpha m_1 r^{-1} + O(r^{-\frac{3}{2}}) = 0. \tag{74}$$

Equating coefficients of $r^{-\frac{1}{2}}$ to zero,

$$(\gamma+1) M^4 f + \alpha z = 0, \tag{75}$$

which was found previously; this excludes the possibility $f$ = constant which was noted at equation (63); and

$$\alpha kg + \tfrac{1}{2}(\gamma+1) M^6 \frac{z^2}{k^2} + \frac{M^2\alpha z^2}{k} + 2M^2\alpha A + \frac{M^2\alpha z^2}{k} + (\gamma-1)\frac{M^2\alpha z^2}{k} - \tfrac{1}{4}z^2 + 2\alpha m_1 = 0. \tag{76}$$

Hence, using (69),

$$g = K_1 z^2, \tag{77}$$

where

$$K_1 = 2K - \frac{1}{\alpha k}\left[(\gamma + \tfrac{3}{2})\frac{M^2\alpha}{k} - \frac{1}{4}\right] = \frac{M^2}{4\alpha^2 k^2}[M^2(3-\gamma) - 4]. \tag{78}$$

From (58), with the additional term $m_1 f'$, $b_1(z)$ and $b_2(z)$ are determined in terms of $m_2(z)$, which is arbitrary,

$$b_1 = \tfrac{1}{3}Kz^3 + B, \tag{79}$$

$$b_2 = -\frac{1}{2k}\int_0^z \zeta m_2'(\zeta)\, d\zeta + \tfrac{1}{2}D + \tfrac{1}{3}(K + 2K_1) z^3 + \frac{M^2 Az}{k}, \tag{80}$$

where $B$ and $D$ are arbitrary constants.

The results are

$$u = -\frac{z}{k} r^{-\frac{1}{2}} + K_1 z^2 r^{-1} + \ldots, \tag{81}$$

$$v = -\alpha u - \left(\frac{z^2}{2k} + A\right) r^{-1} + \left(\frac{Kz^3}{3} + B\right) r^{-\frac{3}{2}} \log r$$
$$- \left\{\frac{1}{2k}\int_0^z \zeta h'(\zeta)\, d\zeta - \tfrac{1}{2}D - \tfrac{1}{3}(K + 2K_1) z^3 - \frac{M^2 Az}{k}\right\} r^{-\frac{3}{2}} + \ldots, \tag{82}$$

$$x = \alpha r - zr^{\frac{1}{2}} + (kKz^2 + M^2 A) \log r - h(z) - \ldots, \tag{83}$$

replacing the arbitrary function $m_2(z)$ by $h(z)$.

It is interesting that the possibility of expansions (48) and (49) implies that $z$ = constant is a characteristic to the second approximation, but not to higher ones; to obtain a unique solution to higher approximations, the nature of these curves must be specified more precisely, as was done above by assuming that they are exactly characteristics.

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The velocities $u$ and $v$ were appropriate for solving the equations of motion in this case, but the velocity potential $\phi$ could have been used and in some problems may be more convenient. The expression for $\phi$ is easily deduced from the above, since $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial r$; it is

$$\phi = -A\log r - 2B(2+\log r)\,r^{-\frac{1}{2}} - C + \frac{z^2}{2k} - \frac{2Kz^3}{3}r^{-\frac{1}{2}}\log r - \left\{\frac{K_1 z^3}{3} - \frac{1}{k}\int_0^z \zeta h'(\zeta)\,d\zeta + D\right\}r^{-\frac{1}{2}} + \ldots, \tag{84}$$

where $C$ is an arbitrary constant. Actually, in this work, the constants $A$, $B$ and $C$ are found to be zero.

## 6. Further terms in the equation of the front shock

In § 3 a first approximation for the equation of the front shock was obtained, but it was assumed to be of the form $x = \alpha r - br^{n+1}$; the possibility of a term $(\log r)^m r^{n+1}$ or other forms was not considered. Now, the arguments are reconsidered making no such assumption, and, in addition, further terms in the equation will be determined.

The equation of a characteristic in the region between the shocks is

$$x = \alpha r - zr^{\frac{1}{2}} + (kKz^2 + M^2A)\log r - h(z) + O(r^{-\frac{1}{2}}\log r), \tag{85}$$

where $z$ is constant, and an error term, of the form expected, has been included for the further terms in the series. For *any* given value of $r$, $x$ is bounded, since the characteristics must lie between the two shocks, hence $z$ and $h(z)$ are bounded and $h(z)$ may be expanded as $h(0) + zh'(0) + O(z^2)$. Then

$$z = \frac{\alpha r - x + M^2A\log r - h(0)}{r^{\frac{1}{2}} + h'(0)} + O(z^2 r^{-\frac{1}{2}}\log r) + O(r^{-1}\log r). \tag{86}$$

Let the equation of the front shock be $x = \alpha r - f(r)$, then immediately $f'(r) = O(r^{-\frac{1}{2}})$, because the shock must always be at an angle less than the local Mach angle to the local flow behind it. The conditions at the shock give

$$\left(\frac{dx}{dr} - \alpha\right)_s = \frac{1}{2}\left(\frac{dx}{dr} - \alpha\right)_c + O\left[\left(\frac{dx}{dr} - \alpha\right)_c^2\right], \tag{87}$$

where suffices $s$ and $c$ indicate values on the shock and on the characteristic respectively. Hence

$$-f'(r) = -\tfrac{1}{4}zr^{-\frac{1}{2}} + \tfrac{1}{2}M^2Ar^{-1} + O(z^2r^{-1}) + O(r^{-\frac{3}{2}}\log r). \tag{88}$$

Substituting the value of $z$ given by (86), with $x = \alpha r - f(r)$, and using

$$z = O(f'r^{\frac{1}{2}}) + O(r^{-\frac{1}{2}})$$

from (88), a first-order ordinary linear differential equation is obtained for $f(r)$:

$$f(r) - 4r^{\frac{1}{2}}(r^{\frac{1}{2}} + h'(0))f'(r) + M^2A(\log r - 2) - h(0) = O(rf'^2\log r) + O(r^{-\frac{1}{2}}\log r). \tag{89}$$

This equation is easily solved by successive approximation; since $f'(r) = O(r^{-\frac{1}{2}})$, (89) gives

$$f(r) - 4rf'(r) = O(\log r) \quad \text{or} \quad d(r^{-\frac{1}{4}}f(r))/dr = O(r^{-\frac{5}{4}}\log r)$$

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and hence
$$f(r) = br^{\frac{1}{4}} + O(\log r),$$
where $b$ is an arbitrary constant. Using this result, (89) may be written as
$$f(r) - 4rf'(r) = -M^2A(\log r - 2) + h(0) + bh'(0)\, r^{-\frac{1}{4}} + O(r^{-\frac{1}{2}} \log r), \tag{90}$$
with solution
$$f(r) = br^{\frac{1}{4}} - M^2A(\log r + 2) + h(0) + \tfrac{1}{2}bh'(0)\, r^{-\frac{1}{4}} + O(r^{-\frac{1}{2}} \log r). \tag{91}$$

Now, the shock conditions are essentially two in number; if the position of the shock is known, the velocity components $u$ and $v$ are determined behind it. In the above, only one condition has been used, namely, the 'angle property'; as the other, it is convenient to take the condition that the component of momentum tangential to the shock is continuous, that is, $v + u \cot(\mu_0 + \epsilon_1) = 0$, or
$$v + \alpha u = uf'(r). \tag{92}$$
From (88) and (91), $z = br^{-\frac{1}{4}} + O(r^{-\frac{1}{2}})$ at the shock; hence from (81) and (82), equation (92) gives
$$-Ar^{-1} + Br^{-\frac{3}{2}} \log r + \frac{1}{2}\left(D - \frac{b^2}{k}\right) r^{-\frac{3}{2}} + O(r^{-\frac{7}{4}})$$
$$= \left[-\frac{br^{-\frac{3}{4}}}{k} + O(r^{-1})\right] [\tfrac{1}{4}br^{-\frac{3}{4}} + O(r^{-1})], \tag{93}$$
for all $r$. Therefore $A = B = 0$ and $b^2 = 2kD$.

The equation of the front shock is
$$x = \alpha r - br^{\frac{1}{4}} - h(0) - \tfrac{1}{2}bh'(0)\, r^{-\frac{1}{4}} + O(r^{-\frac{1}{2}} \log r), \tag{94}$$
where $b = \sqrt{(2Dk)}$.

Further approximations to the values of $u$, $v$ and $p$, at points between the shocks, found in § 4, are now possible. From (81) and (86), using $x - \alpha r = O(r^{\frac{1}{4}})$,
$$u = \frac{x + h(0) - \alpha r}{kr} - \frac{h'(0)}{k} \frac{x + h(0) - \alpha r}{r^{\frac{3}{2}}} + O(r^{-\frac{3}{2}} \log r), \tag{95}$$
$$p = -\rho_0 U^2 u + O(u^2) = -\frac{\rho_0 U^2}{k} \left\{\frac{x + h(0) - \alpha r}{r} - h'(0) \frac{x + h(0) - \alpha r}{r^{\frac{3}{2}}} + O(r^{-\frac{3}{2}} \log r)\right\}. \tag{96}$$
These results show that even to this second approximation the pressure falls linearly between the shocks, although the slope of the pressure signature is now
$$\rho_0 U^2(1 - h'(0)\, r^{-\frac{1}{2}})/kr,$$
and $h'(0)$ depends upon the particular body under consideration.

## 7. Connexion with linearized theory

In this paper a general theory of the flow past a body of revolution has been developed, which is applicable at large distances from the axis. For the special case of a pointed slender body, linearized theory gives a good approximation near the body, and in this section it will be shown how to modify linearized theory to be uniformly valid, for large $r$ as well as small.

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For the flow between the shocks, the results of § 5 written in terms of the variable $y = -h(z)$, with $z = -kF(y)$ say, are

$$u = \frac{F(y)}{r^{\frac{1}{2}}} + \frac{k^2 K_1 F^2(y)}{r} + \dots, \tag{97}$$

$$v = -\alpha u - \frac{\frac{1}{2}kF^2(y)}{r} - \frac{\frac{1}{3}k^3 K F^3(y) \log r}{r^{\frac{3}{2}}} - \frac{\frac{1}{2}\int_{y_0}^{y} F(y')\,dy' - \frac{1}{2}D + \frac{1}{3}k^3(K+2K_1)\,F^3(y) + \dots}{r^{\frac{3}{2}}}, \tag{98}$$

where $$y = x - \alpha r - kF(y)\,r^{\frac{1}{2}} - k^3 K F^2(y) \log r + \dots, \tag{99}$$

and $y_0$ is the zero of $F(y)$, that is, the value of $y$ on the 'dividing' characteristic $C^*$ (which is asymptotic to the straight line $x = \alpha r + y_0$). Linearized theory has

$$\phi = -\frac{1}{2\pi}\int_0^{x-\alpha r} \frac{S'(t)\,dt}{\sqrt{\{(x-t)^2 - \alpha^2 r^2\}}}, \tag{100}$$

where $S(x)$ is the cross-sectional area of the body at a distance $x$ along the axis from the nose. For large $r$, this gives

$$u = \frac{F(\xi)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{\frac{3}{2}}}\right), \tag{101}$$

$$v = -\alpha u - \frac{\frac{1}{2}\int_0^{\xi} F(\xi')\,d\xi'}{r^{\frac{3}{2}}} + O\left(\frac{1}{r^{\frac{5}{2}}}\right), \tag{102}$$

where $\xi = x - \alpha r$ and $$F(\xi) = -\frac{1}{2\pi\sqrt{(2\alpha)}}\int_0^{\xi} \frac{S''(t)\,dt}{\sqrt{(\xi - t)}}. \tag{103}$$

The above results show that linearized theory would have the correct *form* of expansions for $u$ and $v$ at large distances, with only terms in $F^2$, $F^3$, ... neglected, if the arbitrary constant $D$ had the value $-\int_0^{y_0} F(y')\,dy'$. For this case of a slender body, $F$ is small, thus the omission of $F^2$, $F^3$, ... causes only slight errors; the failure of linearized theory at large distances is due to the second modification which it makes to (97) and (98), that is, the replacement of $y$ by $\xi = x - \alpha r$. This suggests that linearized theory will give a good approximation everywhere provided $x - \alpha r$ is replaced therein by $y$, where $y = x - \alpha r - kF(y)\,r^{\frac{1}{2}}$; further, $D$ is determined in terms of $F(y)$ to give

$$b^2 = 2kD = -2k\int_0^{y_0} F(y')\,dy'. \tag{104}$$

From (103), $F(y)$ has a factor $\delta^2$, where $\delta$ is the fineness ratio for the body considered, hence near the body $y = x - \alpha r + O(\delta^2)$, and the procedure suggested above has, from (100),

$$\phi = -\frac{1}{2\pi}\int_0^{y} \frac{S'(t)\,dt}{\sqrt{\{(y-t)\,(y+2\alpha r - t)\}}} \tag{105}$$

$$= -\frac{1}{2\pi}\int_0^{x-\alpha r} \frac{S'(t)\,dt}{\sqrt{\{(x-t)^2 - \alpha^2 r^2\}}} + O(\delta^4). \tag{106}$$

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But linearized theory already neglects terms $O(\delta^4)$, so that the modified theory is just as accurate near the body as well as giving a valid first approximation for large $r$.

The dependence of the various constants, occurring in § 6, upon the Mach number of the undisturbed flow and the fineness ratio of the body may be found from (103) and (104). Since $h(z) = -y$ and $z = -kF(y)$, for any particular shape of body

$$b \propto M^2(M^2-1)^{-\frac{3}{8}}\delta, \quad h'(0) = -(dy/dz)_{z=0} = \{kF'(y_0)\}^{-1} \propto M^{-4}(M^2-1)^{\frac{3}{4}}\delta^{-2}$$

and $h(0) = -y_0$ is independent of $M$ and $\delta$. The strength of the shock, measured by $\alpha - (dx/dr)$, is

$$\tfrac{1}{4}br^{-\frac{3}{4}} - \tfrac{1}{8}bh'(0)\,r^{-\frac{5}{4}} + O(r^{-\frac{3}{2}}\log r),$$

and the strength at the nose (where it is maximum) is found from the theory of flow past a cone to be $O(\delta^4)$, hence the prediction of the shape of the shock is only applicable at distances of order $\delta^{-4}$.

*Example.* This modified theory, valid for all $r$, is now discussed for the special case of the symmetrical parabolic profile $r = 2\delta(x-x^2)$, $0 \leqslant x \leqslant 1$; the body is obtained by revolving an arc of a parabola about a line perpendicular to its axis. In this case, $S''(t) = 8\pi\delta^2(1-6t+6t^2)$ and

$$F(y) = -\frac{4\sqrt{2}}{5\sqrt{\alpha}}\,\delta^2 y^{\frac{1}{2}}(5-20y+16y^2), \tag{107}$$

$$F'(y) = -\frac{2\sqrt{2}}{\sqrt{\alpha}}\,\delta^2 y^{-\frac{1}{2}}(1-12y+16y^2). \tag{108}$$

Hence $F(y) = 0$ at $y = 0$, $(20 \pm \sqrt{80})/32$ and the form of $F(y)$ is as shown (figure 4). The region which applies to the flow at large distances must be $AC$, since earlier considerations have shown that, far from the axis, $F(y)$ increases nearly linearly from negative to positive values, and therefore $y_0$ is the zero of $F(y)$ at which $F'(y)$ is positive. The values of the various constants are

$$b = 0{\cdot}91\,M^2\alpha^{-\frac{3}{4}}\delta, \quad h(0) = -0{\cdot}35 \quad \text{and} \quad h'(0) = 0{\cdot}071\,M^{-4}\alpha^{\frac{3}{4}}\delta^{-2}.$$

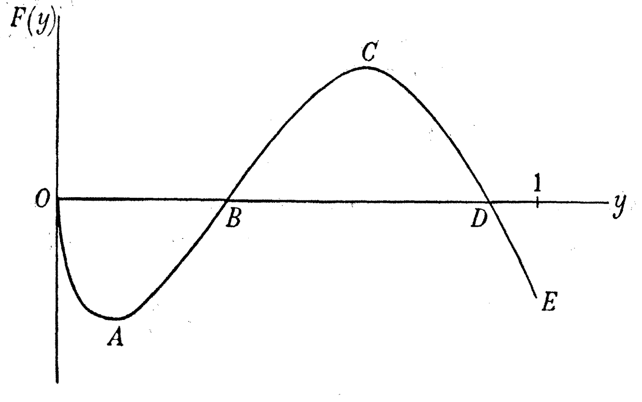


FIGURE 4

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The second zero of $F(y)$ at $D$ suggests immediately that a shock is formed at the rear before the tail is reached because two straight characteristics in the flow could not be accounted for; there is also the possibility of the characteristics corresponding to the regions $CE$ and $OA$ forming envelopes. An envelope of characteristics is given, in terms of a parameter $y$, by

$$kF'(y)\,R^{\frac{1}{2}}+1 = 0, \tag{109}$$

$$X = \alpha R + kF(y)\,R^{\frac{1}{2}}+y. \tag{110}$$

Since the positive root is always taken for $R^{\frac{1}{2}}$, the characteristics actually form an envelope when $F'(y)<0$, that is, for $OA$ and $CE$. Equation (110) shows that if $F(y)<0$, that is, for $DE$, the point $(X, R)$ always lies ahead of the undisturbed Mach cone from the corresponding point on the body. Hence, since the envelope cannot appear in the flow, a shock must be formed before the undisturbed Mach cone from $D$, at a distance $O(\delta^{-4})$ from the body.

The appearance of a third shock as above or due to irregularities in the shape of the body (as seen in photographs of bullets in flight) may be discussed by the methods developed in this paper. It is found that such a shock either joins with the rear shock or if three shocks persist to infinity, the middle one has the equation

$$x = \alpha r + c_0 + d_0 r^{-\frac{1}{2}} + O(r^{-1}\log r),$$

where $c_0$ and $d_0$ are constants depending on the body. In the latter case, the pressure falls linearly from positive to negative values between any two shocks, and this could not be the case for the example considered; the third shock joins with the rear one.

In $OA$, $F(y)<0$, therefore the front envelope is a distance $O(R^{\frac{1}{2}})$ ahead of the undisturbed Mach cone, whereas the shock is a distance $O(R^{\frac{1}{4}})$ ahead. Therefore the front envelope does not occur (the characteristics meet the shock before meeting one another).

## 8. The rear shock wave

In the previous sections, discussions of the rear shock and the flow downstream of it have, in general, been avoided; the results do not follow simply from suitable modifications of the arguments employed at the front shock. In fact, the author cannot give complete results, although some definite answers are found which give support to his conclusions.

The flow behind the rear shock is axisymmetrical, and hence, with the assumption of irrotational motion a solution of the equations for $u$ and $v$ is given in § 5 in terms of a variable $z_1$, an arbitrary function $h_1(z_1)$ and certain arbitrary constants $A_1$, $B_1$, $D_1$, .... However, for this region, $h_1(z_1)$ is not necessarily bounded, and since repeated integrals of $z_1$ with respect to $h_1$ occur in the solution, the expansions are found to be valid only if $h_1/r$, that is, $(x-\alpha r)/r$, is bounded. At the shock where $h_1$ is maximum, it is at most $O(r^{\frac{1}{2}})$, hence the expansions will be valid in a certain neighbourhood of the shock.

Consider characteristics $C$ and $C_1$ through a point of the shock, which belong to the regions ahead of and behind it respectively. Then $C$ is of the form

$$x = \alpha r - z r^{\frac{1}{2}} - h(z) + O(z^2 \log r), \tag{111}$$

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$C_1$ is of the form

$$x = \alpha r - z_1 r^{\frac{1}{4}} + M^2 A_1 \log r - h_1(z_1) + O(z_1^2 \log r), \tag{112}$$

and the equation of the shock is taken to be

$$x = \alpha r + f_1(r). \tag{113}$$

The angle property at the shock gives

$$z + z_1 = -4r^{\frac{1}{2}} f_1'(r) + 2M^2 A_1 r^{-\frac{1}{2}} + O(z^2 r^{-\frac{1}{2}}) + O(z_1^2 r^{-\frac{1}{2}}), \tag{114}$$

and from (111) and (113)

$$z = -\frac{f_1(r) + h(0)}{r^{\frac{1}{2}} + h'(0)} + O(z^2 r^{-\frac{1}{2}} \log r), \tag{115}$$

hence
$$z_1 = \frac{f_1(r) + h(0)}{r^{\frac{1}{2}} + h'(0)} - 4r^{\frac{1}{2}} f_1'(r) + 2M^2 A_1 r^{-\frac{1}{2}} + O(z^2 r^{-\frac{1}{2}} \log r) + O(z_1^2 r^{-\frac{1}{2}}). \tag{116}$$

Equations (115) and (116) are the basis for the conclusion that the rear shock is

$$x - \alpha r = b_1 r^{\frac{1}{4}} + \text{smaller terms}, \tag{117}$$

where $b_1$ is a constant depending on the body. The reason is that (117) is the only form of $f_1(r)$ which makes $z_1$ of appreciably smaller order than $z$, that is, which makes the perturbation velocities and pressure behind the rear shock appreciably smaller than between the shocks. For, from (115), $z = -b_1 r^{-\frac{1}{4}} + O(r^{-\frac{1}{2}})$ and, from (116), $z_1 = O(r^{-\frac{1}{2}})$. This 'small' perturbation behind the shock is expected from comparison with the two-dimensional problem and also the pressure signature obtained from experiments on the flight of bullets show this result. The expression 'appreciably smaller' was used because, for example, $f_1(r) = O(r^{\frac{1}{4}} \log^{\frac{1}{2}} r)$ (which arises as a possibility) gives

$$z = O(r^{-\frac{1}{4}} \log^{\frac{1}{2}} r), \quad z_1 = O(r^{-\frac{1}{4}} \log^{-\frac{1}{2}} r)$$

but is not considered to exhibit the required behaviour.

At the shock, $x - \alpha r = b_1 r^{\frac{1}{4}}$ and $z_1$ is at most $O(r^{-\frac{1}{2}})$, therefore, from (112) $h_1 \sim -b_1 r^{\frac{1}{4}}$ and tends to $-\infty$ as $z_1 \to 0$. The characteristics tend to infinity downstream like $r^{\frac{1}{4}}$, where $r$ is the distance from the axis at which they meet the shock, and become straighter since $z_1 \to 0$; thus the whole field behind the shock influences its shape. This inability to give a definite form for the equation of the shock is not due to the conditions being insufficient to determine it but to the difficulty of completing a solution over the whole field. The arbitrary functions $h(z)$ and $h_1(z_1)$ are treated as known, essentially being determined from the flow near the axis, but for the rear shock, in contrast to the case of the front shock, the precise form of $h_1(z_1)$ must be known before any approximation to the equation can be given.

However, in the special case of a slender body, pointed at both ends, the form of the arbitrary function may be obtained from the modification of the linearized results for this region by substitution of $y_1$, determined by $y_1 = x - \alpha r - kF_1(y_1) r^{\frac{1}{2}}$, for $\xi = x - \alpha r$. Linearized theory has

$$u = -\frac{1}{2\pi} \int_0^1 \frac{S''(t)\, dt}{\sqrt{\{(\xi - t)(\xi - t + 2\alpha r)\}}} \tag{118}$$

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for $\xi > 1$, which, when expanded as before for large $r$, gives

$$u = -\frac{1}{2\pi\sqrt{(2\alpha r)}}\int_0^1 \frac{S''(t)\,dt}{\sqrt{(\xi-t)}} + \frac{1}{8\pi\alpha\sqrt{(2\alpha)}\,r^{\frac{3}{2}}}\int_0^1 S''(t)\sqrt{(\xi-t)}\,dt + \dots.$$

For the flow at large distances from the axis $\xi$ is large, hence this expansion will be valid above some line $x-\alpha r = Kr$. Replacing $\xi$ by $y_1$,

$$F_1(y_1) = -\frac{1}{2\pi\sqrt{(2\alpha)}}\int_0^1 \frac{S''(t)\,dt}{\sqrt{(y_1-t)}}, \tag{119}$$

and it is found from the expansion for $v$ that $A_1 = B_1 = D_1 = 0$. Expanding $F_1(y_1)$ for large $y_1$, using $S(0) = S(1) = S'(0) = S'(1) = 0$,

$$F_1(y_1) = -\frac{V}{y_1^{\frac{5}{2}}} + O(y_1^{-\frac{7}{2}}), \tag{120}$$

where

$$V = \frac{3}{8\pi\sqrt{(2\alpha)}}\int_0^1 S(t)\,dt. \tag{121}$$

In the previous notation this means $h_1(z_1) = -(Vk/z_1)^{\frac{2}{5}} + O(1)$ as $z_1 \to 0$.

Now, the condition for the tangential component of momentum to be continuous through the shock is

$$z_1^2 - z^2 + 2(z_1 - z)\,r^{\frac{1}{2}} f_1'(r) + kD\,r^{-\frac{1}{2}} + O(z_1^{\frac{3}{5}} r^{-\frac{1}{2}}) + O(zr^{-\frac{1}{2}}) = 0 \tag{122}$$

which, with (115) and (116), gives $f_1(r) = br^{\frac{1}{4}} + O(1)$. This shows that at the shock $h_1 \sim -br^{\frac{1}{4}}$, hence $z_1 \sim Vkb^{-\frac{5}{2}} r^{-\frac{5}{8}}$ and from (116), with $A_1 = 0$,

$$f_1(r) = br^{\frac{1}{4}} = h(0) + \frac{2Vk}{3b^{\frac{5}{2}}} r^{-\frac{1}{8}} + \dots. \tag{123}$$

From the equation of a characteristic,

$$x = \alpha r - z_1 r^{\frac{1}{2}} + (Vk/z_1)^{\frac{2}{5}} + O(1)$$

and the maximum value of $z_1$ is $O(r^{-\frac{5}{8}})$, hence

$$z_1 = Vk(x-\alpha r)^{-\frac{5}{2}}[1 + O(r^{-\frac{1}{4}})]$$

and

$$u = -\frac{V}{r^{\frac{1}{2}}(x-\alpha r)^{\frac{5}{2}}} + O(r^{-\frac{11}{8}}), \tag{124}$$

$$p = \frac{\rho_0 U^2 V}{r^{\frac{1}{2}}(x-\alpha r)^{\frac{5}{2}}} + O(r^{-\frac{11}{8}}). \tag{125}$$

Thus the indication of slender body theory, which may be taken as a guide (if an uncertain one) in other cases, is that the equation of the rear shock is (123), with the same $b$ and $h(0)$ as in (94); the pressure directly behind the rear shock is $O(r^{-\frac{9}{8}})$ as against $O(r^{-\frac{3}{4}})$ ahead of it; and it asymptotes to zero from this value as indicated in (125).

## 9. Conclusion

In this section those results that are important from a practical point of view are collected together.

For a general body of revolution of finite length, the equations of the front and rear shocks are $x = \alpha r - br^{\frac{1}{4}} - c - dr^{-\frac{1}{4}} - \dots$, $x = \alpha r + b_1 r^{\frac{1}{4}} + \dots$, respectively, for large $r$,

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where $\alpha = \sqrt{(M^2-1)}$. For constant $r$, the pressure falls approximately *linearly* between the shocks, the slope of the pressure signature being

$$-0{\cdot}58\, M^{-2}\alpha r^{-1}\{1-2b^{-1}\, dr^{-\frac{1}{2}} + O(r^{-\frac{3}{4}}\log r)\} \text{ atmospheres/unit distance,} \qquad (126)$$

where $\gamma$ is taken as $1{\cdot}4$ and the length of the body is taken as the unit of distance. It is important to note that, to a first approximation, this is independent of the particular body considered, depending only on the Mach number $M$ and the distance $r$ from the axis.

For slender bodies, details of the constants may be given. The symmetrical parabolic profile of fineness ratio $\delta$ will be taken as typical of bodies pointed at both ends, and the body with profile $r = \frac{1}{2}\delta x\sqrt{(3-2x)}$, $0 \leqslant x \leqslant 1$, typical of shell-shapes, although the dependence on $M$ and $\delta$ is the same for all slender bodies.

In the first case,

$$b = b_1 = 0{\cdot}91\, M^2\alpha^{-\frac{3}{4}}\,\delta, \quad c = -0{\cdot}35, \quad d = 0{\cdot}032\, M^{-2}\,\alpha^{\frac{3}{4}}\,\delta^{-1}.$$

At a distance $r$, the pressure jump at the front shock is $0{\cdot}53\alpha^{\frac{1}{4}}\delta r^{-\frac{3}{4}} + O(r^{-\frac{5}{4}})$ atmospheres followed by a linear fall at a rate $0{\cdot}58M^{-2}\alpha r^{-1} + O(r^{-\frac{3}{2}})$ atmospheres/unit distance to the value $-0{\cdot}53\alpha^{\frac{1}{4}}\delta r^{-\frac{3}{4}} + O(r^{-\frac{9}{8}})$ atmospheres at the rear shock when it jumps to $0{\cdot}063M^{-3}\alpha^{\frac{11}{8}}\,\delta^{-\frac{1}{2}}r^{-\frac{9}{8}}$ atmospheres and asymptotes to zero as shown by (125).

In the second case,

$$b = 0{\cdot}94M^2\alpha^{-\frac{3}{4}}\delta, \quad c = -0{\cdot}75, \quad d = 0{\cdot}16M^{-2}\alpha^{\frac{3}{4}}\delta^{-1}.$$

The pressure jump at the front shock is $0{\cdot}55\alpha^{\frac{1}{4}}\delta r^{-\frac{3}{4}} + O(r^{-\frac{5}{4}})$ atmospheres with the same rate of fall as before between the shocks, to a value of order $r^{-\frac{3}{4}}$ at the rear shock. It is expected to return to a small value compared with this behind the rear shock, but the exact form is not known for a shell shape.

If a third shock is formed between the two principal ones, it either meets the rear shock (as in the case of the symmetrical parabolic profile) or is asymptotic to a straight line and its strength is $O(r^{-\frac{3}{2}})$; the pressure falls linearly between any two shocks at the same rate as above.

Du Mond, Cohen, Panofsky & Deeds (1946) have given experimental data in good agreement with these predictions; measurements made at various distances $r$ from a bullet moving with supersonic speed, give the distance between the two shocks to be proportional to $r^n$, where $n$ differs from $0{\cdot}25$ by not more than $0{\cdot}04$ for the four sizes of bullet used. Further, from (126), the product of the slope of the pressure signature and the distance $r$ is a constant $0{\cdot}58\sqrt{(M^2-1)}/M^2 = 0{\cdot}252$ for $M = 2$; the experimental value is $0{\cdot}238$. The authors also suggest a theoretical explanation of these observed results, but many unproved assumptions are made and they emphasize that in order to account for the $-\frac{3}{4}$ power law of amplitude it is very definitely necessary to take into account the degradation of energy into heat as the wave propagates; the present paper gives an explanation without such considerations.

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# Surface deformation and friction of metals at light loads

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[Plates 2 to 6]

A study has been made of the interaction of sliding metals when the normal load between them is very small. A new apparatus is described which enables friction to be measured, under controlled conditions of sliding, down to loads of a few milligrams. The deformation of the sliding surfaces has been studied by light and electron microscopy. These techniques have been used to investigate the validity of Amontons's law (proportionality of frictional force to load) over a very wide range of loads, from a few milligrams to several kilograms.

Experimental results on electrolytically polished copper reveal a departure from Amontons's law when the normal load is less than a few grams. A corresponding change is observed in the deformation within the track of sliding. At light loads the friction is low and the damage is slight. This is attributed to the effect of the thin film of oxide formed by contact with the atmosphere. A thicker film of oxide, formed by heating in air, reduces the friction at heavy loads. Results of observations on copper surfaces prepared in various ways show that the influence of surface roughness upon friction is not great.

The results of further experiments show that the friction of silver on silver and of aluminium on aluminium is constant and Amontons's law holds over the whole range of loads. It appears that the thin film of oxide on aluminium is penetrated, even at the lightest loads. The results are discussed in relation to those for copper, and the difference in behaviour of the various metals is attributed to the contrasting properties of their oxides.

Isolated experiments on sapphire and diamond show that the coefficient of friction is low and constant for these non-metals, and a few experiments made on boundary lubricated metals are recorded.

In spite of the deviations from Amontons's law on some metals, the results of all the experiments on dry sliding emphasize the essential similarity of the frictional process and of the basic mechanism over an enormous range of loads.

## Introduction

Recent work on the friction of metals and on the mechanism of sliding has shown that the observed phenomena may be explained in terms of the formation and