irreducible representations of S_n

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This article describes the theory of complex representations of S_n by Young diagrams, as developed by Frobenius, Schur, and Young. The situation for representations in nonzero characteristic is more complicated.

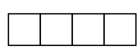
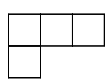
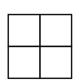
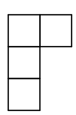
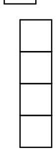
Recall the well-known result that the conjugacy class of any $\sigma \in S_n$ is determined by its cycle type. The number of different cycle structures is simply the number of partitions of n. Therefore the number of conjugacy classes of S_n, and thus the number of irreducible representations of S_n, is just the number of partitions of n.

For example, for S_4 the partitions, and a representative of each conjugacy class, are

4	$(1\ 2\ 3\ 4)$
$3,1$	$(1\ 2\ 3)$
$2,2$	$(1\ 2)(3\ 4)$
$2,1,1$	$(1\ 2)$
$1,1,1,1$	e

The partitions can be represented visually using Young diagrams, and there is an algorithm for explicitly extracting the dimensions of the irreducible representations of S_n from the Young diagrams. Of course, each diagram corresponds to a single irreducible representation. with each square in a given Young diagram the number of squares directly to its right plus the number of squares directly below it, and add 1 for the square itself. Multiply these numbers together for all squares in the diagram, and divide into $n!$. The result is the dimension of an irreducible representation of S_n.

Taking again S_4, we have

4		$\frac{4!}{4\cdot3\cdot2\cdot1} = 1$
$3,1$		$\frac{4!}{4\cdot2\cdot1\cdot1} = 3$
$2,2$		$\frac{4!}{3\cdot2\cdot2\cdot1} = 2$
$2,1,1$		$\frac{4!}{4\cdot2\cdot1\cdot1} = 3$
$1,1,1,1$		$\frac{4!}{4\cdot3\cdot2\cdot1} = 1$

S_4 has two 1-dimensional irreducible representations - ϵ and sgn. One of the 3-dimensional representations is the augmentation of the natural action of S_4 on $\mathbb{C}^4$, and the other is the tensor of that representation with sgn. The irreducible representation of dimension 2 arises from the map $S_4 \to S_4/V_4 \cong S_3$.