example of groups of order pq

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As a specific example, let us classify groups of order 21. Let G be a group of order 21. There is only one Sylow 7-subgroup K so it is normal. The possibility of there being conjugate Sylow 3-subgroups is not ruled out. Let x denote a generator for K, and y a generator for one of the Sylow 3-subgroups H. Then $x^7 = y^3 = 1$, and $yxy^{-1} = x^i$ for some $i < 7$ since K is normal. Now $x = y^3xy^{-3} = y^2x^iy^{-2} = yx^{i^2}y^{-1} = x^{i^3}$, or $i^3 = 1 \mod 7$. This implies $i = 1, 2$, or 4.

Case 1: $yxy^{-1} = x$, so G is abelian and isomorphic to $C_{21} = C_3 \times C_7$.

Case 2: $yxy^{-1} = x^2$, then every product of the elements x, y can be reduced to one in the form x^iy^j, $0 \leq i < 7$, $0 \leq j < 3$. These 21 elements are clearly distinct, so $G = \langle x, y \mid x^7 = y^3 = 1, yx = x^2y \rangle$.

Case 3: $yxy^{-1} = x^4$, then since y^2 is also a generator of H and $y^2xy^{-2} = yx^4y^{-1} = x^{16} = x^2$, we have recovered case 2 above.