representation ring

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Let G be a group and k a field. Consider the class

$$\mathcal{R} = \{X \mid X \text{ is a representation of } G \text{ over } k\}$$

and its subclass $\mathcal{R}_f$ consisting of those representations which are finite-dimensional as vector spaces. We consider a special representation

$$\mathcal{F} = (V, \cdot)$$

where V is a fixed vector space with a basis $\mathcal{B}$ which is in bijective correspondence with G. If $f : \mathcal{B} \to G$ is a required bijection, then we define „$\cdot$" on basis $\mathcal{B}$ by

$$g \cdot b = gf(b)$$

where on the right side we have a multiplication in G. It can be shown that this gives us a well-defined representation and further more, if $X \in \mathcal{R}_f$, then there exists an epimorphism of representations

$$e : \mathcal{F}^n \to X$$

for some $n \in \mathbb{N}$ ($\mathcal{F}$ is a „free" representation). In particular every finite-dimensional representation is a quotient of a direct sum of copies of $\mathcal{F}$. This fact shows that a maximal subclass $\mathcal{X} \subset \mathcal{R}_f$ consisting of pairwise nonisomorphic representations is actually a set (note that $\mathcal{X}$ is never unique). Fix such a set.

Definition. The representation semiring $\overline{R_k(G)}$ of G is defined as a triple $(\mathcal{X}, +, \cdot)$, where $\mathcal{X}$ is a maximal set of pairwise nonisomorphic representations taken from $\mathcal{R}_f$. Addition and multiplication are given by

$$X + Y = Z$$

where Z is a representation in $\mathcal{X}$ isomorphic to the direct sum $X \oplus Y$ and

$$X \cdot Y = Z'$$

where Z' is a representation in $\mathcal{X}$ isomorphic to the tensor product $X \otimes Y$. Note that $\overline{R_k(G)}$ is not a ring, because there are no additive inverses.

The representation ring $R_k(G)$ is defined as `http://planetmath.org/GrothendieckGroupt` Grothendieck ring induced from $\overline{R_k(G)}$. It can be shown that the definition does not depend on the choice of $\mathcal{X}$ (in the sense that it always gives us naturally isomorphic rings).

It is convenient to forget about formal definition which includes the choice of $\mathcal{X}$ and simply write elements of $\overline{R_k(G)}$ as isomorphism classes of representations $[X]$. Thus every element in $R_k(G)$ can be written as a formal difference $[X] - [Y]$. And we can write

$$[X] + [Y] = [X \oplus Y];$$

$$[X][Y] = [X \otimes Y].$$