orbit-stabilizer theorem

Canonical name	OrbitstabilizerTheorem
Date of creation	2013-03-22 12:23:10
Last modified on	2013-03-22 12:23:10
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	22
Author	yark (2760)
Entry type	Theorem
Classification	msc 20M30

Suppose that G is a group `http://planetmath.org/GroupAction`acting on a set X. For each $x \in X$, let Gx be the orbit of x, let G_x be the stabilizer of x, and let $\mathcal{L}_x$ be the set of left cosets of G_x. Then for each $x \in X$ the function $f\colon Gx \to \mathcal{L}_x$ defined by $gx \mapsto gG_x$ is a bijection. In particular,

$$|Gx| = [G : G_x]$$

and

$$|Gx| \cdot |G_x| = |G|$$

for all $x \in X$.

Proof:
If $y \in Gx$ is such that $y = g_1 x = g_2 x$ for some $g_1, g_2 \in G$, then we have $g_2^{-1} g_1 x = g_2^{-1} g_2 x = 1x = x$, and so $g_2^{-1} g_1 \in G_x$, and therefore $g_1 G_x = g_2 G_x$. This shows that f is well-defined.

It is clear that f is surjective. If $gG_x = g' G_x$, then $g = g'h$ for some $h \in G_x$, and so $gx = (g'h)x = g'(hx) = g'x$. Thus f is also injective.