generalized quaternion group

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The groups given by the presentation

$$Q_{4n} = \langle a, b : a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1} \rangle$$

are the *generalized quaternion groups*. Generally one insists that $n > 1$ as the properties of generalized quaternions become more uniform at this stage. However if $n = 1$ then one observes $a = b^2$ so $Q_{4n} \cong \mathbb{Z}_4$. Dihedral group properties are strongly related to generalized quaternion group properties because of their highly related presentations. We will see this in many of our results.

Proposition 1. *1.* $|Q_{4n}| = 4n$.

2. *Q_{4n} is abelian if and only if $n = 1$.*

3. *Every element $x \in Q_{4n}$ can be written uniquely as $x = a^i b^j$ where $0 \leq i < 2n$ and $j = 0, 1$.*

4. *$Z(Q_{4n}) = \langle a^n \rangle \cong \mathbb{Z}_2$.*

5. *$Q_{4n}/Z(Q_{4n}) \cong D_{2n}$.*

Proof. Given the relation $b^{-1}ab = a^{-1}$ (rather treating it as $ab = ba^{-1}$) then as with dihedral groups we can shuffle words in $\{a, b\}$ to group all the $a's$ at the beginning and the $b's$ at the end. So every word takes the form $a^i b^j$. As $|a| = 2n$ and $|b| = 4$ we have $0 \leq i < 2n$ and $0 \leq j < 4$. However we have an added relation that $a^n = b^2$ so we can write $a^i b^2 = a^{i+2}$ and also $a^i b^3 = a^{i+2} b$ so we restrict to $j = 0, 1$. This gives us $4n$ elements of this form which makes the order of Q_{4n} at most $4n$.

As $a^n = b^2$ it follows $[a^n, a^i b^j] = [a^n, b^j] = [b^2, b^j] = 1$. So a^n is central. If we quotient by $\langle a^n \rangle$ then we have the presentation

$$\langle a, b : a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$$

which we recognize as the presentation of the dihedral group. Thus $Q_{4n}/\langle a^n \rangle \cong D_{2n}$. This prove the order of Q_{4n} is exactly $4n$. Moreover, given $a^i b^j \in Z(Q_{4n})$ we have

$$1 = [a^i b^j, b] = b^{-j} a^{-i} b^{-1} a^i b^j b = b^{-j} a^{-i} a^{-i} b^{-1} b^j b = b^{-j} a^{-2i} b^j = a^{2i}.$$

So we have $i = n$. So $a^n b^j = b^{j+2}$. Then $1 = [b^{j+2}, a]$ forces $j = 0, 2$. This means $Z(Q_{4n}) = \langle a^n \rangle = \langle b^2 \rangle$. $\square$

1 Examples

As mentioned, if $n = 1$ then $Q_4 \cong \mathbb{Z}_4$. If $n = 2$ then we have the usual quaternion group Q_8. Because of the genesis of quaternions, this group is often denoted with i, j, k relations as follows:

$$Q_8 = \langle -1, i, j, k : i^2 = j^2 = k^2 = -1, ij = k = -1ji \rangle.$$

These relations are responsible for many useful results such as defining cross products for three-dimensional manipulations, and are also responsible for the most common example of a division ring. As a group, Q_8 is a curious specimen of a p-group in that it has only normal subgroups yet is non-abelian, it has a unique minimal subgroup and cannot be represented faithfully except by a regular representation – thus requiring degree 8. [To see this note that the unique minmal subgroup is necessarily normal, thus if a proper subgroup is the stabilizer of an action, then the minimal normal subgroup is in the kernel so the representation is not faithful.]

A common work around is to use 2×2 matrices over $\mathbb{C}$ but to treat these as matrices over $\mathbb{R}$.

$$-1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A worthwhile additional example is $n = 3$. For this produces a group order 12 which is often overlooked.

2 Subgroup structure

Proposition 2. *Q_{4n} is Hamiltonian – meaning all a non-abelian group whose subgroups are normal – if and only if $n = 2$.*

Proof. As $Q_{4n}/Z(Q_{4n}) \cong D_{2n}$, then if Q_{4n} is Hamiltonian then we require D_{2n} to be as well. However when $n > 2$ we know D_{2n} has non-normal subgroups, for example $\langle ab \rangle$. So we require $n \leq 2$. If $n = 1$ then Q_{4n} is cyclic and so trivially Hamiltonian. When $n = 2$ we have the usual quaternion group of order 8 which is Hamiltonian by direct inspection: the conjugacy classes are $\{1\}$, $\{a^2\}$, $\{a, a^3\}$, $\{b, a^2b\}$ and $\{ab, a^3b\}$, more commonly described by $\{1\}$, $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$ and $\{k, -k\}$. In any case, all subgroups are normal. □

By way of converse it can be shown that the only finite Hamiltonian groups are $A \oplus Q_8$ where A is abelian without an element of order 4. One sees already in $\mathbb{Z}_4 \oplus Q_8$ that the subgroup $\langle(1, i)\rangle$ is conjugate to the distinct subgroup $\langle(1, -i)\rangle$ and so such groups are not Hamiltonian.

Proposition 3. *1. $|a^i| = 2n/i$ for $1 < i \leq 2n$ and $|a^i b| = 4$ for all i.*

2. Every subgroup of Q_{4n} is either cyclic or a generalized quaternion.

3. The normal subgroups of Q_{4n} are either subgroups of $\langle a\rangle$ or $n = 2^i$ and it is maximal subgroups (of index 2) of which there are 2 acyclic ones.

Proof. The order of elements of $\langle a\rangle$ follows from standard cyclic group theory. Now for $a^i b$ we simply compute: $(a^i b)^2 = a^i b a^i b = a^i a^{-i} b^2 = b^2$. So $|a^i b| = 4$.

Now let H be a subgroup of Q_{4n}. If $Z(Q_{4n}) \leq H$ then $H/Z(Q_{4n})$ is a subgroup of D_{2n}. We know the subgroups of D_{2n} are either cyclic or dihedral. If $H/Z(Q_{4n})$ is cyclic then H is cyclic (indeed it is a subgroup of $\langle a\rangle$ or $H = \langle a^i b\rangle$). So assume that $H/Z(Q_{4n})$ is dihedral. Then we have a dihedral presentation $\langle x, y : x^m = 1, y^2 = 1, y^{-1}xy = x^{-1}\rangle$ for $H/Z(Q_{4n})$. Now pullback this presentation to H and we find H is quaternion.

Finally, if H does not contain $Z(Q_{4n})$ then H does not contain an element of the form $a^i b$, so $H \leq \langle a\rangle$ and so it is cyclic.

For the normal subgroup structure, from the relation $b^{-1}ab = a^{-1}$ we see $\langle a\rangle$ is normal. Thus all subgroups of $\langle a\rangle$ are normal as $\langle a\rangle$ is a normal cyclic subgroup. Next suppose H is a normal subgroup not contained in $\langle a\rangle$. Then H contains some $a^i b$, and so H contains $Z(Q_{4n})$. Thus $H/Z(Q_{4n})$ is a normal subgroup of D_{2n}. We know this forces $H/Z(Q_{4n})$ to be contained in $\langle a\rangle/Z(Q_{4n})$, a contradiction on our assumptions on H, or $n = 2^i$ and $H/Z(Q_{4n})$ is a maximal subgroup (of index 2). $\square$

Proposition 4. *Q_{4n} has a unique minimal subgroup if and only if $n = 2^i$.*

Proof. If $p|n$ and $p > 2$ then $a^{2n/p}$ has order p and so the subgroup $\langle a^{2n/p}\rangle$ is of order p, so it is minimal. As the center is also a minimal subgroup of order 2, then we do not have a unique minimal subgroup in these conditions. Thus $n = 2^i$.

Now suppose $n = 2^i$ then Q_{4n} is a 2-group so the minimal subgroups must all be of order 2. So we locate the elements of order 2. We have shown $|a^i b| = 4$ for any i, and furthermore that $(a^i b)^2 = b^2 = a^n$. The only other minimal subgroups will be generated by a^i for some i, and as $|a| = 2^{i+1}$ there is a unique minimal subgroup. $\square$

It can also be shown that any finite group with a unique minimal subgroup is either cyclic of prime power order, or Q_{4n} for some $n = 2^i$. We note that these groups have only regular faithful representations.