Frattini subset

Canonical name	FrattiniSubset
Date of creation	2013-03-22 16:30:56
Last modified on	2013-03-22 16:30:56
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Last modified by	Algeboy (12884)
Numerical id	7
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Entry type	Definition
Classification	msc 20D15
Related topic	FrattiniSubgroup

Suppose A is a set with a binary operation. Then we say a subset S of A *generates* A if finite iterations of products (commonly called words over S) from the set S eventually produce every element of A. We write this property as $A = \langle S \rangle$.

Definition 1. *An element $g \in A$ is said to be a* non-generator *if given a subset S of A such that $A = \langle S \cup \{g\} \rangle$ then in fact $A = \langle S \rangle$. The set of all non-generators is called the* Frattini subset *of A.*

Example 2. *The only non-generator of $\mathbb{Z}$, the group of integers under addition, is 0.*

Proof. Take $n \neq 0$. Without loss of generality, n is positive. Then take $1 < m$ some integer relatively prime to n. Thus by the Euclidean algorithm we know there are integers a, b such that $1 = am + bn$. This shows $1 \in \langle m, n \rangle$ so in fact $\{m, n\}$ generates $\mathbb{Z}$.

However, $\mathbb{Z}$ is not generated by m, as $m > 1$. Therefore n cannot be removed form the generating set $\{m, n\}$ and so indeed the only non-generator of $\mathbb{Z}$ is 0. □

Example 3. *In the ring $\mathbb{Z}_4$, the element* 2 *is a non-generator.*

Proof. Check the possible generating sets directly. □

Example 4. *The set of positive integers under addition, $\mathbb{N}$, has no non-generators.*

Proof. Apply the same proof as done to in Example **??**. □

So we see not all sets with binary operations have non-generators. In the case that a binary operation has an identity then the identity always serves as a non-generator due to the convention that the empty word be defined as the identity. However, without further assumptions on the product, such as associativity, it is not always possible to treat the Frattini subset as a subobject in the category of the orignal object. For example, we have just shown that the Frattini subset of a semi-group need not be a semi-group.

Proposition 5. *In the category of groups, the Frattini subset is a fully invariant subgroup.*

To prove this, we prove the following strong re-characterize the Frattini subset.

Theorem 6. *In a group G, the intersection of all maximal subgroups is the Frattini subset.*

Proof. For a group G, given a non-generator a, and M any maximal subgroup of G. If a is not in M then $\langle M, a\rangle$ is a larger subgroup than M. Thus $G = \langle M, a\rangle$. But a is a non-generator so $G = \langle M\rangle = M$. This contradicts the assumption that M is a maximal subgroup and therefore $a \in M$. So the Frattini subset lies in every maximal subgroup.

In contrast, if a is in all maximal subgroups of G, then given any subset S of G for which $G = \langle S, a\rangle$, then set $M = \langle S\rangle$. If $M = G$, then a is a non-generator. If not, then M lies in some maximal subgroup H of G. Since a lies in all maximal subgroup, a lies in H, and thus H contains $\langle S, a\rangle = G$. As H is maximal, this is impossible. Hence $G = M$ and a is a non-generator. $\square$