Brandt groupoid

Canonical name	BrandtGroupoid
Date of creation	2013-03-22 18:38:32
Last modified on	2013-03-22 18:38:32
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	18
Author	CWoo (3771)
Entry type	Definition
Classification	msc 20L05
Classification	msc 18B40
Related topic	GroupoidCategoryTheoretic
Related topic	ConnectedCategory

Brandt groupoids, like http://planetmath.org/GroupoidCategoryTheoreticcategory theoretic groupoids, are generalizations of groups, where a multiplication is defined, and inverses with respect to the multiplication exist for each element. However, unlike elements of a group, each element in a Brandt groupoid behaves like an arrow, with a source and target, and multiplication of two elements only work when the target of the first element coincides with the source of the second element.

Definition

A *Brandt groupoid* is a non-empty set B, together with a partial binary operation (called a multiplication) $\cdot$ defined on it (we write ab for $a \cdot b$), such that

1. For every $a \in B$, there are unique elements e, f such that ea and af are defined, and is equal to a.
2. If $ae = a$ or $ea = a$ for some $a, e \in B$, then ee is defined, and is equal to e.
3. For $a, b \in B$, ab is defined iff there is an $e \in B$ such that $ae = a$ and $eb = b$.
4. For $a, b, c \in B$ such that ab and bc are defined, then so are $(ab)c$ and $a(bc)$ and they equal.
5. If $ea = af = e$ for some $a, e, f \in B$, then there is a $b \in B$ such that ab and ba are defined and $ab = e$ and $ba = f$.
6. If $ee = e$ and $ff = f$ for some $e, f \in B$, then there is $a \in B$ such that ea and af are defined and are equal to a.

In the definition above, we see several instances of elements e such that $e^2 = ee = e$. Such elements are called *idempotents*. If we let I be the set of all idempotents of B, then $I \neq \varnothing$ by conditions 1 and 2.

Brandt Groupoids versus Categories

Brandt groupoids are intimately related to categories, as we will presently discuss.

The first two conditions above imply that there are two surjective functions $s, t : B \to I$, where $t(a)$ and $s(a)$ are the unique idempotents such that $as(a) = a$ and $t(a)a = a$. In addition, $s(e) = t(e) = e$ for all $e \in I$. Call s the source function, t the target function, and for any $a \in B$, $s(a), t(a)$ the source and the target of a.

The third condition says that ab is defined iff the source of a is the equal to the target of b: $s(a) = t(b)$. The fourth condition is the associativity law for the multiplication. An easy consequence of this condition is that if ab exists, then $s(b) = s(ab)$ and $t(a) = t(ab)$.

Altogether, the first four conditions say that a B is a small category, with I its set of objects, and G the set of morphisms, and composition of morphisms is just the multiplication.

A morphism a in B is said to be an *isomorphism* if there is a morphism b in G such that $ab, ba \in I$. Now, b is uniquely determined by a, so that a is an isomorphism in the category theoretic sense.

Proof. First notice that $s(b) = s(ab) = ab = t(ab) = t(a)$ and $t(b) = t(ba) = ba = s(ba) = s(a)$. If $ac, ca \in I$, then $s(c) = t(a) = s(b)$ and $t(c) = s(a) = t(b)$. So $ab = ac$ and $ba = bc$. As a result, $c = t(c)c = t(b)c = (ba)c = b(ac) = b(ab) = bs(b) = b$. $\square$

b is said to be the inverse of a, and is often written a^{-1}. Condition 5 says that the category B is in fact a `http://planetmath.org/GroupoidCategoryTheoretic`category theoretic groupoid. Thus, a Brandt groupoid is a group if the multiplication is everywhere defined.

Finally, condition 6 says that between every pair of objects, there is a morphism from one to the other, this is equivalent to saying that B is strongly connected. As a result, a Brandt groupoid may be equivalently defined as a small strongly connected groupoid (in the category theoretic sense).

An Example

A Brandt groupoid may be constructed as follows: take a group G and a non-empty set I, set $B := I \times G \times I$, and define multiplication on B as follows:

$$(p, x, q)(r, y, s) = \begin{cases} (p, xy, s) & \text{if } q = r, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then B with the partial multiplication is a Brandt groupoid. The idempotents in B have the form (p, e, p), where $e \in G$ is the group identity. And for any (p,x,q), its source, target, and inverse are (q,e,q) and (p,e,p), (q,x^{-1},p) respectively.

In fact, it may be shown that every Brandt groupoid is isomorphic to one constructed above (for a proof, see `http://planetmath.org/ConstructionOfABrandtGroupoid`he

Remark. A non-trivial Brandt groupoid can not have a zero element, for if $0a = a0 = 0$ for all $a \in B$, then a must be the source and target of 0, but then a would have to be unique by condition 1, which is impossible unless B is trivial. If we adjoin 0 to a Brandt groupoid B, and call $S := B \cup \{0\}$, then S has the structure of a semigroup. Here's how the multiplication is defined on S:

$$ab = \begin{cases} ab & \text{if } ab \text{ is defined in } B, \\ 0 & \text{otherwise, or if either } a = 0 \text{ or } b = 0. \end{cases}$$

Since the multiplication on S is everywhere defined, S is a groupoid. To see that S is a semigroup, we must show that associativity of the multiplication applies everywhere. There are four cases

- If both ab and bc are defined in B, they are certainly defined in S, and the associativity follows from condition 4.
- If neither ab nor bc is defined in B, then $(ab)c = 0c = 0 = a0 = a(bc)$ in S.
- If ab is not defined in B, but bc is, then $s(a) \neq t(b) = t(bc)$, and $(ab)c = 0c = 0 = a(bc)$.
- Similarly, if ab is defined in B but not bc, then $(ab)c = 0 = a(bc)$.

Thus, S is a semigroup (with 0). In fact, Clifford showed that S is completely simple.

References

[1] H. Brandt, *Uber die Axiome des Gruppoids, Vierteljschr. naturforsch. Ges. Zurich 85, Beiblatt (Festschrift Rudolph Fueter)*, pp. 95-104, **MR2**, 218, 1940.

[2] R. H. Bruck, *A Survey on Binary Systems*, Springer-Verlag, New York, 1966.

[3] N. Jacobson, *Theory of Rings*, American Mathematical Society, New York, 1943.

[4] A. H. Clifford, *Matrix Representations of Completely Simple Semigroups*, Amer. J. Math. 70. pp. 521-526, 1948.