alternative proof of condition on a near ring to be a ring

Canonical name	AlternativeProofOfConditionOnANearRingToBeARing
Date of creation	2013-03-22 17:20:06
Last modified on	2013-03-22 17:20:06
Owner	Wkbj79 (1863)
Last modified by	Wkbj79 (1863)
Numerical id	9
Author	Wkbj79 (1863)
Entry type	Proof
Classification	msc 20-00
Classification	msc 16-00
Classification	msc 13-00

Theorem 1. *Let* $(R, +, \cdot)$ *be a near ring with a multiplicative identity* 1 *such that the* $\cdot$ *also left distributes over* $+$*; that is,* $c \cdot (a + b) = c \cdot a + c \cdot b$*. Then* R *is a ring.*

Proof. All that needs to be verified is commutativity of $+$.

Let $a, b \in R$. Consider the expression $(1 + 1)(a + b)$.

We have:

$$\begin{aligned}
(1+1)(a+b) &= (1+1)a + (1+1)b && \text{by left distributivity} \\
&= 1a + 1a + 1b + 1b && \text{by right distributivity} \\
&= a + a + b + b && \text{since 1 is a multiplicative identity}
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
(1+1)(a+b) &= 1(a+b) + 1(a+b) && \text{by right distributivity} \\
&= a + b + a + b && \text{since 1 is a multiplicative identity}
\end{aligned}$$

Thus, $a + a + b + b = a + b + a + b$. Hence:

$$\begin{aligned}
a + b &= 0 + (a+b) + 0 && \text{since 0 is an \texttt{http://planetmath.org/AdditiveIde}} \\
&= (-a + a) + (a + b) + (b + -b) && \text{by definition of \texttt{http://planetmath.org/AdditiveI}} \\
&= -a + (a + a + b + b) + -b \\
&= -a + (a + b + a + b) + -b && \text{since } a + a \\
&= (-a + a) + (b + a) + (b + -b) \\
&= 0 + (b + a) + 0 \\
&= b + a
\end{aligned}$$

□