# rational set

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Given an alphabet $\Sigma$, recall that a regular language $R$ is a certain subset of the free monoid $M$ generated by $\Sigma$, which can be obtained by taking singleton subsets of $\Sigma$, and perform, in a finite number of steps, any of the three basic operations: taking union, string concatenation, and the Kleene star.

The construction of a set like $R$ is still possible without $M$ being finitely generated free.

Let $M$ be a monoid, and $\mathcal{S}_M$ the set of all singleton subsets of $M$. Consider the closure $\mathcal{R}_M$ of $S$ under the operations of union, product, and the formation of a submonoid of $M$. In other words, $\mathcal{R}_M$ is the smallest subset of $M$ such that

- $\varnothing \in \mathcal{R}_M$,
- $A, B \in \mathcal{R}_M$ imply $A \cup B \in \mathcal{R}_M$,
- $A, B \in \mathcal{R}_M$ imply $AB \in \mathcal{R}_M$, where $AB = \{ab \mid a \in A, b \in B\}$,
- $A \in \mathcal{R}_M$ implies $A^* \in \mathcal{R}_M$, where $A^*$ is the submonoid generated by $A$.

**Definition**. A *rational set* of $M$ is an element of $\mathcal{R}_M$.

If $M$ is a finite generated free monoid, then a rational set of $M$ is also called a *rational language*, more commonly known as a *regular language*.

Like regular languages, rational sets can also be represented by regular expressions. A regular expression over a monoid $M$ and the set it represents are defined inductively as follows:

- $\varnothing$ and $a$ are regular expressions, for any $a \in M$, representing sets $\varnothing$ and $\{a\}$ respectively.
- if $a, b$ are regular expressions, so are $a \cup b$, $ab$, and $a^*$. Furthermore, if $a, b$ represent sets $A, B$, then $a \cup b$, $ab$, and $a^*$ represent sets $A \cup B$, $AB$, and $A^*$ respectively.

Parentheses are used, as usual, to avoid ambiguity.

From the definition above, it is easy to see that a set $A \subseteq M$ is rational iff it can be represented by a regular expression over $M$.

Below are some basic properties of rational sets:

1. Any rational set $M$ is a subset of a finitely generated submonoid of $M$. As a result, every rational set over $M$ is finite iff $M$ is locally finite (meaning every finitely generated submonoid of $M$ is actually finite).

2. Rationality is preserved under homomorphism: if $A$ is rational over $M$ and $f : M \to N$ is a homomorphism, then $f(A)$ is rational over $N$.

3. Conversely, if $B \in f(M)$ is rational over $N$, then there is a rational set $A$ over $M$ such that $f(A) = B$. Thus if $f$ is onto, every rational set over $N$ is mapped by a rational set over $M$. If $f$ fails to be onto, the statement becomes false. In fact, inverse homomorphisms generally do not preserve rationality.

# References

[1] S. Eilenberg, *Automata, Languages, and Machines, Vol. A*, Academic Press (1974).