# SL(n;R) is connected

| | |
|---|---|
| Canonical name | SLnRIsConnected |
| Date of creation | 2013-03-22 18:52:05 |
| Last modified on | 2013-03-22 18:52:05 |
| Owner | Stephaninos (23208) |
| Last modified by | Stephaninos (23208) |
| Numerical id | 5 |
| Author | Stephaninos (23208) |
| Entry type | Result |
| Classification | msc 20G15 |

The special feature is that although not every element of $SL(n,\mathbb{R})$ is in the image of the exponential map of $\mathfrak{sl}(n,\mathbb{R})$, $SL(n,\mathbb{R})$ is still a connected Lie group. The proof below is a guideline and should be clarified a bit more at some points, but this was done intentionally.

To illustrate the point, first we show

**Proposition 0.1.** $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \notin \exp \mathfrak{sl}(2,\mathbb{R})$, *but it is in* $SL(2,\mathbb{R})$.

*Proof.* $\det x =: \det \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = 1$, so $x \in SL(2,\mathbb{R})$. We see that $x$ is not diagonalizable, it already is in Jordan normal form. Moreover, it has a double eigenvalue, $-1$. Suppose that $x = \exp X, X \in \mathfrak{sl}(2,\mathbb{R})$, then $\operatorname{tr} X = 0$. Since $x$ had a double eigenvalue, so does $X$, hence the eigenvalues of $X$ both are 0. But this implies the eigenvalues of $x$ are 1. This is a contradiction. $\square$

**Lemma 0.2.** *We have* $\forall x \in SL(n,\mathbb{R}) : x = \exp(X_a)\exp(X_s)$ *with* $X_a^t = -X_a, X_s^t = X_s \in \mathfrak{sl}(n,\mathbb{R})$.

*Proof.* The keyword here is *polar decomposition*. We notice that $x^t x$ is symmetric and positive definite, since $\forall \psi \in \mathbb{R}^n : \langle \psi, x^t x \psi \rangle > 0$, with the standard inner product on $\mathbb{R}^n$. Hence, we can write $x = RP$, with $P = (x^t x)^{\frac{1}{2}}$ and $R = xP^{-1}$. $P$ is well defined, since any real symmetric, positive definite matrix is diagonalizable. It's easy to check that $RR^t = \mathrm{id}_n$, hence $R \in O(n)$. We had $\det P > 0$ and $\det x = 1$, hence $\det(R) > 0 \Rightarrow \det R = 1 \Rightarrow R \in SO(n \ltimes)$ and so $\det P = 1$. Since the choice of positive root is unique, $R$ and $P$ are unique. Moreover, $SO(n)$ is exactly generated by the set $\{X \in GL(n,\mathbb{R}) | X^t = -X\}$ and $\Omega$, the set of real symmetric matrices of determinant 1, by $\{X \in GL(n,\mathbb{R}) | X^t = X, \operatorname{tr} X = 0\}$, we have the wanted statement: $SL(n, \mathbb{R} \subset SO(n) \times \exp \Omega$. $\square$

The reverse inclusion is simply shown: any such combination is trivially in $SL(n,\mathbb{R})$.

**Corollary 0.3.** $SL(n,\mathbb{R})$ *is connected.*

*Proof.* This is now clear from the fact that both $SO(n)$ and $\Omega$ are connected and so $\forall s,t \in [0,1] : \exp sX \exp tY \in SL(n,\mathbb{R})$, a fact easily checked by taking the determinant. So $SL(n,\mathbb{R})$ is path-connected, hence connected. $\square$