associative

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Let (S, ϕ) be a set with binary operation ϕ. ϕ is said to be *associative* over S if

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$$

for all $a, b, c \in S$.

Examples of associative operations are addition and multiplication over the integers (or reals), or addition or multiplication over $n \times n$ matrices.

We can construct an operation which is not associative. Let S be the integers. and define $\nu(a, b) = a^2 + b$. Then $\nu(\nu(a, b), c) = \nu(a^2 + b, c) = a^4 + 2ba^2 + b^2 + c$. But $\nu(a, \nu(b, c)) = \nu(a, b^2 + c) = a + b^4 + 2cb^2 + c^2$, hence $\nu(\nu(a, b), c) \neq \nu(a, \nu(b, c))$.

Note, however, that if we were to take $S = \{0\}$, ν would be associative over S!. This illustrates the fact that the set the operation is taken with respect to is very important.

Example. We show that the division operation over nonzero reals is non-associative. All we need is a counter-example: so let us compare $1/(1/2)$ and $(1/1)/2$. The first expression is equal to 2, the second to $1/2$, hence division over the nonzero reals is not associative.

Remark. The property of being associative of a binary operation can be generalized to an arbitrary n-ary operation, where $n \geq 2$. An n-ary operation ϕ on a set A is said to be *associative* if for any elements $a_1, \dots, a_{2n-1} \in A$, we have

$$\phi(\phi(a_1, \dots, a_n), a_{n+1} \dots, a_{2n-1}) = \cdots = \phi(a_1, \dots, a_{n-1}, \phi(a_n, \dots, a_{2n-1})).$$

In other words, for any $i = 1, \dots, n$, if we set $b_i := \phi(a_1, \dots, \phi(a_i, \dots, a_{i+n-1}), \dots, a_{2n-1})$, then ϕ is associative iff $b_i = b_1$ for all $i = 1, \dots, n$. Therefore, for instance, a ternary operation f on A is associative if $f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e))$.