complete group

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A *complete group* is a group G that is

1. centerless (center $Z(G)$ of G is the trivial group), and

2. any of its automorphism $g\colon G \to G$ is an inner automorphism.

If a group G is complete, then its group of automorphisms, $\mathrm{Aut}(G)$, is isomorphic to G. Here's a quick proof. Define $\phi\colon G \to \mathrm{Aut}(G)$ by $\phi(g) = g^{\#}$, where $g^{\#}(x) = gxg^{-1}$. For $g, h \in G$, $(gh)^{\#}(x) = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1} = (g^{\#}h^{\#})(x)$, so ϕ is a homomorphism. It is onto because every $\alpha \in \mathrm{Aut}(G)$ is inner, ($=g^{\#}$ for some $g \in G$). Finally, if $g^{\#}(x) = h^{\#}(x)$, then $gxg^{-1} = hxh^{-1}$, which means $(h^{-1}g)x = x(h^{-1}g)$, for all $x \in G$. This implies that $h^{-1}g \in Z(G) = \langle e \rangle$, or $h = g$. ϕ is one-to-one.

It can be shown that all symmetric groups on n letters are complete groups, except when $n = 2$ and 6.

References

[1] J. Rotman, *The Theory of Groups, An Introduction*, Allyn and Bacon, Boston (1965).