groups of order pq

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We can use Sylow's theorems to examine a group G of order pq, where p and q are `http://planetmath.org/Prime`primes and $p < q$.

Let n_p and n_q denote, respectively, the number of Sylow p-subgroups and Sylow q-subgroups of G.

Sylow's theorems tell us that $n_q = 1 + kq$ for some integer k and n_q divides pq. But p and q are prime and $p < q$, so this implies that $n_q = 1$. So there is exactly one Sylow q-subgroup, which is therefore normal (indeed, fully invariant) in G.

Denoting the Sylow q-subgroup by Q, and letting P be a Sylow p-subgroup, then $Q \cap P = \{1\}$ and $QP = G$, so G is a semidirect product of Q and P. In particular, if there is only one Sylow p-subgroup, then G is a direct product of Q and P, and is therefore cyclic.

Given $G = Q \rtimes P$, it remains to determine the action of P on Q by conjugation. There are two cases:

Case 1: If p does not divide $q - 1$, then since $n_p = 1 + mp$ cannot equal q we must have $n_p = 1$, and so P is a normal subgroup of G. This gives $G = C_p \times C_q$ a direct product, which is isomorphic to the cyclic group C_{pq}.

Case 2: If p divides $q-1$, then $\mathrm{Aut}(Q) \cong C_{q-1}$ has a unique `http://planetmath.org/Subgroups` P' of order p, where $P' = \{x \mapsto x^i \mid i \in \mathbb{Z}/q\mathbb{Z}, i^p = 1\}$. Let a and b be generators for P and Q respectively, and suppose the action of a on Q by conjugation is $x \mapsto x^{i_0}$, where $i_0 \neq 1$ in $\mathbb{Z}/q\mathbb{Z}$. Then $G = \langle a, b \mid a^p = b^q = 1, aba^{-1} = b^{i_0} \rangle$. Choosing a different i_0 amounts to choosing a different generator a for P, and hence does not result in a new isomorphism class. So there are exactly two isomorphism classes of groups of order pq.