orthogonality relations

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First orthogonality relations: Let $\rho_\alpha\colon G \to V_\alpha$ and $\rho_\beta\colon G \to V_\beta$ be irreducible representations of a finite group G over the field $\mathbb{C}$. Then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\rho_{ij}^{(\alpha)}(g)}\rho_{kl}^{(\beta)}(g) = \frac{\delta_{\alpha\beta}\delta_{ik}\delta_{jl}}{\dim V_\alpha}.$$

We have the following useful corollary. Let χ_1, χ_2 be characters of representations V_1, V_2 of a finite group G over a field k of characteristic 0. Then

$$(\chi_1,\chi_2) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_1(g)}\chi_2(g) = \dim(\mathrm{Hom}(V_1,V_2)).$$

Proof. First of all, consider the special case where $V = k$ with the trivial action of the group. Then $\mathrm{Hom}_G(k,V_2) \cong V_2^G$, the fixed points. On the other hand, consider the map

$$\phi = \frac{1}{|G|}\sum_{g\in G} g\colon V_2 \to V_2$$

(with the sum in $\mathrm{End}(V_2)$). Clearly, the image of this map is contained in V_2^G, and it is the identity restricted to V_2^G. Thus, it is a projection with image V_2^G. Now, the rank of a projection (over a field of characteristic 0) is its trace. Thus,

$$\dim_k \mathrm{Hom}_G(k,V_2) = \dim V_2^G = \mathrm{tr}(\phi) = \frac{1}{|G|}\sum \chi_2(g)$$

which is exactly the orthogonality formula for $V_1 = k$.

Now, in general, $\mathrm{Hom}(V_1,V_2) \cong V_1^*\otimes V_2$ is a representation, and $\mathrm{Hom}_G(V_1,v_2) = (\mathrm{Hom}(V_1,V_2))^G$. Since $\chi_{V_1^*\otimes V_2} = \overline{\chi_1}\chi_2$,

$$\dim_k \mathrm{Hom}_G(V_1,V_2) = \dim_k(\mathrm{Hom}(V_1,V_2))^G = \sum_{g\in G}\overline{\chi_1}\chi_2$$

which is exactly the relation we desired. □

In particular, if V_1, V_2 irreducible, by Schur's Lemma

$$\mathrm{Hom}(V_1,V_2) = \begin{cases} D & V_1 \cong V_2 \\ 0 & V_1 \ncong V_2 \end{cases}$$

where D is a division algebra. In particular, non-isomorphic irreducible representations have orthogonal characters. Thus, for any representation V, the multiplicities n_i in the unique decomposition of V into the http://planetmath.org/DirectSum sum of irreducibles

$$V \cong V_1^{\oplus n_1} \oplus \cdots \oplus V_m^{\oplus n_m}$$

where V_i ranges over irreducible representations of G over k, can be determined in terms of the character inner product:

$$n_i = \frac{(\psi, \chi_i)}{(\chi_i, \chi_i)}$$

where ψ is the character of V and χ_i the character of V_i. In particular, representations over a field of characteristic zero are determined by their character. Note: This is not true over fields of positive characteristic.

If the field k is algebraically closed, the only finite division algebra over k is k itself, so the characters of irreducible representations form an orthonormal basis for the vector space of class functions with respect to this inner product. Since $(\chi_i, \chi_i) = 1$ for all irreducibles, the multiplicity formula above reduces to $n_i = (\psi, \chi_i)$.

Second orthogonality relations: We assume now that k is algebraically closed. Let g, g' be elements of a finite group G. Then

$$\sum_{\chi} \chi(g)\overline{\chi(g')} = \begin{cases} |C_G(g_1)| & g \sim g' \\ 0 & g \nsim g' \end{cases}$$

where the sum is over the characters of irreducible representations, and $C_G(g)$ is the centralizer of g.

Proof. Let $\chi_1, \ldots, \chi_n$ be the characters of the irreducible representations, and let $g_1, \ldots, g_n$ be representatives of the conjugacy classes.

Let A be the matrix whose ijth entry is $\sqrt{|G : C_G(g_j)|}(\overline{\chi_i(g_j)})$. By first orthogonality, $AA^* = |G|I$ (here $*$ denotes conjugate transpose), where I is the identity matrix. Since left http://planetmath.org/MatrixInverseinverses are right , $A^*A = |G|I$. Thus,

$$\sqrt{|G : C_G(g_i)||G : C_G(g_k)|} \sum_{j=1}^{n} \chi_j(g_i)\overline{\chi_j(g_k)} = |G|\delta_{ik}.$$

Replacing g_i or g_k with any conjuagate will not change the expression above. thus, if our two elements are not conjugate, we obtain that $\sum_\chi \chi(g)\overline{\chi(g')} = 0$. On the other hand, if $g \sim g'$, then $i = k$ in the sum above, which reduced to the expression we desired. $\square$

A special case of this result, applied to 1 is that $|G| = \sum_\chi \chi(1)^2$, that is, the sum of the squares of the `http://planetmath.org/Dimension`dimensions of the irreducible representations of any finite group is the order of the group.