semi-direct factor and quotient group

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Theorem. If the group G is a semi-direct product of its subgroups H and Q, then the semi-direct Q is isomorphic to the quotient group G/H.

Proof. Every element g of G has the unique representation $g = hq$ with $h \in H$ and $q \in Q$. We therefore can define the mapping

$$g \mapsto q$$

from G to Q. The mapping is surjective since any element y of Q is the image of ey. The mapping is also a homomorphism since if $g_1 = h_1 q_1$ and $g_2 = h_2 q_2$, then we obtain

$$f(g_1 g_2) = f(h_1 q_1 h_2 q_2) = f(h_1 h_2 q_1 q_2) = q_1 q_2 = f(g_1) f(g_2).$$

Then we see that $\ker f = H$ because all elements $h = he$ of H are mapped to the identity element e of Q. Consequently we get, according to the first isomorphism theorem, the result

$$G/H \cong Q.$$

Example. The multiplicative group $\mathbb{R}^{\times}$ of reals is the semi-direct product of the subgroups $\{1, -1\} = \{\pm 1\}$ and $\mathbb{R}_{+}$. The quotient group $\mathbb{R}^{\times}/\{\pm 1\}$ consists of all cosets

$$x\{\pm 1\} = \{x, -x\}$$

where $x \neq 0$, and is obviously isomorphic with $\mathbb{R}_{+} = \{x \mid x > 0\}$.