proof of fourth isomorphism theorem

Canonical name	ProofOfFourthIsomorphismTheorem
Date of creation	2013-03-22 14:17:38
Last modified on	2013-03-22 14:17:38
Owner	aoh45 (5079)
Last modified by	aoh45 (5079)
Numerical id	9
Author	aoh45 (5079)
Entry type	Proof
Classification	msc 20A05

First we must prove that the map defined by $A \mapsto A/N$ is a bijection. Let θ denote this map, so that $\theta(A) = A/N$. Suppose $A/N = B/N$, then for any $a \in A$ we have $aN = bN$ for some $b \in B$, and so $b^{-1}a \in N \subseteq B$. Hence $A \subseteq B$, and similarly $B \subseteq A$, so $A = B$ and θ is injective. Now suppose S is a subgroup of G/N and $\phi : G \to G/N$ by $\phi(g) = gN$. Then $\phi^{-1}(S) = \{s \in G : sN \in S\}$ is a subgroup of G containing N and $\theta(\phi^{-1}(S)) = \{sN : sN \in S\} = S$, proving that θ is bijective.

Now we move to the given properties:

1. $A \leq B$ iff $A/N \leq B/N$

   If $A \leq B$ then trivially $A/N \leq B/N$, and the converse follows from the fact that θ is bijective.

2. $A \leq B$ implies $|B : A| = |B/N : A/N|$

   Let ψ map the cosets in B/A to the cosets in $(B/N)/(A/N)$ by mapping the coset bA $b \in B$ to the coset $(bN)(A/N)$. Then ψ is well defined and injective because:

   $$\begin{aligned} b_1A = b_2A &\iff b_1^{-1}b_2 \in A \\ &\iff (b_1N)^{-1}(b_2N) = b_1^{-1}b_2N \in A/N \\ &\iff (b_1N)(A/N) = (b_2N)(A/N). \end{aligned}$$

   Finally, ψ is surjective since b ranges over all of B in $(bN)(A/N)$.

3. $\langle A, B\rangle/N = \langle A/N, B/N\rangle$

   To show $\langle A, B\rangle/N \subseteq \langle A/N, B/N\rangle$ we need only show that if $x \in A$ or $x \in B$ then $xN \in \langle A/N, B/N\rangle$. The other cases are dealt with using the fact that $(xy)N = (xN)(yN)$. So suppose $x \in A$ then clearly $xN \in \langle A/N, B/N\rangle$ because $xN \in A/N$. Similarly for $x \in B$. Similarly, to show $\langle A/N, B/N\rangle \subseteq \langle A, B\rangle/N$ we need only show that if $xN \in A/N$ or $xN \in B/N$ then $x \in \langle A, B\rangle$. So suppose $xN \in A/N$, then $xN = aN$ for some $a \in A$, giving $a^{-1}x \in N \subseteq A$ and so $x \in A \subseteq \langle A, B\rangle$. Similarly for $xN \in B/N$.

4. $(A \cap B)/N = (A/N) \cap (B/N)$

   Suppose $xN \in (A \cap B)/N$, then $xN = yN$ for some $y \in (A \cap B)$ and since $N \subseteq (A \cap B)$, $x \in (A \cap B)$. Therefore $x \in A$ and $x \in B$, and so $xN \in (A/N) \cap (B/N)$ meaning $(A \cap B)/N \subseteq (A/N) \cap (B/N)$. Now suppose $xN \in (A/N) \cap (B/N)$. Then $xN = aN$ for some $a \in A$, giving $a^{-1}x \in N \subseteq A$ and so $x \in A$. Similarly $x \in B$, therefore $xN \in (A \cap B)/N$ and $(A/N) \cap (B/N) \subseteq (A \cap B)/N$.

5. $A \trianglelefteq G$ iff $(A/N) \trianglelefteq (G/N)$

   Suppose $A \trianglelefteq G$. Then for any $g \in G$ we have $(gN)(A/N)(gN)^{-1} = (gAg^{-1})/N = A/N$ and so $(A/N) \trianglelefteq (G/N)$.
   Conversely suppose $(A/N) \trianglelefteq (G/N)$. Consider $\sigma\colon g \mapsto (gN)(A/N)$, the composition of the map from G onto G/N and the map from G/N onto $(G/N)/(A/N)$. $g \in \ker \pi$ iff $(gN)(A/N) = (A/N)$ which occurs iff $gN \in A/N$ therefore $gN = aN$ for some $a \in A$. However N is contained in A, so this statement is equivalnet to saying $g \in A$. So A is the kernel of a homomorphism, hence is a normal subgroup of G.