proof of basic theorem about ordered groups

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Property 1:

Consider $ab^{-1} \in G$. Since G can be written as a pairwise disjoint union, exactly one of the following conditions must hold:

$$ab^{-1} \in S \qquad ab^{-1} = 1 \qquad ab^{-1} \in S^{-1}$$

By definition of the ordering relation, $a < b$ if the first condition holds. If the second condition holds, then $a = b$. If the third condition holds, then we must have $ab^{-1} = s^{-1}$ for some $s \in S$. Taking inverses, this means that $ba^{-1} = s$, so $b < a$, or equivalently $a > b$. Hence, one of the following three conditions must hold:

$$a < b \qquad a = b \qquad b < a$$

Property 2:

The hypotheses can be rewritten as

$$ab^{-1} \in S \qquad bc^{-1} \in S$$

Multiplying, and remembering that S is closed under multiplication,

$$ac^{-1} = (ab^{-1})(bc^{-1}) \in S.$$

In other words, $a < c$.

Property 3:

Suppose that $a < b$, so $ab^{-1} = s \in S$. Then

$$s = ab^{-1} = a1b^{-1} = acc^{-1}b^{-1} = (ac)(bc)^{-1}$$

so $ac < bc$.

By the defining property of S, we have $csc^{-1} \in S$. Also,

$$csc^{-1} = cab^{-1}c^{-1} = (ca)(cb)^{-1},$$

hence $(ca)(cb)^{-1} \in S$, so $ca < cb$

Property 4:

By property 3, $a < b$ implies $ac < bc$ and likewise $c < d$ implies $bc < bd$. Then, by property 2, we conclude $ac < bd$.

Property 5:

By the hypothesis, $ab^{-1} = s \in S$. By the defining property, $b^{-1}sb \in S$. Since $b^{-1}sb = b^{-1}a$, we have $b^{-1}a \in S$. In other words, $b^{-1} < a^{-1}$.

Property 6:

By definition, $a < 1$ means that $a1^{-1} \in S$. Since $1^{-1} = 1$ and $a1 = a$, this is equivalent to stating that $a \in S$.