a group of even order contains an element of order 2

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Author	azdbacks4234 (14155)
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Proposition. *Every group of even order contains an element of order* 2.

Proof. Let G be a group of even order, and consider the set $S = \{g \in G : g \neq g^{-1}\}$. We claim that $|S|$ is even; to see this, let $a \in S$, so that $a \neq a^{-1}$; since $(a^{-1})^{-1} = a \neq a^{-1}$, we see that $a^{-1} \in S$ as well. Thus the elements of S may be exhausted by repeatedly selecting an element and it with its inverse, from which it follows that $|S|$ is a `http://planetmath.org/Divisibility`multiple of 2 (i.e., is even). Now, because $S \cap (G \setminus S) = \emptyset$ and $S \cup (G \setminus S) = G$, it must be that $|S| + |G \setminus S| = |G|$, which, because $|G|$ is even, implies that $|G \setminus S|$ is also even. The identity element e of G is in $G \setminus S$, being its own inverse, so the set $G \setminus S$ is nonempty, and consequently must contain at least two distinct elements; that is, there must exist some $b \neq e \in G \setminus S$, and because $b \notin S$, we have $b = b^{-1}$, hence $b^2 = 1$. Thus b is an element of order 2 in G. $\square$

Notice that the above is logically equivalent to the assertion that a group of even order has a non-identity element that is its own inverse.