# submodule

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| Canonical name | Submodule |
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| Entry type | Definition |
| Classification | msc 20-00 |
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| Related topic | SumOfIdeals |
| Related topic | QuotientOfIdeals |
| Defines | R-submodule |
| Defines | generated submodule |
| Defines | generator |
| Defines | sum of submodules |
| Defines | product submodule |
| Defines | quotient of submodules |

Given a ring $R$ and a left $R$-module $T$, a subset $A$ of $T$ is called a (*left*) *submodule* of $T$, if $(A,\,+)$ is a subgroup of $(M,\,+)$ and $ra \in A$ for all elements $r$ of $R$ and $a$ of $A$.

**Examples**

1. The subsets $\{0\}$ and $T$ are always submodules of the module $T$.
2. The set $\{t \in T:\;\; rt = t\;\; \forall r \in R\}$ of all invariant elements of $T$ is a submodule of $T$.
3. If $X \subseteq T$ and $\mathfrak{a}$ is a left ideal of $R$, then the set

$$\mathfrak{a}X := \{\text{finite} \sum_{\nu} a_\nu x_\nu :\;\; a_\nu \in \mathfrak{a},\; x_\nu \in X\; \forall \nu\}$$

is a submodule of $T$. Especially, $RX$ is called the submodule *generated* by the subset $X$; then the elements of $X$ are *generators* of this submodule.

There are some operations on submodules. Given the submodules $A$ and $B$ of $T$, the *sum* $A+B := \{a+b \in T:\; a \in A \wedge b \in B\}$ and the intersection $A \cap B$ are submodules of $T$.

The notion of sum may be extended for any family $\{A_j\, :\;\, j \in J\}$ of submodules: the sum $\sum_{j\in J} A_j$ of submodules consists of all finite sums $\sum_j a_j$ where every $a_j$ belongs to one $A_j$ of those submodules. The sum of submodules as well as the intersection $\bigcap_{j\in J} A_j$ are submodules of $T$. The submodule $RX$ is the intersection of all submodules containing the subset $X$.

If $T$ is a ring and $R$ is a subring of $T$, then $T$ is an $R$-module; then one can consider the *product* and the *quotient* of the left $R$-submodules $A$ and $B$ of $T$:

- $AB := \{\text{finite} \sum_\nu a_\nu b_\nu :\;\; a_\nu \in A,\; b_\nu \in B\; \forall \nu\}$
- $[A:B] := \{t \in T:\; tB \subseteq A\}$

Also these are left $R$-submodules of $T$.