braid group

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Let C_n be the space of unordered n-tuples of distinct points in the complex plane. The braid group B_n is the fundamental group of C_n.

A closed path γ on this space is a set of n paths $\gamma_i : [0,1] \to \mathbb{C}$ with $\gamma_i(t) \neq \gamma_j(t)$, and $\gamma_i(1) = \gamma_{\sigma(i)}(0)$, where σ is some permutation of $\{1, \ldots, n\}$. Drawing the graphs of all these paths in 3 space, what we see is n strands between the $z = 0$ and $z = 1$ planes, possibly tangled, with composition given by stacking these braids on top of each other. Homotopy corresponds to isotopy of the braid, homotopies of the strands such that none of them cross. This is the origin of the name "braid group"

The braid group determines a homomorphism $\phi : B_n \to S_n$, where S_n is the symmetric group on n letters. For $\gamma \in B_n$, we get an element of S_n from map sending $i \mapsto \gamma_i(1)$. This works because of our requirement on the points that the braids start and end, and since our homotopies fix basepoints. The kernel of ϕ consists of the braids that bring each strand to its original order. This kernel gives us the **pure braid group on n strands**, and is denoted by P_n. Hence, we have a short exact sequence

$$1 \to P_n \to B_n \to S_n \to 1.$$

We can also describe braid groups in more generality. Let M be a manifold. The configuration space of n ordered points on M is defined to be $F_n(M) = \{(a_1, \ldots, a_n) \in M^n \mid a_i \neq a_j \text{for} i \neq j\}$. The group S_n acts on $F_n(M)$ by permuting coordinates, and the corresponding quotient space $C_n(M) = F_n(M)/S_n$ is called the configuration space of n unordered points on M. In the case that $M = \mathbb{C}$, we obtain the regular and pure braid groups as $\pi_1(C_n(M))$ and $\pi_1(F_n(M))$ respectively.

The group B_n can be given the following presentation. The presentation was given in Artin's first paper [**?**] on the braid group. Label the braids 1 through n as before. Let σ_i be the braid that twists strands i and $i+1$, with i passing beneath $i+1$. Then the σ_i generate B_n, and the only relations needed are

$$\begin{array}{rcll} \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for } |i-j| \geq 2,\ 1 \leq i,j \leq n-1 \\ \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \leq i \leq n-2 \end{array}$$

The pure braid group has a presentation with

$$\text{generators} a_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \text{ for } 1 \leq i < j \leq n$$

that is, a_{ij} wraps the ith strand around the jth strand, and defining relations

$$a_{rs}^{-1}a_{ij}a_{rs} = \begin{cases} a_{ij} & \text{if } i < r < s < j \text{ or } r < s < i < j \\ a_{rj}a_{ij}a_{rj}^{-1} & \text{if } r < i = s < j \\ a_{rj}a_{sj}a_{ij}a_{sj}^{-1}a_{rj}^{-1} & \text{if } i = r < s < j \\ a_{rj}a_{sj}a_{rj}^{-1}a_{sj}^{-1}a_{ij}a_{sj}a_{rj}a_{sj}^{-1}a_{rj}^{-1} & \text{if } r < i < s < j \end{cases}$$

References

[1] E. Artin *Theorie der Zöpfe*. Abh. Math. Sem. Univ. Hamburg 4(1925), 42-72.

[2] V.L. Hansen *Braids and Coverings*. London Mathematical Society Student Texts 18. Cambridge University Press. 1989.