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core of a subgroup

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Let H be a subgroup of a group G.

The *core* (or *normal interior*, or *normal core*) of H in G is the intersection of all conjugates of H in G:

$$\mathrm{core}_G(H) = \bigcap_{x \in G} x^{-1} H x.$$

It is not hard to show that $\mathrm{core}_G(H)$ is the largest normal subgroup of G contained in H, that is, $\mathrm{core}_G(H) \trianglelefteq G$ and if $N \trianglelefteq G$ and $N \subseteq H$ then $N \subseteq \mathrm{core}_G(H)$. For this reason, some authors denote the core by H_G rather than $\mathrm{core}_G(H)$, by analogy with the notation H^G for the normal closure.

If $\mathrm{core}_G(H) = \{1\}$, then H is said to be *core-free*.

If $\mathrm{core}_G(H)$ is of finite index in H, then H is said to be *normal-by-finite*.

Let $\mathcal{L}$ be the set of left cosets of H in G. By considering the action of G on $\mathcal{L}$ it can be shown that the `http://planetmath.org/QuotientGroup`quotient $G/\mathrm{core}_G(H)$ embeds in the symmetric group $\mathrm{Sym}(\mathcal{L})$. A consequence of this is that if H is of finite index in G, then $\mathrm{core}_G(H)$ is also of finite index in G, and $[G : \mathrm{core}_G(H)]$ divides $[G : H]!$ (the factorial of $[G : H]$). In particular, if a simple group S has a proper subgroup of finite index n, then S must be of finite order dividing $n!$, as the core of the subgroup is trivial. It also follows that a group is virtually abelian if and only if it is abelian-by-finite, because the core of an abelian subgroup of finite index is a normal abelian subgroup of finite index (and the same argument applies if 'abelian' is replaced by any other property that is inherited by subgroups).