a group embeds into its profinite completion if and only if it is residually finite

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Let G be a group.

First suppose that G is residually finite, that is,

$$\mathrm{R}(G) := \bigcap_{N \trianglelefteq_{\mathrm{f}} G} N = 1$$

(where $N \trianglelefteq_{\mathrm{f}} G$ denotes that N is a normal subgroup of finite index in G). Consider the natural mapping of G into its profinite completion $\hat{G}$ given by $g \mapsto (Ng)_{N \trianglelefteq_{\mathrm{f}} G}$. It is clear that the kernel of this map is precisely $\mathrm{R}(G)$, so that it is a monomorphism when G is residually finite.

Now suppose that G embeds into its profinite completion $\hat{G}$ and identify G with a subgroup of $\hat{G}$. Now, a theorem on profinite groups tells us that

$$\bigcap_{N \trianglelefteq_{\mathrm{o}} \hat{G}} N = 1,$$

(where $N \trianglelefteq_{\mathrm{o}} G$ denotes that N is an `http://planetmath.org/TopologicalSpace`open normal subgroup of G) and since open subgroups of a profinite group have finite index, we have that

$$\mathrm{R}(\hat{G}) = 1,$$

so $\hat{G}$ is residually finite. Then G is a subgroup of a residually finite group, so is itself residually finite, as required.

References

[1] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, 2nd ed., Cambridge studies in advanced mathematics, Cambridge University Press, 1999.