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enumerating groups

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1 How many finite groups are there?

The current tables list the number of groups up to order 2000 [Besche, Eick, O'Brien] (2000).

The graph is chaotic – both figuratively and mathematically. Most groups are distributed along the interval at values $2^i m$ where m is odd and i large, for instance $i > 5$. Indeed most groups are actually of order $2^{10} = 1024$. We see this by connecting the dots of certain families of groups.

Most integers are square-free, most groups are not [Mays 1980; Miller 1930; Balas 1966].

An explanation for this distribution is offered by considering nilpotent groups. Nilpotent groups are the product of their Sylow subgroups. So enumerating nilpotent groups asks to enumerating p-groups.

2 How many nilpotent groups are there?

Theorem 1 (Pyber, 1993)**.** *If $g_{nil}(N)$ is the number of nilpotent groups of order $\leq N$ and $g(N)$ the number of groups of order $\leq N$ then*

$$\lim_{N\to\infty} \frac{\log g_{nil}(N)}{\log g(N)} = 1.$$

The proof bounds the number of groups with a given set of Sylow subgroups and involves the Classification of Finite Simple Groups.

Conjecture 2 (Pyber, 1993)**.**

$$\lim_{N\to\infty} \frac{g_{nil}(N)}{g(N)} = 1.$$

If the conjecture is true, then most groups are 2-groups.

3 The Higman and Sims bounds

Theorem 3 (Higman 1960, Sims 1964)**.** *The number of p-groups of order p^n, denoted, $f(p^n)$, satisfies*

$$\frac{2}{27}n^3 + C_1 n^2 \leq \log_p f(p^n) \leq \frac{2}{27}n^3 + C_2 n^{8/3}$$

for constants C_1 and C_2.

This result should be compared to the later work of Neretin on enumerating algebras. The lower bound is the work of Higman and is achieved by constructing a large family of class 2 p-groups (called Φ-class 2 groups as $\Phi(\Phi(P)) = 1$ where Φ is the Frattini subgroup of P).

The $n^{8/3}$ factor has been improved to $o(n^{5/2})$ by M. Newman and Seeley. Sims' suggests that it should be possible to show

$$\log_p f(p, n) \in \frac{2}{27}n^3 + O(n^2)$$

(with a positive leading coefficient) which would prove Pyber's conjecture [Shalev].

S. R. Blackburn's work (1992) on the number of class 3 p-groups provides strong evidence that this claim is true as he demonstrates that class 3 groups also attain this lower bound. Since class 3 groups involve the Jacobi identity (Hall-Witt identity) it is plausible to expect class c, for c less than some fixed bound, will asymptotically achieve the lower bound as well.