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presentation of a group

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A *presentation* of a group G is a description of G in terms of generators and relations (sometimes also known as relators). We say that the group is finitely presented, if it can be described in terms of a finite number of generators and a finite number of defining relations. A collection of group elements $g_i \in G,\ i \in I$ is said to generate G if every element of G can be specified as a product of the g_i, and of their inverses. A relation is a word over the alphabet consisting of the generators g_i and their inverses, with the property that it multiplies out to the identity in G. A set of relations $r_j,\ j \in J$ is said to be defining, if all relations in G can be given as a product of the r_j, their inverses, and the G-conjugates of these.

The standard notation for the presentation of a group is

$$G = \langle g_i \mid r_j \rangle,$$

meaning that G is generated by generators g_i, subject to relations r_j. Equivalently, one has a short exact sequence of groups

$$1 \to N \to F[I] \to G \to 1,$$

where $F[I]$ denotes the free group generated by the g_i, and where N is the smallest normal subgroup containing all the r_j. By the Nielsen-Schreier Theorem, the kernel N is itself a free group, and hence we assume without loss of generality that there are no relations among the relations.

Example. The symmetric group on n elements $1, \ldots, n$ admits the following finite presentation (Note: this presentation is not canonical. Other presentations are known.) As generators take

$$g_i = (i, i+1), \quad i = 1, \ldots, n-1,$$

the transpositions of adjacent elements. As defining relations take

$$(g_i g_j)^{n_{i,j}} = \mathrm{id}, \quad i, j = 1, \ldots n,$$

where

$$\begin{aligned} n_{i,i} &= 1 \\ n_{i,i+1} &= 3 \\ n_{i,j} &= 2, \quad |j-i| > 1. \end{aligned}$$

This means that a finite symmetric group is a Coxeter group.