normality of subgroups of prime index

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Proposition. *If H is a subgroup of a finite group G of index p, where p is the smallest prime dividing the order of G, then H is normal in G.*

Proof. Suppose $H \leq G$ with $|G|$ finite and $|G : H| = p$, where p is the smallest prime divisor of $|G|$, let G act on the set L of left cosets of H in G by left , and let $\varphi : G \to S_p$ be the `http://planetmath.org/node/3820`homomorphism induced by this action. Now, if $g \in \ker\varphi$, then $gxH = xH$ for each $x \in G$, and in particular, $gH = H$, whence $g \in H$. Thus $K = \ker\varphi$ is a normal subgroup of H (being contained in H and normal in G). By the First Isomorphism Theorem, G/K is isomorphic to a subgroup of S_p, and consequently $|G/K| = |G : K|$ must `http://planetmath.org/node/923`divide $p!$; moreover, any divisor of $|G : K|$ must also $|G| = |G : K||K|$, and because p is the smallest divisor of $|G|$ different from 1, the only possibilities are $|G : K| = p$ or $|G : K| = 1$. But $|G : K| = |G : H||H : K| = p|H : K| \geq p$, which $|G : K| = p$, and consequently $|H : K| = 1$, so that $H = K$, from which it follows that H is normal in G. $\square$