normal subgroups of the symmetric groups

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Author	rm50 (10146)
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Theorem 1. *For $n \geq 5$, A_n is the only proper nontrivial normal subgroup of S_n.*

Proof. This is essentially a corollary of the simplicity of the alternating groups A_n for $n \geq 5$. Let $N \trianglelefteq S_n$ be normal. Clearly $N \cap A_n \trianglelefteq A_n$. But A_n is simple, so $N \cap A_n = A_n$ or $N \cap A_n = \{e\}$. In the first case, either $N = A_n$, or else N also contains an `http://planetmath.org/SignatureOfAPermutation`odd permutation, in which case $N = S_n$. In the second case, either $N = \{e\}$ or else N consists solely of one or more odd permutations in addition to $\{e\}$. But if N contains two distinct odd permutations, σ and τ, then either $\sigma^2 \neq e$ or $\sigma\tau \neq e$, and both σ^2 and $\sigma\tau$ are `http://planetmath.org/SignatureOfAPermutation`even, contradicting the assumption that N contains only odd nontrivial permutations. Thus N must be of order 2, consisting of a single odd permutation of order 2 together with the identity.

It is easy to see, however, that such a subgroup cannot be normal. An odd permutation of order 2, σ, has as its cycle decomposition one or more (an odd number, in fact, though this does not matter here) of disjoint transpositions. Suppose wlog that $(1\ 2)$ is one of these transpositions. Then $\tau = (1\ 3)\sigma(1\ 3) = (1\ 3)(1\ 2)(\ldots)(1\ 3)$ takes 2 to 3 and thus is neither σ nor e. So this group is not normal. □

If $n = 1$, S_1 is the trivial group, so it has no nontrivial [normal] subgroups.

If $n = 2$, $S_2 = C_2$, the unique group on 2 elements, so it has no nontrivial [normal] subgroups.

If $n = 3$, S_3 has one nontrivial proper normal subgroup, namely the group generated by $(1\ 2\ 3)$.

S_4 is the most interesting case for $n \leq 5$. The arguments in the theorem above do not apply since A_4 is not simple. Recall that a normal subgroup must be a union of conjugacy classes of elements, and that conjugate elements in S_n have the same cycle type. If we examine the sizes of the various conjugacy classes of S_4, we get

Cycle Type	Size
4	6
3,1	8
2,2	3
2,1,1	6
1,1,1,1	1

A subgroup of S_4 must be of order $1, 2, 3, 4, 6, 8$, or 12 (the factors of $|S_4| = 24$). Since each subgroup must contain $\{e\}$, it is easy to see that the only possible nontrivial normal subgroups have orders 4 and 12. The order 4 subgroup is $H = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, while the order 12 subgroup is A_4. A_4 is obviously normal, being of index 2, and one can easily check that $H \cong V_4$ is also normal in S_4. So these are the only two nontrivial proper normal subgroups of S_4.