local finiteness is closed under extension, proof that

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Let G be a group and N a normal subgroup of G such that N and G/N are both locally finite. We aim to show that G is locally finite. Let F be a finite subset of G. It suffices to show that F is contained in a finite subgroup of G.

Let R be a set of coset representatives of N in G, chosen so that $1 \in R$. Let $r\colon G/N \to R$ be the function mapping cosets to their representatives, and let $s\colon G \to N$ be defined by $s(x) = r(xN)^{-1}x$ for all $x \in G$. Let $\pi\colon G \to G/N$ be the canonical projection. Note that for any $x \in G$ we have $x = r(xN)s(x)$.

Put $A = r(\langle \pi(F) \rangle)$, which is finite as G/N is locally finite. Let $B = s(F \cup AA \cup A^{-1})$, let $C = B \cup B^{-1}$ and let

$$D = \{a^{-1}ca \mid a \in A \text{ and } c \in C\} \subseteq N.$$

Put $H = \langle D \rangle$, which is finite as N is locally finite. Note that $1 \in A \subseteq R$ and $1 \in B \subseteq C \subseteq D \subseteq H \leq N$.

For any $a_1, a_2 \in A$ we have $a_1a_2 = r(a_1a_2N)s(a_1a_2) \in AB$. Note that $D^{-1} = D$, and so every element of H is a product of elements of D. So any element of the form $a^{-1}ha$, where $a \in A$ and $h \in H$, is a product of elements of the form $a^{-1}a_1^{-1}ca_1a$ for $a_1 \in A$ and $c \in C$; but $a_1a = a_2b$ for some $a_2 \in A$ and $b \in B$, so $a^{-1}ha$ is a product of elements of the form $b^{-1}a_2^{-1}ca_2b = b^{-1}(a_2^{-1}ca_2)b \in CDB \subseteq H$, and therefore $a^{-1}ha \in H$.

We claim that $AH \leq G$. Let $a_1, a_2 \in A$ and $h_1, h_2 \in H$. We have $(a_1h_1)(a_2h_2) = a_1a_2(a_2^{-1}h_1a_2)h_2$. But, by the previous paragraph, $a_1a_2 \in AB$ and $a_2^{-1}h_1a_2 \in H$, so $a_1a_2(a_2^{-1}h_1a_2)h_2 \in ABHH \subseteq AH$. Thus $AHAH \subseteq AH$. Also, $(a_1h_1)^{-1} = h_1^{-1}a_1^{-1} \in Ha_1^{-1}$. But $a_1^{-1} = r(a_1^{-1}N)s(a_1^{-1}) \in AB$, so $Ha_1^{-1} \subseteq HAB \subseteq AHAH \subseteq AH$. Thus $(AH)^{-1} \subseteq AH$. It follows that AH is a subgroup of G, and it is clearly finite.

For any $x \in F$ we have $x = r(xN)s(x) \in AB$. So $F \subseteq AH$, which completes the proof.