proof of Scott-Wiegold conjecture

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Suppose the conjecture were false. Then we have some $w \in C_p * C_q * C_r$ with $N(w) = C_p * C_q * C_r$. Let a, b, c denote the of w onto C_p, C_q, C_r respectively. Then a, b, c are all non-trivial as otherwise $N(w)$ would be contained in the kernel of one of the .

For $0° < \theta < 360°$ we say that a spin through θ consists of a unit vector, $\vec{u} \in \mathbb{R}^3$ together with the rotation of $\mathbb{R}^3$ through the angle θ anticlockwise about $\vec{u}$. In we have a single spin through the angle $0°$ and a single spin through $360°$. Thus the set of spins (usually denoted Spin(3)) naturally has the topology of a 3-sphere.

The spin through θ about a unit vector $\vec{u}$ has the same underlying rotation as the spin through $360° - \theta$ about $-\vec{u}$. Hence there are precisely two spins corresponding to each rotation of $\mathbb{R}^3$ about the origin.

is well defined on spins as you can compose the underlying rotations and continuity determines which of the two spins is the correct result. For example a $350°$ spin about $\vec{u}$ composed with a $20°$ spin about $\vec{u}$ is a $350°$ spin about $-\vec{u}$ (not a $10°$ spin about $\vec{u}$ which would be at the other end of the 3-sphere).

Let $\vec{n}$ denote the unit vector $(0, 0, 1)$. Fix an arc, I, on the unit sphere connecting $\vec{n}$ and $-\vec{n}$. Let $\vec{t}$ be a vector on this arc. Let $\vec{u}$ be an arbitrary unit vector. We may define a homomorphism $\phi_{\vec{t},\vec{u}} \colon F_{\{a,b,c\}} \to \mathrm{Spin}(3)$ by:
$\phi_{\vec{t},\vec{u}} \colon a \mapsto$ the spin through $(\frac{p-1}{2})\frac{360°}{p}$ (or $180°$ if $p = 2$) about $\vec{n}$
$\phi_{\vec{t},\vec{u}} \colon b \mapsto$ the spin through $(\frac{q-1}{2})\frac{360°}{q}$ (or $180°$ if $q = 2$) about $\vec{t}$
$\phi_{\vec{t},\vec{u}} \colon c \mapsto$ the spin through $(\frac{r-1}{2})\frac{360°}{r}$ (or $180°$ if $r = 2$) about $\vec{u}$

(Here $F_{\{a,b,c\}}$ denotes the free group on a, b, c).

So $\phi_{\vec{t},\vec{u}}(a)$, $\phi_{\vec{t},\vec{u}}(b)$ and $\phi_{\vec{t},\vec{u}}(c)$ are spins of between $120°$ and $180°$, all having non-trivial underlying rotations.

Let $\tilde{w}$ be a word in $F_{\{a,b,c\}}$ representing w, such that a, b, c occur in it 1 Mod $(2p)$ times, 1 Mod $2q$ times and 1 Mod $(2r)$ times, respectively.

We have a homomorphism $\phi' : C_p * C_q * C_r \to SO(3)$ induced by ϕ. If $\phi_{\vec{t},\vec{u}}(\tilde{w})$ has a trivial underlying rotation for some $\vec{t}$ and $\vec{u}$, then $N(w)$ will only contain elements in the kernel of ϕ'. In particular, we would have $a, b, c \notin N(w)$. So we may assume we have a map:

$$h \colon I \times S^2 \to S^2$$

which maps $(\vec{t}, \vec{u})$ to the unit vector corresponding to $\phi_{\vec{t},\vec{u}}(\tilde{w})$.

By we have $h(\vec{n}, R\vec{u}) = Rh(\vec{n}, \vec{u})$ for any rotation R about $\vec{n}$. Thus $h(\vec{n}, \_) \colon S^2 \to S^2$ maps latitudes to latitudes (possibly rotating them and /

or moving them up or down).

Also $h(\vec{n}, \vec{n}) = -\vec{n}$, as $\phi_{\vec{n},\vec{n}}(a)$, $\phi_{\vec{n},\vec{n}}(b)$ and $\phi_{\vec{n},\vec{n}}(c)$ are spins of between $120°$ and $180°$ anticlockwise about $\vec{n}$, so the sum of the angles will be greater than $360°$. Similarly one may that $h(\vec{n}, -\vec{n}) = \vec{n}$. Thus, as $h(n, \_)$ maps latitudes to latitudes, it must be homotopic to a reflection of S^2.

Again by we have $h(-\vec{n}, R\vec{u}) = Rh(-\vec{n}, \vec{u})$ for all rotations R about $\vec{n}$. Hence $h(-\vec{n}, \_) \colon S^2 \to S^2$ also maps latitudes to latitudes.

Further, $h(-\vec{n}, \vec{n}) = \vec{n}$ and $h(-\vec{n}, -\vec{n}) = -\vec{n}$. Thus $h(-\vec{n}, \_)$ is homotopic to the .

But h gives a homotopy from $h(\vec{n}, \_)$ to $h(-\vec{n}, \_)$, yielding the desired contradiction.