Burnside's Theorem

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Theorem 1 (Burnside's Theorem). *Let G be a simple group, $\sigma \in G$. Then the number of conjugates of σ is not a prime power (unless σ is its own conjugacy class).*

Proofs of this theorem are quite difficult and rely on representation theory. From this we immediately get

Corollary 1. *A group G of order $p^a q^b$, where p, q are prime, cannot be a nonabelian simple group.*

Proof. Suppose it is. Then the center of G is trivial, $\{e\}$, since the center is a normal subgroup and G is simple nonabelian. So if C_i are the nontrivial conjugacy classes, we have from the class equation that

$$|G| = 1 + \sum |C_i|$$

Now, each $|C_i|$ divides $|G|$, but cannot be 1 since the center is trivial. It cannot be a power of either p or q by Burnside's theorem. Thus $pq \mid |C_i|$ for each i and thus $|G| \equiv 1 \pmod{pq}$, which is a contradiction. □

Finally, a corollary of the above is known as the `http://planetmath.org/BurnsidePQTheoremB` p-q Theorem.

Corollary 2. *A group of order $p^a q^b$ is solvable.*