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proof of second isomorphism theorem for rings

Canonical name	ProofOfSecondIsomorphismTheoremForRings
Date of creation	2013-03-22 15:22:39
Last modified on	2013-03-22 15:22:39
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	14
Author	CWoo (3771)
Entry type	Proof
Classification	msc 20-00
Classification	msc 16-00
Classification	msc 13-00
Related topic	ProofOfSecondIsomorphismTheorem

In the context of rings, the Second Isomorphism Theorem can be phrased as follows:

If A is an ideal in a ring R and S is a subring of R , then

- $S + A$ is a subring of R ,
- A is an ideal in $S + A$,
- $S \cap A$ is an ideal in S ,
- There is an isomorphism $S/(S \cap A) \cong (S + A)/A$.

It should be emphasized that the proof of this is exactly the same as the proof of the corresponding statement for groups. Again, the main idea is to use the First Isomorphism Theorem. (It is quite routine to verify the above statements concerning what's a subring of what and what's an ideal in what. The heart of the matter is the isomorphism, and that is what we are up to here.)

Consider the mapping $f : S \rightarrow (S + A)/A$ where $f(s) = s + A$. Note that this is a ring homomorphism. Furthermore, it is surjective (or "onto"): If $(s + a) + A$ is an arbitrary element of $(S + A)/A$ with $s \in S$ and $a \in A$, then $f(s) = s + A = s + (a + A) = (s + a) + A$.

Consequently, the First Isomorphism Theorem tells us that

$$S/\ker(f) \cong (S + A)/A, \text{ where } \ker(f) \text{ denotes the kernel of } f.$$

So it all comes down to showing that $\ker(f) = S \cap A$.

Let $x \in \ker(f)$. Then $A = f(x) = x + A$. So $x \in A$. Moreover, since $\ker(f) \subseteq S$, we have $x \in S$. Thus $x \in S \cap A$ and so $\ker(f) \subseteq S \cap A$.

Conversely, suppose $x \in S \cap A$. Then $x \in A$. So $f(x) = x + A = A$. That is, $x \in \ker(f)$ and so $S \cap A \subseteq \ker(f)$.

Therefore, $\ker(f) = S \cap A$.