normal subgroups form sublattice of a subgroup lattice

Canonical name	NormalSubgroupsFormSublatticeOfASubgroupLattice
Date of creation	2013-03-22 15:48:24
Last modified on	2013-03-22 15:48:24
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Last modified by	CWoo (3771)
Numerical id	5
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Entry type	Example
Classification	msc 20E15

Consider $L(G)$, the subgroup lattice of a group G. Let $N(G)$ be the subset of $L(G)$, consisting of all normal subgroups of G.

First, we show that $N(G)$ is closed under $\wedge$. Suppose H and K are normal subgroups of G. If $x \in H \wedge K = H \cap K$, then for any $g \in G$, $gxg^{-1} \in H$ since H is normal, and $gxg^{-1} \in K$ likewise. So $gxg^{-1} \in H \cap K = H \wedge K$, implying that $H \wedge K$ is normal in G, or $H \wedge K \in N(G)$.

To see that $N(G)$ is closed under $\vee$, let H, K be normal subgroups of G, and consider an element

$$x = x_1 x_2 \cdots x_n \in H \vee K,$$

where $x_i \in H$ or $x_i \in K$. If $g \in G$, then

$$gxg^{-1} = gx_1x_2 \cdots x_n g^{-1} = (gx_1g^{-1})(gx_2g^{-1}) \cdots (gx_ng^{-1}),$$

where each $gx_ig^{-1} \in H$ or K. Therefore, $gxg^{-1} \in H \vee K$, so $H \vee K$ is normal in G and $H \vee K \in N(G)$.

Since $N(G)$ is closed under $\wedge$ and $\vee$, $N(G)$ is a sublattice of $L(G)$.

Remark. If G is finite, it can be shown (Wielandt) that the subnormal subgroups of G form a sublattice of $L(G)$.

References

[1] H. Wielandt *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Zeit. 45, pp. 209-244 (1939)