correspondence between normal subgroups and homomorphic images

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Assume, that G and H are groups. If $f : G \to H$ is a group homomorphism, then the first isomorphism theorem states, that the function $F : G/\ker(f) \to \mathrm{im}(f)$ defined by $F(g\ker(f)) = f(g)$ is a well-defined group isomorphism. Note that $\ker(f)$ is always normal in G.

This leads to the following question: is there a correspondence between normal subgroups of G and homomorphic images of G? We will try to answer this question, but before that, let us introduce some notion.

First of all, homomorphic image $\mathrm{im}(f)$ is not only a subgroup of H. Actually homomorphic image contains also some data about homomorphism. This observation leads to the following definition:

Definition. Let G be a group. Pair (H, f) is called *a homomorphic image of* G iff H is a group and $f : G \to H$ is a surjective group homomorphism. We will say that two homomorphic images (H, f) and (H', f') of G are isomorphic (or equivalent), if there exists a group isomorphism $F : H \to H'$ such that $F \circ f = f'$.

It is easy to see, that this isomorphism relation is actually an equivalence relation and thus we may speak about isomorphism classes of homomorphic images (which will be denoted by $[H, f]$ for homomorphic image (H, f)). Furthermore, if $N \subset G$ is a normal subgroup, then $(G/N, \pi_N)$ is a homomorphic image, where $\pi_N : G \to G/N$ is a projection, i.e. $\pi_N(g) = gN$. Let

$$\mathrm{norm}(G) = \{N \subseteq G \mid N \text{ is normal subgroup}\};$$

$$\mathrm{h.im}(G) = \{[H, f] \mid (H, f) \text{ is a homomorphic image of } G\}.$$

Proposition. Function $T : \mathrm{norm}(G) \to \mathrm{h.im}(G)$ defined by $T(N) = [G/N, \pi_N]$ is a bijection.

Proof. First, we will show, that T is onto. Let (H, f) be a homomorphic image of G. Let $N = \ker(f)$. Then (due to the first isomorphism theorem), there exists a group isomorphism $F : G/N \to H$ defined by $F(gN) = f(g)$. This shows, that

$$f(g) = F(gN) = F(\pi_N(g)) = (F \circ \pi_N)(g)$$

and thus $(G/N, \pi_N)$ is isomorphic to (H, f). Therefore

$$T(N) = [G/N, \pi_N] = [H, f],$$

which completes this part.

Now assume, that $T(N) = T(N')$ for some normal subgroups $N, N' \in \mathrm{norm}(G)$. This means, that $(G/N, \pi_N)$ and $(G/N', \pi_{N'})$ are isomorphic, i.e. there exists a group isomorphism $F : G/N \to G/N'$ such that $F \circ \pi_N = \pi_{N'}$. Let $x \in N' = \ker(\pi_{N'})$ and denote by $e \in G/N'$ the neutral element. Then, we have

$$e = \pi_{N'}(x) = F(\pi_N(x))$$

and (since F is an isomorphism) this is if and only if $x \in \ker(\pi_N) = N$. Thus, we've shown that $N' \subseteq N$. Analogously (after considering F^{-1}) we have that $N \subseteq N'$. Therefore $N = N'$, which shows, that T is injective. This completes the proof. $\square$