representations vs modules

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Let G be a group and k a field. Recall that a pair $(V, \cdot)$ is a representation of G over k, if V is a vector space over k and $\cdot : G \times V \to V$ is a linear group action (compare with parent object). On the other hand we have a group algebra kG, which is a vector space over k with G as a basis and the multiplication is induced from the multiplication in G. Thus we can consider modules over kG. These two concepts are related.

If $\mathbb{V} = (V, \cdot)$ is a representation of G over k, then define a kG-module $\overline{\mathbb{V}}$ by putting $\overline{\mathbb{V}} = V$ as a vector space over k and the action of kG on $\overline{\mathbb{V}}$ is given by

$$(\sum \lambda_i g_i) \circ v = \sum \lambda_i (g_i \cdot v).$$

It can be easily checked that $\overline{\mathbb{V}}$ is indeed a kG-module.

Analogously if M is a kG-module (with action denoted by ,,$\circ$"), then the pair $\underline{M} = (M, \cdot)$ is a representation of G over k, where ,,$\cdot$" is given by

$$g \cdot v = g \circ v.$$

As a simple exercise we leave the following proposition to the reader:

Proposition. Let $\mathbb{V}$ be a representation of G over k and let M be a kG-module. Then

$$\underline{\overline{\mathbb{V}}} = \mathbb{V};$$

$$\overline{\underline{M}} = M.$$

This means that modules and representations are the same concept. One can generalize this even further by showing that $\bar{\cdot}$ and $\underline{\cdot}$ are both functors, which are (mutualy invert) isomorphisms of appropriate categories.

Therefore we can easily define such concepts as ,,direct sum of representations" or ,,tensor product of representations", etc.