characterization of free submonoids

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Defines	intersection of free submonoids is free

Let A be an arbitrary set, let A^* be the free monoid on A, and let e be the identity element (empty word) of A^*. Let M be a submonoid of A^* and let $\mathrm{mgs}(M)$ be its minimal generating set.

We recall the universal property of free monoids: for every mapping $f : A \to M$ with M a monoid, there exists a unique morphism $\phi : A^* \to M$ such that $\phi(a) = f(a)$ for every $a \in A$.

Theorem 1 *The following are equivalent.*

1. *M is a free submonoid of A^*.*
2. *Any equation*

$$x_1 \cdots x_n = y_1 \cdots y_m \ , \ x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathrm{mgs}(M) \qquad (1)$$

   has only the trivial solutions $n = m, x_1 = y_1, \ldots, x_n = y_n$.

3. *For every $w \in A^*$, if $p, q \in M$ exist such that $pw, wq \in M$, then $w \in M$.*

From point **??** of Theorem **??** follows

Corollary 1 *An intersection of free submonoids of A^* is a free submonoid of A^*.*

As a consequence of Theorem **??**, there is no Nielsen-Schreier theorem for monoids. In fact, consider $A = \{a, b\}$ and $Y = \{a, ab, ba\} \subseteq A^*$: then $\mathrm{mgs}(Y^*) = Y$, but $x_1x_2 = y_1y_2$ has a nontrivial solution over Y, namely, $(ab)a = a(ba)$.

We now prove Theorem **??**.

Point **??** *implies point* **??**. Let $f : \mathrm{mgs}(M) \to B$ be a bijection. By the universal property of free monoids, there exists a unique morphism $\phi : B^* \to M$ that extends f^{-1}; such a morphism is clearly surjective. Moreover, any equation $\phi(b_1 \cdots b_n) = \phi(b'_1 \cdots b'_m)$ translates into an equation of the form (**??**), which by hypothesis has only trivial solutions: therefore $n = m$, $b_i = b'_i$ for all i, and ϕ is injective.

Point **??** *implies point* **??**. Suppose the existence of $p, q \in M$ such that $pw, wq \in M$ implies $w \in A^*$ is actually in M. Consider an equation of the form (**??**) which is a counterexample to the thesis, and such that the length of the compared words is minimal: we may suppose x_1 is a prefix of y_1, so that $y_1 = x_1w$ for some $w \in A^*$. Put $p = x_1, q = y_2 \cdots y_m$: then

$pw = y_1$ and $wq = x_2 \cdots x_n$ belong to M by construction. By hypothesis, this implies $w \in M$: then $y_1 \in \mathrm{mgs}(M)$ equals a product $x_1 w$ with $x_1, w \in M$—which, by definition of $\mathrm{mgs}(M)$, is only possible if $w = e$. Then $x_1 = y_1$ and $x_2 \cdots x_n = y_2 \cdots y_m$: since we had chosen a counterexample of minimal length, $n = m, x_2 = y_2, \ldots, x_n = y_n$. Then the original equation has only trivial solutions, and is not a counterexample after all.

Point **??** *implies point* **??**. Let $\phi : B^* \to M$ be an isomorphism of monoids. Then clearly $\mathrm{mgs}(M) \subseteq \phi(B)$; since removing $m = \phi(b)$ from $\mathrm{mgs}(M)$ removes $\phi(b^*)$ from M, the equality holds. Let $w \in A^*$ and let $p, q \in M$ satisfy $pw, wq \in M$: put $x = \phi^{-1}(p)$, $y = \phi^{-1}(q)$, $r = \phi^{-1}(pw)$, $s = \phi^{-1}(wq)$. Then $\phi(xs) = \phi(ry) = pwq \in M$, so $xs = ry$: this is an equality over B^*, and is satisfied only by $r = xu$, $s = uy$ for some u. Then $w = \phi(u) \in M$.

References

[1] M. Lothaire. *Combinatorics on words.* Cambridge University Press 1997.