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nilpotent group

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Defines	nilpotent
Defines	upper central series
Defines	lower central series
Defines	nilpotency class
Defines	nilpotent class

We define the *lower central series* of a group G to be the filtration of subgroups

$$G = G^1 \supset G^2 \supset \cdots$$

defined inductively by:

$$\begin{aligned} G^1 &:= G, \\ G^i &:= [G^{i-1}, G], \quad i > 1, \end{aligned}$$

where $[G^{i-1}, G]$ denotes the subgroup of G generated by all commutators of the form $hkh^{-1}k^{-1}$ where $h \in G^{i-1}$ and $k \in G$. The group G is said to be *nilpotent* if $G^i = 1$ for some i.

Nilpotent groups can also be equivalently defined by means of upper central series. For a group G, the *upper central series* of G is the filtration of subgroups

$$C_0 \subset C_1 \subset C_2 \subset \cdots$$

defined by setting C_0 to be the trivial subgroup of G, and inductively taking C_i to be the unique subgroup of G such that C_i/C_{i-1} is the center of G/C_{i-1}, for each $i > 1$. The group G is nilpotent if and only if $G = C_i$ for some i. Moreover, if G is nilpotent, then the length of the upper central series (i.e., the smallest i for which $G = C_i$) equals the length of the lower central series (i.e., the smallest i for which $G^{i+1} = 1$).

The *nilpotency class* or *nilpotent class* of a nilpotent group is the length of the lower central series (equivalently, the length of the upper central series).

Nilpotent groups are related to nilpotent Lie algebras in that a Lie group is nilpotent as a group if and only if its corresponding Lie algebra is nilpotent. The analogy extends to solvable groups as well: every nilpotent group is solvable, because the upper central series is a filtration with abelian quotients.