proof that a nontrivial normal subgroup of a finite p-group G and the center of G have nontrivial intersection

Canonical name	ProofThatANontrivialNormalSubgroupOfAFinitePgroupGAndTheCenterOfG
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Define G to act on H by conjugation; that is, for $g \in G$, $h \in H$, define

$$g \cdot h = ghg^{-1}$$

Note that $g \cdot h \in H$ since $H \triangleleft G$. This is easily seen to be a well-defined group action.

Now, the set of invariants of H under this action are

$$G_H = \{h \in H \mid g \cdot h = h \ \forall g \in G\} = \{h \in H \mid ghg^{-1} = h \ \forall g \in G\} = H \cap Z(G)$$

The class equation theorem states that

$$|H| = |G_H| + \sum_{i=1}^{r} [G : G_{x_i}]$$

where the G_{x_i} are proper subgroups of G, and thus that

$$|G_H| = |H| - \sum_{i=1}^{r} [G : G_{x_i}]$$

We now use elementary group theory to show that p divides each term on the right, and conclude as a result that p divides $|G_H|$, so that $G_H = H \cap Z(G)$ cannot be trivial.

As G is a nontrivial finite p-group, it is obvious from Cauchy's theorem that $|G| = p^n$ for $n > 0$. Since H and the G_{x_i} are subgroups of G, each either is trivial or has order a power of p, by Lagrange's theorem. Since H is nontrivial, its order is a nonzero power of p. Since each G_{x_i} is a proper subgroup of G and has order a power of p, it follows that $[G : G_{x_i}]$ also has order a nonzero power of p.