characteristic subgroup

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If $(G, *)$ is a group, then H is a *characteristic subgroup* of G (written H char G) if every automorphism of G maps H to itself. That is, if $f \in \mathrm{Aut}(G)$ and $h \in H$ then $f(h) \in H$.

A few properties of characteristic subgroups:

- If H char G then H is a normal subgroup of G.
- If G has only one subgroup of a given cardinality then that subgroup is characteristic.
- If K char H and $H \trianglelefteq G$ then $K \trianglelefteq G$. (Contrast with normality of subgroups is not transitive.)
- If K char H and H char G then K char G.

Proofs of these properties:

- Consider H char G under the inner automorphisms of G. Since every automorphism preserves H, in particular every inner automorphism preserves H, and therefore $g * h * g^{-1} \in H$ for any $g \in G$ and $h \in H$. This is precisely the definition of a normal subgroup.
- Suppose H is the only subgroup of G of order n. In general, `http://planetmath.org/GroupHo` take subgroups to subgroups, and of course isomorphisms take subgroups to subgroups of the same order. But since there is only one subgroup of G of order n, any automorphism must take H to H, and so H char G.
- Take K char H and $H \trianglelefteq G$, and consider the inner automorphisms of G (automorphisms of the form $h \mapsto g * h * g^{-1}$ for some $g \in G$). These all preserve H, and so are automorphisms of H. But any automorphism of H preserves K, so for any $g \in G$ and $k \in K$, $g * k * g^{-1} \in K$.
- Let K char H and H char G, and let ϕ be an automorphism of G. Since H char G, $\phi[H] = H$, so ϕ_H, the restriction of ϕ to H is an automorphism of H. Since K char H, so $\phi_H[K] = K$. But ϕ_H is just a restriction of ϕ, so $\phi[K] = K$. Hence K char G.