general linear group

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Given a vector space V, the *general linear group* $\mathrm{GL}(V)$ is defined to be the group of invertible linear transformations from V to V. The group operation is defined by composition: given $T : V \longrightarrow V$ and $T' : V \longrightarrow V$ in $\mathrm{GL}(V)$, the product TT' is just the composition of the maps T and T'.

If $V = \mathbb{F}^n$ for some field $\mathbb{F}$, then the group $\mathrm{GL}(V)$ is often denoted $\mathrm{GL}(n, \mathbb{F})$ or $\mathrm{GL}_n(\mathbb{F})$. In this case, if one identifies each linear transformation $T : V \longrightarrow V$ with its matrix with respect to the standard basis, the group $\mathrm{GL}(n, \mathbb{F})$ becomes the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$, under the group operation of matrix multiplication.

One also discusses the general linear group on a module M over some ring R. There it is the set of automorphisms of M as an R-module. For example, one might take $\mathrm{GL}(\mathbb{Z} \oplus \mathbb{Z})$; this is isomorphic to the group of two-by-two matrices with integer entries having determinant ± 1. If M is a general R-module, there need not be a natural interpretation of $\mathrm{GL}(M)$ as a matrix group.

The general linear group is an example of a group scheme; viewing it in this way ties together the properties of $\mathrm{GL}(V)$ for different vector spaces V and different fields F. The general linear group is an algebraic group, and it is a Lie group if V is a real or complex vector space.

When V is a finite-dimensional Banach space, $\mathrm{GL}(V)$ has a natural topology coming from the operator norm; this is isomorphic to the topology coming from its embedding into the ring of matrices. When V is an infinite-dimensional vector space, some elements of $\mathrm{GL}(V)$ may not be continuous and one generally looks instead at the set of bounded operators.