existence of maximal subgroups

Canonical name	ExistenceOfMaximalSubgroups
Date of creation	2013-03-22 16:24:54
Last modified on	2013-03-22 16:24:54
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Last modified by	Algeboy (12884)
Numerical id	5
Author	Algeboy (12884)
Entry type	Example
Classification	msc 20E28
Related topic	PropertyOfInfiniteSimpleGroup
Related topic	JordanHolderDecomposition
Related topic	EveryRingHasAMaximalIdeal

Because every finite group is a finite set, every chain of proper subgroups of a finite group has a maximal element and thus every finite group has a maximal subgroup. The same applies to maximal normal subgroups.

However, there are infinite groups, even abelian, with no maximal subgroups and no maximal normal subgroups. The Prüfer group

$$\mathbb{Z}_{p^\infty} = \varprojlim \mathbb{Z}_{p^i}$$

(for any prime p) is an example of an abelian group with no maximal subgroups. As the group is abelian all subgroups are normal so it also has no maximal normal subgroups. Such groups fail to fit the hypothesis of the Jordan-Hölder decomposition theorem as they do not have the ascending chain condition and so we cannot assign a composition series to such groups.

This stands in contrast to the category of unital rings where if one assumes Zorn's lemma (axiom of choice) then one may prove every unital ring has a maximal ideal.