alternative definition of group

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The below theorem gives three conditions that form alternative postulates. It is not hard to show that they hold in the group defined ordinarily.

Theorem. Let the non-empty set G satisfy the following three conditions:
I. For every two elements a, b of G there is a unique element ab of G.
II. For every three elements a, b, c of G the equation $(ab)c = a(bc)$ holds.
III. For every two elements a and b of G there exists at least one such element x and at least one such element y of G that $xa = ay = b$.
Then the set G forms a group.

Proof. If a and b are arbitrary elements, then there are at least one such e_a and such e_b that $e_a a = a$ and $be_b = b$. There are also such x and y that $xb = e_a$ and $ay = e_b$. Thus we have

$$e_a = xb = x(be_b) = (xb)e_b = e_a e_b = e_a(ay) = (e_a a)y = ay = e_b,$$

i.e. there is a unique neutral element e in G. Moreover, for any element a there is at least one couple a', a'' such that $a'a = aa'' = e$. We then see that

$$a' = a'e = a'(aa'') = (a'a)a'' = ea'' = a'',$$

i.e. a has a unique neutralizing element a'.