Young's projection operators

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Associated to a Young tableau with n boxes, we have two elements of the group ring of the permutation group on n symbols. To construct the operators, we first construct the antisymmetrizing operators corresponding to the columns and the symmetrizing operators corresponding to the rows. Then one operator corresponding to the tableau consists of the product of the symmetrizing operators corresponding to the rows multiplied by the product of the antisymmetrizing operators corresponding to the columns and the other consists of the product of the antisymmetrizing operators corresponding to the columns multiplied by the symmetrizing operators corresponding to the rows.

How this works may be illustrated with a simple example. Consider the tableau

$$\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}.$$

Corresponding to the first row, we have the symmetrization operator

$$1 + (1\,2).$$

Corresponding to the second row, we have the symmetrization operator

$$1 + (3\,4)$$

Multiplying these two symmetrization operators (the order does not matter because they involve permutations of different elements) produces

$$1 + (1\,2) + (3\,4) + (1\,2)(3\,4)$$

Corresponding to the first column, we have the antisymmetrization operator

$$1 - (1\,3).$$

Corresponding to the second column, we have the antisymmetrization operator

$$1 - (2\,4).$$

Multiplying these two antisymmetrization operators (the order does not matter because they involve permutations of different elements) produces

$$1 - (1\,3) - (2\,4) + (1\,3)(2\,4).$$

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To obtain one Young projector, we multiply the product of the symmetrization operators by the product of the antisymmetrization operators.

$$(1 + (1\,2) + (3\,4) + (1\,2)(3\,4))\,(1 - (1\,3) - (2\,4) + (1\,3)(2\,4)) =$$

$$
\begin{aligned}
1 \;\; + \;\; & (1\,2) + (3\,4) + (1\,2)(3\,4) - (1\,3) - (1\,2\,3) - (1\,3\,4) - (1\,2\,3\,4) - \\
(2\,4) \;\; - \;\; & (1\,4\,2) - (2\,4\,3) - (1\,4\,3\,2) + (1\,3)(2\,4) + (1\,4\,2\,3) + (1\,3\,2\,4) + (1\,4)(2\,3)
\end{aligned}
$$

To obtain the other projector, we multiply in the other order.

$$(1 - (1\,3) - (2\,4) + (1\,3)(2\,4))\,(1 + (1\,2) + (3\,4) + (1\,2)(3\,4)) =$$

$$
\begin{aligned}
1 \;\; + \;\; & (1\,2) + (3\,4) + (1\,2)(3\,4) - (1\,3) - (1\,3\,2) - (1\,4\,3) - (1\,4\,3\,2) - \\
(2\,4) \;\; - \;\; & (1\,2\,4) - (2\,3\,4) - (1\,2\,3\,4) + (1\,3)(2\,4) + (1\,3\,2\,4) + (1\,4\,2\,3) + (1\,4)(2\,3)
\end{aligned}
$$