semigroup with two elements

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Perhaps the simplest non-trivial example of a semigroup which is not a group is a particular semigroup with two elements. The underlying set of this semigroup is $\{a, b\}$ and the operation is defined as follows:

$$\begin{aligned} a \cdot a &= a \\ a \cdot b &= b \\ b \cdot a &= b \\ b \cdot b &= b \end{aligned}$$

It is rather easy to check that this operation is associative, as it should be:

$$\begin{array}{rcl} a \cdot (a \cdot a) = a \cdot a = & a & = a \cdot a = (a \cdot a) \cdot a \\ a \cdot (a \cdot b) = a \cdot b = & b & = a \cdot b = (a \cdot a) \cdot b \\ a \cdot (b \cdot b) = a \cdot b = & b & = b \cdot b = (a \cdot b) \cdot b \\ b \cdot (a \cdot a) = b \cdot a = & b & = a \cdot a = (a \cdot a) \cdot a \\ a \cdot (b \cdot b) = a \cdot b = & b & = b \cdot b = (a \cdot b) \cdot b \\ b \cdot (a \cdot b) = b \cdot b = & b & = b \cdot b = (b \cdot a) \cdot b \\ b \cdot (b \cdot a) = b \cdot b = & b & = b \cdot a = (b \cdot b) \cdot a \\ b \cdot (b \cdot b) = b \cdot b = & b & = b \cdot b = (b \cdot b) \cdot b \end{array}$$

It is worth noting that this semigroup is commutative and has an identity element, which is a. It is not a group because the element b does not have an inverse. In fact, it is not even a cancellative semigroup because we cannot cancel the b in the equation $a \cdot b = b \cdot b$.

This semigroup also arises in various contexts. For instance, if we choose a to be the truth value "true" and b to be the truth value "false" and the operation $\cdot$ to be the logical connective "and", we obtain this semigroup in logic. We may also represent it by matrices like so:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$