Moufang loop

Canonical name	MoufangLoop
Date of creation	2013-03-22 13:50:29
Last modified on	2013-03-22 13:50:29
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	12
Author	yark (2760)
Entry type	Definition
Classification	msc 20N05

Proposition: Let Q be a nonempty quasigroup.
I) The following conditions are equivalent.

$$\begin{aligned}
(x(yz))x &= (xy)(zx) \quad \text{for all } x, y, z \in Q \quad (1)\\
((xy)z)y &= x(y(zy)) \quad \text{for all } x, y, z \in Q \quad (2)\\
(xz)(yx) &= x((zy)x) \quad \text{for all } x, y, z \in Q \quad (3)\\
((yz)y)x &= y(z(yx)) \quad \text{for all } x, y, z \in Q \quad (4)
\end{aligned}$$

II) If Q satisfies those conditions, then Q has an identity element (i.e., Q is a loop).

For a proof, we refer the reader to the two references. Kunen in [1] shows that that any of the four conditions implies the existence of an identity element. And Bol and Bruck [2] show that the four conditions are equivalent for loops.

Definition: A nonempty quasigroup satisfying the conditions (1)–(4) is called a Moufang quasigroup or, equivalently, a Moufang loop (after Ruth Moufang, 1905–1977).

The 16-element set of unit octonions over $\mathbb{Z}$ is an example of a nonassociative Moufang loop. Other examples appear in projective geometry, coding theory, and elsewhere.

References

[1] Kenneth Kunen, *Moufang Quasigroups*, J. Algebra 83 (1996) 231–234. (A preprint in PostScript format is available from Kunen's website: `http://www.math.wisc.edu/ kunen/moufang.ps`Moufang Quasigroups.)

[2] R. H. Bruck, *A Survey of Binary Systems,* Springer-Verlag, 1958.