supernatural number

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Classification	msc 20E18
Defines	gcd of supernatural numbers
Defines	greatest commond divisor of supernatural numbers
Defines	lcm of supernatural numbers
Defines	least common multiple of supernatural numbers

A supernatural number ω is a formal product

$$\omega = \prod_p p^{n_p},$$

where p runs over all (rational) prime numbers, and the values n_p are each either natural numbers or the symbol ∞.

We note first that by the fundamental theorem of arithmetic, we can view any natural number as a supernatural number. Supernatural numbers form a generalization of natural numbers in two ways: First, by allowing the possibility of infinitely many prime factors, and second, by allowing any given prime to divide ω "infinitely often," by taking that prime's corresponding exponent to be the symbol ∞.

We can extend the usual p-adic to these supernatural numbers by defining, for ω as above, $v_p(\omega) = n_p$ for each p. We can then extend the notion of divisibility to supernatural numbers by declaring $\omega_1 \mid \omega_2$ if $v_p(\omega_1) \leq v_p(\omega_2)$ for all p (where, by definition, the symbol ∞ is considered greater than any natural number). Finally, we can also generalize the notion of the least common multiple (lcm) and greatest common divisor (gcd) for (arbitrarily many) supernatural numbers, by defining

$$\mathrm{lcm}(\{\omega_i\}) = \prod_p p^{\sup(v_p(\omega_i))}$$
$$\gcd(\{\omega_i\}) = \prod_p p^{\inf(v_p(\omega_i))}$$

Note that the supernatural version of the definitions of divisibility, lcm, and gcd carry over exactly from their corresponding notions for natural numbers, though we can now take the gcd or lcm of infinitely many natural numbers (to get a supernatural number).

Supernatural numbers are used to define orders and indices of profinite groups and subgroups, in which case many of the theorems from finite group theory carry over verbatim.

References

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[2] [Ser] Serre, J.-P. (Ion, P., translator) *Galois Cohomology*. Springer, New York, NY. 1997