ping-pong lemma

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Theorem (Ping Pong Lemma). *Let $k \geq 2$ and let G be a group acting on a space X. Suppose we are given a class $\mathcal{M} = \{A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k\}$ of $2k$ pairwise disjoint subsets of X and suppose $y_1, y_2, \ldots, y_k$ are elements of G such that*

$$B_i^c \subseteq y_i(A_i) \quad i = 1, 2, \ldots, k$$

(B_i^c is the complement of B_i in X). Then, the subgroup of G generated by $y_1, y_2, \ldots, y_k$ is free.

Before turning to prove the lemma let's state three simple facts:

Fact 1. *For all $i = 1, \ldots, k$ we have $y_i(A_i^c) \subseteq B_i$ and $y_i^{-1}(B_i^c) \subseteq A_i$*

Proof. $B_i^c \subseteq y_i(A_i) \implies B_i \supseteq y_i(A_i)^c = y_i(A_i^c)$ □

Fact 2. *If $i \neq j$ then $A_j \cup B_j \subseteq A_i^c \cap B_i^c$.*

Proof. A_i and A_j are disjoint therefore $A_j \subseteq A_i^c$. Similarly, $A_j \subseteq B_i^c$ so $A_j \subseteq A_i^c \cap B_i^c$. In the same way, $B_j \subseteq A_i^c \cap B_i^c$ so $A_j \cup B_j \subseteq A_i^c \cap B_i^c$. □

Fact 3. *If $R, S \in \mathcal{M}$ then $R^c \nsubseteq S$*

Proof. Assume by contradiction that $R^c \subseteq S$. Then, $X = R \cup S$ and therefore any element of M intersects with either R or S. However, the elements of M are pairwise disjoint and there are at least 4 elements in M so this is a contradiction. □

Using the above 3 facts, we now turn to the proof of the Ping Pong Lemma:

Proof. Suppose we are given $w = z_n^{\epsilon_n} \cdots z_2^{\epsilon_2} z_1^{\epsilon_1}$ such that $z_\ell \in \{y_1, y_2, \ldots, y_k\}$ and $\epsilon_\ell \in \{-1, +1\}$. and suppose further that w is freely reduced, namely, if $z_i = z_{i+1}$ then $\epsilon_i = \epsilon_{i+1}$. We want to show that $w \neq 1$ in G. Assume by contradiction that $w = 1$. We get a contradiction by giving $R, S \in \mathcal{M}$ such that $w(S^c) \subseteq R$ and therefore contradicting Fact **??** above since $S^c = w(S^c) \subseteq R$.

The set S is chosen as follows. Assume that $z_1 = y_i$ then:

$$S = \begin{cases} A_i & \text{if } \epsilon_1 = 1 \\ B_i & \text{if } \epsilon_1 = -1 \end{cases}$$

Define the following subsets $P_0, P_1, \dots, P_n$ of X:

$$P_0 = S^c; \quad P_1 = z_1^{\epsilon_1}(P_0), \dots, \; P_n = z_n^{\epsilon_n}(P_{n-1}) = w(S^c)$$

To complete the proof we show by induction that for $\ell = 1, 2, \dots, n$ if $z_\ell = y_i$ then:

1. if $\epsilon_\ell = 1$ then $P_\ell \subseteq B_i$.
2. if $\epsilon_\ell = -1$ then $P_\ell \subseteq A_i$.

For $\ell = 1$ the above follows from Fact **??** and the specific choice of P_0. Assume it is true for $\ell - 1$ and assume that $z_\ell = y_i$. We have two cases to check:

1. $z_{\ell-1} \neq z_\ell$: by the induction hypothesis $P_{\ell-1}$ is a subset of $A_j \cup B_j$ for some $j \neq i$. Therefore, by Fact **??** we get that $P_{\ell-1}$ is a subset of $A_i^c \cap B_i^c$. Consequently, we get the following:

   $$P_\ell = z_\ell^{\epsilon_\ell}(P_{\ell-1}) = y_i^{\epsilon_\ell}(P_{\ell-1}) \subseteq y_i^{\epsilon_\ell}(A_i^c \cap B_i^c)$$

   Hence, if $\epsilon_\ell = 1$ then:

   $$P_\ell \subseteq y_i(A_i^c \cap B_i^c) \subseteq y_i(A_i^c) \subseteq B_i$$

   And if $\epsilon_\ell = -1$ then:

   $$P_\ell \subseteq y_i^{-1}(A_i^c \cap B_i^c) \subseteq y_i^{-1}(B_i^c) \subseteq A_i$$

2. $z_{\ell-1} = z_\ell$: by the fact that w is freely reduced we get an equality between $\epsilon_{\ell-1}$ and ϵ_ℓ. Hence, if $\epsilon_\ell = 1$ then $P_{\ell-1} \subseteq B_i \subseteq A_i^c$ and therefore:

   $$P_\ell = z_\ell^{\epsilon_\ell}(P_{\ell-1}) = y_i(P_{\ell-1}) \subseteq y_i(A_i^c) \subseteq B_i$$

   Similiarly, if $\epsilon_\ell = -1$ then $P_{\ell-1} \subseteq A_i \subseteq B_i^c$ and therefore:

   $$P_\ell = z_\ell^{\epsilon_\ell}(P_{\ell-1}) = y_i^{-1}(P_{\ell-1}) \subseteq y_i^{-1}(B_i^c) \subseteq A_i$$

$\square$