Hall subgroup

Canonical name	HallSubgroup
Date of creation	2013-03-22 14:02:02
Last modified on	2013-03-22 14:02:02
Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
Numerical id	22
Author	Algeboy (12884)
Entry type	Definition
Classification	msc 20D20
Related topic	VeeCuhininsTheorem
Related topic	SylowTheorems
Defines	Hall's theorem
Defines	Hall π-subgroup

Let G be a finite group. A subgroup H of G is said to be a *Hall subgroup* if

$$\gcd(|H|, |G/H|) = 1.$$

In other words, H is a Hall subgroup if the order of H and its index in G are coprime. These subgroups are name after Philip Hall who used them to characterize solvable groups.

Hall subgroups are a generalization of Sylow subgroups. Indeed, every Sylow subgroup is a Hall subgroup. According to Sylow's theorem, this means that any group of order $p^k m$, $\gcd(p, m) = 1$, has a Hall subgroup (of order p^k).

A common notation used with Hall subgroups is to use the notion of http://planetmath.org/PiGroupsAndPiGroupsπ-groups. Here π is a set of primes and a Hall π-subgroup of a group is a subgroup which is also a π-group, and maximal with this property.

Theorem 1 (Hall (1928))**.** *A finite group G is solvable iff G has a Hall π-subgroup for any set of primes π.*

The sets of primes π in Hall's theorem can be restricted to the subsets of primes which divide $|G|$. However, this result fails for non-solvable groups.

Example 2. *The group A_5 has no Hall $\{2, 5\}$-subgroup. That is, A_5 has no subgroup of order 20.*

Proof. Suppose that A_5 has a Hall $\{2, 5\}$-subgroup H. As $|A_5| = 60$, it follows that $|H| = 20$. Thus, there are three cosets of H. Since a group always acts on the cosets of a subgroup, it follows that A_5 acts on the three member set C of cosets of H. This induces a non-trivial homomorphism from A_5 to $S_C \cong S_3$ (here, S_C is the symmetric group on C, see http://planetmath.org/GroupActionsAndHomomorphismsthis for more detail). Since A_5 is simple, this homomorphism must be one-to-one, implying that its image must have order at most 6, an impossibility. $\square$

This example can also be proved by direct inspection of the subgroups of A_5. In any case, A_5 is non-abelian simple and therefore it is not a solvable group. Thus, Hall's theorem does not apply to A_5.