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isomorphic groups

Canonical name	IsomorphicGroups
Date of creation	2013-03-22 14:01:58
Last modified on	2013-03-22 14:01:58
Owner	alozano (2414)
Last modified by	alozano (2414)
Numerical id	10
Author	alozano (2414)
Entry type	Definition
Classification	msc 20A05
Defines	isomorphic
Defines	abstractly identical

Two groups $(X_1, *_1)$ and $(X_2, *_2)$ are said to be *isomorphic* if there is a group isomorphism $\psi\colon X_1 \to X_2$.

Next we name a few necessary conditions for two groups X_1, X_2 to be isomorphic (with isomorphism ψ as above).

1. *If two groups are isomorphic, then they have the same cardinality.* Indeed, an isomorphism is in particular a bijection of sets.

2. *If the group X_1 has an element g of order n, then the group X_2 must have an element of the same order.* If there is an isomorphism ψ then $\psi(g) \in X_2$ and $(\psi(g))^n = \psi(g^n) = \psi(e_1) = e_2$ where e_i is the identity elements of X_i. Moreover, if $(\psi(g))^m = e_2$ then $\psi(g^m) = e_2$ and by the injectivity of ψ we must have $g^m = e_1$ so n divides m. Therefore the order of $\psi(g)$ is n.

3. *If one group is cyclic, the other one must be cyclic too.* Suppose X_1 is cyclic generated by an element g. Then it is easy to see that X_2 is generated by the element $\psi(g)$. Also *if X_1 is finitely generated, then X_2 is finitely generated as well.*

4. *If one group is abelian, the other one must be abelian as well.* Indeed, suppose X_2 is abelian. Then

$$\psi(g *_1 h) = \psi(g) *_2 \psi(h) = \psi(h) *_2 \psi(g) = \psi(h *_1 g)$$

   and using the injectivity of ψ we conclude $g *_1 h = h *_1 g$.

Note. Isomorphic groups are sometimes said to be *abstractly identical*, because their “abstract” are completely similar — one may think that their elements are the same but have only different names.