decomposable homomorphisms and full families of groups

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Let $\{G_i\}_{i\in I}, \{H_i\}_{i\in I}$ be two families of groups (indexed with the same set I).

Definition. We will say that a homomorphism

$$f : \bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} H_i$$

is *decomposable* if there exists a family of homomorphisms $\{f_i : G_i \to H_i\}_{i\in I}$ such that

$$f = \bigoplus_{i\in I} f_i.$$

Remarks. For each $j \in I$ and $g \in \bigoplus_{i\in I} G_i$ we will say that $g \in G_j$ if $g(i) = 0$ for any $i \neq j$. One can easily show that any homomorphism

$$f : \bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} H_i$$

is decomposable if and only if for any $j \in I$ and any $g \in \bigoplus_{i\in I} G_i$ such that $g \in G_j$ we have $f(g) \in H_j$. This implies that if f is an isomorphism and f is decomposable, then each homomorphism in decomposition is an isomorphism and

$$\left(\bigoplus_{i\in I} f_i\right)^{-1} = \bigoplus_{i\in I} f_i^{-1}.$$

Also it is worthy to note that composition of two decomposable homomorphisms is also decomposable and

$$\left(\bigoplus_{i\in I} f_i\right) \circ \left(\bigoplus_{i\in I} g_i\right) = \bigoplus_{i\in I} f_i \circ g_i.$$

Definition. We will say that family of groups $\{G_i\}_{i\in I}$ is *full* if each homomorphism

$$f : \bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} G_i$$

is decomposable.

Remark. It is easy to see that if $\{G_i\}_{i\in I}$ is a full family of groups and $I_0 \subseteq I$, then $\{G_i\}_{i\in I_0}$ is also a full family of groups.

Example. Let $\mathcal{P} = \{p \in \mathbb{N} \mid p \text{ is prime}\}$. Then $\{\mathbb{Z}_p\}_{p\in\mathcal{P}}$ is full. Indeed, let

$$f : \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_p \to \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_p$$

be a group homomorphism. Then, for any $q \in \mathcal{P}$ and $a \in \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_p$ such that $a \in \mathbb{Z}_q$ we have that $|a|$ divides q and thus $|f(a)|$ divides q, so it is easy to see that $f(a) \in \mathbb{Z}_q$. Therefore (due to first remark) f is decomposable.

Counterexample. Let G_1, G_2 be two copies of $\mathbb{Z}$. Then $\{G_1, G_2\}$ is not full. Indeed, let

$$f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

be a group homomorphism defined by

$$f(x, y) = (0, x + y).$$

Now assume that $f = f_1 \oplus f_2$. Then we have:

$$(0, 1) = f(1, 0) = (f_1(1), f_2(0))$$

and so $f_2(0) = 1$. Contradiction, since group homomorphisms preserve neutral elements.