A finitely generated group has only finitely many subgroups of a given index

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Let G be a finitely generated group and let n be a positive integer. Let H be a subgroup of G of index n and consider the action of G on the coset space $(G : H)$ by right multiplication. Label the cosets $1, \ldots, n$, with the coset H labelled by 1. This gives a homomorphism $\phi : G \to S_n$. Now, $x \in H$ if and only if $Hx = H$, that is, G fixes the coset H. Therefore, $H = \mathrm{Stab}_G(1) = \{g \in G \mid 1(g\phi) = 1\}$, and this is completely determined by ϕ. Now let X be a finite generating set for G. Then ϕ is determined by the images $x\phi$ of the generators $x \in X$. There are $|S_n| = n!$ choices for the image of each $x \in X$, so there are at most $n!^{|X|}$ homomorphisms $G \to S_n$. Hence, there are only finitely many possibilities for H.

References

[1] M. Hall, Jr., *A topology for free groups and related groups*, Ann. of Math. **52** (1950), no. 1, 127–139.