proof of the Jordan Hölder decomposition theorem

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Let $|G| = N$. We first prove existence, using induction on N. If $N = 1$ (or, more generally, if G is simple) the result is clear. Now suppose G is not simple. Choose a maximal proper normal subgroup G_1 of G. Then G_1 has a Jordan–Hölder decomposition by induction, which produces a Jordan–Hölder decomposition for G.

To prove uniqueness, we use induction on the length n of the decomposition series. If $n = 1$ then G is simple and we are done. For $n > 1$, suppose that

$$G \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

and

$$G \supset G'_1 \supset G'_2 \supset \cdots \supset G'_m = \{1\}$$

are two decompositions of G. If $G_1 = G'_1$ then we're done (apply the induction hypothesis to G_1), so assume $G_1 \neq G'_1$. Set $H := G_1 \cap G'_1$ and choose a decomposition series

$$H \supset H_1 \supset \cdots \supset H_k = \{1\}$$

for H. By the second isomorphism theorem, $G_1/H = G_1G'_1/G'_1 = G/G'_1$ (the last equality is because $G_1G'_1$ is a normal subgroup of G properly containing G_1). In particular, H is a normal subgroup of G_1 with simple quotient. But then

$$G_1 \supset G_2 \supset \cdots \supset G_n$$

and

$$G_1 \supset H \supset \cdots \supset H_k$$

are two decomposition series for G_1, and hence have the same simple quotients by the induction hypothesis; likewise for the G'_1 series. Therefore $n = m$. Moreover, since $G/G_1 = G'_1/H$ and $G/G'_1 = G_1/H$ (by the second isomorphism theorem), we have now accounted for all of the simple quotients, and shown that they are the same.