proof of properties of Hopfian and co-Hopfian groups

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Proposition. A group G is Hopfian if and only if every surjective homomorphism $G \to G$ is an automorphism.

Proof. "$\Rightarrow$" Assume that $\psi : G \to G$ is a surjective homomorpism such that ψ is not an automorphism, which means that $\mathrm{Ker}(\psi)$ is nontrivial. Then (due to the First Isomorphism Theorem) $G/\mathrm{Ker}(\psi)$ is isomorphic to $\mathrm{Im}(\psi) = G$. Contradiction, since G is Hopfian.

"$\Leftarrow$" Assume that G is not Hopfian. Then there exists nontrivial normal subgroup H of G and an isomorphism $\phi : G/H \to G$. Let $\pi : G \to G/H$ be the quotient homomorphism. Then obviously $\pi \circ \phi : G \to G$ is a surjective homomorphism, but $\mathrm{Ker}(\pi \circ \phi) = H$ is nontrivial, therefore $\pi \circ \phi$ is not an automorphism. Contradiction. $\square$

Proposition. A group G is co-Hopfian if and only if every injective homomorphism $G \to G$ is an automorphism.

Proof. "$\Rightarrow$" Assume that $\psi : G \to G$ is an injective homomorphism which is not an automorphism. Therefore $\mathrm{Im}(\psi)$ is a proper subgroup of G, therefore (since $\mathrm{Ker}(\psi) = \{e\}$ and due to the First Isomorphism Theorem) G is isomorphic to its proper subgroup, namely $\mathrm{Im}(\psi)$. Contradiction, since G is co-Hopfian.

"$\Leftarrow$" Assume that G is not co-Hopfian. Then there exists a proper subgroup H of G and an isomorphism $\phi : G \to H$. Let $i : H \to G$ be an inclusion homomorphism. Then $i \circ \phi : G \to G$ is an injective homomorphism which is not onto (because i is not). Contradiction. $\square$