mixed group

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A *mixed group* is a partial groupoid G such that G contains a non-empty subset K, called the *kernel* of G, with the following conditions:

1. if $a, b \in G$, then ab is defined iff $a \in K$,
2. if $a, b \in K$ and $c \in G$, then $(ab)c = a(bc)$,
3. if $a \in K$, then $K \subseteq aK \cap Ka$,
4. if $a \in K$ and $b \in G$ such that $ab = b$, then $ac = c$ for all $c \in G$.

Mixed groups are generalizations of groups, as the following proposition illustrates:

Proposition 1. *If $K = G$, then G is a group.*

Proof. G is a groupoid by condition 1, and a semigroup by condition 2.

Now, by condition 3, given $a \in G$, there is $b \in G$ such that $ba = a$, so that $bc = c$ for all $c \in G$ by condition 4. In other words, b is a left identity of G. Again, by condition 3, for every $a \in G$, there is a $d \in G$ such that $b = da$. So $ad = a(bd) = a(da)d = (ad)^2$, so, by condition 4, $adx = x$ for all $x \in G$. In particular, set $x = a$, we get $a = (ad)a = a(da) = ab$. Hence, b is a two-sided identity, and G is a monoid.

Finally, by condition 3, for every $a \in G$, there are $c, d \in G$, such that $b = ac = da$. So, $c = bc = (da)c = d(ac) = db = d$, showing that a has a two-sided inverse. This means that G is a group. $\square$

For a non-trivial example of a mixed group, let G be a group and H a subgroup of G. Define a new multiplication $\cdot$ on G as follows: $a \cdot b$ is defined iff $a \in H$, and if $a \cdot b$ is defined, it is defined as ab, the group multiplication of a and b. Then $(G, \cdot)$ is a mixed group. Clearly, associativity of $\cdot$ is automatically satisfied. Next, pick any $a \in H$, then, for any $b \in H$, $a^{-1} \cdot b$ and $b \cdot a^{-1}$ are both elements of H, so that $b \in a \cdot H \cap H \cdot a$, and condition 3 is also satisfied. Finally, if $a \in H$ and $b \in G$ such that $a \cdot b = b$, then a is the multiplicative identity of G, clearly $a \cdot c = c$ for all $c \in G$.

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