fundamental homomorphism theorem

Canonical name	FundamentalHomomorphismTheorem
Date of creation	2013-03-22 15:35:06
Last modified on	2013-03-22 15:35:06
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	9
Author	yark (2760)
Entry type	Theorem
Classification	msc 20A05

The following theorem is also true for rings (with ideals instead of normal subgroups) or modules (with submodules instead of normal subgroups).

theorem 1. *Let G, H be groups, $f\colon G \to H$ a homomorphism, and let N be a normal subgroup of G contained in $\ker(f)$. Then there exists a unique homomorphism $h\colon G/N \to H$ so that $h \circ \varphi = f$, where φ denotes the canonical homomorphism from G to G/N.*

Furthermore, if f is onto, then so is h; and if $\ker(f) = N$, then h is injective.

Proof. We'll first show the uniqueness. Let $h_1, h_2\colon G/N \to H$ functions such that $h_1 \circ \varphi = h_2 \circ \varphi$. For an element y in G/N there exists an element x in G such that $\varphi(x) = y$, so we have

$$h_1(y) = (h_1 \circ \varphi)(x) = (h_2 \circ \varphi)(x) = h_2(y)$$

for all $y \in G/N$, thus $h_1 = h_2$.

Now we define $h : G/N \to H$, $h(gN) = f(g)\ \forall\ g \in G$. We must check that the definition is of the given representative; so let $gN = kN$, or $k \in gN$. Since N is a subset of $\ker(f)$, $g^{-1}k \in N$ implies $g^{-1}k \in \ker(f)$, hence $f(g) = f(k)$. Clearly $h \circ \varphi = f$.

Since $x \in \ker(f)$ if and only if $h(xN) = 1_H$, we have

$$\ker(h) = \{xN \mid x \in \ker(f)\} = \ker(f)/N.$$

□

A consequence of this is: If $f\colon G \to H$ is onto with $\ker(f) = N$, then G/N and H are isomorphic.