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order of elements in finite groups

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This article proves two elementary results regarding the orders of group elements in finite groups.

Theorem 1 *Let G be a finite group, and let $a \in G$ and $b \in G$ be elements of G that commute with each other. Let $m = |a|$, $n = |b|$. If $\gcd(m, n) = 1$, then $mn = |ab|$.*

Proof. Note first that

$$(ab)^{mn} = a^{mn}b^{mn} = (a^m)^n(b^n)^m = e_G$$

since a and b commute with each other. Thus $|ab| \leq mn$. Now suppose $|ab| = k$. Then

$$e_G = (ab)^k = (ab)^{km} = a^{km}b^{km} = b^{km}$$

and thus $n|km$. But $\gcd(m, n) = 1$, so $n|k$. Similarly, $m|k$ and thus $mn|k = |ab|$. These two results together imply that $mn = k$.

Theorem 2 *Let G be a finite abelian group. If G contains elements of orders m and n, then it contains an element of order $\operatorname{lcm}(m, n)$.*

Proof. Choose a and b of orders m and n respectively, and write

$$\operatorname{lcm}(m, n) = \prod p_i^{k_i}$$

where the p_i are distinct primes. Thus for each i, either $p_i^{k_i} \mid m$ or $p_i^{k_i} \mid n$. Thus either $a^{m/p_i^{k_i}}$ or $b^{n/p_i^{k_i}}$ has order $p_i^{k_i}$. Let this element be c_i. Now, the orders of the c_i are pairwise coprime by construction, so

$$\left|\prod c_i\right| = \prod |c_i| = \operatorname{lcm}(m, n)$$

and thus $\prod c_i$ is the required element.