corollaries of basic theorem on ordered groups

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Corollary 1 Let G be an ordered group. For all $x \in G$, either $x \leq 1 \leq x^{-1}$ or $x^{-1} \leq 1 \leq x$.

Proof: By conclusion 1, either $x < 1$ or $x = 1$ or $1 < x$. If $x < 1$, then, by conclusion 5, $1^{-1} < x^{-1}$, so $x < 1 < x^{-1}$. If $x = 1$, the conclusion is trivial. If $1 < x$, then, by conclusion 5, $x^{-1} < 1^{-1}$, so $x^{-1} < 1 < x$. Q.E.D.

Corollary 2 Let G be an ordered group and n a strictly positive integer. Then, for all $x, y \in G$, we have $x < y$ if and only if $x^n < y^n$.

Proof: We shall first prove that $x < y$ implies $x^n < y^n$ by induction. If $n = 1$, this is a simple tautology. Assume the conclusion is true for a certain value of n. Then, conclusion 4 allows us to multiply the inequalities $x < y$ and $x^n < y^n$ to obtain $x^{n+1} < y^{n+1}$.

As for the proof that $x^n < y^n$ implies $x < y$, we shall prove the contrapositive statement. Assume that $x < y$ is false. By conclusion 1, it follows that either $x = y$ or $x > y$. If $x = y$, then $x^n = y^n$ so, by conclusion 1 $x^n < y^n$ is false. If $x > y$ then, by what we have already shown, $x^n > y^n$ so $x^n < y^n$ is also false in this case for the same reason. Q.E.D.

Corollary 3 Let G be an ordered group and n a strictly positive integer. Then, for all $x, y \in G$, we have $x = y$ if and only if $x^n = y^n$.

Proof: It is trivial that, if $x = y$, then $x^n = y^n$. Assume that $x^n = y^n$. By conclusion 1 of the main theorem, it is the case that either $x < y$ or $x = y$ or $y < x$. If $x < y$ then, by the preceding corollary, $x^n < y^n$, which is not possible. Likewise, if $y < x$, then we would have $y^n < x^n$, which is also impossible. The only remaining possibility is $x = y$. Q.E.D.

Corollary 4 An ordered group cannot contain any elements of finite order.

Let x be an element of an ordered group distinct from the identity. By definition, if x were of finite order, there would exist an integer such that $x^n = 1$. Since $1 = 1^n$, we would have $x^n = 1^n$ but, by Corollary 3, this would imply $x = 1$, which contradicts our hypothesis.

It is worth noting that, in the context of additive groups of rings, this result states that ordered rings have characteristic zero.