general commutativity

Canonical name	GeneralCommutativity
Date of creation	2014-05-10 21:59:41
Last modified on	2014-05-10 21:59:41
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	10
Author	pahio (2872)
Entry type	Theorem
Classification	msc 20-00
Related topic	CommutativeLanguage
Related topic	GeneralAssociativity
Related topic	AbelianGroup2

Theorem. If the binary operation "$\cdot$" on the set S is commutative, then for each $a_1, a_2, \ldots, a_n$ in S and for each permutation π on $\{1,\,2,\,\ldots,\,n\}$, one has

$$\prod_{i=1}^{n} a_{\pi(i)} \;=\; \prod_{i=1}^{n} a_i. \tag{1}$$

Proof. If $n = 1$, we have nothing to prove. Make the induction hypothesis, that (1) is true for $n = m-1$. Denote

$$\pi^{-1}(m) \;=\; k, \quad \text{i.e.} \quad \pi(k) = m.$$

Then

$$\prod_{i=1}^{m} a_{\pi(i)} \;=\; \prod_{i=1}^{k-1} a_{\pi(i)} \cdot a_{\pi(k)} \cdot \prod_{i=1}^{m-k} a_{\pi(k+i)} \;=\; \left(\prod_{i=1}^{k-1} a_{\pi(i)} \cdot \prod_{i=1}^{m-k} a_{\pi(k+i)}\right) \cdot a_m,$$

where a_m has been moved to the end by the induction hypothesis. But the product in the parenthesis, which exactly the factors $a_1, a_2, \ldots, a_{m-1}$ in a certain , is also by the induction hypothesis equal to $\prod_{i=1}^{m-1} a_i$. Thus we obtain

$$\prod_{i=1}^{m} a_{\pi(i)} \;=\; \prod_{i=1}^{m-1} a_i \cdot a_m \;=\; \prod_{i=1}^{m} a_i,$$

whence (1) is true for $n = m$.

Note. There is mentionned in the Remark of the entry "`http://planetmath.org/node/2148co`
a more general notion of commutativity.