proof of Nielsen-Schreier theorem and Schreier index formula

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While there are purely algebraic proofs of the Nielsen-Schreier theorem, a much easier proof is available through geometric group theory.

Let G be a group which is free on a set X. Any group acts freely on its Cayley graph, and the Cayley graph of G is a $2|X|$-regular tree, which we will call $\mathcal{T}$.

If H is any subgroup of G, then H also acts freely on $\mathcal{T}$ by restriction. Since groups that act freely on trees are free, H is free.

Moreover, we can obtain the rank of H (the size of the set on which it is free). If $\mathcal{G}$ is a finite graph, then $\pi_1(\mathcal{G})$ is free of rank $-\chi(\mathcal{G}) - 1$, where $\chi(\mathcal{G})$ denotes the Euler characteristic of $\mathcal{G}$. Since $H \cong \pi_1(H \backslash \mathcal{T})$, the rank of H is $\chi(H \backslash \mathcal{T})$. If H is of finite index n in G, then $H \backslash \mathcal{T}$ is finite, and $\chi(H \backslash \mathcal{T}) = n\chi(G \backslash \mathcal{T})$. Of course $-\chi(G \backslash \mathcal{T}) + 1$ is the rank of G. Substituting, we obtain the Schreier index formula:

$$\operatorname{rank}(H) = n(\operatorname{rank}(G) - 1) + 1.$$