subsemigroup,, submonoid,, and subgroup

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Defines	subsemigroup
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Let S be a semigroup, and let T be a subset of S.

T is a *subsemigroup* of S if T is closed under the operation of S; that it if $xy \in T$ for all $x, y \in T$.

T is a *submonoid* of S if T is a subsemigroup, and T has an identity element.

T is a *subgroup* of S if T is a submonoid which is a group.

Note that submonoids and subgroups do not have to have the same identity element as S itself (indeed, S may not have an identity element). The identity element may be any idempotent element of S.

Let $e \in S$ be an idempotent element. Then there is a maximal subsemigroup of S for which e is the identity:

$$eSe = \{exe \mid x \in S\}.$$

In addition, there is a maximal subgroup for which e is the identity:

$$\mathcal{U}(eSe) = \{x \in eSe \mid \exists y \in eSe \text{ st } xy = yx = e\}.$$

Subgroups with different identity elements are disjoint. To see this, suppose that G and H are subgroups of a semigroup S with identity elements e and f respectively, and suppose $x \in G \cap H$. Then x has an inverse $y \in G$, and an inverse $z \in H$. We have:

$$e = xy = fxy = fe = zxe = zx = f.$$

Thus intersecting subgroups have the same identity element.