# free product with amalgamated subgroup

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**Definition 1.** Let $G_k$, $k = 0, 1, 2$ be groups and $i_k \colon G_0 \to G_i$, $k = 1, 2$ be monomorphisms. The free product of $G_1$ and $G_2$ with amalgamated subgroup $G_0$, is defined to be a group $G$ that has the following two properties

1. there are homomorphisms $j_k \colon G_k \to G$, $k = 1, 2$ that make the following diagram commute

$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] & \\ G_0 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & G \\ & G_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

2. $G$ is universal with respect to the previous property, that is for any other group $G'$ and homomorphisms $j'_k \colon G_k \to G'$, $k = 1, 2$ that fit in such a commutative diagram there is a unique homomorphism $G \to G'$ so that the following diagram commutes

$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] \arrow[drr, "j'_1"] & & \\ G_0 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & G \arrow[r, "!"] & G' \\ & G_2 \arrow[ur, "j_2"'] \arrow[urr, "j'_2"'] & & \end{tikzcd}$$

It follows by "general nonsense" that the free product of $G_1$ and $G_2$ with amalgamated subgroup $G_0$, if it exists, is "unique up to unique isomorphism." The free product of $G_1$ and $G_2$ with amalgamated subgroup $G_0$, is denoted by $G_1 \bigstar_{G_0} G_2$. The following theorem asserts its existence.

**Theorem 2.** *$G_1 \bigstar_{G_0} G_2$ exists for any groups $G_k$, $k = 0, 1, 2$ and monomorphisms $i_k \colon G_0 \to G_i$, $k = 1, 2$.*

***Sketch of proof.*** Without loss of generality assume that $G_0$ is a subgroup of $G_k$ and that $i_k$ is the inclusion for $k = 1, 2$. Let

$$G_k = \langle (x_{k;s})_{s \in S} \mid (r_{k;t})_{t \in T} \rangle$$

be a presentation of $G_k$ for $k = 1, 2$. Each $g \in G_0$ can be expressed as a word in the generators of $G_k$; denote that word by $w_k(g)$ and let $N$ be the normal closure of $\{w_1(g)w_2(g)^{-1} \mid g \in G_0\}$ in the `http://planetmath.org/FreeProduct`free product $G_1 \bigstar G_2$. Define

$$G_1 \bigstar_{G_0} G_2 := G_1 \bigstar G_2 / N$$

and for $k = 0, 1$ define $j_k$ to be the inclusion into the free product followed by the canonical projection. Clearly (1) is satisfied, while (2) follows from the universal properties of the free product and the quotient group. $\square$

Notice that in the above proof it would be sufficient to divide by the relations $w_1(g)w_2(g)^{-1}$ for $g$ in a generating set of $G_0$. This is useful in practice when one is interested in obtaining a presentation of $G_1 \bigstar_{G_0} G_2$.

In case that the $i_k$'s are not injective the above still goes through verbatim. The group thusly obtained is called a "pushout".

Examples of free products with amalgamated subgroups are provided by Van Kampen's theorem.