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heap

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Defines	semiheap
Defines	heapoid

A *heap* is a non-empty set H with a ternary operation $f : H^3 \to H$, such that

1. $f(f(r,s,t),u,v) = f(r,s,f(t,u,v))$ for any $r,s,t,u,v \in H$, and
2. $f(r,s,s) = f(s,s,r) = r$ for any $r,s \in H$.

Heaps and groups are intimately related. Every group has the structure of a heap:

Given a group G, if we define $f : G^3 \to G$ by

$$f(a,b,c) = ab^{-1}c,$$

then (G,f) is a heap, for $f(f(r,s,t),u,v) = (rs^{-1}t)u^{-1}v = rs^{-1}(tu^{-1}v) = f(r,s,f(t,u,v))$, and $f(r,s,s) = rs^{-1}s = r = ss^{-1}r = f(s,s,r)$.

The associated heap structure on a group is the associated heap of the group.

Conversely, every heap can be derived this way:

Proposition 1. *Given a heap (H,f), then $(H,\cdot)$ is a group for some binary operation $\cdot$ on H, such that $f(a,b,c) = a \cdot b^{-1} \cdot c$.*

Proof. Pick an arbitrary element $r \in H$, and define a binary operation $\cdot$ on H by

$$a \cdot b := f(a,r,b).$$

We next show that $(H,\cdot)$ is a group.

First, $\cdot$ is associative: $(a\cdot b)\cdot c = f(f(a,r,b),r,c) = f(a,r,f(b,r,c)) = a\cdot(b\cdot c)$. This shows that $(H,\cdot)$ is a semigroup. Second, r is an identity with respect to $\cdot$: $a \cdot r = f(a,r,r) = a$ and $r \cdot a = f(r,r,a) = a$, showing that $(H,\cdot)$ is a monoid. Finally, given $a \in H$, the element $b = f(r,a,r)$ is a two-sided inverse of a: $a \cdot b = f(a,r,b) = f(a,r,f(r,a,r)) = f(f(a,r,r),a,r) = f(a,a,r) = r$ and $b \cdot a = f(b,r,a) = f(f(r,a,r),r,a) = f(r,a,f(r,r,a)) = f(r,a,a) = r$, hence $(H,\cdot)$ is a group.

Finally, by a direction computation, we see that $a \cdot b^{-1} \cdot c = af(r,b,r)c = f(a,r,f(r,b,r))c = f(f(a,r,r),b,r)c = f(a,b,r)c = f(f(a,b,r),r,c) = f(a,b,f(r,r,c)) = f(a,b,c)$. □

From the proposition above, we see that any element of H can be chosen, so that the associated group operation turns that element into an identity element for the group. In other words, one can think of a heap as a group where the designation of a multiplicative identity is erased, in much the same way that an affine space is a vector space without the origin (additive identity):

An immediate corollary is the following: for any element r in a heap (H,f), the equation

$$f(x,y,z)=r$$

in three variables x,y,z has exactly one solution in the remaining variable, if two of the variables are replaced by elements of H.

Remarks.

1. A heap is also known as a *flock*, due to its application in affine geometry, or as an *abstract coset*, because, as it can be easily shown, a subset H of a group G is a coset (of a subgroup of G) iff it is a subheap of G considered as a heap (see example above).

   Proof. First, notice that we have two equations

   $$f(ar,as,at)=af(r,s,t) \qquad \text{and} \qquad f(ra,sa,ta)=f(r,s,t)a.$$

   From this, we see that if $H=aK$ or $H=Ka$ for some subgroup K of G, then $f(H,H,H)\subseteq H$, whence H is a subheap of G. On the other hand, suppose that H is a subheap of G, and let $K=\{rs^{-1}\mid r,s\in H\}$. We want to show that K is a subgroup of G (and hence H is a coset of K). Certainly $e=rr^{-1}\in K$. If $rs^{-1}\in K$, then $sr^{-1}=(rs^{-1})^{-1}\in K$. Finally, if rs^{-1} and tu^{-1} are both in K, then $rs^{-1}tu^{-1}=f(r,s,t)u^{-1}$, which is in K because both $f(r,s,t)$ and u are in H. □

2. More generally, a structure H with a ternary operation f satisfying only condition 1 above is known as a *heapoid*, and a heapoid satisfying the condition

   $$f(f(r,s,t),u,v)=f(r,f(u,t,s),v)$$

   is called a *semiheap*. Every heap is a semiheap, for, by Proposition 1 above:

   $$f(r,f(u,t,s),v)=r(ut^{-1}s)^{-1}v=rs^{-1}tu^{-1}v=f(rs^{-1}t,u,v)=f(f(r,s,t),u,v).$$

3. Let (H, f) be a heap. Then (H, f) is a `http://planetmath.org/PolyadicSemigroup`3-group iff $f(u, t, s) = f(s, t, u)$. First, if (H, f) is a 3-group, then f is associative, so $f(r, f(u, t, s), v) = f(r, f(s, t, u), v)$ since a heap is a semiheap. By the corollary above, we get the equation $f(u, t, s) = f(s, t, u)$. On the other hand, the equation shows that f is associative, and together with the corollary, (H, f) is a 3-group.

4. Suppose now that (H, f) is a 3-group such that $f(u, t, s) = f(s, t, u)$. Then (H, f) is a heap iff $f(r, r, r) = r$ for all $r \in H$. The first condition of a heap is automatically satisfied since f is associative. Now, if (H, f) is a heap, then $f(r, r, r) = r$ by condition 2. Conversely, $f(r, s, s) = f(s, s, r) = t$ by the given equation above. So $f(s, t, s) = f(s, f(r, s, s), s) = f(s, r, f(s, s, s)) = f(s, r, s)$. As a 3-group, it has a covering group, so $t = r$ as a result.

References

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[2] H. Prüfer, *Theorie der Abelschen Gruppen*, Math. Z. 20, 166-187, 1924