the derived subgroup is normal

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We are going to prove:
"The derived subgroup (or commutator subgroup) $[G,G]$ is normal in G"

Proof:
We have to show that for each $x \in [G,G]$, gxg^{-1} it is also in $[G,G]$.

Since $[G,G]$ is the subgroup generated by the all commutators in G, then for each $x \in [G,G]$ we have $x = c_1 c_2 \cdots c_m$ –a word of commutators– so $c_i = [a_i, b_i]$ for all i.

Now taking any element of $g \in G$ we can see that

$$\begin{aligned} g[a_i,b_i]g^{-1} &= ga_ib_ia_i^{-1}b_i^{-1}g^{-1} \\ &= ga_ig^{-1}gb_ig^{-1}ga_i^{-1}g^{-1}gb_i^{-1}g^{-1} \\ &= (ga_ig^{-1})(gb_ig^{-1})(ga_ig^{-1})^{-1}(gb_ig^{-1})^{-1} \\ &= [ga_ig^{-1}, gb_ig^{-1}], \end{aligned}$$

that is

$$g[a_i,b_i]g^{-1} = [ga_ig^{-1}, gb_ig^{-1}]$$

so a conjugation of a commutator is another commutator, then for the conjugation

$$\begin{aligned} gxg^{-1} &= gc_1c_2\cdots c_mg^{-1} \\ &= gc_1g^{-1}gc_2g^{-1}g\cdots g^{-1}gc_mg^{-1} \\ &= (gc_1g^{-1})(gc_2g^{-1})\cdots(gc_mg^{-1}) \end{aligned}$$

is another word of commutators, hence gxg^{-1} is in $[G,G]$ which in turn implies that $[G,G]$ is normal in G, QED.