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injection can be extended to isomorphism

Canonical name	InjectionCanBeExtendedToIsomorphism
Date of creation	2013-03-22 18:56:50
Last modified on	2013-03-22 18:56:50
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	10
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Entry type	Theorem
Classification	msc 20A05
Classification	msc 03E20
Related topic	Restriction
Related topic	Cardinality

Theorem. If f is an injection from a set S into a group G, then there exist a group H containing S and a group isomorphism $\varphi\colon H \to G$ such that $\varphi|_S = f$.

Proof. Let M be a set such that $\mathrm{card}(M) \geqq \mathrm{card}(G)$. Because $\mathrm{card}(f(S)) = \mathrm{card}(S)$, we have $\mathrm{card}(M \smallsetminus S) \geqq \mathrm{card}(G \smallsetminus f(S))$, and therefore there exists an injection

$$\psi\colon G \smallsetminus f(S) \to M \smallsetminus S$$

(provided that $G \smallsetminus f(S) \neq \varnothing$; otherwise the mapping $f\colon S \to G$ would be a bijection). Define

$$H \;:=\; S \cup \psi(G \smallsetminus f(S)),$$

$$\varphi(h) \;:=\; \begin{cases} f(h) & \text{for } h \in S, \\ \psi^{-1}(h) & \text{for } h \in H \smallsetminus S. \end{cases}$$

Then apparently, $\varphi\colon H \to G$ is a bijection and $\varphi|_S = f$. Moreover, define the binary operation "$*$" of the set H by

$$h_1 * h_2 \;:=\; \varphi^{-1}(\varphi(h_1)\cdot\varphi(h_2)). \tag{1}$$

We see first that

$$\begin{aligned} (h_1 * h_2) * h_3 \;&=\; \varphi^{-1}\big(\varphi\left(\varphi^{-1}(\varphi(h_1)\cdot\varphi(h_2))\right)\cdot\varphi(h_3)\big) \\ &=\; \varphi^{-1}((\varphi(h_1)\cdot\varphi(h_2))\cdot\varphi(h_3)) \\ &=\; \varphi^{-1}(\varphi(h_1)\cdot(\varphi(h_2)\cdot\varphi(h_3))) \\ &=\; \varphi^{-1}\big(\varphi(h_1)\cdot\varphi\left(\varphi^{-1}(\varphi(h_2)\cdot\varphi(h_3))\right)\big) \\ &=\; h_1 * (h_2 * h_3). \end{aligned}$$

Secondly,

$$h * \varphi^{-1}(e) \;=\; \varphi^{-1}\big(\varphi(h)\cdot\varphi(\varphi^{-1}(e))\big) \;=\; \varphi^{-1}(\varphi(h)) \;=\; h,$$

whence $\varphi^{-1}(e)$ is the right identity element of H. Then,

$$h * \varphi^{-1}\big((\varphi(h))^{-1}\big) \;=\; \varphi^{-1}\left(\varphi(h)\cdot\varphi\left(\varphi^{-1}(\varphi(h)^{-1})\right)\right) \;=\; \varphi^{-1}(e),$$

and accordingly $\varphi^{-1}\big((\varphi(h))^{-1}\big)$ is the right inverse of h in H. Consequently, $(H, *)$ is a group. The equation (1) implies that

$$\varphi(h_1 * h_2) \;=\; \varphi(h_1)\cdot\varphi(h_2),$$

whence φ is an isomorphism from H onto G. Q.E.D.