example of induced representation

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To understand the definition of induced representation, let us work through a simple example in detail.

Let G be the group of permutations of three objects and let H be the subgroup of even permutations. We have

$$G = \{e, (ab), (ac), (bc), (abc), (acb)\}$$

$$H = \{e, (abc), (acb)\}$$

Let V be the one dimensional representation of H. Being one-dimensional, V is spanned by a single basis vector v. The action of H on V is given as

$$ev = v$$

$$(abc)v = \exp(2\pi i/3)v$$

$$(acb)v = \exp(4\pi i/3)v$$

Since H has half as many elements as G, there are exactly two cosets, σ_1 and σ_2 in G/H where

$$\sigma_1 = \{e, (abc), (acb)\}$$

$$\sigma_2 = \{(ab), (ac), (bc)\}$$

Since there are two cosets, the vector space of the induced representation consists of the direct sum of two formal translates of V. A basis for this space is $\{\sigma_1 v, \sigma_2 v\}$.

We will now compute the action of G on this vector space. To do this, we need a choice of coset representatives. Let us choose $g_1 = e$ as a representative of σ_1 and $g_2 = (ab)$ as a representative of σ_2. As a preliminary step, we shall express the product of every element of G with a coset representative as the product of a coset representative and an element of H.

$$e \cdot g_1 = e = g_1 \cdot e$$

$$e \cdot g_2 = (ab) = g_2 \cdot e$$

$$(ab) \cdot g_1 = (ab) = g_2 \cdot e$$

$$(ab) \cdot g_2 = e = g_1 \cdot e$$

$$(bc) \cdot g_1 = (bc) = g_2 \cdot (acb)$$

$$(bc) \cdot g_2 = (abc) = g_1 \cdot (abc)$$

$$(ac) \cdot g_1 = (ac) = g_2 \cdot (abc)$$
$$(ac) \cdot g_2 = (acb) = g_1 \cdot (acb)$$
$$(abc) \cdot g_1 = (abc) = g_1 \cdot (abc)$$
$$(abc) \cdot g_2 = (bc) = g_2 \cdot (acb)$$
$$(acb) \cdot g_1 = (acb) = g_1 \cdot (acb)$$
$$(acb) \cdot g_2 = (ac) = g_2 \cdot (abc)$$

We will now compute of the action of G using the formula $g(\sigma v) = \tau(hv)$ given in the definition.

$$e(\sigma_1 v) = [e \cdot g_1](ev) = \sigma_1 v$$
$$e(\sigma_2 v) = [e \cdot g_2](ev) = \sigma_2 v$$
$$(ab)(\sigma_1 v) = [(ab) \cdot g_1](ev) = \sigma_2 v$$
$$(ab)(\sigma_2 v) = [(ab) \cdot g_2](ev) = \sigma_1 v$$
$$(bc)(\sigma_1 v) = [(bc) \cdot g_1]((acb)v) = \exp(4\pi i/3)\sigma_2 v$$
$$(bc)(\sigma_2 v) = [(bc) \cdot g_2]((abc)v) = \exp(2\pi i/3)\sigma_1 v$$
$$(ac)(\sigma_1 v) = [(ac) \cdot g_1]((abc)v) = \exp(2\pi i/3)\sigma_2 v$$
$$(ac)(\sigma_2 v) = [(ac) \cdot g_2]((acb)v) = \exp(4\pi i/3)\sigma_1 v$$
$$(abc)(\sigma_1 v) = [(abc) \cdot g_1]((abc)v) = \exp(2\pi i/3)(\sigma_1 v)$$
$$(abc)(\sigma_2 v) = [(abc) \cdot g_2]((acb)v) = \exp(4\pi i/3)(\sigma_2 v)$$
$$(acb)(\sigma_1 v) = [(acb) \cdot g_1]((acb)v) = \exp(4\pi i/3)(\sigma_1 v)$$
$$(acb)(\sigma_2 v) = [(acb) \cdot g_2]((abc)v) = \exp(2\pi i/3)(\sigma_2 v)$$

Here the square brackets indicate the coset to which the group element inside the brackets belongs. For instance, $[(ac) \cdot g_2] = [(ac) \cdot (ab)] = [(acb)] = \sigma_1$ since $(acb) \in \sigma_1$.

The results of the calculation may be easier understood when expressed in matrix form

$$e \quad \rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$(ab) \quad \rightarrow \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$
\begin{aligned}
(bc) \quad &\to \quad \begin{pmatrix} 0 & \exp(2\pi i/3) \\ \exp(4\pi i/3) & 0 \end{pmatrix} \\
(ac) \quad &\to \quad \begin{pmatrix} 0 & \exp(4\pi i/3) \\ \exp(2\pi i/3) & 0 \end{pmatrix} \\
(abc) \quad &\to \quad \begin{pmatrix} \exp(2\pi i/3) & 0 \\ 0 & \exp(4\pi i/3) \end{pmatrix} \\
(acb) \quad &\to \quad \begin{pmatrix} \exp(4\pi i/3) & 0 \\ 0 & \exp(2\pi i/3) \end{pmatrix}
\end{aligned}
$$

Having expressed the answer thus, it is not hard to verify that this is indeed a representation of G. For instance, $(acb) \cdot (ac) = (bc)$ and

$$
\begin{pmatrix} \exp(4\pi i/3) & 0 \\ 0 & \exp(2\pi i/3) \end{pmatrix} \begin{pmatrix} 0 & \exp(4\pi i/3) \\ \exp(2\pi i/3) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \exp(2\pi i/3) \\ \exp(4\pi i/3) & 0 \end{pmatrix}
$$