Zeta function of a group

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Let G be a finitely generated group and let $\mathcal{X}$ be a family of finite index subgroups of G. Define

$$a_n(\mathcal{X}) = |\{H \in \mathcal{X} \mid |G : H| = n\}|.$$

Note that these numbers are finite since a finitely generated group has only finitely many subgroups of a given index. We define the **zeta function** of the family $\mathcal{X}$ to be the formal Dirichlet series

$$\zeta_{\mathcal{X}}(s) = \sum_{n=1}^{\infty} a_n(\mathcal{X}) n^{-s}.$$

Two important special cases are the zeta function counting all subgroups and the zeta function counting normal subgroups. Let $\mathcal{S}(G)$ and $\mathcal{N}(G)$ be the families of all finite index subgroups of G and of all finite index normal subgroups of G, respectively. We write $a_n^{\leqslant}(G) = a_n(\mathcal{S}(G))$ and $a_n^{\trianglelefteq}(G) = a_n(\mathcal{N}(G))$ and define

$$\zeta_G^{\leqslant}(s) = \zeta_{\mathcal{S}(G)}(s) = \sum_{H \leqslant_{\mathrm{f}} G} |G : H|^{-s},$$

and

$$\zeta_G^{\trianglelefteq}(s) = \zeta_{\mathcal{N}(G)}(s) = \sum_{N \trianglelefteq_{\mathrm{f}} G} |G : N|^{-s}.$$

If, in addition, G is nilpotent, then $\zeta_G^{\leqslant}$ has a decomposition as a formal Euler product

$$\zeta_G^{\leqslant}(s) = \prod_{p \text{ prime}} \zeta_{G,p}^{\leqslant}(s),$$

where

$$\zeta_{G,p}^{\leqslant}(s) = \sum_{i=0}^{\infty} a_{p^i}^{\leqslant}(G) p^{-is}.$$

An analogous result holds for the normal zeta function $\zeta_G^{\trianglelefteq}$. The result for both $\zeta_G^{\leqslant}$ and $\zeta_G^{\trianglelefteq}$ can be proved using properties of the profinite completion of G. However, a simpler proof for the normal zeta function is provided by the fact that a finite nilpotent group decomposes into a direct product of its Sylow subgroups. These results allow the zeta functions to be expressed in terms of p-adic integrals, which can in turn be used to prove (using some high-powered machinery) that $\zeta^{\leqslant}_{G,p}(s)$ and $\zeta^{\trianglelefteq}_{G,p}(s)$ are rational functions in p and p^{-s}.

In the case when G is a $\mathcal{T}$-group, that is, G is finitely generated, torsion free, and nilpotent, define $\alpha^{\leqslant}_G$ to be the abscissa of convergence of $\zeta^{\leqslant}_G$. That is, $\alpha^{\leqslant}_G$ is the smallest $\alpha \in \mathbb{R}$ such that $\zeta^{\leqslant}_G$ defines a holomorphic function in the right half-plane $\{z \in \mathbb{C} \mid \Re(z) > \alpha\}$. It can then be shown that $\alpha^{\leqslant}_G \leqslant \mathrm{h}(G)$, where $\mathrm{h}(G)$ is the Hirsch number of G. Therefore, if G is a $\mathcal{T}$-group, $\zeta^{\leqslant}_G$ defines a holomorphic function in some right half-plane.

References

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