proof of simplicity of Mathieu groups

Canonical name	ProofOfSimplicityOfMathieuGroups
Date of creation	2013-03-22 18:44:08
Last modified on	2013-03-22 18:44:08
Owner	monster (22721)
Last modified by	monster (22721)
Numerical id	8
Author	monster (22721)
Entry type	Proof
Classification	msc 20D08
Classification	msc 20B20

We give a uniform proof of the simplicity of the Mathieu groups M_{22}, M_{23}, and M_{24}, and the alternating groups A_n (for $n > 5$), assuming the simplicity of $M_{21} \cong PSL(3, \mathbb{F}_4)$ and $A_5 \cong PSL(2, \mathbb{F}_4)$. (Essentially, we are assuming that the simplicity of the projective special linear groups is known.)

Lemma 1. *Let G act transitively on a set S. If H is a normal subgroup of G, then the transitivity classes of the action, restricted to H, form a set of blocks for the action of G.*

Proof. If T, U are any transitivity classes for the restricted action, let $t \in T$, $u \in U$, and $g \in G$ such that $gt = u$. Then $x \mapsto gx$ is a bijective map from T onto U (here we use normality). Hence any element of G maps transitivity classes to transitivity classes. □

Hence it follows:

Corollary 2. *Let G act* `http://planetmath.org/PrimativeTransitivePermutationGroupOnA` *on a set S. If H is a normal subgroup of G, then either H acts transitively on S, or H lies in the kernel of the action. If the action is faithful, then either $H = \{1\}$ or H is transitive.*

Theorem 3. *Let G be a group acting primitively and faithfully on a set S. Let K be the stabilizer of some point $s_0 \in S$, and assume that K is simple. Then if H is a nontrivial proper normal subgroup of G, then G is isomorphic to the semidirect product of H by K. H can be identified with S in such a way that $1 \in H$ is identified with s_0, the action of H becomes left multiplication, and the action of K becomes conjugation.*

Proof. Since $H \cap K$ is a normal subgroup of K, it is either $\{1\}$ or K.

If $H \cap K = K$, then $K \subset H$, and since K is maximal and H is proper, we have $K = H$. Since H is normal and H stabilizes s_0, then H stabilizes every point (since the action is transitive). Since the action is faithful, $K = H = \{1\}$, a contradiction. (This contradiction can also be reached by applying the corollary.)

Therefore, $H \cap K = \{1\}$. So no element of H, other than 1, fixes s_0. Thus H acts freely and transitively on S. For any $g \in G$, if $gs_0 = s$ and $hs = s_0$, then $hgs_0 = s_0$, hence hg is in K. Thus G is generated by H and K. Since H is normal and $H \cap K = \{1\}$, G is the (internal) semidirect product of H by K. □

Now we come to the main theorem from which we will deduce the simplicity results.

Theorem 4. *Let G be a group acting faithfully on a set S. Let $s_0 \in S$ and let K be the stabilizer of s_0. Assume K is simple.*

1. Assume the action of G is doubly transitive, and let H be a nontrivial proper normal subgroup of G. Then H is an elementary abelian p-group for some prime p. Furthermore, K is isomorphic to a subgroup of $GL(n, \mathbb{F}_p)$, and G is isomorphic to a subgroup of $AGL(n, \mathbb{F}_p)$, the group of affine transformations of H.

2. If the action of G is triply transitive and $|S| > 3$, then any nontrivial proper normal subgroup of G is an elementary abelian 2-group.

3. If the action of G is quadruply transitive and $|S| > 4$, then G is simple.

Proof. For part 1, use the identification of H with S given by the previous theorem. Since the action is doubly transitive, the action by conjugation of K is transitive on $H - \{1\}$. Therefore, all non-identity elements of H have the same order, which must therefore be some prime p. Hence H is a p-group. The center $Z(H)$ is nontrivial, and is preserved by all automorphisms. By double transitivity again, there is an automorphism taking any nontrivial element to any other; hence H is abelian. Therefore H is an elementary abelian p-group.

For part 2, we know from part 1 that H is isomorphic to an elementary abelian p-group and K acts as linear transformations of H. Since the action of G is triply transitive, the action of K on the nonzero elements is doubly transitive. However, if $p > 2$, then the linearity of the action disallows double transitivity (if $x \mapsto y$, then $2x \mapsto 2y$ so we do not have complete freedom for any two elements since H some element besides 0, y and $2y$.)

(We note that when $|S| = 3$, we have the example $G = S_3$, $H = A_3$, $K = S_2$.)

Here is an example illustrating part 2. The group $AGL(n, \mathbb{F}_2)$ acts triply transitively on $\mathbb{F}_2^n$, and the stabilizer of a point is $GL(n, \mathbb{F}_2)$, which is simple if $n > 2$. $AGL(n, \mathbb{F}_2)$ contains the normal subgroup of translations, an elementary abelian 2-group.

For part 3, note that the action of $GL(n, \mathbb{F}_2)$ on $\mathbb{F}_2^n$, $n > 2$, is not triply transitive on nonzero elements, so the only conclusion left is that G is simple. $\square$

Corollary 5. *The Mathieu groups M_{21}, M_{22}, M_{23}, and M_{24} are simple.*

Proof. We take it as known that $M_{21} \cong PSL(3, \mathbb{F}_4)$ is simple. Since M_n has M_{n-1} as point stabilizer, and has a triply transitive action on a set of n elements, we may work our way inductively up to M_{24}, using the previous theorem. The `http://planetmath.org/Coset`index of M_{n-1} in M_n is n, which is not a power of 2. Hence in all cases, M_n is simple. □

Corollary 6. *The alternating groups A_n are simple for $n \geq 5$.*

Proof. Since the natural action of A_n on n letters is quadruply transitive for $n \geq 6$, and the point stabilizer of A_n is A_{n-1}, we may apply the theorem to deduce the simplicity of the alternating groups A_n, $n \geq 5$, from the simplicity of $A_5 \cong PSL(2, \mathbb{F}_4) \cong PSL(2, \mathbb{F}_5)$. □