proof of Frobenius reciprocity

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We prove the slightly more general result

Theorem 0.1. *If G is a finite group with subgroup H, α a class function on H and β a class function on G, then*

$$\langle \alpha{\uparrow}_H^G, \beta \rangle_G = \langle \alpha, \beta{\downarrow}_H^G \rangle_H$$

Here we use ${\uparrow}_H^G$ to refer to the `http://planetmath.org/InducedRepresentation`induction to G of a class function on H, and ${\downarrow}_H^G$ to refer to the `http://planetmath.org/RestrictionRepresentation`restriction of a class function on G to one on H.

Proof.

$$\langle \alpha \uparrow_H^G, \beta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}gt \in H}} \alpha(t^{-1}gt) \right) \overline{\beta(g)} = \frac{1}{|G|\,|H|} \sum_{t \in G} \left(\sum_{\substack{g \in G \\ t^{-1}gt \in H}} \alpha(t^{-1}gt) \right) \overline{\beta(g)}$$

Since β is a class function, this is the same as

$$\frac{1}{|G|\,|H|} \sum_{\substack{t \in G \\ g \in G \\ t^{-1}gt \in H}} \alpha(t^{-1}gt)\overline{\beta(t^{-1}gt)} = \frac{1}{|G|\,|H|} \sum_{h \in H} \sum_{\substack{t \in G \\ g \in G \\ t^{-1}gt = h}} \alpha(h)\overline{\beta(h)}$$

Clearly for every $h \in H, t \in G$ there is a unique $g \in G$ with $t^{-1}gt = h$, so every element of H is counted $|G|$ times by the sum. Thus the sum is equal to

$$\frac{|G|}{|G|\,|H|} \sum_{h \in H} \alpha(h)\overline{\beta(h)} = \frac{1}{|H|} \sum_{h \in H} \alpha(h)\overline{\beta(h)} = \langle \alpha, \beta{\downarrow}_H^G \rangle_H$$

$\square$