polyadic semigroup

Canonical name	PolyadicSemigroup
Date of creation	2013-03-22 18:37:47
Last modified on	2013-03-22 18:37:47
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	10
Author	CWoo (3771)
Entry type	Definition
Classification	msc 20N15
Classification	msc 20M99
Synonym	n-semigroup
Synonym	n-group
Defines	n-semigroup
Defines	n-group
Defines	polyadic group
Defines	covering group

Recall that a semigroup is a non-empty set, together with an associative binary operation on it. *Polyadic semigroups* are generalizations of semigroups, in that the associative binary operation is replaced by an associative n-ary operation. More precisely, we have

Definition. Let n be a positive integer at least 2. A *n-semigroup* is a non-empty set S, together with an n-ary operation f on S, such that f is associative:

$$f(f(a_1, \ldots, a_n), a_{n+1}, \ldots, a_{2n-1}) = f(a_1, \ldots, f(a_i, \ldots, a_{i+n-1}), \ldots, f_{2n-1})$$

for every $i \in \{1, \ldots, n\}$. A *polyadic semigroup* is an n-semigroup for some n.

An n-semigroup S (with the associated n-ary operation f) is said to be *commutative* if f is commutative. An element $e \in S$ is said to be an *identity element*, or an *f-identity*, if

$$f(a, e, \ldots, e) = f(e, a, \ldots, e) = \cdots = f(e, e, \ldots, a) = a$$

for all $a \in S$. If S is commutative, then e is an identity in S if $f(a, e, \ldots, e) = a$.

Every semigroup S has an n-semigroup structure: define $f : S^n \to S$ by

$$f(a_1, a_n \ldots, a_n) = a_1 \cdot a_2 \cdots a_n \tag{1}$$

The associativity of f is induced from the associativity of $\cdot$.

Definition. An n-semigroup S is called an *n-group* if, in the equation

$$f(x_1, \ldots, x_n) = a, \tag{2}$$

any $n-1$ of the n variables x_i are replaced by elements of G, then the equation with the remaining one variable has *at least one* solution in that variable. A *polyadic group* is just an n-group for some integer n.

n-groups are generalizations of groups. Indeed, a 2-group is just a group.

Proof. Let G be a 2-group. For $a, b \in G$, we write ab instead of $f(a, b)$. Given $a \in G$, there are $e_1, e_2 \in G$ such that $ae_1 = a$ and $e_2 a = a$. In addition, there are $x, y \in G$ such that $xa = e_2$ and $ay = e_1$. So $e_2 = xa = x(ae_1) = (xa)e_1 = e_2 e_1 = e_2(ay) = (e_2 a)y = ay = e_1$.

Next, suppose $ae_1 = ae_3 = a$. Then the equation $e_2 a = a$ from the previous paragraph as well as the subsequent discussion shows that $e_1 = e_2 = e_3$. This means that, for every $a \in G$, there is a unique $e_a \in G$ such that $e_a a = a e_a = a$. Since $e_a^2 a = e_a(e_a a) = e_a a = a = a e_a = (a e_a) e_a = a e_a^2$, we see that e_a is idempotent: $e_a^2 = e_a$.

Now, pick any $b \in G$. Then there is $c \in G$ such that $b = c e_a$. So $b e_a = (c e_a) e_a = c e_a^2 = c e_a = b$. From the last two paragraphs, we see that $e_a = e_b$. This shows that there is a $e \in G$ such that $ae = ea = a$ for all $a \in G$. In other words, e is the identity with respect to the binary operation f.

Finally, given $a \in G$, there are $b, c \in G$ such that $ab = ca = e$. Then $c = ce = c(ab) = (ca)b = eb = b$. In addition, if $ab_1 = ab_2 = e$, then, from the equation $ca = e$, we get $b_1 = c = b_2$. This shows b is the unique inverse of a with respect to binary operation f. Hence, G is a group. □

Every group has a structure of an n-group, where the n-ary operation f on G is defined by the equation (1) above. Interestingly, Post has proved that, for every n-group G, there is a group H, and an injective function $\phi : G \to H$ with the following properties:

1. $\phi(G)$ generates H
2. $\phi(f(a_1, \ldots, a_n)) = \phi(a_1) \cdots \phi(a_n)$

If we call the group H with the two above properties a *covering group* of G, then Post's theorem states that every n-group has a covering group.

From Post's result, one has the following corollary: an n-semigroup G is an n-group iff equation (2) above has *exactly one* solution in the remaining variable, when $n - 1$ of the n variables are replaced by elements of G.

References

[HB] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, 1966

[EP] E. L. Post, *Polyadic groups*, Trans. Amer. Math. Soc., 48, 208-350, 1940, MR 2, 128

[WD] W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z. 29, 1-19, 1928