Schützenberger graph

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Defines	Schützenberger graph
Defines	left Schützenberger graph
Defines	right Schützenberger graph

Let $(X;T)$ be a presentation for the inverse monoid $\mathrm{Inv}^1\langle X|T\rangle$ [resp. inverse semigroup $\mathrm{Inv}\langle X|T\rangle$]. In what follows, the argument for inverse semigroups and inverse monoids is exactly the same, so we concentrate on the last one.

Given $m \in \mathrm{Inv}^1\langle X|T\rangle$, let $[m]_{\mathcal{R}}$ be the equivalence class of m with respect to the Right Green relation $\mathcal{R}$. The *Right Schützenberger graph* of $[m]_{\mathcal{R}}$ with respect to the presentation $(X;T)$ is defined as the X-inverse word graph $\mathcal{S}\Gamma(X;T;m)$ with vertex and edge set respectively

$$\mathrm{V}(\mathcal{S}\Gamma(X;T;m)) = \left\{v \in \mathrm{Inv}^1\langle X|T\rangle \mid [v]_{\mathcal{R}} = [m]_{\mathcal{R}}\right\},$$

$$\mathrm{E}(\mathcal{S}\Gamma(X;T;m)) = \left\{(v_1, x, v_2) \,|\, v_1, v_2 \in \mathrm{V}(\mathcal{S}\Gamma(X;T;m)),\ x \in \left(X \amalg X^{-1}\right),\ v_2 = v_1 \cdot [x]_\tau\right\},$$

where $\tau = (T \cup \rho_X)^{\mathrm{c}}$, i.e. τ is the congruence generated by T and the Wagner congruence ρ_X, and $[x]_\tau$ is the congruence class of the letter $x \in (X \amalg X^{-1})$ with respect to the congruence τ.

This is a good definition, in fact it can be easily shown that given $m, n \in \mathrm{Inv}^1\langle X|T\rangle$ with $[m]_{\mathcal{R}} = [n]_{\mathcal{R}}$ we have $\mathcal{S}\Gamma(X;T;m) = \mathcal{S}\Gamma(X;T;n)$.

Analogously we can define the *Left Schützenberger graph* using the Left Green relation $\mathcal{L}$ instead of the Right Green relation $\mathcal{R}$, but this notion is not used in literature.

Schützenberger graphs play in combinatorial inverse semigroups theory the role that Cayley graphs play in combinatorial group theory. In fact, if $G = \mathrm{Inv}^1\langle X|T\rangle$ happen to be a group (with identity 1_G), then the Schützenberger graph $\mathcal{S}\Gamma(X;T;1_G)$ of its unique $\mathcal{R}$-class is exactly the Cayley graph of the group G.

References

[1] N. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.

[2] J.B. Stephen, *Presentation of inverse monoids*, J. Pure Appl. Algebra 63 (1990) 81-112.