group actions and homomorphisms

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Notes on group actions and homomorphisms

Let G be a group, X a non-empty set and S_X the symmetric group of X, i.e. the group of all bijective maps on X. $\cdot$ may denote a left group action of G on X.

1. For each $g \in G$ and $x \in X$ we define

   $$f_g \colon X \to X, \quad x \mapsto g \cdot x.$$

   Since $f_{g^{-1}}(f_g(x)) = g^{-1} \cdot (g \cdot x) = x$ for each $x \in X$, $f_{g^{-1}}$ is the inverse of f_g. so f_g is bijective and thus element of S_X. We define $F : G \to S_X, F(g) = f_g$ for all $g \in G$. This mapping is a group homomorphism: Let $g, h \in G, x \in X$. Then

   $$\begin{aligned} F(gh)(x) = f_{gh}(x) = (gh) \cdot x = g \cdot (h \cdot x) \\ = (f_g \circ f_h)(x) = (F(g) \circ F(h))(x) \end{aligned}$$

   for all $x \in X$ implies $F(gh) = F(g) \circ F(h)$. — The same is obviously true for a right group action.

2. Now let $F : G \to S_x$ be a group homomorphism, and let $f : G \times X \to X, (g, x) \mapsto F(g)(x)$ satisfy

   (a) $f(1_G, x) = F(1_g)(x) = x$ for all $x \in X$ and

   (b) $f(gh, x) = F(gh)(x) = (F(g) \circ F(h)(x) = F(g)(F(h)(x)) = f(g, f(h, x))$,

   so f is a group action induced by F.

Characterization of group actions

Let G be a group acting on a set X. Using the same notation as above, we have for each $g \in \ker(F)$

$$F(g) = \mathrm{id}_x = f_g \Leftrightarrow g \cdot x = x, \quad \forall x \in X \Leftrightarrow g \in \cup_{x \in X} G_x \tag{1}$$

and it follows

$$\ker(F) = \bigcap_{x \in X} G_x.$$

Let G act transitively on X. Then for any $x \in X$, X is the orbit $G(x)$ of x. As shown in "conjugate stabilizer subgroups', all stabilizer subgroups of elements $y \in G(x)$ are conjugate subgroups to G_x in G. From the above it follows that

$$\ker(F) = \bigcap_{g \in G} gG_xg^{-1}.$$

For a faithful operation of G the condition $g \cdot x = x,\ \forall x \in X \to g = 1_G$ is equivalent to

$$\ker(F) = \{1_G\}$$

and therefore $F\colon G \to S_X$ is a monomorphism.

For the trivial operation of G on X given by $g \cdot x = x,\ \forall g \in G$ the stabilizer subgroup G_x is G for all $x \in X$, and thus

$$\ker(F) = G.$$

If the operation of G on X is free, then $G_x = \{1_G\},\ \forall x \in X$, thus the kernel of F is $\{1_G\}$–like for a faithful operation. But:

Let $X = \{1, \ldots, n\}$ and $G = S_n$. Then the operation of G on X given by

$$\pi \cdot i := \pi(i), \quad \forall i \in X,\ \pi \in S_n$$

is faithful but not free.