central product of groups

Canonical name	CentralProductOfGroups
Date of creation	2013-03-22 18:49:45
Last modified on	2013-03-22 18:49:45
Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
Numerical id	6
Author	Algeboy (12884)
Entry type	Definition
Classification	msc 20E34
Synonym	central decomposition
Defines	central decomposition
Defines	central product

1 Definitions

A *central decomposition* is a set $\mathcal{H}$ of subgroups of a group G where

1. for $\mathcal{J} \subseteq \mathcal{H}$, $G = \langle \mathcal{J} \rangle$ if, and only if, $\mathcal{J} = \mathcal{H}$, and
2. $[H, \langle \mathcal{H} - \{H\} \rangle] = 1$ for all $H \in \mathcal{H}$.

A group G is *centrally indecomposable* if its only central decomposition is $\{G\}$. A central decomposition is *fully refined* if its members are centrally indecomposable.

Remark 1. *Condition 1 is often relaxed to $G = \langle \mathcal{H} \rangle$ but this has the negative affect of allowing, for example, $\mathbb{R}^2$ to have the central decomposition such sets as $\{\langle (1,0) \rangle, \langle (0,1) \rangle, \langle (1,1) \rangle, 0, \mathbb{R}^2\}$ and in general a decomposition of any possible size. By impossing 1, we then restrict the central decompositions of $\mathbb{R}^2$ to direct decompositions. Furthermore, with condition 1, the meaning of indecomposable is easily had.*

A *central product* is a group $G = \left(\prod_{H\in\mathcal{H}} H\right)/N$ where N is a normal subgroup of $\prod_{H\in\mathcal{H}} H$ and $H \cap N = 1$ for all $H \in \mathcal{H}$.

Proposition 2. *Every finite central decomposition $\mathcal{H}$ is a central product of the members in $\mathcal{H}$.*

Proof. Suppose that $\mathcal{H}$ is a a finite central decomposition of G. Then define $\pi : \prod_{H\in\mathcal{H}} H \to G$ by $(x_H : H \in \mathcal{H}) \mapsto \prod_{H\in\mathcal{H}} x_H$. Then $G = \left(\prod_{H\in\mathcal{H}} H\right)/\ker\pi$. Furthermore, $H \cap \ker\pi = 1$ for each direct factor H of $\prod_{K\in\mathcal{H}} K$. Thus, G is a central product of $\mathcal{H}$. $\square$

2 Examples

1. Every direct product is also a central product and so also every direct decomposition is a central decomposition. The converse is generally false.
2. Let $E = \left\{ \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta, \gamma \in K \right\}$, for a field K. Then G is a centrally indecomposable group.

3. If
$$F = \left\{ \begin{bmatrix} 1 & \alpha_1 & \dots & \alpha_n & \gamma \\ 0 & 1 & 0 & & \beta_1 \\ \vdots & & \ddots & 0 & \vdots \\ & & & 1 & \beta_n \\ & & & & 1 \end{bmatrix} : \alpha_i, \beta_i, \gamma \in K, 1 \leq i \leq n \right\}$$
and
$$H_i = \{A \in F : \alpha_j = 0 = \beta_j \forall j \neq i, 1 \leq j \leq n\}$$
then $\{H_1, \dots, H_n\}$ is a central decomposition of G. Furthermore, each H_i is isomorphic to E and so $\mathcal{H}$ is a fully refined central decomposition.

4. If $D_8 = \langle a, b | a^4, b^2, (ab)^2 \rangle$ – the dihedral group of order 8, and $Q_8 = \langle i, j | i^4, i^2 = j^2, i^j = i^{-1} \rangle$ – the quaternion group of order 8, then $D_8 \circ D_8 = D_8 \times D_8 / \langle (a^2, a^2) \rangle$ is isomorphic to $Q_8 \circ Q_8 = (Q_8 \times Q_8) / \langle (i^2, i^2) \rangle$; yet, D_8 and Q_8 are nonisomorphic and centrally indecomposable. In particular, central decompositions are not unique even up to automorphisms. This is in contrast the well-known Krull-Remak-Schmidt theorem for direct products of groups.

3 History

The name *central product* appears to have been coined by Philip Hall [**?**, Section 3.2] though the principal concept of such a product appears in earlier work (e.g. [**?**, Theorem II]). Hall describes central products as "...the group obtained from the direct product by identifying the centres of the direct factors...". The modern definition clearly out grows this original version as now centers may be only partially identified.

References

[1] P. Hall, *Finite-by-nilpotent groups*, Proc. Camb. Phil. Soc., 52 (1956), 611-616.

[2] B. H. Neumann, and H. Neumann, *A remark on generalized free products*, J. London Math. Soc. 25 (1950), 202-204.