proof of the Burnside basis theorem

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Owner	Algeboy (12884)
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Author	Algeboy (12884)
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Let P be a p-group and $\Phi(P)$ its Frattini subgroup.

Every maximal subgroup Q of P is of index p in P and is therefore normal in P. Thus $P/Q \cong \mathbb{Z}_p$. So given $g \in P$, $g^p \in Q$ which proves $P^p \leq Q$. Likewise, $\mathbb{Z}_p$ is abelian so $[P,P] \leq Q$. As Q is any maximal subgroup, it follows $[P,P]$ and P^p lie in $\Phi(P)$.

Now both $[P,P]$ and P^p are characteristic subgroups of P so in particular $F = [P,P]P^p$ is normal in P. If we pass to $V = P/F$ we find that V is abelian and every element has order p – that is, V is a vector space over $\mathbb{Z}_p$. So the maximal subgroups of P are in a 1-1 correspondence with the hyperplanes of V. As the intersection of all hyperplanes of a vector space is the origin, it follows the intersection of all maximal subgroups of P is F. That is, $[P,P]P^p = \Phi(P)$.