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# bicyclic semigroup

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The *bicyclic semigroup* $\mathcal{C}(p,q)$ is the monoid generated by $\{p,q\}$ with the single relation $pq=1$.

The elements of $\mathcal{C}(p,q)$ are all words of the form $q^n p^m$ for $m,n \geq 0$ (with the understanding $p^0 = q^0 = 1$). These words are multiplied as follows:

$$q^n p^m q^k p^l = \begin{cases} q^{n+k-m} p^l & \text{if } m \leq k, \\ q^n p^{l+m-k} & \text{if } m \geq k. \end{cases}$$

It is apparent that $\mathcal{C}(p,q)$ is simple, for if $q^n p^m$ is an element of $\mathcal{C}(p,q)$, then $1 = p^n(q^n p^m)q^m$ and so $S^1 q^n p^m S^1 = S$.

It is also easy to see that $\mathcal{C}(p,q)$ is an inverse semigroup: the element $q^n p^m$ has inverse $q^m p^n$.

It is useful to picture some further properties of $\mathcal{C}(p,q)$ by arranging the elements in a table:

$$\begin{array}{cccccc} 1 & p & p^2 & p^3 & p^4 & \dots \\ q & qp & qp^2 & qp^3 & qp^4 & \dots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & q^2p^4 & \dots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & q^3p^4 & \dots \\ q^4 & q^4p & q^4p^2 & q^4p^3 & q^4p^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then the elements below any horizontal line drawn through this table form a right ideal and the elements to the right of any vertical line form a left ideal. Further, the elements on the diagonal are all idempotents and their standard ordering is

$$1 > qp > q^2p^2 > q^3p^3 > \cdots.$$