hypergroup

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Defines	hypergroupoid
Defines	hypersemigroup
Defines	left identity
Defines	right identity
Defines	identity
Defines	absolute identity
Defines	left inverse
Defines	right inverse
Defines	inverse
Defines	absolute identity

Hypergroups are generalizations of groups. Recall that a group is set with a binary operation on it satisfying a number of conditions. If this binary operation is taken to be multivalued, then we arrive at a hypergroup. In order to make this precise, we need some preliminary concepts:

Definition. A *hypergroupoid*, or *multigroupoid*, is a non-empty set G, together with a multivalued function $\cdot : G \times G \Rightarrow G$ called the *multiplication* on G.

We write $a \cdot b$, or simply ab, instead of $\cdot(a,b)$. Furthermore, if $ab = \{c\}$, then we use the abbreviation $ab = c$.

A hypergroupoid is said to be *commutative* if $ab = ba$ for all $a, b \in G$. Defining associativity of $\cdot$ on G, however, is trickier:

Given a hypergroupoid G, the multiplication $\cdot$ induces a binary operation (also written $\cdot$) on $P(G)$, the powerset of P, given by

$$A \cdot B := \bigcup \{a \cdot b \mid a \in A \text{ and } b \in B\}.$$

As a result, we have an induced groupoid $P(G)$. Instead of writing $\{a\}B$, $A\{b\}$, and $\{a\}\{b\}$, we write aB, Ab, and ab instead. From now on, when we write $(ab)c$, we mean "first, take the product of a and b via the multivalued binary operation $\cdot$ on G, then take the product of the set ab with the element c, under the induced binary operation on $P(G)$". Given a hypergroupoid G, there are two types of associativity we may define:

Type 1. $(ab)c \subseteq a(bc)$, and

Type 2. $a(bc) \subseteq (ab)c$.

G is said to be *associative* if it satisfies both types of associativity laws. An associative hypergroupoid is called a *hypersemigroup*. We are now ready to formally define a hypergroup.

Definition. A *hypergroup* is a hypersemigroup G such that $aG = Ga = G$ for all $a \in G$.

For example, let G be a group and H a subgroup of G. Let M be the collection of all left cosets of H in G. For $aH, bH \in M$, set

$$aH \cdot bH := \{cH \mid c = ahb,\ h \in H\}.$$

Then M is a hypergroup with multiplication $\cdot$.

If the multiplication in a hypergroup G is single-valued, then G is a `http://planetmath.org/PolyadicSemigroup`2-group, and therefore a group (see `http://planetmath.org/PolyadicSemigroup`proof here).

Remark. A hypergroup is also known as a *multigroup*, although some call a multigroup as a hypergroup with a designated identity element e, as well as a designated inverse for every element with respect e. Actually identities and inverses may be defined more generally for hypergroupoids:

Let G be a hypergroupoid. Identity elements are defined via the following three sets:

1. (set of *left identities*): $I_L(G) := \{e \in G \mid a \in ea \text{ for all } a \in G\}$,

2. (set of *right identities*): $I_R(G) := \{e \in G \mid a \in ae \text{ for all } a \in G\}$, and

3. (set of *identities*): $I(G) = I_L(G) \cap I_R(G)$.

$e \in L(G)$ is called an *absolute identity* if $ea = ae = a$ for all $a \in G$. If $e, f \in G$ are both absolute identities, then $e = ef = f$, so G can have at most one absolute identity.

Suppose $e \in I_L(G) \cup I_R(G)$ and $a \in G$. An element $b \in G$ is said to be a *left inverse* of a with respect to e if $e \in ba$. *Right inverses* of a are defined similarly. If b is both a left and a right inverse of a with respect to e, then b is called an *inverse* of a with respect to e.

Thus, one may say that a multigroup is a hypergroup G with an identity $e \in G$, and a function ${}^{-1} : G \to G$ such that $a^{-1} :={}^{-1} (a)$ is an inverse of a with respect to e.

In the example above, M is a multigroup in the sense given in the remark above. The designated identity is H (in fact, this is the only identity in M), and for every $aH \in M$, its designated inverse is provided by $a^{-1}H$ (of course, this may not be its only inverse, as any bH such that $ahb = e$ for some $h \in H$ will do).

References

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