conjugacy in A_n

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Recall that conjugacy classes in the symmetric group S_n are determined solely by cycle type. In the alternating group A_n, however, this is not always true. A single conjugacy class in S_n that is contained in A_n may split into two distinct classes when considered as a subset of A_n. For example, in S_3, $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are conjugate, since

$$(2\ 3)(1\ 2\ 3)(2\ 3) = (1\ 3\ 2)$$

but these two are not conjugate in A_3 (note that $(2\ 3) \notin A_3$).

Note in particular that the fact that conjugacy in S_n is determined by cycle type means that if $\sigma \in A_n$ then all of its conjugates in S_n also lie in A_n.

The following theorem fully characterizes the behavior of conjugacy classes in A_n:

Theorem 1. *A conjugacy class in S_n splits into two distinct conjugacy classes under the action of A_n if and only if its cycle type consists of* distinct *odd integers. Otherwise, it remains a single conjugacy class in A_n.*

Thus, for example, in S_7, the elements of the conjugacy class of $(1\ 2\ 3\ 4\ 5)$ are all conjugate in A_7, while the elements of the conjugacy class of $(1\ 2\ 3)(4\ 5\ 6)$ split into two distinct conjugacy classes in A_7 since there are two cycles of length 3. Similarly, any conjugacy class containing an even-length cycle, such as $(1\ 2\ 3\ 4)(5\ 6)$, splits in A_7.

We will prove the above theorem by proving the following statements:

- A conjugacy class in S_n consisting solely of even permutations (i.e. that is contained in A_n) either is a single conjugacy class or is the disjoint union of two equal-sized conjugacy classes when considered under the action of A_n.

- If $\sigma \in A_n$, then the elements of the conjugacy class of σ in S_n (which is just all elements of the same cycle type as σ) are conjugate in A_n if and only if σ commutes with some odd permutation.

- $\sigma \in S_n$ does not commute with an odd permutation if and only if the cycle type of σ consists of *distinct* odd integers.

Throughout, we will denote by $\mathcal{C}_S(\sigma)$ the conjugacy class of σ under the action of S_n.

To prove the first statement, note that conjugacy is a transitive action. By the theorem that orbits of a normal subgroup are equal in size when the full group acts transitively, we see that if $\sigma \in A_n$, then $\mathcal{C}_S(\sigma)$ splits into $|S_n : A_n C_{S_n}(\sigma)|$ classes under the action of A_n (recall that $C_G(x)$, the centralizer of x, is simply the stabilizer of x under the conjugation action of G on itself). But since $|S_n : A_n|$ is either 1 or 2, we see that the conjugacy class of σ either remains a single class in A_n or splits into two classes.

Note also that the elements of $\mathcal{C}_S(\sigma)$ are all conjugate in A_n if and only if $A_n C_{S_n}(\sigma) = S_n$, which happens if and only if $C_{S_n}(\sigma) \nsubseteq A_n$, which in turn is the case if and only if some odd permutation is in the centralizer of σ, which means precisely that σ commutes with some odd permutation. This proves the second statement.

To prove the third statement, suppose first that σ does not commute with an odd permutation. Clearly σ commutes with any cycle in its own cycle decomposition, so if σ contains a cycle of even length, that is an odd permutation with which σ commutes. So σ must consist solely of [disjoint] cycles of odd length. If two of these cycles have the same length, say $(a_1\ a_2\ \ldots\ a_{2k+1})$ and $(b_1\ b_2\ \ldots\ b_{2k+1})$, then

$$((a_1\, b_1)\ldots(a_{2k+1}\, b_{2k+1}))(a_1\ a_2\ \ldots\ a_{2k+1})(b_1\ b_2\ \ldots\ b_{2k+1})((a_1\, b_1)\ldots(a_{2k+1}\, b_{2k+1}))^{-1} = (a_1\ a_2\ \ldots\ a_{2k+1})(b_1\ b_2\ \ldots\ b_{2k+1})$$

so the product of $(a_1\ a_2\ \ldots\ a_{2k+1})$ and $(b_1\ b_2\ \ldots\ b_{2k+1})$, and thus σ, commutes with the product of $2k+1$ transpositions, which is an odd permutation. Thus all the cycles in the cycle decomposition of σ must have different [odd] lengths.

To prove the converse, we show that if the cycles in the cycle decomposition all have distinct lengths, then σ commutes precisely with the group generated by its cycles. It follows then that if all the distinct lengths are odd, then σ commutes only with these permutations, which are all even. Choose σ with distinct cycle lengths in its cycle decomposition, and suppose that σ commutes with some element $\tau \in S_n$. Conjugation preserves cycle length, so since τ commutes with σ and σ has all its cycles of distinct lengths, each cycle in τ must commute with each cycle in σ individually.

Now, choose a nontrivial cycle τ_1 of τ, and choose $j \in \tau$ such that σ moves j (we can do this, since σ can have at most one cycle of length 1 and the cycle length of τ is greater than 1). Let σ_1 be the cycle of σ containing j. Then τ_1 commutes with σ_1 since τ commutes with σ, so τ_1 is in the centralizer of

σ_1, and it is not disjoint from σ_1. But the centralizer of a k-cycle ρ consists of products of powers of ρ and cycles disjoint from ρ. Thus τ_1 is a power of σ_1. So each cycle in τ is a power of a cycle in σ, and we are done.