homogeneous space

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Defines	action on cosets
Defines	isotropy subgroup

Overview and definition. Let G be a group acting transitively on a set X. In other words, we consider a homomorphism $\phi : G \to \mathrm{Perm}(X)$, where the latter denotes the group of all bijections of X. If we consider G as being, in some sense, the automorphisms of X, the transitivity assumption means that it is impossible to distinguish a particular element of X from any another element. Since the elements of X are indistinguishable, we call X a *homogeneous space*. Indeed, the concept of a homogeneous space, is logically equivalent to the concept of a transitive group action.

Action on cosets. Let G be a group, $H < G$ a subgroup, and let G/H denote the set of left cosets, as above. For every $g \in G$ we consider the mapping $\psi_H(g) : G/H \to G/H$ with action

$$aH \to gaH, \quad a \in G.$$

Proposition 1 *The mapping $\psi_H(g)$ is a bijection. The corresponding mapping $\psi_H : G \to \mathrm{Perm}(G/H)$ is a group homomorphism, specifying a transitive group action of G on G/H.*

Thus, G/H has the natural structure of a homogeneous space. Indeed, we shall see that every homogeneous space X is isomorphic to G/H, for some subgroup H.

N.B. In geometric applications, the want the homogeneous space X to have some extra structure, like a topology or a differential structure. Correspondingly, the group of automorphisms is either a continuous group or a Lie group. In order for the quotient space X to have a Hausdorff topology, we need to assume that the subgroup H is closed in G.

The isotropy subgroup and the basepoint identification. Let X be a homogeneous space. For $x \in X$, the subgroup

$$H_x = \{h \in G : hx = x\},$$

consisting of all G-actions that fix x, is called the isotropy subgroup at the basepoint x. We identify the space of cosets G/H_x with the homogeneous space by means of the mapping $\tau_x : G/H_x \to X$, defined by

$$\tau_x(aH_x) = ax, \quad a \in G.$$

Proposition 2 *The above mapping is a well-defined bijection.*

To show that τ_x is well defined, let $a, b \in G$ be members of the same left coset, i.e. there exists an $h \in H_x$ such that $b = ah$. Consequently

$$bx = a(hx) = ax,$$

as desired. The mapping τ_x is onto because the action of G on X is assumed to be transitive. To show that τ_x is one-to-one, consider two cosets $aH_x, bH_x,\ a, b \in G$ such that $ax = bx$. It follows that $b^{-1}a$ fixes x, and hence is an element of H_x. Therefore aH_x and bH_x are the same coset.

The homogeneous space as a quotient. Next, let us show that τ_x is equivariant relative to the action of G on X and the action of G on the quotient G/H_x.

Proposition 3 *We have that*

$$\phi(g) \circ \tau_x = \tau_x \circ \psi_{H_x}(g)$$

for all $g \in G$.

To prove this, let $g, a \in G$ be given, and note that

$$\psi_{H_x}(g)(aH_x) = gaH_x.$$

The latter coset corresponds under τ_x to the point gax, as desired.

Finally, let us note that τ_x identifies the point $x \in X$ with the coset of the identity element eH_x, that is to say, with the subgroup H_x itself. For this reason, the point x is often called the basepoint of the identification $\tau_x : G/H_x \to X$.

The choice of basepoint. Next, we consider the effect of the choice of basepoint on the quotient structure of a homogeneous space. Let X be a homogeneous space.

Proposition 4 *The set of all isotropy subgroups $\{H_x : x \in X\}$ forms a single conjugacy class of subgroups in G.*

To show this, let $x_0, x_1 \in X$ be given. By the transitivity of the action we may choose a $\hat{g} \in G$ such that $x_1 = \hat{g}x_0$. Hence, for all $h \in G$ satisfying $hx_0 = x_0$, we have

$$(\hat{g}h\hat{g}^{-1})x_1 = \hat{g}(h(\hat{g}^{-1}x_1)) = \hat{g}x_0 = x_1.$$

Similarly, for all $h \in H_{x_1}$ we have that $\hat{g}^{-1} h \hat{g}$ fixes x_0. Therefore,

$$\hat{g}(H_{x_0})\hat{g}^{-1} = H_{x_1};$$

or what is equivalent, for all $x \in X$ and $g \in G$ we have

$$gH_x g^{-1} = H_{gx}.$$

Equivariance. Since we can identify a homogeneous space X with G/H_x for every possible $x \in X$, it stands to reason that there exist equivariant bijections between the different G/H_x. To describe these, let $H_0, H_1 < G$ be conjugate subgroups with

$$H_1 = \hat{g} H_0 \hat{g}^{-1}$$

for some fixed $\hat{g} \in G$. Let us set

$$X = G/H_0,$$

and let x_0 denote the identity coset H_0, and x_1 the coset $\hat{g}H_0$. What is the subgroup of G that fixes x_1? In other words, what are all the $h \in G$ such that

$$h\hat{g}H_0 = \hat{g}H_0,$$

or what is equivalent, all $h \in G$ such that

$$\hat{g}^{-1} h \hat{g} \in H_0.$$

The collection of all such h is precisely the subgroup H_1. Hence, $\tau_{x_1} : G/H_1 \to G/H_0$ is the desired equivariant bijection. This is a well defined mapping from the set of H_1-cosets to the set of H_0-cosets, with action given by

$$\tau_{x_1}(aH_1) = a\hat{g}H_0, \quad a \in G.$$

Let $\psi_0 : G \to \mathrm{Perm}(G/H_0)$ and $\psi_1 : G \to \mathrm{Perm}(G/H_1)$ denote the corresponding coset G-actions.

Proposition 5 *For all $g \in G$ we have that*

$$\tau_{x_1} \circ \psi_1(g) = \psi_0(g) \circ \tau_{x_1}.$$