# cyclic semigroup

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| Canonical name | CyclicSemigroup |
| Date of creation | 2013-03-22 13:07:30 |
| Last modified on | 2013-03-22 13:07:30 |
| Owner | mclase (549) |
| Last modified by | mclase (549) |
| Numerical id | 6 |
| Author | mclase (549) |
| Entry type | Definition |
| Classification | msc 20M99 |
| Defines | index |
| Defines | period |

A semigroup which is generated by a single element is called a *cyclic semigroup.*

Let $S = \langle x \rangle$ be a cyclic semigroup. Then as a set, $S = \{x^n \mid n > 0\}$.

If all powers of $x$ are distinct, then $S = \{x, x^2, x^3, \ldots\}$ is (countably) infinite.

Otherwise, there is a least integer $n > 0$ such that $x^n = x^m$ for some $m < n$. It is clear then that the elements $x, x^2, \ldots, x^{n-1}$ are distinct, but that for any $j \geq n$, we must have $x^j = x^i$ for some $i$, $m \leq i \leq n-1$. So $S$ has $n-1$ elements.

Unlike in the group case, however, there are in general multiple non-isomorphic cyclic semigroups with the same number of elements. In fact, there are $t$ non-isomorphic cyclic semigroups with $t$ elements: these correspond to the different choices of $m$ in the above (with $n = t+1$).

The integer $m$ is called the *index* of $S$, and $n - m$ is called the *period* of $S$.

The elements $K = \{x^m, x^{m+1}, \ldots, x^{n-1}\}$ are a subsemigroup of $S$. In fact, $K$ is a cyclic group.

A concrete representation of the semigroup with index $m$ and period $r$ as a semigroup of transformations can be obtained as follows. Let $X = \{1, 2, 3, \ldots, m+r\}$. Let

$$\phi = \begin{pmatrix} 1 & 2 & 3 & \ldots & m+r-1 & m+r \\ 2 & 3 & 4 & \ldots & m+r & r+1 \end{pmatrix}.$$

Then $\phi$ generates a subsemigroup $S$ of the full semigroup of transformations $\mathcal{T}_X$, and $S$ is cyclic with index $m$ and period $r$.