condition on a near ring to be a ring

Canonical name	ConditionOnANearRingToBeARing
Date of creation	2013-03-22 17:19:54
Last modified on	2013-03-22 17:19:54
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	14
Author	CWoo (3771)
Entry type	Theorem
Classification	msc 20-00
Classification	msc 16-00
Classification	msc 13-00
Related topic	UnitalRing

Every ring is a near-ring. The converse is true only when additional conditions are imposed on the near-ring.

Theorem 1. *Let $(R, +, \cdot)$ be a near ring with a multiplicative identity 1 such that the $\cdot$ also left distributes over $+$; that is, $c \cdot (a + b) = c \cdot a + c \cdot b$. Then R is a ring.*

In short, a distributive near-ring with 1 is a ring.

Before proving this, let us list and prove some general facts about a near ring:

1. Every near ring has a unique additive identity: if both 0 and $0'$ are additive identities, then $0 = 0 + 0' = 0'$.

2. Every element in a near ring has a unique additive inverse. The additive inverse of a is denoted by $-a$.

   Proof. If b and c are additive inverses of a, then $b + a = 0 = a + c$ and $b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c$. □

3. $-(-a) = a$, since a is the (unique) additive inverse of $-a$.

4. There is no ambiguity in defining "subtraction" $-$ on a near ring R by $a - b := a + (-b)$.

5. $a - b = 0$ iff $a = b$, which is just the combination of the above three facts.

6. If a near ring has a multiplicative identity, then it is unique. The proof is identical to the one given for the first Fact.

7. If a near ring has a multiplicative identity 1, then $(-1)a = -a$.

   Proof. $a + (-1)a = 1a + (-1)a = (1 + (-1))a = 0a = 0$. Therefore $(-1)a = -a$ since a has a unique additive inverse. □

We are now in the position to prove the theorem.

Proof. Set $r = a + b$ and $s = b + a$. Then

$$\begin{aligned}
r - s &= r - (b + a) && \text{substitution} \\
&= r + (-1)(b + a) && \text{by Fact ?? above} \\
&= r + ((-1)b + (-1)a) && \text{by left distributivity} \\
&= r + (-b + (-a)) && \text{by Fact ?? above} \\
&= (a + b) + (-b + (-a)) && \text{substitution} \\
&= ((a + b) + (-b)) + (-a) && \text{additive associativity} \\
&= (a + (b + (-b)) + (-a) && \text{additive associativity} \\
&= (a + 0) + (-a) && -b \text{ is the additive inverse of } b \\
&= a + (-a) && 0 \text{ is the additive identity} \\
&= 0 && \text{same reason as above}
\end{aligned}$$

Therefore, $a + b = r = s = b + a$ by Fact **??** above. □