alternating group has index 2 in the symmetric group, the

Canonical name	AlternatingGroupHasIndex2InTheSymmetricGroupThe
Date of creation	2013-03-22 16:48:49
Last modified on	2013-03-22 16:48:49
Owner	yesitis (13730)
Last modified by	yesitis (13730)
Numerical id	8
Author	yesitis (13730)
Entry type	Proof
Classification	msc 20-00

We prove that the alternating group A_n has index 2 in the symmetric group S_n, i.e., A_n has the same cardinality as its complement $S_n \setminus A_n$. The proof is function-theoretic. Its idea is similar to the proof in the parent topic, but the focus is less on algebraic aspect.

Let $\pi \in S_n \setminus A_n$. Define $\pi : S_n \setminus A_n \to A_n$ by $\pi(\sigma) = \pi\sigma$, where $\pi\sigma$ is the product of π and σ.

One-to-one:

$$\pi(\sigma) = \pi(\delta) \Longrightarrow \sigma = \delta$$

since π^{-1} exists and $\pi^{-1}\pi\sigma = \pi^{-1}\pi\delta$.

Onto: Given $\alpha \in A_n$, there exists an element in $S_n \setminus A_n$, namely $\lambda = \pi^{-1}\alpha$, such that

$$\pi(\alpha) = \lambda.$$

(The element λ is in $S_n \setminus A_n$ because π^{-1} is and the product of an odd permutation and an even permutation is odd.)

The function $\pi : S_n \setminus A_n \to A_n$ is, therefore, a one-to-one correspondence, so both sets $S_n \setminus A_n$ and A_n have the same cardinality.