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# Myhill-Nerode theorem for semigroups

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| Defines | recognizable subset of a semigroup |

Let $S$ be a semigroup. $X \subseteq S$ is *recognizable* if it is union of classes of a congruence $\chi$ such that $S/\chi$ is finite.

In the rest of the entry, $\equiv_X$ will be the syntactic congruence of $X$, and $\mathcal{N}_X$ its Nerode equivalence.

**Theorem 1 (Myhill-Nerode theorem for semigroups)** *Let $S$ be a semigroup and let $X \subseteq S$. The following are equivalent.*

1. *$X$ is recognizable.*
2. *There exist a finite semigroup $T$ and a morphism $\phi : S \to T$ such that $X = \phi^{-1}(\phi(X))$.*
3. *The syntactic semigroup of $X$ is finite.*
4. *There exists an equivalence relation $\sim$ on $S$ such that*
   - *$S/\sim$ is finite and*
   - *$s_1 \sim s_2$ implies $s_1 t \sim s_2 t$.*
5. *The quotient set $S/\mathcal{N}_X$ is finite.*

Theorem **??** generalizes Myhill-Nerode theorem for languages to subsets of generic, not necessarily free, semigroups. In fact, as a consequence of Theorem **??**, a language on a finite alphabet is recognizable in the sense given above if and only if it is recognizable by a DFA.

The equivalence of points **??**, **??** and **??** is attributed to John Myhill, while the equivalence of points **??**, **??** and **??** is attributed to Anil Nerode.

*Proof of Theorem* **??**.

**??** $\Rightarrow$ **??**. Given a congruence $\chi$ such that $|S/\chi| < \infty$ and $X$ is union of classes of $\chi$, choose $T$ as the quotient semigroup $S/\chi$ and $\phi$ as the natural homomorphism mapping $s$ to $[s]_\chi$.

**??** $\Rightarrow$ **??**. Given a morphism of semigroups $\phi : S \to T$ with $T$ finite and $X = \phi^{-1}(\phi(X))$, put $s \chi t$ iff $\phi(s) = \phi(t)$.

Since $\phi$ is a morphism, $\chi$ is a congruence; moreover, $X = \phi^{-1}(\phi(X))$ means that $X$ is union of classes of $\chi$-equivalence. By the maximality property of syntactic congruence, the number of classes of $\equiv_X$ does not exceed the number of classes of $\chi$, which in turn does not exceed the number of elements of $T$.

**??** $\Rightarrow$ **??**. Straightforward from $X$ being union of classes of $\equiv_X$.

**??** $\Rightarrow$ **??**. Straightforward from $\equiv_X$ satisfying the second requirement.

**??** $\Rightarrow$ **??**. Straightforward from the maximality property of Nerode equivalence.

**??** $\Rightarrow$ **??**. Let

$$S/\mathcal{N}_X = \{[\xi_1]_{\mathcal{N}_X}, \ldots, [\xi_K]_{\mathcal{N}_X}\} \ . \tag{1}$$

Since $s_1 \equiv_X s_2$ iff $\ell s_1 \mathcal{N}_X \ell s_2$ for every $\ell \in S$, determining $[s]_{\equiv_X}$ is the same as determining $[\ell s]_{\mathcal{N}_X}$ as $\ell$ varies in $S$, *plus the class* $[s]_{\mathcal{N}_X}$. (This additional class takes into account the possibility that $s \notin [\ell s]_{\mathcal{N}_X}$ for any $\ell \in S$, which cannot be excluded *a priori*: do not forget that $S$ is not required to be a monoid.)

But since $s_1 \mathcal{N}_X s_2$ implies $s_1 t \mathcal{N}_X s_2 t$, if $[\ell_1]_{\mathcal{N}_X} = [\ell_2]_{\mathcal{N}_X}$ then $[\ell_1 s]_{\mathcal{N}_X} = [\ell_2 s]_{\mathcal{N}_X}$ as well. To determine $[s]_{\equiv_X}$ it is thus sufficient to determine the $[\xi_i s]_{\mathcal{N}_X}$ for $1 \leq i \leq K$, plus $[s]_{\mathcal{N}_X}$.

We can thus identify the class $[s]_{\equiv_X}$ with the sequence $([\xi_1 s]_{\mathcal{N}_X}, \ldots, [\xi_K s]_{\mathcal{N}_X}, [s]_{\mathcal{N}_X})$. Then the number of classes of $\equiv_X$ cannot exceed that of $(K+1)$-ples of classes of $\mathcal{N}_X$, which is $K^{K+1}$. $\square$

# References


A. de Luca and S. Varricchio. *Finiteness and Regularity in Semigroups and Formal Languages.* Springer Verlag, Heidelberg 1999.