order of a profinite group

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Defines	index of a profinite group

Let G be a profinite group, and let H be any closed subgroup. We define the *of H in G* by

$$[G : H] = \mathrm{lcm}(\{[G/N : HN/N]\}),$$

where N runs over all open (and hence of finite index) subgroups of G, and where lcm is taken in the sense of the least common multiple of supernatural numbers.

In particular, we can define the *order of a profinite group* to be the index of the identity subgroup in G:

$$|G| := [G : \{e\}].$$

Some examples of orders of profinite groups:

- $G = \mathbb{Z}_p$, the ring of p-adic integers. Since every finite quotient of $\mathbb{Z}_p$ is cyclic of p^n elements (for some n), and every such group occurs as a quotient, we have $|G| = \mathrm{lcm}(p^n)$, where n runs over all natural numbers. Thus $|G| = p^\infty$.

- $G = \widehat{\mathbb{Z}}$. Since $G \approx \prod_p \mathbb{Z}_p$, we have $|G| = \prod_p |\mathbb{Z}_p| = \prod_p p^\infty$. This example illustrates the limitations of this concept: Despite being "relatively small" in of profinite groups, $\widehat{\mathbb{Z}}$ has the largest possible profinite order.

References

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