cycle notation

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The cycle notation is a useful convention for writing down a permutations in terms of its constituent cycles. Let S be a finite set, and

$$a_1, \ldots, a_k, \quad k \geq 2$$

distinct elements of S. The expression $(a_1, \ldots, a_k)$ denotes the cycle whose action is

$$a_1 \mapsto a_2 \mapsto a_3 \ldots a_k \mapsto a_1.$$

Note there are k different expressions for the same cycle; the following all represent the same cycle:

$$(a_1, a_2, a_3, \ldots, a_k) = (a_2, a_3, \ldots, a_k, a_1), = \ldots = (a_k, a_1, a_2, \ldots, a_{k-1}).$$

Also note that a 1-element cycle is the same thing as the identity permutation, and thus there is not much point in writing down such things. Rather, it is customary to express the identity permutation simply as () or (1).

Let π be a permutation of S, and let

$$S_1, \ldots, S_k \subset S, \quad k \in \mathbb{N}$$

be the orbits of π with more than 1 element. For each $j = 1, \ldots, k$ let n_j denote the cardinality of S_j. Also, choose an $a_{1,j} \in S_j$, and define

$$a_{i+1,j} = \pi(a_{i,j}), \quad i \in \mathbb{N}.$$

We can now express π as a product of disjoint cycles, namely

$$\pi = (a_{1,1}, \ldots a_{n_1,1})(a_{2,1}, \ldots, a_{n_2,2}) \ldots (a_{k,1}, \ldots, a_{n_k,k}).$$

By way of illustration, here are the 24 elements of the symmetric group on $\{1, 2, 3, 4\}$ expressed using the cycle notation, and grouped according to their conjugacy classes:

$$
\begin{aligned}
&(),\\
&(12),\ (13),\ (14),\ (23),\ (24),\ (34)\\
&(123),\ (213),\ (124),\ (214),\ (134),\ (143),\ (234),\ (243)\\
&(12)(34),\ (13)(24),\ (14)(23)\\
&(1234),\ (1243),\ (1324),\ (1342),\ (1423),\ (1432)
\end{aligned}
$$