Cayley's theorem for semigroups

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Let X be a set. We can define on X^X, the set of functions from X to itself, a structure of semigroup by putting $f \otimes g = g \circ f$. Such semigroup is actually a monoid, whose identity element is the identity function of X.

Theorem 1 (Cayley's theorem for semigroups) *For every semigroup* $(S, \cdot)$ *there exist a set* X *and an injective map* $\phi : S \to X^X$ *which is a morphism of semigroups from* $(S, \cdot)$ *to* $(X^X, \otimes)$.

In other words, every semigroup is isomorphic to a semigroup of transformations of some set. This is an extension of Cayley's theorem on groups, which states that every group is isomorphic to a group of *invertible* transformations of some set.

Proof of Theorem **??**. The argument is similar to the one for Cayley's theorem on groups. Let $X = S$, the set of elements of the semigroup.

First, suppose $(S, \cdot)$ is a monoid with unit e. For $s \in S$ define $f_s : S \to S$ as

$$f_s(x) = x \cdot s \ \forall x \in S \,. \tag{1}$$

Then for every $s, t, x \in S$ we have

$$\begin{aligned} f_{s \cdot t}(x) &= x \cdot (s \cdot t) \\ &= (x \cdot s) \cdot t \\ &= f_t(x \cdot s) \\ &= f_t(f_s(x)) \\ &= (f_t \circ f_s)(x) \\ &= (f_s \otimes f_t)(x) \,, \end{aligned}$$

so $\phi(s) = f_s$ is a homomorphism of monoids, with $f_e = \mathrm{id}_S$. This homomorphism is injective, because if $f_s = f_t$, then $s = f_s(e) = f_t(e) = t$.

Next, suppose $(S, \cdot)$ is a semigroup but not a monoid. Let $e \notin S$. Construct a monoid $(M, *)$ by putting $M = S \cup \{e\}$ and defining

$$s * t = \begin{cases} s \cdot t & \text{if } s, t \in S, \\ s & \text{if } s \in S, t = e, \\ t & \text{if } s = e, t \in S, \\ e & \text{if } s = t = e. \end{cases}$$

Then $(M, *)$ is isomorphic to a submonoid of $(M^M, \otimes)$ as by (**??**). For $s \in S$ put $g_s = f_s|_S$: then $g_s \in S^S$ for every s, $g_{s \cdot t} = f_{s * t}|_S$, and $(S, \cdot)$ is isomorphic to $(\Sigma, \otimes)$ with $\Sigma = \{g_s \mid s \in S\}$. $\square$

Observe that the theorem remains valid if $f \otimes g$ is defined as $f \circ g$. In this case, the morphism ϕ is defined by $f_s(x) = s \cdot x \, \forall x \in S$.