free product

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Definition

Let G be a group, and let $(A_i)_{i\in I}$ be a family of http://planetmath.org/Subgroupsubgroups of G. Then G is said to be a *free product* of the subgroups A_i if given any group H and a http://planetmath.org/GroupHomomorphismhomomorphism $f_i\colon A_i \to H$ for each $i \in I$, there is a unique homomorphism $f\colon G \to H$ such that $f|_{A_i} = f_i$ for all $i \in I$. The subgroups A_i are then called the *free factors* of G.

If G is the free product of $(A_i)_{i\in I}$, and $(K_i)_{i\in I}$ is a family of groups such that $K_i \cong A_i$ for each $i \in I$, then we may also say that G is the free product of $(K_i)_{i\in I}$. With this definition, every family of groups has a free product, and the free product is unique up to isomorphism.

The free product is the coproduct in the category of groups.

Construction

Free groups are simply the free products of infinite cyclic groups, and it is possible to generalize the construction given in the free group article to the case of arbitrary free products. But we will instead construct the free product as a http://planetmath.org/QuotientGroupquotient of a free group.

Let $(K_i)_{i\in I}$ be a family of groups. For each $i \in I$, let X_i be a set and $\gamma_i\colon X_i \to K_i$ a function such that $\gamma_i(X_i)$ generates K_i. The X_i should be chosen to be pairwise disjoint; for example, we could take $X_i = K_i \times \{i\}$, and let γ_i be the obvious bijection. Let F be a free group freely generated by $\bigcup_{i\in I} X_i$. For each $i \in I$, the subgroup $\langle X_i\rangle$ of F is freely generated by X_i, so there is a homomorphism $\phi_i\colon \langle X_i\rangle \to K_i$ extending γ_i. Let N be the normal closure of $\bigcup_{i\in I} \ker\phi_i$ in F.

Then it can be shown that F/N is the free product of the family of subgroups $(\langle X_i\rangle N/N)_{i\in I}$, and $K_i \cong \langle X_i\rangle N/N$ for each $i \in I$.