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example of a non-fully invariant subgroup

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Every fully invariant subgroup is characteristic, but some characteristic subgroups need not be fully invariant. For example, the center of a group is characteristic but not always fully invariant. We pursue a single example.

Recall the dihedral group of order $2n$, denoted D_{2n}, can be considered as the symmetries of a regular n-gon. If we consider a regular hexagon, so $n = 6$, and label the vertices counterclockwise from 1 to 6 we can then encode each symmetry as a permutation on 6 points. So a rotation by $\pi/3$ can be encoded as the permutation $\rho = (123456)$ and the reflection fixing the axis through the vertices 1 and 4 can be encoded as $\phi = (26)(35)$. Indeed these two permutations generate a permutation group isomorphic to D_{12}.

The center of a dihedral group of order $2n$ is trivial if n is odd, and of order 2 if $n > 2$ is even (if $n = 2$ it is the entire group $D_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, see the remark below). Specifically, if ρ is a rotation of order n, and $n = 2m$, then $\langle \rho^m \rangle$ is the center of D_{2n}. (Note this is the only rotation or order 2, and in particular it is always a rotation by π.) So when $n = 6$, the center is $\langle (14)(25)(36) \rangle$.

Now fix $n = 6$ and note the following assignment of generators determines an endomorphism $f : D_{12} \to D_{12}$:

$$(123456) \mapsto (26)(35), \quad (26)(35) \mapsto (14)(25)(36).$$

Note that image $K := \langle (26)(35), (14)(25)(36) \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, as $(14)(25)(36)$ is central in D_{12} and the generators of K are distinct elements of order 2. [This can be proved with the relations of the dihedral group.]

Remark 1. *Geometrically we note that the kernel of the homomorphism is $\langle \rho^2 \rangle$ – the group of rotations of order 3. So if we quotient by the kernel we are identifying the three inscribed (non-square) rectangles of the hexagon (1245, 2356 and 3461). The symmetry group of a non-square rectangle is none other than $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, sometimes called D_4.*

Now the center is mapped via f to the subgroup $\langle (26)(35) \rangle$ so $f(Z(D_{12}))$ is not contained in $Z(D_{12})$ proving $Z(D_{12})$ is not fully-invariant.

Of course the example applies without serious modification to the dihedral groups on $2m$-gons, where $m > 1$ is odd. Here a generally offending endomorphism may be described with a composition of maps (the first leaves the center invariant, the second swaps the basis of the image of the first thus moving the image of the center):

$$\rho \mapsto \rho^m \mapsto \phi, \qquad \phi \mapsto \phi \mapsto \rho^m.$$

As m is odd and the center, $\langle \rho^m \rangle$, has order 2, it follows $\langle \rho^m \rangle$ maps to $\langle \rho^m \rangle$ under the first map, and then can be interchanged with a reflection to violate the condition of full invariance. If m is even then the center lies in the kernel of the first map so no such trick can be played.