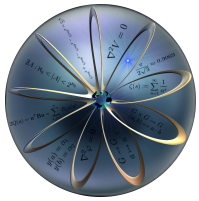

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estimation of index of intersection subgroup

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Theorem. If H_1, H_2, ..., H_n are subgroups of G, then

$$\left[G : \bigcap_{i=1}^{n} H_i\right] \leqq \prod_{i=1}^{n} [G : H_i].$$

Proof. We prove here only the case $n = 2$; the general case may be handled by the induction.

Let $H_1 \cap H_2 := K$. Let R be the set of the right cosets of K and R_i the set of the right cosets of H_i $(i = 1, 2)$. Define the relation ϱ from R to $R_1 \times R_2$ as

$$\varrho := \{(Kx, (H_1x, H_2x)) \vdots\ x \in G\}.$$

We then have the `http://planetmath.org/Equivalent3`equivalent conditions

$$\begin{gathered} Kx = Ky, \\ xy^{-1} \in K, \\ xy^{-1} \in H_1 \quad \wedge \quad xy^{-1} \in H_2, \\ H_1x = H_1y \quad \wedge \quad H_2x = H_2y, \\ (H_1x, H_2x) = (H_1y, H_2y), \end{gathered}$$

whence ϱ is a mapping and injective, $\varrho : R \to R_1 \times R_2$. i.e. it is a bijection from R onto the subset $\{\varrho(x) \vdots\ x \in R\}$ of $R_1 \times R_2$. Therefore,

$$\operatorname{card}(R) \leqq \operatorname{card}(R_1 \times R_2) = \operatorname{card}(R_1) \cdot \operatorname{card}(R_2).$$

As a consequence one obtains the

Theorem (Poincaré). The index of the intersection of finitely many subgroups with finite `http://planetmath.org/Coset`indices is finite.