permutation representation

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Let G be a group, and S any finite set on which G acts.
That means that for any $g, h \in G$; $\mathbf{v}, \mathbf{w} \in S$

- $g\mathbf{v} \in V$,
- $(gh)\mathbf{v} = g(h\mathbf{v})$,
- $e\mathbf{v} = \mathbf{v}$.

Notice that we almost have what it takes to make S a representation of G, but S is no vector space. We can however obtain a G-module (a vector space carrying a representation of G) as follows.

Let $S = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$. And let $\mathbb{C}S = \mathbb{C}[\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n]$ be the vector space generated by S over $\mathbb{C}$. in other words, $\mathbb{C}S$ is made of all formal linear combinations $c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \cdots + c_n\mathbf{s}_n$ with $c_j \in \mathbb{C}$. The sum is defined coordinate-wise as is scalar multiplication.

Then the action of G in S can be extended linearly to $\mathbb{C}S$ as

$$g(c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \cdots + c_n\mathbf{s}_n) = c_1(g\mathbf{s}_1) + c_2(g\mathbf{s}_2) + \cdots + c_n(g\mathbf{s}_n)$$

and then the map $\rho : G \to GL(\mathbb{C}S)$ where ρ is such that $\rho(g)(\mathbf{v}) = g\mathbf{v}$ makes $\mathbb{C}S$ into a G-module. The G-module $\mathbb{C}S$ is known as the *permutation representation* associated with S.

Example.
If $G = S_n$ acts on $S = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}\}$, then

$$\mathbb{C}S = \{c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n}\}.$$

If $\sigma \in S_n$, the action becomes

$$\sigma(c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n}) = c_1\sigma(\mathbf{1}) + c_2\sigma(\mathbf{2}) + \cdots + c_n\sigma(\mathbf{n}).$$

Since S forms a basis for this space, we can compute the matrices corresponding to the defining permutation and we will see that the corresponding permutation matrices are obtained.

References. Bruce E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions.* 2a Ed. 2000. Graduate Texts in Mathematics. Springer.