characterization of full families of groups

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Proposition. Let $\mathcal{G} = \{G_k\}_{k\in I}$ be a family of groups. Then $\mathcal{G}$ is full if and only if for any $i, j \in I$ such that $i \neq j$ we have that any homomorphism $f : G_i \to G_j$ is trivial.

Proof. ,,$\Rightarrow$" Assume that $f : G_i \to G_j$ is a nontrivial group homomorphism. Then define

$$h : \bigoplus_{k\in I} G_k \to \bigoplus_{k\in I} G_k$$

as follows: if $t \in I$ is such that $t \neq i$ and $g \in \bigoplus_{k\in I} G_k$ is such that $g \in G_t$, then $h(g) = g$. If $g \in \bigoplus_{k\in I} G_k$ is such that $g \in G_i$, then $h(g)(j) = f(g(i))$ and $h(g)(k) = 0$ for $k \neq j$. This values uniquely define h and one can easily check that h is not decomposable. $\square$

,,$\Leftarrow$" Assume that for any $i, j \in I$ such that $i \neq j$ we have that any homomorphism $f : G_i \to G_j$ is trivial. Let

$$h : \bigoplus_{k\in I} G_k \to \bigoplus_{k\in I} G_k$$

be any homomorphism. Moreover, let $i \in I$ and $g \in \bigoplus_{k\in I} G_k$ be such that $g \in G_i$. We wish to show that $h(g) \in G_i$.

So assume that $h(g) \notin G_i$. Then there exists $j \neq i$ such that $0 \neq h(g)(j) \in G_j$. Let

$$\pi : \bigoplus_{k\in I} G_k \to G_j$$

be the projection and let

$$u : G_i \to \bigoplus_{k\in I} G_k$$

be the natural inclusion homomorphism. Then $\pi \circ u : G_i \to G_j$ is a nontrivial group homomorphism. Contradiction. $\square$

Corollary. Assume that $\{G_k\}_{k\in I}$ is a family of nontrivial groups such that G_i is periodic for each $i \in I$. Moreover assume that for any $i, j \in I$ such that $i \neq j$ and any $g \in G_i$, $h \in G_j$ orders $|g|$ and $|h|$ are realitvely prime (which implies that I is countable). Then $\{G_k\}_{k\in I}$ is full.

Proof. Assume that $i \neq j$ and $f : G_i \to G_j$ is a group homomorphism. Then $|f(g)|$ divides $|g|$ for any $g \in G_i$. But $f(g) \in G_j$, so $|g|$ and $|f(g)|$ are relatively prime. Thus $|f(g)| = 1$, so $f(g) = 0$. Therefore f is trivial, which (due to proposition) completes the proof. $\square$