$C_{mn} \cong C_m \times C_n$ when m, n are relatively prime

Canonical name	CmncongCmtimesCnWhenMNAreRelativelyPrime
Date of creation	2013-03-22 17:59:46
Last modified on	2013-03-22 17:59:46
Owner	yesitis (13730)
Last modified by	yesitis (13730)
Numerical id	8
Author	yesitis (13730)
Entry type	Proof
Classification	msc 20A05

We show that C_{mn}, $\gcd(m,n)=1$, is isomorphic to $C_m \times C_n$, where C_r denotes the cyclic group of order r for any positive integer r.

Let $C_m = \langle x \rangle$ and $C_n = \langle y \rangle$. Then the external direct product $C_m \times C_n$ consists of elements (x^i, y^j), where $0 \le i \le m-1$ and $0 \le j \le n-1$.

Next, we show that the group $C_m \times C_n$ is cyclic. We do so by showing that it is generated by an element, namely (x, y): if (x, y) generates $C_m \times C_n$, then for each $(x^i, y^j) \in C_m \times C_n$, we must have $(x^i, y^j) = (x, y)^k$ for some $k \in \{0, 1, 2, \ldots, mn-1\}$. Such k, if exists, would satisfy

$$\begin{aligned} k &\equiv i \ (mod\, m) \\ k &\equiv j \ (mod\, n). \end{aligned}$$

Indeed, by the Chinese Remainder Theorem, such k exists and is unique modulo mn. (Here is where the relative primality of m, n comes into play.) Thus, $C_m \times C_n$ is generated by (x, y), so it is cyclic.

The order of $C_m \times C_n$ is mn, so is the order of C_{mn}. Since cyclic groups of the same order are isomorphic, we finally have $C_{mn} \cong C_m \times C_n$.