a representation which is not completely reducible

Canonical name	ARepresentationWhichIsNotCompletelyReducible
Date of creation	2013-03-22 13:31:47
Last modified on	2013-03-22 13:31:47
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Last modified by	bwebste (988)
Numerical id	5
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Entry type	Example
Classification	msc 20C15

If G is a finite group, and k is a field whose characteristic does divide the order of the group, then Maschke's theorem fails. For example let V be the regular representation of G, which can be thought of as functions from G to k, with the G action $g \cdot \varphi(g') = \varphi(g^{-1}g')$. Then this representation is not completely reducible.

There is an obvious trivial subrepresentation W of V, consisting of the constant functions. I claim that there is no complementary invariant subspace to this one. If W' is such a subspace, then there is a homomorphism $\varphi : V \to V/W' \cong k$. Now consider the characteristic function of the identity $e \in G$

$$\delta_e(g) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

and $\ell = \varphi(\delta_e)$ in V/W'. This is not zero since δ generates the representation V. By G-equivarience, $\varphi(\delta_g) = \ell$ for all $g \in G$. Since

$$\eta = \sum_{g \in G} \eta(g)\delta_g$$

for all $\eta \in V$,

$$W' = \varphi(\eta) = \ell \left(\sum_{g \in G} \eta(g) \right).$$

Thus,

$$\ker \varphi = \{\eta \in V | \sum_{\in G} \eta(g) = 0\}.$$

But since the characteristic of the field k divides the order of G, $W \leq W'$, and thus could not possibly be complementary to it.

For example, if $G = C_2 = \{e, f\}$ then the invariant subspace of V is spanned by $e + f$. For characteristics other than 2, $e - f$ spans a complementary subspace, but over characteristic 2, these elements are the same.