homomorphic image of group

Canonical name	HomomorphicImageOfGroup
Date of creation	2013-03-22 18:56:27
Last modified on	2013-03-22 18:56:27
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	14
Author	pahio (2872)
Entry type	Theorem
Classification	msc 20A05
Classification	msc 08A05
Related topic	GroupHomomorphism
Related topic	CorrespondenceBetweenNormalSubgroupsAndHomomorphicImages

Theorem. The homomorphic image of a group is a group. More detailed, if f is a homomorphism from the group $(G,\,*)$ to the groupoid $(\Gamma,\,\star)$, then the groupoid $(f(G),\,\star)$ also is a group. Especially, the isomorphic image of a group is a group.

Proof. Let $\alpha,\,\beta,\,\gamma$ be arbitrary elements of the image $f(G)$ and $a,\,b,\,c$ some elements of G such that $f(a)=\alpha,\ f(b)=\beta,\ f(c)=\gamma$. Then

$$\alpha\star\beta \;=\; f(a)\star f(b) \;=\; f(a*b) \;\in\; f(G),$$

whence $f(G)$ is closed under "$\star$", and we, in fact, can speak of a groupoid $(f(G),\,\star)$.

Secondly, we can calculate

$$\begin{aligned}(\alpha\star\beta)\star\gamma &= (f(a)\star f(b))\star f(c)\\ &= f(a*b)\star f(c)\\ &= f((a*b)*c)\\ &= f(a*(b*c))\\ &= f(a)\star f(b*c)\\ &= f(a)\star(f(b)\star f(c))\\ &= \alpha\star(\beta\star\gamma),\end{aligned}$$

whence the associativity is in in the groupoid $(f(G),\,\star)$.

Let e be the identity element of $(G,\,*)$ and $f(e)=\varepsilon$. Then

$$\varepsilon\star\alpha \;=\; f(e)\star f(a) \;=\; f(e*a) \;=\; f(a) \;=\; \alpha,$$

$$\alpha\star\varepsilon \;=\; f(a)\star f(e) \;=\; f(a*e) \;=\; f(a) \;=\; \alpha,$$

and therefore ε is an identity element in $f(G)$.

If $f(a^{-1})=\alpha'$, then

$$\alpha\star\alpha' \;=\; f(a)\star f(a^{-1}) \;=\; f(a*a^{-1}) \;=\; f(e) \;=\; \varepsilon,$$

$$\alpha'\star\alpha \;=\; f(a^{-1})\star f(a) \;=\; f(a^{-1}*a) \;=\; f(e) \;=\; \varepsilon.$$

Thus any element α of $f(G)$ has in $f(G)$ an inverse.

Accordingly, $(f(G),\,\star)$ is a group.

Remark 1. If $(G, *)$ is Abelian, the same is true for $(f(G), \star)$.

Remark 2. Analogically, one may prove that the homomorphic image of a ring is a ring.

Example. If we define the mapping f from the group $(\mathbb{Z}, +)$ to the groupoid $(\mathbb{Z}_9, \cdot)$ by

$$f(n) \;:=\; \langle 4\rangle^n,$$

then f is homomorphism:

$$f(m+n) \;=\; \langle 4\rangle^{m+n} \;=\; \langle 4\rangle^m\langle 4\rangle^n \;=\; f(m)f(n).$$

The image $f(\mathbb{Z})$ consists of powers of the `http://planetmath.org/Congruences`residue class $\langle 4\rangle$, which are

$$\langle 4\rangle, \;\; \langle 16\rangle = \langle 7\rangle, \;\; \langle 64\rangle = \langle 1\rangle.$$

These apparently form the cyclic group of order 3.