planetmath.org

Math for the people, by the people.

# subgroup

| | |
|---|---|
| Canonical name | Subgroup |
| Date of creation | 2013-03-22 12:02:10 |
| Last modified on | 2013-03-22 12:02:10 |
| Owner | Daume (40) |
| Last modified by | Daume (40) |
| Numerical id | 18 |
| Author | Daume (40) |
| Entry type | Definition |
| Classification | msc 20A05 |
| Related topic | Group |
| Related topic | Ring |
| Related topic | FreeGroup |
| Related topic | Cycle2 |
| Related topic | Subring |
| Related topic | GroupHomomorphism |
| Related topic | QuotientGroup |
| Related topic | ProperSubgroup |
| Related topic | SubmonoidSubsemigroup |
| Related topic | ProofThatGInGImpliesThatLangleGRangleLeG |
| Related topic | AbelianGroup2 |
| Related topic | EssentialSubgroup |
| Related topic | PGroup4 |
| Defines | trivial subgroup |

**Definition:**
Let $(G, *)$ be a group and let $K$ be subset of $G$. Then $K$ is a subgroup of $G$ defined under the same operation if $K$ is a group by itself (with respect to $*$), that is:

- $K$ is closed under the $*$ operation.
- There exists an identity element $e \in K$ such that for all $k \in K$, $k * e = k = e * k$.
- Let $k \in K$ then there exists an inverse $k^{-1} \in K$ such that $k^{-1} * k = e = k * k^{-1}$.

The subgroup is denoted likewise $(K, *)$. We denote $K$ being a subgroup of $G$ by writing $K \leq G$.

In addition the notion of a subgroup of a semigroup can be defined in the following manner. Let $(S, *)$ be a semigroup and $H$ be a subset of $S$. Then $H$ is a subgroup of $S$ if $H$ is a subsemigroup of $S$ and $H$ is a group.

**Properties:**

- The set $\{e\}$ whose only element is the identity is a subgroup of any group. It is called the trivial subgroup.
- Every group is a subgroup of itself.
- The null set $\{\}$ is never a subgroup (since the definition of group states that the set must be non-empty).

There is a very useful theorem that allows proving a given subset is a subgroup.

**Theorem:**
If $K$ is a nonempty subset of the group $G$. Then $K$ is a *subgroup* of $G$ if and only if $s, t \in K$ implies that $st^{-1} \in K$.

**Proof:** First we need to show if $K$ is a subgroup of $G$ then $st^{-1} \in K$. Since $s, t \in K$ then $st^{-1} \in K$, because $K$ is a group by itself.
Now, suppose that if for any $s, t \in K \subseteq G$ we have $st^{-1} \in K$. We want to show that $K$ is a subgroup, which we will accomplish by proving it holds the group axioms.

Since $tt^{-1} \in K$ by hypothesis, we conclude that the identity element is in $K$: $e \in K$. (Existence of identity)

Now that we know $e \in K$, for all $t$ in $K$ we have that $et^{-1} = t^{-1} \in K$ so the inverses of elements in $K$ are also in $K$. (Existence of inverses).

Let $s, t \in K$. Then we know that $t^{-1} \in K$ by last step. Applying hypothesis shows that

$$s(t^{-1})^{-1} = st \in K$$

so $K$ is closed under the operation. *QED*

**Example:**

- Consider the group $(\mathbb{Z}, +)$. Show that$(2\mathbb{Z}, +)$ is a subgroup.

  The subgroup is closed under addition since the sum of even integers is even.

  The identity 0 of $\mathbb{Z}$ is also on $2\mathbb{Z}$ since 2 divides 0. For every $k \in 2\mathbb{Z}$ there is an $-k \in 2\mathbb{Z}$ which is the inverse under addition and satisfies $-k + k = 0 = k + (-k)$. Therefore $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$.

  Another way to show $(2\mathbb{Z}, +)$ is a subgroup is by using the proposition stated above. If $s, t \in 2\mathbb{Z}$ then $s, t$ are even numbers and $s - t \in 2\mathbb{Z}$ since the difference of even numbers is always an even number.

**See also:**

- Wikipedia, `http://www.wikipedia.org/wiki/Subgroup`subgroup