alternative definition of a quasigroup

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In the parent entry, a quasigroup is defined as a set, together with a binary operation on it satisfying two formulas, both of which using existential quantifiers. In this entry, we give an alternative, but equivalent, definition of a quasigroup using only universally quantified formulas. In other words, the class of quasigroups is an equational class.

Definition. A *quasigroup* is a set Q with three binary operations $\cdot$ (multiplication), $\backslash$ (*left division*), and $/$ (*right division*), such that the following are satisfied:

- $(Q,\cdot)$ is a groupoid (not in the category theoretic sense)
- (left division identities) for all $a,b\in Q$, $a\backslash(a\cdot b)=b$ and $a\cdot(a\backslash b)=b$
- (right division identities) for all $a,b\in Q$, $(a\cdot b)/b=a$ and $(a/b)\cdot b=a$

Proposition 1. *The two definitions of a quasigroup are equivalent.*

Proof. Suppose Q is a quasigroup using the definition given in the `http://planetmath.org/LoopAn` entry. Define $\backslash$ on Q as follows: for $a,b\in Q$, set $a\backslash b:=c$ where c is the unique element such that $a\cdot c=b$. Because c is unique, $\backslash$ is well-defined. Now, let $x=a\cdot b$ and $y=a\backslash x$. Since $a\cdot y=x=a\cdot b$, and y is uniquely determined, this forces $y=b$. Next, let $x=a\backslash b$, then $a\cdot x=b$, or $a\cdot(a\backslash b)=b$. Similarly, define $/$ on Q so that a/b is the unique element d such that $d\cdot b=a$. The verification of the two right division identities is left for the reader.

Conversely, let Q be a quasigroup as defined in this entry. For any $a,b\in Q$, let $c=a\backslash b$ and $d=b/a$. Then $a\cdot c=a\cdot(a\backslash b)=b$ and $d\cdot a=(b/a)\cdot a=b$. $\square$