transversals / lifts / sifts

Canonical name	TransversalsLiftsSifts
Date of creation	2013-03-22 15:53:52
Last modified on	2013-03-22 15:53:52
Owner	Algeboy (12884)
Last modified by	Algeboy (12884)
Numerical id	11
Author	Algeboy (12884)
Entry type	Definition
Classification	msc 20K27
Related topic	SchreiersLemma
Related topic	ExampleOfSchreiersLemma
Defines	transversal
Defines	lift
Defines	sift

Definition 1. *Given a group G and a subgroup H of G, a* transversal *of H in G is a subset $T \subseteq G$ such that for every $g \in G$ there exists a unique $t \in T$ such that $Hg = Ht$.*

Typically one insists $1 \in T$ so that the coset H is described uniquely by $H1$. However no standard terminology has emerged for transversals of this sort.

An alternative definition for a transversal is to use functions and homomorphisms in a method more conducive to a categorical setting. Here one replaces the notion of a transversal as a subset of G and instead treats it as a certain type of map $T : G/H \to G$. Since H is generally not normal in G, G/H simply means the set of cosets, and T is therefore a function not a homomorphism. We only require that T satisfy the following property: Given the canonical projection map $\pi : G \to G/H$ given by $g \mapsto Hg$ (this is generally not a homomorphism either, and so both π and T are simply functions between sets) then $\pi T = 1_{G/H}$. It follows immediately that the image of T in G is a transversal in the original sense of the term.

Remark 2. *Because it is customary in group theory to write actions to the right of elements many times it is preferable to write $T\pi = 1_{G/H}$ to match the right side notation.*

When H is a normal subgroup of G our terminology adjusts from transversals to *lifts*.

Definition 3. *Given a group G and a homomorphism $\pi : G \to Q$, a* lift *of Q to G is a function $f : Q \to G$ such that $\pi f = 1_Q$.*

It follows that π must be an epimorphism if it has a lift. Once again it is nearly always requested that $f(1) = 1$ but this restriction is generally not part of the definition.

Because both lifts and transversals are injective mappings it is common to use the word lift/transversal for the image and the map with the context of the use providing any necessary clarification.

Definition 4. *Given a group G and a homomorphism $\pi : G \to Q$, a* splitting map *of Q to G is a* homomorphism *$f : Q \to G$ such that $\pi f = 1_Q$.*

So we see a gradual progression in the definitions: We always have a group G and a set Q, and the maps $\pi : G \to Q$, $f : Q \to G$ satisfying

$$\pi f = 1_Q.$$

It follows, f is injective and π is surjective.

- f is a transversal if $Q = G/H$ for some subgroup H. Here π and f are simply functions.

- f is a lift if Q is a group. Here π is a homomorphism and f a function.

- f is a splitting map if Q is group and both π and f are homomorphisms.

Finally we arrive at a stronger requirement for transversals and lifts which makes greater use of the group structure involved.

Definition 5. *Given a group $G = \langle S \rangle$, there is a natural map $\pi : F(S) \to G$ from the free group on S onto G. A* lift *is a map $l : G \to F(S)$ such that $\pi l = 1_G$. Furthermore a* sift *is a lift $s : G \to F(S)$ with the added condition that $sg = g$ for all $g \in S$.*

Although a general sift is no more than a map that writes the elements of G as reduced words in S, in many cases the sifts have the added property of providing the words in a canonical form. This occurs when $G = T_0 \cdots T_{n-1}$ where T_i is a transversal of G^i/G^{i+1}. In such a case every element in G has a unique decomposition as a word $t_0 t_1 \cdots t_{n-1}$ for unique $t_i \in T_i$.