category of paths on a graph

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A nice class of illustrative examples of some notions of category theory is provided by categories of paths on a graph.

Let G be an undirected graph. Denote the set of vertices of G by "V" and denote the set of edges of G by "E".

A path of the graph G is an ordered tuplet of vertices $(x_1, x_2, \dots x_n)$ such that, for all i between 1 and $n-1$, there exists an edge connecting x_i and x_{i+i}. As a special case, we allow trivial paths which consist of a single vertex — soon we will see that these in fact play an important role as identity elements in our category.

In our category, the vertices of the graph will be the objects and the morphisms will be paths; given two of these objects a and b, we set $\mathrm{Hom}(a, b)$ to be the set of all paths $(x_1, x_2, \dots x_n)$ such that $x_1 = a$ and $x_n = b$. Given an object a, we set $1_a = (a)$, the trivial path mentioned above.

To finish specifying our category, we need to specify the composition operation. This operation will be the concatenation of paths, which is defined as follows: Given a path $(x_1, x_2, \dots, x_n) \in \mathrm{Hom}(a, b)$ and a path $(y_1, y_2, \dots, y_m) \in \mathrm{Hom}(a, b)$, we set

$$a \circ b = (x_1, x_2, \dots x_n, y_2, \dots, y_m).$$

(Remember that $x_n = y_1 = b$.) To have a bona fide category, we need to check that this choice satisfies the defining properties (A1 - A3 in the entry `http://planetmath.org/node/965`category). This is rather easily verified.

A1: Given a morphism $(x_1, x_2, \dots x_n)$, it can only belong to $\mathrm{Hom}(a, b)$ if $x_1 = a$ and $x_n = b$, hence $\mathrm{Hom}(a, b) \cup \mathrm{Hom}(c, d) = \emptyset$ unless $a = c$ and $b = d$.

A2: Suppose that we have four objects a, b, c, d and three morphisms, $(x_1, x_2, \dots x_n) \in \mathrm{Hom}(a, b)$, $(y_1, y_2, \dots y_m) \in \mathrm{Hom}(b, c)$, and $(z_1, z_2, \dots z_k) \in \mathrm{Hom}(c, d)$. Then, by the definition of the operation $\circ$ given above,

$$\begin{aligned}
&((x_1, x_2, \dots, x_n) \circ (y_1, y_2, \dots, y_m)) \circ (z_1, z_2, \dots, z_k) \\
&\qquad = (x_1, x_2, \dots, x_n, y_2, \dots, y_m) \circ (z_1, z_2, \dots, z_k) \\
&\qquad = (x_1, x_2, \dots, x_n, y_2, \dots, y_m, z_2, \dots, z_k) \\
&(x_1, x_2, \dots, x_n) \circ ((y_1, y_2, \dots, y_m) \circ (z_1, z_2, \dots, z_k)) \\
&\qquad = (x_1, x_2, \dots, x_n) \circ (y_1, y_2, \dots, y_m, z_2, \dots, z_k) \\
&\qquad = (x_1, x_2, \dots, x_n, y_2, \dots, y_m, z_2, \dots, z_k).
\end{aligned}$$

Since these two quantities are equal, the operation is associative.

A3: It is easy enough to check that paths with a single vertex act as identity elements:

$$(x_1) \circ (x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$$
$$(x_1, x_2, \ldots, x_n) \circ (x_n) = (x_1, x_2, \ldots, x_n)$$

It is also possible to consider the equivalence class of paths modulo retracing. To introduce this category, we first define a binary relation $\approx$ on the class of paths as follows: Let A and B be any two paths such that the right endpoint of A is the same as the left endpoint of B, i.e. $A \in \mathrm{Hom}(a, b)$ and $B \in \mathrm{Hom}(b, c)$ for some vertices a, b, c of our graph. Let d be any vertex which shares an edge with d. Then we set $A \circ B \approx A \circ (c, d, c) \circ B$.

Let $\sim$ be the smallest equivalence relations which contains $\approx$. We will call this equivalence relation *retracing*.

As defined above, it may not intuitively obvious what this equivalence amounts to. To this end, we may consider a different description. Define the *reversal* of a path to be the path obtained by reversing the order of the vertices traversed:

$$(x_1, x_2, \ldots, x_{n-1}, x_n)^{-1} = (x_n, x_{n-1}, \ldots, x_2, x_1)$$

Then we may show that two paths are equivalent under retracing if they may both be obtained from a third path by inserting terms of the form XX^{-1}. In symbols, we claim that $A \sim B$ if there exists an integer $n0$ and paths $X_1, \ldots X_{n+1}, Y_1, \ldots Y_{n-1}, Z_1, \ldots Z_n$ such that

$$A = X_1 \circ X_1^{-1} \circ Z_1 \circ X_2 \circ X_2^{-1} \circ \cdots \circ X_{n-1} \circ X_{n-1}^{-1} \circ Z_n \circ X_n \circ X_n^{-1} \circ Z_n \circ X_{n+1} \circ X_{n+1}^{-1}$$

and

$$B = Y_1 \circ Y_1^{-1} \circ Z_1 \circ Y_2 \circ Y_2^{-1} \circ \cdots \circ Y_{n-1} \circ Y_{n-1}^{-1} \circ Z_n \circ Y_n \circ Y_n^{-1} \circ Z_n \circ Y_{n+1} \circ Y_{n+1}^{-1}$$

This characterization explains the choice of the term "retracing" — we do not change the equivalence class of the path if we happen to wander off somewhere in the course of following the path but then backtrack and pick the path up again where we left off on our digression.

Rather than presenting a detailed formal proof, we will sketch how the two definitions may be shown to be equivalent.