a subgroup of index 2 is normal

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Lemma. *Let $(G, \cdot)$ be a group and let H be a subgroup of G of index 2. Then H is normal in G.*

Proof. Let G be a group and let H be an index 2 subgroup of G. By definition of index, there are only two left cosets of H in G, namely:

$$H, \quad g_1 H$$

where g_1 is any element of G which is not in H. Notice that if $g_1,\ g_2$ are two elements in G which are not in H then $g_1 \cdot g_2$ belongs to H. Indeed, the coset $g_1 g_2 H \neq g_1 H$ (because $g_1 g_2 = g_1 h$ would immediately yield $g_2 = h \in H$) and so $g_1 g_2 H = H$ and $g_1 g_2 \in H$.

Let $h \in H$ be an arbitrary element of H and let $g \in G$. If $g \in H$ then $ghg^{-1} \in H$ and we are done. Otherwise, assume that $g \notin H$. Thus $gh \notin H$ and by the remark above $ghg^{-1} = (gh)g^{-1} \in H$, as desired. $\square$