wreath product

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Defines	wreath product

Let A and B be groups, and let B act on the set Γ. Define the action of B on the direct product A^Γ by

$$bf(\gamma) := f(b^{-1}\gamma),$$

for any $f \in A^\Gamma$ and $\gamma \in \Gamma$. The *wreath product* of A and B according to the action of B on Γ, denoted $A \wr_\Gamma B$, is the semidirect product of groups $A^\Gamma \rtimes B$.

Let us pause to unwind this definition. The elements of $A \wr_\Gamma B$ are ordered pairs (f, b), where $f \in A^\Gamma$ and $b \in B$. The group operation is given by

$$(f, b)(f', b') = (fbf', bb').$$

Note that by definition of the action of B on A^Γ,

$$(fbf')(\gamma) = f(\gamma)f'(b^{-1}\gamma).$$

The action of B on Γ in the semidirect product permutes the elements of a tuple $f \in A^\Gamma$, and the group operation defined on A^Γ gives pointwise multiplication. To be explicit, suppose Γ is an n-tuple, and let $(f, b),\ (f', b') \in A \wr_\Gamma B$. Let b_i denote $b^{-1}(i)$. Then

$$\begin{aligned}
(f,b)(f',b') &= \big((f(1),\ f(2),\ \ldots,\ f(n)),\ b\big)\big((f'(1),\ f'(2),\ \ldots,\ f'(n)),\ b'\big) \\
&= \big((f(1),\ f(2),\ \ldots,\ f(n))\big(f'(b_1),\ f'(b_2),\ \ldots,\ f'(b_n)\big),\ bb'\big)(*) \\
&= \big((f(1)f'(b_1),\ f(2)f'(b_2),\ \ldots,\ f(n)f'(b_n)),\ bb'\big).
\end{aligned}$$

Notice the permutation of the indices in (*).

A bit amount of thought to understand this slightly messy notation will be illuminating, and might also shed some light on the choice of terminology.