examples of primitive groups that are not doubly transitive

Canonical name	ExamplesOfPrimitiveGroupsThatAreNotDoublyTransitive
Date of creation	2013-03-22 17:22:37
Last modified on	2013-03-22 17:22:37
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	7
Author	rm50 (10146)
Entry type	Example
Classification	msc 20B15

The group $\mathcal{D}_{2n}, n \geq 3$, the dihedral group of order $2n$, is the symmetry group of the regular n-gon. (Note that we use the more common notation $\mathcal{D}_{2n}$ for this group rather than $\mathcal{D}_n$).

$\mathcal{D}_{2n}$ is clearly not doubly transitive for $n \geq 4$, since it preserves "adjacency" in the vertices. Thus, for example, clearly no element of $\mathcal{D}_{2n}$ can take $(1,2)$ to $(1,3)$. ($\mathcal{D}_{2\cdot 3} = \mathcal{D}_6$, the symmetry group of the triangle, is, however, doubly transitive).

We show that for p prime, $\mathcal{D}_{2p}$ is primitive. To prove this, we need only verify that any block containing two distinct elements is the entire set of vertices. Number the vertices consecutively $\{0,\dots,p-1\}$, and let r be the element of $\mathcal{D}_{2n}$ that takes each vertex into its successor $\pmod{p}$. Now, suppose a block contains two distinct elements a,b; assume wlog that $b \neq 0$. Iteratively apply r^{b-a} to these elements to get

$$\begin{array}{cc} a & b \\ b & 2b-a \\ 2b-a & 3b-a \\ \cdots & \cdots \end{array}$$

Since blocks are either equal or disjoint, we see that the block in question contains a,b, and $nb-a$ for each n. But $a \neq b$, so $nb-a$ runs through all `http://planetmath.org/ResidueSystems`residues $\pmod{p}$ and thus the block contains each vertex. Thus D_{2p} is primitive.

For nonprime n, $\mathcal{D}_{2n}$ is not primitive. In this case, if d is a divisor of n, then the set of vertices that are multiples of d form a block.