derived subgroup

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Defines	commutator
Defines	derived series
Defines	second derived subgroup

Let G be a group. For any $a, b \in G$, the element $a^{-1}b^{-1}ab$ is called the *commutator of a and b*.

The commutator $a^{-1}b^{-1}ab$ is sometimes written $[a, b]$. (Usage varies, however, and some authors instead use $[a, b]$ to represent the commutator $aba^{-1}b^{-1}$.) If A and B are subsets of G, then $[A, B]$ denotes the subgroup of G generated by $\{[a, b] \mid a \in A \text{ and } b \in B\}$. This notation can be further extended by recursively defining $[X_1, \ldots, X_{n+1}] = [[X_1, \ldots, X_n], X_{n+1}]$ for subsets $X_1, \ldots, X_{n+1}$ of G.

The subgroup of G generated by all the commutators in G (that is, the smallest subgroup of G containing all the commutators) is called the *derived subgroup*, or the *commutator subgroup*, of G. Using the notation of the previous paragraph, the derived subgroup is denoted by $[G, G]$. Alternatively, it is often denoted by G', or sometimes $G^{(1)}$.

Note that a and b commute if and only if the commutator of $a, b \in G$ is trivial, i.e.,

$$a^{-1}b^{-1}ab = 1.$$

Thus, in a fashion, the derived subgroup measures the degree to which a group fails to be abelian.

Proposition 1 *The derived subgroup $[G, G]$ is normal (in fact, fully invariant) in G, and the factor group $G/[G, G]$ is abelian. Moreover, G is abelian if and only if $[G, G]$ is the trivial subgroup.*

The factor group $G/[G, G]$ is the largest abelian `http://planetmath.org/QuotientGroup`quotie of G, and is called the abelianization of G.

One can of course form the derived subgroup of the derived subgroup; this is called the *second derived subgroup*, and denoted by G'' or $G^{(2)}$. Proceeding inductively one defines the n^{th} derived subgroup $G^{(n)}$ as the derived subgroup of $G^{(n-1)}$. In this fashion one obtains a sequence of subgroups, called the *derived series* of G:

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

Proposition 2 *The group G is solvable if and only if the derived series terminates in the trivial group $\{1\}$ after a `http://planetmath.org/Finite`finite number of steps.*

The derived series can also be continued transfinitely—see the article on the transfinite derived series.