pure subgroup

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Defines	pure submodule
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Definition. A *pure subgroup* H of an abelian group G is

1. a subgroup of G, such that
2. $H \cap mG = mH$ for all $m \in \mathbb{Z}$.

The second condition says that for any $h \in H$ such that $h = ma$ for some integer m and some $a \in G$, then there exists $b \in H$ such that $h = mb$. In other words, if h is divisible in G by an integer, then it is divisible in H by that same integer. Purity in abelian groups is a relative notion, and we denote $H <_p G$ to mean that H is a pure subgroup of G.

Examples. All groups mentioned below are abelian groups.

1. For any group, two trivial examples of pure subgroups are the trivial subgroup and the group itself.
2. Any `http://planetmath.org/DivisibleGroup`divisible subgroup or any direct summand of a group is pure.
3. *The* torsion subgroup (= the subgroup of all torsion elements) of any group is pure.
4. If $K <_p H$, $H <_p G$, then $K <_p G$.
5. If $H = \bigcup_{i=1}^{\infty} H_i$ with $H_i \leq H_{i+1}$ and $H_i <_p G$, then $H <_p G$.
6. In Z_{n^2}, $\langle n \rangle$ is an example of a subgroup that is not pure.
7. In general, $\langle m \rangle <_p Z_n$ if $\gcd(s,t) = 1$, where $s = \gcd(m,n)$ and $t = n/s$.

Remark. This definition can be generalized to modules over commutative rings.

Definition. Let R be a commutative ring and $\mathcal{E} \colon 0 \to A \to B \to C \to 0$ a short exact sequence of R-modules. Then $\mathcal{E}$ is said to be *pure* if it remains exact after tensoring with any R-module. In other words, if D is any R-module, then

$$D \otimes \mathcal{E} \colon 0 \to D \otimes A \to D \otimes B \to D \otimes C \to 0,$$

is exact.

Definition. Let N be a submodule of M over a ring R. Then N is said to be a *pure submodule* of M if the exact sequence

$$0 \to N \to M \to M/N \to 0$$

is a pure exact sequence.

From this definition, it is clear that H is a pure subgroup of G iff H is a pure $\mathbb{Z}$-submodule of G.

Remark. N is a pure submodule of M over R iff whenever a finite sum

$$\sum r_i m_i = n \in N,$$

where $m_i \in M$ and $r_i \in R$ implies that

$$n = \sum r_i n_i$$

for some $n_i \in N$. As a result, if I is an ideal of R, then the purity of N in M means that $N \cap IM = IN$, which is a generalization of the second condition in the definition of a pure subgroup above.