an associative quasigroup is a group

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Proposition 1. *Let G be a set and $\cdot$ a binary operation on G. Write ab for $a \cdot b$. The following are equivalent:*

1. *$(G, \cdot)$ is an associative quasigroup.*
2. *$(G, \cdot)$ is an associative loop.*
3. *$(G, \cdot)$ is a group.*

Proof. We will prove this in the following direction (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2)**.** Let $x \in G$, and $e_1, e_2 \in G$ such that $xe_1 = x = e_2x$. So $xe_1^2 = xe_1 = x$, which shows that $e_1^2 = e_1$. Let $a \in G$ be such that $e_1a = x$. Then $e_2e_1a = e_2x = x = e_1a$, so that $e_2e_1 = e_1 = e_1^2$, or $e_2 = e_1$. Set $e = e_1$. For any $y \in G$, we have $ey = e^2y$, so $y = ey$. Similarly, $ye = ye^2$ implies $y = ye$. This shows that e is an identity of G.

(2) $\Rightarrow$ (3)**.** First note that all of the group axioms are automatically satisfied in G under $\cdot$, except the existence of an (two-sided) inverse element, which we are going to verify presently. For every $x \in G$, there are unique elements y and z such that $xy = zx = e$. Then $y = ey = (zx)y = z(xy) = ze = z$. This shows that x has a unique two-sided inverse $x^{-1} := y = z$. Therefore, G is a group under $\cdot$.

(3) $\Rightarrow$ (1)**.** Every group is clearly a quasigroup, and the binary operation is associative.

This completes the proof. □

Remark. In fact, if $\cdot$ on G is flexible, then every element in G has a unique inverse: for $z(xz) = (zx)z = ez = z = ze$, so by left division (by z), we get $xz = e = xy$, and therefore $z = y$, again by left division (by x). However, G may no longer be a group, because associativity may longer hold.