groups with abelian inner automorphism group

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The inner automorphism group $\mathrm{Inn}(G)$ is isomorphic to the central quotient of G, $G/Z(G)$. if $\mathrm{Inn}(G)$ is abelian, one cannot conclude that G itself is abelian. For example, let $G = \mathcal{D}_8$, the dihedral group of symmetries of the square.

$$G = \langle r, s \mid r^4 = s^2 = 1, rs = sr^3 \rangle$$

and $Z(G) = \{1, r^2\}$. Representatives of the cosets of $Z(G)$ are $\{1, r, s, rs\}$; these define a group of order 4 that is isomorphic to the `http://planetmath.org/Klein4Group`Klein 4-group V_4. Thus the central quotient is abelian, but the group itself is not.

However, if the central quotient is *cyclic*, it does follow that G is abelian. For, choose a representative x in G of a generator for $G/Z(G)$. Each element of G is thus of the form $x^a z$ for $z \in Z(G)$. So given $g, h \in G$,

$$gh = x^{a_1} z_1 x^{a_2} z_2 = x^{a_1} x^{a_2} z_1 z_2 = x^{a_1+a_2} z_1 z_2 = x^{a_2} x^{a_1} z_2 z_1 = x^{a_2} z_2 x^{a_1} z_1 = hg$$

where the various manipulations are justified by the fact that the $z_i \in Z(G)$ and that powers of x commute with themselves.

Finally, note that if $\mathrm{Inn}(G)$ is non-trivial, then G is nonabelian, for $\mathrm{Inn}(G)$ nontrivial implies that for some $g \in G$, conjugation by g is not the identity, so there is some element of G with which g does not commute. So by the above argument, $\mathrm{Inn}(G)$, if non-trivial, cannot be cyclic (else G would be abelian).