examples of infinite simple groups

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Let X be a set and let $f : X \to X$ be a function. Define

$$C(f) = \{x \in X | f(x) \neq x\}.$$

Throughout, we will say that $f : X \to X$ is a *permutation on* X iff f is a bijection and $C(f)$ is a finite set.

For permutation $f : X \to X$, the set $C(f)$ will play the role of a ,,bridge" between the infinite world and the finite world.

Let $S(X)$ denote the group of all permutations on X (with composition as a multiplication). For $f \in S(X)$, subset $A \subset X$ will be called *f-finite* iff A is finite and $C(f) \subseteq A$. This is equivalent to the fact, that A is finite and if $f(x) \neq x$, then $x \in A$.

It is easy to see, that if $f \in S(X)$ and A is f-finite, then $f(A) = A$. Thus, we have well defined permutation (on a finite set) $f_A : A \to A$ by the formula $f_A(x) = f(x)$.

Lemma. For any subset $A \subseteq X$ and any $f, g \in S(X)$ such that A is f-finite and g-finite we have that A is $f \circ g$-finite and

$$(f \circ g)_A = f_A \circ g_A.$$

Proof. Assume, that A is f-finite and g-finite. Let $x \in X$ be such that $(f \circ g)(x) \neq x$. Assume, that $x \notin A$. Then $f(x) = g(x) = x$ and thus $(f \circ g)(x) = x$. Contradiction. Thus $x \in A$, so $C(f \circ g) \subseteq A$ and since A is finite, then A is $f \circ g$-finite. Finally, the equality

$$(f \circ g)_A = f_A \circ g_A$$

holds, because $(f \circ g)_A$ is well definied (since A is $f \circ g$-finite) and the operation $(\cdot)_A$ does not change the formulas of functions. $\square$

Now we can talk about the sign of a permutation. For $f \in S(X)$ define

$$\mathrm{sgn}(f) = \mathrm{sgn}(f_A).$$

It can be easily checked, that sgn is well defined (indeed, sign depends only on those $x \in X$ for which $f(x) \neq x$). Furthermore, it follows directly from the definition, that

$$\mathrm{sgn} : S(X) \to \{-1, 1\}$$

is a group homomorphism (in $\{-1, 1\}$ we have standard multiplication). Define

$$A(X) = \ker(\mathrm{sgn}).$$

Briefly speaking, $A(X)$ is the subgroup of even permutations on a set X (a.k.a. the alternating group for the set X).

Now, we shall prove the following proposition, using the fact, that for any finite set X with at least 5 elements, the group $A(X)$ is simple (this is well known fact).

Proposition. If X is an infinite set, then $A(X)$ is a simple group.

Proof. Assume, that $A(X)$ is not simple and let $N \subseteq A(X)$ be a proper, nontrivial, normal subgroup. For a subset $Y \subseteq X$ define

$$N_Y = \{f_Y \mid f \in N \text{ and } C(f) \subseteq Y\}.$$

Note, that

$$A(Y) = \{f_Y \mid f \in A(X) \text{ and } C(f) \subseteq Y\}.$$

Obviously $N_Y \subseteq A(Y)$ is a subgroup (due to lemma) of $A(Y)$. We will show, that it is normal. Let $f_Y \in N_Y$ and $g_Y \in A(Y)$. We have to show, that $g_Y \circ f_Y \circ g_Y^{-1} \in N_Y$. Of course

$$g \circ f \circ g^{-1} \in N,$$

because N is normal (here f, g correspond to f_Y, g_Y). It follows from lemma (note, that Y is $g \circ f \circ g^{-1}$-finite), that

$$g_Y \circ f_Y \circ g_Y^{-1} = (g \circ f \circ g^{-1})_Y \in N_Y,$$

which shows, that N_Y is normal. To obtain the contradiction, we need to show, that there exists $Y \subseteq X$ with at least 5 elements, such that N_Y is nontrivial and proper (because in this case $A(Y)$ is simple).

Let $f \in N$ be such that $f \neq \mathrm{id}_X$ and let $g \in A(X)$ be such that $g \notin N$. Let Y be any f-finite and g-finite subset of X with at least 5 elements (such subset exists). Then N_Y is nontrivial, because $f_Y \in N_Y$ is nontrivial.

Now assume, that $g_Y \in N_Y$, i.e. assume, that there exists $h \in N$ with $C(h) \subseteq Y$, such that $g_Y = h_Y$. Then (due to lemma) Y is $h \circ g^{-1}$-finite, and since $g_Y = h_Y$ we have that for any $x \in Y$ the following holds:

$$(h \circ g^{-1})(x) = x.$$

On the other hand, for $x \in X \backslash Y$ we have $g(x) = h(x) = x$. This shows, that $h = g$, but $h \in N$ and $g \notin N$. Contradiction. Thus $g_Y \notin N_Y$, so N_Y is proper.

This completes the proof. □

Remark. This proposition shows, that the class of simple groups is actually a proper class, i.e. it is not a set. Therefore studying infinite simple groups can be very difficult.