semidirect product of groups

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The goal of this exposition is to carefully explain the correspondence between the notions of external and internal semi–direct products of groups, as well as the connection between semi–direct products and short exact sequences.

Naturally, we start with the construction of semi–direct products.

Definition 1. Let H and Q be groups and let $\theta : Q \longrightarrow \mathrm{Aut}(H)$ be a group homomorphism. The *semi–direct product* $H \rtimes_\theta Q$ is defined to be the group with underlying set $\{(h, q) \mid h \in H,\ q \in Q\}$ and group operation $(h, q)(h', q') := (h\theta(q)h', qq')$.

We leave it to the reader to check that $H \rtimes_\theta Q$ is really a group. It helps to know that the inverse of (h, q) is $(\theta(q^{-1})(h^{-1}), q^{-1})$.

For the remainder of this article, we omit θ from the notation whenever this map is clear from the context.

Set $G := H \rtimes Q$. There exist canonical monomorphisms $H \longrightarrow G$ and $Q \longrightarrow G$, given by

$$\begin{aligned} h &\mapsto (h, 1_Q), \qquad && h \in H \\ q &\mapsto (1_H, q), \qquad && q \in Q \end{aligned}$$

where 1_H (resp. 1_Q) is the identity element of H (resp. Q). These monomorphisms are so natural that we will treat H and Q as subgroups of G under these inclusions.

Theorem 2. *Let $G := H \rtimes Q$ as above. Then:*

- *H is a normal subgroup of G.*
- *$HQ = G$.*
- *$H \cap Q = \{1_G\}$.*

Proof. Let $p : G \longrightarrow Q$ be the projection map defined by $p(h, q) = q$. Then p is a homomorphism with kernel H. Therefore H is a normal subgroup of G.

Every $(h, q) \in G$ can be written as $(h, 1_Q)(1_H, q)$. Therefore $HQ = G$.

Finally, it is evident that $(1_H, 1_Q)$ is the only element of G that is of the form $(h, 1_Q)$ for $h \in H$ and $(1_H, q)$ for $q \in Q$. $\square$

This result motivates the definition of internal semi–direct products.

Definition 3. Let G be a group with subgroups H and Q. We say G is the *internal semi–direct product* of H and Q if:

- H is a normal subgroup of G.
- $HQ = G$.
- $H \cap Q = \{1_G\}$.

We know an external semi–direct product is an internal semi–direct product (Theorem **??**). Now we prove a converse (Theorem **??**), namely, that an internal semi–direct product is an external semi–direct product.

Lemma 4. *Let G be a group with subgroups H and Q. Suppose $G = HQ$ and $H \cap Q = \{1_G\}$. Then every element g of G can be written uniquely in the form hq, for $h \in H$ and $q \in Q$.*

Proof. Since $G = HQ$, we know that g can be written as hq. Suppose it can also be written as $h'q'$. Then $hq = h'q'$ so $h'^{-1}h = q'q^{-1} \in H \cap Q = \{1_G\}$. Therefore $h = h'$ and $q = q'$. $\square$

Theorem 5. *Suppose G is a group with subgroups H and Q, and G is the internal semi–direct product of H and Q. Then $G \cong H \rtimes_\theta Q$ where $\theta : Q \longrightarrow \mathrm{Aut}(H)$ is given by*

$$\theta(q)(h) := qhq^{-1}, \; q \in Q, \, h \in H.$$

Proof. By Lemma **??**, every element g of G can be written uniquely in the form hq, with $h \in H$ and $q \in Q$. Therefore, the map $\phi : H \rtimes Q \longrightarrow G$ given by $\phi(h, q) = hq$ is a bijection from G to $H \rtimes Q$. It only remains to show that this bijection is a homomorphism.

Given elements (h, q) and (h', q') in $H \rtimes Q$, we have

$$\phi((h, q)(h', q')) = \phi((h\theta(q)(h'), qq')) = \phi(hqh'q^{-1}, qq') = hqh'q' = \phi(h, q)\phi(h', q').$$

Therefore ϕ is an isomorphism. $\square$

Consider the external semi–direct product $G := H \rtimes_\theta Q$ with subgroups H and Q. We know from Theorem **??** that G is isomorphic to the external semi–direct product $H \rtimes_{\theta'} Q$, where we are temporarily writing θ' for the conjugation map $\theta'(q)(h) := qhq^{-1}$ of Theorem **??**. But in fact the two maps θ and θ' are the same:

$$\theta'(q)(h) = (1_H, q)(h, 1_Q)(1_H, q^{-1}) = (\theta(q)(h), 1_Q) = \theta(q)(h).$$

In summary, one may use Theorems **??** and **??** to pass freely between the notions of internal semi–direct product and external semi–direct product.

Finally, we discuss the correspondence between semi–direct products and split exact sequences of groups.

Definition 6. An exact sequence of groups

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} Q \longrightarrow 1.$$

is *split* if there exists a homomorphism $k : Q \longrightarrow G$ such that $j \circ k$ is the identity map on Q.

Theorem 7. *Let G, H, and Q be groups. Then G is isomorphic to a semi–direct product $H \rtimes Q$ if and only if there exists a split exact sequence*

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} Q \longrightarrow 1.$$

Proof. First suppose $G \cong H \rtimes Q$. Let $i : H \longrightarrow G$ be the inclusion map $i(h) = (h, 1_Q)$ and let $j : G \longrightarrow Q$ be the projection map $j(h, q) = q$. Let the splitting map $k : Q \longrightarrow G$ be the inclusion map $k(q) = (1_H, q)$. Then the sequence above is clearly split exact.

Now suppose we have the split exact sequence above. Let $k : Q \longrightarrow G$ be the splitting map. Then:

- $i(H) = \ker j$, so $i(H)$ is normal in G.
- For any $g \in G$, set $q := k(j(g))$. Then $j(gq^{-1}) = j(g)j(k(j(g)))^{-1} = 1_Q$, so $gq^{-1} \in \operatorname{Im} i$. Set $h := gq^{-1}$. Then $g = hq$. Therefore $G = i(H)k(Q)$.
- Suppose $g \in G$ is in both $i(H)$ and $k(Q)$. Write $g = k(q)$. Then $k(q) \in \operatorname{Im} i = \ker j$, so $q = j(k(q)) = 1_Q$. Therefore $g = k(q) = k(1_Q) = 1_G$, so $i(H) \cap k(Q) = \{1_G\}$.

This proves that G is the internal semi–direct product of $i(H)$ and $k(Q)$. These are isomorphic to H and Q, respectively. Therefore G is isomorphic to a semi–direct product $H \rtimes Q$. □

Thus, not all normal subgroups $H \subset G$ give rise to an (internal) semi–direct product $G = H \rtimes G/H$. More specifically, if H is a normal subgroup of G, we have the canonical exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1.$$

We see that G can be decomposed into $H \rtimes G/H$ as an internal semi–direct product if and only if the canonical exact sequence splits.