examples of semigroups

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Examples of semigroups are numerous. This entry presents some of the most common examples.

1. The set $\mathbb{Z}$ of integers with multiplication is a semigroup, along with many of its subsets (subsemigroups):
   (a) The set of non-negative integers
   (b) The set of positive integers
   (c) $n\mathbb{Z}$, the set of all integral multiples of an integer n
   (d) For any prime p, the set of $\{p^i \mid i \geq n\}$, where n is a non-negative integer
   (e) The set of all composite integers
2. $\mathbb{Z}_n$, the set of all integers modulo an integer n, with integer multiplication modulo n. Here, we may find examples of nilpotent and idempotent elements, relative inverses, and eventually periodic elements:
   (a) If $n = p^m$, where p is prime, then every non-zero element containing a factor of p is nilpotent. For example, if $n = 16$, then $6^4 = 0$.
   (b) If $n = 2p$, where p is an odd prime, then p is a non-trivial idempotent element ($p^2 = p$), and since $2^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem, we see that $a = 2^{p-2}$ is a relative inverse of 2, as $2 \cdot a \cdot 2 = 2$ and $a \cdot 2 \cdot a = a$
   (c) If $n = 2^m p$, where p is an odd prime, and $m > 1$, then 2 is eventually periodic. For example, $n = 96$, then $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 32$, $2^8 = 64$, etc...
3. The set $M_n(R)$ of $n \times n$ square matrices over a ring R, with matrix multiplication, is a semigroup. Unlike the previous two examples, $M_n(R)$ is not commutative.
4. The set $E(A)$ of functions on a set A, with functional composition, is a semigroup.
5. Every group is a semigroup, as well as every monoid.
6. If R is a ring, then R with the ring multiplication (ignoring addition) is a semigroup (with 0).

7. *Group with Zero.* A semigroup S is called a *group with zero* if it contains a zero element 0, and $S - \{0\}$ is a subgroup of S. In R in the previous example is a division ring, then R with the ring multiplication is a group with zero. If G is a group, by adjoining G with an extra symbol 0, and extending the domain of group multiplication $\cdot$ by defining $0 \cdot a = a \cdot 0 = 0 \cdot 0 := 0$ for all $a \in G$, we get a group with zero $S = G \cup \{0\}$.

8. As mentioned earlier, every monoid is a semigroup. If S is not a monoid, then it can be embedded in one: adjoin a symbol 1 to S, and extend the semigroup multiplication $\cdot$ on S by defining $1 \cdot a = a \cdot 1 = a$ and $1 \cdot 1 = 1$, we get a monoid $M = S \cup \{1\}$ with multiplicative identity 1. If S is already a monoid with identity 1, then adjoining $1'$ to S and repeating the remaining step above gives us a new monoid with identity $1'$. However, 1 is no longer an identity, as $1' = 1 \cdot 1'$.