proof that all cyclic groups of the same order are isomorphic to each other

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The following is a proof that all cyclic groups of the same order are isomorphic to each other.

Proof. Let G be a cyclic group and g be a generator of G. Define $\varphi\colon \mathbb{Z} \to G$ by $\varphi(c) = g^c$. Since $\varphi(a+b) = g^{a+b} = g^a g^b = \varphi(a)\varphi(b)$, φ is a group homomorphism. If $h \in G$, then there exists $x \in \mathbb{Z}$ such that $h = g^x$. Since $\varphi(x) = g^x = h$, φ is surjective.

Note that $\ker\varphi = \{c \in \mathbb{Z} : \varphi(c) = e_G\} = \{c \in \mathbb{Z} : g^c = e_G\}$.

If G is infinite, then $\ker\varphi = \{0\}$, and φ is injective. Hence, φ is a group isomorphism, and $G \cong \mathbb{Z}$.

If G is finite, then let $|G| = n$. Thus, $|g| = |\langle g\rangle| = |G| = n$. If $g^c = e_G$, then n divides c. Therefore, $\ker\varphi = n\mathbb{Z}$. By the first isomorphism theorem, $G \cong \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$.

Let H and K be cyclic groups of the same order. If H and K are infinite, then, by the above , $H \cong \mathbb{Z}$ and $K \cong \mathbb{Z}$. If H and K are finite of order n, then, by the above , $H \cong \mathbb{Z}_n$ and $K \cong \mathbb{Z}_n$. In any case, it follows that $H \cong K$. $\square$