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quaternion group

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The quaternion group, or quaternionic group, is a noncommutative group with eight elements. It is traditionally denoted by Q (not to be confused with $\mathbb{Q}$) or by Q_8. This group is defined by the presentation

$$\{i, j; i^4, i^2j^2, iji^{-1}j\}$$

or, equivalently, defined by the multiplication table

$\cdot$	1	i	j	k	$-i$	$-j$	$-k$	-1
1	1	i	j	k	$-i$	$-j$	$-k$	-1
i	i	-1	k	$-j$	1	$-k$	j	$-i$
j	j	$-k$	-1	i	k	1	$-i$	$-j$
k	k	j	$-i$	-1	$-j$	i	1	$-k$
$-i$	$-i$	1	$-k$	j	-1	k	$-j$	i
$-j$	$-j$	k	1	$-i$	$-k$	-1	i	j
$-k$	$-k$	$-j$	i	1	j	$-i$	-1	k
-1	-1	$-i$	$-j$	$-k$	i	j	k	1

where we have put each product xy into row x and column y. The minus signs are justified by the fact that $\{1, -1\}$ is subgroup contained in the center of Q. Every subgroup of Q is normal and, except for the trivial subgroup $\{1\}$, contains $\{1, -1\}$. The dihedral group D_4 (the group of symmetries of a square) is the only other noncommutative group of order 8.

Since $i^2 = j^2 = k^2 = -1$, the elements i, j, and k are known as the imaginary units, by analogy with $i \in \mathbb{C}$. Any pair of the imaginary units generate the group. Better, given $x, y \in \{i, j, k\}$, any element of Q is expressible in the form $x^m y^n$.

Q is identified with the group of units (invertible elements) of the ring of quaternions over $\mathbb{Z}$. That ring is not identical to the group ring $\mathbb{Z}[Q]$, which has dimension 8 (not 4) over $\mathbb{Z}$. Likewise the usual quaternion algebra is not quite the same thing as the group algebra $\mathbb{R}[Q]$.

Quaternions were known to Gauss in 1819 or 1820, but he did not publicize this discovery, and quaternions weren't rediscovered until 1843, with Hamilton.