abelian group is divisible if and only if it is an injective object

Canonical name	AbelianGroupIsDivisibleIfAndOnlyIfItIsAnInjectiveObject
Date of creation	2013-03-22 18:48:15
Last modified on	2013-03-22 18:48:15
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	8
Author	joking (16130)
Entry type	Theorem
Classification	msc 20K99

Proposition. Abelian group A is divisible if and only if A is an injective object in the category of abelian groups.

Proof. ,,$\Leftarrow$" Assume that A is not divisible, i.e. there exists $a \in A$ and $n \in \mathbb{N}$ such that the equation $nx = a$ has no solution in A. Let $B =< a >$ be a cyclic subgroup generated by a and $i : B \to A$ the canonical inclusion. Now there are two possibilities: either B is finite or infinite.

If B is infinite, then let $H = \mathbb{Z}$ and let $f : B \to H$ be defined on generator by $f(a) = n$. Now A is injective, thus there exists $h : H \to A$ such that $h \circ f = i$. Thus

$$n \cdot h(1) = h(1) + \cdots + h(1) = h(1 + \cdots + 1) = h(n) = h(f(a)) = i(a) = a.$$

Contradiction with definition of $n \in \mathbb{N}$ and $a \in A$.

If B is finite, then let $k = |B|$ (note that n does not divide k) and let $H = \mathbb{Z}_{n \cdot k}$. Furtheremore define $f : B \to H$ on generator by $f(a) = n$ (note that in this case f is a well defined homomorphism). Again injectivity of A implies existence of $h : H \to A$ such that $h \circ f = i$. Similarly we get contradiction:

$$n \cdot h(1) = h(1) + \cdots + h(1) = h(1 + \cdots + 1) = h(n) = h(f(a)) = i(a) = a.$$

This completes first implication.

,,$\Rightarrow$" This implication is proven `http://planetmath.org/ExampleOfInjectiveModule`here.
$\square$

Remark. It is clear that in the category of abelian groups $\mathcal{AB}$, a group A is projective if and only if A is free. This is since $\mathcal{AB}$ is equivalent to the category of $\mathbb{Z}$-modules and projective modules are direct summands of free modules. Since $\mathbb{Z}$ is a principal ideal domain, then every submodule of a free module is free, thus projective $\mathbb{Z}$-modules are free.