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free product with amalgamated subgroup

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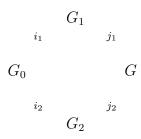
Synonym amalgamated free product Related topic AmalgamationProperty Related topic CategoricalPullback

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Defines pushout of groups

Definition 1. Let G_k , k = 0, 1, 2 be groups and i_k : $G_0 \to G_i$, k = 1, 2 be monomorphisms. The free product of G_1 and G_2 with amalgameted subgroup G_0 , is defined to be a group G that has the following two properties

1. there are homomorphisms $j_k \colon G_k \to G$, k = 1, 2 that make the following diagram commute



2. G is universal with respect to the previous property, that is for any other group G' and homomorphisms $j'_k \colon G_k \to G'$, k = 1, 2 that fit in such a commutative diagram there is a unique homomorphism $G \to G'$ so that the following diagram commutes

It follows by "general nonsense" that the free product of G_1 and G_2 with amalgamated subgroup G_0 , if it exists, is "unique up to unique isomorphism." The free product of G_1 and G_2 with amalgamated subgroup G_0 , is denoted by $G_1 \bigstar_{G_0} G_2$. The following theorem asserts its existence.

Theorem 2. $G_1 \bigstar_{G_0} G_2$ exists for any groups G_k , k = 0, 1, 2 and monomorphisms $i_k : G_0 \to G_i$, k = 1, 2.

Sketch of proof. Without loss of generality assume that G_0 is a subgroup of G_k and that i_k is the inclusion for k = 1, 2. Let

$$G_k = \langle (x_{k;s})_{s \in S} \mid (r_{k;t})_{t \in T} \rangle$$

be a presentation of G_k for k=1,2. Each $g \in G_0$ can be expressed as a word in the generators of G_k ; denote that word by $w_k(g)$ and let N be the normal closure of $\{w_1(g)w_2(g)^{-1} \mid g \in G_0\}$ in the http://planetmath.org/FreeProductfree product $G_1 \bigstar G_2$. Define

$$G_1 \bigstar_{G_0} G_2 := G_1 \bigstar G_2/N$$

and for k = 0, 1 define j_k to be the inclusion into the free product followed by the canonical projection. Clearly (1) is satisfied, while (2) follows from the universal properties of the free product and the quotient group.

Notice that in the above proof it would be sufficient to divide by the relations $w_1(g)w_2(g)^{-1}$ for g in a generating set of G_0 . This is useful in practice when one is interested in obtaining a presentation of $G_1 \bigstar_{G_0} G_2$.

In case that the i_k 's are not injective the above still goes through verbatim. The group thusly obtained is called a "pushout".

Examples of free products with amalgamated subgroups are provided by Van Kampen's theorem.