unity

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Entry type	Definition
Classification	msc 20-00
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Synonym	multiplicative identity
Synonym	characterization of unity
Related topic	ZeroDivisor
Related topic	RootOfUnity
Related topic	ZeroRing
Related topic	NonZeroDivisorsOfFiniteRing
Related topic	OppositePolynomial
Defines	non-zero unity
Defines	nonzero unity

The *unity* of a ring $(R, +, \cdot)$ is the multiplicative identity of the ring, if it has such. The unity is often denoted by e, u or 1. Thus, the unity satisfies

$$e \cdot a \;=\; a \cdot e \;=\; a \quad \forall a \in R.$$

If R consists of the mere 0, then 0 is its unity, since in every ring, $0 \cdot a = a \cdot 0 = 0$. Conversely, if 0 is the unity in some ring R, then $R = \{0\}$ (because $a = 0 \cdot a = 0 \;\forall a \in R$).

Note. When considering a ring R it is often mentioned that "...having $1 \neq 0$" or that "...with non-zero unity", sometimes only "...with unity" or "...with "; all these exclude the case $R = \{0\}$.

Theorem. An element u of a ring R is the unity iff u is an idempotent and regular element.

Proof. Let u be an idempotent and regular element. For any element x of R we have

$$ux \;=\; u^2x \;=\; u(ux),$$

and because u is no left zero divisor, it may be cancelled from the equation; thus we get $x = ux$. Similarly, $x = xu$. So u is the unity of the ring. The other half of the is apparent.