# transitive actions are primitive if and only if stabilizers are maximal subgroups

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**Theorem 1.** *If $G$ is transitive on the set $A$, then $G$ is primitive on $A$ if and only if for each $a \in A$, $G_a$ is a maximal subgroup of $G$. Here $G_a = \mathrm{Stab}_G(a)$ is the stabilizer of $a \in A$.*

*Proof.* First claim that if $G$ is transitive on $A$ and $B \subset A$ is a `http://planetmath.org/BlockSyste` with $a \in B$, then $G_B = \{\sigma \in G \mid \sigma(B) = B\}$ is a subgroup of $G$ containing $G_a$. It is obvious that $G_B$ is a subgroup, since

$$\sigma \in G_B \Rightarrow \sigma(B) = B \Rightarrow \sigma^{-1}(\sigma(B)) = \sigma^{-1}(B) \Rightarrow B = \sigma^{-1}(B) \Rightarrow \sigma^{-1} \in G_B$$
$$\sigma, \tau \in G_B \Rightarrow (\sigma\tau)(B) = \sigma(\tau(B)) = \sigma(B) = B \Rightarrow \sigma\tau \in G_B$$

But also, if $\sigma \in G_a$ for $a \in B$, then $\sigma(a) = a$, so $\sigma(B) \cap B \neq \emptyset$ and thus $\sigma(B) = B$ since $B$ is a block system and thus $\sigma \in G_B$. This proves the claim.

To prove the theorem, note that for each $a \in A$, there is by the claim a $1 - 1$ correspondence between  containing $a$ and subgroups of $G$ containing $G_a$. Thus, $G$ is primitive on $A$ if and only if all blocks are either of size 1 or equal to $A$, if and only if any group containing $G_a$ is either $G_a$ itself or $G$, if and only if for all $a \in A$, $G_a$ is maximal in $G$. $\square$