groups in field

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If $(K, +, \cdot)$ is a field, then

- $(K, +)$ is the *additive group of the field*,
- $(K \setminus \{0\}, \cdot)$ is the *multiplicative group of the field*.

Both of these groups are Abelian.

The former has always as a subgroup

$$\{n \cdot 1 \colon n \in \mathbb{Z}\},$$

the group of the multiples of unity. This is, apparently, isomorphic to the additive group $\mathbb{Z}$ or $\mathbb{Z}_p$ depending on whether the http://planetmath.org/Characteristiccharacter of the field is 0 or a prime number p.

The multiplicative group of any field has as its subgroup the set E consisting of all roots of unity in the field. The group E has the subgroup $\{1, -1\}$ which reduces to $\{1\}$ if the of the field is two.

Example 1. The additive group $(\mathbb{R}, +)$ of the reals is isomorphic to the multiplicative group $(\mathbb{R}_+, \cdot)$ of the positive reals; the isomorphy is implemented e.g. by the isomorphism mapping $x \mapsto 2^x$.

Example 2. Suppose that the of K is not 2 and denote the multiplicative group of K by K^*. We can consider the four functions $f_i\colon K^* \to K^*$ defined by $f_0(x) := x,\ f_1(x) := -x,\ f_2(x) := x^{-1},\ f_3(x) := -x^{-1}$. The composition of functions is a binary operation of the set $G = \{f_0, f_1, f_2, f_3\}$, and we see that G is isomorphic to Klein's 4-group.

Note. One may also speak of the *additive group* of any *ring*. Every ring contains also its group of units.