abelian groups of order 120

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Here we present an application of the fundamental theorem of finitely generated abelian groups.

Example (Abelian groups of order 120):

Let G be an abelian group of order $n = 120$. Since the group is finite it is obviously finitely generated, so we can apply the theorem. There exist $n_1, n_2, \ldots, n_s$ with

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_s\mathbb{Z}$$

$$\forall i, n_i \geq 2; \quad n_{i+1} \mid n_i \text{ for } 1 \leq i \leq s-1$$

Notice that in the case of a finite group, r, as in the statement of the theorem, must be equal to 0. We have

$$n = 120 = 2^3 \cdot 3 \cdot 5 = \prod_{i=1}^{s} n_i = n_1 \cdot n_2 \cdot \ldots \cdot n_s$$

and by the divisibility properties of n_i we must have that every prime divisor of n must divide n_1. Thus the possibilities for n_1 are the following

$$2 \cdot 3 \cdot 5, \quad 2^2 \cdot 3 \cdot 5, \quad 2^3 \cdot 3 \cdot 5$$

If $n_1 = 2^3 \cdot 3 \cdot 5 = 120$ then $s = 1$. In the case that $n_1 = 2^2 \cdot 3 \cdot 5$ then $n_2 = 2$ and $s = 2$. It remains to analyze the case $n_1 = 2 \cdot 3 \cdot 5$. Now the only possibility for n_2 is 2 and $n_3 = 2$ as well.

Hence if G is an abelian group of order 120 it must be (**up to isomorphism**) one of the following:

$$\mathbb{Z}/120\mathbb{Z}, \quad \mathbb{Z}/60\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Also notice that they are all non-isomorphic. This is because

$$\mathbb{Z}/(n \cdot m)\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \Leftrightarrow \gcd(n, m) = 1$$

which is due to the Chinese Remainder theorem.