properties of group commutators and commutator subgroups

Canonical name	PropertiesOfGroupCommutatorsAndCommutatorSubgroups
Date of creation	2013-03-22 15:30:50
Last modified on	2013-03-22 15:30:50
Owner	GrafZahl (9234)
Last modified by	GrafZahl (9234)
Numerical id	11
Author	GrafZahl (9234)
Entry type	Theorem
Classification	msc 20F12
Related topic	NormalSubgroup
Defines	Hall-Witt identity
Defines	three subgroup lemma

The purpose of this entry is to collect properties of http://planetmath.org/node/2812group commutators and commutator subgroups. Feel free to add more theorems!

Let G be a group.

Theorem 1. *Let $x, y \in G$, then $[x,y]^{-1} = [y,x]$.*

Proof. Direct computation yields

$$[x,y]^{-1} = (x^{-1}y^{-1}xy)^{-1} = y^{-1}x^{-1}yx = [y,x].$$

□

Theorem 2. *Let X, Y be subsets of G, then $[X,Y] = [Y,X]$.*

Proof. By Theorem **??**, the elements from $[X,Y]$ or $[Y,X]$ are products of commutators of the form $[x,y]$ or $[y,x]$ with $x \in X$ and $y \in Y$. □

Theorem 3 (Hall–Witt identity)**.** *Let $x, y, z \in G$, then*

$$y^{-1}[x,y^{-1},z]yz^{-1}[y,z^{-1},x]zx^{-1}[z,x^{-1},y]x = 1.$$

Proof. This is mainly a brute-force calculation. We can easily calculate the first factor $y^{-1}[x,y^{-1},z]y$ explicitly using theorem **??**:

$$\begin{aligned}
&y^{-1}[x,y^{-1},z]y \\
=&y^{-1}[y^{-1},x]z^{-1}[x,y^{-1}]zy \\
=&y^{-1}yx^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy \\
=&x^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy.
\end{aligned}$$

Let $h_1 := x^{-1}y^{-1}xz^{-1}x^{-1}$, the "first half" of $y^{-1}[x,y^{-1},z]y$. Let h_2 be the element obtained from h_1 by the cyclic shift $S\colon x \mapsto y \mapsto z \mapsto x$, and h_3 be the element obtained from h_2 by S. We have

$$h_2^{-1} = (y^{-1}z^{-1}yx^{-1}y^{-1})^{-1} = yxy^{-1}zy$$

which gives us

$$y^{-1}[x,y^{-1},z]y = h_1h_2^{-1},$$

and, by applying S twice

$$\begin{aligned}
z^{-1}[y,z^{-1},x]z &= h_2h_3^{-1}, \\
x^{-1}[z,x^{-1},y]x &= h_3h_1^{-1}.
\end{aligned}$$

In total, we have

$$y^{-1}[x, y^{-1}, z]yz^{-1}[y, z^{-1}, x]zx^{-1}[z, x^{-1}, y]x = h_1h_2^{-1}h_2h_3^{-1}h_3h_1^{-1} = 1.$$

□

Theorem 4 (Three subgroup lemma)**.** *Let N be a normal subgroup of G. Furthermore, let X, Y and Z be subgroups of G, such that $[X, Y, Z]$ and $[Y, Z, X]$ are contained in N. Then $[Z, X, Y]$ is contained in N as well.*

Proof. The group $[Z, X, Y]$ is generated by all elements of the form $[z, x^{-1}, y]$ with $x \in X$, $y \in Y$ and $z \in Z$. Since N is normal, $y^{-1}[x, y^{-1}, z]y$ and $x^{-1}[z, x^{-1}, y]x$ are elements of N. The Hall–Witt identity then implies that $x^{-1}[z, x^{-1}, y]x$ is an element of N as well. Again, since N is normal, $[z, x^{-1}, y] \in N$ which concludes the proof. □

Theorem 5. *For any $x, y, z \in G$ we have*

$$\begin{aligned}
[xy, z] &= [x, z]^y[y, z] \\
[x, yz] &= [x, z][x, y]^z \\
[x, y]^z &= [x^z, y^z] \\
[x^z, y] &= [x, y^{z^{-1}}]
\end{aligned}$$

where a^b denotes $b^{-1}ab$

Proof. By expanding:

$$\begin{aligned}
[xy, z] &= y^{-1}x^{-1}z^{-1}xyz \\
&= y^{-1}x^{-1}z^{-1} \cdot xz \cdot z^{-1}x^{-1} \cdot xyz \\
&= y^{-1}[x, z] \cdot y \cdot y^{-1} \cdot z^{-1}x^{-1} \cdot xyz \\
&= [x, z]^y \cdot y^{-1}z^{-1}yz \\
&= [x, z]^y[y, z]
\end{aligned}$$

The other identities are proved similarly. □