full families of Hopfian (co-Hopfian) groups

Canonical name	FullFamiliesOfHopfiancoHopfianGroups
Date of creation	2013-03-22 18:36:05
Last modified on	2013-03-22 18:36:05
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	7
Author	joking (16130)
Entry type	Theorem
Classification	msc 20A99

Proposition. Let $\{G_i\}_{i\in I}$ be a full family of groups. Then each G_i is Hopfian (co-Hopfian) if and only if $\bigoplus_{i\in I} G_i$ is Hopfian (co-Hopfian).

Proof. ,,⇒" Let

$$f : \bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} G_i$$

be a surjective (injective) homomorphism. Since $\{G_i\}_{i\in I}$ is full, then there exists family of homomorphisms $\{f_i : G_i \to G_i\}_{i\in I}$ such that

$$f = \bigoplus_{i\in I} f_i.$$

Of course since f is surjective (injective), then each f_i is surjective (injective). Thus each f_i is an isomorphism, because each G_i is Hopfian (co-Hopfian). Therefore f is an isomorphism, because

$$f^{-1} = \bigoplus_{i\in I} f_i^{-1}. \ \square$$

,,⇐" Fix $j \in I$ and assume that $f_j : G_j \to G_j$ is a surjective (injective) homomorphism. For $i \in I$ such that $i \neq j$ define $f_i : G_i \to G_i$ to be any automorphism of G_i. Then

$$\bigoplus_{i\in I} f_i : \bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} G_i$$

is a surjective (injective) group homomorphism. Since $\bigoplus_{i\in I} G_i$ is Hopfian (co-Hopfian) then $\bigoplus_{i\in I} f_i$ is an isomorphism. Thus each f_i is an isomorphism. In particular f_j is an isomorphism, which completes the proof. $\square$

Example. Let $\mathcal{P} = \{p \in \mathbb{N} \mid p \text{ is prime}\}$ and $\mathcal{P}_0$ be any subset of $\mathcal{P}$. Then

$$\bigoplus_{p\in P_0} \mathbb{Z}_p$$

is both Hopfian and co-Hopfian.

Proof. It is easy to see that $\{\mathbb{Z}_p\}_{p\in\mathcal{P}}$ is full, so $\{\mathbb{Z}_p\}_{p\in\mathcal{P}_0}$ is also full. Moreover for any $p \in P_0$ the group $\mathbb{Z}_p$ is finite, so both Hopfian and co-Hopfian. Therefore (due to proposition)

$$\bigoplus_{p\in P_0} \mathbb{Z}_p$$

is both Hopfian and co-Hopfian. $\square$