nonabelian group

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A group is said to be *nonabelian*, or *noncommutative*, if has elements which do not commute, that is, if there exist a and b in the group such that $ab \neq ba$. Equivalently, a group is nonabelian if there exist a and b in the group such that the commutator $[a, b]$ is not equal to the identity of the group. There exist many natural nonabelian groups, with order as small as 6. While any group for which the square map is a homomorphism is abelian, there exist nonabelian groups of order as small as 27 for which the cube map is a homomorphism.

In the first section we give a way to visualize the group of rotations of a sphere and prove that it is nonabelian. This should be readable by an undergraduate student in algebra. In the second section, we discuss groups admitting a cube map and show that there are small nonabelian examples. The second section is somewhat more technical than the first and will require more facility with group theory, especially working with finitely presented groups and the commutator calculus.

1 Concrete examples of nonabelian groups

Although most number systems we use are abelian by design, there exist quite natural nonabelian groups. Perhaps the simplest example to visualize is given by the group of rotations of a sphere.[1] We can compose two rotations by performing them in sequence, and we can invert a rotation by rotating in the opposite direction, so rotations do form a group. To follow what rotation does to the sphere, imagine that inside is suspended a copy of Marshall Hall's classic text *The Theory of Groups*. We will keep track of three pieces of information, namely, the directions that the front cover, the spine, and the bottom of the book face. When the sphere is in the identity position, the front cover faces the reader, the spine faces the left, and the bottom of the book is oriented downward.

In preparation for verifying that the group is not abelian, we define two rotations, F and R. First, let F (for "flip") be the rotation which takes the point at the very top of the sphere and moves it forward through an angle of π. For example, if we start with the sphere in the identity position and

[1] The treatment we give here is informal. For a more formal treatment of the group of rotations, consult the entries "`http://planetmath.org/RotationMatrix`Rotation matrix" and "`http://planetmath.org/DimensionOfTheSpecialOrthogonalGroup`Dimension of the special orthogonal group".

then perform F, the front cover will face away from the reader, the spine will remain to the left, and the bottom of the book will be oriented upward. Second, let R (for "rotate") be the rotation which takes the point at the very top of the sphere and moves it left through an angle of $\frac{\pi}{2}$. If we start with the sphere in the identity position and then perform R, the front cover will continue to face the reader, the spine will face downward, and the bottom of the book will be oriented to the right.

We now verify that the group of rotations is not abelian. If we start with the sphere in the identity position and perform FR, that is, first F, then R, then the front cover will face away from the reader, the spine will face downward, and the bottom will be oriented to the left. On the other hand, if we start with the sphere in the identity position and perform RF, then while the front cover will face away from the reader, the spine will face *upward*, and the bottom will be oriented to the *right*. So it matters in which order we perform F and R, that is, $FR \neq RF$, proving that the group is not abelian.

Since every rotation in three-dimensional Euclidean space can be decomposed as a finite sequence of reflections and rotations in the Euclidean plane, one might hope that we can find finite nonabelian groups arising from objects in the plane, and in fact we can. For each regular polygon, there is an associated group, the dihedral group D_{2n}, which is the group of symmetries of the polygon. (Here n denotes the number of the sides of the polygon, and $2n$ gives the number of elements of the group of symmetries.) It is generated by two elements, F (for "flip") and R (for "rotate"). These elements can be defined by analogy with the F and R above; for full details, consult the entry "`http://planetmath.org/DihedralGroup`Dihedral group", where flips are labelled instead by M (for "mirror"). If $n \geq 3$ (so we are dealing with an actual polygon here), it is possible to show that $FR \neq RF$. Moreover, every group with order 1, p, or p^2, where p is a prime, is abelian. Thus the smallest possible order for a nonabelian group is 6. But $D_{2 \cdot 3}$ has 6 elements and is nonabelian, so it is the smallest possible nonabelian group.

2 Small nonabelian groups admitting a cube map

When we say that a group admits $x \mapsto x^n$, we mean that the function φ defined on the group by the formula $\varphi(x) = x^n$ is a homomorphism, that is, that is, that for any x and y in the group,

$$(xy)^n = \varphi(xy) = \varphi(x)\varphi(y) = x^n y^n.$$

If a group admits $x \mapsto x^2$, then for any x and y we have that $(xy)^2 = x^2y^2$. Multiplying on the left by x^{-1} and on the right by y^{-1} yields the identity $yx = xy$. Thus all such groups are abelian. Moreover, the generalized commutativity and associativity laws for abelian groups imply that an abelian group admits all maps $x \mapsto x^n$. It is therefore reasonable to wonder whether the converse holds. In fact it is possible for a nonabelian group to admit $x \mapsto x^3$. The smallest order for such a group is 27. It is beyond the scope of this entry to prove that 27 is the smallest possible order, but we will give an explicit example.

Let G be the group with presentation

$$G = \langle a, b, c \mid a^3, b^3, c^3, [a,c], [b,c], [a,b]c\rangle.$$

This can be realized concretely as the group of upper-triangular matrices over $\mathbb{Z}/3\mathbb{Z}$ with 1s on the diagonal, but for simplicity we shall work directly with the presentation.

The first three relators tell us that each generator of the group has order 3. The next two tell us that c is central — since it commutes with the other two generators and commutes with itself, it must therefore commute with everything. The final relator is perhaps the most interesting. We can interpret it as the rewrite rule

$$ba \mapsto abc,$$

that is,

"when b moves past a it turns into bc."

Thus given an element of G we can always write it in the normal form $a^j b^k c^\ell$, where $0 \leq j, k, \ell < 3$, and all such elements are distinct. This proves that the cardinality of G is 27. Moreover, we also observe that

$$ba = abc \neq ab,$$

so G is not abelian.

It remains to check that G admits the cube map. We will prove the simpler statement that G has exponent 3. Given x in G, we first normalize it, so $x = a^j b^k c^\ell$. Since c is in the center of G,

$$x^3 = (a^j b^k)^3 c^{3\ell} = (a^j b^k)^3 = a^j (b^k a^j)^2 b^k.$$

To normalize the word $b^k a^j$, we push each b past all of the as. Since pushing b past a *single* a turns it into bc, pushing it past a^j turns it into bc^j, that is,

$$b^k a^j = b^{k-1} a^j b c^j.$$

By induction it follows that

$$b^k a^j = a^j b^k c^{jk}.$$

Applying this result to x^3, we get that

$$x^3 = a^j (a^j b^k c^{jk})^2 b^k = a^{2j} (b^k a^j) b^{2k} c^{2jk} = a^{3j} b^{3k} c^{3jk} = 1.$$

Since x^3 is trivial for any x, it follows that G admits the cube map.

The other nonabelian group of order 27 has exponent 9 and also admits the cube map. This will be described in an attached entry.