cycically reduced

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Defines	cyclic reduction
Defines	cyclic conjugation

Let $M(X)$ be a free monoid with involution $^{-1}$ on X. A word $w \in M(X)$ is said to be *cyclically reduced* if every cyclic conjugate of it is reduced. In other words, w is cyclically reduced iff w is a reduced word and that if $w = uvu^{-1}$ for some words u and v, then $w = v$.

For example, if $X = \{a, b, c\}$, then words such as

$$c^{-1}bc^2a \qquad \text{and} \qquad abac^2ba^2$$

are cyclically reduced, where as words

$$a^2bca^{-1} \qquad \text{and} \qquad cb^2b^3c$$

are not, the former is reduced, but of the form $a(abc)a^{-1}$, while the later is not even a reduced word.

Remarks. The concept of cyclically reduced words carries over to words in groups. We consider words in a group G.

- If a word is cyclically reduced, so is its inverse and all of its cyclic conjugates.
- A word v is a *cyclic reduction* of a word w if $w = uvu^{-1}$ for some word u, and v is cyclically reduced. Clearly, every word and its cyclic reduction are conjugates of each other. Furthermore, any word has a unique cyclic reduction.
- Every group G has a presentation $\langle S|R\rangle$ such that

  1. R is cyclically reduced (meaning every element of R is cyclically reduced),
  2. closed under inverses (meaning if $u \in R$, then $u^{-1} \in R$), and
  3. closed under cyclic conjugation (meaning any cyclic conjugate of an element in R is in R).

  Furthermore, if G is finitely presented, R above can be chosen to be finite.

  Proof. Every group G has a presentation $\langle S|R'\rangle$. There is an isomorphism from $F(S)/N(R')$ to G, where $F(S)$ is the free group freely generated by S, and $N(R')$ is the normalizer of the subset $R' \subseteq F(S)$ in $F(S)$. Let R'' be the set of all cyclic reductions of words in R'. Then

$N(R'') = N(R')$, since any word not cyclically reduced in R' is conjugate to its cyclic reduction in R'', and hence in $N(R'')$. Next, for each $u \in R''$, toss in its inverse and all of its cyclic conjugates. The resulting set R is still cyclically reduced. Furthermore, R satisfies the remaining conditions above. Finally, $N(R) = N(R'')$, as any cyclic conjugate v of a word w is clearly a conjugate of w. Therefore, G has presentation $\langle S|R\rangle$.

If G is finitely presented, then S and R' above can be chosen to be finite sets. Therefore, R'' and R are both finite. R is finite because the number of cyclic conjugates of a word is at most the length of the word, and hence finite. $\square$