the kernel of a group homomorphism is a normal subgroup

Canonical name	TheKernelOfAGroupHomomorphismIsANormalSubgroup
Date of creation	2013-03-22 17:20:34
Last modified on	2013-03-22 17:20:34
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Last modified by	alozano (2414)
Numerical id	8
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Entry type	Theorem
Classification	msc 20A05
Related topic	KernelOfAGroupHomomorphism
Related topic	NaturalProjection

In this entry we show the following simple lemma:

Lemma 1. *Let G and H be groups (with group operations $*_G$, $*_H$ and identity elements e_G and e_H, respectively) and let $\Phi : G \to H$ be a group homomorphism. Then, the kernel of Φ, i.e.*

$$\operatorname{Ker}(\Phi) = \{g \in G : \Phi(g) = e_H\},$$

is a normal subgroup of G.

Proof. Let G, H and Φ be as in the statement of the lemma and let $g \in G$ and $k \in \operatorname{Ker}(\Phi)$. Then, $\Phi(k) = e_H$ by definition and:

$$\begin{aligned}
\Phi(g *_G k *_G g^{-1}) &= \Phi(g) *_H \Phi(k) *_H \Phi(g^{-1}) \\
&= \Phi(g) *_H (e_H) *_H \Phi(g^{-1}) \\
&= \Phi(g) *_H \Phi(g^{-1}) \\
&= \Phi(g) *_H \Phi(g)^{-1} \\
&= e_H,
\end{aligned}$$

where we have used several times the properties of group homomorphisms and the properties of the identity element e_H. Thus, $\Phi(gkg^{-1}) = e_H$ and $gkg^{-1} \in G$ is also an element of the kernel of Φ. Since $g \in G$ and $k \in \operatorname{Ker}(\Phi)$ were arbitrary, it follows that $\operatorname{Ker}(\Phi)$ is normal in G. □

Conversely:

Lemma 2. *Let G be a group and let K be a normal subgroup of G. Then there exists a group homomorphism $\Phi : G \to H$, for some group H, such that the kernel of Φ is precisely K.*

Proof. Simply set H equal to the quotient group G/K and define $\Phi : G \to G/K$ to be the natural projection from G to G/K (i.e. Φ sends $g \in G$ to the coset gK). Then it is clear that the kernel of Φ is precisely formed by those elements of K. □

Although the first lemma is very simple, it is very useful when one tries to prove that a subgroup is normal.

Example. Let F be a field. Let us prove that the special linear group $\mathrm{SL}(n, F)$ is normal inside the general linear group $\mathrm{GL}(n, F)$, for all $n \geq 1$.

By the lemmas, it suffices to construct a homomorphism of $\mathrm{GL}(n, F)$ with $\mathrm{SL}(n, F)$ as kernel. The determinant of matrices is the homomorphism we are looking for. Indeed:

$$\det : \mathrm{GL}(n, F) \to F^{\times}$$

is a group homomorphism from $\mathrm{GL}(n, F)$ to the multiplicative group $F^{\times}$ and, by definition, the kernel is precisely $\mathrm{SL}(n, F)$, i.e. the matrices with determinant $= 1$. Hence, $\mathrm{SL}(n, F)$ is normal in $\mathrm{GL}(n, F)$.