proof of second isomorphism theorem for groups

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First, we shall prove that HK is a subgroup of G: Since $e \in H$ and $e \in K$, clearly $e = e^2 \in HK$. Take $h_1, h_2 \in H, k_1, k_2 \in K$. Clearly $h_1k_1, h_2k_2 \in HK$. Further,

$$h_1k_1h_2k_2 = h_1(h_2h_2^{-1})k_1h_2k_2 = h_1h_2(h_2^{-1}k_1h_2)k_2$$

Since K is a normal subgroup of G and $h_2 \in G$, then $h_2^{-1}k_1h_2 \in K$. Therefore $h_1h_2(h_2^{-1}k_1h_2)k_2 \in HK$, so HK is closed under multiplication.

Also, $(hk)^{-1} \in HK$ for $h \in H$, $k \in K$, since

$$(hk)^{-1} = k^{-1}h^{-1} = h^{-1}hk^{-1}h^{-1}$$

and $hk^{-1}h^{-1} \in K$ since K is a normal subgroup of G. So HK is closed under inverses, and is thus a subgroup of G.

Since HK is a subgroup of G, the normality of K in HK follows immediately from the normality of K in G.

Clearly $H \cap K$ is a subgroup of G, since it is the intersection of two subgroups of G.

Finally, define $\phi\colon H \to HK/K$ by $\phi(h) = hK$. We claim that ϕ is a surjective homomorphism from H to HK/K. Let h_0k_0K be some element of HK/K; since $k_0 \in K$, then $h_0k_0K = h_0K$, and $\phi(h_0) = h_0K$. Now

$$\ker(\phi) = \{h \in H \mid \phi(h) = K\} = \{h \in H \mid hK = K\}$$

and if $hK = K$, then we must have $h \in K$. So

$$\ker(\phi) = \{h \in H \mid h \in K\} = H \cap K$$

Thus, since $\phi(H) = HK/K$ and $\ker \phi = H \cap K$, by the First Isomorphism Theorem we see that $H \cap K$ is normal in H and that there is a canonical isomorphism between $H/(H \cap K)$ and HK/K.