alternating group is a normal subgroup of the symmetric group

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Theorem 1. *The alternating group A_n is a normal subgroup of the symmetric group S_n*

Proof. Define the epimorphism $f : S_n \to \mathbb{Z}_2$ by : $\sigma \mapsto 0$ if σ is an even permutation and : $\sigma \mapsto 1$ if σ is an odd permutation. Hence, A_n is the kernel of f and so it is a normal subgroup of the domain S_n. Furthermore $S_n/A_n \cong \mathbb{Z}_2$ by the first isomorphism theorem. So by Lagrange's theorem

$$|S_n| = |A_n||S_n/A_n|.$$

Therefore, $|A_n| = n!/2$. That is, there are $n!/2$ many elements in A_n $\square$

Remark. What we have shown in the theorem is that, in fact, A_n has index 2 in S_n. In general, if a subgroup H of G has index 2, then H is normal in G. (Since $[G : H] = 2$, there is an element $g \in G - H$, so that $gH \cap H = \varnothing$ and thus $gH = Hg$).