finite nilpotent groups

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The study of finite nilpotent groups mostly centers around the study of p-groups. This is because of the following two theorems.

Theorem 1. `http://planetmath.org/Finite`*Finite p-groups are nilpotent.*

Proof. From the class equation we know the center of a finite p-group is non-trivial. Thus by induction the upper central series of a p-group P terminates at P. So P is nilpotent. $\square$

Example. Infinite p-groups may not always be nilpotent. In the extreme there are counterexamples like the Tarski monsters T_p. These are infinite p-groups in which every proper subgroup has order p. Therefore given any two non-trivial elements x, y in which $y \notin \langle x \rangle$ generate T_p. In particular, the only central element is 1 so that the upper central series is trivial and therefore T_p is not nilpotent.

Indeed, Tarski monsters are not in fact solvable groups which is a weaker property than nilpotent. $\square$

Example. Some infinite p-groups are nilpotent. Indeed, some infinite p-groups are even abelian such as $\mathbb{Z}_p^\infty$ – the countable dimension vector space over the field $\mathbb{Z}_p$ – and the Prüfer group $\mathbb{Z}_{p^\infty}$ – the inductive limit of $\mathbb{Z}_{p^n}$. $\square$

Theorem 2. *Let G be a finite group. Then all the following are equivalent.*

1. *G is nilpotent.*
2. *Every Sylow subgroup of G is normal.*
3. *For every prime $p \big| |G|$, there exists a unique Sylow p-subgroup of G.*
4. *G is the direct product of its Sylow subgroups.*

For the proof recal the following consequence of the Sylow theorems:

Proposition 3. *If G is a finite group and P a Sylow p-subgroup of G then*

$$N_G(N_G(P)) = N_G(P).$$

(See Subgroups Containing The Normalizers Of Sylow Subgroups Normalize Themselves)

Now we prove Theorem **??**

Proof. (**??**) implies (**??**). Suppose that G is nilpotent and that P is a Sylow p-subgroup of G. Then as G is nilpotent, every subgroup of G is subnormal in G, meaning, if H is properly contained in G then $N_G(H)$ properly contains H. Thus $N_G(N_G(P))$ is larger than $N_G(P)$ or $N_G(P) = G$. However because P is a Sylow p-subgroup we know $N_G(P) = N_G(N_G(P))$ so we conclude $N_G(P) = G$. Therefore every Sylow p-subgroup of G is normal in G.

(**??**) implies (**??**). Suppose every Sylow subgroup of G is normal in G. Then by the Sylow theorems we know that for every prime p dividing $|G|$ there is exactly one Sylow p-subgroup of G – as all Sylow p-subgroups are conjugate and here by assumption all are also normal.

(**??**) implies (**??**). Suppose that there is a unique Sylow p-subgroup of G for every $p \big| |G|$. Then by the Sylow theorems every Sylow subgroup of G is normal in G. Furthermore, if P and Q are two distinct Sylow subgroups then they they are Sylow subgroups for different primes so that by Lagrange's theorem their intersection is trivial. Let $P_1, \ldots, P_k$ the Sylow subgroups of G. Then as each P_i is normal in G we have $G = P_1 \cdots P_k$ and we have also demonstrated $P_1 \cdots P_i \cap P_{i+1} = 1$ for $2 \leq i \leq k$ therefore G is the direct product of $P_1, \ldots, P_k$.

(**??**) implies (**??**). Suppose that G is a product of its Sylow subgroups. Then as every Sylow subgroup is a p-group, G is a product of nilpotent groups so G itself is nilpotent. $\square$