semilattice decomposition of a semigroup

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A semigroup S has a *semilattice decomposition* if we can write $S = \bigcup_{\gamma\in\Gamma} S_\gamma$ as a disjoint union of subsemigroups, indexed by elements of a semilattice Γ, with the additional condition that $x \in S_\alpha$ and $y \in S_\beta$ implies $xy \in S_{\alpha\beta}$.

Semilattice decompositions arise from homomorphims of semigroups onto semilattices. If $\phi\colon S \to \Gamma$ is a surjective homomorphism, then it is easy to see that we get a semilattice decomposition by putting $S_\gamma = \phi^{-1}(\gamma)$ for each $\gamma \in \Gamma$. Conversely, every semilattice decomposition defines a map from S to the indexing set Γ which is easily seen to be a homomorphism.

A third way to look at semilattice decompositions is to consider the congruence ρ defined by the homomorphism $\phi\colon S \to \Gamma$. Because Γ is a semilattice, $\phi(x^2) = \phi(x)$ for all x, and so ρ satisfies the constraint that $x \mathrel{\rho} x^2$ for all $x \in S$. Also, $\phi(xy) = \phi(yx)$ so that $xy \mathrel{\rho} yx$ for all $x, y \in S$. A congruence ρ which satisfies these two conditions is called a *semilattice congruence*.

Conversely, a semilattice congruence ρ on S gives rise to a homomorphism from S to a semilattice S/ρ. The ρ-classes are the components of the decomposition.