proof of Cauchy's theorem

Canonical name	ProofOfCauchysTheorem
Date of creation	2013-03-22 12:23:30
Last modified on	2013-03-22 12:23:30
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	8
Author	yark (2760)
Entry type	Proof
Classification	msc 20E07
Classification	msc 20D99

Let G be a finite group, and suppose p is a prime divisor of $|G|$. Consider the set X of all p-tuples $(x_1, \ldots, x_p)$ for which $x_1 \cdots x_p = 1$. Note that $|X| = |G|^{p-1}$ is a multiple of p. There is a natural group action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on X under which $m \in \mathbb{Z}/p\mathbb{Z}$ sends the tuple $(x_1, \ldots, x_p)$ to $(x_{m+1}, \ldots, x_p, x_1, \ldots, x_m)$. By the Orbit-Stabilizer Theorem, each orbit contains exactly 1 or p tuples. Since $(1, \ldots, 1)$ has an orbit of cardinality 1, and the orbits partition X, the cardinality of which is divisible by p, there must exist at least one other tuple $(x_1, \ldots, x_p)$ which is left fixed by every element of $\mathbb{Z}/p\mathbb{Z}$. For this tuple we have $x_1 = \ldots = x_p$, and so $x_1^p = x_1 \cdots x_p = 1$, and x_1 is therefore an element of order p.

References

[1] James H. McKay. *Another Proof of Cauchy's Group Theorem*, American Math. Monthly, 66 (1959), p119.