centralizer of a k-cycle

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Owner	rm50 (10146)
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Author	rm50 (10146)
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Theorem 1. *Let σ be a k-cycle in S_n. Then the centralizer of σ is*

$$C_{S_n}(\sigma) = \{\sigma^i \tau \mid 0 \leq i \leq k-1, \tau \in S_{n-k}\}$$

where S_{n-k} is the subgroup of S_n consisting of those permutations that fix all elements appearing in σ.

Proof. This is fundamentally a counting argument. It is clear that σ commutes with each element in the set given, since σ commutes with powers of itself and also commutes with disjoint permutations. The size of the given set is $k \cdot (n-k)!$. However, the number of conjugates of σ is the index of $C_{S_n}(\sigma)$ in S_n by the orbit-stabilizer theorem, so to determine $|C_{S_n}(\sigma)|$ we need only count the number of conjugates of σ, i.e. the number of k-cycles.

In a k-cycle $(a_1 \ \ldots \ a_k)$, there are n choices for a_1, $n-1$ choices for a_2, and so on. So there are $n(n-1)\cdots(n-k+1)$ choices for the elements of the cycle. But this counts each cycle k times, depending on which element appears as a_1. So the number of k-cycles is

$$\frac{n(n-1)\cdots(n-k+1)}{k}$$

Finally,

$$n! = |S_n| = \frac{n(n-1)\cdots(n-k+1)}{k}|C_{S_n}(\sigma)|$$

so that

$$|C_{S_n}(\sigma)| = \frac{k \cdot n!}{n(n-1)\cdots(n-k+1)} = k \cdot (n-k)!$$

and we see that the elements in the list above must exhaust $C_{S_n}(\sigma)$. $\square$