modules over decomposable rings

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Let R_1, R_2 be two, nontrivial, unital rings and $R = R_1 \oplus R_2$. If M_1 is a R_1-module and M_2 is a R_2-module, then obviously $M_1 \oplus M_2$ is a R-module via $(r,s)\cdot(m_1,m_2) = (r\cdot m_1, s\cdot m_2)$. We will show that every R-module can be obtain in this way.

Proposition. If M is a R-module, then there exist submodules $M_1, M_2 \subseteq M$ such that $M = M_1 \oplus M_2$ and for any $r \in R_1$, $s \in R_2$, $m_1 \in M_1$ and $m_2 \in M_2$ we have

$$(r,s)\cdot m_1 = (r,0)\cdot m_1 \qquad (r,s)\cdot m_2 = (0,s)\cdot m_2,$$

i.e. ring action on M_1 (respectively M_2) does not depend on R_2 (respectively R_1).

Proof. Let $e = (1,0) \in R$ and $f = (0,1) \in R$. Of course both e, f are idempotents and $(1,1) = e + f$. Moreover $ef = fe = 0$ and e, f are central, i.e. $e, f \in \{c \in R \mid \forall_{x\in R}\ cx = xc\}$. We will use e, f to construct submodules M_1, M_2. More precisely, let $M_1 = eM$ and $M_2 = fM$. Because e, f are central, then it is clear that both M_1 and M_2 are submodules. We will show that $M_1 + M_2 = M$. Indeed, let $m \in M$. Then we have

$$m = (1,1)\cdot m = (e+f)\cdot m = e\cdot m + f\cdot m.$$

Thus $M_1 + M_2 = M$. Furthermore, assume that $m \in M_1 \cap M_2$. Then there exist $m_1, m_2 \in M$ such that

$$e\cdot m_1 = m = f\cdot m_2$$

and therefore

$$e\cdot m_1 - f\cdot m_2 = 0.$$

Now, after multiplying both sides by e we obtain that

$$0 = (ee)\cdot m_1 - (ef)\cdot m_2 = e\cdot m_1 - 0\cdot m_2 = e\cdot m_1 = m,$$

thus $M_1 \cap M_2 = 0$. This shows that $M = M_1 \oplus M_2$. To finish the proof, we need to show that the ring action on M_1 does not depend on R_2 (the other case is analogous). But this is clear, since for any $(r,s) \in R$ and $m \in M$ we have

$$(r,s)\cdot(e\cdot m) = \big((r,s)(1,0)\big)\cdot m = (r,0)\cdot m = \big((r,0)(1,0)\big)\cdot m = (r,0)\cdot(e\cdot m).$$

This completes the proof. $\square$