proof of general associativity

Canonical name	ProofOfGeneralAssociativity
Date of creation	2014-05-11 15:12:33
Last modified on	2014-05-11 15:12:33
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	10
Author	pahio (2872)
Entry type	Proof
Classification	msc 20-00
Related topic	PowerSet
Related topic	Cardinality

We suppose that "$\cdot$" is an associative binary operation of the set S.

Let $f_1\colon\ S \to 2^S$ be the mapping

$$f_1(a_1) \;:=\; \{a_1\} \quad \forall a_1 \in S.$$

We define recursively the mapping

$$f_n\colon\ \underbrace{S \times S \times \ldots \times S}_{n} \to 2^S$$

such that

$$f_n(a_1, \ldots, a_n) \;:=\; \bigcup_{r=1}^{n-1} f_r(a_1, \ldots, a_r) \cdot f_{n-r}(a_{r+1}, \ldots, a_n) \tag{1}$$

for $n = 2,\, 3,\, 4,\, \ldots$

For example,

$$f_2(a_1, a_2) \;=\; \{a_1\} \cdot \{a_2\} \;=\; \{a_1 \cdot a_2\},$$

$$f_3(a_1, a_2, a_3) \;=\; \{a_1{\cdot}(a_2{\cdot}a_3)\} \cup \{(a_1{\cdot}a_2){\cdot}a_3\} \;=\; \{a_1{\cdot}(a_2{\cdot}a_3),\, (a_1{\cdot}a_2){\cdot}a_3\} \;=\; \{(a_1{\cdot}a_2){\cdot}a_3\};$$

the last equality due to the associativity. It's clear that always

$$|f_1(a_1)| \;=\; 1, \qquad |f_2(a_1, a_2)| \;=\; 1, \qquad |f_3(a_1, a_2, a_3)| \;=\; 1.$$

We shall show by induction that

$$|f_n(a_1, a_2, \ldots, a_n)| \;=\; 1 \tag{2}$$

for each positive integer n. This means that all groupings of the n fixed elements using parentheses in forming the products with "$\cdot$" yield one single element.

We make the induction hypothesis, that (2) is true for all $n < k$.

Now let z and z' be arbitrary elements of $f_k(a_1, \ldots, a_k)$. Then there exist the elements x, y, x', y' of S and the integers $r, s \in \{1, \ldots, k{-}1\}$ such that

$$z \;=\; x \cdot y, \quad x \in f_r(a_1, \ldots, a_r), \quad y \in f_{k-r}(a_{r+1}, \ldots, a_k),$$

$$z' \;=\; x' \cdot y', \quad x' \in f_s(a_1, \ldots, a_s), \quad y' \in f_{k-s}(a_{s+1}, \ldots, a_k).$$

If specially $r = s$, then, by the induction hypothesis, $x = x'$ and $y = y'$, whence $z = x \cdot y = x' \cdot y' = z'$. If on the contrary, $r \neq s$, e.g. $r < s$, then the induction hypothesis guarantees the existence of an element v of S such that

$$f_{s-r}(a_{r+1}, \ldots, a_s) = \{v\}$$

and

$$x' = x \cdot v, \quad y = v \cdot y'.$$

Since "$\cdot$" is associative, we have

$$z = x \cdot y = x \cdot (v \cdot y') = (x \cdot v) \cdot y' = x' \cdot y' = z'.$$

Thus the equation (2) is in force for $n = k$.