group socle

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The *socle* of a group is the subgroup generated by all minimal normal subgroups. Because the product of normal subgroups is a subgroup, it follows we can remove the word "generated" and replace it by "product." So the socle of a group is now the product of its minimal normal subgroups. This description can be further refined with a few observations.

Proposition 1. *If M and N are minimal normal subgroups then M and N centralize each other.*

Proof. Given two distinct minimal normal subgroup M and N, $[M, N]$ is contained in N and M as both are normal. Thus $[M, N] \leq M \cap N$. But M and N are distinct minimal normal subgroups and $M \cap N$ is normal so $M \cap N = 1$ thus $[M, N] = 1$. □

Proposition 2. *The socle of a finite group is a direct product of minimal normal subgroups.*

Proof. Let S be the socle of G. We already know S is the product of its minimal normal subgroups, so let us assume $S = N_1 \cdots N_k$ where each N_i is a distinct minimal normal subgroup of G. Thus $N_1 \cap N_2 = 1$ and $N_1 N_2$ clearly contains N_1 and N_2. Now suppose we extend this to a subsquence $N_{i_1} = N_1, N_{i_2} = N_2, N_{i_3}, \ldots, N_{i_j}$ where

$$N_{i_k} \cap (N_{i_1} \cdots N_{i_{k-1}}) = 1$$

for $1 \leq k < j$ and $N_i \leq N_{i_1} \cdots N_{i_j}$ for all $1 \leq i \leq i_j$. Then consider N_{i_j+1}.

As N_{i_j+1} is a minimal normal subgroup and $N_{i_1} \cdots N_{i_j}$ is a normal subgroup, N_{i_j+1} is either contained in $N_{i_1} \cdots N_{i_j}$ or intersects trivially. If N_{i_j+1} is contained in $N_{i_1} \cdots N_{i_j}$ then skip to the next N_i, otherwise set it to be $N_{i_{j+1}}$. The result is a squence $N_{i_1}, \ldots, N_{i_j}$ of minimal normal subgroups where $S = N_{i_1} \cdots N_{i_s}$ and

$$N_{i_j} \cap (N_{i_1} \cdots N_{i_{j-1}}) = 1, \quad 1 \leq j \leq s.$$

As we have already seen distinct minimal normal subgroups centralize each other we conclude that $S = N_{i_1} \times \cdots \times N_{i_s}$. □

Proposition 3. *A minimal normal subgroup is characteristically simple, so if it is finite then it is a product of isomorphic simple groups.*

Proof. If M is a minimal normal subgroup of G and $1 < C < M$ is characteristic in M, then C is normal in G which contradicts the minimality of M. Thus M is characteristically simple. □

Corollary 4. *The socle of a finite group is a direct product of simple groups.*

Proof. As each N_{i_j} is characteristically simple each N_{i_j} is a direct product of isomorphic simple groups, thus S is a direct product simple groups. □