congruence

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Defines	quotient semigroup

Let S be a semigroup. An equivalence relation $\sim$ defined on S is called a *congruence* if it is preserved under the semigroup operation. That is, for all $x, y, z \in S$, if $x \sim y$ then $xz \sim yz$ and $zx \sim zy$.

If $\sim$ satisfies only $x \sim y$ implies $xz \sim yz$ (resp. $zx \sim zy$) then $\sim$ is called a *right congruence* (resp. *left congruence*).

Example. *Suppose $f : S \to T$ is a semigroup homomorphism. Define $\sim$ by $x \sim y$ iff $f(x) = f(y)$. Then it is easy to see that $\sim$ is a congruence.*

If $\sim$ is a congruence, defined on a semigroup S, write $[x]$ for the equivalence class of x under $\sim$. Then it is easy to see that $[x] \cdot [y] = [xy]$ is a well-defined operation on the set of equivalence classes, and that in fact this set becomes a semigroup with this operation. This semigroup is called the *quotient of S by $\sim$* and is written $S/\sim$.

Thus semigroup are related to homomorphic images of semigroups in the same way that normal subgroups are related to homomorphic images of groups. More precisely, in the group case, the congruence is the coset relation, rather than the normal subgroup itself.