proof of counting theorem

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Let N be the cardinality of the set of all the couples (g, x) such that $g \cdot x = x$. For each $g \in G$, there exist $\mathrm{stab}_g(X)$ couples with g as the first element, while for each x, there are $|G_x|$ couples with x as the second element. Hence the following equality holds:

$$N = \sum_{g \in G} \mathrm{stab}_g(X) = \sum_{x \in X} |G_x|.$$

From the orbit-stabilizer theorem it follows that:

$$N = |G| \sum_{x \in X} \frac{1}{|G(x)|}.$$

Since all the x belonging to the same orbit $G(x)$ contribute with

$$|G(x)| \frac{1}{|G(x)|} = 1$$

in the sum, then $\sum_{x \in X} 1/|G(x)|$ precisely equals the number of distinct orbits s. We have therefore

$$\sum_{g \in G} \mathrm{stab}_g(X) = |G| s,$$

which proves the theorem.