proof that dimension of complex irreducible representation divides order of group

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Theorem Let G be a finite group and V an irreducible complex representation of finite dimension d. Then d divides $|G|$.

Proof: Given any α in the group ring of G (denoted $\mathbb{Z}G$) we may define a sequence of submodules of $\mathbb{Z}G$ (regarded as a module over $\mathbb{Z}$) by A_i equals the $\mathbb{Z}$ linear span of $\{1, \alpha, \alpha^2, \cdots, \alpha^i\}$.

$\mathbb{Z}G$ is Noetherian as a module over $\mathbb{Z}$ so we must have $A_i = A_{i-1}$ for some i. Hence α^i may be expressed as a $\mathbb{Z}$ linear combination of lower powers of α. In other α solves a monic polynomial of degree i with coefficients in $\mathbb{Z}$.

Given a conjugacy class C in G, we may set $\phi_C = \sum_{g \in C} g$. Then ϕ_C is central in $\mathbb{Z}G$, as given $h \in G$, we have:

$$\phi_C h = h \sum_{g \in C} h^{-1} g h = h \sum_{g \in C} g = h \phi_C$$

Hence applying ϕ_C to V induces a $\mathbb{C}G$ linear map $V \to V$. By Schur's lemma this must be multiplication by some complex number λ_C. Then λ_C is an algebraic integer as it solves the same monic polynomial as ϕ_C.

Also any $g \in G$ has finite order so the map it induces on V must have eigenvalues which are roots of unity and hence algebraic integers. Hence the sum of the eigenvalues, $\chi_V(g)$, must also be an algebraic integer.

Now V is irreducible so,

$$|G| = \sum_{g \in G} \chi_V(g) \chi_V(g)^* = \sum_{C \subset G} \operatorname{tr}(\phi_C) \chi_V(C)^* = d \sum_{C \subset G} \lambda_C \chi_V(C)^*$$

Therefore $|G|/d$ is both rational and an algebraic integer. Hence it is an integer and d divides $|G|$.