blocks of permutation groups

Canonical name	BlocksOfPermutationGroups
Date of creation	2013-03-22 17:19:05
Last modified on	2013-03-22 17:19:05
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	15
Author	rm50 (10146)
Entry type	Topic
Classification	msc 20B05
Defines	trivial block
Defines	block
Defines	block system
Defines	conjugate block

Throughout this article, A is a set and G is a permutation group on A.

A *block* is a subset B of A such that for each $\sigma \in G$, either $\sigma \cdot B = B$ or $(\sigma \cdot B) \cap B = \emptyset$, where $\sigma \cdot B = \{\sigma(b) \mid b \in B\}$. In other words, if $\sigma \cdot B$ intersects B, then $\sigma \cdot B = B$.

Note that for any such permutation group, each of $\emptyset$, A, and every element of A is a block. These are called *trivial blocks*.

It is obvious that if $H \subset G$ are permutation groups on A, then any block of G is also a block of H.

Blocks are closed under finite intersection:

Theorem. *If $B_1, B_2 \subset A$ are blocks of G, then $B = B_1 \cap B_2$ is a block of G.*

Proof. Choose $\sigma \in G$. Note that $\sigma \cdot (B_1 \cap B_2) = (\sigma \cdot B_1) \cap (\sigma \cdot B_2)$. Thus if $(\sigma \cdot B) \cap B \neq \emptyset$, then

$$(\sigma \cdot B) \cap B = (\sigma \cdot (B_1 \cap B_2)) \cap (B_1 \cap B_2) = (\sigma \cdot B_1 \cap B_1) \cap (\sigma \cdot B_2 \cap B_2)$$

is nonempty, and thus $\sigma \cdot B_i \cap B_i \neq \emptyset$ for $i = 1, 2$. But B_1 and B_2 are blocks, so that $\sigma \cdot B_i = B_i$ for $i = 1, 2$. Thus

$$\sigma \cdot B = \sigma \cdot (B_1 \cap B_2) = (\sigma \cdot B_1) \cap (\sigma \cdot B_2) = B_1 \cap B_2 = B$$

and B is a block. □

We show, as a corollary to the following theorem, that blocks themselves are permuted by the action of the group.

Theorem. *If $H \subset G$ are permutation groups on A, $B \subset A$ is a block of H, and $\sigma \in G$, then $\sigma \cdot B$ is a block of $\sigma H \sigma^{-1}$.*

Proof. Choose $\tau \in H$ and assume that

$$((\sigma \tau \sigma^{-1})\sigma \cdot B) \cap \sigma \cdot B \neq \emptyset$$

Then, applying σ^{-1} to this equation, we see that

$$(\tau \cdot B) \cap B \neq \emptyset$$

But B is a block of H, so $\tau \cdot B = B$. Multiplying by σ, we see that

$$\sigma \cdot (\tau \cdot B) = \sigma \cdot B$$

and thus

$$(\sigma \tau \sigma^{-1})\sigma \cdot B = \sigma \cdot B$$

and the result follows. □

Corollary. *If B is a block of G, $\sigma \in G$, then $\sigma \cdot B$ is also a block of G.*

Proof. Set $G = H$ in the above theorem. $\square$

Definition. If B is a block of G, $\sigma \in G$, then B and $\sigma \cdot B$ are *conjugate blocks*. The set of all blocks conjugate to a given block is a *block system*.

It is clear from the fact that B is a block that conjugate blocks are either equal or disjoint, so the action of G permutes the blocks of G. Then if G acts transitively on A, the union of any nontrivial block and its conjugates is A.

Theorem. *If G is finite and G acts transitively on A, then the size of a nonempty block divides the order of G.*

Proof. Since G acts transitively, A is finite as well. All conjugates of the block have the same size; since the action is transitive, the union of the block and all its conjugates is A. Thus the size of the block divides the size of A. Finally, by the orbit-stabilizer theorem, the order of G is divisible by the size of A. $\square$