characterization of a Kleene algebra

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Let A be an idempotent semiring with a unary operator $*$ on A. The following are equivalent

1. $ac + b \leq c$ implies $a^*b \leq c$,
2. $ab \leq b$ implies $a^*b \leq b$.

Proof. $(1 \Rightarrow 2)$. Assume $ab \leq b$. So $ab + b = b$. Then $(ab + b) + b = ab + (b + b) = ab + b = b$, which implies $ab + b \leq b$. By 1, this means $a^*b \leq b$ as desired. $(2 \Rightarrow 1)$. Assume $ac + b \leq c$. Since $0 \leq b$, we get $ac = ac + 0 \leq ac + b \leq c$. Consequently $a^*c \leq c$. □

From the above observation, we see that we get an equivalent definition of a Kleene algebra if the axioms

$$ac + b \leq c \text{ implies } a^*b \leq c \qquad \text{and} \qquad ca + b \leq c \text{ implies } ba^* \leq c$$

are replaced by

$$ab \leq b \text{ implies } a^*b \leq b \qquad \text{and} \qquad ba \leq b \text{ implies } ba^* \leq b.$$

Let A be a Kleene algebra. Some of the interesting properties of * on A are the following:

1. $0^* = 1$.
2. $a^n \leq a^*$ for all non-negative integers n.
3. $1^*a \leq a$.
4. $1^* = 1$.
5. $1 + a^* = a^*$.
6. $a \leq b$ implies $a^* \leq b^*$.
7. $a^*a^* = a^*$.
8. $a^{**} = a^*$.
9. $1 + aa^* = a^*$.
10. $1 + a^*a = a^*$.
11. $ac \leq cb$ implies $a^*c \leq cb^*$.
12. $cb \leq ac$ implies $cb^* \leq a^*c$.
13. $(ab)^*a = a(ba)^*$.
14. $1 + a(ba)^*b = (ab)^*$.
15. If $c = c^2$ and $1 \leq c$, then $c^* = c$.
16. $(a + b)^* = a^*(ba^*)^*$.

Remarks.

- Properties 9 and 10 imply that the axioms $1+aa^* \leq a^*$ and $1+a^*a \leq a^*$ of a Kleene algebra can be replaced by these stronger ones.

- Though in the example of Kleene algebras formed by regular expressions,
$$a^* = \bigcup \{a^i \mid i = 0, 1, 2, \ldots\},$$
and we see that a^* is the least upper bound of the a^i's, this is not a property of a general Kleene algebra. A counterexample of this can be found in Kozen's article, see below. He calls a Kleene algebra K *-*continuous* if $a^* \leq b$ whenever $a^i \leq b$ for any $i \in \mathbb{N} \cup \{0\}$, and $a, b \in K$.

References

[1] D. Kozen, *http://www.cs.cornell.edu/ kozen/papers/kacs.ps**On Kleene Algebras and Closed Semirings* (1990).