n-divisible group

Canonical name	NdivisibleGroup
Date of creation	2013-03-22 17:27:30
Last modified on	2013-03-22 17:27:30
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	5
Author	CWoo (3771)
Entry type	Definition
Classification	msc 20K99
Synonym	n-divisible group
Defines	n-divisible
Defines	n-divisible

Let n be a positive integer and G an abelian group. An element $x \in G$ is said to be divisible by n if there is $y \in G$ such that $x = ny$.

By the unique factorization of $\mathbb{Z}$, write $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ where each p_i is a prime number (distinct from one another) and m_i a positive integer.

Proposition 1. *If x is divisible by n, then x is divisible by $p_1, p_2, \ldots, p_k$.*

Proof. If x is divisible by n, write $x = ny$, where $y \in G$. Since p_i divides n, write $n = p_i t_i$ where t_i is a positive integer. Then $x = p_i t_i(y) = p_i(t_i y)$. Since $t_i y \in G$, x is divisible by p_i. $\square$

Definition. An abelian group G such that every element is divisible by n is called an n-divisible group. Clearly, every group is 1-divisible.

For example, the subset $D \subseteq \mathbb{Q}$ of all decimal fractions is 10-divisible. D is also 2 and 5-divisible. In general, we have the following:

Proposition 2. *If G is n-divisible, it is also n^s-divisible for every non-negative integer s.*

Proposition 3. *Suppose p and q are coprime, then G is p-divisible and q-divisible iff it is pq-divisible.*

Proof. This follows from proposition 1 and the fact that if $p|n$, $q|n$ and $\gcd(p, q) = 1$, then $pq|n$. $\square$

Proposition 4. *G is n-divisible iff G is p-divisible for every prime p dividing n.*

Proof. Suppose G is n-divisible. By proposition 1, every element $x \in G$ is divisible by p, so that G is p-divisible. Conversely, suppose G is p-divisible for every $p|n$. Write $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. Then if G is $p_i^{m_i}$-divisible for every $i = 1, \ldots, k$. Since $p_i^{m_i}$ and $p_j^{m_j}$ are coprime, G is n-divisible by induction and proposition 3. $\square$

Remark. G is a divisible group iff G is p-divisible for every prime p.