free products and group actions

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Owner	rm50 (10146)
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Author	rm50 (10146)
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Theorem 1. *(See Lang, Exercise 54 p. 81) Suppose* $G_1, \ldots G_n$ *are subgroups of* G *that generate* G*. Suppose further that* G *acts on a set* S *and that there are subsets* $S_1, S_2, \ldots S_n \subset S$*, and some* $s \in S - \cup S_i$ *such that for each* $1 \leq i \leq n$*, the following holds for each* $g \in G_i, g \neq e$*:*

- $g(S_j) \subset S_i$ *if* $j \neq i$*, and*
- $g(s) \in S_i$*.*

Then $G = G_1 \star \ldots \star G_n$ *(where* $\star$ *denotes the free product).*

Proof: Any $g \in G$ can be written $g = g_1 g_2 \ldots g_k$ with $g_i \in G_{j_i}, j_i \neq j_{i+1}, g_i \neq e$, since the G_i generate G. Thus there is a surjective homomorphism $\phi : \coprod G_i \twoheadrightarrow G$ (since $\coprod G_i$, as the coproduct, has this universal property). We must show $\ker \phi$ is trivial. Choose $g_1 g_2 \ldots g_k$ as above. Then $g_k(s) \in S_{j_k}$, $g_{k-1}(g_k(s)) \in S_{j_{k-1}}$, and so forth, so that $g_1(g_2(\ldots(g_k(s)\ldots) \in S_{j_1}$. But $e(s) = s \notin S_{j_1}$. Thus $\phi(g_1 g_2 \ldots g_k) \neq e$, and ϕ is injective.