proof of example of medial quasigroup

Canonical name	ProofOfExampleOfMedialQuasigroup
Date of creation	2013-03-22 16:27:35
Last modified on	2013-03-22 16:27:35
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	8
Author	rspuzio (6075)
Entry type	Proof
Classification	msc 20N05

We shall proceed by first showing that the algebraic systems defined in the http://planetmath.org/MedialQuasigroupparent entry are quasigroups and then showing that the medial property is satisfied.

To show that the system is a quasigroup, we need to check the solubility of equations. Let x and y be two elements of G. Then, by definition of $\cdot$, the equation $x \cdot z = y$ is equivalent to

$$f(x) + g(z) + c = y.$$

This is equivalent to

$$g(z) = y - c - f(x).$$

Since g is an automorphism, there will exist a unique solution z to this equation.

Likewise, the equation $z \cdot x = y$ is equivalent to

$$f(z) + g(x) + c = y$$

which, in turn is equivalent to

$$f(z) = y - c - g(x),$$

so we may also find a unique z such that $z \cdot x = y$. Hence, $(G, \cdot)$ is a quasigroup.

To check the medial property, we use the definition of $\cdot$ to conclude that

$$\begin{aligned} (x \cdot y) \cdot (z \cdot w) &= (f(x) + g(y) + c) \cdot (f(z) + g(w) + c) \\ &= f(f(x) + g(y) + c) + g(f(z) + g(w) + c) + c \end{aligned}$$

Since f and g are automorphisms and the group is commutative, this equals

$$f(f(x)) + f(g(y)) + g(f(z)) + g(g(w)) + f(c) + g(c) + c.$$

Since f and g commute this, in turn, equals

$$f(f(x)) + g(f(y)) + f(g(z)) + g(g(w)) + f(c) + g(c) + c.$$

Using the commutative and associative laws, we may regroup this expression as follows:

$$(f(f(x)) + f(g(z)) + f(c)) + (g(f(y)) + g(g(w)) + g(c)) + c$$

Because f and g are automorphisms, this equals

$$f(f(x) + g(z) + c) + g(f(y) + g(w) + c) + c$$

By defintion of $\cdot$, this equals

$$f(x \cdot z) + g(y \cdot z) + c,$$

which equals $(x \cdot z) \cdot (y \cdot z)$, so we have

$$(x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot z).$$

Thus, the medial property is satisfied, so we have a medial quasigroup.