Leech lattice

Canonical name	LeechLattice
Date of creation	2013-03-22 18:43:23
Last modified on	2013-03-22 18:43:23
Owner	monster (22721)
Last modified by	monster (22721)
Numerical id	7
Author	monster (22721)
Entry type	Definition
Classification	msc 20D08
Classification	msc 20B25
Classification	msc 11H56
Classification	msc 11H06
Classification	msc 51E10
Related topic	BinaryGolayCode
Related topic	MiracleOctadGenerator

The Leech lattice is the unique http://planetmath.org/EvenLatticeeven unimodular lattice of http://planetmath.org/Dimension2dimension 24 having no elements of norm 2. Its http://planetmath.org/EquivalentCodeautomorphism group is the largest Conway group Co_0 (sometimes denoted by $\cdot 0$). The quotient of Co_0 by its center is called Co_1, a sporadic simple group.

The construction of the Leech lattice below depends on the existence of the extended binary Golay code $\mathcal{G}_{24}$ (for a construction of the latter, see miracle octad generator).

1 Construction of the Leech lattice

Let $\Omega = \{1, 2, \ldots, 24\}$ and assume we have constructed the Golay $\mathcal{G}_{24}$ on Ω. The Leech lattice Λ is the set of all points

$$\frac{1}{\sqrt{8}}(x_1, x_2, \ldots, x_{24})$$

in $\mathbb{R}^{24}$ where each x_i is an integer, such that

- For some integer m, we have $x_i \equiv x_j \equiv m \pmod 2$ for all $i, j \in \Omega$;
- For any integer n, the set of coordinates $\{i \in \Omega : x_i \equiv n \pmod 4\}$ is in $\mathcal{G}_{24}$;
- $\sum_{i \in \Omega} x_i \equiv 4m \pmod 8$.

2 Properties of the Leech lattice

1. The Leech lattice Λ is an unimodular lattice; in other words:

- The set Λ spans all of $\mathbb{R}^{24}$ as an $\mathbb{R}$-vector space.
- For any $x, y \in \Lambda$, the scalar product $x \cdot y$ is an integer.
- For any $x \in \Lambda$, the norm $x \cdot x$ is an even integer.
- The volume of the fundamental parallelogram of Λ is 1.

2. Let $\Lambda(n) = \{x \in \Lambda : x \cdot x = 2n\}$. Then $|\Lambda(0)| = 1$, $|\Lambda(1)| = 0$, $|\Lambda(2)| = 196560$, $|\Lambda(3)| = 16773120$, $|\Lambda(4)| = 398034000$.

3. The http://planetmath.org/EquivalentCodeautomorphism group $\mathrm{Aut}(\Lambda)$ is the largest Conway group Co_0 with order $8\,315\,553\,613\,086\,720\,000 = 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

4. The group Co_0 acts http://planetmath.org/LeftActiontransitively on the sets $\Lambda(2)$, $\Lambda(3)$, $\Lambda(4)$. For $n = 2, 3$, the imprimitivity blocks of the action of Co_0 on $\Lambda(n)$ are the sets $\{x, -x\}$ where $x \in \Lambda(n)$. The imprimitivity blocks of the action of Co_0 on $\Lambda(4)$ are sets of 48 vectors called . Any two distinct vectors in a are either or orthogonal, and are http://planetmath.org/QuotientGroupcongruent modulo 2Λ.

5. Any vector in Λ is modulo 2Λ to a vector in $\Lambda(n)$ for one of $n = 0, 2, 3, 4$. The imprimitivity blocks of the action of Co_0 on these sets account for all http://planetmath.org/EquivalenceClassclasses of $\Lambda/2\Lambda$:

$$1 + |\Lambda(2)|/2 + |\Lambda(3)|/2 + |\Lambda(4)|/48 = 2^{24} = |\Lambda/2\Lambda|.$$

References

[1] J. H. Conway and N. J. A. Sloane. Sphere Packings, Lattices, and Groups. Springer-Verlag, 1999.