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centralizer

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Let G be a group. The *centralizer* of an element $a \in G$ is defined to be the set

$$C(a) = \{x \in G \mid xa = ax\}$$

Observe that, by definition, $e \in C(a)$, and that if $x, y \in C(a)$, then $xy^{-1}a = xy^{-1}a(yy^{-1}) = xy^{-1}yay^{-1} = xay^{-1} = axy^{-1}$, so that $xy^{-1} \in C(a)$. Thus $C(a)$ is a subgroup of G. For $a \neq e$, the subgroup is non-trivial, containing at least $\{e, a\}$.

To illustrate an application of this concept we prove the following lemma.

Lemma:

There exists a bijection between the right cosets of $C(a)$ and the conjugates of a.

Proof:

If $x, y \in G$ are in the same right coset, then $y = cx$ for some $c \in C(a)$. Thus $y^{-1}ay = x^{-1}c^{-1}acx = x^{-1}c^{-1}cax = x^{-1}ax$. Conversely, if $y^{-1}ay = x^{-1}ax$ then $xy^{-1}a = axy^{-1}$ and $xy^{-1} \in C(a)$ giving x, y are in the same right coset. Let $[a]$ denote the conjugacy class of a. It follows that $|[a]| = [G : C(a)]$ and $|[a]| \mid |G|$.

We remark that $a \in Z(G) \iff C(a) = G \iff |[a]| = 1$, where $Z(G)$ denotes the center of G.

Now let G be a p-group, i.e. a finite group of order p^n, where p is a prime and n is a positive integer. Let $z = |Z(G)|$. Summing over elements in distinct conjugacy classes, we have $p^n = \sum |[a]| = z + \sum_{a \notin Z(G)} |[a]|$ since the center consists precisely of the conjugacy classes of cardinality 1. But $|[a]| \mid p^n$, so $p \mid z$. However, $Z(G)$ is certainly non-empty, so we conclude that every p-group has a non-trivial center.

The groups $C(gag^{-1})$ and $C(a)$, for any g, are isomorphic.