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general associativity

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Related topic	FactorsWithMinusSign
Related topic	IdealOfElementsWithFiniteOrder
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Related topic	Characteri
Defines	power
Defines	multiple
Defines	even power
Defines	odd power
Defines	even multiple
Defines	odd multiple

If an associative binary operation of a set S is denoted by "$\cdot$", the associative law in S is usually expressed as

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

or leaving out the dots, $(ab)c = a(bc)$. Thus the common value of both may be denoted as abc. With four elements of S we can , using only the associativity, as follows:

$$(ab)(cd) = a(b(cd)) = a((bc)d) = (a(bc))d = ((ab)c)d$$

So we may denote the common value of those five expressions as $abcd$.

Theorem. The expression formed of elements $a_1, a_2, \ldots, a_n$ of S . The common value is denoted by $a_1 a_2 \ldots a_n$.

Note. The n elements can be joined, without changing their , in $\frac{(2n-2)!}{n!(n-1)!}$ ways (see the Catalan numbers).

The theorem is proved by induction on n. The cases $n = 3$ and $n = 4$ have been stated above.

Let $n \in \mathbb{Z}_+$. The expression $aa \ldots a$ with n equal "factors" a may be denoted by a^n and called a *power* of a. If the associative operation is denoted "additively", then the "sum" $a+a+\cdots+a$ of n equal elements a is denoted by na and called a *multiple* of a; hence in every ring one may consider powers and multiples. According to whether n is an even or an odd number, one may speak of *even powers*, *odd powers*, *even multiples*, *odd multiples*.

The following two laws can be proved by induction:

$$a^m \cdot a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

In notation:

$$ma+na = (m+n)a,$$

$$n(ma) = (mn)a$$

Note. If the set S together with its operation is a group, then the notion of multiple na resp. power a^n can be extended for negative integer and zero values of n by means of the inverse and identity elements. The above laws remain in .