proof of Cayley's theorem

Canonical name	ProofOfCayleysTheorem
Date of creation	2013-03-22 12:30:50
Last modified on	2013-03-22 12:30:50
Owner	Evandar (27)
Last modified by	Evandar (27)
Numerical id	5
Author	Evandar (27)
Entry type	Proof
Classification	msc 20B99

Let G be a group, and let S_G be the permutation group of the underlying set G. For each $g \in G$, define $\rho_g : G \to G$ by $\rho_g(h) = gh$. Then ρ_g is invertible with inverse $\rho_{g^{-1}}$, and so is a permutation of the set G.

Define $\Phi : G \to S_G$ by $\Phi(g) = \rho_g$. Then Φ is a homomorphism, since

$$(\Phi(gh))(x) = \rho_{gh}(x) = ghx = \rho_g(hx) = (\rho_g \circ \rho_h)(x) = ((\Phi(g))(\Phi(h)))(x)$$

And Φ is injective, since if $\Phi(g) = \Phi(h)$ then $\rho_g = \rho_h$, so $gx = hx$ for all $x \in X$, and so $g = h$ as required.

So Φ is an embedding of G into its own permutation group. If G is finite of order n, then simply numbering the elements of G gives an embedding from G to S_n.