proof of fundamental theorem of finitely generated abelian groups

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Every finitely generated abelian group A is a direct sum of its cyclic subgroups, i.e.

$$A = C_{m_1} \oplus C_{m_2} \oplus \ldots \oplus C_{m_k} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z},$$

where $1 < m_1 \mid m_2 \mid \ldots \mid m_k$. The numbers m_i are uniquely determined as well as the number of $\mathbb{Z}$'s, which is the rank of an abelian group.

Proof. Let G be an abelian group with n generators. Then for a free group F_n, G is isomorphic to the quotient group F_n/A. Now F_n and A contain a basis $f_1, \ldots, f_n$ and $a_1, \ldots, a_k$ satisfying $a_i = m_i f_i$ for all $1 \leq i \leq k$. As $G \cong F_n/A$, it suffices to show that F_n/A is a direct sum of its cyclic subgroups $\langle f_1 + A \rangle$.

It is clear that F_n/A is generated by its subgroups $\langle f_i + A \rangle$. Assume that the zero element of F_n/A can be written as a form $A = l_1 f_1 + \ldots + l_n f_n + A$. It follows that $l_1 f_1 + \ldots + l_n f_n = a \in A$. As we write a as a linear combination of that basis and using $a_i = m_i f_i$ we get the equations

$$l_1 f_1 + \ldots + l_n f_n = s_1 a_1 + \ldots s_k a_k = s_1 m_1 f_1 + \ldots + s_k m_k f_k.$$

As every element can be represented uniquely as a linear combination of its free generators f_1, we have $l_i = s_i m_i$ for every $1 \leq i \leq k$ and $l_j = 0$ for every $k < j \leq n$.

This means that every element $l_i f_i$ belongs to A, so $l_i f_i + A = A$. Therefore the zero element has a unique representation as a sum of the elements of the subgroup $\langle f_i + A \rangle$.

References

[1] P. Paajanen: *Ryhmäteoria*. Lecture notes, Helsinki university, Finland (fall 2008)