commensurable subgroups

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0.1 Definition

Definition - Let G be a group. Two subgroups $S_1, S_2 \subseteq G$ are said to be **commensurable**, in which case we write $S_1 \sim S_2$, if $S_1 \cap S_2$ has finite index both in S_1 and in S_2, i.e. if $[S_1 : S_1 \cap S_2]$ and $[S_2 : S_1 \cap S_2]$ are both finite.

This can be interpreted informally in the following : S_1 and S_2 are commensurable if their intersection $S_1 \cap S_2$ is "big" in both S_1 and S_2.

0.2 Commensurability is an equivalence relation

- of subgroups is an equivalence relation. In particular, if $S_1 \sim S_2$ and $S_2 \sim S_3$, then $S_1 \sim S_3$.

: Let S_1, S_2 and S_3 be subgroups of a group G.

- Reflexivity: we have that $S_1 \sim S_1$, since $[S_1 : S_1] = 1$.
- Symmetry: is clear from the definition.
- Transitivity: if $S_1 \sim S_2$ and $S_2 \sim S_3$, then one has

$$\begin{aligned}
[S_1 : S_1 \cap S_3] &\leq [S_1 : S_1 \cap S_2 \cap S_3] \\
&= [S_1 : S_1 \cap S_2][S_1 \cap S_2 : S_1 \cap S_2 \cap S_3] \\
&\leq [S_1 : S_1 \cap S_2][S_2 : S_2 \cap S_3] \\
&< \infty.
\end{aligned}$$

Similarly, we can prove that $[S_3 : S_1 \cap S_3] < \infty$ and therefore $S_1 \sim S_3$. $\square$

0.3 Examples:

- All non-zero subgroups of $\mathbb{Z}$ are commensurable with each other.
- All conjugacy classes of the general linear group $GL(n; \mathbb{Z})$, seen as a subgroup of $GL(n; \mathbb{Q})$, are commensurable with each other.

References

[1] A. Krieg, , Mem. Amer. Math. Soc., no. 435, vol. 87, 1990.