quasicyclic group

Canonical name	QuasicyclicGroup
Date of creation	2013-03-22 15:35:22
Last modified on	2013-03-22 15:35:22
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	19
Author	yark (2760)
Entry type	Definition
Classification	msc 20F50
Classification	msc 20K10
Synonym	quasi-cyclic group
Synonym	Prüfer group
Defines	quasicyclic
Defines	quasi-cyclic
Defines	Prüfer p-group

Let p be a prime number. The *p-quasicyclic group* (or *Prüfer p-group*, or *p^∞ group*) is the p-primary component of $\mathbb{Q}/\mathbb{Z}$, that is, the unique maximal http://planetmath.org/PGroup4p-subgroup of $\mathbb{Q}/\mathbb{Z}$. Any http://planetmath.org/Groupgroup isomorphic to this will also be called a p-quasicyclic group.

The p-quasicyclic group will be denoted by $\mathbb{Z}(p^\infty)$. Other notations in use include $\mathbb{Z}[p^\infty]$, $\mathbb{Z}/p^\infty\mathbb{Z}$, $\mathbb{Z}_{p^\infty}$ and C_{p^∞}.

$\mathbb{Z}(p^\infty)$ may also be defined in a number of other (equivalent) ways (again, up to isomorphism):

- $\mathbb{Z}(p^\infty)$ is the group of all p^n-th complex roots of 1, for $n \in \mathbb{N}$.
- $\mathbb{Z}(p^\infty)$ is the injective hull of $\mathbb{Z}/p\mathbb{Z}$ (viewing abelian groups as $\mathbb{Z}$-http://planetmath.org/Modulemodules).
- $\mathbb{Z}(p^\infty)$ is the direct limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$.

A *quasicyclic group* (or *Prüfer group*) is a group that is p-quasicyclic for some prime p.

The http://planetmath.org/Subgroupsubgroup structure of $\mathbb{Z}(p^\infty)$ is particularly simple: all proper subgroups are finite and cyclic, and there is exactly one of order p^n for each non-negative integer n. In particular, this means that the subgroups are linearly ordered by inclusion, and all subgroups are fully invariant. The quasicyclic groups are the only infinite groups with a linearly ordered subgroup lattice. They are also the only infinite solvable groups whose proper subgroups are all finite.

Quasicyclic groups are locally cyclic, http://planetmath.org/DivisibleGroupdivisible and co-Hopfian.

Every infinite locally cyclic p-group is isomorphic to $\mathbb{Z}(p^\infty)$.