Schur's lemma

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Schur's lemma is a fundamental result in representation theory, an elementary observation about irreducible modules, which is nonetheless noteworthy because of its profound applications.

Lemma (Schur's lemma)**.** *Let G be a finite group and let V and W be irreducible G-modules. Then, every G-module homomorphism $f : V \to W$ is either invertible or the trivial zero map.*

Proof. Note that both the kernel, $\ker f$, and the image, $\operatorname{im} f$, are G-submodules of V and W, respectively. Since V is irreducible, $\ker f$ is either trivial or all of V. In the former case, $\operatorname{im} f$ is all of W — also because W is irreducible — and hence f is invertible. In the latter case, f is the zero map. □

One of the most important consequences of Schur's lemma is the following.

Corollary. *Let V be a finite-dimensional, irreducible G-module taken over an algebraically closed field. Then, every G-module homomorphism $f : V \to V$ is equal to a scalar multiplication.*

Proof. Since the ground field is algebraically closed, the linear transformation $f : V \to V$ has an eigenvalue; call it λ. By definition, $f - \lambda 1$ is not invertible, and hence equal to zero by Schur's lemma. In other words, $f = \lambda$, a scalar. □