FFT Library

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1 Revision History

Date	Version	Notes
Date 1	1.0	Notes
Date 2	1.1	Notes

2 Reference Material

This section records information for easy reference.

2.1 Table of Units

Throughout this document SI (Système International d'Unités) is employed as the unit system. In addition to the basic units, several derived units are used as described below. For each unit, the symbol is given followed by a description of the unit and the SI name.

symbol	unit	SI
V	voltage	volt
W	power	Watt ($\mathrm{W} = \mathrm{J\,s^{-1}}$)
Hz	frequency	Hertz
dB	decibel	Decibel

[Only include the units that your CA actually uses. If there are no units for your problem, like for a general purpose library, you should still include the heading, with the content "not applicable" (or similar). —SS]

2.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the heat transfer literature and with existing documentation for solar water heating systems. The symbols are listed in alphabetical order.

symbol	unit	description
A_C	m^2	coil surface area
A_{in}	m^2	surface area over which heat is transferred in

[Use your problems actual symbols. The si package is a good idea to use for units. —SS]

2.3 Abbreviations and Acronyms

symbol	description
A	Assumption
DD	Data Definition
GD	General Definition
GS	Goal Statement
IM	Instance Model
LC	Likely Change
PS	Physical System Description
R	Requirement
SRS	Software Requirements Specification
FFT	FFT
T	Theoretical Model
DFT	
IDFT	

[Add any other abbreviations or acronyms that you add —SS]

Contents

3 Introduction

[This CA template is based on ?. It will get you started, but you will have to make changes. Any changes to section headings should be approved by the instructor, since that implies a deviation from the template. Although the bits shown below do not include type information, you may need to add this information for your problem. —SS]

[Feel free to change the appearance of the report by modifying the LaTeX commands. —SS]

3.1 Purpose of Document

This document will be used as a starting point for subsequent development phases, including writing the design specification and the software verification and validation plan. The design document will show how the requirements are to be realized, including decisions on the numerical algorithms and programming environment. The verification and validation plan will show the steps that will be used to increase confidence in the software documentation and the implementation.

3.2 Scope of the Family

The scope of the requirements includes specifing the input data, specifying the number of radix of FFT. Given the full information of input and parameters, the FFT Library will take advantage of FFT algorithms to do DFT or IDFT calcualtion effectively.

3.3 Characteristics of Intended Reader

Reviewers of this documentation should have a strong knowledge in Complex Variables Functions as well as have an understanding of differential equaltions.

3.4 Organization of Document

4 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

4.1 Potential System Contexts

Figure 1 shows the system context. A circle represents an external entity outside the software, the user in this case. A rectangle represents the software system itself (FFT). Arrows are used to show the data flow between the system and its environment.

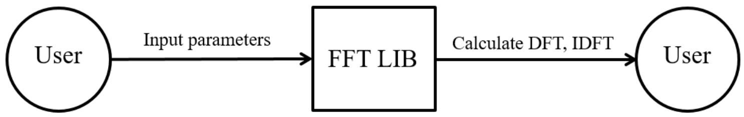

Figure 1: System Context

- User Responsibilities:
  - Provide the input data to the system, ensuring no errors in the data entry
  - Make sure invoking the right functions according to different input data type
- FFT Responsibilities:
  - Determine if the inputs satisfy the required data type and format
  - Calculate the required outputs

4.2 Potential User Characteristics

The end user of FFT should have an understanding of undergraduate Level 1 Calculus and Physics.

4.3 Potential System Constraints

[You may not have any system constraints —SS]

5 Commonalities

5.1 Background Overview

5.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- Disceret Fourier Transform(DFT): A transform method to convert a signal from its original domain (often time or space) to a representation in the frequency domain.
- Inverse Disceret Fourier Transform(IDFT): A inverse transform method of DFT.
- Fast Fourier Transform(FFT): A fast algorithm computes the discrete Fourier transform (DFT) of a sequence.
- Inverse Fast Fourier Transform(IFFT): A fast algorithm computes the inverse discrete Fourier transform (IDFT) of a sequence.

5.3 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given. [Modify the examples below for your problem, and add additional definitions as appropriate. —SS]

Number	DD1
Label	**A symbol represents** $e^{-2\pi i/N}$
Symbol	W_N
SI Units	None
Equation	$W_N = e^{-2\pi i/N}$
Description	W_N is a simpler term to represent $e^{-2\pi i/N}$. Readers can analyse complex-structure equations more directly and focus more on transformation process.
Sources	Commently Admitted
Ref. By	IM??

Number	DD2
Label	**Power Spectral Density (PSD)**
Symbol	$X[k]$
SI Units	$\mathrm{W\,Hz^{-1}}$
Equation	$X[k] = X[0] + X[1] + X[2] + ..., (k = 0, 1, 2, 3...)$
Description	Power spectral density (PSD) shows the strength of the variations(energy) as a function of frequency. In other words, it shows at which frequencies variations are strong and at which frequencies variations are weak.
Sources	`http://www.cygres.com/OcnPageE/Glosry/SpecE.html`
Ref. By	IM??

Number	DD3
Label	**Amplitude Function**
Symbol	$x(n)$
SI Units	dB
Equation	Usually obtained from detector or other equipment directly.
Description	The amplitude of a periodic variable is a measure of its change over a single period (such as time or spatial period).
Sources	https://en.wikipedia.org/wiki/Amplitude
Ref. By	IM??

Number	DD4
Label	**The Odd Terms OF Power Spectral Density**
Symbol	$X[k]$
SI Units	$\mathrm{W\,Hz^{-1}}$
Equation	$X_E[k] = X[1] + X[3] + ..., (k = 1, 3, 5...)$
Description	The odd term set of the complete PSD but keeping the same physical meaning as PSD.
Sources	
Ref. By	IM??

Number	DD5
Label	**The Even Terms OF Power Spectral Density**
Symbol	$X_O[k]$
SI Units	$\mathrm{W\,Hz^{-1}}$
Equation	$X_O[k] = X[0] + X[2] + ..., (k = 0, 2, 4...)$
Description	The even term set of the complete PSD but keeping the same physical meaning as PSD.
Sources	
Ref. By	IM??

5.4 Goal Statements

Given the input data array of $x(n)$ or $X[k]$, radix r, the goal statements are:

GS1: Complete radix-2 FFT and IFFT when input is real data.

GS2: Complete radix-2 FFT and IFFT when input is complex data.

GS3: Complete radix-3 FFT and IFFT when input is real data.

GS4: Complete radix-3 FFT and IFFTwhen input is complex data.

5.5 Theoretical Models

This section focuses on the general equations and laws that FFT is based on. [Modify the examples below for your problem, and add additional models as appropriate. —SS]

Number	T1
Label	**Discrete Fourier Transform(DFT)**
Equation	$X[k] = \sum_{n=0}^{N-1} x(n)\ e^{-2\pi ni/N}$
Description	The above equation is the defination of Discrete Fourier Transform, which transforms a sequence of N complex numbers $x(0)$, $x(1)$, $x(2)$, $x(3)$... into another sequence of complex numbers, $X[0]$, $X[1]$, $X[2]$, $X[3]$, $X[4]$...In the equation, $x(n)$ is amplitude in time domain and $e^{-2\pi i/N}$ is an important term in DFT algorithm.
Source	https://en.wikipedia.org/wiki/Discrete_Fourier_transform
Ref. By	GD??

Number	T2
Label	**Inverse Discrete Fourier Transform(IDFT)**
Equation	$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X[k]\ e^{2\pi ni/N}$
Description	The above equation is the defination of Inverse Discrete Fourier Transform, which transforms a sequence of N complex numbers $X[0]$, $X[1]$, $X[2]$, $X[3]$, $X[4]$... into another sequence of complex numbers, $x(0)$, $x(1)$, $x(2)$, $x(3)$...In the equation, $X[k]$ is amplitude in time domain and $e^{-2\pi i/N}$ is an important term in IDFT algorithm, while N is the length of sequence.
Source	https://en.wikipedia.org/wiki/Discrete_Fourier_transform
Ref. By	GD??

Number	T3
Label	**Euler's Formula**
Equation	$e^{ix} = cosx + isinx$
Description	Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. e is the base of the natural logarithm, i is the imaginary unit, and cos and sin are the trigonometric functions cosine and sine respectively, with the argument x given in radians.
Source	https://en.wikipedia.org/wiki/Euler%27s_formula
Ref. By	GD??

Number	T4
Label	**Euler's Formula**
Equation	$e^{ix} = cosx + isinx$
Description	Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. e is the base of the natural logarithm, i is the imaginary unit, and cos and sin are the trigonometric functions cosine and sine respectively, with the argument x given in radians.
Source	https://en.wikipedia.org/wiki/Euler%27s_formula
Ref. By	GD??

6 Variabilities

6.1 Assumptions

A1: The number of input of radix-2 FFT and IFFT shoud be $2^s(s \in \mathbb{Z}$, s $< 20)$ [T4].

A2: The number of input of radix-3 FFT and IFFT should be $3^s(s \in \mathbb{Z}$, s $< 20)$ [T4].

A3: The length of $X[k]$sequence is always equal to the length of sequence $x(n)$ [T4].

A4: The coefficient of every term in $x(n)$ is an integer for both complex and real number [T4].

A5: When retrieving the complex number FFT function, some terms are still real numbers. Users enter 0 as an coefficient for complex terms under this occation [T4].

A6: Assume the strides for $X[n]$ and $x(n)$ are both 1.

A7: The length of input sequence is known [T4].

6.2 Calculation

- Theory module to address [T4]:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x(n)W_N^{kn} \\ &= \sum_{even} x(n)W_N^{kn} + \sum_{odd} x(n)W_N^{kn} \end{aligned}$$

Define n = 2r and n = 2r + 1, r = 0, 1, 2, ..., N/2 -1.

$$\begin{aligned} X[k] &= \sum_{r=0}^{N/2-1} x(2r)W_N^{2kr} + \sum_{r=0}^{N/2-1} x(2r+1)W_N^{(2r+1)k} \\ &= \sum_{r=0}^{N/2-1} x(2r)(W_N^2)^{kr} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_N^2)^{kr} \end{aligned}$$

According to Eular's Formula, $W_N^a = W_{N/a}$,

$$
\begin{aligned}
X[k] &= \sum_{r=0}^{N/2-1} x(2r)(W_N^2)^{kr} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_N^2)^{kr} \\
&= \sum_{r=0}^{N/2-1} x(2r)(W_{N/2})^{kr} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_{N/2})^{kr} \\
&= X_e[k] + W_N^k X_o[k] \\
S
\end{aligned}
$$

We can use FFT butterfly diagram to analysis a FFT computing process and we can clearly tell how the sequence is divided and computed. Here we will have a simplest N = 8, radix-2, butterfly diagram as example:

Figure 2

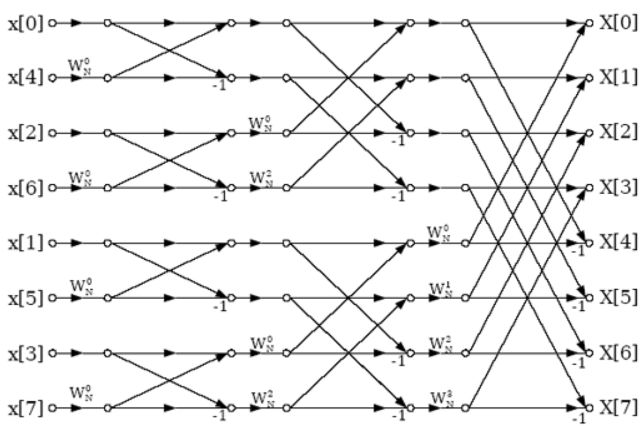

Figure 2: Radix-2 8-Point FFT

From the diagram, there were twice times of division of even and odd sequences. The first time division spilt the even and odd sequences. The second time, we took even and odd sequences retrieved from last division as two new sequences and split both of them into the even and odd sequences again. So far from the lowest level, we can consider our original sequence as a four-piece sequences. And then we can compute the sequence from the lowest level. Thus, we can develop our algorithm to a bigger domain. Unless the number of terms in a sequence is in pattern of $2^s (s \in \mathbb{Z}$, s < 20), the sequence can be reduced until the smallest piece is consist of two terms.

- Theory module to address [T4]:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \\
&= \sum_{n=3^r} x(n) W_N^{kn} + \sum_{n=3^r+1} x(n) W_N^{kn} + \sum_{n=3^r+2} x(n) W_N^{kn} \\
&= \sum x_1(n) W_N^{kn} + \sum x_2(n) W_N^{kn} + \sum x_3(n) W_N^{kn}()
\end{aligned}
$$

Define n = 3r, n = 3r + 1 and n = 3r +2, r = 0, 1, 2, ..., N/3 -1.

$$
\begin{aligned}
X[k] &= \sum_{r=0}^{N/3-1} x(3r) W_N^{3kr} + \sum_{r=0}^{N/3-1} x(3r+1) W_N^{(3r+1)k} + \sum_{r=0}^{N/3-1} x(3r+2) W_N^{(3r+2)k} \\
&= \sum_{r=0}^{N/3-1} x(3r)(W_N^3)^{kr} + W_N^k \sum_{r=0}^{N/3-1} x(3r+1)(W_N^3)^{kr} + W_N^{2k} \sum_{r=0}^{N/3-1} x(3r+2)(W_N^3)^{kr}
\end{aligned}
$$

According to Eular's Formula, $W_N^a = W_{N/a}$,

$$
\begin{aligned}
X[k] &= \sum_{r=0}^{N/3-1} x(3r)(W_N^3)^{kr} + W_N^k \sum_{r=0}^{N/3-1} x(3r+1)(W_N^3)^{kr} + W_N^{2k} \sum_{r=0}^{N/3-1} x(3r+2)(W_N^3)^{kr} \\
&= \sum_{r=0}^{N/3-1} x(3r)(W_{N/3})^{kr} + W_N^k \sum_{r=0}^{N/3-1} x(3r+1)(W_{N/3})^{kr} + W_N^{2k} \sum_{r=0}^{N/3-1} x(3r+2)(W_{N/3})^{kr} \\
&= X_1[k] + W_N^k X_2[k] + W_N^{2k} X_3[k]
\end{aligned}
$$

We can use FFT butterfly diagram to analysis a FFT computing process and we can clearly tell how the sequence is divided and computed. Here we will have a simplest N = 9, radix-3, butterfly diagram as example:

Figure 3

From the diagram, there were twice times of division of even and odd sequences. The first time division spilt the even and odd sequences. The second time, we took even

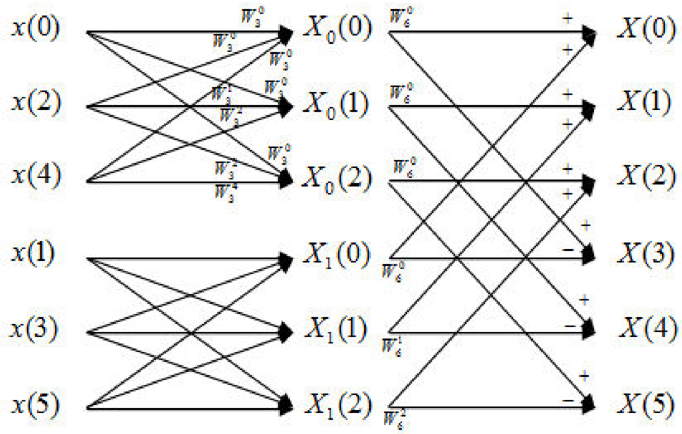

Figure 3: Radix-3 9-Point FFT

and odd sequences retrieved from last division as two new sequences and split both of them into the even and odd sequences again. So far from the lowest level, we can consider our original sequence as a four-piece sequences. And then we can compute the sequence from the lowest level. Thus, we can develop our algorithm to a bigger domain. Unless the number of terms in a sequence is in pattern of $2^s(s \in \mathbb{Z}$, s $< 20)$, the sequence can be reduced until the smallest piece is consist of two terms.

- Inverse Fast Fourier Transform(IFFT): A fast algorithm computes the inverse discrete Fourier transform (IDFT) of a sequence.

6.3 Output

7 Traceability Matrices and Graphs

[You will have to add tables. —SS]

8 Appendix

[Your report may require an appendix. For instance, this is a good point to show the values of the symbolic parameters introduced in the report. —SS]

8.1 Symbolic Parameters

[The definition of the requirements will likely call for SYMBOLIC_CONSTANTS. Their values are defined in this section for easy maintenance. —SS]