

# FFT Library

Yuzhi Zhao

October 2, 2017

# Contents

# 1 Revision History

Date	Version	Notes
Date 1	1.0	Notes
Date 2	1.1	Notes

## 2 Reference Material

This section records information for easy reference.

### 2.1 Table of Units

Throughout this document SI (Système International d'Unités) is employed as the unit system. In addition to the basic units, several derived units are used as described below. For each unit, the symbol is given followed by a description of the unit and the SI name.

symbol	unit	SI
V	voltage	volt
W	power	Watt ( $W = J s^{-1}$ )
Hz	frequency	Hertz
dB	decibel	Decibel

### 2.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the heat transfer literature and with existing documentation for solar water heating systems. The symbols are listed in alphabetical order.

symbol	unit	description
$A_C$	$m^2$	coil surface area
$A_{in}$	$m^2$	surface area over which heat is transferred in

[Use your problems actual symbols. The si package is a good idea to use for units. —SS]

## 2.3 Abbreviations and Acronyms

symbol	description
A	Assumption
DD	Data Definition
GS	Goal Statement
IM	Instance Model
SRS	Software Requirements Specification
FFT	Fast Fourier Transform
T	Theoretical Model
DFT	Discrete Fourier Transform
IDFT	Invers Discrete Transform

## 3 Introduction

[This CA template is based on ?. It will get you started, but you will have to make changes. Any changes to section headings should be approved by the instructor, since that implies a deviation from the template. Although the bits shown below do not include type information, you may need to add this information for your problem. —SS]

[Feel free to change the appearance of the report by modifying the LaTeX commands. —SS]

### 3.1 Purpose of Document

This document will be used as a starting point for subsequent development phases, including writing the design specification and the software verification and validation plan. The design document will show how the requirements are to be realized, including decisions on the numerical algorithms and programming environment. The verification and validation plan will show the steps that will be used to increase confidence in the software documentation and the implementation.

### 3.2 Scope of the Family

The scope of the requirements includes specifying the input data, specifying the number of radix of FFT. Given the full information of input and parameters, the FFT Library will take advantage of FFT algorithms to do DFT or IDFT calculation effectively.

### 3.3 Characteristics of Intended Reader

Reviewers of this documentation should have a strong knowledge in Complex Variables Functions as well as have an understanding of differential equations.

### 3.4 Organization of Document

The organization of this document follows the template for an SRS for scientific computing software proposed by ? and ?. The presentation follows the standard pattern of presenting goals, theories, definitions, and assumptions. For readers that would like a more bottom up approach, they can start reading the instance models in Section ?? and trace back to find any additional information they require. The instance models provide the algebraic equations that model the FFT algorithm.

## 4 General System Description

This section identifies the interfaces between the system and its environment, describes the potential user characteristics and lists the potential system constraints.

### 4.1 Potential System Contexts

Figure ?? shows the system context. A circle represents an external entity outside the software, the user in this case. A rectangle represents the software system itself (FFT). Arrows are used to show the data flow between the system and its environment.

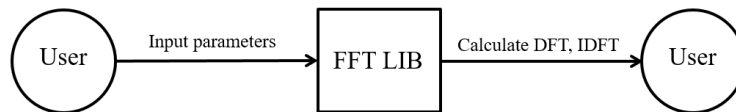


Figure 1: System Context

- User Responsibilities:
  - Provide the input data to the system, ensuring no errors in the data entry
  - Make sure invoking the right functions according to different input data type
- FFT Responsibilities:
  - Determine if the inputs satisfy the required data type and format
  - Calculate the required outputs

## 4.2 Potential User Characteristics

The end user of FFT should have an understanding of undergraduate Level 1 Calculus and Physics.

## 4.3 Potential System Constraints

Not Applicable.

# 5 Commonalities

## 5.1 Background Overview

## 5.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- Discrete Fourier Transform(DFT): A transform method to convert a signal from its original domain (often time or space) to a representation in the frequency domain.
- Inverse Discrete Fourier Transform(IDFT): A inverse transform method of DFT.
- Fast Fourier Transform(FFT): A fast algorithm computes the discrete Fourier transform (DFT) of a sequence.
- Inverse Fast Fourier Transform(IFFT): A fast algorithm computes the inverse discrete Fourier transform (IDFT) of a sequence.

## 5.3 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

Number	DD1
Label	<b>A symbol represents</b> $e^{-2\pi i/N}$
Symbol	$W_N$
SI Units	None
Equation	$W_N = e^{-2\pi i/N}$
Description	$W_N$ is a simpler term to represent $e^{-2\pi i/N}$ . Readers can analyse complex-structure equations more directly and focus more on transformation process.
Sources	Commently Admitted
Ref. By	IM??

Number	DD2
Label	<b>Power Spectral Density (PSD)</b>
Symbol	$X[k]$
SI Units	W Hz <sup>-1</sup>
Equation	$X[k] = X[0] + X[1] + X[2] + \dots, (k = 0, 1, 2, 3\dots)$
Description	Power spectral density (PSD) shows the strength of the variations(energy) as a function of frequency. In other words, it shows at which frequencies variations are strong and at which frequencies variations are weak.
Sources	<a href="http://www.cygres.com/OcnPageE/Glosry/SpecE.html">http://www.cygres.com/OcnPageE/Glosry/SpecE.html</a>
Ref. By	IM??



Number	DD3
Label	<b>Amplitude Function</b>
Symbol	$x(n)$
SI Units	dB
Equation	Usually obtained from detector or other equipment directly.
Description	The amplitude of a periodic variable is a measure of its change over a single period (such as time or spatial period).
Sources	<a href="https://en.wikipedia.org/wiki/Amplitude">https://en.wikipedia.org/wiki/Amplitude</a>
Ref. By	IM??

Number	DD4
Label	<b>The Odd Terms OF Power Spectral Density</b>
Symbol	$X[k]$
SI Units	W Hz <sup>-1</sup>
Equation	$X_E[k] = X[1] + X[3] + \dots, (k = 1, 3, 5\dots)$
Description	The odd term set of the complete PSD but keeping the same physical meaning as PSD.
Sources	
Ref. By	IM??

Number	DD5
Label	<b>The Even Terms OF Power Spectral Density</b>
Symbol	$X_O[k]$
SI Units	$\text{W Hz}^{-1}$
Equation	$X_O[k] = X[0] + X[2] + \dots, (k = 0, 2, 4, \dots)$
Description	The even term set of the complete PSD but keeping the same physical meaning as PSD.
Sources	
Ref. By	IM??

## 5.4 Goal Statements

Given the input data array of  $x(n)$  or  $X[k]$ , radix r, the goal statements are:

GS1: Complete radix-2 FFT and IFFT when input is real data.

GS2: Complete radix-2 FFT and IFFT when input is complex data.

GS3: Complete radix-3 FFT and IFFT when input is real data.

GS4: Complete radix-3 FFT and IFFT when input is complex data.

## 5.5 Theoretical Models

This section focuses on the general equations and laws that FFT is based on.

Number	T1
Label	<b>Discrete Fourier Transform(DFT)</b>
Equation	$X[k] = \sum_{n=0}^{N-1} x(n) e^{-2\pi ni/N}$
Description	The above equation is the definition of Discrete Fourier Transform, which transforms a sequence of N complex numbers $x(0), x(1), x(2), x(3), \dots$ into another sequence of complex numbers, $X[0], X[1], X[2], X[3], X[4], \dots$ . In the equation, $x(n)$ is amplitude in time domain and $e^{-2\pi ni/N}$ is an important term in DFT algorithm.
Source	<a href="https://en.wikipedia.org/wiki/Discrete_Fourier_transform">https://en.wikipedia.org/wiki/Discrete_Fourier_transform</a>
Ref. By	GD??

Number	T2
Label	<b>Inverse Discrete Fourier Transform(IDFT)</b>
Equation	$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{2\pi ni/N}$
Description	The above equation is the defination of Inverse Discrete Fourier Transform, which transforms a sequence of N complex numbers $X[0], X[1], X[2], X[3], X[4]...$ into another sequence of complex numbers, $x(0), x(1), x(2), x(3)...$ . In the equation, $X[k]$ is amplitude in time domain and $e^{-2\pi i/N}$ is an important term in IDFT algorithm, while N is the length of sequence.
Source	<a href="https://en.wikipedia.org/wiki/Discrete_Fourier_transform">https://en.wikipedia.org/wiki/Discrete_Fourier_transform</a>
Ref. By	GD??

Number	T3
Label	<b>Euler's Formula</b>
Equation	$e^{ix} = \cos x + i \sin x$
Description	Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. $e$ is the base of the natural logarithm, $i$ is the imaginary unit, and $\cos$ and $\sin$ are the trigonometric functions cosine and sine respectively, with the argument $x$ given in radians.
Source	<a href="https://en.wikipedia.org/wiki/Euler%27s_formula">https://en.wikipedia.org/wiki/Euler%27s_formula</a>
Ref. By	GD??

Number	T4
Label	<b>Periodicity Of <math>W_N^{kn}</math></b>
Equation	$W_N^{kn} = W_N^{k(n+N)}$
Description	$W_N^{kn}$ is a periodic function and the period is N. Using Euler's Formula can prove this.
Source	<a href="https://en.wikipedia.org/wiki/Euler%27s_formula">https://en.wikipedia.org/wiki/Euler%27s_formula</a>
Ref. By	GD??

Number	T5
Label	<b>A Transform Of <math>W_N^{kn}</math></b>
Equation	$W_N^{akn} = W_{N/a}^{kn}$
Description	a is a coefficient. This transform is frequently used in FFT algorithms.
Source	
Ref. By	GD??

## 5.6 Instance Models

Number	IM1
Label	<b>Radix-2 FFT Calculation</b>
Input	$a_n, b_n, N, r$ from IM?? The input is constrained so that $N \leq 20, r = 2$ .
Output	$X[k], k = (0, 1, \dots, N)$ , in format 'a + bi'
Description	$X[k]$ is the power spectral density ( $\text{W Hz}^{-1}$ ), a is the real number part, b is the coefficient part of complex number part.
Sources	?
Ref. By	IM??

### Detailed Radix-2 FFT Algorithm

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \\
&= \sum_{\text{even}} x(n) W_N^{kn} + \sum_{\text{odd}} x(n) W_N^{kn}
\end{aligned}$$

Define  $n = 2r$  and  $n = 2r + 1, r = 0, 1, 2, \dots, N/2 - 1$ .

$$\begin{aligned}
X[k] &= \sum_{r=0}^{N/2-1} x(2r) W_N^{2kr} + \sum_{r=0}^{N/2-1} x(2r+1) W_N^{(2r+1)k} \\
&= \sum_{r=0}^{N/2-1} x(2r) (W_N^2)^{kr} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1) (W_N^2)^{kr}
\end{aligned}$$

According to Euler's Formula,  $W_N^a = W_{N/a}$ ,

$$\begin{aligned}
X[k] &= \sum_{r=0}^{N/2-1} x(2r)(W_N^2)^{kr} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_N^2)^{kr} \\
&= \sum_{r=0}^{N/2-1} x(2r)(W_{N/2})^{kr} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1)(W_{N/2})^{kr} \\
&= X_e[k] + W_N^k X_o[k]
\end{aligned}$$

$S$

We can use FFT butterfly diagram to analysis a FFT computing process and we can clearly tell how the sequence is divided and computed. Here we will have a simplest  $N = 8$ , radix-2, butterfly diagram as example:

Figure ??

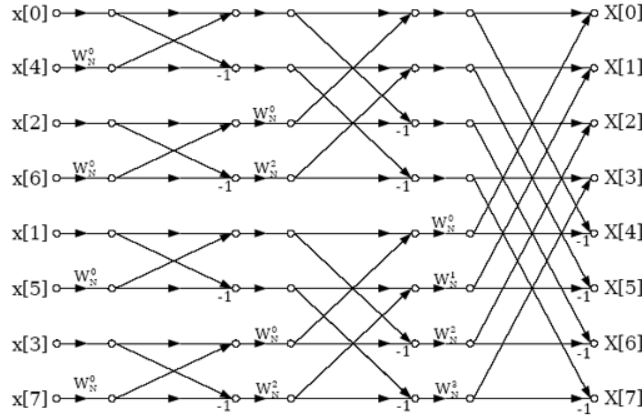


Figure 2: Radix-2 8-Point FFT

From the diagram, there were twice times of division of even and odd sequences. The first time division split the even and odd sequences. The second time, we took even and odd sequences retrieved from last division as two new sequences and split both of them into the even and odd sequences again. So far from the lowest level, we can consider our original sequence as a four-piece sequences. And then we can compute the sequence from the lowest level. Thus, we can develop our algorithm to a bigger domain. Unless the number of terms in a sequence is in pattern of  $2^s$  ( $s \in \mathbb{Z}$ ,  $s < 20$ ), the sequence can be reduced until the smallest piece is consist of two terms.

Radix-2 IFFT implements the same algorithm.

Number	IM2
Label	<b>Radix-3 FFT Calculation</b>
Input	$a_n, b_n, N, r$ from IM?? The input is constrained so that $N \leq 20, r = 3$ .
Output	$X[k], k = (0, 1, \dots, N)$ , in format 'a + bi'
Description	$X[k]$ is the power spectral density ( $\text{W Hz}^{-1}$ ), a is the real number part, b is the coefficient part of complex number part.
Sources	?
Ref. By	IM??

### Detailed Radix-3 FFT Algorithm

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \\
&= \sum_{n=3^r} x(n) W_N^{kn} + \sum_{n=3^r+1} x(n) W_N^{kn} + \sum_{n=3^r+2} x(n) W_N^{kn} \\
&= \sum x_1(n) W_N^{kn} + \sum x_2(n) W_N^{kn} + \sum x_3(n) W_N^{kn} ()
\end{aligned}$$

Define  $n = 3r, n = 3r + 1$  and  $n = 3r + 2, r = 0, 1, 2, \dots, N/3 - 1$ .

$$\begin{aligned}
X[k] &= \sum_{r=0}^{N/3-1} x(3r) W_N^{3kr} + \sum_{r=0}^{N/3-1} x(3r+1) W_N^{(3r+1)k} + \sum_{r=0}^{N/3-1} x(3r+2) W_N^{(3r+2)k} \\
&= \sum_{r=0}^{N/3-1} x(3r) (W_N^3)^{kr} + W_N^k \sum_{r=0}^{N/3-1} x(3r+1) (W_N^3)^{kr} + W_N^{2k} \sum_{r=0}^{N/3-1} x(3r+2) (W_N^3)^{kr}
\end{aligned}$$

According to Euler's Formula,  $W_N^a = W_{N/a}$ ,

$$\begin{aligned}
X[k] &= \sum_{r=0}^{N/3-1} x(3r)(W_N^3)^{kr} + W_N^k \sum_{r=0}^{N/3-1} x(3r+1)(W_N^3)^{kr} + W_N^{2k} \sum_{r=0}^{N/3-1} x(3r+2)(W_N^3)^{kr} \\
&= \sum_{r=0}^{N/3-1} x(3r)(W_{N/3})^{kr} + W_N^k \sum_{r=0}^{N/3-1} x(3r+1)(W_{N/3})^{kr} + W_N^{2k} \sum_{r=0}^{N/3-1} x(3r+2)(W_{N/3})^{kr} \\
&= X_1[k] + W_N^k X_2[k] + W_N^{2k} X_3[k]
\end{aligned}$$

We can use FFT butterfly diagram to analysis a FFT computing process and we can clearly tell how the sequence is divided and computed. Here we will have a simplest  $N = 9$ , radix-3, butterfly diagram as example:

Figure ??

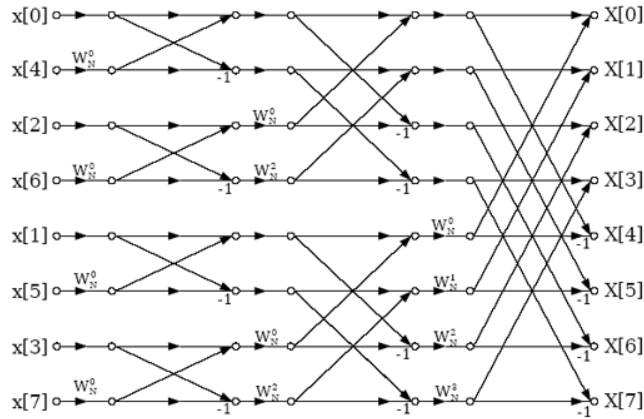


Figure 3: Radix-3 9-Point FFT

From the diagram, there were two times of division of sequences. Each division split sequence into three parts. The first sequence is the sum of all terms whose sequence number can be divided by 3 without the rest. The second sequence is the sum of all terms whose sequence number can be divided by 3 with rest of 1. The third sequence is the sum of all terms whose sequence number can be divided by 3 with the rest of 2. The first time division spilt the sequence into three parts following the rules as mentioned. The second time, we implement the same rule to split these three sequences so that each sequence is divided into three sequences. So far from the lowest level, we can consider our original sequence as a nine-piece sequences. And then we can compute the sequence from the lowest level. Thus, we can develop our algorithm to a bigger domain. Unless the number of terms in a sequence



is in pattern of  $3^s (s \in \mathbb{Z}, s < 20)$ , the sequence can be reduced until the smallest piece is consist of three terms.

Radix-3 IFFT implements the same algorithm.

## 6 Variabilities

### 6.1 Assumptions

- A1: The number of input of radix-2 FFT and IFFT should be  $2^s (s \in \mathbb{Z}, s < 20)$  [T??].
- A2: The number of input of radix-3 FFT and IFFT should be  $3^s (s \in \mathbb{Z}, s < 20)$  [T??].
- A3: The length of  $X[k]$  sequence is always equal to the length of sequence  $x(n)$  [T??].
- A4: The coefficient of every term in  $x(n)$  is an integer for both complex and real number [T??].
- A5: When retrieving the complex number FFT function, some terms are still real numbers. Users enter 0 as an coefficient for complex terms under this occation [T??].
- A6: Assume the strides for  $X[n]$  and  $x(n)$  are both 1.
- A7: The length of input sequence is known [T??].

### 6.2 Calculation

Not Applicable.

### 6.3 Output

Not Applicable.

## 7 Traceability Matrices and Graphs

The purpose of the traceability matrices is to provide easy references on what has to be additionally modified if a certain component is changed. Every time a component is changed, the items in the column of that component that are marked with an “X” should be modified as well. Table ?? shows the dependencies of theoretical models, general definitions, data definitions, and instance models with each other. Table ?? shows the dependencies of instance models, requirements, and data constraints on each other. Table ?? shows the dependencies of theoretical models, general definitions, data definitions, instance models, and likely changes on the assumptions.

	T??	T??	T??	T??	T??	GD??	GD??	DD??	DD??	DD??	DD??	IM??	IM??
T??													
T??			X										
T??													
T??			X										
T??													
GD??													
GD??	X												
DD??				X									
DD??				X									
DD??													
DD??								X					
IM??					X	X	X				X		
IM??					X		X		X	X			X
IM??		X											
IM??		X	X				X	X	X		X		

Table 1: Traceability Matrix Showing the Connections Between Items of Different Sections

	IM??	IM??	IM??	IM??	??	R??	R??
IM??		X				X	X
IM??	X			X		X	X
IM??						X	X
IM??		X				X	X
R??							
R??						X	
R??					X		
R??	X	X				X	X
R??	X						
R??		X					
R??			X				
R??				X			
R??			X	X			
R??		X					
R??		X					

Table 2: Traceability Matrix Showing the Connections Between Requirements and Instance Models

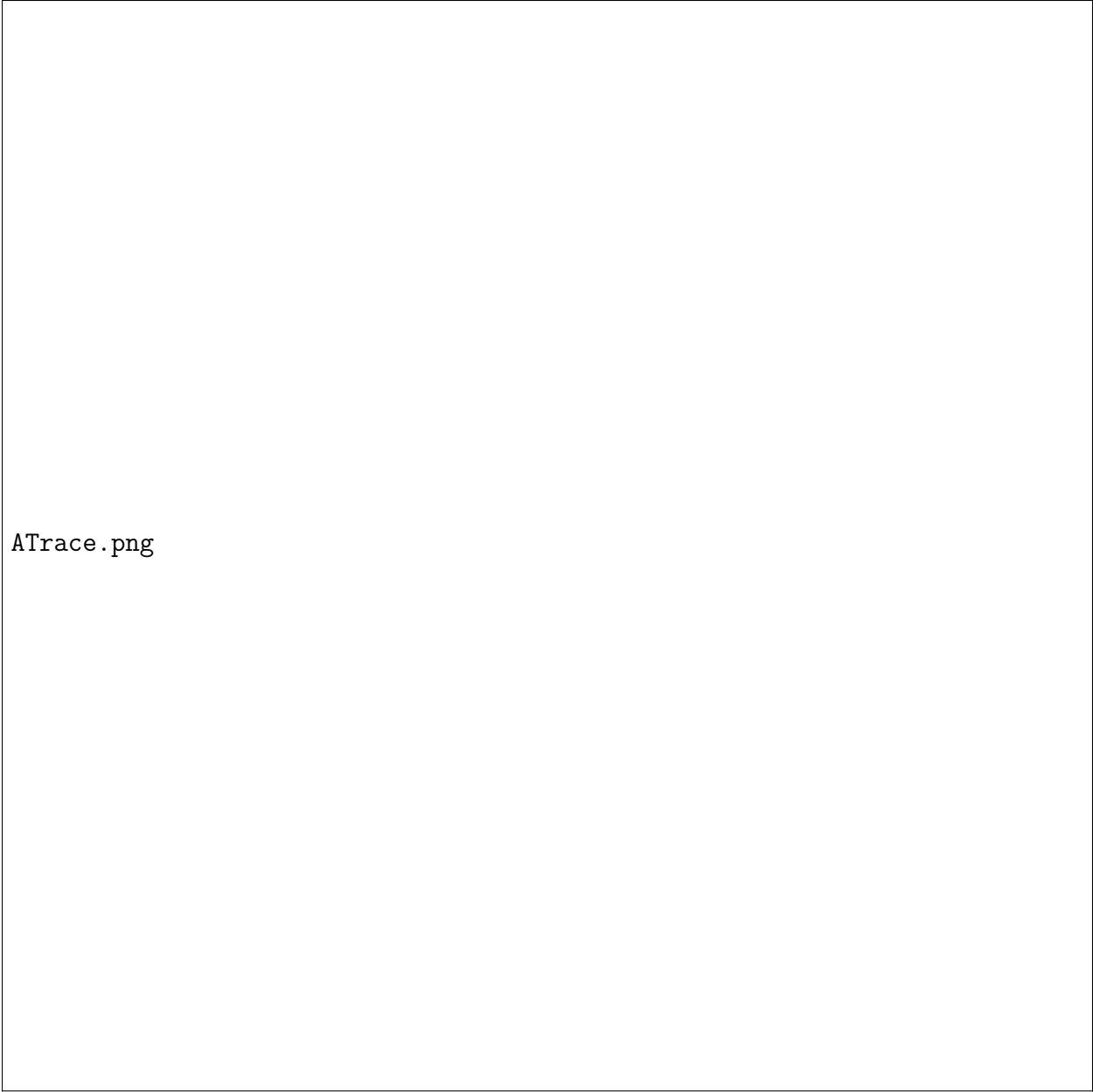
	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??	A??
T??	X																		
T??																			
T??																			
T??																			
T??																			
GD??		X																	
GD??			X	X	X	X													
DD??							X	X	X										
DD??			X	X						X									
DD??																			
DD??																			
IM??											X	X		X	X	X			X
IM??												X	X			X	X	X	
IM??														X					X
IM??													X					X	
LC??				X															
LC??								X											
LC??									X										
LC??											X								
LC??												X							
LC??															X				

Table 3: Traceability Matrix Showing the Connections Between Assumptions and Other Items

The purpose of the traceability graphs is also to provide easy references on what has to be additionally modified if a certain component is changed. The arrows in the graphs represent dependencies. The component at the tail of an arrow is depended on by the component at the head of that arrow. Therefore, if a component is changed, the components that it points to should also be changed. Figure ?? shows the dependencies of theoretical models, general definitions, data definitions, instance models, likely changes, and assumptions on each other. Figure ?? shows the dependencies of instance models, requirements, and data constraints on each other.

NOTE: Building a tool to automatically generate the graphical representation of the matrix by scanning the labels and reference can be future work.

[You will have to add tables. —SS]



ATrace.png

Figure 4: Traceability Matrix Showing the Connections Between Items of Different Sections

## 8 Appendix

[Your report may require an appendix. For instance, this is a good point to show the values of the symbolic parameters introduced in the report. —SS]

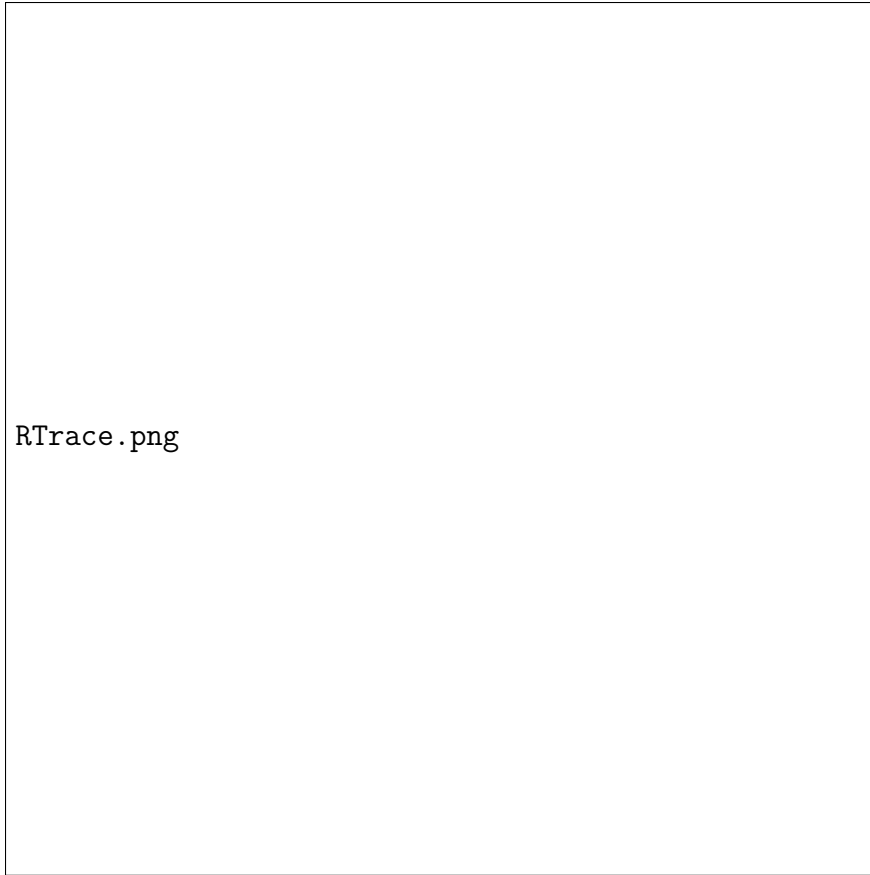


Figure 5: Traceability Matrix Showing the Connections Between Requirements, Instance Models, and Data Constraints

## 8.1 Symbolic Parameters

[The definition of the requirements will likely call for `SYMBOLIC_CONSTANTS`. Their values are defined in this section for easy maintenance. —SS]