COMP 330

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2018-9-24

DATE:9-24 LECTURE 9

Consider $L = \{a^n b^n | n \geq 0\}$, the language is not regular, as finite automata can't do unbounded counting.
Suppose that M recognizes L (M is a DFA)
We assume that M has k states, we take the word $a^m b^m \quad (m > k)$, according to the pigeon hole theorem, such a string must hit the same state twice as it has more "a" than the number of states.
Suppose that there is such a loop with length k, then we can say that the word $a^{m+k} b^m \quad a^{m+2k} b^m \quad a^{m+3k} b^m$ should also be accepted, while there are not in the language.

Pumping Lemma:

THEOREM: For any regular language L, $\exists p > 0$, such that $\forall w \in L, |w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and $|xy| \leq P, |y| \geq 1$,
then we can say that $\forall i \in \mathbb{N}, xy^i z \in L$
NOTE: x, y could be ε

REMARK:

- L is regular $\implies$ L can be pumped
- L can't be pumped $\implies$ L is not regular

Contrapositive Version:

Suppose that $L \in \Sigma^*$, such that, $\forall p > 0, \exists w \in L$ with $|w| \geq P$ such that $\forall x, y, z \in \Sigma^*$ such that $w = xyz$ and $|xy| \leq P, |y| \geq 1$, $\exists i \in \mathbb{N}, xy^i z \notin L$,
then, we can say L is not regular.

We can consider it as a game: Demon: $\forall$ Me: $\exists$

- (1): Demon chooses a random number p, I won't know what he will choose
- (2): I choose a string w with $|w| > p$, this $|w|$ tends to be represented in terms of p, which means I do operation according to Demon's choice
- (3): Demon choose x, y, z randomly, with $|xy| \leq P, |y| \geq 1$
- (4): I choose i, so that $xy^i z \notin L$

EXAMPLE: $L = \{a^n b^n | n \geq 0\}$

- Demon choose number p
- I will choose word $a^p b^p$, which is obviously in the language
- as Demon has to choose x, y, z randomly, with $|xy| \leq P, |y| \geq 1$, so his xy must contains "a" only, suppose $|y| = k$
- I choose $i = 2$, then the $xy^2 z = a^{p+k} b^p$, obviously, $a^{p+k} b^p \notin L$

Some tricks:

L may be hard to pump.

- If R is a regular language, then $\bar{R}$ is a regular language
(as R can be recognized by a DFA, then we just need to change the accept state into decline state and change the decline state into the accept state)
- If R is a regular language and L is a regular language then $R \cap L$ is a regular language.
(as union is regular, thus we can use union and complement as well as de morgen's law to prove intersection is regular as well)

EXAMPLE: $L = \{a^m b^n | m \neq n\}$
Suppose that L is regular, then $\bar{L}$ is regular. Then $\bar{L} \cap a^* b^* = \{a^n b^n | n \geq 0\}$ is regular, and this is not regular.

DATE:9-26 LECTURE 10

EXAMPLE 1

$$L = \{a^{2^n} | a, aa, aaaa, \ldots\}$$

- Demon choose number p
- I choose $|w| = 2^m > p$, obviously a^{2^m} is in the language L
- as Demon has to choose x, y, z randomly, with $|xy| \leq p, |y| \geq 1$, so his $|y| \leq P$
- I choose $i = 2$, then the $xy^2 z = a^{2^m} \cdot a^{|y|} = a^{2^m + |y|}$.
So, $|xy^2 z| = 2^m + |y| \leq 2^m + p < 2^m + 2^m = 2^{m+1}$
And we know that $|xy^2 z| > 2^m$, so $xy^2 z$ is not in L

EXAMPLE 2

$$L = \{a^q | q \text{ is a prime number}\}$$

- Demon choose number p
- I choose $|w| = n > p$, where n is prime number as well. obviously a^n is in the language L
- as Demon has to choose x, y, z randomly, with $|xy| \leq p, |y| \geq 1$, so his $y = a^k$ where $0 < k \leq p$
- Now, $xy^i z = a^{n+(i-1)k}$, so I choose $i = n + 1$, then we have $xy^i z = a^{n+(i-1)k} = a^{n+nk}$ which is not a prime.

EXAMPLE 3

$$\Sigma = \{a, b\} \qquad L = \{w|\text{the number of a is different from the number of b}\}$$

- Demon choose number p
- I choose $w = a^p b^{p!+p}$. Obviously w is in the language L
- as Demon has to choose x, y, z randomly, with $|xy| \leq p, |y| \geq 1$, so his $y = a^k$ where $0 < k \leq p$
- Now, $xy^iz = a^{p+(i-1)k}b^{p!+p}$, so I choose $i = \frac{p!}{k} + 1$, then we have $xy^iz = a^{p!+p}b^{p!+p}$ which is not a prime.

OR:

$$\bar{L} = \{w|\text{the number of a is the same as the number of b}\}$$

So, we can use $\bar{L} \cap a^*b^* = \{a^nb^n|n \geq 0\}$, which is not a regular language.
So, L is not regular as well.

EXAMPLE 4

$$\Sigma = \{a, b\} \qquad L = \{a^ib^j|gcd(i, j) = 1\}$$

We will show $\bar{L}$ is not regular.

- Demon choose number p
- I choose $q > p + 1$ and q is a prime . Obviously a^qb^q is in the language $\bar{L}$
- as Demon has to choose x, y, z randomly, with $|xy| \leq p, |y| \geq 1$, so his $y = a^k$ where $0 < k \leq p$
- Now, $xy^iz = a^{q+(i-1)k}b^q$, so I choose $i = 0$, then we have $xy^iz = a^{q-k}b^q$
  As $q - k > p + 1 - k \geq 1$ and q is a prime, so we have $gcd(q - k, q) = 1$, so $xy^iz \notin \bar{L}$
  Thus $\bar{L}$ is not regular so is L