Numerical ODE (10 points)

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In this course, we mainly focus on the following general *Initial-Value-Problem* for a first order ODE.

$$\begin{cases} x' = f(x,t) \\ x(a) = x_0 \end{cases} \qquad or \qquad \begin{cases} \dfrac{dx(t)}{dt} = f\Big(x(t), t\Big) \\ \\ x(a) = x_0 \end{cases}$$

In many applications, the closed-form solution for the above IVP may be very complicated and difficult to evaluate or there is no closed-form solution. So, we want a numerical solution.

A numerical algorithm for solving an ODE produces a sequence of points $(t_i, x_i), \quad i = 0, 1, 2, \ldots$ where x_i is an approximation to the true value $x(t_i)$, while the mathematical solution is a continuous function $x(t)$

NOTE: But you can do polynomial interpolation on the points we have already calculated to get a continuous form.

Euler's Method:

We would like to find approximate values of the solution to the IVP over the interval $[a, b]$. Use $n+1$ points $t_0, t_1, t_2, \ldots, t_n$ to equally partition $[a, b]$. We call $h = t_{i+1} - t_i = \dfrac{b-a}{n}$ **Size Step**.

Suppose that we have already obtained x_i, an approximation to $x(t_i)$. We would like to get x_{i+1}, an approximation to $x(t_{i+1})$. We use the Taylor Expansion:

$$x(t_{i+1}) \approx x(t_i) + (t_{i+1} - t_i)x'(t_i) = x(t_i) + h \cdot f\Big(x(t_i), t_i\Big)$$

leads to the formula of Euler's Method:

$$x_{i+1} = x_i + h \cdot f(t_i, x_i)$$

Algorithm 1 Euler's Method:

Input: $f(x,t), [a,b], x_0, n$ $\triangleright$ x_0 is value of initial point

$h \leftarrow \dfrac{b-a}{n}$

$t_0 \leftarrow a$

for $i = 0 : n-1$ **do**

  $x_{i+1} \leftarrow x_i + h \cdot f(t_i, x_i)$

  $t_{i+1} = t_i + h$

end for

Error Analysis for Euler's Method:

By Taylor Expansion, we have:

$$x(t_{i+1}) = x(t_i) + h \cdot f(t_i, x(t_i)) + \frac{1}{2}h^2 x''(z_{i+1}) \qquad z_{i+1} \in [t_i, t_{i+1}] \tag{1}$$

Euler's Methods gives us:

$$x_{i+1} = x_i + h \cdot f(t_i, x_i) \tag{2}$$

We use (1) – (2), then we have:

$$x(t_{i+1}) - x_{i+1} = \Big[x(t_i) - x_i\Big] + h \cdot \Big[f(t_i, x(t_i)) - f(t_i, x_i)\Big] + \frac{1}{2} \cdot h^2 x''(z_{i+1}) \qquad z_{i+1} \in [t_i, t_{i+1}]$$

$x(t_{i+1}) - x_{i+1}$ is the error at t_{i+1}. This is called the **global error** at t_{i+1}. It arises from two sources:

- The **Local Truncation Error**: $\frac{1}{2} \cdot h^2 x''(z_{i+1}) \qquad O(h^2)$
- The **Propagation Error:** $\Big[x(t_i) - x_i\Big] + h \cdot \Big[f(t_i, x(t_i)) - f(t_i, x_i)\Big]$. This is due to the accumulated effects of all local truncation errors at $t_1, t_2, \ldots t_i$

When we do computing on computers, there will be **rounding errors** as well.

Trapezoidal Euler's Method:

For $x'(t) = f\Big(t, x(t)\Big)$, we integral both sides:

$$\int_{t_i}^{t_{i+1}} x'(t)\ dt = \int_{t_i}^{t_{i+1}} f\Big(t, x(t)\Big)\ dt \implies x(t_{i+1}) - x_{t_i} = \int_{t_i}^{t_{i+1}} f\Big(t, x(t)\Big)\ dt \tag{3}$$

We apply Trapezoid Rule to the right-hand-side, then we have:

$$x(t_{i+1}) - x_{t_i} \approx \frac{1}{2} \cdot h \cdot \Bigg[f(t_i, x(t_i) + f\Big(t_{i+1}, x(t_{i+1})\Big)\Bigg]$$

This leads to scheme:

$$x_{i+1} = x_i + \frac{1}{2} \cdot h \cdot \Bigg[f(t_i, x_i) + f\Big(t_{i+1}, x_{i+1}\Big)\Bigg]$$

But, there is another question, the Right-Hand-Side involves x_{i+1}, but x_{i+1} is what we want to calculate, to solve this problem, we use **Euler's Method** to compute x_{i+1} on the RHS, leading to the formula of the **Trapezoidal Euler's Method**

$$\begin{cases} \widehat{x}_{i+1} = x_i + h \cdot f(t_i, x_i) \\ \\ x_{i+1} = x_i + \frac{1}{2} \cdot h \cdot \left[f(t_i, x_i) + f\Big(t_{i+1}, \widehat{x}_{i+1}\Big) \right] \end{cases}$$

The Local Truncation Error is $O(h^3)$

Midpoint Euler's Method:

It is really similar to the **Trapezoidal Euler's Method**.

According to Equation(3), we have:

$$x(t_{i+1}) - x_{t_i} = \int_{t_i}^{t_{i+1}} f\Big(t, x(t)\Big)\ dt \tag{4}$$

Then, we apply **Midpoint Rule** on the RHS, then we have:

$$x(t_{i+1}) - x_{t_i} \approx h \cdot f(t_i + \frac{1}{2} \cdot h, x_{i+1/2})$$

This leads to scheme:

$$x_{i+1} = x_i + h \cdot f\Big(t_i + \frac{1}{2} \cdot h,\ x_{i+1/2}\Big)$$

Then, we use **Midpoint Rule** to get $x_{1+1/2}$, which gives us the formula of the **Midpoint Euler's Method**:

$$\begin{cases} x_{i+1/2} = x_i + \frac{1}{2} h \cdot f\Big(t_i,\ x_i\Big) \\ \\ x_{i+1} = x_i + h \cdot f\Big(t_i + \frac{1}{2} \cdot h,\ x_{i+1/2}\Big) \end{cases}$$

The Local Truncation Error is $O(h^3)$