

Solving Non-linear Equations (20 points)

Name: Yuhao Wu
ID Number: 260711365

2018-11-30

Question: Why we need iterative methods?

If f is a general nonlinear function, there is no formula exists for finding roots of $f(x) = 0$, **iterative methods** will be used to compute **approximate roots**

Three algorithms:

Bisection algorithm:

Algorithm 1 Bisection algorithm

```
1: Input:  $f(x)$   $[a, b]$  with  $f(a) \cdot f(b) < 0$  the tolerance  $\delta$ 
2:  $c \leftarrow \frac{a+b}{2}$  ▷  $c$  is the root we will return
3:  $error\ bound \leftarrow \frac{|b-a|}{2}$ 
4: while  $error\ bound > \delta$  do
5:   if  $f(c) = 0$  then
6:     we say  $c$  is the root, stop.
7:   else
8:     if  $f(a) \cdot f(c) < 0$  then
9:        $b \leftarrow c$ 
10:    else
11:       $a \leftarrow c$ 
12:    end if
13:  end if
14:   $c \leftarrow \frac{a+b}{2}$ 
15:   $error\ bound \leftarrow \frac{error\ bound}{2}$ 
16: end while
17: return  $c$ 
```

Newton's Method:

Algorithm 2 Newton's algorithm

```

1: Input:  $f(x)$   $f'(x)$ ,  $x$ ,  $xtol$ ,  $ftol$ ,  $n_{max}$   $\triangleright x \rightarrow$  initial point
2:  $\triangleright n_{max} \rightarrow$  maximum number of iterations
3: for  $n = 1 : n_{max}$  do
4:
5:    $d \leftarrow \frac{f(x)}{f'(x)}$ 
6:
7:    $x \leftarrow x - d$ 
8:   if  $|d| \leq xtol$  or  $|f(x)| \leq ftol$  then
9:      $root \leftarrow x$ 
10:    stop
11:   end if
12:
13: end for
14:  $root \leftarrow x$ 

```

Stop Criteria:

- $|x_{n+1} - x_n| \leq xtol$
- $|f_{n+1}| \leq ftol$
- The maximum number of iteration reached

Newton's Method's Iteration formula:

Given a point x_0 , we do Taylor Expansion about x_0 :

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{f''(z)}{2}(x - x_0)^2$$

where $z \in (x, x_0)$

We use $\ell(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$ as an approximation of $f(x)$. In order to solve $f(x) = 0$, we try to solve $\ell(x) = 0$

$$\ell(x) = 0 \implies f(x_0) + f'(x_0) \cdot (x - x_0) = 0 \implies x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We choose $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ as our approximate root of $f(x) = 0$

If this x_1 is not a good approximate root, we continue this iteration. In general, we have the Newton iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Secant Method:

Sometimes, the derivative of a function is really hard or even impossible to compute, then, we try to use **divided difference** to replace $f'(x_n)$. Then we get our secant iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{use } \mathbf{divided\ difference} \text{ to replace } f'(x_n) \implies \boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n)}$$

Algorithm 3 Secant's algorithm

```

1: Input:  $f(x), x_0, x_1, xtol, ftol, n_{max}$   $\triangleright x_0, x_1 \rightarrow$  initial point
2:  $\triangleright n_{max} \rightarrow$  maximum number of iterations
3: for  $n = 1 : n_{max}$  do
4:
5:    $d \leftarrow \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_1)$ 
6:
7:    $x_0 \leftarrow x_1$ 
8:    $f(x_0) \leftarrow f(x_1)$ 
9:    $x_1 \leftarrow x_1 - d$ 
10:   $fx_1 = f(x_1)$ 
11:
12:  if  $|d| \leq xtol$  or  $|fx_1| \leq ftol$  then
13:     $root \leftarrow x_1$ 
14:    stop
15:  end if
16:
17: end for
18:  $root \leftarrow x_1$ 

```

Comparison of the Three Methods:

- The Bisection Method:
 - Advantage: Simple, guaranteed to converge, applicable to non-smooth functions
 - Disadvantage: Linear convergence
- Newton's Method:
 - Advantage: Quadratic convergence
 - Disadvantage: need to compute f' may not converge
- The Secant Method:
 - Advantage: Super-Linear convergence no need to compute f'
 - Disadvantage: slower than NM may not converge

Convergence Analysis:

Quadratic Convergence: A sequence $\{x_n\}$ is said to have quadratic convergence to x if:

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x|}{|x_n - x|^2} = c$$

where c is some finite non-zero constant.

Suppose r is a root of $f(x) = 0$ and x_n converges to r .

We do Taylor series expansion about x_n

$$f(r) = f(x_n) + (r - x_n)f'(x_n) + \frac{(r - x_n)^2}{2}f''(z_n)$$

where z_n is between r and x_n .

Divide both sides by $f'(x_n)$ and use the fact r is the root of $f(x) = 0$, then we have:

$$0 = \frac{f(x_n)}{f'(x_n)} + (r - x_n) + (r - x_n)^2 \cdot \frac{f''(z_n)}{2f'(x_n)}$$

As we use Newton's Method, we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, then we have:

$$0 = x_n - x_{n+1} + (r - x_n) + (r - x_n)^2 \cdot \frac{f''(z_n)}{2f'(x_n)} \implies r - x_{n+1} = c_n \cdot (r - x_n)^2, \quad c_n = -\frac{f''(z_n)}{2f'(x_n)}$$

As we said before, $x_n \rightarrow r$ and z_n is between x_n and r , thus we have $z_n \rightarrow r$ and then, we have:

$$\lim_{n \rightarrow \infty} \frac{|r - x_{n+1}|}{|r - x_n|^2} = \lim_{n \rightarrow \infty} |c_n| = \left| \frac{f''(r)}{2f'(r)} \right| = c$$

Show that $\lim_{n \rightarrow \infty} \left| \frac{f(x_{n+1})}{f(x_n)^2} \right| = c$

We expand the Taylor Series as following:

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) \cdot f'(x_n) + \frac{(x_{n+1} - x_n)^2}{2} \cdot f''(z_n) \text{ where } z_n \text{ is between } x_n \text{ and } x_{n+1}$$

According to Newton Iteration, we have:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \implies x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

Thus, we have:

$$f(x_{n+1}) = f(x_n) - f(x_n) + \frac{1}{2} \cdot \frac{f(x_n)^2}{f'(x_n)^2} \cdot f''(z_n)$$

$$\frac{f(x_{n+1})}{f(x_n)^2} = \frac{1}{2} \cdot \frac{f''(z_n)}{f'(x_n)^2}$$

As, we know, z_n is between x_n and x_{n+1} , when $n \rightarrow \infty$, we have x_n and x_{n+1} converges to root r , which means, $z_n \rightarrow r$

Thus, we have:

$$\lim_{n \rightarrow \infty} \frac{|f(x_{n+1})|}{|f(x_n)^2|} = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{|f''(z_n)|}{|f'(x_n)|^2} = \frac{1}{2} \cdot \frac{|f''(r)|}{|f'(r)|^2} = c \neq 0$$

Some Questions about this Chapter:

1 Under which cases, NM and SM are not converge?

Answer: the initial point is not close to the root r

2 Do we need to know how to derive the convergence rate of Bisection is linear? (No.)