Polynomial Interpolation (25 points)

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Definition:

A function S is called a spline of degree k if

- The domain of S is an interval $[a, b]$
- $S, S', S'', \ldots, S^{(k)}$ are continuous on $[a, b]$
- There are points t_i (the knots of S) such that $a = t_0 < t_1 < \ldots < t_n = b$ and such that S is a polynomial of degree at most k on each $[t_i, t_{i+1}]$

Linear Spline:

Given $n+1$ points $(t_0, y_0), (t_1, y_1), \ldots, (t_n, y_n)$, obviously, we can write:

$$S(n) = \begin{cases} S_0(x) & t_0 \le x \le t_1 \\ S_1(x) & t_1 \le x \le t_2 \\ \vdots & \\ S_{n-1}(x) & t_{n-1} \le x \le t_n \end{cases}$$

where $S_i(x) = y_i + m_i \cdot (x - t_i)$, $\quad m_i = \dfrac{y_{i+1} - y_i}{t_{i+1} - t_i} \qquad t_i \le x \le t_{i+1}$

Algorithm for evaluating $S(x)$

Algorithm 1 Algorithm for evaluating $S(x)$

Input: given $x, (t_i, y_i)$ *and* m_i ▷ x is the point we want to evaluate
▷ $(t_i, y_i), i = 0, 1, \ldots, n$ is the given knots

for $k = 0 : n-1$ **do** ▷ this is to find which interval x is in
  if $x - t_{i+1} \le 0$ **then**
    Exit Loop
  end if
end for

$S \leftarrow y_i + m_i \cdot (x - t_i)$

In fact, if we choose to use *Binary Search* to find the exact interval, this will makes things more efficient.

Cubic Spline:

Disadvantage of Linear Spline:

For a Linear Spline, we generally don't have S' to be continuous, so its graph lacks of smoothness.

Disadvantage of Quadratic Spline:

For a Quadratic Spline, we generally don't have S'' to be continuous, so the curvature of its graph change abruptly at each **knot** .

So, the most frequently used splines are cubic spline.

Given $n+1$ points $(t_0, y_0), (t_1, y_1), \dots, (t_n, y_n)$, obviously, we can write:

$$S(n) = \begin{cases} S_0(x) & t_0 \le x \le t_1 \\ S_1(x) & t_1 \le x \le t_2 \\ \vdots \\ S_{n-1}(x) & t_{n-1} \le x \le t_n \end{cases}$$

where $S_i(x)$ is a cubic polynomial on $[t_i, t_{i+1}]$

- **Number Of Unknowns:** Each S_i has 4 unknowns. So, there are total $4n$ unknowns.
- **Number Of Conditions:** $S(t_i) = y_i$ for $i = 0, 1, 2, \dots n$, leads to $n+1$ conditions.
  $S_{i-1}^{(k)}(t_i) = S_i^{(k)}(t_i)$ for $i = 0, 1, \dots, n$ and $k = 0, 1, 2$ gives us $3(n-1)$ conditions.
  Total $4n-2$ conditions.

In order to get unique solution, we add two more extra conditions.

$$S''(t_0) = S''(t_n) = 0$$

EXAMPLE:

x	-1	0	1
y	1	2	-1

$$S(x) = \begin{cases} S_0(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 \\ \\ S_1(x) = b_0 + b_1 \cdot x + b_2 \cdot x^2 + b_3 \cdot x^3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = a_1 + 2a_2 \cdot x + 3a_3 \cdot x^2 \\ \\ S_1'(x) = b_1 + 2b_2 \cdot x + 3b_3 \cdot x^2 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 2a_2 + 6a_3 \cdot x \\ \\ S_1''(x) = 2b_2 + 6b_3 \cdot x \end{cases}$$

$$S_0(-1) = a_0 - a_1 + a_2 - a_3 = 1$$
$$S_0(0) = S_1(0) = 2 \implies a_0 = b_0 = 2$$
$$S_1(1) = b_0 + b_1 + b_2 + b_3 = -1$$

$$S_0'(0) = S_1'(0) \implies a_1 = b_1$$

$$S_0''(0) = S_1''(0) \implies a_2 = b_2$$

$$S_0''(-1) = 0 \implies 2a_2 = 6a_3$$
$$S_1''(1) = 0 \implies 2b_2 + 6b_3 = 0$$

Thus, we have

$$\begin{cases} a_0 = 2 \quad a_1 = -1 \quad a_2 = -3 \quad a_3 = -1 \\ \\ b_0 = 2 \quad b_1 = -1 \quad b_2 = -3 \quad b_3 = 1 \end{cases}$$

Least Square Approximation:

Data Fitting By a Straight Line:

Suppose the data are thought to conform to a linear relationship:

$$y = a \cdot x + b$$

We want to solve the following optimization problem:

$$\min_{a,b} \phi(a,b) \quad where\ \phi(a,b) = \sum_{k=0}^{m} \Big(ax_k + b - y_k \Big)^2$$

From Calculus, the conditions that

$$\frac{\partial \phi}{\partial a} = 0 \qquad \frac{\partial \phi}{\partial b} = 0$$

are necessary at the minimum.

$$\frac{\partial \phi}{\partial a} = 0 \implies 2 \cdot \sum_{k=0}^{m} \Big(ax_k + b - y_k \Big) \cdot x_k = 0 \implies a \cdot \sum_{k=0}^{m} x_k^2 + b \cdot \sum_{k=0}^{m} x_k = \sum_{k=0}^{m} y_k \cdot x_k$$

$$\frac{\partial \phi}{\partial b} = 0 \implies 2 \cdot \sum_{k=0}^{m} \Big(ax_k + b - y_k \Big) = 0 \implies a \cdot \sum_{k=0}^{m} x_k + (m+1) \cdot b = \sum_{k=0}^{m} y_k$$

which is to solve the linear equations as follows:

$$\begin{bmatrix} \sum_{k=0}^{m} x_k^2 & \cdot \sum_{k=0}^{m} x_k \\ \\ \sum_{k=0}^{m} x_k & (m+1) \end{bmatrix} \cdot \begin{bmatrix} a \\ \\ b \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{m} y_k \cdot x_k \\ \\ \sum_{k=0}^{m} y_k \end{bmatrix}$$

Data Fitting by a General Linear Family of Functions:

Suppose that the data are thought to conform to a relationship like:

$$y = \sum_{j=0}^{n} c_j \cdot g_j(x)$$

where the functions $g_0, g_1, \ldots, g_n$ (called basis function) are known, the coefficients $c_0, c_1, \ldots, c_n$ are to be determined.

Then we need to solve the least square problem:

$$\min_{c_0, c_1, \ldots, c_n} \phi(c_0, c_1, \ldots, c_n) \quad where\ \phi(c_0, c_1, \ldots, c_n) = \sum_{k=0}^{m} \Big(\sum_{j=0}^{n} c_j \cdot g_j(x_k) - y_k \Big)^2 \tag{1}$$

Now, we calculate the first derivative:

$$\frac{\partial \phi}{\partial c_i} = \sum_{k=0}^{m} 2 \times \left[\sum_{j=0}^{n} c_j \cdot g_j(x_k) - y_k \right] \cdot g_i(x_k) = 0 \qquad i = 0, 1, 2, \ldots, n$$

Now, we have:

$$\sum_{k=0}^{m} \left[\sum_{j=0}^{n} c_j \cdot g_j(x_k) - y_k \right] \cdot g_i(x_k) = 0 \implies \sum_{j=0}^{m} \left[\sum_{k=0}^{n} \cdot g_j(x_k) \cdot g_i(x_k) \right] \cdot c_j = \sum_{k=0}^{m} g_i(x_k) \cdot y_k \tag{2}$$

Normal Equations

Another way of thinking:

For Equation(1), we can easily write it into the form like:

$$A = \begin{bmatrix} g_0(x_0) & g_1(x_0) & \ldots & g_n(x_0) \\ g_0(x_0) & g_1(x_1) & \ldots & g_n(x_1) \\ \vdots & \vdots & \ldots & \vdots \\ g_0(x_m) & g_1(x_m) & \ldots & g_n(x_m) \end{bmatrix} \qquad c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ v_n \end{bmatrix} \qquad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Then our problem changes into

$$\min_c ||A \cdot c - y||_2^2$$

$A \cdot c$ can be considered as $A \cdot c = c_0 \cdot \mathbf{a}_0 + c_1 \cdot \mathbf{a}_1 + \ldots + c_n \cdot \mathbf{a}_n$ where $\mathbf{a}_i$ is the column vector of matrix A. We can assume that all the column vector of the A are linearly independent, then this constructs a Hyper-Plane of dimension m.

In order to minimize the $||A \cdot c - y||_2^2$, geometrically, we need $A \cdot c - y$ to be perpendicular to the Hyper-Plane, which means to perpendicular to every basis of vector space.

Thus, we have:

$$Ac - y \perp \mathbf{a}_j \quad j = 0, 1, \ldots, n \implies \mathbf{a}_j^T (Ac - y) = 0 \implies A^T (A \cdot c - y) = 0 \implies A^T A \cdot c = A^T \cdot y$$

If Basic Function is Exponential Function:

$$y(x) = k \cdot e^{-\lambda \cdot x}$$

x	x_0	x_1	$\dots$	x_m
y	y_0	y_1	$\dots$	y_m

Then we take ln both sides, which gives us $\ln y(x) = \ln k - \lambda \cdot x$

Now, we just need to calculate

$$\min \sum_{i=0}^{m} (\ln k - \lambda \cdot x_i - \ln y_i)^2$$

which is the same as straight line case.

EXAMPLE of Straight Line Data Fitting:

TEXTBOOK PAGE 498 EXAMPLE:

x	4	7	11	13	17
y	2	0	2	6	7

Straight Line can be considered as $y = ax + b$ where $g_0(x) = x \quad g_1(x) = b$

Thus we can build matrix A, c as following:

$$A = \begin{bmatrix} 4 & 1 \\ 7 & 1 \\ 11 & 1 \\ 13 & 1 \\ 17 & 1 \end{bmatrix} \qquad c = \begin{bmatrix} a \\ b \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \\ 7 \end{bmatrix}$$

Now we want to find c such that

$$\min_c ||A \cdot c - y||$$

$$A^T \cdot A \cdot c = A^T \cdot y$$

which gives us:

$$\begin{bmatrix} 644 & 52 \\ 52 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 227 \\ 17 \end{bmatrix}$$