Study guide: Time-dependent problems and variational forms

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Time-dependent problems

- So far: used the finite element framework for discretizing in space
- What about $u_t = u_{xx} + f$?
    1. Use finite differences in time to obtain a set of recursive spatial problems
    2. Solve the spatial problems by the finite element method

Example: diffusion problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha \nabla^2 u + f(\boldsymbol{x}, t), & \boldsymbol{x} \in \Omega, t \in (0, T] \\
u(\boldsymbol{x}, 0) &= I(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \\
\frac{\partial u}{\partial n} &= 0, & \boldsymbol{x} \in \partial\Omega, \ t \in (0, T]
\end{aligned}
$$

A Forward Euler scheme; ideas

$$
[D_t^+ u = \alpha \nabla^2 u + f]^n, \quad n = 1, 2, \ldots, N_t - 1
$$

Solving wrt u^{n+1}:

$$
u^{n+1} = u^n + \Delta t \left(\alpha \nabla^2 u^n + f(\boldsymbol{x}, t_n) \right)
$$

- $u^n = \sum_j c_j^n \psi_j \in V$, $u^{n+1} = \sum_j c_j^{n+1} \psi_j \in V$
- Compute u^0 from I
- Compute u^{n+1} from u^n by solving the PDE for u^{n+1} at each time level

A Forward Euler scheme; stages in the discretization

- $u_e(\boldsymbol{x}, t)$: exact solution of the PDE problem

- $u_e^n(\boldsymbol{x})$: exact solution of time-discrete problem (after applying a finite difference scheme in time)

- $u_e^n(\boldsymbol{x}) \approx u^n = \sum_{j \in \mathcal{I}_s} c_j^n \psi_j$ = solution of the time- and space-discrete problem (after applying a Galerkin method in space)

$$\frac{\partial u_e}{\partial t} = \alpha \nabla^2 u_e + f(\boldsymbol{x}, t)$$

$$u_e^{n+1} = u_e^n + \Delta t \left(\alpha \nabla^2 u_e^n + f(\boldsymbol{x}, t_n) \right)$$

$$u_e^n \approx u^n = \sum_{j=0}^{N} c_j^n \psi_j(\boldsymbol{x}), \quad u_e^{n+1} \approx u^{n+1} = \sum_{j=0}^{N} c_j^{n+1} \psi_j(\boldsymbol{x})$$

$$R = u^{n+1} - u^n - \Delta t \left(\alpha \nabla^2 u^n + f(\boldsymbol{x}, t_n) \right)$$

A Forward Euler scheme; weighted residual (or Galerkin) principle

$$R = u^{n+1} - u^n - \Delta t \left(\alpha \nabla^2 u^n + f(\boldsymbol{x}, t_n) \right)$$

The weighted residual principle:

$$\int_\Omega R w \, \mathrm{d}x = 0, \quad \forall w \in W$$

results in

$$\int_\Omega \left[u^{n+1} - u^n - \Delta t \left(\alpha \nabla^2 u^n + f(\boldsymbol{x}, t_n) \right) \right] w \, \mathrm{d}x = 0, \quad \forall w \in W$$

Galerkin: $W = V$, $w = v$

A Forward Euler scheme; integration by parts

Isolating the unknown u^{n+1} on the left-hand side:

$$\int_\Omega u^{n+1} \psi_i \, \mathrm{d}x = \int_\Omega \left[u^n + \Delta t \left(\alpha \nabla^2 u^n + f(\boldsymbol{x}, t_n) \right) \right] v \, \mathrm{d}x$$

Integration by parts of $\int \alpha (\nabla^2 u^n) v \, \mathrm{d}x$:

$$\int_\Omega \alpha (\nabla^2 u^n) v \, \mathrm{d}x = - \int_\Omega \alpha \nabla u^n \cdot \nabla v \, \mathrm{d}x + \underbrace{\int_{\partial\Omega} \alpha \frac{\partial u^n}{\partial n} v \, \mathrm{d}x}_{=0 \quad \Leftarrow \quad \partial u^n / \partial n = 0}$$

Variational form:

$$\int_\Omega u^{n+1} v \, \mathrm{d}x = \int_\Omega u^n v \, \mathrm{d}x - \Delta t \int_\Omega \alpha \nabla u^n \cdot \nabla v \, \mathrm{d}x + \Delta t \int_\Omega f^n v \, \mathrm{d}x, \quad \forall v \in V$$

New notation for the solution at the most recent time levels

- u and `u`: the spatial unknown function to be computed

- u_1 and `u_1`: the spatial function at the previous time level $t - \Delta t$

- u_2 and `u_2`: the spatial function at $t - 2\Delta t$

- This new notation gives close correspondence between code and math

$$\int_\Omega uv \, \mathrm{d}x = \int_\Omega u_1 v \, \mathrm{d}x - \Delta t \int_\Omega \alpha \nabla u_1 \cdot \nabla v \, \mathrm{d}x + \Delta t \int_\Omega f^n v \, \mathrm{d}x$$

or shorter

$$(u, v) = (u_1, v) - \Delta t (\alpha \nabla u_1, \nabla v) + \Delta t (f^n, v)$$

Deriving the linear systems

- $u = \sum_{j=0}^N c_j \psi_j(\boldsymbol{x})$

- $u_1 = \sum_{j=0}^N c_{1,j} \psi_j(\boldsymbol{x})$

- $\forall v \in V$: for $v = \psi_i$, $i = 0, \ldots, N$

Insert these in

$$(u, \psi_i) = (u_1, \psi_i) - \Delta t (\alpha \nabla u_1, \nabla \psi_i) + \Delta t (f^n, \psi_i)$$

and order terms as matrix-vector products $(i = 0, \ldots, N)$:

$$\sum_{j=0}^N \underbrace{(\psi_i, \psi_j)}_{M_{i,j}} c_j = \sum_{j=0}^N \underbrace{(\psi_i, \psi_j)}_{M_{i,j}} c_{1,j} - \Delta t \sum_{j=0}^N \underbrace{(\nabla \psi_i, \alpha \nabla \psi_j)}_{K_{i,j}} c_{1,j} + \Delta t (f^n, \psi_i)$$

Structure of the linear systems

$$Mc = Mc_1 - \Delta t K c_1 + \Delta t f$$

$$
\begin{aligned}
M &= \{M_{i,j}\}, \quad M_{i,j} = (\psi_i, \psi_j), \quad i, j \in \mathcal{I}_s \\
K &= \{K_{i,j}\}, \quad K_{i,j} = (\nabla \psi_i, \alpha \nabla \psi_j), \quad i, j \in \mathcal{I}_s \\
f &= \{(f(\boldsymbol{x}, t_n), \psi_i)\}_{i \in \mathcal{I}_s} \\
c &= \{c_i\}_{i \in \mathcal{I}_s} \\
c_1 &= \{c_{1,i}\}_{i \in \mathcal{I}_s}
\end{aligned}
$$

Computational algorithm

1. Compute M and K.

2. Initialize u^0 by either interpolation or projection

3. For $n = 1, 2, \ldots, N_t$:

   (a) compute $b = Mc_1 - \Delta t K c_1 + \Delta t f$

   (b) solve $Mc = b$

   (c) set $c_1 = c$

Initial condition:

- Either interpolation: $c_{1,j} = I(\boldsymbol{x}_j)$ (finite elements)

- Or projection: solve $\sum_j M_{i,j} c_{1,j} = (I, \psi_i)$, $i \in \mathcal{I}_s$

Comparing P1 elements with the finite difference method; ideas

- P1 elements in 1D

- Uniform mesh on $[0, L]$ with cell length h

- No Dirichlet conditions: $\psi_i = \varphi_i$, $i = 0, \ldots, N = N_n - 1$

- Have found formulas for M and K at the element level

- Have assembled the global matrices

- Have developed corresponding finite difference operator formulas

- M: $h[D_t^+ (u + \frac{1}{6} h^2 D_x D_x u)]_i^n$

- K: $h[\alpha D_x D_x u]_i^n$

Comparing P1 elements with the finite difference method; results

Diffusion equation with finite elements is equivalent to

$$[D_t^+ (u + \frac{1}{6} h^2 D_x D_x u) = \alpha D_x D_x u + f]_i^n$$

Can lump the mass matrix by Trapezoidal integration and get the standard finite difference scheme

$$[D_t^+ u = \alpha D_x D_x u + f]_i^n$$

Discretization in time by a Backward Euler scheme

Backward Euler scheme in time:

$$[D_t^- u = \alpha \nabla^2 u + f(\boldsymbol{x}, t)]^n$$

$$u_{\mathrm{e}}^n - \Delta t \left(\alpha \nabla^2 u_{\mathrm{e}}^n + f(\boldsymbol{x}, t_n) \right) = u_{\mathrm{e}}^{n-1}$$

$$u_{\mathrm{e}}^n \approx u^n = \sum_{j=0}^{N} c_j^n \psi_j(\boldsymbol{x}), \quad u_{\mathrm{e}}^{n+1} \approx u^{n+1} = \sum_{j=0}^{N} c_j^{n+1} \psi_j(\boldsymbol{x})$$

The variational form of the time-discrete problem

$$\int_\Omega \left(u^n v + \Delta t \alpha \nabla u^n \cdot \nabla v \right) \mathrm{d}x = \int_\Omega u^{n-1} v \, \mathrm{d}x + \Delta t \int_\Omega f^n v \, \mathrm{d}x, \quad \forall v \in V$$

or

$$(u, v) + \Delta t (\alpha \nabla u, \nabla v) = (u_1, v) + \Delta t (f^n, \psi_i)$$

The linear system: insert $u = \sum_j c_j \psi_i$ and $u_1 = \sum_j c_{1,j} \psi_i$,

$$(M + \Delta t K) c = M c_1 + \Delta t f$$

Calculations with P1 elements in 1D

Can interpret the resulting equation system as

$$[D_t^- (u + \frac{1}{6} h^2 D_x D_x u) = \alpha D_x D_x u + f]_i^n$$

Lumped mass matrix (by Trapezoidal integration) gives a standard finite difference method:

$$[D_t^- u = \alpha D_x D_x u + f]_i^n$$

Dirichlet boundary conditions

Dirichlet condition at $x = 0$ and Neumann condition at $x = L$:

$$u(\boldsymbol{x}, t) = u_0(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \partial \Omega_D$$

$$-\alpha \frac{\partial}{\partial n} u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \partial \Omega_N$$

Forward Euler in time, Galerkin's method, and integration by parts:

$$\int_\Omega u^{n+1}v\,\mathrm{d}x = \int_\Omega (u^n - \Delta t\alpha\nabla u^n\cdot\nabla v)\,\mathrm{d}x + \Delta t\int_\Omega fv\,\mathrm{d}x - \Delta t\int_{\partial\Omega_N} gv\,\mathrm{d}s, \quad \forall v \in V$$

Requirement: $v = 0$ on $\partial\Omega_D$

Boundary function

$$u^n(\boldsymbol{x}) = u_0(\boldsymbol{x}, t_n) + \sum_{j\in\mathcal{I}_s} c_j^n \psi_j(\boldsymbol{x})$$

$$\sum_{j\in\mathcal{I}_s}\left(\int_\Omega \psi_i\psi_j\,\mathrm{d}x\right)c_j^{n+1} = \sum_{j\in\mathcal{I}_s}\left(\int_\Omega (\psi_i\psi_j - \Delta t\alpha\nabla\psi_i\cdot\nabla\psi_j)\,\mathrm{d}x\right)c_j^n -$$

$$\int_\Omega (u_0(\boldsymbol{x}, t_{n+1}) - u_0(\boldsymbol{x}, t_n) + \Delta t\alpha\nabla u_0(\boldsymbol{x}, t_n)\cdot\nabla\psi_i)\,\mathrm{d}x$$

$$+ \Delta t\int_\Omega f\psi_i\,\mathrm{d}x - \Delta t\int_{\partial\Omega_N} g\psi_i\,\mathrm{d}s, \quad i \in \mathcal{I}_s$$

Finite element basis functions

- $B(\boldsymbol{x}, t_n) = \sum_{j\in I_b} U_j^n\varphi_j$

- $\psi_i = \varphi_{\nu(j)}$, $j \in \mathcal{I}_s$

- $\nu(j)$, $j \in \mathcal{I}_s$, are the node numbers corresponding to all nodes without a Dirichlet condition

$$u^n = \sum_{j\in I_b} U_j^n\varphi_j + \sum_{j\in\mathcal{I}_s} c_{1,j}\varphi_{\nu(j)},$$

$$u^{n+1} = \sum_{j\in I_b} U_j^{n+1}\varphi_j + \sum_{j\in\mathcal{I}_s} c_j\varphi_{\nu(j)}$$

$$\sum_{j\in\mathcal{I}_s}\left(\int_\Omega \varphi_i\varphi_j\,\mathrm{d}x\right)c_j = \sum_{j\in\mathcal{I}_s}\left(\int_\Omega (\varphi_i\varphi_j - \Delta t\alpha\nabla\varphi_i\cdot\nabla\varphi_j)\,\mathrm{d}x\right)c_{1,j} -$$

$$\sum_{j\in I_b}\int_\Omega \left(\varphi_i\varphi_j(U_j^{n+1} - U_j^n) + \Delta t\alpha\nabla\varphi_i\cdot\nabla\varphi_j U_j^n\right)\,\mathrm{d}x$$

$$+ \Delta t\int_\Omega f\varphi_i\,\mathrm{d}x - \Delta t\int_{\partial\Omega_N} g\varphi_i\,\mathrm{d}s, \quad i \in \mathcal{I}_s$$

Modification of the linear system; the raw system

- Drop boundary function

- Compute as if there are not Dirichlet conditions

- Modify the linear system to incorporate Dirichlet conditions

- $\mathcal{I}_s$ holds the indices of all nodes $\{0, 1, \ldots, N = N_n - 1\}$

$$\sum_{j \in \mathcal{I}_s} \left(\underbrace{\int_\Omega \varphi_i \varphi_j \, \mathrm{d}x}_{M_{i,j}} \right) c_j = \sum_{j \in \mathcal{I}_s} \left(\underbrace{\int_\Omega \varphi_i \varphi_j \, \mathrm{d}x}_{M_{i,j}} - \Delta t \underbrace{\int_\Omega \alpha \nabla \varphi_i \cdot \nabla \varphi_j \, \mathrm{d}x}_{K_{i,j}} \right) c_{1,j}$$

$$\underbrace{+\Delta t \int_\Omega f \varphi_i \, \mathrm{d}x - \Delta t \int_{\partial \Omega_N} g \varphi_i \, \mathrm{d}s}_{f_i}, \quad i \in \mathcal{I}_s$$

Modification of the linear system; setting Dirichlet conditions

$$Mc = b, \quad b = Mc_1 - \Delta t K c_1 + \Delta t f$$

For each k where a Dirichlet condition applies, $u(x_k, t_{n+1}) = U_k^{n+1}$,

- set row k in M to zero and 1 on the diagonal: $M_{k,j} = 0$, $j \in \mathcal{I}_s$, $M_{k,k} = 1$

- $b_k = U_k^{n+1}$

Or apply the slightly more complicated modification which preserves symmetry of M

Modification of the linear system; Backward Euler example

Backward Euler discretization in time gives a more complicated coefficient matrix:

$$Ac = b, \quad A = M + \Delta t K, \quad b = Mc_1 + \Delta t f$$

- Set row k to zero and 1 on the diagonal: $M_{k,j} = 0$, $j \in \mathcal{I}_s$, $M_{k,k} = 1$

- Set row k to zero: $K_{k,j} = 0$, $j \in \mathcal{I}_s$

- $b_k = U_k^{n+1}$

Observe: $A_{k,k} = M_{k,k} + \Delta t K_{k,k} = 1 + 0$, so $c_k = U_k^{n+1}$

Analysis of the discrete equations

The diffusion equation $u_t = \alpha u_{xx}$ allows a (Fourier) wave component

$$u = A_{\mathrm{e}}^n e^{ikx}, \quad A_{\mathrm{e}} = e^{-\alpha k^2 \Delta t}$$

Numerical schemes often allow the similar solution

$$u_q^n = A^n e^{ikx}$$

- A: amplification factor to be computed

- How good is this A compared to the exact one?

Handy formulas

$$[D_t^+ A^n e^{ikq\Delta x}]^n = A^n e^{ikq\Delta x} \frac{A - 1}{\Delta t},$$

$$[D_t^- A^n e^{ikq\Delta x}]^n = A^n e^{ikq\Delta x} \frac{1 - A^{-1}}{\Delta t},$$

$$[D_t A^n e^{ikq\Delta x}]^{n+\frac{1}{2}} = A^{n+\frac{1}{2}} e^{ikq\Delta x} \frac{A^{\frac{1}{2}} - A^{-\frac{1}{2}}}{\Delta t} = A^n e^{ikq\Delta x} \frac{A - 1}{\Delta t},$$

$$[D_x D_x A^n e^{ikq\Delta x}]_q = -A^n \frac{4}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right)$$

Amplification factor for the Forward Euler method; results

Introduce $p = k\Delta x / 2$ and $C = \alpha \Delta t / \Delta x^2$:

$$A = 1 - 4C \frac{\sin^2 p}{\underbrace{1 - \frac{2}{3}\sin^2 p}_{\text{from } M}}$$

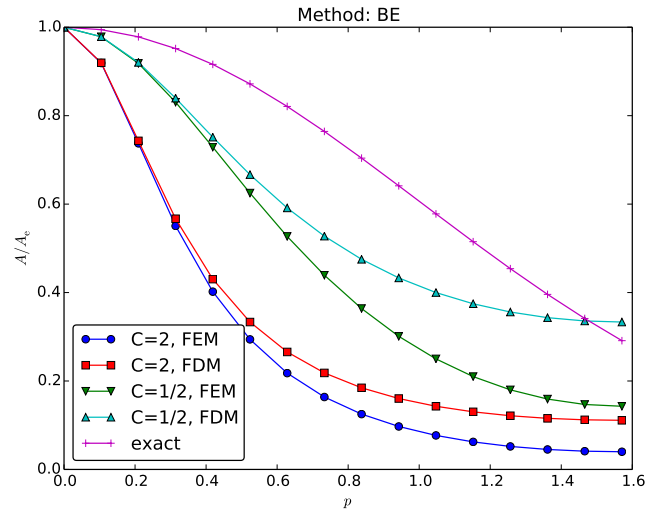(See notes for details)
Stability: $|A| \leq 1$:

$$C \leq \frac{1}{6} \quad \Rightarrow \quad \Delta t \leq \frac{\Delta x^2}{6\alpha}$$

Finite differences: $C \leq \frac{1}{2}$, so finite elements give a *stricter* stability criterion for this PDE!
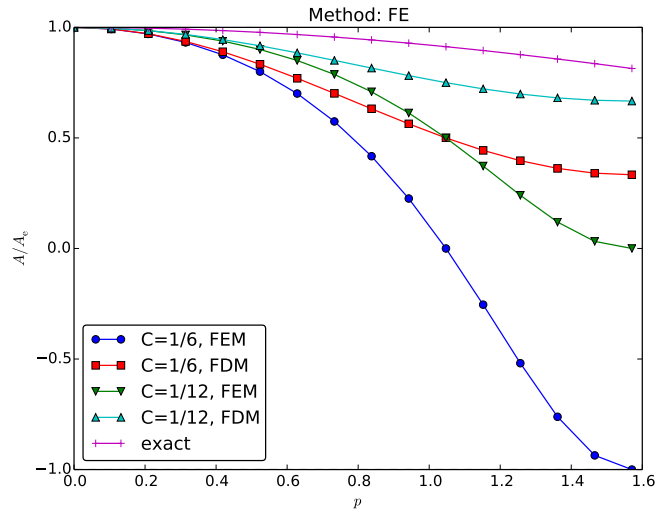
Amplification factor for the Backward Euler method; results

Coarse meshes:

$$A = \left(1 + 4C\frac{\sin^2 p}{1 + \frac{2}{3}\sin^2 p}\right)^{-1} \quad \text{(unconditionally stable)}$$

Amplification factors for smaller time steps; Forward Euler

Amplification factors for smaller time steps; Backward Euler