Cooling schemes

$$\frac{dT}{dt} = -k(T - T_s), \qquad T(0) = T_0 \tag{1}$$

For simplicity we change the variable to $\tilde{T} = T - T_s$, and this gives:

$$\frac{d\tilde{T}}{dt} = -k(\tilde{T}), \qquad \tilde{T}(0) = T_0 - T_s \tag{2}$$

with corresponding exact solution:

$$\tilde{T}_e = (T_0 - T_s)e^{-kt} \tag{3}$$

This has an exact discrete solution:

$$\tilde{T}_e(t) = \tilde{T}_e(n\Delta t) = (T_0 - T_s)e^{-kn\Delta t} = (T_0 - T_s)\left(e^{-k\Delta t}\right)^n \tag{4}$$

Applying the Forward Euler scheme:

$$\begin{aligned} \frac{u^{n+1} - u^n}{dt} &= -ku^n \\ u^{n+1} &= u^n(1 - kdt) \\ u^n &= u^0(1 - kdt)^n \end{aligned} \tag{5}$$

Backward Euler:

$$\begin{aligned} \frac{u^n - u^{n-1}}{dt} &= -ku^n \\ u^n &= u^{n-1}\frac{1}{1 + kdt} \\ u^n &= u^0\left(\frac{1}{1 + kdt}\right)^n \end{aligned} \tag{6}$$

Crank-Nicolson:

$$\begin{aligned} \frac{u^{n+1} - u^n}{dt} &= -ku^{n+\frac{1}{2}} = -k\frac{1}{2}(u^{n+1} + u^n) \\ u^{n+1} &= u^n\left(\frac{1 - \frac{kdt}{2}}{1 + \frac{kdt}{2}}\right) \\ u^n &= u^0\left(\frac{1 - \frac{kdt}{2}}{1 + \frac{kdt}{2}}\right)^n \end{aligned} \tag{7}$$

θ-rule:

$$\begin{aligned} \frac{u^{n+1} - u^n}{dt} &= -k(\theta u^{n+1} + (1 - \theta)u^n) \\ u^{n+1} &= u^n\frac{1 - (1 - \theta)kdt}{1 + \theta kdt} \\ u^n &= u^0\left(\frac{1 - (1 - \theta)kdt}{1 + \theta kdt}\right)^n \end{aligned} \tag{8}$$