Study guide: Computing with variational forms for systems of PDEs

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Systems of differential equations

Consider $m+1$ unknown functions: $u^{(0)},\ldots,u^{(m)}$ governed by $m+1$ differential equations:

$$
\begin{aligned}
\mathcal{L}_0(u^{(0)},\ldots,u^{(m)}) &= 0\\
&\vdots\\
\mathcal{L}_m(u^{(0)},\ldots,u^{(m)}) &= 0,
\end{aligned}
$$

Goals.

- How do we derive variational formulations of systems of differential equations?
- How do we apply the finite element method?

Variational forms: treat each PDE as a scalar PDE

- First approach: treat each equation as a scalar equation
- For equation no. i, use test function $v^{(i)}\in V^{(i)}$

$$
\begin{aligned}
\int_\Omega \mathcal{L}^{(0)}(u^{(0)},\ldots,u^{(m)})v^{(0)}\,\mathrm{d}x &= 0\\
&\vdots\\
\int_\Omega \mathcal{L}^{(m)}(u^{(0)},\ldots,u^{(m)})v^{(m)}\,\mathrm{d}x &= 0
\end{aligned}
$$

Terms with second-order derivatives may be integrated by parts, with Neumann conditions inserted in boundary integrals.

$$V^{(i)} = \text{span}\{\varphi_0^{(i)}, \ldots, \varphi_{N_i}^{(i)}\},$$

$$u^{(i)} = B^{(i)}(\boldsymbol{x}) + \sum_{j=0}^{N_i} c_j^{(i)} \varphi_j^{(i)}(\boldsymbol{x}),$$

Can derive m coupled linear systems for the unknowns $c_j^{(i)}$, $j = 0, \ldots, N_i$, $i = 0, \ldots, m$.

Variational forms: treat the PDE system as a vector PDE

- Second approach: work with vectors (and vector notation)
- $\boldsymbol{u} = (u^{(0)}, \ldots, u^{(m)})$
- $\boldsymbol{v} = (u^{(0)}, \ldots, u^{(m)})$
- $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} = V^{(0)} \times \cdots \times V^{(m)}$
- Note: if $\boldsymbol{B} = (B^{(0)}, \ldots, B^{(m)})$ is needed for nonzero Dirichlet conditions, $\boldsymbol{u} - \boldsymbol{B} \in \boldsymbol{V}$ (not $\boldsymbol{u}$ in $\boldsymbol{V}$)
- $\boldsymbol{\mathcal{L}}(\boldsymbol{u}) = 0$
- $\boldsymbol{\mathcal{L}}(\boldsymbol{u}) = (\mathcal{L}^{(0)}(\boldsymbol{u}), \ldots, \mathcal{L}^{(m)}(\boldsymbol{u}))$

The variational form is derived by taking the *inner product* of $\boldsymbol{\mathcal{L}}(\boldsymbol{u})$ and $\boldsymbol{v}$:

$$\int_\Omega \boldsymbol{\mathcal{L}}(\boldsymbol{u}) \cdot \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

- Observe: this is a scalar equation (!).
- Can derive m independent equation by choosing m independent $\boldsymbol{v}$
- E.g.: $\boldsymbol{v} = (v^{(0)}, 0, \ldots, 0)$ recovers (**??**)
- E.g.: $\boldsymbol{v} = (0, \ldots, 0, v^{(m)}$ recovers (**??**)

A worked example

$$
\begin{aligned}
&\mu\nabla^2 w = -\beta \\
&\kappa\nabla^2 T = -\mu||\nabla w||^2 \quad (= \mu\nabla w\cdot\nabla w)
\end{aligned}
$$

- Unknowns: $w(x,y)$, $T(x,y)$
- Known constants: μ, β, κ
- Application: fluid flow in a straight pipe, w is velocity, T is temperature
- Ω: cross section of the pipe
- Boundary conditions: $w=0$ and $T=T_0$ on $\partial\Omega$
- Note: T depends on w, but w does not depend on T (one-way coupling)

Identical function spaces for the unknowns

Let $w, (T-T_0)\in V$ with test functions $v\in V$.

$$V = \mathrm{span}\{\varphi_0(x,y),\ldots,\varphi_N(x,y)\},$$

$$w = \sum_{j=0}^{N} c_j^{(w)}\varphi_j, \quad T = T_0 + \sum_{j=0}^{N} c_j^{(T)}\varphi_j$$

Variational form of each individual PDE

Inserting (**??**) in the PDEs, results in the residuals

$$
\begin{aligned}
R_w &= \mu\nabla^2 w + \beta \\
R_T &= \kappa\nabla^2 T + \mu||\nabla w||^2
\end{aligned}
$$

Galerkin's method: make residual orthogonal to V,

$$
\begin{aligned}
\int_\Omega R_w v \,\mathrm{d}x = 0 \quad \forall v\in V \\
\int_\Omega R_T v \,\mathrm{d}x = 0 \quad \forall v\in V
\end{aligned}
$$

Integrate by parts and use $v=0$ on $\partial\Omega$ (Dirichlet conditions!):

$$
\begin{aligned}
\int_\Omega \mu\nabla w\cdot\nabla v\,\mathrm{d}x &= \int_\Omega \beta v\,\mathrm{d}x \quad \forall v\in V \\
\int_\Omega \kappa\nabla T\cdot\nabla v\,\mathrm{d}x &= \int_\Omega \mu\nabla w\cdot\nabla w\, v\,\mathrm{d}x \quad \forall v\in V
\end{aligned}
$$

Compound scalar variational form

- Test vector function $\boldsymbol{v} \in \boldsymbol{V} = V \times V$
- Take the inner product of $\boldsymbol{v}$ and the system of PDEs (and integrate)

$$\int_\Omega (R_w, R_T) \cdot \boldsymbol{v} \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

With $\boldsymbol{v} = (v_0, v_1)$:

$$\int_\Omega (R_w v_0 + R_T v_1) \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

$$\int_\Omega (\mu \nabla w \cdot \nabla v_0 + \kappa \nabla T \cdot \nabla v_1) \,\mathrm{d}x = \int_\Omega (\beta v_0 + \mu \nabla w \cdot \nabla w \, v_1) \,\mathrm{d}x, \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

Choosing $v_0 = v$ and $v_1 = 0$ gives the variational form (**??**), while $v_0 = 0$ and $v_1 = v$ gives (**??**).

Alternative inner product notation

$$\begin{aligned} \mu(\nabla w, \nabla v) &= (\beta, v) \quad \forall v \in V \\ \kappa(\nabla T, \nabla v) &= \mu(\nabla w \cdot \nabla w, v) \quad \forall v \in V \end{aligned}$$

Decoupled linear systems

$$\begin{aligned} \sum_{j=0}^{N} A_{i,j}^{(w)} c_j^{(w)} &= b_i^{(w)}, \quad i = 0, \ldots, N \\ \sum_{j=0}^{N} A_{i,j}^{(T)} c_j^{(T)} &= b_i^{(T)}, \quad i = 0, \ldots, N \\ A_{i,j}^{(w)} &= \mu(\nabla \varphi_j, \nabla \varphi_i) \\ b_i^{(w)} &= (\beta, \varphi_i) \\ A_{i,j}^{(T)} &= \kappa(\nabla \varphi_j, \nabla \varphi_i) \\ b_i^{(T)} &= (\mu \nabla w_- \cdot (\sum_k c_k^{(w)} \nabla \varphi_k), \varphi_i) \end{aligned}$$

Matrix-vector form (alternative notation):

$$\begin{aligned} \mu K c^{(w)} &= b^{(w)} \\ \kappa K c^{(T)} &= b^{(T)} \end{aligned}$$

where

$$
\begin{aligned}
K_{i,j} &= (\nabla\varphi_j, \nabla\varphi_i)\\
b^{(w)} &= (b_0^{(w)}, \ldots, b_N^{(w)})\\
b^{(T)} &= (b_0^{(T)}, \ldots, b_N^{(T)})\\
c^{(w)} &= (c_0^{(w)}, \ldots, c_N^{(w)})\\
c^{(T)} &= (c_0^{(T)}, \ldots, c_N^{(T)})
\end{aligned}
$$

First solve the system for $c^{(w)}$, then solve the system for $c^{(T)}$

Coupled linear systems

- Pretend two-way coupling, i.e., need to solve for w and T simultaneously
- Want to derive *one system* for $c_j^{(w)}$ and $c_j^{(T)}$, $j = 0, \ldots, N$
- The system is nonlinear because of $\nabla w \cdot \nabla w$
- Linearization: pretend an iteration where $\hat{w}$ is computed in the previous iteration and set $\nabla w \cdot \nabla w \approx \nabla\hat{w} \cdot \nabla w$ (so the term becomes linear in w)

$$
\begin{aligned}
\sum_{j=0}^{N} A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^{N} A_{i,j}^{(w,T)} c_j^{(T)} &= b_i^{(w)}, \quad i = 0, \ldots, N,\\
\sum_{j=0}^{N} A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^{N} A_{i,j}^{(T,T)} c_j^{(T)} &= b_i^{(T)}, \quad i = 0, \ldots, N,\\
A_{i,j}^{(w,w)} &= \mu(\nabla\varphi_j, \varphi_i)\\
A_{i,j}^{(w,T)} &= 0\\
b_i^{(w)} &= (\beta, \varphi_i)\\
A_{i,j}^{(w,T)} &= \mu(\nabla w_- \cdot \nabla\varphi_j), \varphi_i)\\
A_{i,j}^{(T,T)} &= \kappa(\nabla\varphi_j, \varphi_i)\\
b_i^{(T)} &= 0
\end{aligned}
$$

Alternative notation for coupled linear system

$$
\begin{aligned}
\mu K c^{(w)} &= b^{(w)}\\
L c^{(w)} + \kappa K c^{(T)} &= 0
\end{aligned}
$$

L is the matrix from the $\nabla w_- \cdot \nabla$ operator: $L_{i,j} = A_{i,j}^{(w,T)}$.

Corresponding block form:

$$
\left(\begin{array}{cc} \mu K & 0 \\ L & \kappa K \end{array}\right)\left(\begin{array}{c} c^{(w)} \\ c^{(T)} \end{array}\right) = \left(\begin{array}{c} b^{(w)} \\ 0 \end{array}\right)
$$

Different function spaces for the unknowns

- Generalization: $w \in V^{(w)}$ and $T \in V^{(T)}$, $V^{(w)} \neq V^{(T)}$
- This is called a *mixed finite element method*

$$
\begin{aligned}
V^{(w)} &= \text{span}\{\varphi_0^{(w)}, \ldots, \varphi_{N_w}^{(w)}\} \\
V^{(T)} &= \text{span}\{\varphi_0^{(T)}, \ldots, \varphi_{N_T}^{(T)}\}
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega \mu \nabla w \cdot \nabla v^{(w)} \,\mathrm{d}x &= \int_\Omega \beta v^{(w)} \,\mathrm{d}x \quad \forall v^{(w)} \in V^{(w)} \\
\int_\Omega \kappa \nabla T \cdot \nabla v^{(T)} \,\mathrm{d}x &= \int_\Omega \mu \nabla w \cdot \nabla w \, v^{(T)} \,\mathrm{d}x \quad \forall v^{(T)} \in V^{(T)}
\end{aligned}
$$

Take the inner product with $\boldsymbol{v} = (v^{(w)}, v^{(T)})$ and integrate:

$$
\int_\Omega (\mu \nabla w \cdot \nabla v^{(w)} + \kappa \nabla T \cdot \nabla v^{(T)}) \,\mathrm{d}x = \int_\Omega (\beta v^{(w)} + \mu \nabla w \cdot \nabla w \, v^{(T)}) \,\mathrm{d}x,
$$

valid $\forall \boldsymbol{v} \in \boldsymbol{V} = V^{(w)} \times V^{(T)}$.