# INF5620 Lecture: Analysis of finite difference schemes for diffusion processes

Hans Petter Langtangen[1,2]

Center for Biomedical Computing, Simula Research Laboratory[1]

Department of Informatics, University of Oslo[2]

Aug 18, 2014

## Properties of the solution

The PDE

$$u_t = \alpha u_{xx} \tag{1}$$

admits solutions

$$u(x, t) = Qe^{-\alpha k^2 t} \sin(kx) \tag{2}$$

Observations from this solution:

- The initial shape $I(x) = Q \sin kx$ undergoes a damping $\exp(-\alpha k^2 t)$
- The damping is very strong for short waves (large $k$)
- The damping is weak for long waves (small $k$)
- Consequence: $u$ is smoothened with time

# Example

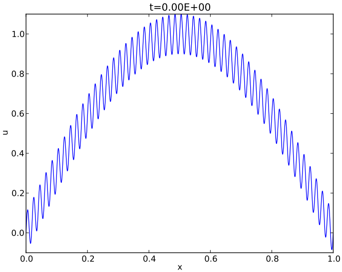
Test problem:

$$\begin{aligned} u_t &= u_{xx}, & x \in (0,1),\ \ t \in (0, T] \\ u(0,t) &= u(1,t) = 0, & t \in (0, T] \\ u(x,0) &= \sin(\pi x) + 0.1 \sin(100\pi x) \end{aligned}$$

Exact solution:

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + 0.1 e^{-\pi^2 10^4 t} \sin(100\pi x) \qquad (3)$$

# Visualization of the damping in the diffusion equation



# Damping of a discontinuity; problem and model

**Problem.**

Two pieces of a material, at different temperatures, are brought in contact at $t = 0$. Assume the end points of the pieces are kept at the initial temperature. How does the heat flow from the hot to the cold piece?

**Solution.**

Assume a 1D model is sufficient (insulated rod):

$$u(x,0) = \begin{cases} U_L, & x < L/2 \\ U_R, & x \geq L/2 \end{cases}$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = U_L,\ u(L,t) = U_R$$

# Damping of a discontinuity; Backward Euler simulation

Movie

# Damping of a discontinuity; Forward Euler simulation

Movie

# Damping of a discontinuity; Crank-Nicolson simulation

Movie

# Fourier representation

Represent $I(x)$ as a Fourier series

$$I(x) \approx \sum_{k \in K} b_k e^{ikx} \tag{4}$$

The corresponding sum for $u$ is

$$u(x,t) \approx \sum_{k \in K} b_k e^{-\alpha k^2 t} e^{ikx}\,. \tag{5}$$

Such solutions are also accepted by the numerical schemes, but with an amplification factor $A$ different from $\exp\left(-\alpha k^2 t\right)$:

$$u_q^n = A^n e^{ikq\Delta x} = A^n e^{ikx} \tag{6}$$

# Analysis of the finite difference schemes

Stability:

- $|A| < 1$: decaying numerical solutions (as we want)
- $A < 0$: *oscillating* numerical solutions (as we do not want)

Accuracy:

- Compare numerical and exact amplification factor: $A$ vs $A_e = \exp\left(-\alpha k^2 \Delta t\right)$

# Analysis of the Forward Euler scheme

$$[D_t^+ u = \alpha D_x D_x u]^n_q$$

Inserting

$$u^n_q = A^n e^{ikq\Delta x}$$

leads to

$$A = 1 - 4C\sin^2\left(\frac{k\Delta x}{2}\right), \quad C = \frac{\alpha\Delta t}{\Delta x^2} \tag{7}$$

The complete numerical solution is

$$u^n_q = (1 - 4C\sin^2 p)^n e^{ikq\Delta x}, \quad p = k\Delta x/2 \tag{8}$$

## Results for stability

We always have $A \leq 1$. The condition $A \geq -1$ implies

$$4C \sin^2 p \leq 2$$

The worst case is when $\sin^2 p = 1$, so a sufficient criterion for stability is

$$C \leq \frac{1}{2} \tag{9}$$

or:

$$\Delta t \leq \frac{\Delta x^2}{2\alpha} \tag{10}$$

**Implications of the stability result.**

Less favorable criterion than for $u_{tt} = c^2 u_{xx}$: halving $\Delta x$ implies time step $\frac{1}{4}\Delta t$ (not just $\frac{1}{2}\Delta t$ as in a wave equation). Need very small time steps for fine spatial meshes!

# Analysis of the Backward Euler scheme

$$[D_t^- u = \alpha D_x D_x u]_q^n$$

$$u_q^n = A^n e^{ikq\Delta x}$$

$$A = (1 + 4C\sin^2 p)^{-1} \tag{11}$$

$$u_q^n = (1 + 4C\sin^2 p)^{-n} e^{ikq\Delta x} \tag{12}$$

# Stability

We see from (11) that $|A| < 1$ for all $\Delta t > 0$ and that $A > 0$ (no oscillations).

# Analysis of the Crank-Nicolson scheme

The scheme

$$[D_t u = \alpha D_x D_x \overline{u}^x]_q^{n+\frac{1}{2}}$$

leads to

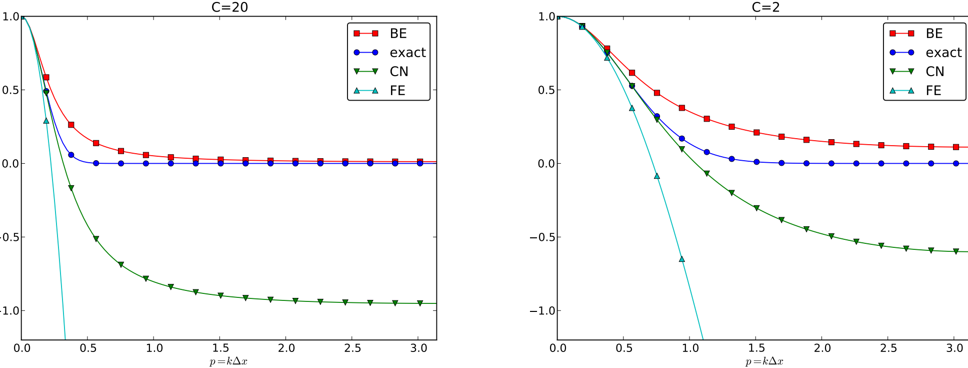
$$A = \frac{1 - 2C\sin^2 p}{1 + 2C\sin^2 p} \tag{13}$$

$$u_q^n = \left(\frac{1 - 2C\sin^2 p}{1 + 2C\sin^2 p}\right)^n e^{ikp\Delta x} \tag{14}$$

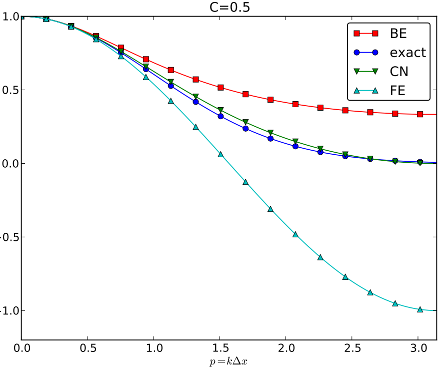
# Stability

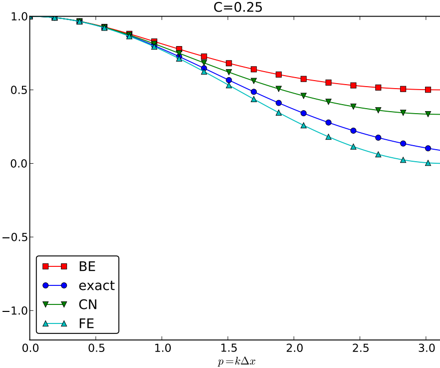
The criteria $A > -1$ and $A < 1$ are fulfilled for any $\Delta t > 0$.

# Summary of accuracy of amplification factors; large time steps

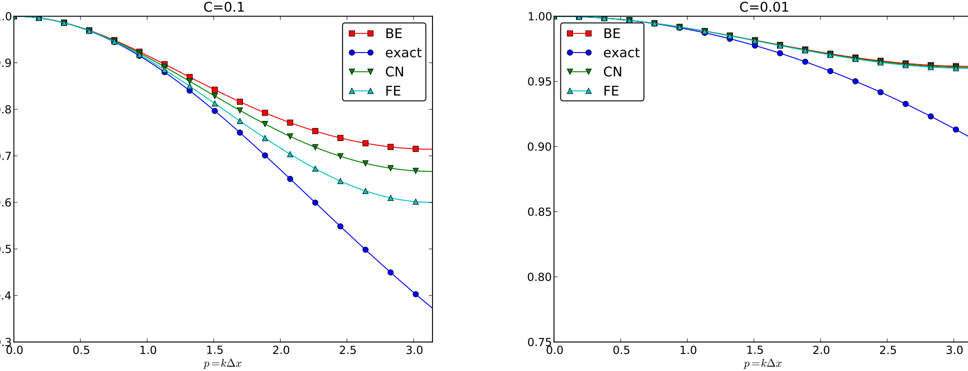


# Summary of accuracy of amplification factors; time steps around the Forward Euler stability limit



# Summary of accuracy of amplification factors; small time steps



# Observations

- Crank-Nicolson gives oscillations and not much damping of short waves for increasing $C$.
- These waves will manifest themselves as high frequency oscillatory noise in the solution.
- All schemes fail to dampen short waves enough

The problems of correct damping for $u_t = u_{xx}$ is partially manifested in the similar time discretization schemes for $u'(t) = -\alpha u(t)$.