ANISOTROPIC SCALING OF THE X-RAY TRANSFORM

In ASTRA, the voxel size is assumed to be 1 in each direction. If in practice, a voxel has dimensions $\Delta\boldsymbol{x} = (a, b, c) > 0$, i.e. the volume grid has the nodes $\boldsymbol{x_j} = \boldsymbol{x_0} + \boldsymbol{j}\Delta\boldsymbol{x}$, we rescale $\tilde{\boldsymbol{x}}_{\boldsymbol{j}} = \boldsymbol{x_j}/\Delta\boldsymbol{x}$ to get a grid with isotropic spacing equal to 1. By restricting our considerations to single axis tilting, we can reduce the scaling to the following two-dimensional case. Let thus

$$\mathcal{P}f(\boldsymbol{\theta}, \boldsymbol{v}) = \int_{\mathbb{R}} f(t\boldsymbol{\omega} + \boldsymbol{v})\,\mathrm{d}t, \quad \boldsymbol{\theta} \in \mathscr{S}^1,\ \boldsymbol{v} \in \boldsymbol{\theta}^{\perp},$$

be the 2D X-ray transform and

$$\mathcal{P}^* g(\boldsymbol{x}) = \int_{\mathscr{S}^1} g(\boldsymbol{\theta}, \Pi_{\boldsymbol{\theta}}\boldsymbol{x})\,\mathrm{d}\boldsymbol{\theta}$$

the corresponding backprojection with $\Pi_{\boldsymbol{\theta}}\boldsymbol{x} = \boldsymbol{x} - \langle \boldsymbol{x}, \boldsymbol{\theta}\rangle\boldsymbol{\theta}$. Note that the following arguments can also be applied if only a part of the sphere is covered.

Let $D = \mathrm{diag}(a, b)$ be the anisotropic scaling and

$$\mathcal{S}_D f(\boldsymbol{x}) = f(D\boldsymbol{x}) = f_D(\boldsymbol{x}) \tag{1}$$

be the corresponding scaling operator. If the old samples were $f_{\boldsymbol{j}} = f(\boldsymbol{x_j})$ with $f_{\boldsymbol{j}+\boldsymbol{1}} = f(\boldsymbol{x_j} + \Delta\boldsymbol{x})$, the new samples are $\tilde{f}_{\boldsymbol{j}} = f_D(\tilde{\boldsymbol{x}}_{\boldsymbol{j}})$ with

$$\tilde{f}_{\boldsymbol{j}+\boldsymbol{1}} = f_D(\tilde{\boldsymbol{x}}_{\boldsymbol{j}} + \boldsymbol{1}) = f(\boldsymbol{x_j} + D\boldsymbol{1}) = f_{\boldsymbol{j}+\boldsymbol{1}},$$

i.e. the transformed function lives on a grid with spacing $\boldsymbol{1} = (1, 1, 1)$. Now we can write

$$\begin{aligned}
\mathcal{P}f(\boldsymbol{\theta}, \boldsymbol{v}) &= \int_{\mathbb{R}} f_D(tD^{-1}\boldsymbol{\theta} + D^{-1}\boldsymbol{v})\,\mathrm{d}t \\
&= |D^{-1}\boldsymbol{\theta}|^{-1} \int_{\mathbb{R}} f_D\left(t\frac{D^{-1}\boldsymbol{\theta}}{|D^{-1}\boldsymbol{\theta}|} + D^{-1}\boldsymbol{v}\right)\mathrm{d}t \\
&= |D^{-1}\boldsymbol{\theta}|^{-1} \int_{\mathbb{R}} f_D(t\tilde{\boldsymbol{\theta}} + \Pi_{\tilde{\boldsymbol{\theta}}} D^{-1}\boldsymbol{v})\,\mathrm{d}t \\
&= |D^{-1}\boldsymbol{\theta}|^{-1} \mathcal{P} f_D(\tilde{\boldsymbol{\theta}}, \Pi_{\tilde{\boldsymbol{\theta}}} D^{-1}\boldsymbol{v}) \\
&= \mathcal{T}_D \mathcal{P} \mathcal{S}_D f(\boldsymbol{\theta}, \boldsymbol{v}),
\end{aligned}$$

and thus

$$\mathcal{P} = \mathcal{T}_D \mathcal{P} \mathcal{S}_D \tag{2}$$

with

$$\mathcal{T}_D g(\boldsymbol{\theta}, \boldsymbol{v}) = |D^{-1}\boldsymbol{\theta}|^{-1} g(\tilde{\boldsymbol{\theta}}, \Pi_{\tilde{\boldsymbol{\theta}}} D^{-1}\boldsymbol{v}), \quad \tilde{\boldsymbol{\theta}} = \frac{D^{-1}\boldsymbol{\theta}}{|D^{-1}\boldsymbol{\theta}|}. \tag{3}$$

This implicates that in the rescaled setting, the projection directions are changed from $\boldsymbol{\theta} = (\cos\varphi, \sin\varphi)$ to $\tilde{\boldsymbol{\theta}} = (\cos\tilde{\varphi}, \sin\tilde{\varphi})$ with

$$(\cos\tilde{\varphi}, \sin\tilde{\varphi}) = \frac{(\cos\varphi, \tau\sin\varphi)}{\sqrt{\cos^2\varphi + \tau^2\sin^2\varphi}}, \quad \tau = \frac{a}{b}, \tag{4}$$

i.e. the new angles $\tilde{\varphi}$ depend on the ratio τ between the x and y scalings.

Moreover, to acquire the projections at a detector grid with nodes $\boldsymbol{v}_k = \boldsymbol{v}_0 + k\Delta\boldsymbol{v}$, $\Delta\boldsymbol{v} = \Delta v(-\sin\varphi, \cos\varphi)$, we not only need to rescale $\tilde{\boldsymbol{v}}_k = D^{-1}\boldsymbol{v}_k$ but also project onto the new perpendicular component. This induces the new stepping

$$\begin{aligned}\Delta\boldsymbol{u} &= \Delta\tilde{\boldsymbol{v}} - \langle\Delta\tilde{\boldsymbol{v}},\, \tilde{\boldsymbol{\theta}}\rangle\tilde{\boldsymbol{\theta}}\\ &= \Delta v(-a^{-1}\sin\varphi, b^{-1}\cos\varphi) - \Delta v\left(-a^{-1}\sin\varphi\cos\tilde{\varphi} + b^{-1}\cos\varphi\sin\tilde{\varphi}\right)(\cos\tilde{\varphi}, \sin\tilde{\varphi})\\ &= \Delta v\left(a^{-1}\sin\varphi\sin\tilde{\varphi} + b^{-1}\cos\varphi\cos\tilde{\varphi}\right)(-\sin\tilde{\varphi}, \cos\tilde{\varphi})\\ &= \Delta u\left(-\sin\tilde{\varphi}, \cos\tilde{\varphi}\right),\end{aligned}$$

i.e. the grid spacing is scaled by the factor

$$\left(a^{-1}\sin\varphi\sin\tilde{\varphi} + b^{-1}\cos\varphi\cos\tilde{\varphi}\right) = \langle\tilde{D}^{-1}\boldsymbol{\theta},\, \tilde{\boldsymbol{\theta}}\rangle, \quad \tilde{D} = \operatorname{diag}(b, a)$$

which depends on the current direction.

The backprojection can also be computed with the isotropic operator by taking the adjoint of (2):

$$\mathcal{P}^* = \mathcal{S}_D^*\mathcal{P}^*\mathcal{T}_D^*. \tag{5}$$

We calculate these adjoints in the following. As for $\mathcal{S}_D$, it is easy to see that

$$\langle\mathcal{S}_D f,\, h\rangle = \int_{\mathbb{R}^2} f(D\boldsymbol{x})\, h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = (\det D)^{-1}\int_{\mathbb{R}^2} f(\boldsymbol{y})\, h(D^{-1}\boldsymbol{y})\, \mathrm{d}\boldsymbol{y},$$

which implies

$$\mathcal{S}_D^* f(\boldsymbol{x}) = (\det D)^{-1} f(D^{-1}\boldsymbol{x}). \tag{6}$$

Regarding $\mathcal{T}_D$, we start with

$$\begin{aligned}\langle\mathcal{T}_D g,\, k\rangle &= \int_{\mathscr{S}^1}\int_{\boldsymbol{\theta}^\perp} \mathcal{T}_D g(\boldsymbol{\theta}, \boldsymbol{v})\, k(\boldsymbol{\theta}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{\theta}\\ &= \int_{\mathscr{S}^1}\int_{\boldsymbol{\theta}^\perp} |D^{-1}\boldsymbol{\theta}|^{-1}\, g(\tilde{\boldsymbol{\theta}}, \Pi_{\tilde{\boldsymbol{\theta}}} D^{-1}\boldsymbol{v})\, k(\boldsymbol{\theta}, \boldsymbol{v})\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{\theta}.\end{aligned} \tag{7}$$

In the first step, we reparametrize the $\boldsymbol{\theta}$ integral and observe that

$$\int_{\mathscr{S}^1} |D^{-1}\boldsymbol{\theta}|^{-1}\, u\left(\frac{D^{-1}\boldsymbol{\theta}}{|D^{-1}\boldsymbol{\theta}|}\right)\, \mathrm{d}\boldsymbol{\theta} = \int_0^{2\pi} |D^{-1}\boldsymbol{\theta}(\varphi)|^{-1}\, u\left(\frac{D^{-1}\boldsymbol{\theta}(\varphi)}{|D^{-1}\boldsymbol{\theta}(\varphi)|}\right)\, |\boldsymbol{\theta}'(\varphi)|\mathrm{d}\varphi. \tag{8}$$

To rephrase this integral in terms of $\boldsymbol{\omega} = \frac{D^{-1}\boldsymbol{\theta}}{|D^{-1}\boldsymbol{\theta}|}$, we need to calculate $|\boldsymbol{\omega}'(\varphi)|$. Using the abbreviation $\boldsymbol{r}(\varphi) = D^{-1}\boldsymbol{\theta}(\varphi) = (a^{-1}\cos\varphi, b^{-1}\sin\varphi)$, it can be immediately seen that

$$\omega_1' = -\frac{r_2(r_1 r_2' - r_2 r_1')}{|\boldsymbol{r}|^3} = -(ab)^{-1}\frac{r_2}{|\boldsymbol{r}|^3}$$

and likewise

$$\omega_2' = (ab)^{-1}\frac{r_1}{|\boldsymbol{r}|^3}.$$

Thereby it follows that

$$|\boldsymbol{\omega}'(\varphi)| = (\det D)^{-1}|\boldsymbol{r}|^{-2} = (\det D)^{-1}|D^{-1}\boldsymbol{\theta}|^{-2},$$

and from the definition of $\boldsymbol{\omega}$ we conclude that $|D\boldsymbol{\omega}| = |D^{-1}\boldsymbol{\theta}|^{-1}$. Hence, we can rewrite (8) as

$$\int_{\mathscr{S}^1} |D^{-1}\boldsymbol{\theta}|^{-1}\, u\left(\frac{D^{-1}\boldsymbol{\theta}}{|D^{-1}\boldsymbol{\theta}|}\right)\, \mathrm{d}\boldsymbol{\theta} = \det D\int_{\mathscr{S}^1} |D\boldsymbol{\omega}|^{-1}\, u(\boldsymbol{\omega})\, \mathrm{d}\boldsymbol{\omega}. \tag{9}$$

Inserting this into (7) yields

$$\langle \mathcal{T}_D g,\, k\rangle = \det D \int_{\mathscr{S}^1} \int_{(D\boldsymbol{\omega})^\perp} |D\boldsymbol{\omega}|^{-1}\, g(\boldsymbol{\omega}, \Pi_{\boldsymbol{\omega}} D^{-1}\boldsymbol{v})\, k\left(\frac{D\boldsymbol{\omega}}{|D\boldsymbol{\omega}|}, \boldsymbol{v}\right)\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{\omega} \tag{10}$$

For the substitution of the inner integral, we observe that

$$(D\boldsymbol{\omega})^\perp = \operatorname{span}\left\{\left(-a^{-1}\omega_2, b^{-1}\omega_1\right)\right\} = \operatorname{span}\{\boldsymbol{p}\}$$

and observe

$$\int_{(D\boldsymbol{\omega})^\perp} h(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} = \int_{\mathbb{R}} h(\lambda \boldsymbol{p})\, |\boldsymbol{p}|\, \mathrm{d}\lambda,$$

$$\int_{\boldsymbol{\omega}^\perp} H(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}} H(\lambda \boldsymbol{q})\, \mathrm{d}\lambda,$$

where $\boldsymbol{q} = D\boldsymbol{p} = (-\omega_2, \omega_1)$. Both right hand sides coincide if we choose

$$H(\boldsymbol{y}) = h(D^{-1}\boldsymbol{y})\, |D^{-1}\boldsymbol{q}| = h(D^{-1}\boldsymbol{y})\, |\tilde{D}^{-1}\boldsymbol{\omega}|, \quad \tilde{D} = \operatorname{diag}(b, a).$$

Applying this result to (10) finally gives the form

$$\langle \mathcal{T}_D g,\, k\rangle = \det D \int_{\mathscr{S}^1} \int_{\boldsymbol{\omega}^\perp} |\tilde{D}^{-1}\boldsymbol{\omega}|^{-1}\, |D\boldsymbol{\omega}|^{-1}\, g(\boldsymbol{\omega}, \boldsymbol{y})\, k\left(\tilde{\boldsymbol{\omega}}, \Pi_{\tilde{\boldsymbol{\omega}}} D\boldsymbol{y}\right)\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{\omega} \tag{11}$$

with $\tilde{\boldsymbol{\omega}} = D\boldsymbol{\omega}/|D\boldsymbol{\omega}|$, which allows us to read off the adjoint operator

$$\mathcal{T}_D^* k(\boldsymbol{\omega}, \boldsymbol{y}) = \det D\, |\tilde{D}^{-1}\boldsymbol{\omega}|^{-1}\, |D\boldsymbol{\omega}|^{-1}\, k\left(\tilde{\boldsymbol{\omega}}, \Pi_{\tilde{\boldsymbol{\omega}}} D\boldsymbol{y}\right). \tag{12}$$

Apart from the adaption of the direction vectors, the backprojection must be weighted with the dirction-dependent factor

$$\begin{aligned} \mu(\boldsymbol{\omega}) &= |\tilde{D}^{-1}\boldsymbol{\omega}|^{-1}\, |D\boldsymbol{\omega}|^{-1} \\ &= \left[\cos^2\psi \left(\sin^2\psi + a^2/b^2 \cos^2\psi\right) + \sin^2\psi \left(\cos^2\psi + b^2/a^2 \sin^2\psi\right)\right]^{-1/2} \end{aligned}$$

for $\boldsymbol{\omega} = (\cos\psi, \sin\psi)$ while the detector grid scaling is independent of the direction. The $(\det D)$ factor cancels with the factor of $\mathcal{S}_D^*$ and can be ignored.

In summary, we have the following recipes for forward and backward projections, respectively.

0.1 Algorithm (Forward projection with anisotropic scaling).

Given:

- *Data on 2D grid with spacing* $\Delta\boldsymbol{x} = (a, b) > 0$
- *1D detector grid with spacing* $\eta > 0$
- *Projection angles* $\varphi_k,\ k = 1, \ldots, p$

Computation:

- *Calculate auxiliary quantities* $\alpha = 1/a,\ \beta = 1/b$.
- *For* $1 \le k \le p$*:*
    - ∗ *Set* $l_k = (\alpha^2 \cos^2\varphi_k + \beta^2 \sin^2\varphi_k)^{1/2}$.
    - ∗ *Set* $\boldsymbol{p}_k = (\alpha\cos\varphi_k, \beta\sin\varphi_k)/l_k$.
    - ∗ *Determine* $\tilde{\varphi}_k$ *with* $(\cos\tilde{\varphi}_k, \sin\tilde{\varphi}_k) = \boldsymbol{p}_k$.
    - ∗ *Scale detector spacing to* $\eta_k = (\alpha\sin\varphi_k\sin\tilde{\varphi}_k + \beta\cos\varphi_k\cos\tilde{\varphi}_k)\,\eta$.
    - ∗ *Perform projection with one angle* $\tilde{\varphi}_k$ *and detector spacing* η_k.
    - ∗ *Multiply the result with* l_k^{-1}.

Result: *Projections for angles φ_k and detector grid spacing η using scaled 2D grid input.* □

0.2 Algorithm (Backprojection with anisotropic scaling).

Given:

- *Projection data on 1D grid with spacing $\eta > 0$*
- *2D volume grid with spacing $\Delta\boldsymbol{x} = (a, b) > 0$*
- *Projection angles φ_k, $k = 1, \dots, p$*

Computation:

- *Calculate auxiliary quantities $\alpha = 1/a$, $\beta = 1/b$, $\tau = a/b$.*
- *For $1 \le k \le p$:*
  - *Set $L_k = (a^2 \cos^2 \varphi_k + b^2 \sin^2 \varphi_k)^{1/2}$.*
  - *Set $\boldsymbol{q}_k = (a \cos \varphi_k, b \sin \varphi_k)/L_k$.*
  - *Determine $\tilde{\psi}_k$ with $(\cos \tilde{\psi}_k, \sin \tilde{\psi}_k) = \boldsymbol{q}_k$.*
  - *Scale detector spacing to $\eta_k = (a \sin \varphi_k \sin \tilde{\psi}_k + b \cos \varphi_k \cos \tilde{\psi}_k)\, \eta$.*
  - *Calculate weight $\mu_k = \left[\cos^2 \varphi_k \left(\sin^2 \varphi_k + \tau^2 \cos^2 \varphi_k\right) + \sin^2 \varphi_k \left(\cos^2 \varphi_k + \tau^{-2} \sin^2 \varphi_k\right)\right]^{-1/2}$.*
  - *Multiply k-th projection data with μ_k.*
  - *Perform backprojection with one angle $\tilde{\psi}_k$ and detector spacing η_k.*
  - *Add the result to backprojection from previous step, weighted according to a numerical integration formula, e.g. $(\tilde{\psi}_{k+1} - \tilde{\psi}_{k-1})/2$ for trapezoidal rule.*

Result: *Backprojection for angles φ_k and detector grid spacing η using a scaled 2D grid volume.* □