COMMON REPRESENTATION OF ACQUISITION GEOMETRIES IN TOMOGRAPHY

During data acquisition, detector and sample move relative to each other. Thus, the position of a specific point on the detector varies with time or some other parameter, like a rotation angle. Furthermore, we consider detectors which are surface-, curve- or point-like and can be regarded as manifolds in $\mathbb{R}^D$ parameterized over an open set in $\mathbb{R}^d$ with typically $d < D$. Hence, we use the following representation:
Let $T \subset \mathbb{R}^p$ be a set of (time, angle, ...) parameters and $U \subset \mathbb{R}^d$ a set of detector parameters, e.g. x and y coordinates on a flat 2D detector. We call a mapping

$$X : T \times U \longrightarrow \mathbb{R}^D \tag{1}$$

the *detector trace parametrization* and $X(T \times U) \subset \mathbb{R}^D$ the *detector trace*. For fixed $t \in T$, we define the *detector parametrization*

$$X_t = X(t, \cdot) : U \longrightarrow \mathbb{R}^D \tag{2}$$

and call $X_t(U) \subset \mathbb{R}^D$ the detector surface (at time t).

To model the very general situation of directional input (like rays) to the detector, we further define the *directional field*

$$N : T \times U \longrightarrow \mathscr{S}^{D-1}. \tag{3}$$

For $t \in T$ and $u \in U$, the value $N(t, u) \in \mathscr{S}^{D-1}$ stands for the orientation of the detector at the point $X(t, u)$ caused by e.g. collimators. Often, $N(t, u)$ is the unit normal to the detector surface at $X(t, u)$.

In the case that one detector point "sees" more than one incoming direction, we additionally define the *detector pupil*

$$P : T \times U \longrightarrow \mathrm{P}(\mathscr{S}^{D-1}) \tag{4}$$

which maps t, u to the subset $P(t, u) \subset \mathscr{S}^{D-1}$ of directions seen by the detector point at $X(t, u)$. This covers, for example, the case of PET where the detector pixels register not only perpendicular photons but also photons coming in at an angle.

Finally, if the incoming "radiation" is not travelling along straight lines, we need to provide this information, too. However, we will restrict ourselves to the case where we can uniquely trace such a ray back from a detector point through the sample. We thus define a mapping

$$\gamma : T \times U \times \mathscr{S}^{D-1} \longrightarrow C^1_{\mathrm{pw}}\big([0, \infty), \mathbb{R}^d\big) \tag{5}$$

where for $t \in T$, $u \in u$ and $\theta \in \mathscr{S}^{D-1}$, the function $\gamma(t, u, \theta)$ represents the ray arriving at the detector point $X(t, u)$ from the direction θ. The subscript "pw" stands for "piecewise" with the restriction that the lengths of such pieces must be bounded from below by a positive constant. If $P(t, u) = \{N(t, u)\}$, we write

$$\tilde{\gamma} : T \times U \longrightarrow C^1_{\mathrm{pw}}\big([0, \infty), \mathbb{R}^d\big), \quad \tilde{\gamma}(t, u) = \gamma\big(t, u, N(t, u)\big). \tag{6}$$

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Let now $\Omega \subset \mathbb{R}^D$ be a suitable set and $\big(\mathscr{H}_j, \langle\cdot,\,\cdot\rangle_j\big)$, $j = 1, 2$ be Hilbert spaces. In this setting, we can study forward operators of the form

$$\mathcal{A} : \mathscr{H}_1(\Omega) \longrightarrow \mathscr{H}_2(T \times U) \tag{7}$$

$$\mathcal{A}f(t,u) = \int_{P(t,u)} \int_{\gamma(t,u,\theta)\cap\Omega} f \,\mathrm{d}\gamma\,\mathrm{d}\theta. \tag{8}$$

which maps a pair of parameters (t, u) to the integral of f along all rays arriving at $X(t, u)$. For L^2 spaces, we want to calculate the adjoint operator:

$$\begin{aligned}
\langle \mathcal{A}f,\, g\rangle_2 &= \int_T \int_U \mathcal{A}f(t,u)\, g(t,u)\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_T \int_U \int_{P(t,u)} \int_{\gamma(t,u,\theta)\cap\Omega} f\,\mathrm{d}\gamma\,\mathrm{d}\theta\, g(t,u)\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_T \int_U \int_{P(t,u)} \int_0^\infty [f\circ\gamma(t,u,\theta)](s)\, |\gamma(t,u,\theta)'(s)|\, g(t,u)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}u\,\mathrm{d}t. \qquad (9)
\end{aligned}$$

We first consider the case where $P(t, u) = \{N(t, u)\}$, i.e. there is exactly one ray arriving at each detector point $X_t(u)$ with incoming direction $N_t(u)$. For fixed $t \in T$, we assume that for each $x \in \Omega$, there is a unique ray $\tilde{\gamma}_t(u) = \tilde{\gamma}_t(u; x)$ containing the point x exactly once, i.e. $\tilde{\gamma}_t(u; x)(s(x)) = x$. Thus, we can define a mapping

$$\Gamma_t : \Omega \to [0,\infty) \times U, \quad x \mapsto (s,u) \text{ with } \tilde{\gamma}_t(u)(s) = x,$$

which is the inverse of the mapping $(s, u) \mapsto \tilde{\gamma}_t(u)(s)$. The point $X_t\big(u(x)\big)$ can be interpreted as the projection of x to the detector surface along the corresponding ray, and $s(x)$ is the arc length along the ray from x to its projection. Now we can rewrite the integral (9) as

$$\begin{aligned}
\langle \mathcal{A}f,\, g\rangle_2 &= \int_T \int_{\Gamma_t(\Omega)} f\big(\Gamma_t^{-1}(s,u)\big)\, \big|\partial_s \Gamma_t^{-1}(s,u)\big|\; g(t,u)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_T \int_{\Gamma_t(\Omega)} f\big(\Gamma_t^{-1}(s,u)\big)\, \big|\big[\partial\Gamma_t\big(\Gamma_t^{-1}(s,u)\big)\big]_1\big|^{-1}\; g(t,u)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_T \int_\Omega f(x)\, \big|\big[\partial\Gamma_t(x)\big]_1\big|^{-1}\; |\det \partial\Gamma_t(x)|\, g\big(t, \Pi_{\gamma_t}(x)\big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $[\partial\Gamma_t]_1$ stands for the first column of the Jacobian of Γ_t and $\Pi_{\gamma_t}(x) = [\Gamma_t(x)]_2$ is the u component of $(s, u) = \Gamma_t(x)$. Hence, the adjoint operator can be written as

$$\mathcal{A}^* g(x) = \int_T \big|\big[\partial\Gamma_t(x)\big]_1\big|^{-1}\; |\det \partial\Gamma_t(x)|\, g\big(t, \Pi_{\gamma_t}(x)\big)\,\mathrm{d}t \tag{10}$$

which is equal to

$$\mathcal{A}^* g(x) = \int_T \big|\partial_s \Gamma_t^{-1}\big(\Gamma_t(x)\big)\big|\; \big|\det \partial\Gamma_t^{-1}\big(\Gamma_t(x)\big)\big|^{-1} g\big(t, \Pi_{\gamma_t}(x)\big)\,\mathrm{d}t \tag{11}$$

Example 0.1 (Parallel geometry)**.** We consider a flat 2D detector moving on the unit circle in the x-y plane, oriented to the center of the circle:

$$\begin{aligned}
&T = [0, \pi), \quad U = [-l_y/2, l_y/2] \times [-l_z/2, l_z/2] \\
&X(\varphi, u) = \theta(\varphi) + R(\varphi)\cdot(u_1, 0, u_2)^{\mathrm{T}}
\end{aligned}$$

with

$$\theta(\varphi) = (-\sin\varphi, \cos\varphi, 0)^{\mathrm{T}}, \quad R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

The directional mapping is given by $N(\varphi, u) = -\theta(\varphi)$, which is also the canonical normal $n = \eta_1 \times \eta_2$ with $\eta_j = \partial_{u_j} X / |\partial_{u_j} X|$. It is $P(\varphi, u) = \{N(\varphi, u)\}$.
Finally, the rays are straight lines, and their formula is

$$\tilde{\gamma}_\varphi(u) = \big(s \mapsto X(\varphi, u) + (1 - S)N(\varphi, u),\ s > 0\big).$$

They start at the far end of the object support Ω (see below) and continue to the detector, hence the integration domain of the s integral can be extended to $\mathbb{R}$.

We consider L^2 functions in the unit ball in $\mathbb{R}^3$, i.e. $\Omega = B_1$ and the parallel X-ray transform

$$\begin{aligned} \mathcal{P} : L^2(B_1) &\to L^2(T \times U) \\ \mathcal{P} f(\varphi, u) &= \int_{\tilde{\gamma}_\varphi(u)} f \, \mathrm{d}\gamma \\ &= \int_0^\infty f\big(X(\varphi, u) + (1 - s)N(\varphi, u)\big) \, \mathrm{d}s \\ &= \int_{\mathbb{R}} f\big(s\theta(\varphi) + R(\varphi) \cdot (u_1, 0, u_2)^{\mathrm{T}}\big) \, \mathrm{d}s, \end{aligned}$$

which is the classical transform in parametrized form. To compare the well-known backprojection with the adjoint obtained with (10), we observe that we can write $x \in \Omega$ as

$$\begin{aligned} x &= \langle \theta(\varphi), x \rangle \, \theta(\varphi) + \big(x - \langle \theta(\varphi), x \rangle \, \theta(\varphi)\big) \\ &= \langle \theta(\varphi), x \rangle \, \theta(\varphi) + \langle \theta^\perp(\varphi), x \rangle \, \theta^\perp(\varphi) + \langle e_z, x \rangle \, e_z \\ &= \langle \theta(\varphi), x \rangle \, \theta(\varphi) + R(\varphi)\big(\langle \theta^\perp(\varphi), x \rangle, 0, \langle e_z, x \rangle\big)^{\mathrm{T}} \end{aligned}$$

with $\theta^\perp(\varphi) = (\cos\varphi, \sin\varphi, 0)^{\mathrm{T}}$, the first column of $R(\varphi)$. This means that for $s = 1 - \langle \theta(\varphi), x \rangle$ and $u = \big(\langle \theta^\perp(\varphi), x \rangle, \langle e_z, x \rangle\big)^{\mathrm{T}}$, it is $x = \gamma\big(\varphi, u, N(\varphi, u)\big)(s)$, and thus the mapping Γ_φ can be explicitly determined as

$$\Gamma_\varphi(x) = \big(1 - \langle \theta(\varphi), x \rangle, \langle \theta^\perp(\varphi), x \rangle, \langle e_z, x \rangle\big)^{\mathrm{T}}$$

The Jacobian of this coordinate transform is apparently a column permutation of $R(\varphi)$, hence the additional factors in the integral (10) are one, and we can conclude that

$$\mathcal{P} g(x) = \int_0^\pi g\big(\varphi, \langle \theta^\perp(\varphi), x \rangle, \langle e_z, x \rangle\big) \, \mathrm{d}\varphi,$$

which is the parametrized form of the well-known backprojection formula. □

Example 0.2 (Fan beam geometry). Here, we consider the case of 2D functions on $\Omega = B_1 \subset \mathbb{R}^2$. The detector is a segment of a circle with radius $r > 1$ and detects rays coming from a point source on the opposite side of the same circle. We parametrize the detector as follows:

$$T = [0, 2\pi), \quad U = [-\psi_0/2, \psi_0/2]$$
$$X(\varphi, \psi) = r\theta(\varphi + \psi), \quad \theta(\varphi) = (\cos\varphi, \sin\varphi).$$

The directional field of the detector is given by the normalized line from a detector point to the source located at $x_{\mathrm{S}} = -r\theta(\varphi)$. Without normalization, we acquire

$$\begin{aligned}\tilde{N}(\varphi, \psi) &= X(\varphi, \psi) - x_{\mathrm{S}} \\ &= -r\big(\theta(\varphi + \psi) + \theta(\varphi)\big) \\ &= -r\big(\cos(\varphi + \psi) + \cos\varphi, \sin(\varphi + \psi) + \sin\varphi\big) \\ &= -\sqrt{2(1 + \cos\psi)}\, r\theta\big(\varphi + \delta(\psi)\big), \quad \tan\delta = \frac{1 - \cos\psi}{1 + \cos\psi} = \tan^2(\psi/2)\end{aligned}$$

as one can check using trigonometric identities. Thus, the normalized vector field is

$$N(\varphi, \psi) = -\theta\big(\varphi + \delta(\psi)\big). \tag{13}$$

Again, as in the parallel geometry, we have $P(\varphi, \psi) = \{N(\varphi, \psi)\}$, and the rays are lines from the far end of the object to the detector. We select a virtual point beyond the source as its starting point:

$$\tilde{\gamma}(\varphi, \psi) = s \mapsto X(\varphi, \psi) + (2r - s)N(\varphi, \psi).$$

This leads to the transform

$$\begin{aligned}\mathcal{D} : L^2(B_1) &\longrightarrow L^2\big([0, 2\pi) \times [-\psi_0/2, \psi_0/2]\big) \\ \mathcal{D}f(\varphi, \psi) &= \int_{\tilde{\gamma}(\varphi,\psi)} f \,\mathrm{d}\gamma \\ &= \int_0^\infty f\big(r\theta(\varphi + \psi) + (s - 2r)\theta(\varphi + \delta(\psi))\big)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} f\big(r\theta(\varphi + \psi) + s\theta(\varphi + \delta(\psi))\big)\mathrm{d}s,\end{aligned}$$

which is the well-known divergent beam transform in our parametrization. In order to determine the adjoint, we need to find Γ_φ as the inverse of $(\psi, s) \mapsto \tilde{\gamma}_\varphi(\psi)(s)$, i.e. for $x \in \mathbb{R}^2$, we need to solve

$$r\theta(\varphi + \psi) + s\theta(\varphi + \delta(\psi)) = x$$

for ψ and s. To find this coordinate change, we consider for $\theta = \theta(\varphi)$ the line

$$\gamma_x(\sigma) = -r\theta + \sigma(x + r\theta)$$

from the source point ($\sigma = 0$) through x ($\sigma = 1$). The intersection with the circle $|x| = r$ determines the detector point reached by the ray, from which the parameters σ and ψ can be calculated. It is

$$\begin{aligned}|\gamma_x(\sigma)|^2 - r^2 &= r^2 + \sigma^2|x + r\theta|^2 - 2rt\big(r + \langle x,\, \theta\rangle\big) - r^2 \\ &= \sigma\left(\sigma|x + r\theta|^2 - 2r\big(r + \langle x,\, \theta\rangle\big)\right).\end{aligned}$$

The second zero of this expression besides $\sigma = 0$ is apparently

$$\sigma_x = \frac{2r\langle x + r\theta,\, \theta\rangle}{|x + r\theta|^2}, \tag{14}$$

and the corresponding curve point is

$$\begin{aligned}\gamma_x(\sigma_x) &= \frac{2r\langle x+r\theta,\,\theta\rangle}{|x+r\theta|^2}\,x + r\,\frac{2\langle x+r\theta,\,r\theta\rangle - |x+r\theta|^2}{|x+r\theta|^2}\,\theta\\ &= |x+r\theta|^{-2}\big(2r\langle x+r\theta,\,\theta\rangle\,x + r(|x|^2-r^2)\,\theta\big)\end{aligned}$$

Setting this expression equal to $r\theta(\varphi+\psi)$ yields the following alternative ways of acquiring ψ:

$$\psi+\varphi = \arccos\left(\frac{2\langle x+r\theta,\,r\theta\rangle\,x_1 + (|x|^2-r^2)\,r\cos\varphi}{|x+r\theta|^2}\right) \tag{15}$$

$$= \arcsin\left(\frac{2\langle x+r\theta,\,r\theta\rangle\,x_2 + (|x|^2-r^2)\,r\sin\varphi)}{|x+r\theta|^2}\right) \tag{16}$$

$$= \arctan\left(\frac{2\langle x+r\theta,\,r\theta\rangle\,x_2 + (|x|^2-r^2)\,r\sin\varphi}{2\langle x+r\theta,\,r\theta\rangle\,x_1 + (|x|^2-r^2)\,r\cos\varphi}\right). \tag{17}$$

Using the mapping

$$\Gamma_\varphi^{-1}(\sigma,\psi) = r\theta(\varphi+\psi) + \sigma\theta\big(\varphi+\delta(\psi)\big),$$

we can calculate the first factor $|\partial_\sigma\Gamma_\varphi^{-1}|$ in (11) as

$$|\partial_\sigma\Gamma_\varphi^{-1}(\sigma,\psi)| = |\theta\big(\varphi+\delta(\psi)\big)| = 1,$$

On the other hand, due to rotational symmetry, we can consider the case $\varphi=0$ and calculate the determinant of the Jacobian $\partial(\sigma,\psi)/\partial x$. After lengthy derivations, one can finally acquire

$$\det\frac{\partial(\sigma,\psi)}{\partial x} = -\frac{4\langle r\theta,\,x+r\theta\rangle}{|x+r\theta|^4},$$

which is invariate under simultaneous rotations of x and θ as expected. Therefore, the adjoint is given as

$$\mathcal{D}^*g(x) = \int_0^{2\pi}\frac{4\langle r\theta(\varphi),\,x+r\theta(\varphi)\rangle}{|x+r\theta(\varphi)|^4}\,g(\varphi,\Pi_{\gamma_\varphi}x)\,\mathrm{d}\varphi \tag{18}$$

with the projection $\Pi_{\gamma_\varphi}x=\psi$ given by one of the formulas (15)–(17).

$\square$